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Worst-case equilibria in first-price auctions

Vitali Gretschko
School of Business and Economics, University of Muenster

Helene Mass
Institute for Microeconomics, University of Bonn

The usual analysis of bidding in first-price auctions assumes that bidders know the distribution of valuations. We analyze first-price auctions in which bidders do not know the precise distribution of their competitors' valuations, but only the mean of the distribution. We propose a novel equilibrium solution concept based on worst-case reasoning. We find an essentially unique and efficient worst-case equilibrium of the first-price auction that has appealing properties from both the bidders' and the seller's point of view.

Keywords. Auctions, worst-case equilibria, uncertainty.

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1. Introduction

Consider a bidder preparing a bid for a first-price auction. If her valuation for the auctioned object is private and independently distributed from the valuation of her opponents, the only relevant information for an optimal bid is the bid distribution of her competitors. Ideally, the bidder has access to data from prior auctions to estimate the bid distribution. However, many important auction environments are either one shot or infrequent. For example, auctions in mergers and acquisitions are usually one shot. Spectrum auctions and auctions for sports rights take place infrequently in a changing market environment. Moreover, data on prior auctions is not always revealed by the sellers.

Without data, an alternative way for a bidder to prepare for the auction is to build a model. Within the model, she optimizes her bid against the bid distribution generated by her competitors' bidding strategies and the distribution of their valuations. Typical analyses of first-price auctions assume that the distributions of valuations are common knowledge. Bidding strategies can then be derived by assuming that all bidders choose

Vitali Gretschko: vitali.gretschko@wiwi.uni-muenster.de
Helene Mass: hmass@uni-bonn.de
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optimal bids given the valuation distributions and the other bidders' strategies, that is, assuming that the competitors play a (Bayes–) Nash equilibrium.

In this paper, we analyze first-price auctions without a commonly known distribution of valuations. Bidders consider all distributions of their competitors' valuations with the same mean and lower and upper bound on the support, as feasible. A bidder preparing for such an auction faces two sources of uncertainty: the distribution of valuations and the bidding strategies. Thus, the bidder cannot directly apply Nash equilibrium reasoning, since this merely resolves the uncertainty about bidding strategies given a unique distribution of valuations.

We assume that the bidder resolves the two uncertainties jointly by building a mental model. Specifically, she first resolves the uncertainty about the valuation distribution by choosing a feasible belief about it. To resolve the uncertainty about bidding strategies, she additionally chooses a symmetric bidding strategy for her competitors. Then, given the belief and the assumed bidding strategy, she can calculate a (hypothetical) bid distribution for her competitors and choose a payoff-maximizing bid. We assume that the choice of bidding strategy in the mental model is not arbitrary: we require that if everyone were to share the beliefs of the bidder whose mental model we are considering, the bidding strategy would form a symmetric Nash equilibrium.

Given that the set of feasible distributions is large, so is the set of feasible mental models. To choose between mental models, a bidder prepares for the worst case: she considers the mental model that gives her the worst possible payoff, provided that she chooses the optimal bid in that mental model.

We take an equilibrium perspective on worst-case mental models. That is, we require that if all bidders choose worst-case mental models and bids within their models, the bidders' best-reply to their competitors. We call the resulting equilibrium worst-case equilibrium.

Our main finding is a characterization of a worst-case equilibrium in the first-price auction. In particular, we show that a worst-case equilibrium exists. The payoff of a bidder is minimal in a mental model, in which her competitors bid as high as possible. In this case, it is a best reply of this bidder to also bid as high as possible. Bidders, therefore, share the same beliefs about the strategies of their competitors in the worst case. Overall, existence is then a consequence of the structure of best replies in a first-price auction. Bidders with valuations below the mean of the distribution bid their valuation, while bidders with valuations above the mean bid below their valuation and outbid any bidder with a lower valuation. Thus, the worst-case equilibrium is efficient.

The worst-case equilibrium described above has several appealing properties. First, it is unique for bidders with valuations above the mean. That is, in any other worst-case equilibrium, those bidders must have the same beliefs and strategies. As bidders with valuations below the mean always earn zero payoff in any worst-case equilibrium, we say that the worst-case equilibrium is essentially unique, i.e., it is payoff unique. Second, if a bidder bids according to the worst-case equilibrium strategy, she will outbid all bidders with lower valuations, irrespective of the mental models they choose. Third, the bid distributions generated from the strategies in the worst-case equilibrium coupled with any true distribution of valuations first-order stochastically dominate the bid distributions generated in the same way from any other mental model with an efficient equilibrium. Fourth, from the seller's point of view, the worst-case equilibrium maximizes revenue over all mental models. Moreover, seller revenue in the worst-case equilibrium of the first-price auction is higher than revenue in a second-price auction for any true distribution of valuations.

1.1 *Related literature*

There are several approaches to analyzing the behavior of a bidder whose information about the distribution of her competitors' valuations is consistent with multiple priors. One way is to model the situation as a game of incomplete information with a noncommon prior endowing the bidders with subjective beliefs from the set of feasible distributions. A justification for such an approach is that bidders learn equilibrium play from frequent interaction and from their observation of the empirical bid distributions. Thus, Nash equilibrium with a correct common prior emerges as a steady state. Fudenberg and Levine (1993) conceptualize such learning by introducing a self-confirming equilibrium (SCE), in which a bidder's strategy is a best response to her beliefs about her competitors, and beliefs are consistent with observed bids. Indeed, Esponda (2008) shows that if, in a private-value first-price auction, the seller reveals the two largest bids, then all SCE are Nash equilibria. However, if only the highest bid is revealed, the set of SCE is larger than the set of Nash equilibria. Similarly, Dekel, Fudenberg, and Levine (2004) show that without a common prior, not all SCE are Nash equilibria. Moreover, learning takes time; thus, SCE is not a good approximation for infrequent or one-shot auctions.

Another strand of the literature assumes that bidders are ambiguity averse and have maximin preferences. They evaluate each bid with respect to their competitors' strategies and the valuation distribution that gives them the worst payoff given their bid (Bose, Ozdenoren, and Pape (2006), Bodoh-Creed (2012), Di Tillio, Kos, and Messner (2016), Lang and Wambach (2013), Lo (1998)). For example, Auster and Kellner (2022) analyze the Dutch auction with ambiguity and ambiguity-averse bidders. They use maximin preferences to evaluate bidding strategies. To account for the dynamic setting, they assume prior-by-prior updating of the beliefs, which may lead to dynamic inconsistencies. Thus, ambiguity aversion solves the problem of a multiplicity of priors by focusing on the maximin, giving rise to sharp predictions about bidding behavior. By contrast, in this paper we consider Bayesian bidders who maximize expected utility within their mental models. Thus, our contribution is to provide a useful solution for first-price auctions under ambiguity without dropping the assumption of expected utility maximization.

Knightian preferences, as introduced by Bewley (2002), have recently been used in the analysis of auctions (Chiesa, Micali, and Zhu (2015)). In particular, Koçyiğit, Iyengar, Kuhn, and Wiesemann (2020) introduce the concept of a Knightian Nash equilibrium. A profile of strategies constitutes a Knightian Nash equilibrium if, given the bidding strategies of her competitors, the equilibrium bid of a bidder yields a higher payoff than any other bid coupled with any of the feasible beliefs about her competitors' valuation distributions. Knightian Nash equilibria are not necessarily unique. Koçyiğit et al. (2020) sidestep this issue by analyzing the equilibrium that gives the lowest seller revenue. Thus, their analysis can be viewed as a robustness analysis for different mechanisms. By contrast, we take the viewpoint of the bidders rather than that of the seller. Our worst-case concept yields an essentially unique equilibrium.

All the papers described above consider uncertainty only with respect to valuations, not with respect to strategies. Three papers also consider first-price auctions with strategic uncertainty: Kasberger and Schlag (2023), Kasberger (2020), and Mass (2023). They derive strategies that minimize the maximal loss for any admissible strategy of the competitors. However, unlike in our work, the resulting strategies do not form an equilibrium.

Bergemann, Brooks, and Morris (2017) analyze the first-price auction when the seller is uncertain about the distribution of the bidders' valuations. Bidders have a common prior and know the information structure. Bergemann, Brooks, and Morris (2017) derive bounds on the seller revenue under any information structure among the bidders. In Bergemann, Brooks, and Morris (2019), the authors extend their results to all standard auctions. In contrast to their work, we do not assume that the bidders have a common prior or analyze the seller's problem.

A recent literature focuses on the problem of a mechanism designer who does not have precise beliefs about the participants in the mechanism. The papers in this literature either assume that the participants in the mechanism have a common prior (Azar, Chen, and Micali (2012), Bergemann, Brooks, and Morris (2017)), analyze mechanisms with a single participant (Bergemann and Schlag (2008, 2011), Carrasco, Luz, Kos, Messner, Monteiro, and Moreira (2018), Carroll (2015), Pınar and Kızılkale (2017)), or focus on dominant-strategy incentive-compatible mechanisms (Allouah and Besbes (2020)). All of them sidestep the issue of how participants behave if they face uncertainty about their competitors. By contrast, we focus on bidder behavior under uncertainty and provide a solution concept for such situations. However, we do not consider optimal mechanism design; rather we focus on the first-price auction.

The idea that bidders form a valuation-dependent belief and best reply to their competitors' strategies given this belief within a mental model is related to the work of Gagnon-Bartsch, Pagnozzi, and Rosato (2021), who consider bidders with beliefs biased toward their own valuations. This assumption captures the idea that bidders overestimate how similar their tastes are to others'. In Gagnon-Bartsch, Pagnozzi, and Rosato (2021), each bidder forms a mental model by assigning her own biased belief to all of her competitors, who she assumes play a Nash equilibrium in this hypothetical game of incomplete information. Unlike in our work, the bidders' beliefs about their competitors' strategies are not generically correct. However, the authors find, as we do, that the first-price auction generates a higher revenue than the second-price auction.

2. Model

2.1 *Environment*

There are n risk-neutral and ex ante symmetric bidders competing in a first-price sealed-bid auction for one indivisible object. Before the auction starts, each bidder

$i \in \{1, \dots, n\} = I$ privately observes her valuation $\theta_i \in \Theta = \{0 = \theta^1, \theta^2, \dots, \theta^{m-1}, 1 = \theta^m\}$, with $\theta^k > \theta^l$ for $k > l$. For any vector $(v_1, \dots, v_n)$, we denote by v_{-i} the vector $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$. The valuations are independently distributed with a common distribution. We call this distribution the *true distribution* of valuations; we assume that it is unknown to the bidders. However, it is common knowledge that the mean of this distribution is μ. Hence, every bidder knows that the probability mass function of a bidder's valuations is an element of

$$\mathcal{F}_\mu = \left\{ f : \Theta \to [0, 1] \,\middle|\, \sum_{k=1}^{m} f(\theta^k) = 1 \text{ and } \sum_{k=1}^{m} \theta^k f(\theta^k) = \mu \right\}.$$

In other words, $\mathcal{F}_\mu$ is the set of all probability mass functions over the set Θ with mean μ. We focus on the symmetric case. That is, from the point of view of a bidder, all her competitors have the same distribution.

The assumption that the mean characterizes the set of feasible distributions is frequently used to analyze mechanisms under uncertainty about the distribution of valuations (distributional uncertainty). (For example, see Carrasco et al. (2018), Wolitzky (2016), Azar and Micali (2013), or Pınar and Kızılkale (2017).) There are several ways in which this assumption is interpreted in the literature. First, bidders may have only a limited amount of data for a nonparametric estimation of the true distribution. Second, bidders may acquire information about their own valuations before the auction and may be uncertain about each other's information acquisition technologies. Third, the restriction of the set of possible distributions may be viewed as what makes the problem interesting. Fourth, bidders may learn that their competitors' valuations lie in some neighborhood, but may not be able to quantify the error.

In a first-price auction, the bidders submit bids $b \in \mathbb{R}^+$; the bidder with the highest bid wins the object and pays her bid. Ties are broken in favor of bidders with higher valuations (efficient tie-breaking). Thus, the payoff of bidder i with valuation θ_i and bid b_i, given that the other bids are b_{-i} and valuations are θ_{-i}, is denoted by

$$u(\theta_i, \theta_{-i}, b_i, b_{-i}) = \begin{cases} \theta_i - b_i & \text{if } b_i > \max\limits_{j \neq i} b_j \\ \theta_i - b_i & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i > \max\limits_{j \neq i} \{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i < \max\limits_{j \neq i} \{\theta_j \mid b_j = b_i\} \\ \dfrac{1}{p}(\theta_i - b_i) & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i = \max\limits_{j \neq i} \{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i < \max\limits_{j \neq i} b_j, \end{cases}$$

where θ_j denotes the valuation of bidder j with bid b_j for $j \in \{1, \dots, n\}$ and $p = \#\{j \mid \theta_j = \theta_i \wedge b_j = b_i\}$.

We assume an efficient tie-breaking rule, since this simplifies notation. With a random tie-breaking rule, all results are similar; the main difference is that we need to assume a discrete bid grid (which may be arbitrarily fine) in order to ensure equilibrium existence. With such a bid grid, the equilibrium strategies and beliefs under a random tie-breaking rule differ from those under an efficient tie-breaking rule by at most one step in the bid grid.

A *symmetric (mixed) strategy* β maps each bidder's valuations to a distribution of bids,

$$\begin{aligned}\beta : \Theta &\to \Delta\mathbb{R}^+\\ \theta^k &\mapsto \beta(\theta^k) = G_k,\end{aligned}$$

where $\Delta\mathbb{R}^+$ is the set of all cumulative distribution functions on $\mathbb{R}^+$ and G_k denotes the cumulative distribution function of bids from a bidder with valuation θ^k. That is, $G_k(s)$ is the probability that a bidder with valuation θ^k bids s or lower.

A *pure strategy* for a bidder with valuation θ^k is a mapping

$$\begin{aligned}\beta : \Theta &\to \mathbb{R}^+\\ \theta^k &\mapsto \beta(\theta^k)\end{aligned}$$

from the set of valuations to the set of bids. A pure strategy can be interpreted as a distribution of bids that puts probability weight 1 on a single bid. These definitions involve an abuse of notation, since, in the case of a pure strategy, $\beta(\theta^k)$ denotes an element in $\mathbb{R}^+$, while in the case of a (mixed) strategy, $\beta(\theta^k)$ denotes an element in $\Delta\mathbb{R}^+$. However, in the following discussion, it will always be clear whether β is a pure or a mixed strategy. In addition, we use the notation G_k instead of $\beta(\theta^k)$ in the case of mixed strategies.

2.2 *Worst-case mental models*

To choose a bid, a bidder builds a mental model of the situation. A mental model is a vehicle for a bidder to form a belief about the bid distribution of her competitors and choose an optimal bid. For her mental model, bidder i with valuation θ^k chooses a feasible belief $f^k \in \mathcal{F}_\mu$ about her competitors' valuations. To reason about her competitors' behavior, she assumes that they employ a symmetric bidding strategy β. Given the belief f^k and the assumed bidding strategy β, bidder i can calculate a (hypothetical) bid distribution for her competitors and thereby determine the expected payoff of each bid she may place.

Denote by $\tilde{b}_j^\beta$ the random variable representing the bid of bidder $j \neq i$ given β. That is, $\text{Prob}(\tilde{b}_j^\beta \leq s) = \sum_{l=1}^m f^k(\theta^l)G_l(s)$. Denote by $\tilde{\theta}_{-i}$ the random vector of the valuations of bidders other than bidder i. The expected payoff of bidder i with valuation θ^k, belief f^k, and bid b, given that her competitors employ the symmetric bidding strategy β, is then given by

$$U(\theta^k, f^k, b, \beta) = \mathbb{E}_{\tilde{\theta}_{-i}, \tilde{b}_{-i}^\beta}\big[u(\theta^k, \tilde{\theta}_{-i}, b, \tilde{b}_{-i}^\beta)\big].$$

The bidder can now choose a bid b to maximize $U(\theta^k, f^k, b, \beta)$.

In a mental model, the bidder does not choose an arbitrary symmetric β for her competitors. The chosen bidding strategy β reflects that for all possible valuations, the competitors also best-reply to β and a feasible belief. Thus, a bidder with valuation θ^k not only chooses a feasible belief f^k about the valuations of her competitors, but also assigns a belief to every valuation. If these beliefs are the same across bidders, β constitutes a symmetric Nash equilibrium. Formally, a bidder building a mental model considers a mapping

$$\varphi : \Theta \to \mathcal{F}_\mu$$
$$\theta^l \mapsto f^l,$$

assigning to each valuation θ^l a feasible belief $\varphi(\theta^l) = f^l$, which describes the symmetric and valuation-dependent belief of all bidders. Denote by $\mathcal{G}_\varphi = (I, \mathbb{R}^+, \varphi, u)$ the resulting game of incomplete information. Note that the beliefs are the same across bidders, but not necessarily the same across valuations. That is, bidder i and bidder j with valuation θ^k will have the same belief, but bidder i will not necessarily have the same belief with valuation θ^k as with valuation θ^l. The assumption that every bidder maximizes her utility given these beliefs yields that β is a symmetric Nash equilibrium in $\mathcal{G}_\varphi$.

Definition 1. A bidding strategy β constitutes a *symmetric Nash equilibrium* in undominated strategies of $\mathcal{G}_\varphi$ if for every $k \in \{1, \dots, m\}$ and for every bid $b \in \operatorname{supp}(\beta(\theta^k))$, it holds that

$$b \in \arg\max_{\hat{b} \in \mathbb{R}^+} U\big(\theta^k, \varphi\big(\theta^k\big), \hat{b}, \beta\big) \quad \text{and} \quad b \leq \theta^k.$$

We call β an efficient equilibrium if for all $\theta^k, \theta^l \in \Theta$ with $\theta^k > \theta^l$, for all $b \in \operatorname{supp}(\beta(\theta^k))$, and for all $b' \in \operatorname{supp}(\beta(\theta^l))$, it holds that $b \geq b'$. Together with the efficient tie-breaking rule, an efficient equilibrium implies that the bidder with the highest valuation will win the object in equilibrium.

Any bid strictly above the own valuation induces a payoff of at most zero and is, therefore, weakly dominated by bidding the valuation. Allowing for dominated bids may lead to implausible equilibria. For example, a bidder with a valuation strictly below μ can believe that all other bidders have a strictly higher valuation. In this case, there exist equilibria in which such a bidder wins with probability 0 when placing a bid above her valuation. To simplify exposition, in what follows we write “Nash equilibrium” as shorthand for “Nash equilibrium in undominated strategies.”

We define a mental model as a (hypothetical) game of incomplete information with beliefs φ and a Nash equilibrium β of this game.

Definition 2. A pair $(\mathcal{G}_\varphi, \beta)$ is a *mental model* if β is a Nash equilibrium of $\mathcal{G}_\varphi$. If β is an efficient equilibrium, we speak of a *mental model with an efficient equilibrium*. Denote by $U^k(\mathcal{G}_\varphi, \beta)$ the equilibrium expected payoff of a bidder with valuation θ^k in $(\mathcal{G}_\varphi, \beta)$; that is, $U^k(\mathcal{G}_\varphi, \beta) = U(\theta^k, \varphi(\theta^k), \beta(\theta^k), \beta)$.

Given that the set of feasible beliefs $\mathcal{F}_\mu$ is large, so is the set of feasible mental models. To choose between mental models, the bidder prepares for the worst case. That is, she is pessimistic and constructs a mental model that gives her the worst possible expected payoff given her valuation and provided that she best-replies in her mental model.

Definition 3. A mental model $(\mathcal{G}_{\varphi^w}, \beta^w)_k$ is a *worst-case mental model* for a bidder with valuation θ^k for $k \in \{1, \dots, m\}$ if

$$U^k\big((\mathcal{G}_{\varphi^w}, \beta^w)_k\big) = \min_{(\mathcal{G}_\varphi, \beta) \text{ is a mental model}} U^k(\mathcal{G}_\varphi, \beta),$$

that is, if the mental model minimizes the bidder's expected payoff among all mental models.

Worst-case mental models are not necessarily unique, as a bidder with a valuation θ^k may achieve her worst payoff in different mental models. Moreover, the worst case is defined with respect to the bidder's valuation. Thus, bidders with different valuations may, in principle, arrive at different worst-case mental models.

2.3 *Worst-case equilibria*

Up to this point we have taken the perspective of a single bidder building a mental model of the situation, conditional on her valuation, making certain assumptions about her competitors. In particular, she assumes beliefs for her competitors, that they choose a symmetric bidding strategy, and that this bidding strategy forms a Nash equilibrium given the beliefs. The bidder is pessimistic and constructs a worst-case mental model.

Next we take an equilibrium perspective on worst-case mental models. That is, we require that if all bidders choose worst-case mental models and bids within their models, the bidders best-reply to their competitors. As bidding strategies within mental models constitute a Nash equilibrium, this can be achieved by requiring that all bidders choose the same worst-case mental model.

Definition 4. A mental model $(\mathcal{G}_{\varphi^*}, \beta^*)$ is a *symmetric worst-case equilibrium* if it is the worst-case mental model for all $\theta^k \in \Theta$. That is, it minimizes the expected payoff over all mental models $(\mathcal{G}_\varphi, \beta)$ for all valuations:

$$U^k(\mathcal{G}_{\varphi^*}, \beta^*) = \min_{(\mathcal{G}_\varphi, \beta) \text{ is a mental model}} U^k(\mathcal{G}_\varphi, \beta).$$

Worst-case equilibrium captures the idea that all bidders assume that they prepare for the auction in the same way, that is, by considering worst-case mental models given their valuations. The worst-case equilibrium is not self-defeating: if bidders were informed about all of their competitors' beliefs and bidding strategies, they would not change their bids.

Definition 4 differs from Definition 3 in that the equilibrium worst-case mental model does not depend on the bidders' valuations. Bidders with all valuations choose the same worst-case mental model. Therefore, as bidding strategies within a mental model form a Nash equilibrium, all bidders best-reply to their competitors' bidding strategies.

It is not straightforward that a worst-case equilibrium exists, since bidders with different valuations could obtain their worst payoff in different mental models. A more permissive equilibrium definition would merely require that each bidder best-replies to the bidding strategies of her competitors given her beliefs in some worst-case mental model. Such a definition would allow bidders with different valuations to choose different worst-case mental models as long as each bidder best-replies to her beliefs and the strategies of the other bidders. However, as we will show below, a worst-case equilibrium in the sense of Definition 4 always exists, and, therefore, it is not limiting to require all bidders to choose the same worst-case mental model.

We illustrate the concept by means of a simple example.

Example 1. Consider two bidders with valuations $\Theta = \{0, \frac{1}{2}, 1\}$ and a set of feasible probability mass functions containing two functions $\{f_a = (\frac{1}{8}, \frac{1}{4}, \frac{5}{8}), f_b = (0, \frac{1}{2}, \frac{1}{2})\}$. This setup leads to $2^3 = 8$ different mental models. However, as a bidder with valuation 0 always bids 0 in any mental model, only *four* of the models lead to different behaviors and, thus, different payoffs. Consider first the mental model in which $\varphi(0) = f_a$, $\varphi(\frac{1}{2}) = f_a$, and $\varphi(1) = f_a$, which results in a standard symmetric first-price auction with a common belief f_a. The unique symmetric Nash equilibrium bidding function has the following properties. A bidder with valuation 0 bids 0, a bidder with valuation $\frac{1}{2}$ mixes her bids on $[0, \frac{1}{3}]$, and a bidder with valuation 1 mixes her bids on $[\frac{1}{3}, \frac{3}{4}]$. The expected payoffs of the bidders are 0 for a bidder with valuation 0, $\frac{1}{16}$ for a bidder with valuation $\frac{1}{2}$, and $\frac{1}{4}$ for a bidder with valuation 1.

The beliefs and the payoffs in the three other relevant mental models are

$$\varphi = (f_a, f_b, f_a), \qquad U^0 = 0, \qquad U^{\frac{1}{2}} = 0, \qquad U^1 = \frac{3}{16}$$

$$\varphi = (f_a, f_b, f_b), \qquad U^0 = 0, \qquad U^{\frac{1}{2}} = 0, \qquad U^1 = \frac{1}{4}$$

$$\varphi = (f_a, f_a, f_b), \qquad U^0 = 0, \qquad U^{\frac{1}{2}} = \frac{1}{16}, \qquad U^1 = \frac{1}{3}.$$

For a bidder with valuation 0, each of these mental models is a worst-case mental model. For a bidder with valuation $\frac{1}{2}$, the models with $\varphi = (f_a, f_b, f_a)$ and $\varphi = (f_a, f_b, f_b)$ are worst-case mental models. For a bidder with valuation 1, the model with $\varphi = (f_a, f_b, f_a)$ is a worst-case mental model. Thus, the worst-case equilibrium in the example is the mental model with $\varphi = (f_a, f_b, f_a)$. ◊

For our general setup, we show below that a worst-case equilibrium exists (Proposition 1). It is efficient and unique for bidders with valuations above the mean (Proposition 2). Moreover, the worst-case equilibrium has some appealing properties compared with other mental models a bidder might consider. If a bidder bids according to the worst-case equilibrium bidding strategy, she will outbid all bidders with lower valuations, irrespective of the mental models they choose (Proposition 3). The bid distributions generated from the strategies in the worst-case equilibrium and some true valuation distribution first-order stochastically dominate the bid distributions generated in the same way from any other mental model with an efficient equilibrium (Proposition 4). From the point of view of the seller, the worst-case equilibrium maximizes revenue over all mental models. Moreover, seller revenue in the worst-case equilibrium of the first-price auction is higher than revenue in a second-price auction for any true distribution of valuations (Proposition 5).

3. Worst-case equilibrium of the first-price auction

In this section, we characterize the beliefs and strategies in a worst-case equilibrium and show that it is essentially unique. We denote by $[\underline{b}_k^*, \overline{b}_k^*]$ the support of the bid distribution of a bidder with valuation θ^k in the worst-case strategy β^*. We denote the worst-case belief of a bidder with valuation θ^k by $\varphi^*(\theta^k) = f^{k,*} = (f^{k,*}(\theta^1), \dots, f^{k,*}(\theta^m))$. More generally, for any belief φ, we use the notation $\varphi(\theta^k) = f^k = (f^k(\theta^1), \dots, f^k(\theta^m))$, with $f^k(\theta^l)$ denoting the probability that a bidder with valuation θ^k assigns to one of the other $n-1$ bidders having a valuation θ^l.

We start with a heuristic construction of the equilibrium to explain the underlying intuition. We proceed in two steps, considering (i) bidders with valuations below μ and (ii) bidders with valuations above μ. To organize the argument and highlight the main insights we will present "observations" without formal proofs. At the end of the section, we will summarize the results in our main proposition, Proposition 1. The formal proof of the proposition can be found in the Appendix.

(i) Bidders with valuations below μ If a bidder's valuation is below the mean, it is feasible for her to believe that she has the lowest valuation. In any equilibrium with such a belief, the bidder places a bid equal to her valuation. In this case, she earns a payoff of 0 in equilibrium. Therefore, the belief that one has the lowest valuation must lead to the worst case. To calculate a feasible belief for bidders with valuations $\theta^k \le \mu$, consider θ^z, the lowest valuation strictly greater than μ. The belief that puts strictly positive weight only on θ^k and θ^z induces a best reply of $\beta(\theta^k) = \theta^k$ in every mental model, i.e., for $\theta^k \le \mu$, it holds that $\underline{b}_k = \overline{b}_k = \theta^k$. The probability weight is determined by

$$f^{k,*}(\theta^k) + f^{k,*}(\theta^z) = 1$$

$$f^{k,*}(\theta^k)\theta^k + f^{k,*}(\theta^z)\theta^z = \mu.$$

The unique solution is given by

$$f^{k,*}(\theta^k) = \frac{\theta^z - \mu}{\theta^z - \theta^k}, \qquad f^{k,*}(\theta^z) = \frac{\mu - \theta^k}{\theta^z - \theta^k}.$$

Observation 1. Whenever $\theta^k \le \mu$, the worst-case bidding strategy is $\beta^*(\theta^k) = \theta^k$. Let θ^z denote the lowest valuation strictly larger than μ. A worst-case belief is

$$f^{k,*}(\theta^k) = \frac{\theta^z - \mu}{\theta^z - \theta^k}, \qquad f^{k,*}(\theta^z) = \frac{\mu - \theta^k}{\theta^z - \theta^k}, \qquad f^{k,*}(\theta^l) = 0 \quad \text{for all } l \neq k, z.$$

(ii) Bidders with valuations above μ Now consider the belief and bidding strategy of a bidder with valuation $\theta^k > \mu$. It is infeasible for such a bidder to believe that she has the lowest valuation. Thus, she earns a positive payoff in any mental model. We will show below that bidders whose valuation θ^k is above μ play a mixed strategy without atoms on the interval $[\underline{b}_k^*, \overline{b}_k^*]$ according to a continuous bid distribution G_k^*. Given the bidding strategy, there are two levers to minimize a bidder's payoff: reducing the winning probability and increasing the best reply to her competitors' equilibrium bid.

To minimize the winning probability, it is optimal to maximize the probability weight on θ^k while respecting that the expected value of the valuations is μ. To see that this minimizes the equilibrium payoff of a bidder with valuation θ^k, observe that in a mixed-strategy equilibrium, she is indifferent between all bids in her bidding interval. In particular, her equilibrium payoff is

$$\big(f^{k,*}(\theta^1)+\cdots+f^{k,*}(\theta^{k-1})\big)^{n-1}\big(\theta^k-\overline{b}_{k-1}^*\big).$$

This expression is minimized given μ if the probability weight on θ^k is maximized in her belief. Putting probability weight on types higher than θ^k would require an increase in probability weight on types below θ^k in order to preserve the mean. However, if we simply maximized the probability weight on θ^k, we would distribute the probability weight between valuations θ^k and 0. This would cause a bidder with valuation θ^k to bid 0. Thus, to increase the best reply, the worst-case belief should put just enough probability weight on valuations strictly below θ^k. This is achieved by making a bidder with valuation θ^k exactly indifferent between her equilibrium bids and the bids of the bidders with valuations lower than θ^k. This observation leads to the following system of equations.

Observation 2. The conditions

$$\sum_{l=1}^{k} f^k(\theta^l)=1 \quad \text{and} \quad \sum_{l=1}^{k} f^k(\theta^l)\theta^l=\mu$$

and the equations

$$\begin{aligned}&\left(\sum_{l=1}^{k-1} f^k(\theta^l)\right)^{n-1}\big(\theta^k-\overline{b}_{k-1}^*\big)\\&\quad=\left(\sum_{l=1}^{h} f^k(\theta^l)\right)^{n-1}\big(\theta^k-\overline{b}_{h}^*\big)\quad \text{for all } h\in\{1,\ldots,k-2\}\end{aligned} \tag{1}$$

constitute a system of linear equations. The worst-case belief $(f^{k,*}(\theta^1),\ldots,f^{k,*}(\theta^m))$ is the unique nonnegative solution to this system.

It remains to characterize the bidding functions and their support. The upper endpoint of the bidding interval of a bidder with valuation θ^k is obtained from the equation

$$\big(f^{k,*}(\theta^1)+\cdots+f^{k,*}(\theta^{k-1})\big)^{n-1}\big(\theta^k-\bar{b}_{k-1}^*\big)=\theta^k-\overline{b}_k^*.$$

Recall that bidders with valuations below μ bid their valuation. Thus, we can calculate the endpoints of the intervals inductively, starting with bidders with valuation 0. In particular, the system of equations in (1) depends only on the bid intervals of bidders with lower valuations. Thus, there is no circularity between beliefs and bid distribution in the derivation. The bid distribution G_k^* is defined to be such that every bidder with valuation θ^k is indifferent between every bid in her bidding interval given her belief and the other bidders' strategies; i.e., for every $s \in [\overline{b}_{k-1}^*, \overline{b}_k^*]$, it holds that

$$\begin{aligned}&\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-1}) + f^{k,*}(\theta^k)G_k^*(s)\big)^{n-1}(\theta^k - s)\\&= \big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-1})\big)^{n-1}(\theta^k - \bar{b}_{k-1}^*).\end{aligned}$$

Rearranging yields our last observation.

Observation 3. For every $s \in [\overline{b}_{k-1}^*, \overline{b}_k^*]$, $G_k^*(s)$ is given by

$$G_k^*(s) = \frac{\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-1})\big)\big((\theta^k - \overline{b}_{k-1}^*)^{\frac{1}{n-1}} - (\theta^k - s)^{\frac{1}{n-1}}\big)}{f^{k,*}(\theta^k)(\theta^k - s)^{\frac{1}{n-1}}}. \tag{2}$$

That is, the bid distribution and $\overline{b}_k^*$ make a bidder with valuation θ^k indifferent between all bids in $[\underline{b}_k^*, \overline{b}_k^*]$.

We are now in a position to state our main proposition.

Proposition 1. *The mental model $(\mathcal{G}_{\varphi^*}, \beta^*)$ constitutes a symmetric worst-case equilibrium of the first-price auction. It holds that*

- $\beta^*(\theta^k) = \theta^k$ *whenever* $\theta^k \le \mu$
- $\beta^*(\theta^k) = G_k^*$ *whenever* $\theta^k > \mu$.

That is, all bidders with valuations below μ bid their valuation and all bidders with valuations above μ play a mixed strategy on the interval $[\underline{b}_k^*, \overline{b}_k^*]$ with $\underline{b}_k^* = \overline{b}_{k-1}^*$, according to a continuous bid distribution G_k^* given by (2). In particular, the worst-case equilibrium is efficient.

Let $\theta^z > \mu$ be the lowest valuation strictly larger than μ. A worst-case belief of a bidder with valuation $\theta^k \le \mu$ is given by

$$f^{k,*}(\theta^k) = \frac{\theta^z - \mu}{\theta^z - \theta^k}, \qquad f^{k,*}(\theta^z) = \frac{\mu - \theta^k}{\theta^z - \theta^k}, \qquad f^{k,*}(\theta^l) = 0 \quad \textit{for all } l \ne k, z,$$

while that of a bidder with valuation $\theta^k > \mu$ is given by

$$f^{k,*}(\theta^l) > 0 \quad \textit{for all } l \le k, \qquad f^{k,*}(\theta^l) = 0 \quad \textit{for all } l > k,$$

with $f^{k,*}(\theta^l)$ for $l \le k$ being the unique solution to the system of equations (1).

All proofs can be found in the Appendix. We illustrate the worst-case equilibrium with two bidders for $\Theta = \{0, 0.25, 0.5, 0.75, 1\}$ and $\mu = 0.5$ in Table 1.

TABLE 1. Worst-case equilibrium with two bidders for $\Theta = \{0, 0.25, 0.5, 0.75, 1\}$ and $\mu = 0.5$.

Valuation	$f_0^{k,*}$	$f_{0.25}^{k,*}$	$f_{0.5}^{k,*}$	$f_{0.75}^{k,*}$	$f_1^{k,*}$	Bid
0	$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	0
0.25	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{4}$
0.5	0	0	1	0	0	$\frac{1}{2}$
0.75	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{3}{11}$	$\frac{5}{11}$	0	$G_{0.75} = (12 - 24s)(20s - 15)^{-1}$ on $[\frac{1}{2}, \frac{27}{44}]$
1	$\frac{102}{353}$	$\frac{34}{353}$	$\frac{68}{353}$	$\frac{60}{353}$	$\frac{89}{353}$	$G_1 = (162 - 264s)(89s - 89)^{-1}$ on $[\frac{27}{44}, \frac{251}{353}]$

4. Properties of the worst-case equilibrium

In this section, we derive several properties of the worst-case equilibrium. We argue why these properties are appealing and reinforce the worst-case equilibrium as a solution concept. A solution concept is most useful if it selects a unique equilibrium. Thus, we start by showing that the worst-case equilibrium is unique in the following sense.

PROPOSITION 2. *For any mental model $(\mathcal{G}_\varphi, \beta)$ such that for all $1 \le k \le m$,*

$$U^k(\mathcal{G}_\varphi, \beta) = U^k(\mathcal{G}_{\varphi^*}, \beta^*), \tag{3}$$

it holds that $\beta(\theta^k) = \beta^*(\theta^k)$ and $\varphi(\theta^k) = \varphi^*(\theta^k)$ for all $\theta^k > \mu$.

The worst-case equilibrium yields unique bidding strategies and unique beliefs for all bidders with valuations above the mean. For bidders with valuations strictly below the mean, there are several beliefs and strategies that induce the worst-case payoff of 0. In particular, if such a bidder believes that the lowest possible valuation in her mental model is strictly larger than her own valuation, many different bidding strategies can constitute an equilibrium. One can get rid of this multiplicity by assuming that a bidder needs to place a positive probability on her own valuation in her mental model. In this case, in every worst-case equilibrium, a bidder with a valuation below the mean believes that her competitors with the same valuation bid their valuation; otherwise, she would not obtain the worst-case payoff of 0.

In what follows, we first compare the mental models that arise as part of the worst-case equilibrium to all other potential mental models. We then restrict attention to mental models with efficient Nash equilibria. We close the section by considering properties of the worst-case equilibrium from the point of view of the seller.

4.1 *Comparison to all potential mental models*

If the bidder is uncertain whether she has constructed the correct mental model, she may want to make sure that she will at least win the object whenever all her competitors' valuations are lower than hers. This is an appealing property. Ex post, a bidder may be remorseful after losing to a bidder with a lower valuation, as she could have imitated that bidder's strategy ex ante and won the auction. Losing to bidders with a higher valuation is different, because the imitation of their strategies is not always feasible. The following proposition establishes that if a bidder follows the bidding strategy from the worst-case equilibrium, she will win against all competitors with lower valuations, even if her competitors follow bidding strategies from other mental models.

Proposition 3. *For any mental model* $(\mathcal{G}_\varphi, \beta)$*, it holds for all* $\theta^k, \theta^l \in \Theta$ *with* $\theta^k > \theta^l$*, for all* $b \in \operatorname{supp}(\beta^*(\theta^k))$*, and for all* $b' \in \operatorname{supp}(\beta(\theta^l))$ *that*

$$b \geq b'.$$

That is, if each of the competitors bids according to the bidding strategy in some mental model, a bidder who follows the strategy from the worst-case equilibrium always outbids all bidders with lower valuations.

To establish the result, we demonstrate that the upper bound of the support of the bidding strategy for each valuation is weakly higher in the worst-case equilibrium than any other mental model. We arrive at the result because the worst-case equilibrium is efficient and the upper bound of the bid support of a bidder with valuation θ^{k-1} is equal to the lower bound of the bid support of a bidder with valuation θ^k. Moreover, this shows that the worst-case equilibrium is, in a sense, the unique mental model that satisfies Proposition 3.

4.2 *Comparison to mental models with efficient equilibria*

In this section, we compare the worst-case equilibrium to other mental models with efficient equilibria. Restricting our attention to efficient equilibria allows us to derive sharper results about the worst-case equilibrium. The focus on efficient equilibria is similar in spirit to the earlier restriction to symmetric Nash equilibria: to narrow down the possibilities when building a mental model, a bidder may reasonably conjecture that her competitors will choose from efficient equilibria.

We focus on the empirical bid distributions that arise from equilibrium bidding strategies in the mental models and the true valuation distribution. As we show in the following proposition, the worst-case equilibrium generates a bid distribution that (first-order stochastically) dominates any bid distribution generated by any efficient equilibrium bidding function and any true valuation distribution.

Proposition 4. *For every mental model* $(\mathcal{G}_\varphi, \beta)$ *with an efficient equilibrium and every* $f = (f(\theta^1), \dots, f(\theta^m)) \in \mathcal{F}_\mu$*, denote by*

$$\mathcal{B}_f^\beta(s) = f(\theta^1)G_1(s) + \cdots + f(\theta^m)G_m(s) \tag{4}$$

the empirical bid distribution generated if bidders bid according to β *and the true valuation distribution is* f*, where* G_k *is the bid distribution of a bidder with valuation* θ^k *for all* $1 \leq k \leq m$*. It holds that* $\mathcal{B}_f^{\beta^*}$ *first-order stochastically dominates* $\mathcal{B}_f^\beta$*.*

We show that the bid distribution of every valuation in the worst-case equilibrium bidding strategy β^* hazard-rate dominates the bid distribution in the bidding strategy

β of any other mental model. First-order stochastic dominance follows directly for the respective empirical bid distributions evaluated at the same true valuation distribution.

There are mental models that lead to inefficient equilibria of the first-price auction. If we allow for inefficient equilibria, there is no result similar to Proposition 4. In particular, there exist inefficient equilibria and true valuation distributions such that the worst-case equilibrium bid distribution does not dominate the empirical bid distribution in those equilibria. However, for any inefficient equilibrium, there exists a true valuation distribution such that the worst-case equilibrium bid distribution dominates the bid distribution in that equilibrium.

4.3 *Properties from the seller's point of view*

Now taking the viewpoint of the seller, we compare the revenue from the worst-case equilibrium with the revenue of a first-price auction in any other mental model, as well as with the expected revenue of the second-price auction. Revenue is a function of the true valuation distribution and the bidding strategies of the bidders. That is, for a true valuation distribution f and bidding strategies β, define n independent random variables $B_1, \dots, B_n$ with B_i distributed according to $\mathcal{B}_f^\beta$. Denote by $B_{(1)}$ the first-order statistic of $B_1, \dots, B_n$. The revenue in a first-price auction is given by $\mathbb{E}[B_{(1)}]$. In a second-price auction, we focus on the dominant-strategy equilibrium, in which bidders bid their valuations. In this case, the revenue is the second-order statistic of the valuations implied by the true valuation distribution f.

Proposition 5. *Let $(\mathcal{G}_\varphi, \beta)$ be a mental model with an efficient equilibrium. The revenue in the worst-case equilibrium of the first-price auction is higher than*

(i) the revenue generated in $(\mathcal{G}_\varphi, \beta)$ in a first-price auction and

(ii) the efficient equilibrium of the second-price auction.

The revenue comparison is a direct consequence of Proposition 4: if the bid distribution of the worst-case equilibrium dominates the bid distribution that arises from any mental model with an efficient equilibrium, the worst-case equilibrium also generates higher revenue than any other profile of mental models with an efficient equilibrium. The comparison with the second-price auction works in a similar fashion. In a second-price auction, it is a weakly dominant strategy to bid one's valuation. Thus, the revenue of the second-price auction is independent of the bidders' beliefs but is determined by the true distribution of valuations. However, if every bidder in the first-price auction has the true distribution as her belief, the resulting mental model is revenue-equivalent to the efficient equilibrium of the second-price auction. Again by way of Proposition 4, this equilibrium must yield a lower revenue than the worst-case equilibrium.

5. Conclusion

We analyze first-price auctions with bidders whose information about their competitors is consistent with all distributions having the same mean. To deal with such uncertainty, we introduce the notion of a worst-case equilibrium, which is based on the idea that bidders build mental models of the situation and are pessimistic. If all bidders reason in the same way, they evaluate the situation with the same mental model.

Our results can be extended to the case in which the bidders' information is consistent with all distributions satisfying $E(h(\theta)) = \mu$ as long as h is a strictly increasing function. In particular, the results can be extended to the case where the bidders' information is consistent with some moment of the distribution other than the mean. In such a case, there is a threshold, depending on h and μ, such that bidders with valuations below this threshold bid their valuation. The proofs for this case follow the same steps as the proofs of our original results, from which they can easily be adapted.

The notion of a worst-case equilibrium can be adapted to any game of incomplete information. We chose the first-price auction because it is arguably a leading example of a game of incomplete information, appearing widely in the literature and textbooks. An interesting question for further research is which games of incomplete information admit a worst-case equilibrium.

Appendix

A.1 *Proof of Proposition 1*

The proof proceeds through four lemmas. Lemma 1 shows that the system of linear equations defining the worst-case beliefs has a unique solution. Lemma 2 establishes that the proposed strategies form a Nash equilibrium. Lemma 3 introduces a useful technical tool for the proof that the proposed equilibrium is worst case. Lemma 4 establishes that the proposed equilibrium is worst case.

Lemma 1. *For every $\theta^k > \mu$, the system of linear equations defined by (1) and*

$$\sum_{l=1}^{k} f^{k,*}(\theta^l) = 1 \quad \textit{and} \quad \sum_{l=1}^{k} f^{k,*}(\theta^l)\theta^l = \mu$$

has a unique solution. In the unique solution, all coordinates are strictly positive.

Proof. We show for every $\theta^k > \mu$ that the matrix corresponding to the system of equations has rank k by applying Gaussian elimination and obtaining a row echelon form. The conditions in (1) can also be summarized as

$$\left(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^h)\right)\left(\sqrt[n-1]{\theta^k - \overline{b}_h^*} - \sqrt[n-1]{\theta^k - \overline{b}_{h+1}^*}\right) - f^{k,*}(\theta^{h+1})\sqrt[n-1]{\theta^k - \overline{b}_{h+1}^*} = 0$$

for all $h \in \{1, \ldots, k-2\}$. In order to obtain an upper triangular matrix, we eliminate the variables $f^{k,*}(\theta^2), \ldots, f^{k,*}(\theta^k)$. We eliminate the variable $f^{k,*}(\theta^k)$ by multiplying the equation

$$\sum_{l=1}^{k} f^{k,*}(\theta^l) = 1$$

by $-\theta^k$ and adding it to

$$\sum_{l=1}^{k} f^{k,*}(\theta^l)\theta^l = \mu.$$

Multiplying the resulting equation by -1 gives

$$\sum_{l=1}^{k-1} f^{k,*}(\theta^l)(\theta^k - \theta^l) = \theta^k - \mu,$$

which eliminates the variable $f^{k,*}(\theta^k)$. Moreover, the coefficients $(\theta^k - \theta^l)$ are strictly positive. Next, we use the transformed conditions given by

$$\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^h)\big)\big(\sqrt[n-1]{\theta^k - \overline{b}_h^*} - \sqrt[n-1]{\theta^k - \overline{b}_{h+1}^*}\big) - f^{k,*}(\theta^{h+1})\sqrt[n-1]{\theta^k - \overline{b}_{h+1}^*} = 0$$

for all $h \in \{1, \ldots, k-2\}$ to eliminate the variables $f^{k,*}(\theta^2), \ldots, f^{k,*}(\theta^{k-1})$. We show by induction that in every elimination step, all coefficients are strictly positive. In particular, this implies that none of the coefficients is equal to 0. We therefore obtain an upper triangular matrix after applying Gaussian elimination. We start the induction by showing that in the equation that is obtained after eliminating $f^{k,*}(\theta^{k-1})$, all coefficients are strictly positive. The variable $f^{k,*}(\theta^{k-1})$ is eliminated by multiplying the condition given by

$$\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-2})\big)\big(\sqrt[n-1]{\theta^k - \overline{b}_{k-2}^*} - \sqrt[n-1]{\theta^k - \overline{b}_{k-1}^*}\big) - f^{k,*}(\theta^{k-1})\sqrt[n-1]{\theta^k - \overline{b}_{k-1}^*} = 0$$

by the factor

$$\frac{\theta^k - \theta^{k-1}}{\sqrt[n-1]{\theta^k - \overline{b}_{k-1}^*}}$$

and adding it to the equation

$$\sum_{l=1}^{k-1} f^{k,*}(\theta^l)(\theta^k - \theta^l) = \theta^k - \mu.$$

This gives the equation

$$\sum_{l=1}^{k-2} f^{k,*}(\theta^l)(\theta^k - \theta^l) + \sum_{l=1}^{k-2} f^{k,*}(\theta^l)\frac{(\theta^k - \theta^{k-1})\big(\sqrt[n-1]{\theta^k - \overline{b}_{k-2}^*} - \sqrt[n-1]{\theta^k - \overline{b}_{k-1}^*}\big)}{\sqrt[n-1]{\theta^k - \overline{b}_{k-1}^*}} = \theta^k - \mu,$$

where all coefficients are strictly positive. Now turning our attention to the induction step, we assume that the variables $f^{k,*}(\theta^{k-1}), \ldots, f^{k,*}(\theta^{h+1})$ have been eliminated and that in the resulting equation

$$\sum_{l=1}^{h} c_l f^{k,*}(\theta^l) = c,$$

all coefficients c and c_l for $1 \le l \le h$ are strictly positive. Now we have to eliminate the variable $f^{k,*}(\theta^h)$ using the condition

$$\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{h-1})\big)\big(\sqrt[n-1]{\theta^k - \overline{b}^*_{h-1}} - \sqrt[n-1]{\theta^k - \overline{b}^*_h}\big) - f^{k,*}(\theta^h)\sqrt[n-1]{\theta^k - \overline{b}^*_h} = 0.$$

We multiply this equation by the factor

$$\frac{c_h}{\sqrt[n-1]{\theta^k - \overline{b}^*_h}}$$

and add it to the equation

$$\sum_{l=1}^{h} c_l f^{k,*}(\theta^l) = c. \tag{5}$$

This gives the equation

$$\sum_{l=1}^{h-1} f^{k,*}(\theta^l) c_l + \sum_{l=1}^{h-1} f^{k,*}(\theta^l) \frac{c_h\big(\sqrt[n-1]{\theta^k - \overline{b}^*_{h-1}} - \sqrt[n-1]{\theta^k - \overline{b}^*_h}\big)}{\sqrt[n-1]{\theta^k - \overline{b}^*_h}} = c, \tag{6}$$

in which all coefficients are strictly positive. We conclude that the system of k linear equations with k variables given by

$$\sum_{l=1}^{k} f^{k,*}(\theta^l) = 1$$

$$\sum_{l=1}^{k} f^{k,*}(\theta^l)\theta^l = \mu$$

$$\left(\sum_{l=1}^{k-1} f^{k,*}(\theta^l)\right)^{n-1}\big(\theta^k - \overline{b}^*_{k-1}\big) = \left(\sum_{l=1}^{h} f^{k,*}(\theta^l)\right)^{n-1}\big(\theta^k - \overline{b}^*_h\big) \quad \text{for all } h \in \{1, \dots, k-2\}$$

can be rearranged into a system of linear equations such that the resulting matrix has rank k and, therefore, this system of equations has a unique solution.

We use an inductive argument to show that all coordinates in the unique solution are strictly positive. Since, in the linear equation obtained after performing Gaussian elimination, all coefficients are strictly positive, it holds that $f^{k,*}(\theta^1)$ is strictly positive. Assume it has been shown that $f^{k,*}(\theta^1), \dots, f^{k,*}(\theta^{h-1})$ are strictly positive. It follows from (5) that

$$c_h f^{k,*}(\theta^h) = c - \sum_{l=1}^{h-1} c_l f^{k,*}(\theta^l).$$

Since we have already established that c_h is strictly positive, we need to show that

$$c - \sum_{l=1}^{h-1} c_l f^{k,*}(\theta^l) > 0.$$

It follows from (6) that

$$c - \sum_{l=1}^{h-1} c_l f^{k,*}(\theta^l) = \sum_{l=1}^{h-1} f^{k,*}(\theta^l) c_l + \sum_{l=1}^{h-1} f^{k,*}(\theta^l) \frac{c_h\left(\sqrt[n-1]{\theta^k - \overline{b}_{h-1}^*} - \sqrt[n-1]{\theta^k - \overline{b}_h^*}\right)}{\sqrt[n-1]{\theta^k - \overline{b}_h^*}} - \sum_{l=1}^{h-1} c_l f^{k,*}(\theta^l)$$

$$= \sum_{l=1}^{h-1} f^{k,*}(\theta^l)\left(\frac{c_h\left(\sqrt[n-1]{\theta^k - \overline{b}_{h-1}^*} - \sqrt[n-1]{\theta^k - \overline{b}_h^*}\right)}{\sqrt[n-1]{\theta^k - \overline{b}_h^*}}\right) > 0,$$

where the last inequality follows from the induction hypothesis. We conclude that all coordinates in the unique solution are strictly positive. □

LEMMA 2. *The bidding strategy β^*, as specified in Proposition 1, constitutes a Nash equilibrium of the first-price auction given by $\mathcal{G}_{\varphi^*}$.*

PROOF. We have to check, for every valuation $\theta^k \in \Theta$, that there does not exist a bid $b \notin \text{supp}(G_k^*)$ that induces a higher expected payoff for a bidder with valuation θ^k than the equilibrium payoff. Fix a valuation θ^k. If $\theta^k \leq \mu$, bidding above the equilibrium bid θ^k is not feasible since it implies bidding above the own valuation. Bidding below θ^k does not constitute a profitable deviation since in the belief of a bidder with valuation θ^k, the lowest bid placed by her competitors with positive probability is θ^k. If $\theta^k > \mu$, we will consider three different sets of possible bids outside the support of G_k^*.

First, we consider all bids above the upper bound of the support of G_k^*. Let $b > \overline{b}_k^*$. Since $f^{k,*}(\theta^l) = 0$ for all $l > k$, it holds that

$$U(\theta^k, \varphi^*(\theta^k), b, \beta^*) = \theta^k - b < \theta^k - \overline{b}_k^* = U^k(\mathcal{G}_{\varphi^*}, \beta^*).$$

Thus, bids above the upper bound of the support of G_k^* can be excluded as deviating bids.

Second, we consider all bids of the form $\overline{b}_l^*$ for $l \in \{1, \dots, k\}$. By construction of the worst-case equilibrium, β^* makes a bidder indifferent between all these bids and the bids in the support of G_k^*, as can be seen in (1). Thus, none of the bids of the form $\overline{b}_l^*$ induces a higher expected payoff than the equilibrium payoff. Third, we can exclude bids b with $\theta^{l-1} = \beta^*(\theta^{l-1}) < b < \beta^*(\theta^l) = \theta^l$ for $0 < \theta^l \leq \mu$, since there is no probability mass between $\beta^*(\theta^{l-1})$ and $\beta^*(\theta^l)$.

Finally, we consider bids $b \in (\overline{b}_{l-1}^*, \overline{b}_l^*)$ for $l < k$ and $\theta^l > \mu$, i.e., bids that are in the bidding interval of a bidder with a strictly lower valuation but are not endpoints. We will proceed in two steps. First, we show that the payoff from bidding $b \in (\overline{b}_{l-1}^*, \overline{b}_l^*)$ has a unique critical point. It then follows that the payoff from bidding b is either less than (or equal to) the payoff from bidding $\overline{b}_{l-1}^*$ for all $b \in (\overline{b}_{l-1}^*, \overline{b}_l^*)$ or greater than (or equal to) the payoff from bidding $\overline{b}_{l-1}^*$ for all $b \in (\overline{b}_{l-1}^*, \overline{b}_l^*)$. This is a direct consequence of the fact that the payoff from bidding $\overline{b}_{l-1}^*$ is the same as the payoff from bidding $\overline{b}_l^*$ and the payoff function has a unique critical point. Second, we show for a particular $b \in (\overline{b}_{l-1}^*, \overline{b}_l^*)$ that the payoff from bidding b is less than or equal to the payoff from bidding $\overline{b}_{l-1}^*$ or $\overline{b}_l^*$.

We start by showing that the payoff from bidding some $b \in (\overline{b}_{l-1}^*, \overline{b}_l^*)$ has a unique critical point. The payoff for a bidder with valuation θ^k from bidding $b \in (\overline{b}_{l-1}^*, \overline{b}_l^*)$ is

$$\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l) G_l^*(b)\big)^{n-1}\big(\theta^k - b\big). \tag{7}$$

The payoff as a function of b is continuous on $[\overline{b}_{l-1}^*, \overline{b}_l^*]$ and differentiable on $(\overline{b}_{l-1}^*, \overline{b}_l^*)$. Thus, it attains a maximum and minimum in $[\overline{b}_{l-1}^*, \overline{b}_l^*]$. Whereas, by construction of the equilibrium beliefs, the payoffs at $\overline{b}_{l-1}^*$ and at $\overline{b}_l^*$ are the same, the derivative of the payoff necessarily is 0 at each critical point. Denote by g_l^* the density of G_l^*. We analyze the solution of

$$\begin{aligned}(n-1)&\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l) G_l^*(b)\big)^{n-2} f^{k,*}(\theta^l) g_l^*(b)\big(\theta^k - b\big) \\ &- \big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l) G_l^*(b)\big)^{n-1} = 0.\end{aligned} \tag{8}$$

This is equivalent to

$$\theta^k - b - \frac{\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l) G_l^*(b)\big)^{n-1}}{(n-1)\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l) G_l^*(b)\big)^{n-2} f^{k,*}(\theta^l) g_l^*(b)} = 0,$$

and, therefore, to

$$\theta^k - b - \frac{f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1})}{(n-1) f^{k,*}(\theta^l) g_l^*(b)} - \frac{G_l^*(b)}{(n-1) g_l^*(b)} = 0. \tag{9}$$

Thus, the left-hand side of (8) has the same number of zeros as the left-hand side of (9). We now show that the left-hand side of (9) has a unique zero. To do so, we take the derivative of the left-hand side of (9) with respect to b and show that it does not change signs. Using

$$G_l^*(b) = \frac{\big(f^{l,*}(\theta^1) + \cdots + f^{l,*}(\theta^{l-1})\big)\big((\theta^l - \overline{b}_{l-1}^*)^{\frac{1}{n-1}} - (\theta^l - b)^{\frac{1}{n-1}}\big)}{f^{l,*}(\theta^l)(\theta^l - b)^{\frac{1}{n-1}}} \tag{10}$$

and

$$g_l^*(b) = \frac{\big(f^{l,*}(\theta^1) + \cdots + f^{l,*}(\theta^{l-1})\big)(\theta^l - \overline{b}_{l-1}^*)^{\frac{1}{n-1}}}{f^{l,*}(\theta^l)(\theta^l - b)^{\frac{n}{n-1}}(n-1)}, \tag{11}$$

we get the following expression for the left-hand side of (9):

$$\begin{aligned}\theta^k - b &- \frac{f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1})}{(n-1) f^{k,*}(\theta^l)} \frac{f^{l,*}(\theta^l)(\theta^l - b)^{\frac{n}{n-1}}(n-1)}{\big(f^{l,*}(\theta^1) + \cdots + f^{l,*}(\theta^{l-1})\big)(\theta^l - \overline{b}_{l-1}^*)^{\frac{1}{n-1}}} \\ &- \frac{\big(f^{l,*}(\theta^1) + \cdots + f^{l,*}(\theta^{l-1})\big)\big((\theta^l - \overline{b}_{l-1}^*)^{\frac{1}{n-1}} - (\theta^l - b)^{\frac{1}{n-1}}\big)}{f^{l,*}(\theta^l)(\theta^l - b)^{\frac{1}{n-1}}(n-1)}\end{aligned}$$

$$\cdot \frac{f^{l,*}(\theta^l)(\theta^l - b)^{\frac{n}{n-1}}(n-1)}{\big(f^{l,*}(\theta^1) + \cdots + f^{l,*}(\theta^{l-1})\big)(\theta^l - \overline{b}^*_{l-1})^{\frac{1}{n-1}}}$$

$$= \theta^k - b - \frac{(\theta^l - b)^{\frac{n}{n-1}}\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1})\big)f^{l,*}(\theta^l)}{(\theta^l - \overline{b}^*_{l-1})^{\frac{1}{n-1}}\big(f^{l,*}(\theta^1) + \cdots + f^{l,*}(\theta^{l-1})\big)f^{k,*}(\theta^l)}$$

$$- (\theta^l - b) + \frac{(\theta^l - b)^{\frac{n}{n-1}}}{(\theta^l - \overline{b}^*_{l-1})^{\frac{1}{n-1}}}.$$

This gives a derivative of

$$\frac{n}{n-1}\frac{(\theta^l - b)^{\frac{1}{n-1}}\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1})\big)f^{l,*}(\theta^l)}{(\theta^l - \overline{b}^*_{l-1})^{\frac{1}{n-1}}\big(f^{l,*}(\theta^1) + \cdots + f^{l,*}(\theta^{l-1})\big)f^{k,*}(\theta^l)} - \frac{n}{n-1}\frac{(\theta^l - b)^{\frac{1}{n-1}}}{(\theta^l - \overline{b}^*_{l-1})^{\frac{1}{n-1}}}$$

$$= \frac{n}{n-1}\frac{(\theta^l - b)^{\frac{1}{n-1}}\Big(\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1})\big)f^{l,*}(\theta^l) - \big(f^{l,*}(\theta^1) + \cdots + f^{l,*}(\theta^{l-1})\big)f^{k,*}(\theta^l)\Big)}{(\theta^l - \overline{b}^*_{l-1})^{\frac{1}{n-1}}\big(f^{l,*}(\theta^1) + \cdots + f^{l,*}(\theta^{l-1})\big)f^{k,*}(\theta^l)}.$$

Since $\theta^l - b > 0$ for all $b \in (\overline{b}^*_{l-1}, \overline{b}^*_l)$, the derivative does not change signs. Thus, the payoff from bidding $b \in (\overline{b}^*_{l-1}, \overline{b}^*_l)$ has a unique critical point. It remains to show that this critical point is a minimum.

Choose $b \in (\overline{b}^*_{l-1}, \overline{b}^*_l)$ such that

$$\sqrt[n-1]{\theta^k - \overline{b}^*_{l-1}} - \sqrt[n-1]{\theta^k - b} = \sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - \overline{b}^*_l}. \tag{12}$$

Suppose for the sake of contradiction that the critical point is a maximum. In this case, it holds that

$$\begin{aligned}&\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l)G_l(b)\big)^{n-1}(\theta^k - b)\\ &\quad\geq \big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l)\big)^{n-1}(\theta^k - \overline{b}^*_l)\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l)G_l(b)\big)^{n-1}(\theta^k - b)\\ &\quad\geq \big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1})\big)^{n-1}(\theta^k - \overline{b}^*_{l-1}).\end{aligned} \tag{14}$$

Both inequalities (13) and (14) hold with equality if $k = l$.

Rearranging (13) gives

$$\begin{aligned}&\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l)G_l(b)\big)\sqrt[n-1]{\theta^k - b}\\ &\geq \big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1}) + f^{k,*}(\theta^l)\big)\sqrt[n-1]{\theta^k - \overline{b}^*_l}\\ \Leftrightarrow\quad &\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{l-1})\big)\Big(\sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - \overline{b}^*_l}\Big)\end{aligned}$$

$$
\begin{aligned}
&\geq f^{k,*}(\theta^l)\sqrt[n-1]{\theta^k-\overline{b}_l^*}-f^{k,*}(\theta^l)G_l(b)\sqrt[n-1]{\theta^k-b} \\
\Leftrightarrow\quad & f^{k,*}(\theta^1)+\cdots+f^{k,*}(\theta^{l-1}) \\
&\geq \frac{f^{k,*}(\theta^l)\sqrt[n-1]{\theta^k-\overline{b}_l^*}-f^{k,*}(\theta^l)G_l(b)\sqrt[n-1]{\theta^k-b}}{\sqrt[n-1]{\theta^k-b}-\sqrt[n-1]{\theta^k-\overline{b}_l^*}}. \qquad (15)
\end{aligned}
$$

Rearranging (14) in the same way gives

$$
\Leftrightarrow f^{k,*}(\theta^1)+\cdots+f^{k,*}(\theta^{l-1})\leq \frac{f^{k,*}(\theta^l)G_l(b)\sqrt[n-1]{\theta^k-b}}{\sqrt[n-1]{\theta^k-\overline{b}_{l-1}^*}-\sqrt[n-1]{\theta^k-b}}. \qquad (16)
$$

Again, both inequalities (15) and (16) hold with equality if $k=l$.

If we show that

$$
\frac{f^{k,*}(\theta^l)G_l(b)\sqrt[n-1]{\theta^k-b}}{\sqrt[n-1]{\theta^k-\overline{b}_{l-1}^*}-\sqrt[n-1]{\theta^k-b}}<\frac{f^{k,*}(\theta^l)\sqrt[n-1]{\theta^k-\overline{b}_l^*}-f^{k,*}(\theta^l)G_l(b)\sqrt[n-1]{\theta^k-b}}{\sqrt[n-1]{\theta^k-b}-\sqrt[n-1]{\theta^k-\overline{b}_l^*}} \qquad (17)
$$

for $k>l$, we get a contradiction to inequalities (13) and (14).

Using (12) and rearranging, we get

$$
2G_l(b)\sqrt[n-1]{\theta^k-b}<\sqrt[n-1]{\theta^k-\overline{b}_l^*}. \qquad (18)
$$

Using (18) with equality for $k=l$, rearranging for $G_l(b)$, substituting in (17) for $k>l$, and rearranging yields

$$
\frac{(\theta^k-b)}{(\theta^l-b)}<\frac{(\theta^k-\overline{b}_l^*)}{(\theta^l-\overline{b}_l^*)},
$$

which is equivalent to

$$
\theta^k(\overline{b}_l^*-b)>\theta^l(\overline{b}_l^*-b).
$$

The last inequality is obviously true as $\theta^k>\theta^l$. Thus, the assumption that the critical point of the payoff function is a maximum leads to a contradiction. Summing up, for a bidder with valuation θ^k, it is not a profitable deviation to choose $b\in(\overline{b}_{l-1}^*,\overline{b}_l^*)$. □

To establish that the proposed equilibrium is worst case, we use the following lemma.

Lemma 3. *Let $f\in\mathcal{F}_\mu$ and let $\tilde{f}$ be a function $\tilde{f}:\Theta\to[0,1]$ such that $\sum_{l=1}^m\tilde{f}(\theta^l)=1$. Let $(\delta_1,\dots,\delta_m)$ be a vector of real numbers such that $\tilde{f}(\theta^l)=f(\theta^l)+\delta_l$ for all $1\leq l\leq m$ and it holds for at least one $1\leq l\leq m$ that $\delta_l\neq 0$. Assume that for all $1\leq l\leq m$, it holds that $\sum_{h=1}^l\delta_h\leq 0$. Then $\sum_{l=1}^m\theta^l\tilde{f}(\theta^l)>\mu$, i.e., $\tilde{f}\notin\mathcal{F}_\mu$.*

PROOF. Let f, $\tilde{f}$, and $(\delta_1, \dots, \delta_m)$ be as in Lemma 3. It holds that

$$\sum_{l=1}^{m} \theta^l \big(\tilde{f}(\theta^l) - f(\theta^l)\big) = \sum_{l=1}^{m} \theta^l \delta_l = \sum_{l=1}^{m-1} (\theta^l - \theta^{l+1}) \left(\sum_{h=1}^{l} \delta_h\right) + \theta^m \left(\sum_{l=1}^{m} \delta_l\right) > 0.$$

Since $\sum_{l=1}^m \tilde{f}(\theta^l) = 1$ and $\tilde{f}(\theta^l) = f(\theta^l) + \delta_l$, it follows that $\sum_{=1}^m \delta_l = 0$. Therefore, $\sum_{l=1}^m \theta^l \tilde{f}(\theta^l) > \mu$. □

Intuitively, the statement is based on the fact that if the conditions of the lemma are fulfilled, the distribution $\tilde{f}$ first-order stochastically dominates the distribution f and, therefore, they cannot have the same mean.

LEMMA 4. *For every mental model $(\mathcal{G}_\varphi, \beta)$ of the first-price auction, it holds for all $k \in \{1, \dots, m\}$ that*

$$U^k(\mathcal{G}_\varphi, \beta) \geq U^k(\mathcal{G}_{\varphi^*}, \beta^*) \tag{19}$$

and $\overline{b}_k \leq \overline{b}_k^*$.

PROOF. As preparation for the proof of this lemma, we will show the following claim. We will then use the claim in an inductive argument.

CLAIM 1. *Let $(\mathcal{G}_\varphi, \beta)$ be a mental model such that*

$$U^k(\mathcal{G}_\varphi, \beta) \leq U^k(\mathcal{G}_{\varphi^*}, \beta^*) \tag{a}$$

for some k with $\theta^k > \mu$ and

$$\overline{b}_l \leq \overline{b}_l^* \tag{b}$$

for all $1 \leq l \leq k-1$, where $\overline{b}_l = \sup\{b \mid b \in \operatorname{supp}(G_l)\}$. For $1 \leq l \leq m$, define δ_l by $f^k(\theta^l) = f^{k,*}(\theta^l) + \delta_l$. Then $\delta_l = 0$ for all $1 \leq l \leq m$. That is, $f^k(\theta^l) = f^{k,*}(\theta^l)$ for all $1 \leq l \leq m$.

PROOF. Let $(\mathcal{G}_\varphi, \beta)$ be a mental model that fulfills the conditions (a) and (b). Recall that the infimum of the support of $\beta(\theta^k)$ is denoted by $\underline{b}_k$. The expected payoff of a bidder with valuation θ^k in the mental model $(\mathcal{G}_\varphi, \beta)$ is the expected payoff of bidding $\underline{b}_k$, which is given by

$$\left(\sum_{l=1}^{m} f^k(\theta^l) G_l(\underline{b}_k)\right)^{n-1} (\theta^k - \underline{b}_k) \tag{20}$$

(some of the expressions of the form $G_l(\underline{b}_k)$ may be zero). Note that there may be an atom in the bid distribution G_l at $\underline{b}_k$. If $f^k(\theta^l) = 0$, this does not matter for the expression in (20). If $f^k(\theta^l) > 0$, a bidder with valuation θ^k would not tie with positive probability at $\underline{b}_k$. In this case, the probability that a bidder with valuation θ^k wins against a bidder with valuation θ^l is given by the limit $b \to \underline{b}_k$ from above and is equal to $G_l(\underline{b}_k)$.

Recall that in the worst-case equilibrium $(\mathcal{G}_{\varphi^*}, \beta^*)$, the lower endpoint of the bidding interval of a bidder with valuation θ^k coincides with the upper endpoint of the bidding interval of a bidder with valuation θ^{k-1}, denoted by $\overline{b}^*_{k-1}$. Thus, the expected payoff of a bidder with valuation θ^k in the worst-case equilibrium $(\mathcal{G}_{\varphi^*}, \beta^*)$ is given by

$$\left(\sum_{l=1}^{k-1} f^{k,*}(\theta^l)\right)^{n-1} (\theta^k - \overline{b}^*_{k-1}).$$

Let $(\delta_1, \dots, \delta_m)$ be such that $f^k(\theta^l) = f^{k,*}(\theta^l) + \delta_l$ for $1 \le l \le m$.

Since β in $(\mathcal{G}_\varphi, \beta)$ is an equilibrium bidding function, it must hold that the expected payoff in (20) is at least as high as the expected payoff of deviating to a bid outside the support of the bidder's bidding strategy. In particular, it must hold that deviating to bidding 0 does not induce a higher payoff. Thus,

$$\left(\sum_{l=1}^{m} f^k(\theta^l) G_l(\underline{b}_k)\right)^{n-1} (\theta^k - \underline{b}_k) \ge \left(f^k(\theta^1)\right)^{n-1} \theta^k.$$

It follows that

$$\begin{aligned}\left(f^{k,*}(\theta^1) + \delta_1\right)^{n-1} \theta^k = \left(f^k(\theta^1)\right)^{n-1} \theta^k &\le \left(\sum_{l=1}^{m} f^k(\theta^l) G_l(\underline{b}_k)\right)^{n-1} (\theta^k - \underline{b}_k) \\ &\le \left(\sum_{l=1}^{k-1} f^{k,*}(\theta^l)\right)^{n-1} (\theta^k - \overline{b}^*_{k-1}) = \left(f^{k,*}(\theta^1)\right)^{n-1} \theta^k,\end{aligned}$$

where the second inequality follows from the fact that $U^k(\mathcal{G}_\varphi, \beta) \le U^k(\mathcal{G}_{\varphi^*}, \beta^*)$ and the second equality follows from (1). We conclude that $\delta_1 \le 0$. We now extend the argument and show for all $1 \le h \le k-1$ that $\sum_{l=1}^{h} \delta_l \le 0$. Let $1 \le h \le k-1$ and let

$$\overline{b}_t := \max_{l \le h} \overline{b}_l.$$

Since we assume an efficient tie-breaking rule and $k > h$, it follows that a bidder with valuation θ^k wins against all valuations less than or equal to θ^h if she deviates to $\overline{b}_t$. It follows that the expected utility from deviating equals

$$\left(\sum_{l=1}^{h} f^k(\theta^l) + \sum_{l=h+1}^{m} f^k(\theta^l) \hat{G}_l(\overline{b}_t)\right)^{n-1} (\theta^k - \overline{b}_t),$$

where $\hat{G}_l(\overline{b}_t)$ denotes the probability that a bidder with valuation θ^k wins against a bidder with valuation θ^l at $\overline{b}_t$ (which may be different from $G_l(\overline{b}_t)$ if there is an atom in the bid distribution for valuation θ^l and $l > h$).

Since deviating to bid $\overline{b}_t$ cannot yield a higher payoff for a bidder with valuation θ^k, it holds that

$$\left(\sum_{l=1}^{m} f^k(\theta^l) G_l(\underline{b}_k)\right)^{n-1} (\theta^k - \underline{b}_k) \ge \left(\sum_{l=1}^{h} f^k(\theta^l) + \sum_{l=h+1}^{m} f^k(\theta^l) \hat{G}_l(\overline{b}_t)\right)^{n-1} (\theta^k - \overline{b}_t), \quad (21)$$

from which it follows that

$$\begin{aligned}
&\left(\sum_{l=1}^{h} f^{k,*}(\theta^l) + \delta_l + \sum_{l=h+1}^{m} f^k(\theta^l)\hat{G}_l(\overline{b}_t)\right)^{n-1} (\theta^k - \overline{b}_t) \\
&\quad = \left(\sum_{l=1}^{h} f^k(\theta^l) + \sum_{l=h+1}^{m} f^k(\theta^l)\hat{G}_l(\overline{b}_t)\right)^{n-1} (\theta^k - \overline{b}_t) \\
&\quad \leq \left(\sum_{l=1}^{m} f^k(\theta^l) G_l(\underline{b}_k)\right)^{n-1} (\theta^k - \underline{b}_k)^{n-1} \leq \left(\sum_{l=1}^{k-1} f^{k,*}(\theta^l)\right)^{n-1} (\theta^k - \overline{b}^*_{k-1}) \\
&\quad = \left(\sum_{l=1}^{t} f^{k,*}(\theta^l)\right)^{n-1} (\theta^k - \overline{b}^*_t),
\end{aligned}$$

where the first inequality follows from (21) and the second inequality follows from condition (a). Since $h \leq k-1$ and $1 \leq t \leq h \leq k-1$, it follows by assumption that $\overline{b}_t \leq \overline{b}^*_t$. Hence, it must hold that

$$\begin{aligned}
&\sum_{l=1}^{h} f^{k,*}(\theta^l) + \delta_l + \sum_{l=h+1}^{m} f^k(\theta^l)\hat{G}_l(\overline{b}_t) \leq \sum_{l=1}^{t} f^{k,*}(\theta^l) \\
&\quad \Leftrightarrow \quad \sum_{l=1}^{h} \delta_l + \sum_{l=t+1}^{h} f^{k,*}(\theta^l) + \sum_{l=h+1}^{m} f^k(\theta^l)\hat{G}_l(\overline{b}_t) \leq 0.
\end{aligned}$$

Since

$$\sum_{l=t+1}^{h} f^{k,*}(\theta^l) + \sum_{l=h+1}^{m} f^k(\theta^l)\hat{G}_l(\overline{b}_k) \geq 0,$$

it follows that

$$\sum_{l=1}^{h} \delta_l \leq 0.$$

We have shown for all $1 \leq h \leq k-1$ that $\sum_{l=1}^{h} \delta_l \leq 0$. Let $h \geq k$. By the construction of $(f^{k,*}(\theta^1), \dots, f^{k,*}(\theta^m))$, it holds that $f^{k,*}(\theta^l) = 0$ for all $l > k$. Hence, $\delta_l \geq 0$ for all $k+1 \leq l \leq m$ and, therefore, $\sum_{l=h+1}^{m} \delta_l \geq 0$. Since $\sum_{l=1}^{m} \delta_l = 0$, it follows that $\sum_{l=1}^{h} \delta_l \leq 0$. Thus, there does not exist any $1 \leq h \leq m$ such that $\sum_{l=1}^{h} \delta_l > 0$. It follows from Lemma 3 that $\delta_l = 0$ for all $1 \leq l \leq m$, as otherwise the belief $(f^k(\theta^1), \dots, f^k(\theta^m))$ would violate the mean constraint $\sum_{l=1}^{k} f^k(\theta^l)\theta^l = \mu$. □

Having proved the claim, we proceed with the proof of the statement in (19). Let $(\mathcal{G}_\varphi, \beta)$ be a mental model. We will show the statement by proving the following two statements simultaneously by induction:

(i) For every $k \in \{1, \dots, m\}$, it holds that

$$U^k(G_{\varphi^*}, \beta^*) \le U^k(\mathcal{G}_\varphi, \beta).$$

(ii) For every $k \in \{1, \dots, m\}$, it holds that $\overline{b}_k \le \overline{b}_k^*$.

Recall that we only consider equilibria in which bidders never bid above their own valuation and, hence, a bidder with valuation zero bids 0 (Definition 1). Thus, both statements are trivially true for $k = 1$, since $\theta^1 = 0$ and, therefore,

$$0 = U^1(G_{\varphi^*}, \beta^*) \le U^1(\mathcal{G}_\varphi, \beta) = 0$$

and $0 = \overline{b}_1 \le \overline{b}_1^* = 0$. Assume that both statements have been shown for $k - 1$; we have to prove both statements for k. We begin with statement (i). If $\theta^k \le \mu$, the statement is trivially true, because then a bidder with valuation θ^k obtains the lowest possible payoff of 0 in the worst-case equilibrium. Statement (ii) is also true, since a bidder with valuation θ^k with $\theta^k \le \mu$ bids her valuation in the worst-case equilibrium, and this is the highest possible undominated bid. Assume that $\theta^k > \mu$ and statement (i) is true for all $l < k$.

Let $(\delta_1, \dots, \delta_m)$ be such that $f^k(\theta^l) = f^{k,*}(\theta^l) + \delta_l$ for $1 \le l \le m$. In this case, the conditions (a) and (b) of Claim 1 are fulfilled, and it holds that $\delta_l = 0$ for all $1 \le l \le m$. Thus, given the belief f^k, a bidder with valuation θ^k believes that she has the highest valuation. By the induction hypothesis, $\overline{b}_l \le \overline{b}_{k-1}^*$ for $l < k$. Thus, if a bidder with valuation θ^k bids $\overline{b}_{k-1}^*$ instead of $\underline{b}_k$, she wins against all lower valuations. Therefore, she obtains at least a payoff of

$$\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-1})\big)\big(\theta^k - \overline{b}_{k-1}^*\big). \tag{22}$$

Her payoff from bidding her equilibrium bid $\underline{b}_k$ must be at least as high. By the construction of the worst-case equilibrium, it holds that $\overline{b}_{k-1}^* = \underline{b}_k^*$. Thus, the expression (22) is equal to the payoff of a bidder with valuation θ^k in the worst-case equilibrium. Thus, statement (i) is true for k. Hence, we have shown the induction step for statement (i).

It is left to show the induction step for statement (ii). It holds that

$$\left(\sum_{l=1}^m f^k(\theta^l) G_l(\underline{b}_k)\right)^{n-1} (\theta^k - \underline{b}_k) = \left(\sum_{l=1}^m f^k(\theta^l) \hat{G}_l(\overline{b}_k)\right)^{n-1} (\theta^k - \overline{b}_k)$$

and

$$\left(\sum_{l=1}^{k-1} f^{k,*}(\theta^l)\right)^{n-1} (\theta^k - \overline{b}_{k-1}^*) = \left(\sum_{l=1}^{k} f^{k,*}(\theta^l)\right)^{n-1} (\theta^k - \overline{b}_k^*) = \theta^k - \overline{b}_k^*,$$

where, as above, $\hat{G}_l(\overline{b}_k)$ denotes the probability with which a bidder with valuation θ^k wins against a bidder with valuation θ^l at $\overline{b}_k$.

Since we have shown that

$$\left(\sum_{l=1}^m f^k(\theta^l) G_l(\underline{b}_k)\right)^{n-1} (\theta^k - \underline{b}_k) \ge \left(\sum_{l=1}^{k-1} f^{k,*}(\theta^l)\right)^{n-1} (\theta^k - \overline{b}_{k-1}^*),$$

it follows that

$$\theta^k - \overline{b}_k^* \leq \left(\sum_{l=1}^{m} f^k(\theta^l) \hat{G}_l(\overline{b}_k) \right)^{n-1} (\theta^k - \overline{b}_k).$$

Since

$$\sum_{l=1}^{m} f^k(\theta^l) \hat{G}_l(\overline{b}_k) \leq 1,$$

we conclude that $\overline{b}_k \leq \overline{b}_k^*$. □

A.2 *Proof of Proposition 2*

We prove that whenever

$$U^k(G_{\varphi^*}, \beta^*) = U^k(\mathcal{G}_\varphi, \beta) \tag{23}$$

for all $k \in \{1, \dots, m\}$, it follows that $G_k(b) = G_k^*(b)$ and $f^k = f^{k,*}$ for all $k \in \{1, \dots, m\}$ with $\theta^k > \mu$.

Let $(\mathcal{G}_\varphi, \beta)$ be a mental model such that

$$U^k(G_{\varphi^*}, \beta^*) = U^k(\mathcal{G}_\varphi, \beta)$$

for all $k \in \{1, \dots, m\}$. First, we show that $f^k = f^{k,*}$ for all $k \in \{1, \dots, m\}$ with $\theta^k > \mu$. Let $k \in \{1, \dots, m\}$ with $\theta^k > \mu$. We have shown that $\overline{b}_l \leq \overline{b}_l^*$ for all $1 \leq l \leq m$. Thus, conditions (a) and (b) of Claim 1 are fulfilled and it holds that $f^k = f^{k,*}$.

Second, we show for all $k \in \{1, \dots, m\}$ with $\theta^k > \mu$ that $\overline{b}_k \geq \overline{b}_k^*$ in order to conclude that $\overline{b}_k^* = \overline{b}_k$. Assume that $\overline{b}_k < \overline{b}_k^*$ for some $k \in \{1, \dots, m\}$ with $\theta^k > \mu$. Since, in the equilibrium (G_{φ^*}, β^*), a bidder with valuation θ^k wins with probability 1 if bidding $\overline{b}_k^*$ and a bidder with valuation θ^k obtains the same utility as in the mental model $(\mathcal{G}_\varphi, \beta)$, it must hold that a bidder with valuation θ^k does not win with probability 1 if bidding $\overline{b}_k$ in the mental model $(\mathcal{G}_\varphi, \beta)$. Since there is no probability weight on valuations above k, there must exist a valuation θ^t with $t < k$ and $\overline{b}_k < \overline{b}_t$. If there would be an atom at $\overline{b}_k$ in the bid distribution of a type $\theta^l \leq \theta^k$, it would be possible that $\overline{b}_k = \overline{b}_t$. However, since $f^k = f^{k,*}$, in the belief of type θ^k, there is positive probability weight on every type $\theta^l \leq \theta^k$. Thus a bidder with valuation θ^k would slightly overbid this atom.

Let $U(\theta^k, \varphi(\theta^k), b, \beta)$ denote the expected utility of a bidder with valuation θ^k in the equilibrium $(\mathcal{G}_\varphi, \beta)$ if she bids b (where b may be a deviating bid). We use the notation $U(\theta^k, \varphi^*(\theta^k), b, \beta^*)$ analogously for the equilibrium $(\mathcal{G}_{\varphi^*}, \beta^*)$.

By assumption,

$$U(\theta^k, \varphi(\theta^k), \overline{b}_k, \beta) = U^k(\mathcal{G}_\varphi, \beta) = U^k(\mathcal{G}_{\varphi^*}, \beta^*) = U(\theta^k, \varphi^*(\theta^k), \overline{b}_k^*, \beta^*).$$

Note that if $\overline{b}_k$ is the supremum but not the maximum of the support of the bid distribution for valuation θ^k, the expected utility at $\overline{b}_k$ may not be the equilibrium utility if there is an atom at $\overline{b}_k$ in the bid distribution of some other bidder with valuation θ^l. If $l > k$, an atom does not influence the expected utility for valuations θ^k, since a bidder with valuation θ^k assigns zero probability to type θ^l. The case $l \leq k$ has been excluded above. Bidding above $\overline{b}_k$ is not a dominated bid since $\overline{b}_k < \overline{b}_t \leq \theta^t < \theta^k$. By the construction of the worst-case equilibrium, it holds that

$$U\big(\theta^k, \varphi^*\big(\theta^k\big), \overline{b}_k^*, \beta^*\big) = U\big(\theta^k, \varphi^*\big(\theta^k\big), \overline{b}_t^*, \beta^*\big).$$

Combined, this gives

$$U\big(\theta^k, \varphi\big(\theta^k\big), \overline{b}_k, \beta\big) = U\big(\theta^k, \varphi^*\big(\theta^k\big), \overline{b}_t^*, \beta^*\big).$$

Since $\overline{b}_l \leq \overline{b}_l^*$ for all $1 \leq l \leq m$, a bidder with valuation θ^k would win not only against all bidders with lower valuations when deviating to $\overline{b}_t^*$, but also against all other bidders with valuation θ^k, since $\overline{b}_k < \overline{b}_t \leq \overline{b}_t^*$. This implies that bidding $\overline{b}_t^*$ in the mental model $(\mathcal{G}_\varphi, \beta)$ induces a higher winning probability than in the worst-case equilibrium. Since $f^k = f^{k,*}$, it also induces a higher expected payoff. It follows that

$$U\big(\theta^k, \varphi\big(\theta^k\big), \overline{b}_k, \beta\big) \geq U\big(\theta^k, \varphi\big(\theta^k\big), \overline{b}_t^*, \beta\big) > U\big(\theta^k, \varphi^*\big(\theta^k\big), \overline{b}_t^*, \beta^*\big),$$

which leads to a contradiction. The first inequality holds since the expected payoff in equilibrium cannot be lower than the expected payoff from deviating to $\overline{b}_t^*$. We conclude that $\overline{b}_k^* = \overline{b}_k$.

Third, we show that for all $k \in \{1, \dots, m\}$ with $\theta^k > \mu$, it holds that $G_k(\overline{b}_{k-1}^*) = 0$; i.e., in the mental model $(\mathcal{G}_\varphi, \beta)$, in the bid distribution of a bidder with valuation θ^k there does not exist a positive mass of bids that lies below $\overline{b}_{k-1}^*$. Assume that $G_k(\overline{b}_{k-1}^*) > 0$ for some $k \in \{1, \dots, m\}$ with $\theta^k > \mu$. If a bidder with valuation θ^k bids $\overline{b}_{k-1}^*$ in the equilibrium $(\mathcal{G}_\varphi, \beta)$, she wins against all bidders with valuations lower than θ^k with probability 1. Thus, her winning probability is given by

$$f^k\big(\theta^1\big) + \cdots + f^k\big(\theta^{k-1}\big) + f^k\big(\theta^k\big)G_k\big(\overline{b}_{k-1}^*\big),$$

and since we have established that $f^{k,*} = f^k$ for all k with $\theta^k > \mu$, this is equal to

$$f^{k,*}\big(\theta^1\big) + \cdots + f^{k,*}\big(\theta^{k-1}\big) + f^{k,*}\big(\theta^k\big)G_k\big(\overline{b}_{k-1}^*\big).$$

We conclude that $U^k(\mathcal{G}_\varphi, \beta) > U^k(\mathcal{G}_{\varphi^*}, \beta^*)$, which leads to a contradiction.

Fourth, we show that $\underline{b}_k \leq \underline{b}_k^*$ for all $k \in \{1, \dots, m\}$ with $\theta^k > \mu$. Assume that $\underline{b}_k > \underline{b}_k^*$ for some $k \in \{1, \dots, m\}$ with $\theta^k > \mu$. It follows that $\underline{b}_k > \underline{b}_k^* = \overline{b}_{k-1}^* = \overline{b}_{k-1}$. Valuations below θ^k cannot bid above $\overline{b}_{k-1}^*$, and valuations above θ^k bid between $\overline{b}_{k-1}^*$ and $\overline{b}_k$ with probability 0, since we have established that $G_l(\overline{b}_k) = G_l(\overline{b}_k^*) = 0$ for $l > k$. Thus, there is a gap in the bid distribution between $\overline{b}_{k-1}^*$ and $\underline{b}_k$ given the strategy β. This cannot occur in equilibrium since for a sufficiently small ϵ, a bidder with valuation θ^k could deviate from $\underline{b}_k + \epsilon$ to a bid b with $\overline{b}_{k-1}^* < b < \underline{b}_k$ and strictly increase her expected payoff since for any $\epsilon' > 0$, one can find $\epsilon > 0$ such that the winning probability decreases by less than ϵ' when deviating. It follows that the assumption $\underline{b}_k > \underline{b}_k^*$ leads to a contradiction.

Finally we show that $G_k(b) = G_k^*(b)$ for all $k \in \{1, \ldots, m\}$ with $\theta^k > \mu$. Observe that G_k cannot have an atom at $\overline{b_k}$, as otherwise a bidder with valuation θ^k would gain by slightly overbidding $\overline{b}_k$. Let $b \in [\underline{b}_k, \overline{b}_k]$. We have established that $\overline{b}_k = \overline{b}_k^*$ and $f^k = f^{k,*}$ for all $\theta^k \in \Theta$ with $\theta^k > \mu$, from which it follows that

$$\begin{aligned}&\big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-1}) + f^{k,*}(\theta^k) G_k(b)\big) \sqrt[n-1]{(\theta^k - b)}\\&= \big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^k)\big) \sqrt[n-1]{(\theta^k - \overline{b}_k)}\\&= \big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^k)\big) \sqrt[n-1]{(\theta^k - \overline{b}_k^*)}\\&= \big(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-1}) + f^{k,*}(\theta^k) G_k^*(b)\big) \sqrt[n-1]{(\theta^k - b)}.\end{aligned}$$

Therefore, $G_k(b)$ and $G_k^*(b)$ are solutions of the same linear equation and, hence, are equal. □

A.3 *Proof of Proposition 3*

Proof. Let $(\mathcal{G}_\varphi, \beta)$ be a mental model. Fix a valuation $\theta^k \in \Theta$. Let $b \in \operatorname{supp}(\beta^*(\theta^k))$ and let $b' \in \operatorname{supp}(\beta(\theta^l))$ for $1 \le l < k$. By definition, $b \ge \overline{b}_{k-1}^*$ and $b' \le \overline{b}_l$. As shown in the proof of Lemma 4, it holds that $\overline{b}_l \le \overline{b}_{k-1}^*$, from which it follows that

$$b' \le \overline{b}_l \le \overline{b}_{k-1}^* \le b.$$ □

A.4 *Proof of Proposition 4*

Proof. Let $(\mathcal{G}_\varphi, \beta)$ be a mental model and let $f = (f(\theta^1), \ldots, f(\theta^m)) \in \mathcal{F}_\mu$. We have to show that

$$\mathcal{B}_f^{\beta^*}(s) = f(\theta^1) G_1^*(s) + \cdots + f(\theta^m) G_m^*(s) \tag{24}$$

first-order stochastically dominates

$$\mathcal{B}_f^{\beta}(s) = f(\theta^1) G_1(s) + \cdots + f(\theta^m) G_m(s). \tag{25}$$

Let s be an arbitrary bid. By Lemma 4, $\overline{b}_m^* \ge \overline{b}_k$ for all $k \in \{1, \ldots, m\}$. Therefore, if $s > \overline{b}_m^*$, then $\mathcal{B}_f^{\beta}(s) = \mathcal{B}_f^{\beta^*}(s) = 1$. If $s \in [\theta^{l-1}, \theta^l)$ for $\theta^l \le \mu$, then

$$\mathcal{B}_f^{\beta^*}(s) = f(\theta^1) + \cdots + f(\theta^{l-1}).$$

Since θ^{l-1} is the highest possible bid for a bidder with valuation θ^{l-1}, it holds that $G_k(\theta^l) = 1$ for all $k \le l-1$. It follows that

$$\begin{aligned}\mathcal{B}_f^{\beta}(s) &\ge \mathcal{B}_f^{\beta}(\theta^{l-1}) \ge f(\theta^1) G_1(\theta^{l-1}) + \cdots + f(\theta^{l-1}) G_{j-1}(\theta^{l-1})\\&= f(\theta^1) + \cdots + f(\theta^{l-1}) = \mathcal{B}_f^{\beta^*}(s).\end{aligned}$$

Thus, we can assume that $s \in [\underline{b}_k^*, \overline{b}_k^*]$ for some $k \in \{1, \dots, m\}$ with $\theta^k > \mu$. Since $(\mathcal{G}_{\varphi^*}, \beta^*)$ is a mental model with an efficient equilibrium, it follows that

$$\mathcal{B}_f^{\beta^*}(s) = f(\theta^1) + \cdots + f(\theta^{k-1}) + f(\theta^k) G_k^*(s).$$

Since $\overline{b}_m^*$ is the highest possible bid in any mental model, three cases are relevant: (i) $s \in (\bar{b}_m, \bar{b}_m^*]$, (ii) $s \in (\bar{b}_h, \underline{b}_{h+1})$, and (iii) $s \in [\underline{b}_{h-1}, \overline{b}_h]$ for some $1 \leq h \leq m$. Case (i) is immediate. For case (ii), observe that by Lemma 2, $\overline{b}_l \leq \overline{b}_l^*$ for all $1 \leq l \leq m$ and, therefore, $h \geq k$. We can immediately conclude that $\mathcal{B}_f^{\beta}(s) \geq \mathcal{B}_f^{\beta^*}(s)$. Thus, consider case (iii). Since β is an efficient Nash equilibrium, it holds that $\underline{b}_h \leq \overline{b}_{h-1}$ and

$$\mathcal{B}_f^{\beta}(s) = f(\theta^1) + \cdots + f(\theta^{h-1}) + f(\theta^h) G_h(s).$$

As shown in Lemma 4, $\overline{b}_l \leq \overline{b}_l^*$ for all $1 \leq l \leq m$. It follows that $h \geq k$. If $h > k$, we can immediately conclude that $\mathcal{B}_f^{\beta}(s) \geq \mathcal{B}_f^{\beta^*}(s)$. If $h = k$, we have to show that $G_k(s) \geq G_k^*(s)$. In order to do so, we will show that the bid distribution G_k^* dominates the bid distribution G_k in terms of the reverse hazard rate g/G.

It holds that

$$G_k^*(s) = \frac{\left(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-1})\right)\left(\left(\theta^k - \overline{b}_{k-1}^*\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)}{f^{k,*}(\theta^k)\left(\theta^k - s\right)^{\frac{1}{n-1}}},$$

from which it follows that the reverse hazard rate of this bid distribution is given by

$$\begin{aligned} &\frac{f^{k,*}(\theta^k)\left(\theta^k - s\right)^{\frac{1}{n-1}}}{\left(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-1})\right)\left(\left(\theta^k - \overline{b}_{k-1}^*\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)} \\ &\quad \cdot \frac{\left(f^{k,*}(\theta^1) + \cdots + f^{k,*}(\theta^{k-1})\right)\left(\theta^k - \overline{b}_{k-1}^*\right)^{\frac{1}{n-1}}}{f^{k,*}(\theta^k)\left(\theta^k - s\right)^{\frac{n}{n-1}}(n-1)} \\ &= \frac{\left(\theta^k - \overline{b}_{k-1}^*\right)^{\frac{1}{n-1}}}{\left(\left(\theta^k - \overline{b}_{k-1}^*\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)\left(\theta^k - s\right)(n-1)}. \end{aligned}$$

For G_k consider two cases. First, $f^k(\theta^k) = 0$. In this case, it follows that $\bar{b}_k = \bar{b}_{k-1}$ and we can immediately conclude that $\mathcal{B}_f^{\beta}(s) \geq \mathcal{B}_f^{\beta^*}(s)$. Second, $f^k(\theta^k) > 0$. In this case, G_k is atomless. If G_k were to have an atom and $f^k(\theta^k) > 0$, a bidder with valuation θ^k would slightly outbid the atom instead of mixing her bids. Thus, we obtain that the reverse hazard rate of the bid distribution G_k is given by

$$\frac{\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}}}{\left(\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)\left(\theta^k - s\right)(n-1)}.$$

We therefore have to show that

$$\frac{(\theta^k - \overline{b}^*_{k-1})^{\frac{1}{n-1}}}{(\theta^k - \overline{b}^*_{k-1})^{\frac{1}{n-1}} - (\theta^k - s)^{\frac{1}{n-1}}} \geq \frac{(\theta^k - \overline{b}_{k-1})^{\frac{1}{n-1}}}{(\theta^k - \overline{b}_{k-1})^{\frac{1}{n-1}} - (\theta^k - s)^{\frac{1}{n-1}}}$$

$$\Leftrightarrow \quad (\theta^k - \overline{b}^*_{k-1})^{\frac{1}{n-1}}\big((\theta^k - \overline{b}_{k-1})^{\frac{1}{n-1}} - (\theta^k - s)^{\frac{1}{n-1}}\big)$$
$$\geq (\theta^k - \overline{b}_{k-1})^{\frac{1}{n-1}}\big((\theta^k - \overline{b}^*_{k-1})^{\frac{1}{n-1}} - (\theta^k - s)^{\frac{1}{n-1}}\big),$$

which reduces to

$$(\theta^k - \overline{b}^*_{k-1})^{\frac{1}{n-1}}(\theta^k - s)^{\frac{1}{n-1}} \leq (\theta^k - \overline{b}_{k-1})^{\frac{1}{n-1}}(\theta^k - s)^{\frac{1}{n-1}}. \tag{26}$$

Since $\overline{b}^*_{k-1} \geq \overline{b}_{k-1}$ by the first part of the proof, the last statement is obviously true. □

A.5 *Proof of Proposition 5*

Proof. Both properties are a direct consequence of Proposition 4.

(i) Following the same steps as in the proof of Proposition 4, we can show that $\mathcal{B}_f^{\beta^*}$ first-order stochastically dominates $\mathcal{B}_f^{\beta}$. It follows directly that the expectation of the first-order statistic of n random variables drawn from $\mathcal{B}_f^{\beta^*}$ is larger than the expectation of the first-order statistic of n variables drawn from $\mathcal{B}_f^{\beta}$.

(ii) Riley (1989) provides a revenue-equivalence principle that holds for first- and second-price auctions with discrete valuations. His argument does not rely on any tie-breaking rule and is based on two properties of any equilibrium of the first-price auction: monotonicity (that is, that each bidder's bids are weakly increasing in her valuation) and continuity (that is, that there are no gaps or atoms in the bid distribution). Both properties hold with an efficient tie-breaking rule. It follows that for any true valuation distribution, the revenue in the second-price auction is equal to the revenue in the first-price auction where the true valuation distribution is a common prior. Since a first-price auction with a common prior together with the unique efficient symmetric equilibrium constitutes a profile of mental models, the statement follows from part (i). □

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