Theoretical Economics 18 (2023), 1511–1546 1555-7561/20231511

Attack and interception in networks

Francis Bloch
Université Paris 1 and Paris School of Economics

Kalyan Chatterjee
Department of Economics, Pennsylvania State University

Bhaskar Dutta
Department of Economics, University of Warwick and Ashoka University

This paper studies a game of attack and interception in a network where a single attacker chooses a target and a path, and each node chooses a level of protection. We show that the Nash equilibrium of the game exists and is unique. We characterize equilibrium attack paths and attack distributions as a function of the underlying network and target values. We show that adding a link or increasing the value of a target may harm the attacker—a comparative statics effect that is reminiscent of Braess's paradox in transportation economics. Finally, we contrast the Nash equilibrium with the equilibrium of a variant of the model: one where all nodes cooperate in interception.

Keywords. Network interdiction, networks, attack and defense, inspection games.

JEL classification. C72, D85, K42.

1. Introduction

Networks are often used to transport bombs, drugs, and other contraband goods. Preventing or stopping the transportation of illegal and dangerous objects on networks has long been the goal of army and police forces as well as customs agents. In this paper, we analyze a game between an attacker who chooses a target node in the network and nodes who try to deter the attack.

This issue is connected to the vast literature in operations research on network interdiction. Models of network interdiction involve two players: the *interdictor*, who changes the structure of the network (for example, placing detection devices, destroying links, or limiting capacities on links), and the *evader*, who uses the network to transport objects from a source to a sink. Typically, the game played has a Stackelberg structure:

Francis Bloch: francis.bloch@univ-paris1.fr
Kalyan Chatterjee: kchatterjee@psu.edu
Bhaskar Dutta: b.dutta@warwick.ac.uk

© 2023 The Authors. Licensed under the Creative Commons Attribution-NonCommercial License 4.0.
Available at https://econtheory.org. https://doi.org/10.3982/TE5122

the interdictor moves first and the evader moves second. The objectives of the two players are diametrically opposed: the evader wants to maximize the flow of goods or minimize the length of the path between the source and the sink, whereas the interdictor wants to minimize the evader's objective.

The problem that we focus on in this paper is different in two important respects. First, we assume that the attacker endogenously chooses its target. In models of network interdiction, the source and sink are exogenously given. While this assumption is easily justified in some contexts, for instance, troop movements or delivery of goods to specific locations, it is less likely to be justified for terrorist organizations or drug smugglers. They typically choose their target destination(s) endogenously, at least partly to make it more difficult for their opponent(s) to prevent successful attacks. Second, we consider a decentralized structure where each node only cares about the possible damage to itself, whereas in models of network interdiction, a single defender controls all the nodes.

Formally, we model a $(n+1)$-person game between an attacker A and n defenders with different values. The attacker and defenders are connected in an exogenously given network. The attacker chooses a target and an attack path from its location to the target. Each defender can invest in an interception technology to stop the attack at quadratic cost. The objectives of the attacker and the nodes are diametrically opposed. If the attack is successful, the attacker captures the value of the target, and the target loses its value. However, the game is not zero sum because the nodes also incur a cost of interception that does not appear in the attacker's payoff. All players choose their strategies simultaneously. The appropriate solution concept for this strategic situation is Nash equilibrium.

The game we study involves complex strategic interactions between the $n+1$ players. The attacker's choice of targets and attack paths depends on the interception investments of the nodes. The attacker is less likely to attack a node with higher interception investment and to choose instead an attack path where nodes choose lower interception investments. Each target's interception investment depends both on the attacker's strategy and on the interception investments of the other nodes. Given the assumption that nodes are only affected by attacks that target them, a node's interception investment is increasing in the probability of being attacked, and nodes that are never attacked never choose to protect. Interception investments of different nodes are strategic substitutes. If other nodes intercept more, a target node's incentive to intercept is reduced as the node is less likely to be reached on any attack path. The pattern of externalities across target nodes depends on the attacker's choice of attack paths.

Our main result shows that the game always admits a *unique* Nash equilibrium. We identify conditions under which this unique equilibrium is in pure strategies. Essentially, a pure strategy equilibrium exists if and only if one target node has sufficiently high value so that it is worthwhile to attack even if the node is heavily defended. Otherwise, the unique equilibrium involves the attacker mixing over targets and attack paths. Our analysis also casts light on the equilibrium strategies. We show that when a generic condition of target values is satisfied, the attacker only uses *one* path to attack a node in the support even if there are possibly multiple paths of attack. We characterize the equilibrium attack tree, using the fact that if a node is preceded by several targets, it will always be attacked through the preceding target with lowest value. Equilibrium attack probabilities are characterized as the unique solution to a system of nonlinear equations. The difficulty in fully characterizing equilibrium strategies stems from the difficulty in identifying the set of nodes that are attacked with positive probability. Once the support of the attack distribution is given, equilibrium attack paths and probabilities are easily characterized.

We then go on to describe some comparative statics results. First, we describe how the equilibrium outcomes change when there is a *small* increase in the value of a target node. It turns out that the changes in equilibrium payoff to the attacker can be counterintuitiv: the attacker's equilibrium payoff may be strictly lower when the value of a node increases. Second, we construct examples showing that an analogue of Braess's paradox can occur in our model: the addition of a new link may actually decrease the expected payoff to the attacker. We also show that the attacker cannot benefit from committing not to attack certain targets on a line.

Next, we look at an alternative formulation where a single *centralized* agency coordinates interception. We focus on the line and show that the equilibrium is unique. We also identify conditions under which all nodes in the line will be attacked in equilibrium and compare the equilibrium outcomes under the cooperative and non-cooperative formulations. Not surprisingly, the attacker is worse off in the case of centralized defense. We show that all nodes but the first target are better off in the case of centralized defense, but that the first target may either be better off or worse off. Hence, the first target may need to be subsidized to accept participation in a centralized defense scheme. We also show that if an extra link is added from node 0 to any other node, then, in contrast to the benchmark case, the attacker is never worse off.

Our analysis applies to many situations where an attacker chooses both the target and route in the network, while defenders protect the nodes. A prime example is a terrorist bombing attack: the terrorist chooses the route that the bomb will follow and the place where the bomb will explode, while security agents inspect checked luggage (and suspicious transit luggage) at every airport. The Pan Am 103 crash over Lockerbie on December 21, 1988 illustrates this situation: the bomb was planted in luggage checked in Gozo (Malta) on Malta Airlines, and was then transferred to Frankfurt and London, where it was put on the Pan Am flight to New York. The bomb was designed to explode as soon as the airplane reached an altitude of 28,000 feet, which would happen only on the transatlantic flight. This attack clearly showed that terrorists chose the attack route as well as the target, taking into account the degree of protection in the different airports, and realizing that inspection at the initial point in Malta and at transfer airports would be lax. It also highlighted the importance of cooperation among airport security authorities to foil terrorist attacks.[1]

Another example is drug smuggling from Mexico into the United States, as studied by Dell (2015): drug lords choose the road and the U.S. entry point of drug packages, while local police authorities act to disrupt drug traffic. Dell's (2015) analysis shows

[1]Heal and Kunreuther (2005) propose a simple model to study terrorist attacks and interdependence of airport security protection efforts.

that the election of new mayors from the National Action Party (Partido Acción Nacional (PAN)), with a strong stance on crime, led drug lords to divert traffic routes away from some municipalities. The recent wave of irregular migration into Europe can also be analyzed using our theoretical model. Migrants choose both the destination and transit route, while European governments try to prevent migrants from crossing borders. Migration routes into Europe have been documented, and the choice of destination and transit routes has been extensively studied.[2] Finally, our model can be used to explain the strategies chosen by a law enforcement authority to infiltrate a terrorist or criminal network. The nodes of the network are terrorists or criminals, who invest in protecting themselves from being captured by the police. The attacker is an infiltrator who moves in the network so as to arrest a terrorist or criminal. The value of a node is interpreted as the importance of the agent in the terrorist or criminal network.[3]

2. Related literature

Our paper is related to two strands of the literature: the operations research literature on network interdiction and the economics literature on attack and defense in networks. The extensive literature on network interdiction originated with Wollmer's 1964 characterization of the arc to be removed to minimize the flow between a source and a sink in the network. Three types of problems have been considered. In short path interdiction (Golden (1978)), the objective of the evader is to minimize the length of the path between the source and the sink. In most reliable path interdiction (Washburn and Wood (1995)), the interdictor places detection devices on the edges, and the objective of the evader is to minimize the probability of detection. In network flow interdiction (McMasters and Mustin (1970), Ghare, Montgomery, and Turner (1971), and Fulkerson and Harding (1977)), edges are capacitated and the objective of the evader is to maximize the flow between the source and the sink. The applications range from the disruption of enemy troop movement (McMasters and Mustin (1970) and Ghare, Montgomery, and Turner (1971)) to drug smuggling (Wood (1993) and Washburn and Wood (1995)) and the detection of nuclear material (Morton, Pan, and Saeger (2007)). The literature is very clearly summarized by Collado and Papp (2012) and surveyed by Smith and Song (2020).

A major point of departure between the operations research literature and our paper stems from the questions asked and methods used. The literature on network interdiction focuses attention on the complexity of the integer and linear programming problems involved in network interdiction, and studies algorithms to find exact or approximate solutions. Instead, we study attack and interception as a game, provide an exact characterization of equilibrium, and compute the comparative statics effects of changes in the parameters of the problem.

Economists, using formal game theoretic models, have also contributed to the literature on conflict in networks. Some of these papers focus on network design and

[2] See Dustmann, Fasani, Frattini, Minale, and Schönberg (2017) for an overview of the economics of refugee migration into Europe, and references to the choice of migration routes.

[3] We thank an anonymous referee for suggesting this application of our model.

emphasize trade-offs between connectivity and ease of external disruptions. The literature started with Dziubiński and Goyal (2013), who model attacks on infrastructure. The problem is modeled as a two-person–two-stage game between a designer (D) and an adversary (A). In the first stage, D builds a network among n links and chooses which nodes to defend. There is a cost of both building links and defending nodes. In the second stage, A observes the network and defense strategy of D, and attacks up to k unprotected nodes. The objective of D is to maximize the value of the residual network at minimum cost; the objective of A is to minimize the value of the residual network. The paper focuses on the subgame perfect equilibria of the game. The optimum strategy for D depends uon the relative costs of defense and building links. If it is relatively cheaper to build links, then D will build denser networks and leave several nodes unprotected. The higher is link cost relative to cost of defense, the optimum strategy is to build a sparser network but defend relatively more nodes.

Dziubiński and Goyal (2017) also model attack and defense in infrastructure networks, but the network is exogenously given. The sequential structure of the game and payoff functions remain similar. However, the arbitrariness of the network requires new conceptual tools. The emphasis shifts on the key nodes that need to be defended to ensure efficient functionality. Goyal and Vigier (2014) model hacking and cybersecurity. The defender moves first, constructs the network, and chooses an allocation of defense resources to defend nodes. The attacker then chooses an attack strategy, and decides how to spread through the network by using successful resources. The paper uses a Tullock contest function to model the outcome between the defender and attacker at any node. Successful attacks travel from node to node in the network, the "contagion" representing the spread of computer viruses. Cerdeiro, Dziubiński, and Goyal (2017) modificed Goyal and Vigier. They consider a $(n+2)$-player–three-stage game between a designer (D), n vertices (V), and an intelligent adversary (A). In the first stage, D builds a network g on the n vertices. The network is observed by the nodes in V that simultaneously decide whether to invest in (costly) defense or not. This determines the set P of protected nodes. Finally, A observes (g, P) and chooses a mixed attack strategy. A realization of this lottery is the node to infect, i in V. The infection spreads and eliminates all the unprotected nodes reachable from i in G via a path that lies entirely in $V - P$. The net payoff to a node is an increasing function of the size of its surviving component minus its protection cost if any. The gross payoff to D is the value of the residual network, while the payoff to A is the negative of the residual network.

Finally, Bloch, Dutta, and Dziubiński (2020) study network design in a different game of attack and defense, where the objective of the defender is to hide an object inside the network.

While our paper considers, as in the earlier literature, a game played between an attacker and defender(s) on a network, our model is very different from existing models. The most striking difference is that in all previous papers, the adversary can *directly* attack target nodes, whereas we explicitly take into account paths of attack as an attack may be intercepted before reaching the target. That is, we allow for a significantly broader class of networks in which there may not be any link from the adversary to a specific target: the only path from the adversary to the target must include other potential targets. The consequences of this issue will become clearer once we have described the model in the next section. Second, except for Cerdeiro, Dziubiński, and Goyal (2017), all the other papers model the problem as a two-person problem, where D and A are the two players. This conforms to our analysis in Section 6, where D chooses the level of defense at all nodes. In our benchmark model, each node decides on its expenditure on defense *independently*. Of course, there is strategic interaction between the difference nodes. A target node that has a predecessor on the attack path benefits from the defense outlay of the predecessor since that will reduce the probability that the adversary can successfully avoid capture. Third, all previous papers move *after* observing the pattern of defenses, whereas we assume that the adversary and defenders move simultaneously.

3. The game

We consider a game between an attacker (player 0) and n target nodes, who we also call defenders. There is an exogenously given network $\mathcal{G}$ describing the communication possibilities among the $(n+1)$ players. We let $g_{ij} = 1$ if there exists a direct communication link between players i and j, and $g_{ij} = 0$ otherwise. A *path* p between i and j is a sequence of distinct nodes $i_0, i_1, \dots, i_m$ such that $i_0 = i$, $i_m = j$, and $g(i_k, i_{k+1}) = 1$ for $k = 0, \dots, m-1$. A path thus describes a sequence of moves along the network, where none of the locations is visited twice. Without loss of generality, we suppose that the network $\mathcal{G}$ is connected, so that the attacker can reach any target through some path in the network.

For each defender i, there is a *value* b_i that captures the strategic or symbolic importance of a terrorist target or the profitability of the market for drugs or contraband goods. If the attacker succeeds in attacking target i, she receives the positive value b_i. On the other hand, the defender i loses a value d_i. We normalize all values b_i and d_i to be smaller than 1.

The defenders invest in a technology to intercept the attack at the target. We let x_i denote the probability that defender i intercepts the attack, and assume that the technology allowing defender i to intercept with probability x_i has cost $c(x_i)$.

Throughout the paper, we make two simplifying assumptions that result in considerable simplicity of exposition. We assume that for each i,

(i) $b_i = d_i$

(ii) $c(x_i) = x_i^2/2$.

While these assumptions greatly simplify the exposition, they are not necessary for our results. We show in the Appendix that it is straightforward to extend the results to cost functions that are strictly increasing and strictly convex. Moreover, additional genericity assumptions on the vector d (analogous to those we have made on the vector b) also allow us to drop the assumption that $b_i = d_i$ for all i.

In the baseline model, we assume that defenders choose their interception investments non-cooperatively and simultaneously. The attacker chooses a target i and an attack path p from 0 to i. Let $\mathcal{P}$ denote the set of all paths originating at 0 and let $i(p)$

denote the terminal node of path p. Given any path p and $i \in p$, the set of *predecessors* of i in p is $P(p, i) = \{j \in p | j \text{ lies on the path between 0 and } i\}$. Fix a vector of interception investments $x = (x_1, \dots, x_n)$. For any node i contained in the path p, we let $\alpha_i(p)$ denote the probability that the attack along path p reaches node i,

$$\alpha_i(p) = \prod_{j \in P(p,i)} (1 - x_j),$$

so that the probability that the attack with target i along path p is successful is given by

$$\beta_i(p) = \alpha_i(p)(1 - x_i).$$

As we will soon establish, the attacker often uses a *mixed strategy*, choosing a probability distribution π over all paths p in $\mathcal{P}$. The expected utility of the attacker is then given by

$$U = \sum_{p \in \mathcal{P}} \pi(p)\beta_i(p)b_i$$

for targets i in the support of the attacker's mixed strategy.

The expected payoff of defender i is given by

$$V_i = \sum_{p \in \mathcal{P} | i(p) = i} \pi(p)\beta_i(p)(-b_i) - \frac{x_i^2}{2}.$$

Our objective is then to characterize the *Nash equilibria* of the game of attack and interception, where the attacker selects a probability distribution over $\mathcal{P}$ to maximize her expected utility U and each defender i chooses an interception investment x_i to maximize his expected payoff V_i.

Notice that the game incorporates two important features. First, each defender only cares about attacks targeted at him. This assumption implies that a defender who is not in the support of the attacker's strategy does not invest in interception. Second, defenders invest in interception at the target and not along communication links; the interception occurs at the vertices of the graph $\mathcal{G}$ and not at the edges of the graph. Hence, defenders do not discriminate between different paths leading to them.

4. Equilibrium analysis

We now turn to the analysis of the game. We will use the following generic assumption on payoffs.

Assumption 1. *For any two defenders i, j, $b_i \neq b_j$.*

We first note that a Nash equilibrium of the game of attack and interception always exists by appealing to the Debreu–Fan–Glicksberg fixed point theorem.

Theorem 1. *The game of attack and interception on a network always admits a Nash equilibrium in mixed strategies.*

We next consider equilibria in *pure* strategies, where the attacker selects a unique target in the transportation network. In an equilibrium in pure strategies, where the attacker attacks defender i, the other defenders do not invest in an interception technology. Hence, this equilibrium exists when the value of target i is sufficiently high, so that the attacker has no incentive to deviate and attack any other (unprotected) node. The following proposition summarizes the conditions under which a pure strategy equilibrium exists.

Proposition 1. *The game admits an equilibrium in pure strategies if and only if there exists a defender i such that*

(i) $b_i(1-b_i) \geq b_j$ *for all j such that there is a path p, $i \notin P(p, j)$*

(ii) $b_i \geq b_j$ *for all j such that for all paths p, $i \in P(p, j)$.*

Proposition 1 shows that an equilibrium in pure strategies only occurs in very asymmetric situations, when one of the targets has a value that is much larger than the value of any other target. In all other situations, the equilibrium involves a *mixed* strategy of the attacker. We use Δ to denote the set of targets that are attacked with positive probability. We also fix a vector of interception investments $x = (x_1, \dots, x_n)$ satisfying the restriction that $x_i = 0$ if $i \notin \Delta$.

On any path p, for any two nodes $i, j \in p \cap \Delta$, we will say that j is an *immediate* predecessor of i in p if $j \in P(p, i)$ and $(P(p, i) \setminus \{j\}) \cap \Delta \subset P(p, j) \cap \Delta$. In words, j is an immediate predecessor of i in p if j is a predecessor of i along path p and there are no other predecessors of i in $p \cap \Delta$.

We start by establishing two preliminary lemmas on the equilibrium attack strategies.

Lemma 1. *Fix the interception investments x and consider any best response of the attacker to x.*

(i) If $i, j \in \Delta$ and j is an immediate predecessor of i in p, then

$$b_i(1-x_i) = b_j.$$

(ii) If $i, j \in \Delta$ and there exist attack paths p and p' such that i and j are the first targets in paths p and p', then

$$b_i(1-x_i) = b_j(1-x_j).$$

Lemma 1 is a direct consequence of the fact that the attacker must be indifferent between any target in the support Δ. We use this lemma to derive another preliminary result on the equilibrium strategy of the attacker.

Lemma 2. *For any two paths p and p', if i is attacked along paths p and p', then $P(p, i) \cap \Delta = P(p', i) \cap \Delta$.*

Lemma 2 shows that, without loss of generality, we can assume that any defender i is attacked from a single path p in equilibrium. If the attacker uses two paths p and p' to reach a defender i, the two paths only differ on nodes that are not attacked and, hence, never invest in the interception technology. Hence, the probability that the attack is successful is identical along the two paths, $\beta_i(p) = \beta_i(p')$.

Lemma 2 allows us to simplify notation considerably by identifying any attack path p chosen by the attacker with the final target along the path $i(p)$. Instead of characterizing the equilibrium strategy of the attacker as a probability distribution π over *paths*, we can characterize it as a probability distribution q over *targets* in Δ. Moreover, since every target is attacked through only one path, the attacker is connected to all targets in Δ though a *tree* $\mathcal{T} \subseteq \mathcal{G}$ rooted at 0. We call $\mathcal{T}$ the *equilibrium attack tree*. Notice that when there are two targets i and j such that j is a predecessor of i in $\mathcal{T}$, the interpretation is that the attacker uses a path going through j to reach i, and with probability q_j attacks the intermediary target j and with probability q_i attacks the final target i.

Using this notation, we define the expected utility of the attacker as

$$U(q, x_1, \dots, x_n) = \sum_{i=1}^{n} \beta_i q_i b_i \tag{1}$$

and the expected utility of defender i as

$$V_i(q, x_1, \dots, x_n) = -\alpha_i(1 - x_i)q_i b_i - \frac{x_i^2}{2}. \tag{2}$$

The equilibrium interception investment of defender i is then

$$x_i^* = \alpha_i q_i b_i. \tag{3}$$

We introduce one last piece of notation. Since there is effectively only one path to each target, we can now define predecessors of targets without reference to paths. So $P(i)$ will denote the set of predecessors in $\mathcal{T}$. For any target $i \in \Delta$, we let $\delta(i)$ denote the *number of targets* in Δ along the unique path connecting 0 to i in $\mathcal{T}$. So if a node i is the first node attacked on an attack path, then $\delta(i) = 0$. We let Δ_m be the set of all targets with $\delta(i) = m$:

$$\Delta_m = \{i \in \Delta | \delta(i) = m\}.$$

The set Δ_0 is thus the set of *first targets*, whereas Δ_m, $m \geq 1$, denotes a set of targets that are reached through other targets in the attack tree.

We next prove a simple lemma, which allows us to characterize the equilibrium attack tree $\mathcal{T}$ for any fixed support Δ.

Lemma 3. *Suppose that $i \in \Delta_m$ with $m \geq 1$. Let $k \in \Delta$ be the immediate predecessor of i on the unique path from 0 to i in $\mathcal{T}$. Let*

$$J = \{j \in \Delta \text{ there exists a path from } j \text{ to } i \text{ in } \mathcal{G} \text{ that does not intersect } \Delta\}.$$

Then $b_k < b_j$ for all $j \in J$, $j \neq k$.

Lemma 3 shows that target i will be attacked from the target k with the *lowest* value among all the targets for which there exists a path to i that does not intersect Δ. The intuition underlying Lemma 3 is easy to grasp. If there are two targets j and k for which there is a path to i that does not intersect Δ, because the attacker is indifferent among the two targets, the probability of a successful attack at the target with lower value must be higher than the probability of a successful attack at the target with higher value. But this implies that the probability of reaching i is higher through the target with lower value and, hence, the equilibrium attack path must reach i through this target.

For a fixed support Δ, using Lemma 3, we can fully characterize the equilibrium attack tree. Any target that can be reached directly by the attacker must be attacked directly. Any target that can only be reached through other targets will be attacked from the target with lowest value.

Suppose now that the support Δ and attack tree $\mathcal{T}$ are given. The equilibrium attack probabilities q_i, interception investments x_i^*, and equilibrium utility of the attacker U can be computed as solutions to the system of nonlinear equations

$$x_i^* = 1 - \frac{U}{b_i} \quad \text{if } i \in \Delta_0 \tag{4}$$

$$q_i = \frac{1}{b_i}\left(1 - \frac{U}{b_i}\right) \quad \text{if } i \in \Delta_0 \tag{5}$$

$$x_i^* = 1 - \frac{b_{k(i)}}{b_i} \quad \text{if } i \notin \Delta_0 \tag{6}$$

$$q_i = \frac{b_{k(i)}}{b_i U}\left(1 - \frac{b_{k(i)}}{b_i}\right) \quad \text{if } i \notin \Delta_0 \tag{7}$$

$$\sum_i q_i = 1, \tag{8}$$

where $k(i)$ denotes the immediate predecessor of i along the attack path.

To understand these formulas, note that the equilibrium investment levels are obtained from the equations guaranteeing that the attacker is indifferent among the targets in the support, so that

$$b_i(1 - x_i) = U \quad \text{for } i \in \Delta_0$$
$$b_i(1 - x_i) = b_{k(i)} \quad \text{for } i \notin \Delta_0,$$

providing two different expressions, whether i is the first target along an attack path or not. This system of linear equations characterizes the unique equilibrium investment levels of defenders in the support as a function of U.

The optimal choice of interception investments must satisfy (3):

$$x_i^* = q_i \alpha_i b_i \quad \text{for all } i.$$

Recalling that $U = b_i \alpha_i (1 - x_i^*)$, we obtain

$$x_i^* = q_i \frac{U}{1 - x_i^*}$$

so that

$$q_i = \frac{x_i^*(1 - x_i^*)}{U}.$$

Replacing x_i^* with the values in (4) and (6) gives the expressions for the equilibrium attack probabilities, forming a system of nonlinear equations in U.

Because all probabilities q_i are strictly decreasing functions of U, (8) has a unique solution, establishing that the equilibrium utility of the attacker U is unique. This, in turn, implies that once the equilibrium support Δ and equilibrium attack tree $\mathcal{T}$ are fixed, (4)–(8) uniquely determine the equilibrium attack probabilities and interception investments.

The preceding analysis shows that once the equilibrium support Δ is given, the other elements of the equilibrium strategies (attack tree, attack probabilities, and interception investments) are easily characterized. The difficult part of the equilibrium analysis is the identification of the support Δ of targets over which the attacker randomizes. The following lemma provides a partial characterization of the support.

Lemma 4. *For any equilibrium (q, x^*), where Δ is the support of q, consider the following cases:*

(i) Suppose that $i \in \Delta$. Then, if $j \in \Delta \cap P(i)$, $b_j < b_i$.

(ii) Suppose that $i \in \Delta$, and that there exists a path between i and j in $\mathcal{G}$ that does not intersect Δ. Then, if $b_j > b_i$, we must have $j \in \Delta$.

Lemma 4 first shows that along an equilibrium attack path, the values of targets must be increasing: any defender who is preceded by another defender with higher value cannot be attacked in equilibrium. The second part of Lemma 4 identifies the final targets of the equilibrium attack tree: it shows that any target j that has a higher value than a target i and can be reached through i must be attacked when i is attacked. Using Lemma 4, we obtain the main result of this section.

Theorem 2. *Given Assumption 1, the game of attack and interception admits a unique Nash equilibrium.*

Theorem 2 shows that there is a unique equilibrium support Δ and, hence, a unique equilibrium. To prove that the support is unique, suppose, by contradiction, that there were two different equilibrium supports Δ and Δ'. We first show that as one target is attacked under Δ but not under Δ', the equilibrium utility of the attacker must be strictly higher under Δ' than Δ. (The argument is immediate if there is a target that is directly attacked under Δ, but not under Δ', and can be extended to targets that are not directly attacked using a simple recursive argument based on Lemma 4.)

The fact that equilibrium utility is at least as large under Δ as under Δ' and that the supports are different implies that the support Δ' must be strictly contained in the support Δ.

We next show that for any defender who is directly attacked under Δ', the equilibrium interception strategy must be lower under Δ' than under Δ. (Again, the argument is immediate if one considers a target that is directly attacked under both equilibria, and requires a recursive argument based on Lemma 4 when the target is not directly attacked under Δ.)

Next we show, using Lemma 2, that the same attack paths must be used in the two equilibria. Together with the fact that the attacker's equilibrium utility is strictly larger in the equilibrium with support Δ', this implies that all interception investments are lower under Δ' than under Δ. As interception investments are lower, by (3), equilibrium attack probabilities must be lower under Δ' than under Δ. This final observation leads to a contradiction, as the sum of probabilities is equal to 1 in both equilibria and the support Δ' is strictly contained in Δ.

Theorem 2 is based on a proof by contradiction and does not provide a constructive argument to characterize the equilibrium support Δ. Lemma 4 only provides a partial characterization of the equilibrium support. The full identification of the equilibrium support is the most complex part of the equilibrium analysis and we do not have any simple algorithm that can be used to perform this task. When the underlying transportation network $\mathcal{G}$ is a tree, because once a node is attacked, all targets with increasing value following that node are also attacked, the identification of the support amounts to characterizing the first targets on any path from the source. This is a simpler problem, and efficient algorithms can be used to identify the first targets. In the Appendix, we consider the simplest case—when the underlying network is a line—and provide a full, analytical characterization of the equilibrium support Δ.

5. Comparative statics

In this section, we discuss the consequences of changes in the parameters of the model on the equilibrium outcomes. We first analyze the effect of a small increase in the value attached to a single node. Then we go on to construct examples showing that the addition of a link to $\mathcal{G}$ may actually lower the equilibrium value of the attacker, an effect that is reminiscent of Braess's paradox. Finally, we analyze the effect of the elimination of targets and show that there is no commitment value for the attacker on a line.

5.1 *Change in value of a target*

Suppose that the value of a target increases. As long as the values of the attacker and defender are positively correlated, this increase will both make the target more valuable to the attacker and increase the defense outlays of the target node. Hence, the total effect on the attacker's probability of attacking the target and on the attacker's utility is ambiguous. In addition, the increase in the value of a target will have ripple effects on the entire network, as it changes the entire vector of attack probabilities and possibly the support of the attacker's strategy. We consider changes that are small enough so that the support of the attacker's mixed strategy remains identical, and characterize situations under which the increase in the value of a target results in an increase or a decrease in the value of the attacker.

Formally, let Δ be the support of equilibrium attacks. Given $\mathcal{G}$ and the vector of values $b = (b_1, \dots, b_n)$, the equilibrium attack probability vector q and the interception investments x are determined as solutions to a system of equations given by (5), (7), (4), and (6). Hence, except for nongeneric values $(b_1, \dots, b_n)$, for any $i \in \Delta$, there is sufficiently small $\gamma > 0$ such that an increase in b_i by γ will leave the equilibrium support unchanged at Δ. Throughout this subsection, we assume such a small change.

We first study the effect of an increase in the value of a node that is not a first target.

Proposition 2. *Consider an increase in b_i with $i \in \Delta \setminus \Delta_0$. Then the interception investment at node i, x_i, goes up, the interception investments at all immediate successors of i, x_l for $l \succ i$, go down, and the interception investments at all other nodes that are not first targets, x_j for $j \notin \Delta_0$, remain constant.*

In addition, the following statements hold:

- *If $\frac{b_{k(i)}(2b_{k(i)}-b_i)}{b_i^3} + \sum_{l,l\succ i} \frac{b_l-2b_i}{b_l^2} > 0$, then the attacker's utility U increases, the interception investments at all first targets, x_j for $j \in \Delta_0$, decrease, and the attack probabilities q_j decrease for all nodes j that are not equal to i or immediate successors of i.*
- *If, on the other hand, $\frac{b_{k(i)}(2b_{k(i)}-b_i)}{b_i^3} + \sum_{l,l\succ i} \frac{b_l-2b_i}{b_l^2} < 0$, then the attacker's utility U decreases, the interception investments at all first targets, x_j for $j \in \Delta_0$, increase, and the attack probabilities q_j increase for all nodes j that are not equal to i or immediate successors of i.*

Next we analyze the effect of an increase in the value of a first target.

Proposition 3. *Consider an increase in b_i with $i \in \Delta_0$. Then the interception investments at all immediate successors of i, x_l for $l \succ i$, go down, and the interception investments at all other nodes that are not first targets, x_j for $j \notin \Delta_0$, remain constant.*

In addition, the following statements hold:

- *If $\sum_{l,l\succ i} \frac{b_l-2b_i}{b_l^2} > \frac{1}{8}$, then the attacker's utility U increases, the interception investments at all first targets different from i, x_j for $j \neq i$, $j \in \Delta_0$, decrease, and the attack probabilities q_j decrease for all nodes j that are not equal to i or immediate successors of i.*
- *If $\sum_{l,l\succ i} \frac{b_l-2b_i}{b_l^2} < -1$, then the attacker's utility U decreases, the interception investments at all first targets, x_j for $j \in \Delta_0$, increase, and the attack probabilities q_j increase for all nodes j that are not equal to i or immediate successors of i.*

Propositions 2 and 3 provide a characterization of the comparative statics effects of an increase in b_i on all equilibrium outcomes, except for the probability of attack of node i and of the immediate successors of i. First, the interception investments at all nodes that are not first targets are easily characterized: interception investments at node i increase, decrease for the immediate successors of i, and remain constant for all other targets. Depending on the parameters, an increase in the value of a target b_i

can either increase or decrease the attacker's utility. In the first case, the interception investments at first nodes decrease, the sum of attack probabilities at node i and its immediate successor go up, and all other attack probabilities decrease. In the second case, the comparative statics effects on the interception investments at first nodes and equilibrium probabilities are reversed.

5.2 *Adding a link*

We next consider the effect of the addition of a link to the network $\mathcal{G}$. The addition of a link is a discrete change that can affect the equilibrium support. This makes the derivation of general comparative statics results on the addition of a link in the network complicated. In an earlier working paper (Bloch, Chatterjee, and Dutta (2021)), we derive such results under the assumption that the equilibrium support remains unchanged after the addition of the link. In particular, we show that the attacker may be harmed by the addition of a link in the network—a result that echoes Braess's paradox (Braess, Nagurney, and Wakolbinger (2005)) on the effect of the addition of a road on total congestion. We illustrate this effect in our framework through two simple examples.

Our first example focuses on the addition of a link between two nodes $i, j \in \Delta \setminus \Delta_0$.

Example 1. Let $\mathcal{G} = \{01, 12, 13\}$ be a line with three target nodes. Choose initial values $b_1 = 0.30$, $b_2 = 0.31$, and $b_3 = 0.62$.

Using (7) to compute q_2 and q_3, we derive

$$1 - q_1 = \frac{1}{U}\left[\frac{b_1}{b_2}\left(1 - \frac{b_1}{b_2}\right) + \frac{b_2}{b_3}\left(1 - \frac{b_2}{b_3}\right)\right].$$

In addition, $U = (1 - q_1 b_1) b_1$. We thus obtain q_1 and U as the solutions to a nonlinear system of two equations:

$$q_1 = 0.0487 \quad \text{and} \quad U = 0.2956.$$

Suppose now that the link 13 is added to $\mathcal{G}$. Then, following the same steps, we obtain

$$q_1 = 0.0494, \qquad U = 0.2955. \qquad \diamondsuit$$

So, the attacker is worse off after the addition of the new link between 1 and 3. In the next example, we show that a similar phenomenon can occur when a link is added between 0 and a node in the support.

Example 2. Let $\mathcal{G} = \{01, 12\}$ be a line with two target nodes. Choose $b_1 = 0.25$ and $b_2 = 0.80$. Again, using (7) and $1 - q_1 = q_2$, we obtain

$$q_1 = 0.0575, \qquad U = 0.2456.$$

Suppose now the link 02 is added to $\mathcal{G}$. Then both 1 and 2 become first targets, and using (5),

$$q_1 = 0.1281, \qquad U = 0.2420. \qquad \diamondsuit$$

The fact that the addition of a new link may hurt the attacker is at first glance counterintuitive since the attacker could choose not to use the new link. However, the very existence of the new link changes the incentive structure of the problem, modifying the incentives for the attacker and the equilibrium defenses of the targets.[4] As we remarked earlier, this result is reminiscent of Braess's paradox in transportation economics, where congestion may increase with the addition of a new link in the transportation network. As in the case of congestion in transportation networks, the equilibrium behavior of all players is affected by the change in the infrastructure network, causing the targets to increase their defense spending and making the attacks less likely to succeed.

5.3 *Eliminating a target*

As a final comparative static exercise, we analyze the attacker's incentive to reduce her action space by committing not to attack a specific target. The elimination of targets changes the incentives of the defenders, and results in a change in the attack probabilities and possibly in the equilibrium support of the attacker. While in general, the attacker may benefit from committing not to attack a target, we show below that the attacker has no incentive to eliminate a target when the underlying network is a line and targets are increasing in value.

Proposition 4. *Suppose that $\mathcal{G}$ is a line and that for all $i \in \{1, \dots, n-1\}$, $b_{i+1} > b_i$. Then the attacker's utility decreases if he commits not to attack a specific target.*

Proposition 4 shows that there is no commitment value for the attacker on a line where successive nodes increase in value. Note that since target values are increasing, the equilibrium support must be a first target i_0 and *all* successors of i_0. We prove the proposition by contradiction. Suppose the equilibrium expected payoff of the attacker is U before commitment and U' after commitment, and $U' > U$. Suppose the first target after commitment is i_0'. Of course, it is possible that $i_0' = i_0$ if the attacker commits not to attack a target $j > i_0$. We know from (7) that equilibrium attack probabilities are decreasing in equilibrium expected payoff of the attacker. We use this fact to show that the probability of a successful attack at i_0' actually goes down. This shows that U' cannot exceed U.

6. Cooperation in interception

In the benchmark model, each of the nodes chooses independently the interception investment. We now consider a model where nodes cooperate in their decisions. The game then becomes a two-person game played between an attacker A and a single defender D whose objective is to minimize the sum of losses of all the nodes.

A major difference between the centralized and decentralized models is that the single defender D protects all nodes on attack paths, including nodes outside the support,

[4] It is of course well known that in a game, a player may be hurt by an expansion of her set of actions. This is exactly what is happening in our game when a new link is added and the attacker's value goes down.

as she internalizes the positive externality of defense of a node on all the targets attacked through that node. Furthermore, because nodes outside the support are protected, it is possible to construct examples to show that the attacker no longer selects a unique path to attack a given target.

The analysis of the game thus becomes more complex, and we restrict attention to lines so as to characterize equilibrium. We first show that there is a unique equilibrium. As a first step, we show that equilibrium interception investments are decreasing along the line.

Lemma 5. *For all $i = 1, \dots, n-1$, the equilibrium defense investments y_i, y_{i+1} satisfy $y_i \geq y_{i+1}$ with equality holding if and only if node i is not attacked in equilibrium.*

Suppose i and $i+1$ are successive nodes on the line. Then $\beta_{i+1} = \alpha_i(1-y_i)(1-y_{i+1})$. Hence, the value of β_{i+1} is invariant to the *relative* magnitudes of y_i and y_{i+1}. However, β_i will certainly be higher the larger is y_i. This is the intuition underlying the lemma.

Proposition 5. *Suppose that $\mathcal{G}$ is a line. Then there is a unique equilibrium of the game where nodes cooperate in interception.*

Let us consider a line with an increasing sequence of target values $b_1 < \cdots < b_n$. Let $(y_1, \dots, y_n)$ be the interception investments, let Δ be the support of equilibrium, and let r_i be the probability of attack of a target $i \in \Delta$. The objective of the defender is to minimize the loss function

$$\mathcal{L} = \sum_{i \in \Delta} \prod_{k \leq i} (1-y_k) r_i b_i + \sum_{i \in N} \frac{y_i^2}{2}.$$

The best response of the defender, for all $i \in N$, is given by the first-order conditions

$$y_i = \prod_{k<i} (1-y_k) \sum_{j \in \Delta, j \geq i} \prod_{i<l \leq j} (1-y_l) b_j r_j.$$

Our next results compare the equilibria of the game where nodes cooperate and when interception investments are chosen independently. We first compare the equilibrium supports. According to Lemma 4, all nodes in the increasing sequence following the first target are attacked with positive probability in the decentralized model. Example 3 shows that this is not necessarily the case when the nodes cooperate.

Example 3. Consider a line with three nodes, labeled 1, 2, and 3 with $b_1 = 0.5$, $b_2 = 0.6$, and $b_3 = 0.8$. In the decentralized model, all three nodes belong to the support. In the centralized model, only nodes 1 and 3 are attacked with positive probability. The equilibrium interception investments are given by $y_1 = 0.5$ and $y_2 = y_3 = y = 0.20943$. With these interception levels, the attacker is indifferent between attacking nodes 1 and 3, and does not have an incentive to attack node 2. The description of equilibrium is completed by computing the attack probabilities $r_1 = 0.33772$ and $r_3 = 0.66228$. $\Diamond$

In Example 3, the equilibrium support is larger in the decentralized model than in the centralized model. This may not always be the case, as shown in the next example.

Example 4. Consider a line with four nodes with $b_1 = 0.24$, $b_2 = 0.241$, $b_3 = 0.242$, and $b_4 = 0.50$. In the decentralized model, the equilibrium is in pure strategies with b_4 being the sole target under attack. In the centralized model, both nodes 1 and 4 will be attacked in equilibrium. The equilibrium is characterized by the interception investments $y_1 = 0.24$ and $y_2 = y_3 = y_4 = y = 0.21703$, which make the attacker indifferent between attacking nodes 1 and 4. The equilibrium attack probabilities are given by $r_1 = 0.06839$ and $r_4 = 0.93161$. ♢

Examples 3 and 4 show that there is no general inclusion result comparing the equilibrium support in the decentralized and centralized models. In our next proposition, we provide some welfare comparisons between the two situations, assuming that all nodes are in the support of the attacker's equilibrium strategy in both the centralized and decentralized cases.

To avoid confusion, let $\beta_i^c = (1 - y_1) \cdots (1 - y_i)$ and $\beta_i = (1 - x_1) \cdots (1 - x_i)$. Also, let $\mathcal{L}_i = \beta_i^c b_i r_i + y_i^2/2$ denote the equilibrium expected loss suffered by node i in the centralized model, let $\mathcal{M}_i = \beta_i b_i q_i + x_i^2/2$ be the equilibrium expected loss suffered by node i in the decentralized model, and let $\mathcal{M} = \sum_{i=1}^n \mathcal{M}_i$.

Proposition 6. *Let $\mathcal{G}$ be a line on n nodes and suppose that all n nodes are attacked in both the centralized and decentralized models. Then the following relationships are true:*

(i) $\mathcal{M}_i > \mathcal{L}_i$ *for* $i = 2, 3, \ldots, n - 1$ *and* $\mathcal{M}_n = \mathcal{L}_n$

(ii) $\mathcal{M} > \mathcal{L}$

(iii) A's expected payoff is strictly higher in the benchmark model.

This result shows that nodes 2 to $n - 1$ always benefit from coordinated defense, while node n's expected loss is the same under both the centralized and decentralized interception cases. The attacker always prefers the decentralized scenario, whereas the defenders collectively prefer the centralized scenario.

The only player whose welfare cannot easily be compared in the two cases is the first target, node 1. The node is more heavily defended in the centralized case and so is less heavily attacked. But node 1 also spends more on interception and this increases its cost. We now demonstrate that the net effect on node 1's expected payoff can go either way.

Example 5. Consider three nodes on a line with $b_1 = 0.5$ and $b_3 = 0.8$. First, assume that $b_2 = 0.65$. The equilibrium attack probability of the first target in the decentralized model is given by $q_1 = 0.247$, resulting in an expected payoff of $\mathcal{M}_1 = 0.116$. On the other hand, $r_1 = (\frac{1-b_2}{b_2})^2 = 0.29$ and $\mathcal{L}_1 = 0.197$.

So $\mathcal{M}_1 < \mathcal{L}_1$. However, choose now $b_2 = 0.78$. Then $r_1 = 0.0795$, $q_1 = 0.374$, and

$$\mathcal{M}_1 = 0.170 > 0.145 = \mathcal{L}_1.$$

So node 1's expected loss in the centralized case can be either higher or lower than in the decentralized case. ◊

Example 5 thus shows that the first target can be asked to increase its interception level in the centralized model to a point that decreases her payoff compared to the decentralized scenario. This opens up the possibility that other nodes have to subsidize node 1 (in case $\mathcal{L}_1 > \mathcal{M}_1$) so as to establish coordinated interception. Of course, part (ii) of Proposition 6 shows that the aggregate benefits of nodes 2 to $n-1$ are sufficiently large to more than compensate node 1's loss if any.

6.1 *Adding a link in the centralized model*

We now show that if an additional link $0i$ is added to a line network g, then the attacker can never be worse off. This illustrates another difference between the decentralized and centralized frameworks since Example 2 demonstrated that an additional link from 0 can indeed make the attacker worse off when nodes do not coordinate defense allocations.

Given any line network g, let $U^*(g)$ be the equilibrium payoff of the attacker. As before, Δ and Δ' represent the supports of the attacker's equilibrium mixed strategy corresponding to networks g and g', respectively.

Proposition 7. *Let g be a line network on $\{0, \dots, n\}$, with $i_{k,k+1} \in g$ for $k = 0, \dots, n-1$. Suppose $g' = g \cup \{0i\}$ for some $i > 1$. If $1 \in \Delta$,*[5] *then*

$$U^*(g') \geq U^*(g) \quad \text{with equality only if } 1 \in \Delta' \text{ and } \{i, i+1, \dots, n\} \cap \Delta' = \emptyset.$$

For any network, define

$$\Delta(1) = \{j \in \Delta | 1 \text{ is on the attack path to } j\}.$$

That is, $\Delta(1)$ is the set of nodes that are attacked through node 1 in equilibrium. The key intuition in understanding the difference between the implications of adding a link to cooperative and non-cooperative cases is that in the former, nodes also take into account the probabilities of attack of its successors. We show that

$$y_1 = b_i \sum_{j \in \Delta(1)} r_j.$$

So when $1 \in \Delta$, $y_1 = b_1$, and the attacker's utility is $(1 - b_1)b_1$, an additional link allows the attacker to choose attack paths that do not include node 1. This will reduce the equilibrium defense on node 1 and improve the expected payoff for the attacker.

[5] This assumption simplifies the proof, but is not essential.

Appendix A: Proofs

Proof of Theorem 1. Reinterpret the game as a game with continuous strategy spaces, where the attacker chooses a point q in the $n-1$ dimensional simplex and every defender i chooses an investment $x_i \in \Re_+$. We will use the Debreu–Fan–Glicksberg theorem (Debreu (1952), Fan (1952), and Glicksberg (1952)) to prove existence of an equilibrium in pure strategies of this game.

First note that because $b_i \leq 1$, the strategy space of defender i can be restricted to $[0, 1]$, so that the strategy spaces of all players are compact, convex subsets of Euclidean spaces. Second, an immediate inspection of (1) and (2) shows that the payoffs of the players are continuous in the product of the strategies $(q, x_1, \dots, x_n)$. The payoff of the attacker given by (1) is linear and, hence, quasi-concave in q. Given that the quadratic cost function is convex, the payoff of any defender i given by (2) is a concave function of x_i and, hence, is quasi-concave. All assumptions of the Debreu–Fan–Glicksberg theorem are thus satisfied, and the game admits an equilibrium in pure strategies, which is a mixed strategy Nash equilibrium of the original game of attack and interception. □

Proof of Proposition 1. Suppose that there exists a defender i whose value satisfies the condition. Consider the strategy profile where the attacker chooses $q_i = 1$ and some path to i, defender i chooses $x_i = b_i$, and all defenders $j \neq i$ choose $b_j = 0$. The expected payoff to the attacker is $U = b_i(1 - b_i)$.

Suppose 0 deviates and chooses q' such that $q'_j > 0$, where $j \neq i$ as well as some attack path p to j. We know that $x_j = 0$. We compute the expected payoff of the attacker and show that it is lower than the payoff obtained in the pure strategy $q_i = 1$:

If $i \notin P(p, j)$, then

$$U(q', x) = b_j \leq b_i(1 - b_i).$$

If $i \in P(p, j)$, then

$$U(q', x) = b_j(1 - b_i) < b_i(1 - b_i) \quad \text{since } b_i > b_j.$$

Any defender $j \neq i$ is attacked with probability 0 and, hence, optimally chooses not to invest in the interception technology. Finally defender i chooses x_i to maximize

$$V_i = -b_i(1 - x_i) - \frac{1}{2}x_i^2,$$

resulting in the optimal decision $x_i^* = b_i$.

Suppose now that the game admits an equilibrium in pure strategies, where the attacker chooses $q_i = 1$. As we just argued, defender i then optimally chooses a detection probability $x_i^* = b_i$, resulting in an expected payoff $U = b_i(1 - b_i)$ for the attacker. Moreover, for all $j \neq i$, $x_j^* = 0$. For any j such that there exists a path p such that $i \notin P(p, j)$, a deviation to j gives 0 a payoff of b_j. For all paths p, $i \in P(p, j)$, 0 gets a payoff of $b_j(1 - b_i)$ by deviating. This establishes the necessity of the conditions. □

Proof of Lemma 1. Both equalities follow from the fact that the attacker must be indifferent between attacking any two nodes in Δ. The first equality stems from the observation that if the attacker attacks the two targets i and j, he must receive the same expected payoff by attacking i and j,

$$b_i \beta_i(p) = b_j \beta_j(p),$$

but because j and i are successive targets along path p,

$$\beta_i(p) = \beta_j(p)(1 - x_i).$$

The second equality stems from the fact that if i and j are the first targets on two paths p and p', then $\alpha_i(p) = \alpha_j(p') = 1$, so that indifference implies

$$b_i \beta_i(p) = b_i(1 - x_i) = b_j \beta_j(p') = b_j(1 - x_j). \qquad \square$$

Proof of Lemma 2. Let i be the first target that is attacked along the paths p and p', and has two different predecessors, $j \in p \cap \Delta$ and $k \in p' \cap \Delta$. From Lemma 1, as j and k are both immediate predecessors of i, we must have

$$b_j = b_i(1 - x_i) = b_k,$$

contradicting the fact that $b_j \neq b_k$ for two different defenders j and k. $\square$

Proof of Lemma 3. Suppose by contradiction that there is $j \in J$ with $b_j < b_k$. Since there is a path from j to i that does not intersect Δ, $x_l^* = 0$ for all targets l on this path that are distinct from j and i. Hence, by attacking i through j, the attacker would secure a probability of reaching i of β_j, so that

$$\alpha_i \geq \beta_j. \tag{9}$$

Furthermore, since k and j are both in Δ,

$$\beta_k b_k = \beta_j b_j.$$

So $b_j < b_k$ implies that $\beta_j > \beta_k$. Furthermore, as the attacker chooses to attack i from k, $\beta_k = \alpha_i$. But using inequality (9), we have

$$\beta_j > \alpha_i \geq \beta_j,$$

a contradiction that completes the proof of the lemma. $\square$

Proof of Lemma 4. To prove the first statement, suppose by contradiction that $b_j \geq b_i$. Because j precedes i, $\beta_j = \alpha_j(1 - x_j^*) \geq \alpha_i > \alpha_i(1 - x_i^*) = \beta_i$. Hence,

$$\beta_j b_j > \beta_i b_j \geq \beta_i b_i,$$

contradicting the fact that $i \in \Delta$.

To prove the second statement, suppose by contradiction that $j \notin \Delta$. Then $x_j^* = 0$. Let $p = \{i_1, \ldots, i_K\}$, where (i) $i_1 = i$, $i_K = j$, and $i_k i_{k+1} \in \mathcal{G}$. Suppose for all $k \in p$ and $k \neq i$, $x_k^* = 0$ since $k \notin \Delta$. Then $\beta_i = \beta_j$.

However, because $j \notin \Delta$ and $i \in \Delta$, we must have

$$b_i \beta_i \geq b_j \beta_j,$$

resulting in a contradiction. □

Proof of Theorem 2. We first consider the case analyzed in Proposition 1 that describes the conditions under which there can be a pure strategy equilibrium with only i being attacked in equilibrium.

So suppose there is i such that

(i) $b_i(1 - b_i) \geq b_j$ for all j such that $i \notin P(j)$

(ii) $b_i \geq b_j$ for all j such that $i \in P(j)$.

Suppose by contradiction that there is an equilibrium (q, x^*) such that $q_j > 0$ for some $j \neq i$. Then $x_i^* < b_i$.

Let $i \in P(j)$. Then $\beta_i \leq \beta_j$ and $b_i > b_j$. So

$$\beta_i b_i > \beta_j b_j.$$

This shows that a deviation to attacking i with probability 1 must be profitable.

Next suppose $i \notin P(j)$. Then $\beta_i > (1 - b_i)$ and so a deviation to attacking i with probability 1 gives

$$\beta_i b_i > (1 - b_i) b_i \geq b_j > \beta_j b_j.$$

This establishes uniqueness of equilibrium when the conditions for a pure strategy equilibrium are satisfied.

Next consider the case where there is no pure strategy equilibrium. The strategy of the proof is the following. We suppose that there are two equilibria E and E' with supports Δ and Δ'. We first prove that $\Delta = \Delta'$. We then show that the sequence of targets on equilibrium paths have to be equal in the two equilibria and finally establish that the equilibrium probabilities over the targets have to be equal.

Step 1: The support of targets are equal, $\Delta = \Delta'$. Suppose by contradiction that there exists $i \in \Delta$ such that $i \notin \Delta'$.

Claim 1. *The equilibrium utility in the two equilibria must satisfy* $U' > U$.

Proof. Suppose first that there exists a path to i in $\mathcal{G}$ that does not intersect Δ'. By attacking i along the path, the attacker obtains a payoff b_i. As $i \notin \Delta'$, we must have $U' \geq b_i$. In addition, as i is attacked with positive probability in equilibrium E, $x_i^* > 0$ and $U = \alpha_i b_i (1 - x_i^*) < b_i$, establishing that $U' > U$.

Suppose next that all paths to i in $\mathcal{G}$ intersect Δ'. This is, in particular, true for the equilibrium attack path p to i in E. Let j be the last point in Δ' along path p. By Lemma 4,

$b_j < b_i$. But then, as $i \notin \Delta'$, and there is a path between j and i that does not intersect Δ', by the second part of Lemma 4, $i \in \Delta'$, a contradiction. □

Claim 2. *The supports must satisfy* $\Delta' \subset \Delta$.

Proof. Suppose by contradiction that there exists $i \in \Delta'$, $i \notin \Delta$. Applying the same argument as in the proof of Claim 1, $U > U'$, contradicting the fact that $U' > U$. □

Claim 3. *For any* $i \in \Delta'_0$, $q'_i \leq \alpha_i q_i$.

Proof. Pick a target i that is a first target along some attack path in the equilibrium E'. As $\Delta' \subset \Delta$, $i \in \Delta$. Suppose first that i is a first target in equilibrium E as well, $i \in \Delta_0$. Then, by Claim 1,

$$U' = b_i\bigl(1 - x_i^{*'}\bigr) > U = b_i\bigl(1 - x_i^*\bigr),$$

and by (3) and the fact that $\alpha_i = \alpha'_i = 1$,

$$x_i^{*'} = q'_i b_i, \qquad x_i^* = q_i b_i,$$

yielding the result.

Next suppose that $i \in \Delta_m$ for some $m \geq 1$. Let p be the path in $\mathcal{T}$ from 0 to i in E. Consider the equilibrium path p' in $\mathcal{T}'$ from 0 to i in E'.[6]

Consider equilibrium E. If there is no node in Δ along the path p', then by deviating and attacking i along that path, the attacker obtains a payoff $b_i(1 - x_i^*) > b_i \alpha_i (1 - x_i^*)$ as $\alpha_i < 1$ because there is another node attacked with positive probability before i on path p. This shows that the path p' must intersect Δ.

Let j be the last point in Δ along the path p'. As $i \in \Delta'_0$, $j \notin \Delta'$. We consider two cases. Suppose first that j is on the equilibrium attack path p. If there was a node k between j and i in p, the attacker would have a profitable deviation by attacking i directly from j along path p', resulting in an expected utility $b_i \beta_j (1 - x_i^*) > b_i \alpha_i (1 - x_i^*)$, as $\alpha_i < \beta_j$. Hence, j is the immediate predecessor of i along path p. By Lemma 1 and (3),

$$b_j = b_i\bigl(1 - x_i^*\bigr) \quad \text{and} \quad x_i^* = \alpha_i q_i b_i.$$

Note that $j \notin \Delta'$ and there exists a path to j that does not intersect Δ'. So

$$b_j \leq U' = b_i\bigl(1 - x_i^{*'}\bigr), \quad \text{with } x_i^{*'} = q'_i b_i.$$

Hence, $q'_i \leq \alpha_i q_i$.

Next suppose that j is not on the equilibrium attack path p to i and let k be the last target preceding i on the equilibrium path p. Because p' is a path from j to i that does not intersect Δ, by Lemma 3 and Lemma 1,

$$b_k < b_j \quad \text{and} \quad b_k = b_i\bigl(1 - x_i^*\bigr).$$

[6]Of course, both paths p and p' are in $\mathcal{G}$.

Hence,

$$b_i(1-\alpha_i q_i b_i) = b_k < b_j \le b_i\big(1-q_i' b_i\big).$$

This establishes $q_i' < \alpha_i q_i$. □

Claim 4. *Suppose that $m \ge 1$. Then, for any $i \in \Delta_m'$, if j is the predecessor of i on the equilibrium path p' in E', j is also the predecessor of i on the equilibrium path p in E.*

Proof. Pick a target $i \in \Delta_m'$. We first claim that $i \notin \Delta_0$. Suppose to the contrary that $i \in \Delta_0$. Then there is a path to i that does not intersect Δ. As $\Delta' \subset \Delta$, the path does not intersect Δ' either. But then because $i \in \Delta_m'$ and $m \ge 1$, $b_i(1-x_i^{*'}) > b_i \alpha_i'(1-x_i^{*'})$, so that the attacker has a profitable deviation, establishing a contradiction.

Hence, $i \in \Delta \setminus \Delta_0$. We claim that the immediate predecessor of i on the two equilibrium attack paths must be identical. Suppose by contradiction that j is the immediate predecessor of i on the equilibrium path p' and $k \neq j$ the immediate predecessor of i on the equilibrium path p.

For all l on p' between j and i, as $l \notin \Delta'$, by the second part of Lemma 4, $b_l < b_j$. Hence by the first part of Lemma 4, as $b_j \in \Delta$, $b_l \notin \Delta$. Hence the subpath of p' joining j and i does not intersect Δ. Because $\Delta' \subset \Delta$, the subpath of p joining k and i does not intersect Δ' either. But then, by Lemma 3, we obtain

$$b_j < b_k \quad \text{and} \quad b_k < b_j,$$

a contradiction. Hence the predecessor of i on the two equilibrium attack paths p and p' must be identical. □

Claim 5. *For all $i \in \Delta'$, $q_i' \le q_i$.*

Proof. By Claim 3, the statement is true whenever $i \in \Delta_0'$. Now consider $i \in \Delta_m'$ with $m \ge 1$. By Claim 4, i has a common set of predecessors $i_0, \dots, i_{m-1}$ in the two equilibria.

By Lemma 1, for all $m \ge 1$,

$$b_{i_{m-1}} = b_{i_m}\big(1-\alpha_{i_m}' q_{i_m}' b_{i_m}\big) = b_{i_m}(1-\alpha_{i_m} q_{i_m} b_{i_m}),$$

so that

$$\alpha_{i_m}' q_{i_m}' = \alpha_{i_m} q_{i_m}.$$

Now recall that $\alpha_{i_0}' = \alpha_{i_0} = 1$. We now prove by induction that $\alpha_{i_m}' \ge \alpha_{i_m}$ for all $m \ge 1$. Let $m=1$. By Claim 3, $q_{i_0}' \le q_{i_0}$ so that

$$\alpha_{i_1}' = 1 - b_{i_0} q_{i_0}' \ge 1 - b_{i_0} q_{i_0} = \alpha_{i_1}.$$

Consider then the inductive step. Suppose that $\alpha_{i_{m-1}} \le \alpha_{i_{m-1}}'$. Recall that $\alpha_{i_m} = \alpha_{i_{m-1}}(1-\alpha_{i_{m-1}} q_{i_{m-1}} b_{i_{m-1}})$ and $\alpha_{i_m}' = \alpha_{i_{m-1}}'(1-\alpha_{i_{m-1}}' q_{i_{m-1}}' b_{i_m})$. As $\alpha_{i_{m-1}} q_{i_{m-1}} = \alpha_{i_{m-1}}' q_{i_{m-1}}'$, $\frac{\alpha_{i_m}}{\alpha_{i_m}'} = \frac{\alpha_{i_{m-1}}}{\alpha_{i_{m-1}}'}$. By the induction hypothesis, $\frac{\alpha_{i_{m-1}}}{\alpha_{i_{m-1}}'} \le 1$ and, hence, $\alpha_{i_m} \le \alpha_{i_m}'$.

Finally, using the fact that $\alpha'_{i_m} q'_{i_m} = \alpha_{i_m} q_{i_m}$ for all $m \geq 1$,

$$q'_i = q'_{i_m} \leq q_{i_m} = q_i,$$

concluding the proof of the claim. □

As a final argument for Step 1, note that by Claim 5,

$$\sum_{i \in \Delta'} q'_i \leq \sum_{i \in \Delta'} q_i.$$

Furthermore, by Claim 2,

$$\sum_{i \in \Delta'} q_i < \sum_{i \in \Delta} q_i,$$

so that

$$\sum_{i \in \Delta'} q'_i < \sum_{i \in \Delta} q_i,$$

contradicting the fact that

$$\sum_{i \in \Delta'} q'_i = 1 = \sum_{i \in \Delta} q_i.$$

Step 2: For any target $i \in \Delta = \Delta'$, the sequence of preceding targets is the same in the attack paths p and p'.

Claim 6. *The sets of first targets are identical,* $\Delta_0 = \Delta'_0$.

Proof. Let $i \in \Delta_0$ and suppose $i \in \Delta'_m$ with $m \geq 1$. If there is a path to i that does not intersect Δ', then the attacker has a profitable deviation by attacking i directly under E'. Hence all paths to i must intersect Δ'. But then because $\Delta = \Delta'$, the equilibrium path p to i must intersect Δ, contradicting the fact that $i \in \Delta_0$. Hence, $\Delta_0 \subseteq \Delta'_0$. Reversing the role of Δ_0 and Δ'_0, the same argument shows that $\Delta'_0 \subseteq \Delta_0$ □

Claim 7. *For any* $i \in \Delta_m$, $m \geq 1$, *the preceding target is the same on path* p *and on path* p'.

Proof. Suppose by contradiction that i has two different preceding targets j and k on the paths p and p'. Because $\Delta = \Delta'$, the path from j to i does not intersect Δ' and the path from k to i does not intersect Δ. Hence, by Lemma 3, $b_k < b_j$ and $b_j < b_k$, a contradiction. □

Step 3: For any target $i \in \Delta = \Delta'$, the attack probabilities are the same, $q_i = q'_i$. For a fixed set of targets Δ and an attack tree $\mathcal{T}$, letting U denote the attacker's equilibrium utility, we characterize equilibrium attack probabilities and defense investments as the solutions to the system of equations

$$b_i \prod_{k \preceq i} (1 - x_k) = U$$

$$x_i = \prod_{k \prec i} (1 - x_k) q_i b_i$$

$$\sum_i q_i = 1,$$

where the first equations capture the attacker's indifference over targets, the second equations capture the nodes' best response defense investments, and the last equation guarantees that probabilities belong to the simplex. The attacker's indifference conditions take a different form for first targets and subsequent targets:

$$x_i = 1 - \frac{U}{b_i} \quad \text{if } i \in \Delta_0$$

$$x_i = 1 - \frac{b_k}{b_i} \quad \text{if } i \notin \Delta_0 \text{ with } k(i) \text{ the immediate predecessor of } i.$$

To compute equilibrium probabilities, we multiply each of the equations defining equilibrium defense investments by $(1 - x_i)$ to obtain

$$x_i(1 - x_i) = b_i \prod_{k \preceq i} (1 - x_k) q_i = U q_i,$$

and replacing x_i, we obtain

$$q_i = \frac{1}{b_i}\left(1 - \frac{U}{b_i}\right) \quad \text{if } i \in \Delta_0$$

$$q_i = \frac{b_k}{b_i U}\left(1 - \frac{b_k}{b_i}\right) \quad \text{if } i \notin \Delta_0 \text{ with } k(i) \text{ the immediate predecessor of } i.$$

Let $F(U) \equiv \sum_i q_i(U)$. Since, for every i, $q_i(U)$ is a strictly decreasing function, $F(U)$ is strictly decreasing. Hence the equation $F(U) = 1$ has at most one solution. This concludes the proof of the theorem. □

Proof of Proposition 2. Consider the equations defining the equilibrium distribution over targets and equilibrium interception investments, (4)–(8). From (6), x_i goes up, x_l goes down, and x_j remains constant for any other $j \notin \Delta_0$. Now let $F(U) = \sum_i q_i(U)$. Differentiating F with respect to b_i and using (7),

$$\frac{\partial F}{\partial b_i} = \frac{1}{U}\left[\frac{b_{k(i)}(2b_{k(i)} - b_i)}{b_i^3} + \sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2}\right].$$

Since

$$\frac{dF}{db_i} = \frac{\partial F}{\partial b_i} + \frac{\partial F}{\partial U}\frac{\partial U}{\partial b_i} = 0,$$

then

$$\frac{\partial U}{\partial b_i} = -\frac{\dfrac{\partial F}{\partial b_i}}{\dfrac{\partial F}{\partial U}}.$$

Hence, if $\frac{\partial F}{\partial b_i} > 0$, then $\frac{\partial U}{\partial b_i} > 0$. Next we use (4) to deduce that x_j decreases for all $j \in \Delta_0$ and use (5) to conclude that q_j decreases for all $j \in \Delta_0$ and (7) to show that q_j decreases for all $j \notin \Delta_0$ which is not equal to i or an immediate successor of i. While we conclude that the sum of probabilities $q_i + \sum_{l \succ i} q_l$ must go up, we cannot deduce whether q_i goes up or not.

Similar reasoning gives the opposite result in the case where $\frac{\partial F}{\partial b_i} < 0$. □

Proof of Proposition 3. As in the proof of Proposition 2, a change in b_i does not affect the defense investments of nodes in Δ that are not immediate successors of i by (6). Next, using (5) and (7), we compute

$$\frac{\partial F}{\partial b_i} \equiv G(U) = \frac{2U - b_i}{b_i^2} + \frac{1}{U} \sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2}.$$

Observe that, as opposed to the case of Proposition 2, the sign of $\frac{\partial F}{\partial b_i}$ depends on the value of U. Hence, there is no necessary and sufficient condition on the target values under which the sign of $\frac{\partial F}{\partial b_i}$ can be established, and we look instead for sufficient conditions.

Consider first the case where $\sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2} > 0$. Then the function $G(\cdot)$ is a convex function of U, with $\lim_{U \to 0} G(U) = +\infty$, $G(b_i) > 0$ and $\frac{\partial G}{\partial U} = \frac{2}{b_i^2} - \sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2 U^2}$. Hence, $G'(U) < 0$ whenever $U < U^* = b_i \sqrt{\frac{\sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2}}{2}}$ and $G'(0) > 0$ whenever $U > U^*$. A sufficient condition for $G(U) > 0$ is, thus, that $G(U^*) > 0$. We verify that $G(U^*) > 0$ if and only if $\sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2} > \frac{1}{8}$.

Next suppose that $\sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2} < 0$. Then the function $G(\cdot)$ is an increasing function of U with $\lim_{U \to 0} G(U) = -\infty$. In addition, as $U \le \min_{j \in \Delta} b_j$, $U \le b_i$. Hence, a sufficient condition for $G(U) < 0$ is $G(b_i) < 0$. We verify that $G(b_i) > 0$ if and only if $\sum_{l, l \succ i} \frac{b_l - 2b_i}{b_l^2} < -1$.

Once the sign of $\frac{\partial F}{\partial b_i}$ is established, we follow the same steps as in the proof of Proposition 2 to compute the sign of $\frac{\partial U}{\partial b_i}$ and the comparative statics effects of an increase in b_i on the equilibrium investment levels of first targets different from i and equilibrium attack probabilities for all nodes but i and its immediate successors. □

Proof of Proposition 4. Consider the line graph G with n targets and where $b_i < b_{i+1}$ for all $= 1, \ldots, n-1$.

Suppose the first target in the equilibrium support is i_0. Then the equilibrium support includes all successors of i_0. Let the attacker commit not to attack some potential target j. If $j < i_0$, then the equilibrium remains unchanged, so henceforth $j \ge i_0$.

Case 1: Suppose $j = i_0$. Let (q, x, U) and (q', x', U') be the equilibrium vectors and expected payoffs of the attacker before and after the commitment, respectively. Let the new first target in the support of the attacker's equilibrium mixed strategy be i'_0.

Suppose $U' > U$. Then, from (7), $q'_i < q_i$ for all $i > i'_0$. Hence,

$$q_{i'_0} > \sum_{i=i_0}^{i'_0} q_i,$$

so

$$U' = \beta'_{i'_0} b_{i'_0} < \left(1 - \sum_{i=i_0}^{i'_0} q_i\right) b_{i'_0}.$$

We now calculate $\beta_{i'_0} = \prod_{i=i_0}^{i'_0}(1 - x_i)$,

$$x_{i_0} = q_{i_0} b_{i_0}, \quad \text{and} \quad x_i = \alpha_i q_i b_i \quad \text{for } i > i_0.$$

So

$$\begin{aligned}
\beta_{i'_0} &= \prod_{i=i_0}^{i'_0}(1 - x_i) \\
&> 1 - \sum_{i=i_0}^{i'_0} x_i \\
&= 1 - q_{i_0} b_{i_0} - (1 - q_{i_0} b_{i_0}) q_{i_0+1} b_{i_0+1} - (1 - q_{i_0} b_{i_0})(1 - q_{i_0+1} b_{i_0+1}) q_{i_0+2} b_{i_0+2} - \cdots \\
&> 1 - \sum_{i=i_0}^{i'_0} q_i b_i \\
&> \beta'_{i_0}.
\end{aligned}$$

This contradicts $U' > U$ and so establishes that a commitment not to attack the first node in the equilibrium does not increase the attacker's equilibrium payoff.

Case 2: Suppose $j > i_0$. If the first target now switches to some $k < i_0$, then $U' = (1 - x'_k) b_k < b_k \leq U$, where the last inequality follows from the fact that the attacker does not deviate from the original equilibrium and attack k.

So assume initially that the first target in the equilibrium support remains i_0.

Assume again that $U' > U$. Then, using (7), it follows that

$$q'_i < q_i \quad \text{for all } i \in \{i_0 + 1, \ldots, j - 1\} \cup \{j + 2, \ldots, n\}.$$

Of course, $q'_j = 0$. We now show that $q'_{j+1} = \frac{b_{j-1}}{b_{j+1}U'}(1 - \frac{b_{j-1}}{b_{j+1}}) < q_j + q_{j+1}$. In fact, we prove the stronger statement that $\hat{q}_{j+1} \equiv \frac{b_{j-1}}{b_{j+1}U}(1 - \frac{b_{j-1}}{b_{j+1}}) < q_j + q_{j+1}$.[7] Now

$$
\begin{aligned}
&q_j + q_{j+1} - \hat{q}_{j+1} \\
&\quad = \frac{b_{j-1}}{b_j U}\left(1 - \frac{b_{j-1}}{b_j}\right) + \frac{b_j}{b_{j+1}U}\left(1 - \frac{b_j}{b_{j+1}}\right) - \frac{b_{j-1}}{b_{j+1}U}\left(1 - \frac{b_{j-1}}{b_{j+1}}\right) \\
&\quad = \frac{1}{b_j^2 b_{j+1}^2 U}\left[b_{j-1}(b_j - b_{j-1})b_{j+1}^2 + b_j(b_{j+1} - b_j)b_j^2 - b_{j-1}(b_{j+1} - b_{j-1})b_j^2\right] \\
&\quad = \frac{1}{b_j^2 b_{j+1}^2 U}\left[b_{j-1}(b_j - b_{j-1})b_{j+1}^2 + b_j^2(b_j - b_{j-1})(b_{j+1} - b_j - b_{j-1})\right] \\
&\quad = \frac{1}{b_j^2 b_{j+1}^2 U}\left[(b_j - b_{j-1})(b_{j+1} - b_j)(b_j^2 + b_{j-1}(b_j + b_{j+1})\right] \\
&\quad > 0.
\end{aligned}
$$

Putting all this together, $\sum_{i>i_0} q'_i < \sum_{i>i_0} q_i$ and, hence, $q'_{i_0} > q_{i_0}$. Then

$$x'_{i_0} = q'_{i_0} b_{i_0} > q_{i_0} b_{i_0}.$$

This contradicts $U' > U$.

Finally, if the first target is $i'_0 > i_0$, then the arguments in Case 1 supplemented with the arguments in the previous paragraph conclude the proof of the proposition.

This concludes the proof of the proposition. □

Proof of Lemma 5. Suppose the lemma is false and for some $i < n$, $y_{i+1} > y_i$.

Suppose first that $r_i > 0$. Then consider a change in the strategy of the centralized defender such that $y'_i = y_{i+1}$ and $y'_{i+1} = y_i$. Then $\beta'_i < \beta_i$ and $\beta'_{i+1} = \beta_{i+1}$, while cost remains the same. Clearly, D gains from the move.

If $r_i = 0$, then consider a change in strategies where D equalizes investment of nodes i and $i+1$, so that $y'_i = y'_{i+1} = \frac{y_i + y_{i+1}}{2}$. It is easy to check that D gains again from that deviation, as $\beta'_{i+1} < \beta_{i+1}$. □

Proof of Proposition 5. We first show that all equilibria must have the same support on the line. Consider two equilibria with supports Δ and Δ'.

Let m be such that $b_m > b_i$ for all $i \neq m$. It is clear that $m \in \Delta \cap \Delta'$. Moreover, no node $(m+k) \in \Delta$ since $\beta_m b_m > \beta_m b_{m+k}$. Similarly, $(m+k) \notin \Delta'$. Hence, both supports have the same final target m.

Now suppose that the equilibria have two different supports $\Delta \neq \Delta'$. As the two equilibria have the same final target, there exists a target l such that all targets $k \geq l$ are in both supports Δ and Δ', and there exists a target i preceding l in Δ that does not belong

[7]Note that $\hat{q}_{j+1} > q'_{j+1}$ since $U < U'$.

to Δ' with no target between i and l in Δ'. We distinguish between two cases: (i) there exists a target preceding l in Δ' and (ii) there is no target preceding l in Δ'.

Case 1: There exists a target preceding l in Δ'. Let j be the target preceding i in Δ'. By construction, $j < i$. Because there is no target between j and l in Δ', $b_j = (1 - y'_l)^{l-j} b_l$. We also note that $b_j \leq \prod_{j<k\leq l}(1 - y_k) b_l$. Furthermore, because $y_k \geq y_l$ for all $k < l$, and there exists a node i attacked between k and l, $\prod_{j<k\leq l}(1 - y_k) b_l < (1 - y_l)^{l-j} b_l$. Hence,

$$\left(1 - y'_l\right)^{l-j} = b_j < (1 - y_l)^{l-j},$$

so that $y'_l > y_l$. Finally, because $i \in \Delta$ but $i \notin \Delta'$,

$$b_i = (1 - y_l)^{l-i} b_l \leq \left(1 - y'_l\right)^{l-i} b_l,$$

so that $y'_l \leq y_l$, a contradiction.

Case 2: There is no target preceding l in Δ'. We then compute y'_l as the solution to the equation

$$y'_l = \left(1 - y'_l\right)^{l-1} b_l.$$

Similarly, letting j denote the first target attacked in Δ, we have

$$y_j = (1 - y_j)^{j-1} b_j = (1 - y_j)^{j-1} \prod_{j<k\leq l} (1 - y_k).$$

But as $y_j \geq y_k$ for all $k > j$ with one strict inequality because there is at least one target before l in Δ,

$$(1 - y_j)^{l-1} < y_j < (1 - y_l)^{l-1} b_l.$$

As the function $g(y) = y - (1 - y)^{l-1} b_l$ is increasing, we conclude that $y_j > y'_l$ so that

$$(1 - y_l)^{l-1} b_l > y_j > y'_l = \left(1 - y'_l\right)^{l-1} b_l,$$

showing that $y'_l > y_l$. But as $i \in \Delta$ but $i \notin \Delta'$, $y'_l \leq y_l$, a contradiction.

Finally, we show that for a fixed support Δ, there is a unique equilibrium attack distribution. Let $i = 1$ be the first target. For any two consecutive targets $i - 1$ and i, let $d(i)$ denote the length of the path between i and $i - 1$. The equilibrium attack probabilities and interception investments can be computed as the solutions to the system of equations

$$y_1 = b_1 (1 - y_1)^{d(1)-1} \tag{10}$$

$$b_i (1 - y_i)^{d(i)} = b_{i-1} \quad \text{for } i \in \Delta, i \neq 1 \tag{11}$$

$$y_i = b_i \sum_{j\geq i} r_j \prod_{k\neq i} (1 - y_k) \quad \text{for } i \in \Delta, i \neq 1 \tag{12}$$

$$\sum_i r_i = 1. \tag{13}$$

Notice that (10) and (11) uniquely determine the investment values y_i for all $i \in \Delta$. Given the defense investments, (12) and (13) uniquely determine the equilibrium attack probabilities r_i for all $i \in \Delta$, completing the proof of the proposition. $\square$

Proof of Proposition 6. (i) Since $\mathcal{M}_i = -V_i$, it follows that

$$\mathcal{M}_i = x_i - \frac{x_i^2}{2}.$$

We recall that from the first-order condition of the defender,

$$y_i = \prod_{k<i}(1-y_k) \sum_{j\in\Delta, j\geq i} \prod_{i<l\leq j}(1-y_l)b_j r_j.$$

Using the indifference condition of the attacker over all targets on the line,

$$\begin{aligned} y_i &= \alpha_i r_i b_i + \alpha_i \sum_{j\in\Delta, j>i} \beta_j r_j b_j \quad \text{for } i<n \\ &= \alpha_i r_i b_i \quad \text{for } i=n, \end{aligned}$$

so

$$\begin{aligned} \mathcal{L}_i &= \alpha_i r_i b_i (1-y_i) + \frac{y_i^2}{2} \\ &= y_i(1-y_i) + \frac{y_i^2}{2} - \alpha_i \sum_{j\in\Delta, j>i} \beta_j r_j b_j \\ &< y_i - \frac{y_i^2}{2} \quad \text{for } i<n \\ &= y_i - \frac{y_i^2}{2} \quad \text{for } i=n. \end{aligned}$$

As all nodes are attacked on the line, by the attacker's indifference condition, $x_i = y_i = 1 - \frac{b_{i-1}}{b_i}$ for $i > 1$. So we obtain $\mathcal{M}_i > \mathcal{L}_i$ for $i = 2, 3, \ldots, n-1$ and $\mathcal{M}_n = \mathcal{L}_n$.

(ii) Rewrite $\mathcal{L}$ and $\mathcal{M}$ as

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^{n} \beta_i^c r_i b_i + \sum_{i=1}^{n} \frac{y_i^2}{2} \\ \mathcal{M} &= \sum_{i=1}^{n} \beta_i q_i b_i + \sum_{i=1}^{n} \frac{x_i^2}{2}. \end{aligned}$$

From the attacker's equilibrium condition,

$$\beta_i^c b_i = \beta_j^c b_j \quad \text{and} \quad \beta_i b_i = \beta_j b_j \quad \text{for all } i, j$$

$$
\begin{aligned}
\mathcal{M} - \mathcal{L} &= \beta_1 b_1 - \beta_1^c b_1 + \frac{x_1^2}{2} - \frac{y_1^2}{2} \\
&= (1 - q_1 b_1) b_1 + \frac{(q_1 b_1)^2}{2} - (1 - b_1) b_1 - \frac{(b_1)^2}{2} \\
&= \frac{(q_1 b_1)^2}{2} - q_1 b_1^2 + \frac{(b_1)^2}{2} \\
&= \frac{(b_1)^2}{2} \left(1 + q_1^2 - 2q_1\right) \\
&> 0 \quad \text{for all } q_1 \in (0, 1).
\end{aligned}
$$

(iii) This follows immediately since $\beta_1^c b_1 = (1 - b_1) b_1 < (1 - q_1 b_1) b_1 = \beta_1 b_1$. □

Proof of Proposition 7. First, for any arbitrary network $\tilde{g}$, we know from the defender's first-order condition that

$$
y_1 = r_1 b_1 + \sum_{j \in \tilde{\Delta}(1)} \prod_{1 < l \leq j} (1 - y_l) b_j r_j.
$$

Second, if $1 \in \tilde{\Delta}$, then from the attacker's first-order condition,

$$
b_1 = \prod_{1 < l \leq j} (1 - y_l) b_j \quad \text{for all } j \in \tilde{\Delta}(1).
$$

Hence, if $1 \in \tilde{\Delta}$,

$$
y_1 = b_1 \sum_{j \in \tilde{\Delta}(1)} r_j. \tag{14}
$$

Since $1 \in \Delta$ and $\Delta(1) = \Delta$ on the line network, (14) implies $y_1 = b_1$ and so

$$
U^*(g) = (1 - b_1) b_1. \tag{15}
$$

Equation (15) implies that if $1 \in \tilde{\Delta}$, then

$$
U^*(g) \geq U^*(\tilde{g}) \tag{16}
$$

with equality holding only if $\tilde{\Delta}(1) = \tilde{\Delta}$.

To complete the proof, suppose that $1 \notin \Delta'$ or $\{i, i+1, \ldots, n\} \cap \Delta' \neq \emptyset$.

If $\Delta'(1) = \emptyset$, then $y_1' = 0$ and

$$
U^*\left(g'\right) \geq b_1 > (1 - b_1) b_1 = U^*(g).
$$

So suppose $1 \notin \Delta'$ but $\Delta'(1) \neq \emptyset$.

Let i be the node closest to 1 that is in $\Delta'(1)$. Then

$$
b_i \left(1 - y_i'\right)^{i-1} \geq b_1 \quad \text{since } \beta_i' = \left(1 - y_i'\right)^i \text{ and } y_1' = y_i'.
$$

If $i \in \Delta(1)$, then from the attacker's equilibrium condition,

$$b_i \prod_{1<l\leq i} (1-y_l) = b_1.$$

Hence,

$$(1-y_i')^{i-1} \geq \prod_{1<l\leq i} (1-y_l).$$

But we know from Lemma 5 that $b_1 = y_1 > y_l$ for all $l > 1$. So

$$(1-y_i') > (1-y_1).$$

Hence, since $b_i \in \Delta$ and $b_i > b_1$,

$$U^*(g') = (1-y_i')^i b_i > (1-y_1)b_1 = U^*(g).$$

A slight modification of the argument will again show that $y_i' < y_1$ if $i \notin \Delta(1)$ and, hence, establish that $U^*(g') > U^*(g)$.

Finally, if $1 \in \Delta'$ but $\{i, i+1, \dots, n\} \cap \Delta' \neq \emptyset$, then $\Delta'(1)$ is a strict subset of Δ'. Hence, $\sum_{j\in\Delta'(1)} r_j < 1$. Equation (14) now establishes that

$$U^*(g') > U^*(g). \qquad \square$$

Appendix B: Robustness checks

We suppose that the values of the targets are different for the attacker (b_i) and for the defender (d_i), and that the cost of the interception technology is $c(x_i)$ with $c'(\cdot) > 0$ and $c''(\cdot) > 0$. We check that the main results of the analysis remain true in this more general setting. First notice that (2) will now become

$$V_i(q, x_1, \dots, x_n) = -\alpha_i(1-x_i)q_i d_i - c_i(x_i),$$

so that instead of (3), the optimum level of x_i now satisfies

$$c_i'(x_i) = \alpha_i q_i d_i.$$

Of course, x_i is increasing in q_i.

The proofs of Lemmas 1 and 2 are unaffected: they follow from the observation that the attacker's expected utilities from attacking different nodes in the support of his equilibrium strategy must be equal. In the statement of Proposition 1, statement (i) will become

$$b_i(1-d_i) \geq b_j \quad \text{for all } j \text{ such that there is a path } p, i \notin P(p, j)$$

and the qualitative result does not change.

The unicity of equilibrium (Theorem 2) also remains true. The proof requires a small adaptation. Lemmas 3 and 4 follow straightaway since these only depend upon A's equilibrium behavior and the fact that $x_j^* = 0$ if j is not in the support of the attacker's equilibrium strategy. We now proceed to show that for a fixed support and attack path, there is a unique equilibrium attack probability distribution q.

Fix some q_i as a "numeraire" probability for some node i. Then $U = (1 - x_i)b_i$. Since x_i is increasing in q_i, U is decreasing in q_i.

For all $j \in \Delta_0$, $j \neq i$,

$$U = (1 - x_j)b_j.$$

Then q_j is decreasing in U and, hence, increasing in q_i.

Also, $\forall j \notin \Delta_0$,

$$\begin{aligned} x_j &= \prod_{k<i}(1 - x_k)q_j d_j \\ &= \frac{U}{b_j} q_j d_j, \end{aligned}$$

so

$$q_j = \frac{b_j x_j}{d_j U}.$$

So q_j is strictly increasing in q_i. Hence, as before, there must be a unique vector q since the probabilities add up to 1.

We now come to the results of Section 6 where target nodes cooperate in defense. First notice that the proof of Lemma 5 only depends on strict convexity of the cost function. Moreover, the proof of Proposition 4 relies on Lemma 5. Hence, the proof goes through with the weaker assumptions.

Second, consider the proof of Proposition 5. Note that $x_i = y_i$ for all $i > 1$ continues to hold since this comes from the attacker's equilibrium condition. It will also follow from the defenders' first-order conditions that

$$\begin{aligned} c'(y_i) &= \alpha_i r_i d_i + \alpha_i \sum_{j \in \Delta, j > i} \beta_j r_j d_j \quad \text{for } i < n \\ &= \alpha_i r_i d_i \quad \text{for } i = n, \end{aligned}$$

so

$$\begin{aligned} \mathcal{M}_i &= c'(x_i)(1 - x_i) + c(x_i) \\ \mathcal{L}_i &< c'(y_i)(1 - y_i) + c(y_i) \quad \text{for } i = 2, \dots, n-1 \\ \mathcal{L}_i &= c'(y_i)(1 - y_i) + c(y_i) \quad \text{for } i = n. \end{aligned}$$

Hence, (i) follows since $x_i = y_i$.

Appendix C: Equilibrium support on a line

Proposition 8. *Let $\mathcal{G}$ be a line and let $b_1, \dots, b_n$ be the increasing sequence of targets along the line. Let b_i be the first target such that*

$$\sum_{j=i+1}^{n}\left(1-\frac{b_{j-1}}{b_j}\right)\frac{b_{j-1}}{b_i b_j} \leq 1.$$

Then the equilibrium support is $\Delta = \{b_i, \dots, b_n\}$.

Proof of Proposition 8. In a line with first target b_i, the equilibrium distribution is given by

$$q_i = \frac{1}{b_i}\left(1-\frac{U}{b_i}\right)$$
$$q_j = \frac{b_{j-1}}{b_j U}\left(1-\frac{b_{j-1}}{b_j}\right) \quad \text{for } j > i.$$

Let

$$F(U) \equiv q_i(U) + \sum_{j>i} q_j(U) = \frac{1}{b_i}\left(1-\frac{U}{b_i}\right) + \sum_{j>i}\frac{b_{j-1}}{b_j U}\left(1-\frac{b_{j-1}}{b_j}\right).$$

We compute

$$F(b_i) = \sum_{j=i+1}^{m}\left(1-\frac{b_{j-1}}{b_j}\right)\frac{b_{j-1}}{b_i b_j}$$

and

$$F(b_{i-1}) = \frac{1}{b_i}\left(1-\frac{b_{i-1}}{b_i}\right) + \sum_{j>i}\frac{b_{j-1}}{b_j b_{i-1}}\left(1-\frac{b_{j-1}}{b_j}\right)$$
$$= \sum_{j=i}^{m}\left(1-\frac{b_{j-1}}{b_j}\right)\frac{b_{j-1}}{b_{i-1} b_j}.$$

Given that b_i is the first target such that $\sum_{j=i+1}^{m}(1-\frac{b_{j-1}}{b_j})\frac{b_{j-1}}{b_i b_j} \leq 1$, we have

$$F(b_i) \leq 1 < F(b_{i-1}).$$

At an equilibrium, we must have $F(U) = 1$. The function $F(U)$ is strictly decreasing in U and, hence, $b_{i-1} < U \leq b_i$. As $b_{i-1} < U$, for any target b_j in the increasing sequence with $j < i$, $b_j < U$ and, hence, $b_j \notin \Delta$. As $b_i \geq U$, for any $j \geq i$, $0 \leq q_j$, and, hence, every target $j \geq i$ is attacked with positive probability so that $b_j \in \Delta$, completing the proof of the proposition. □

Proposition 8 identifies the first target in a line. If $b_n(1-b_n) \geq b_{n-1}$, then there is no value $i < n$ such that $\sum_{j=i+1}^{n}(1-\frac{b_{j-1}}{b_j})\frac{b_{j-1}}{b_i b_j} \leq 1$. In this case, the equilibrium is a pure strategy equilibrium. Otherwise, if $b_n(1-b_n) < b_{n-1}$, then there exists a unique target i that is the first target such that $\sum_{j=i+1}^{n}(1-\frac{b_{j-1}}{b_j})\frac{b_{j-1}}{b_i b_j} \leq 1$, and, hence, the first target in the equilibrium attack path. Interestingly, the condition identifying target i as a first target not only depends on the value of the target, but also on the value of all subsequent targets in the path. The identification of the first target thus requires the recursive computation of the sum $\sum_{j=i+1}^{n}(1-\frac{b_{j-1}}{b_j})\frac{b_{j-1}}{b_i b_j}$ for all targets i in the increasing subsequence.

One case of interest is the case where the difference in values among two consecutive targets along the line is equal. Suppose $b_i = i\epsilon$ with $0 < \epsilon < \frac{1}{n}$. The value of the first target is then given by the first value i such that

$$(1+\epsilon)i + H_{i+1} \geq n - H_n,$$

where $H_j = \sum_{k=1}^{j}\frac{1}{k}$ is the jth harmonic number.

References

Bloch, Francis, Kalyan Chatterjee, and Bhaskar Dutta (2021), "Attack and interception in networks." Ashoka University working paper. [1524]

Bloch, Francis, Bhaskar Dutta, and Marcin Dziubiński (2020), "A game of hide and seek in networks." *Journal of Economic Theory*, 190, 105119. [1515]

Braess, Dietrich, Anna Nagurney, and Tina Wakolbinger (2005), "On a paradox of traffic planning." *Transportation Science*, 39, 446–450. [1524]

Cerdeiro, Diego A., Marcin Dziubiński, and Sanjeev Goyal (2017), "Individual security, contagion, and network design." *Journal of Economic Theory*, 170, 182–226. [1515, 1516]

Collado, Ricardo A. and David Papp (2012), "Network interdiction—models, applications, unexplored directions." Rutcor Res Rep, RRR4, Rutgers University, New Brunswick, NJ. [1514]

Debreu, Gerard (1952), "A social equilibrium existence theorem." *Proceedings of the National Academy of Sciences*, 38, 886–893. [1529]

Dell, Melissa (2015), "Trafficking networks and the Mexican drug war." *American Economic Review*, 105, 1738–1779. [1513]

Dustmann, Christian, Francesco Fasani, Tommaso Frattini, Luigi Minale, and Uta Schönberg (2017), "On the economics and politics of refugee migration." *Economic Policy*, 32, 497–550. [1514]

Dziubiński, Marcin and Sanjeev Goyal (2013), "Network design and defence." *Games and Economic Behavior*, 79, 30–43. [1515]

Dziubiński, Marcin and Sanjeev Goyal (2017), "How do you defend a network?" *Theoretical Economics*, 12, 331–376. [1515]

Fan, Ky (1952), "Fixed-point and minimax theorems in locally convex topological linear spaces." *Proceedings of the National Academy of Sciences of the United States of America*, 38, 121. [1529]

Fulkerson, Delbert Ray and Gary C. Harding (1977), “Maximizing the minimum source-sink path subject to a budget constraint.” *Mathematical Programming*, 13, 116–118. [1514]

Ghare, Prabhakar M., Douglas C. Montgomery, and Wayne C. Turner (1971), “Optimal interdiction policy for a flow network.” *Naval Research Logistics Quarterly*, 18, 37–45. [1514]

Glicksberg, Irving L. (1952), “A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points.” *Proceedings of the American Mathematical Society*, 3, 170–174. [1529]

Golden, Bruce (1978), “A problem in network interdiction.” *Naval Research Logistics Quarterly*, 25, 711–713. [1514]

Goyal, Sanjeev and Adrien Vigier (2014), “Attack, defence, and contagion in networks.” *The Review of Economic Studies*, 81, 1518–1542. [1515]

Heal, Geoffrey and Howard Kunreuther (2005), “Ids models of airline security.” *Journal of Conflict Resolution*, 49, 201–217. [1513]

McMasters, Alan W. and Thomas M. Mustin (1970), “Optimal interdiction of a supply network.” *Naval Research Logistics Quarterly*, 17, 261–268. [1514]

Morton, David P., Feng Pan, and Kevin J. Saeger (2007), “Models for nuclear smuggling interdiction.” *IIE Transactions*, 39, 3–14. [1514]

Smith, J. Cole and Yongjia Song (2020), “A survey of network interdiction models and algorithms.” *European Journal of Operational Research*, 283, 797–811. [1514]

Washburn, Alan and Kevin Wood (1995), “Two-person zero-sum games for network interdiction.” *Operations Research*, 43, 243–251. [1514]

Wollmer, Richard (1964), “Removing arcs from a network.” *Operations Research*, 12, 934–940. [1514]

Wood, R. Kevin (1993), “Deterministic network interdiction.” *Mathematical and Computer Modelling*, 17, 1–18. [1514]

Co-editor Marina Halac handled this manuscript.

Manuscript received 15 December, 2021; final version accepted 1 September, 2022; available online 1 December, 2022.