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Contests with sequential entry and incomplete information

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This paper provides a general study of a contest modeled as a multiplayer incomplete-information, all-pay auction with sequential entry. The contest consists of multiple periods. Players arrive and exert efforts sequentially to compete for a prize. They observe the efforts made by their earlier opponents, but not those of their contemporaneous or future rivals. We establish the existence and uniqueness of a symmetric perfect Bayesian equilibrium (PBE) and fully characterize the equilibrium. Based on the equilibrium result, we show that a later mover always secures a larger ex ante expected payoff. Further, we endogenize the timing of moves and show that all players choose to move in the last period in the unique equilibrium that survives iterated elimination of strictly dominated strategies (IESDS).

Keywords. Contest with sequential entry, all-pay auction, later-mover advantage, endogenous timing.

JEL classification. C72, D43, D44, D82, L13.

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1. Introduction

Many competitive activities resemble a contest, in which contenders strive to leapfrog and their efforts are nonrefundable regardless of win or loss. Such phenomena are widespread in socioeconomic contexts, ranging from electoral campaigns (Snyder (1989)), lobbying (Che and Gale (1998), Baye, Kovenock, and De Vries (1993)), internal labor markets inside firms (Lazear and Rosen (1981), Rosen (1986), Green and Stokey (1983)), and sporting events (Brown (2011)) to R&D races (Loury (1979), Lee and Wilde (1980), Taylor (1995), Fullerton and McAfee (1999), Che and Gale (2003)).

Contest-like competitions in practice are often inherently sequential, in that contenders enter and act in succession. Firms may enter a race successively for an innovative technology. Consider, for instance, the recent race to develop Coronavirus vaccines. Moderna/NIH, China's CanSino Biologics, and the University of Oxford/AstraZeneca PLC took the lead in entry.[1] Promising results in early trials sparked strong enthusiasm and encouraged a massive global effort, with more than 200 candidates jumping on the bandwagon. In an R&D project, a firm often has to decide on the intensity of its efforts (e.g., the number of trials) before research progress materializes due to budget requirements and resource planning, which cannot later be flexibly adjusted. Further, firms' actions are often subject to disclosure requirements or leaked to competitors. For instance, EU countries typically require mandatory disclosure of firms' R&D activities (La Rosa and Liberatore (2014)). In the United States, the Honest Leadership and Open Government Act of 2007 amended the Lobbying Disclosure Act of 1995, which strengthened public disclosure requirements regarding lobbying activities and funding. On Taskcn, a leading crowdsourcing platform, a participant is given access to earlier submissions (Liu, Yang, Adamic, and Chen (2014)), and an earlier entrant is fully aware of the information spillover to future contenders.

Dynamic interactions arise in such scenarios. Later movers condition their actions on prior moves, and an earlier mover shapes their strategies in anticipation of future opponents' reactions. These complicate strategical analysis of the contest game. The complexity can be further compounded when the contest allows for richer timing architectures: For instance, multiple players can enter and act in a single period simultaneously; they observe prior actions but not contemporaneous actions, which embeds simultaneous competitions in a dynamic structure. Consider a biopharmaceutical firm that entered the race for Coronavirus vaccines in mid-2020; its research can presumably leverage the efforts of pioneers, but not those of the many entrants that flooded into the arena within the short time window. A full-fledged analysis involves substantial analytical subtlety, which confines the majority of previous studies to limited settings, for example, a two-player, two-period structure.[2,3]

Sequential moves have spawned two related classical questions in oligopoly theory (Amir and Stepanova (2006)). The first concerns players' comparative payoffs with respect to their timing positions; that is, the earlier- versus later-mover advantage (see,

[1] See https://www.nytimes.com/2020/05/20/health/coronavirus-vaccines.html.

[2] See, for example, Dixit (1987), Baik and Shogren (1992), and Hoffmann and Rota-Graziosi (2012).

[3] Hinnosaar (2024) provides a remarkable exception.

e.g., Gal-Or (1987); Dowrick (1986); Dixit (1987)). The second views the timing architecture of an oligopoly as the endogenously determined outcome of players' strategic choices (see, e.g., Hamilton and Slutsky (1990); Amir (1995); Morgan (2003)), which addresses the classical Cournot/Stackelberg debate. The conventional wisdom obtained in the usual duopolistic settings, however, does not readily extend under more general sequential structures and deserves to be reexamined. Shinkai (2000), for instance, considers a three-firm, three-period model. He shows that players' payoffs can be nonmonotone along the sequence, which precludes a convenient answer in general to the question regarding early- or later-mover advantage in oligopoly.

We consider a general contest game with sequential entry that imposes no restrictions on the number of players and accommodates a full spectrum of timing architectures. Analogous to standard static all-pay auction models (e.g., Moldovanu and Sela (2001) and Moldovanu, Sela, and Shi (2007)), ex ante symmetric players strive for a commonly valued prize and the highest bidder wins; players' private types (abilities) are independently and identically distributed, with higher ability yielding lower marginal effort cost. The contest proceeds in multiple periods, and multiple players can be clustered in a single period; all players within each period act simultaneously and they observe earlier moves. A fully sequential contest and the standard simultaneous benchmark boil down to special cases of our model. The unrestricted timing architecture introduces substantial game-theoretical subtleties that would be absent in the usual duopolistic settings.[4,5] The literature has yet to provide an equilibrium analysis of this game, and our paper fills the gap. The equilibrium result further enables us to tackle the two aforementioned classical questions.

Findings and implications: Summary Our paper first conducts a comprehensive equilibrium analysis of the contest game with sequential entry described above. To meet the analytical challenges posed by the dynamic interactions, we take advantage of the recursive property of the payoff structure and convert the game into one that resembles a simultaneous-move, all-pay auction with an *endogenously* determined prize. The pseudo-prize is shaped by players' ability distribution function and can be expressed as a function of a player's bid. Our model does not impose specific requirements on the curvature of players' ability distribution. This may cause irregularity in their payoff functions and, therefore, discontinuity in their bidding strategies. Despite the nuance, we establish that there exists a unique symmetric perfect Bayesian equilibrium (PBE) in the game and provide a complete equilibrium characterization (Theorem 1). The

[4]In a simple two-player, sequential-move contest, the second mover, upon observing the first mover's effort, either simply matches the earlier effort or stays inactive. This property greatly simplifies the equilibrium analysis. This, however, no longer holds when a third player is introduced to the contest. Imagine a simple case with three players and fully sequential moves. Now the second mover cannot simply match the earlier effort, which allows him to defeat the first mover but may not be optimal given the threat from the third. The optimal response depends on his expectation of the future competition. The literature has yet to provide an equilibrium analysis of this game, and our paper fills the gap.

[5]Segev and Sela (2014) and Jian, Li, and Liu (2017) allow for multiple players but assume a fully sequential structure. As previously noted, Hinnosaar (2024) provides a remarkable exception to the literature that allows for an unrestricted timing architecture but assumes a lottery contest, which differs from our setting.

equilibrium result enables three applications that shed light on the fundamentals of the contest game with sequential entry.

We first investigate whether a player who moves later would receive a higher (lower) payoff than his earlier opponents. We establish that a payoff monotonicity arises: Regardless of the prevailing contest architecture, a player ends up with a higher ex ante expected payoff in a later timing position vis-à-vis an earlier one (Theorem 2). Our result thus provides a formal argument for an unambiguous later-mover advantage in the context of multiplayer contests.

We then allow players to simultaneously commit to the timing of their moves prior to the contest, which endogenizes the timing architecture of the contest. It is worth noting that despite the inherent overlap, the above-mentioned analysis—which establishes a later-mover advantage—does not address a player's timing choice. The later-mover advantage is obtained by comparing players' ex ante expected payoff across different periods under a given timing architecture. A player's timing choice, however, affects the timing architecture of the contest; as a result, the analysis requires that we compare a player's equilibrium expected payoffs across different timing architectures. We formally verify that all players choose the last period for their moves, which constitutes the unique equilibrium that survives iterated elimination of strictly dominated strategies (Theorem 3). A fully simultaneous contest arises when each player makes autonomous timing choices.

Finally, we generalize the model to allow for a hybrid payment rule that involves both winner-pay and all-pay elements. Specifically, the winner of the contest is obliged to pay the full cost of his own effort, while a loser may only pay a fraction of that. Our analysis can readily be adapted to accommodate this extension to characterize the equilibrium, as in Theorem 1. The main implications of the equilibrium are summarized in Theorem 4 and are consistent with the insights obtained in Theorems 2 and 3. This indicates that our main predictions do not rely on the all-pay feature of the baseline model.

Link to the literature This paper belongs to the small but burgeoning literature on sequential contests. Dixit (1987), Baik and Shogren (1992), Morgan (2003), and Hoffmann and Rota-Graziosi (2012) all consider complete-information Tullock contests in which two players move sequentially. Morgan and Várdy (2007) adopt a similar framework but assume that the follower has to bear a small cost to observe the leader's effort. Glazer and Hassin (2000) allow for three-period sequential plays. Analysis of multi-player sequential contests involves substantial technical difficulties, because standard backward induction is to no avail. Kahana and Klunover (2018) apply an "inverted best response" approach to a fully sequential lottery contest with multiple symmetric players. Hinnosaar (2024) allows for a general setup that imposes no restrictions on the prevailing timing architecture. Remarkably, he generalizes and formalizes Dixit's thesis that earlier players exert strictly higher efforts and are rewarded with strictly higher payoffs, which results from the strategic substitutability of efforts in a symmetric sequential lottery contest.

Our paper examines a radically contrasting game theoretical context (i.e., all-pay auctions) and provides a general and comprehensive analysis that imposes no restrictions on timing architectures and allows for a broader class of ability distribution functions. All-pay auctions do not generate a continuous and well-behaved best-response correspondence, unlike a lottery or a Tullock contest. Our results diverge from that of Hinnosaar (2024): We establish a later-mover advantage. Segev and Sela (2014) and Jian, Li, and Liu (2017) both consider fully sequential incomplete-information all-pay auctions. Segev and Sela (2014), assuming concave distribution functions, investigate how ex ante heterogeneous players' expected highest effort depends on the number of players and ability distributions. Jian, Li, and Liu (2017), assuming that players' type distribution function takes a power functional form, compare ex ante symmetric players' winning probabilities with respect to the order of moves. Konrad and Leininger (2007) consider two-stage multiplayer complete-information all-pay auctions. They show that, as in simultaneous-move contests, only the player with the lowest cost ends up with a positive expected payoff, while the payoff depends on his own timing position vis-à-vis those of the others.

This paper contributes to the extensive literature on players' comparative payoffs with respect to their timing positions in sequential-move games—such as Gal-Or (1985, 1987), Dixit (1987), Dowrick (1986), Daughety (1990), Deneckere and Kovenock (1992), Amir and Grilo (1999), Van Damme and Hurkens (1999, 2004), Amir and Stepanova (2006), and von Stengel (2010), among many others—in various contexts, ranging from quantity/capacity to price-setting competitions.[6] As stated above, this strand of the literature typically focuses on duopolistic rivalry. Shinkai (2000) extends the framework to a three-firm, three-period setting and illuminates the nuance caused by the more extensive sequence. To the best of our knowledge, our paper and Hinnosaar's (2024) are the few exceptions in the literature that examine earlier-/later-mover advantage under an unrestricted timing architecture.

Our analysis adds to the literature on endogenous timing in oligopoly, such as Hamilton and Slutsky (1990), Mailath (1993), Amir (1995), and Amir and Stepanova (2006). A handful of studies explore this issue in contest settings, including Baik and Shogren (1992), Leininger (1993), and Morgan (2003). All of these studies consider two-player models. Konrad and Leininger (2007) allow for multiple contestants, but impose a two-period structure.

The rest of the article is organized as follows. Section 2 sets up the model. Section 3 characterizes the equilibrium. Section 4 further delves into the fundamentals of this contest game and applies our equilibrium results to the extended settings. Section 5 concludes. Proofs are relegated to the Appendix.

2. The model

A contest involves $N \geq 2$ ex ante identical risk-neutral players, indexed by $i \in \mathcal{N} \equiv \{1, \dots, N\}$. The players arrive sequentially and each exerts effort upon arrival to compete for a prize with unit value. The contest proceeds in $T \geq 1$ period(s), and the players

[6] Kempf and Rota-Graziosi (2010) consider a setting in which two jurisdictions set tax rates and endogenize leadership in tax competitions.

are accordingly partitioned into T groups. Denote by $\mathcal{N}_t$ the set of players in period t, and let $n_t := |\mathcal{N}_t| \geq 1$ indicate the number of players in $\mathcal{N}_t$. A player observes the efforts sunk by his earlier opponents but not those in contemporaneous or future periods. The architecture of the contest is fully described by a vector $\boldsymbol{n} := (n_1, \dots, n_T)$, with $N = \sum_{t=1}^{T} n_t$.[7] The contest is fully sequential with $\boldsymbol{n} = (1, \dots, 1)$, while it degenerates to a fully simultaneous one with $\boldsymbol{n} = (N)$.

A player i, when exerting an effort (or, interchangeably, a bid) $b^i \geq 0$, incurs a cost $c(b^i) = b^i/a^i$, where $a^i > 0$ measures one's ability and is privately known.[8] Abilities are drawn independently from an interval $(0, 1]$ according to a common distribution function $F(\cdot)$. We assume that $F(\cdot)$ admits a positive and continuous density $f(\cdot) \equiv F'(\cdot)$ and is piecewise analytic on $[\delta, 1]$ for all $\delta \in (0, 1)$.

Winner selection mechanism and payoffs The competition is modeled as an all-pay auction. The player with the highest effort wins. Specifically, a player $i \in \mathcal{N}_t$, when exerting an effort $b^i \geq 0$, is the *sole winner* if and only if (i) his effort is greater than or equal to those in earlier periods (i.e., $b^i \geq b^j$ for $j \in \bigcup_{k=1}^{t-1} \mathcal{N}_k$) and (ii) his effort is strictly larger than those in contemporaneous and future periods (i.e., $b^i > b^j$ for $j \in \bigcup_{k=t}^{T} \mathcal{N}_k \setminus \{i\}$). In the event that (i) multiple players in period t place the same highest bid and (ii) no future players match that, the prize is randomly distributed among them. To put it more formally, fixing a set of effort entries $\boldsymbol{b} \equiv (b^1, \dots, b^N)$, contestant i's winning probability is[9]

$$p^i(\boldsymbol{b}) := \begin{cases} 1, & \text{if } b^i \geq \max\limits_{j \in \bigcup_{k=1}^{t-1} \mathcal{N}_k} \{b^j\} \text{ and } b^i > \max\limits_{j \in \bigcup_{k=t}^{T} \mathcal{N}_k \setminus \{i\}} \{b^j\}, \\ 1/m, & \text{if } b^i \geq \max\limits_{j \in \bigcup_{k=1}^{t-1} \mathcal{N}_k} \{b^j\}, b^i > \max\limits_{j \in \bigcup_{k=t+1}^{T} \mathcal{N}_k} \{b^j\}, \\ & \text{and } b^j \text{ is among the } m \text{ highest of } \{b^j\}_{j \in \mathcal{N}_t} \text{ with a tie}, \\ 0, & \text{if } b^i < \max\limits_{j \in \bigcup_{k=1}^{t} \mathcal{N}_k \setminus \{i\}} \{b^j\} \text{ or } b^i \leq \max\limits_{j \in \bigcup_{k=t+1}^{T} \mathcal{N}_k} \{b^j\}, \end{cases} \tag{1}$$

and his ex post payoff, for a given ability level a^i, is

$$p^i(\boldsymbol{b}) - b^i/a^i, \quad \text{for all } i \in \mathcal{N}. \tag{2}$$

Equilibrium concept We consider the solution concept of perfect Bayesian equilibrium (PBE) for the contest game with sequential entry throughout the paper. We focus on the symmetric equilibrium in which all players in the same period adopt the same bidding strategy.

[7] We ignore periods in which no players enter.

[8] We follow the tradition in the contest literature and accommodate player heterogeneity in their cost functions (e.g., Moldovanu and Sela (2001, 2006); Moldovanu, Sela, and Shi (2007); Brown and Minor (2014)). It is noteworthy that the model is isomorphic to an alternative setting in which effort is interpreted as bid in the auction literature: Players value the prize differently but bear the same effort costs. All of our results remain qualitatively unchanged under this model specification.

[9] The tie-breaking rule in (1) is asymmetric, which is commonly assumed in the literature (see, e.g., Andreoni, Che, and Kim (2007), Simon and Zame (1990), Maskin and Riley (2000)). The asymmetry ensures well-defined best responses and the existence of an equilibrium, as in an asymmetric Bertrand duopoly game.

More formally, we fix an effort profile $(b^j)_{j\in\bigcup_{k=1}^{t-1}\mathcal{N}_k}$. Define $\beta_t := \max_{j\in\bigcup_{k=1}^{t-1}\mathcal{N}_k}\{b^j\}$ for $t\in\{2,\dots,T\}$, and let $\beta_1\equiv 0$. In words, β_t is the maximum effort in the contest prior to period t. A symmetric PBE is denoted by $\{b_t^*(a;\beta_t)\}_{t=1}^T$, where a is a player's ability and $b_t^*(a;\beta_t)$ is the equilibrium bidding strategy for a player in period t. It is noteworthy that the information available to a period-t player i is summarized by his own type a^i and the highest effort $\beta_t\equiv\max_{j\in\bigcup_{k=1}^{t-1}\mathcal{N}_k}\{b^j\}$ instead of the bidding history prior to period t, that is, $(b^j)_{j\in\bigcup_{k=1}^{t-1}\mathcal{N}_k}$: Only the maximum previous bid matters to a player in an all-pay auction with sequential entry (see Equation (1)), so β_t can be viewed as a sufficient statistic for $(b^j)_{j\in\bigcup_{k=1}^{t-1}\mathcal{N}_k}$.

Besides the usual restrictions for PBE, we require that each player not update his belief about contemporaneous and future rivals upon observing *past* effort levels and his own type, either on or off the equilibrium path. This condition is sensible because players' abilities are independently distributed.[10] Further, players' payoffs in our setting do not depend on their beliefs about earlier movers' abilities. As a result, we do not specify a belief system explicitly to define the PBE.

3. Equilibrium analysis

In this section, we first lay out the fundamentals of the analysis and then formally characterize the equilibrium.

3.1 *Preliminaries of equilibrium analysis*

This section sets up important primitives that lay the foundation for our equilibrium analysis. We first introduce several pieces of notation that pave the way for our analysis and discussion. We then present four preliminary results (Lemmas 1 to 4) that underpin the main equilibrium results. Lemma 1 depicts the fundamentals of the bidding strategies in a hypothetical symmetric PBE and enables subsequent analysis that relies on the recursive nature of this contest game with sequential entry. Lemma 2 narrows the set of equilibrium efforts. Lemmas 3 and 4 identify a potential discontinuity in the equilibrium bidding strategies.

Fixing a contest architecture $\boldsymbol{n}\equiv(n_1,\dots,n_T)$, we define a sequence of functions $\{Q_t(b), a_t^*(\beta), \tilde{\pi}_t(b,a)\}_{t=1}^T$, recursively, as follows:

$$Q_T(b)\equiv 1,\qquad Q_{t-1}(b):=Q_t(b)F^{n_t}\big(a_t^*(b)\big),\quad \forall b\in[0,1], \tag{3}$$

$$a_t^*(\beta):=\max\big\{0<a\leq 1:\tilde{\pi}_t(b,a)\leq 0,\forall b\in[\beta,1]\big\}, \tag{4}$$

$$\tilde{\pi}_t(b,a):=Q_t(b)F^{n_t-1}(a)-b/a. \tag{5}$$

The sequence of functions $\{Q_t(b), a_t^*(\beta), \tilde{\pi}_t(b,a)\}_{t=1}^T$ is key to equilibrium characterization, and their implications will be revealed as the analysis unfolds. As a head

[10] Note that for all totally mixed small perturbations of beliefs, a player's beliefs about contemporaneous and future rivals must be equal to the prior, which implies that this belief satisfies the consistency requirement under the solution concept of sequential equilibrium.

start, the function $a_t^*(\beta)$ allows us to derive the threshold ability above (below) which a period-t player stays active (inactive) in equilibrium. Let $b \geq 0$ be the realized highest effort by the end of period $t \in T$. Then $Q_t(b)$ gives the probability of effort b's exceeding all subsequent efforts in a symmetric equilibrium. The function $\tilde{\pi}_t(b, a)$ lays a foundation for equilibrium payoff characterization; it plays a critical role in identifying the relevant range of equilibrium efforts and potential discontinuity in players' bidding strategies.

It can be verified from the above definition that $a_t^*(\beta)$ and $Q_t(b)$ are well-defined and satisfy the following properties:

(a) $a_t^*(\beta)$ is continuous, piecewise differentiable, and weakly increasing on $[0, 1]$, satisfying $a_t^*(\beta) \geq \beta$ for all $t \in \mathcal{T}$; and

(b) $Q_t(b)$ is continuous, piecewise differentiable, and strictly increasing on $[0, 1]$, with $Q_t(0) = 0$ and $Q_t(1) = 1$ for all $t \in \mathcal{T} \setminus \{T\}$.[11]

We hereby present four lemmas that pave the way for our equilibrium result. Lemma 1 delineates useful properties of players' bidding strategy in a hypothetical symmetric PBE.

Lemma 1 (Properties of Equilibrium Bidding Strategy). *Consider a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$, and suppose that a symmetric PBE exists. A period-t player's equilibrium bidding strategy $b_t^*(a; \beta_t)$ satisfies the following properties:*

(i) $b_t^*(a; \beta_t)$ is increasing in a on $(0, 1)$;

(ii) $b_t^*(a; \beta_t) = 0$ for $a \leq a_t^*(\beta_t)$ and $b_t^*(a; \beta_t) \geq \beta_t$ for $a > a_t^*(\beta_t)$;

(iii) $b_t^*(a; \beta_t)$ strictly increases with a on $(a_t^*(\beta_t), 1)$ if $n_t \geq 2$.

Lemma 1(i) is intuitive: A stronger player tends to bid more aggressively. Lemma 1(ii) reveals the nature of $a_t^*(\cdot)$: A period-t player would stay active (inactive) in equilibrium if his ability a_t exceeds (falls short of) the threshold $a_t^*(\beta_t)$. Recall that $a_t^*(\cdot)$ increases with its arguments and β_t is the maximum effort prior to period-t, which implies that higher earlier effort elevates the threshold for active bidding, thereby discouraging future competition. By Lemma 1(iii), when a period t involves two or more players, one's effort strictly increases with his ability provided that he is willing to place a positive bid, that is, $a > a_t^*(\beta_t)$. Note that the strict monotonicity does not necessarily hold in the case with $n_t = 1$. To see this, consider a two-player, sequential-move contest $\boldsymbol{n} = (1, 1)$. The later mover simply matches the first mover's effort β_2 irrespective of his own ability, provided that it exceeds $a_2^*(\beta_2) = \beta_2$, that is, $b_2^*(a; \beta_2) = \beta_2$ for $a > a_2^*(\beta_2)$.

Recall that $b \geq 0$ denotes the realized highest effort by the end of period $t \in \mathcal{T}$, which leads to an eventual win if and only if it exceeds all subsequent efforts from period $t + 1$. Lemma 1(ii) allows us to derive the equilibrium probability of this event. By Lemma 1, b ends up as the eventual winning effort if and only if all subsequent players stay inactive (i.e., every period-ℓ player's ability falls below the threshold $a_\ell^*(b)$, $\forall \ell \in \{t + 1, \ldots, T\}$),

[11] See the proof of Lemma 8 in the Appendix for more details.

which occurs with a probability of $\Pi_{\ell=t+1}^{T} F^{n_\ell}(a_\ell^*(b))$; otherwise, at least one player in later periods would stay active and exert an effort above b. Notably, this probability can be expressed recursively, which boils down to the function $Q_t(b)$ in (3). We formally establish this fact in the Appendix (see Lemma 7 in Appendix A.1).

The property of $Q_t(b)$ enables us to exploit the recursive nature of this contest game with sequential entry, which simplifies the equilibrium analysis. Consider a period-t player with ability $a > 0$. In a symmetric PBE, by exerting an arbitrary effort $b \geq \beta_t$, he earns an expected payoff[12]

$$\pi_t(b, a; \beta_t) := Q_t(b) F^{n_t - 1}\big((b_t^*)^{-1}(b; \beta_t)\big) - b/a. \tag{6}$$

He wins with a probability $Q_t(b)F^{n_t-1}((b_t^*)^{-1}(b; \beta_t))$: As stated above, the effort b allows him to beat future opponents with a probability $Q_t(b)$ and prevail over his contemporaneous competitors with a probability $F^{n_t-1}((b_t^*)^{-1}(b; \beta_t))$.

The strategic interactions between a period-t player and his future opponents are encapsulated in the provisional winning probability function $Q_t(\cdot)$: His equilibrium bidding strategy can be solved for as if he competed in a static contest for a prize of a value $Q_t(b)$, which technically dissolves the *dynamic* linkages between contestants across different periods. Despite the analogy, the pseudo-prize value, $Q_t(b)$, endogenously depends on the player's own effort b, so the equilibrium bidding strategy fundamentally differs from that in a standard static contest.

3.1.1 *The set of equilibrium efforts* We now set out to narrow the set of equilibrium efforts. Consider a period-t player with ability $a > 0$ who faces contemporaneous competition, that is, $n_t \geq 2$. Recall the function $\tilde{\pi}_t(b, a)$ in (5):

$$\tilde{\pi}_t(b, a) \equiv Q_t(b) F^{n_t-1}(a) - b/a.$$

Fixing $\beta_t \geq 0$, we define

$$S_t(a; \beta_t) := \big\{b \in [\beta_t, 1] : \tilde{\pi}_t(b, a) > \tilde{\pi}_t\big(b', a\big), \forall b' \in (b, 1]\big\}.$$

We will subsequently establish that any efforts outside those contained in $S_t(a; \beta_t)$ must be suboptimal. Note that the set $S_t(a; \beta_t)$ is type-dependent and shrinks as ability a increases, that is, $S_t(a'; \beta_t) \subseteq S_t(a; \beta_t)$ for $a < a'$. Further, recall the piecewise analyticity of the ability distribution $F(\cdot)$. This implies that $Q_t(b)$ and, therefore, $\tilde{\pi}_t(b, a)$, are continuous and piecewise analytic with respect to b on $[\beta_t, 1]$; $S_t(a; \beta_t)$, in turn, can be expressed as the union of finitely many disjoint intervals. For ease of exposition, we denote by $m_t(a; \beta_t) \in \mathbb{N}_+$ the number of disjoint intervals included in $S_t(a; \beta_t)$. Then $S_t(a; \beta_t)$ can be written as

$$\begin{aligned} S_t(a; \beta_t) = \big[s_t^1(a; \beta_t), e_t^1(a; \beta_t)\big) \cup \big[s_t^2(a; \beta_t), e_t^2(a; \beta_t)\big) \cup \cdots \\ \cup \big[s_t^{m_t(a;\beta_t)}(a; \beta_t), e_t^{m_t(a;\beta_t)}(a; \beta_t)\big], \end{aligned}$$

[12]The inverse function $(b_t^*)^{-1}(b; \beta_t)$ is well-defined by Lemma 1(iii) for $b \geq \beta_t$ and $n_t \geq 2$. Note that there is no need to specify $(b_t^*)^{-1}(b; \beta_t)$ for the case of $n_t = 1$ given that $F^{n_t-1}(a) = 1$ for all $a \in (0, 1)$.

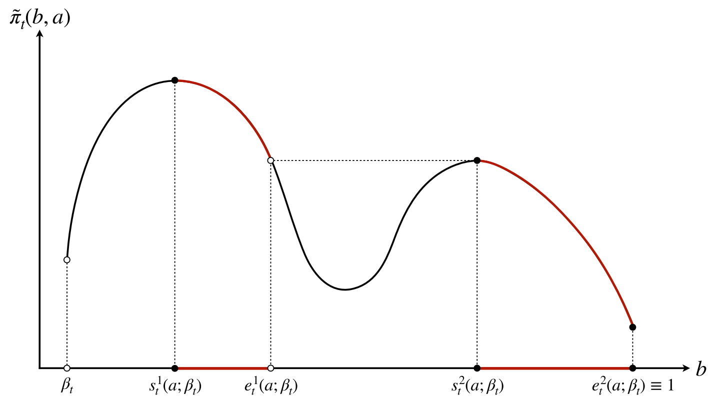


FIGURE 1. Illustration of $S_t(a; \beta_t)$.

with $e_t^m(a; \beta_t) < s_t^{m+1}(a; \beta_t)$ for $1 \leq m \leq m_t(a; \beta_t) - 1$ and $e_t^{m_t(a;\beta_t)}(a; \beta_t) \equiv 1$. Figure 1 graphically illustrates the set $S_t(a; \beta_t)$ for the case of $m_t(a; \beta_t) = 2$. The following can be obtained.

LEMMA 2 (Set of Equilibrium Efforts). *Consider a contest with sequential entry* $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ *and suppose that a symmetric PBE exists. A period-t player's equilibrium effort must be contained within the above-defined set* $S_t(a; \beta_t)$*, provided that it exceeds* β_t*. That is,* $b_t^*(a, \beta_t) \in S_t(a; \beta_t)$ *if* $b_t^*(a, \beta_t) \geq \beta_t$ *is continuous in the neighborhood of* a*.*

We verify this claim by the following argument. Suppose, to the contrary, that $b_t^*(a, \beta_t) \notin S_t(a; \beta_t)$. By definition, there exists some effort $b' > b_t^*(a, \beta_t)$ such that $\tilde{\pi}_t(b', a) \geq \tilde{\pi}_t(b_t^*(a, \beta_t), a)$. We claim that the player's payoff would strictly exceed the value of the constructed function $\tilde{\pi}_t(b', a)$ when he deviates from $b_t^*(a, \beta_t)$ to b'. This is because the higher effort b' increases not only the probability of outbidding his future opponents but also that of beating the contemporaneous ones. More specifically, let a' be the maximum ability such that $b_t^*(a, \beta_t) \leq b'$. Because the equilibrium bidding strategy $b_t^*(a, \beta_t)$ is strictly increasing and continuous around a, we can conclude that $a' > a$. In a symmetric PBE, the player's actual expected payoff from the deviation ends up as $Q_t(b')F^{n_t-1}(a') - b'/a$: In other words, he behaves as if he has an ability a', which allows him to defeat his contemporaneous opponents with a probability $F^{n_t-1}(a')$. This payoff strictly exceeds $\tilde{\pi}_t(b', a) \equiv Q_t(b')F^{n_t-1}(a) - b'/a$, and thus overshadows the equilibrium payoff $\tilde{\pi}_t(b_t^*(a, \beta_t), a)$. Contradiction ensues.

3.1.2 *Equilibrium effort of threshold ability type and potential discontinuity* The set $S_t(a; \beta_t)$ is constructed as the union of a finite number of disjoint intervals, which alludes to the possibility of discontinuity in a hypothetical equilibrium. The following two lemmas shed light on these possibilities and show that any discontinuity in a player's equilibrium bidding strategy, whenever it exists, must arise at the end points of these intervals.

We first establish that the smallest element in $S_t(a; \beta_t)$ is indeed the bid a player of the threshold ability $a = a_t^*(\beta_t)$ tends to place in equilibrium.

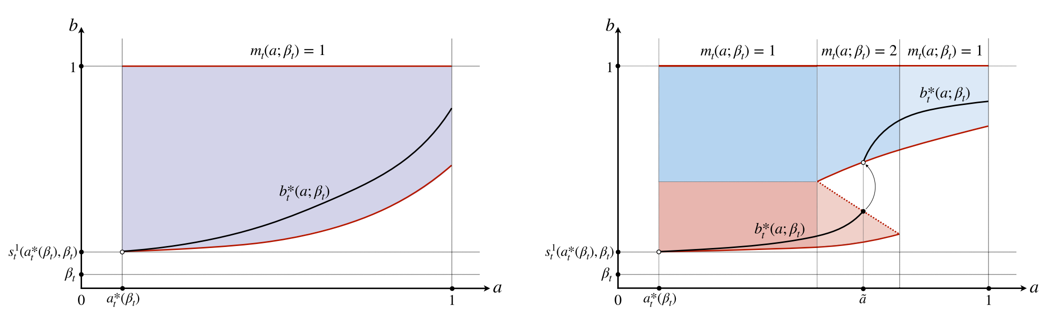
Lemma 3 (Equilibrium Effort at $a_t^*(\beta_t)$). *Consider a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and suppose that a symmetric PBE exists. If $n_t \geq 2$ and $a_t^*(\beta_t) < 1$, then $\lim_{a \searrow a_t^*(\beta_t)} b_t^*(a; \beta_t) = s_t^1(a_t^*(\beta_t); \beta_t)$.*

Because $b_t^*(a; \beta_t) = 0$ for $a \leq a_t^*(\beta_t)$, $\lim_{a \searrow a_t^*(\beta_t)} b_t^*(a; \beta_t) > 0$ indicates a discontinuity in bidding at $a_t^*(\beta_t)$. Such discontinuity does not come as a surprise, since it concerns itself with the behavior of the player of threshold ability who decides to outbid β_t. To see this, recall the two-player, sequential-move example in Footnote 4: The second mover matches the earlier effort when his type exceeds the threshold and remains inactive otherwise. Discontinuity in bidding thus arises for the second mover when his type equals the earlier bid.

The next lemma nevertheless suggests the possibility of discontinuity in a player's equilibrium bidding strategy when his ability exceeds the threshold $a_t^*(\beta_t)$, which stems from the dynamic nature of the game. Such discontinuity can even occur for the first mover. In Section 3.3, we demonstrate that such discontinuity may indeed emerge in equilibrium, but can only arise if the distribution function contains both concave and convex parts. For notational convenience, we use $b_t^*(a - 0; \beta_t)$ and $b_t^*(a + 0; \beta_t)$ to denote the left and right limits of $b_t^*(a; \beta_t)$, respectively. We have the following.

Lemma 4 (Potential Discontinuity of Players' Bidding Strategy). *Consider a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and suppose that a symmetric PBE exists. Fix a period t with $n_t \geq 2$ and a player's ability $\tilde{a} \in (a_t^*(\beta_t), 1]$. If $b_t^*(\tilde{a} - 0; \beta_t) = e_t^m(\tilde{a}; \beta_t)$ for some $1 \leq m \leq m_t(\tilde{a}; \beta_t) - 1$, then $b_t^*(\tilde{a} + 0; \beta_t) = s_t^{m+1}(\tilde{a}; \beta_t)$.*

By Lemma 1, a period-t player, for a given β_t, would increase his effort as his ability ascends. When the effort is in the interior of the set $S_t(a; \beta_t)$, the player's bidding strategy would gradually increase with a for $a > a_t^*(\beta_t)$, as Figure 2(a) illustrates. Recall that the set of eligible efforts $S_t(a; \beta_t)$ shrinks as a ascends (see Figure 2). When the player's effort reaches the end of some interval in the set $S_t(a; \beta_t)$ (i.e., $e_t^m(a; \beta_t)$), he would refrain from exerting an effort in the "undesirable" region $(e_t^m(a; \beta_t), s_t^{m+1}(a; \beta_t))$; his effort jumps directly to the lower bound of the next adjacent interval in $S_t(a; \beta_t)$, that is, $s_t^{m+1}(a; \beta_t)$. This scenario is depicted in Figure 2(b).



(a) Continuous equilibrium bidding strategy (b) Discontinuous equilibrium bidding strategy

Figure 2. Illustration of equilibrium bidding strategy $b_t^*(a; \beta_t)$.

To understand why the boundary of the set $S_t(a;\beta_t)$ can be played in the equilibrium, it is useful to further inspect the constructed function (5) and the equilibrium payoff function (6). Consider a symmetric PBE. When a player's effort increases, he ends up with a higher probability of outperforming his contemporaneous opponents; such an *equilibrium effect* is nevertheless omitted in the expression of (5). Suppose that all other players' bidding strategies contain a jump from $b^\sharp$ to $b^{\sharp\sharp}$ when one's ability increases. All efforts between $b^\sharp$ and $b^{\sharp\sharp}$ would yield the same probability of winning the contemporaneous competition. As a result, the aforementioned equilibrium effect dissolves around the jump.

The jump predicted in Lemma 4 and Figure 2(b) is impossible in a Bayesian Nash equilibrium of a *static* all-pay auction. The discontinuity, if it exists, largely stems from the dynamic interaction in the game, which is captured by the provisional winning probability function $Q_t(\cdot)$ in the interim expected payoff (6).

3.2 *Equilibrium result: Existence, uniqueness, and characterization*

We are ready to verify the existence and uniqueness of a symmetric PBE in the contest game with sequential entry under an arbitrary contest architecture $\boldsymbol{n} \equiv (n_1, \dots, n_T)$ and fully characterize it. Let $q_t(b) := Q_t'(b)$.

Theorem 1 (Equilibrium of Contests With Sequential Entry). *Consider a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \dots, n_T)$. There exists a unique symmetric PBE $\{b_t^*(a;\beta_t)\}_{t=1}^T$ of the contest game, which is fully characterized as follows:*[13]

(i) If $n_t = 1$, then

$$b_t^*(a;\beta_t)\begin{cases} = 0, & \text{if } a \le a_t^*(\beta_t), \\ = \beta_t, & \text{if } a_t^*(\beta_t) < a \le a_t^{**}(\beta_t), \\ \in \arg\max_{b>\beta_t}\big[Q_t(b) - b/a\big], & \text{if } a > a_t^{**}(\beta_t), \end{cases} \tag{7}$$

where $a_t^*(\beta_t)$ is defined in (4) and can be simplified as $a_t^*(\beta_t) = \min_{b\ge\beta_t} b/Q_t(b)$, and $a_t^{**}(\beta_t) := \sup_{a_t^*(\beta_t)\le a\le 1}\{a : Q_t(\beta_t) - \beta_t/a > Q_t(b') - b'/a, \forall b' \in (\beta_t, 1]\}$.

(ii) If $n_t \ge 2$, then $b_t^*(a;\beta_t) = 0$ for $a \le a_t^*(\beta_t)$. For $a > a_t^*(\beta_t)$, $b_t^*(a;\beta_t)$ increases continuously and is governed by the following differential equation:

$$\begin{aligned} &(n_t-1)aF^{n_t-2}(a)f(a)Q_t\big(b_t^*(a;\beta_t)\big) \\ &\quad + aF^{n_t-1}(a)q_t\big(b_t^*(a;\beta_t)\big)\big(b_t^*\big)'(a;\beta_t) - \big(b_t^*\big)'(a;\beta_t) = 0, \end{aligned} \tag{8}$$

with the initial condition $b_t^*(a_t^*(\beta_t)+0;\beta_t) = s_t^1(a_t^*(\beta_t);\beta_t)$. When $b_t^*(\tilde a;\beta_t) = e_t^m(\tilde a;\beta_t)$ for some $\tilde a \in (0,1)$ and $1 \le m \le m_t(\tilde a;\beta_t) - 1$, $b_t^*(a;\beta_t)$ jumps to $s_t^{m+1}(\tilde a;\beta_t)$ at $a = \tilde a$ and then increases continuously from $\tilde a$ again according to (8), with the initial condition $b_t^*(\tilde a + 0;\beta_t) = s_t^{m+1}(\tilde a;\beta_t)$.

[13] The symmetric PBE is unique in the sense that if there exist two symmetric PBE of the contest game—denoted by $\{b_t^*(a;\beta_t)\}_{t=1}^T$ and $\{b_t^{**}(a;\beta_t)\}_{t=1}^T$—then for all $t \in \mathcal{T}$ and $\beta_t \ge 0$, the collection of ability a such that $b_t^*(a;\beta_t) = b_t^{**}(a;\beta_t)$ has F-measure one.

Theorem 1 establishes the existence and uniqueness of a symmetric PBE in contests with sequential entry. A player's equilibrium bidding strategy depends on the number of contemporaneous opponents. Theorem 1(i) considers a scenario in which a single player arrives in a period, while Theorem 1(ii) addresses the case in which multiple players are clustered in one set $\mathcal{N}_t$. It is straightforward to observe that a player would remain inactive if he is of low ability (i.e., $a \leq a_t^*(\beta_t)$) in either scenario, as predicted in Lemma 1(ii). The predictions diverge between the two scenarios when the player's ability is sufficiently high. With $n_t = 1$, the player matches the highest prior bid β_t when his ability remains in an intermediate range (i.e., $a_t^*(\beta_t) < a < a_t^{**}(\beta_t)$) while he strictly outbids β_t if his ability exceeds the cutoff $a_t^{**}(\beta_t)$. In contrast, with $n_t \geq 2$, he strictly outbids β_t whenever his ability exceeds $a_t^*(\beta_t)$; contemporaneous competition compels him to step up effort to avoid a tie.

A closer look at Lemma 1 and Theorem 1 allows us to identify the set of players who constantly stay inactive, that is, exerting zero effort irrespective of their own types and previous efforts. Recall from Lemma 1(ii) that a period-t player would be completely discouraged if his ability falls below $a_t^*(\beta_t)$. Obviously, he would do so if $a_t^*(\beta_t) = 1$ for all $\beta_t \in [0, 1]$, which is equivalent to $\tilde{\pi}_t(b, 1) \leq 0$ for all $b \in [0, 1]$ by (4). This condition, together with (5) and $F(1) = 1$, implies $Q_t(b) \leq b$ for all $b \in [0, 1]$. Let $\mathcal{T}_0 := \{t \in \mathcal{T} : Q_t(b) \leq b, \forall b \in [0, 1]\}$. It can be verified that $t' \in \mathcal{T}_0$ if $t \in \mathcal{T}_0$ and $t' < t$. Define $t_0 := \max \mathcal{T}_0$; it is obvious to infer $0 \leq t_0 \leq T - 1$.[14] The following result naturally ensues.

Proposition 1 (Players who Always Remain Inactive). *Consider a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$. In the unique symmetric PBE $\{b_t^*(a; \beta_t)\}_{t=1}^T$ of the contest game, all players in periods 1 through t_0 choose to stay inactive regardless of their ability and the previous maximum bid, that is, $b_t^*(a; \beta_t) = 0$ for all $a \in (0, 1]$, $\beta_t \in [0, 1]$, and $t \in \mathcal{T}_0$.*

Proposition 1 states that players who arrive in early periods (i.e., $t \leq t_0$) always stay inactive, regardless of their own types. These players obviously receive zero expected payoff in equilibrium, which alludes to a disadvantage of being earlier movers in a contest. Players who arrive subsequently, in contrast, exert positive efforts with positive probabilities.

We illustrate our equilibrium results in more specific settings.

Corollary 1 (Players who Always Remain Inactive With Concave/Convex Ability Distributions). *Consider a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$. The following statements hold in the unique symmetric PBE:*

(i) Suppose that $F(\cdot)$ is continuous, twice differentiable, strictly concave, and satisfies $\lim_{a \searrow 0}[f(a)a] = 0$. Then $t_0 = 0$.

(ii) Suppose that $F(\cdot)$ is continuous, twice differentiable, and weakly convex. Then $t_0 = T - 1$.

[14] In the case that $\mathcal{T}_0$ is an empty set, we let $t_0 = 0$.

By Corollary 1(i), with a concave ability distribution $F(\cdot)$, all players exert positive efforts in equilibrium with positive probabilities. To understand the logic, consider a simple two-player, sequential-move contest $(1, 1)$ and focus on the first mover. His future opponent (i.e., the second mover) will either match his effort or stay inactive. The first mover thus ends up with an expected payoff $F(b) - b/a$ when he exerts an effort b, where $F(b)$ is the probability of his defeating the second mover. A concave ability distribution ensures that the first mover's expected payoff is concave in his effort, which in turn, implies that he tends to increase his effort *gradually* as his ability ascends. To put this intuitively, a concave ability distribution implies milder future competition, since the player who moves in the second period is likely to be mediocre, which compels the first mover to participate actively. That is, the marginal return of sinking the first unit of effort exceeds the associated marginal cost, that is, $F'(0) > 1$.

In contrast, with a convex ability distribution $F(\cdot)$, all players who arrive prior to period T stay inactive in equilibrium regardless of their own types. Period-T players behave as if they are participating in a simultaneous all-pay auction with n_T players, where the standard result in static all-pay auctions applies. We again resort to the two-player, sequential-move contest $(1, 1)$ to elaborate on the intuition. Recall that the first mover receives an expected payoff $F(b) - b/a$ when he sinks an effort of b, which is convex with a convex CDF $F(\cdot)$, and thus its maximizer is either zero or a sufficiently large effort. Intuitively, a convex ability distribution implies intense future competition, because high-ability players are likely to emerge in later periods. This disincentivizes early players, since inaction allows them to avert futile investment. In response, the players—except for those from the last period—choose to drop out of the competition.

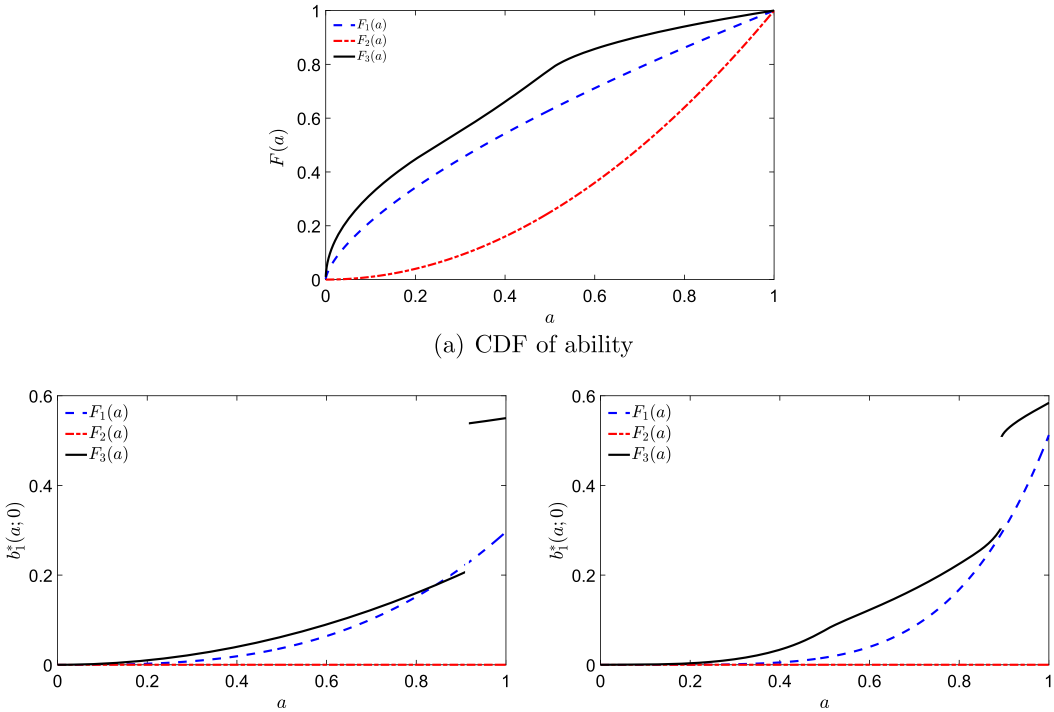
3.3 *Discussion: Discontinuity in equilibrium strategies*

The result of Theorem 1 can readily be adapted to derive the equilibrium in the concave/convexity case. Further, recall that Lemma 4 alludes to the possibility of discontinuous equilibrium bidding strategies. However, such discontinuity arises in neither of the cases laid out above (i.e., concave or convex ability distributions). Next, we demonstrate that discontinuity may indeed emerge under irregular distribution, for example, to be concave in some regions and convex in others.

Consider the following three ability distributions: (i) $F_1(a) = a^{2/3}$; (ii) $F_2(a) = a^2$; and (iii)

$$F_3(a) = \begin{cases} \sqrt{a}, & a < \dfrac{1}{4}, \\ \dfrac{(a+0.75)^2}{2}, & \dfrac{1}{4} \leq a \leq \dfrac{1}{2}, \\ 0.3837\sqrt{a - 0.4764} + 0.7224, & a > \dfrac{1}{2}. \end{cases}$$

Note that $F_3(\cdot)$ is convex in a on $[\frac{1}{4}, \frac{1}{2}]$ and concave on $(0, \frac{1}{4})$ and $(\frac{1}{2}, 1]$ (see Figure 3(a)). Again, consider a simple two-player, sequential-move contest $(n_1, n_2) = (1, 1)$. Figure 3(b) illustrates the first mover's equilibrium bidding strategy under each distribution. A jump in the bidding function with respect to the first mover's ability, a, arises



(a) CDF of ability

(b) Contest $(1, 1)$

(c) Contest $(2, 1)$

Figure 3. First mover's bidding strategy in two-period, sequential-move contests under different ability distributions.

under $F_3(\cdot)$. Recall that the first-mover's winning probability is given by $F_3(b)$ and his expected payoff is $F_3(b) - b/a$. The curvature of the CDF of ability captures the magnitude of the marginal return on his effort. In the convex region of the ability distribution, the first mover enjoys increasing marginal returns on his effort. As a result, his bid would not fall in the region of $[\frac{1}{4}, \frac{1}{2}]$. Moreover, as shown by Figure 3(c), the discontinuity would persist when an additional player is added in the first period, which yields a simultaneous competition ($n_1 = 2$).[15]

4. Discussions and extensions

In this section, we apply our equilibrium results to further delve into the fundamentals of this contest game with sequential entry and demonstrate the versatility of our approach. First, we formally establish the monotonicity of players' ex ante expected payoffs with respect to their timing positions in the general setting. Second, we endogenize players' moving order in the contest. Finally, we allow for a general payment rule, such that each player may not bear the full cost of his effort.

[15] Note that the jump in the first mover's bidding strategy under $F_3(\cdot)$ is not driven by the kinks in the CDF's derivatives; rather it is caused by the change in the concavity/convexity of the CDF. More formally, we can construct an example of a distribution function such that all derivatives are differentiable on $(0, 1)$ and a jump emerges in the equilibrium bidding function.

4.1 *Monotone payoff ranking*

We now formally address the following research question: Holding fixed the contest architecture, does a player benefit from being an earlier/later mover? Answering this question requires that we compare players' expected payoffs with respect to their timing positions. We establish that players' equilibrium expected payoffs can be ranked monotonically.

Let Π_t^* denote a period-t player's equilibrium expected payoff, with $t \in \mathcal{T}$, in a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \dots, n_T)$. Recall that $\mathcal{T}_0 \equiv \{t \in \mathcal{T} : Q_t(b) \leq b, \forall b \in [0, 1]\}$ indicates the set of the periods in which players always stay inactive in equilibrium, with $t_0 \equiv \max \mathcal{T}_0$. The following result can be obtained.

Theorem 2 (Later-Mover Advantage in Contests With Sequential Entry). *Consider a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \dots, n_T)$. A player's expected payoff is higher than those of all earlier movers in the unique symmetric PBE. To put this formally,* $0 = \Pi_1^* = \cdots = \Pi_{t_0}^* < \Pi_{t_0+1}^* < \cdots < \Pi_T^*$.

Recall that $t_0 = 0$ under a concave ability distribution and $t_0 = T - 1$ under a convex distribution. The following result can immediately be obtained.

Corollary 2 (Later-Mover Advantage With Concave/Convex Ability Distributions). *Consider a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \dots, n_T)$. The following statements hold in the unique symmetric PBE:*

(i) Suppose that $F(\cdot)$ is continuous, twice differentiable, strictly concave, and satisfies $\lim_{a \searrow 0}[f(a)a] = 0$. Then $0 < \Pi_1^* < \cdots < \Pi_T^*$.

(ii) Suppose that $F(\cdot)$ is continuous, twice differentiable, and weakly convex. Then $\Pi_1^* = \cdots = \Pi_{T-1}^* = 0 < \Pi_T^*$.

Theorem 2 and Corollary 2 formally establish later-mover advantage in a multi-player all-pay auction with sequential entry. We sketch the proof as follows. For ease of exposition, let us consider a fully sequential contest, with $N = T$ (one entrant per period). Recall that the profile of equilibrium bidding strategies is denoted by $\boldsymbol{b}^* := \{b_1^*(a; \beta_1), \dots, b_T^*(a; \beta_T)\}$. Fix an arbitrary period $\tau \in \{t_0 + 1, \dots, T - 1\}$. We conduct the following thought experiment. Let us modify the period-$(\tau + 1)$ player's bidding strategy from $b_{\tau+1}^*(a; \beta_{\tau+1})$ to

$$b_{\tau+1}^{\dagger}(a; \beta_{\tau+1}) := b_{\tau}^*(a; \beta_{\tau}).$$

In other words, he hypothetically ignores the period-τ player's effort and replicates the latter's equilibrium strategy (*not his effort*). Denote players' expected payoffs under the constructed strategy profile $\boldsymbol{b}^{\dagger} := \{b_1^*(a; \beta_1), \dots, b_{\tau}^*(a; \beta_{\tau}), b_{\tau+1}^{\dagger}(a; \beta_{\tau+1}), b_{\tau+2}^*(a; \beta_{\tau+2}), \dots, b_T^*(a; \beta_T)\}$ by $(\Pi_1^{\dagger}, \dots, \Pi_T^{\dagger})$.

The key is to show that the period-τ player would be strictly better off with the period-$(\tau + 1)$ player's hypothetical deviation, that is, $\Pi_{\tau}^* < \Pi_{\tau}^{\dagger}$. The intuition is as follows. A later mover, ceteris paribus, tends to be more aggressive in competition than an earlier mover: The former needs to beat a smaller number of future opponents for a win than the latter, which encourages the later mover. Thus, when the period-$(\tau+1)$ player deviates and replicates his immediate predecessor's strategy, he would be less likely to outperform the latter. This obviously benefits the period-τ player.

To fix ideas, consider a period-τ player, with ability $a_\tau > a_\tau^{**}(\beta_\tau)$, for a given β_τ. By Theorem 1, he would exert an effort strictly above β_τ. When the period-$(\tau+1)$ player mimics the period-τ player, the former can defeat the latter if and only if the period-$(\tau+1)$ player is of a higher type, which occurs with probability $1-F(a_\tau)$. Under the equilibrium strategy profile, in contrast, the period-$(\tau+1)$ player outperforms the period-τ player as long as the former chooses to stay active: He does so whenever his ability exceeds the threshold $a_{\tau+1}^*(\beta_{\tau+1}) = a_{\tau+1}^*(b_\tau^*(a_\tau;\beta_\tau))$, which occurs with a probability of $1-F(a_{\tau+1}^*(b_\tau^*(a_\tau;\beta_\tau)))$. We formally show in the Appendix that $a_\tau > a_{\tau+1}^*(b_\tau^*(a_\tau;\beta_\tau))$:[16] In other words, the period-$(\tau+1)$ player behaves less aggressively when he mimics his immediate predecessor.

To complete the proof, first note that $\Pi_{\tau+1}^\dagger \le \Pi_{\tau+1}^*$ by the definition of PBE. We further have $\Pi_\tau^\dagger \le \Pi_{\tau+1}^\dagger$ by the construction of $b_{\tau+1}^\dagger = b_\tau^*$.[17] Combining these inequalities yield $\Pi_\tau^* < \Pi_\tau^\dagger \le \Pi_{\tau+1}^\dagger \le \Pi_{\tau+1}^*$, which concludes that a period-$(\tau+1)$ player receives a higher equilibrium payoff than a period-τ player.

Our prediction stands in sharp contrast to that of Hinnosaar (2024). He establishes that an earlier mover exerts a higher effort and secures a larger expected payoff. We nevertheless observe the opposite monotone payoff ranking in our setting. Hinnosaar (2024) considers a lottery contest, in which earlier and later efforts can be strategic substitutes near the equilibrium. In a lottery contest, one is tempted to preempt future opponents. However, this does not occur in an all-pay auction: The later mover is awarded an information advantage since he can observe previous efforts; the winner-selection mechanism of an all-pay auction allows him to outbid earlier opponents by simply matching their efforts. As a result, strategic complementarity could arise in our bidding game, which as we establish, discourages early bidders. Our study thus complements that by Hinnosaar (2024).

4.2 *Endogenous timing*

Our equilibrium results enable us to explore how the architecture of the contest game could arise endogenously. Let the contest be preceded by a timing-choice stage, in which players simultaneously commit to the timing of their moves. Each player picks one from $L \ge 2$ available periods, denoted by $\mathcal{L} := \{1,\dots,L\}$, before he learns his realized type and acts accordingly. Before the contest begins, the architecture $\tilde{\boldsymbol{n}}$ is announced publicly, and each player learns his own ability privately. The contest with

[16]We show in the proof of Theorem 2 that $a_\tau > a_{\tau+1}^*(b_\tau^*(a_\tau;\beta_\tau))$ holds for an arbitrary contest architecture.

[17]Note that the strict inequality may hold (i.e., $\Pi_\tau^\dagger < \Pi_{\tau+1}^\dagger$) due to the tie-breaking rule, despite the fact that period-τ and period-$(\tau+1)$ players employ the same strategy. To see this, consider a fully sequential contest $(1,1,1)$ with a concave ability distribution and let the third mover replicate the second mover's equilibrium strategy. In the event that players 2 and 3 choose to match player 1's effort, player 3 wins, which occurs with a positive probability and results in the strict inequality.

sequential entry takes place as described in Section 2 thereafter, and Theorem 1 fully characterizes the unique PBE of the contest subgame.[18]

It is noteworthy that the later-mover advantage established in Theorem 2 does not imply that choosing a late period is a dominant strategy for each player. To be more specific, the later-mover advantage is obtained by comparing different players' expected payoffs with respect to their timing positions under a predetermined contest architecture. With endogenous timing of moves, however, a player's autonomous timing choice would reshape the resultant contest architecture and affect all players' equilibrium payoffs. Understanding a player's timing choice requires that we compare a given player's equilibrium payoffs across *different* contest architectures. The subsequent analysis takes up this challenge.

The analysis begins with players' equilibrium winning probabilities. Fix an arbitrary contest architecture $\boldsymbol{n} \equiv (n_1, \dots, n_T)$, with $n_t \geq 1$ for $t \in \{1, \dots, T\}$. Consider a period-t player of ability $a \in (0, 1)$ and denote by $\mathrm{WP}_t^*(a; \boldsymbol{n})$ his expected equilibrium winning probability in the unique symmetric PBE. The following lemma can be obtained.

Lemma 5. *Consider two arbitrary contest architectures $\boldsymbol{n}' \equiv (n_1', \dots, n_{T'}')$—with $n_t' \geq 1$, $t \in \{1, \dots, T'\}$, $T' \geq 2$, and $\sum_{t=1}^{T'} n_t' = N$—and $\boldsymbol{n}'' \equiv (n_1'', \dots, n_{T''}'')$—with $n_t'' \geq 1$, $t \in \{1, \dots, T''\}$, $T'' \geq 2$, and $\sum_{t=1}^{T''} n_t'' = N$. For almost every $a \in (0, 1)$, we have*

$$\max\{\mathrm{WP}_1^*(a; \boldsymbol{n}'), \mathrm{WP}_1^*(a; \boldsymbol{n}'')\} < F^{N-1}(a) < \min\{\mathrm{WP}_{T'}^*(a; \boldsymbol{n}'), \mathrm{WP}_{T''}^*(a; \boldsymbol{n}'')\}.$$

That is, for almost every a, a player is more likely to win when acting in the last period of the contest than being one of the first movers, regardless of the prevailing contest architecture. The comparison is bridged through $F^{N-1}(a)$, which is a player's equilibrium winning probability in a simultaneous contest.

This inequality paves the way for a comparison of equilibrium payoffs. We invoke the standard payoff-equivalence argument for direct mechanisms. A period-t player's equilibrium payoff in a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \dots, n_T)$—which we denote by $\Pi_t^*(\boldsymbol{n})$—can be pinned down by his equilibrium expected winning probability as follows:

$$\Pi_t^*(\boldsymbol{n}) = \mathbb{E}\left[\frac{1}{a}\int_0^a \mathrm{WP}_t^*(x; \boldsymbol{n})\,dx\right] = \int_0^1 \int_0^a \frac{\mathrm{WP}_t^*(x; \boldsymbol{n})}{a}\,dx\,dF(a).$$

We further define $\Pi^{\mathrm{SIM}} := \int_0^1 \int_0^a \frac{1}{a} F^{N-1}(a)\,dx\,dF(a)$, which is one's expected payoff in a simultaneous contest. Lemma 5 can then be translated into a comparison of equilibrium payoffs:

$$\max\{\Pi_1^*(\boldsymbol{n}'), \Pi_1^*(\boldsymbol{n}'')\} < \Pi^{\mathrm{SIM}} < \min\{\Pi_{T'}^*(\boldsymbol{n}'), \Pi_{T''}^*(\boldsymbol{n}'')\}. \tag{9}$$

By this inequality, we are ready to explore players' incentives in their timing choices. The following result ensues.

[18]Theorem 1 is established under the assumption that each period possesses at least one player. With endogenous timing, this assumption may not be satisfied due to the possibility that no players choose to move in a certain period. In such a scenario, we can simply remove these periods and relabel the rest to invoke Theorem 1.

Lemma 6 (Strictly Dominated Strategy With Endogenous Moving Order). *For every player, choosing to move in period* 1 *is strictly dominated by choosing to move in period* L.

Lemma 6 implies that the equilibrium in the timing-choice stage is solvable by iterated elimination of strictly dominated strategies (IESDS).

Theorem 3 (Unique Equilibrium With Endogenous Moving Order). *All players' choosing to move in the last period constitutes a Nash equilibrium of the first-stage game that uniquely survives IESDS.*

When players are allowed to pick the timing of their moves, all players will choose the last period and a simultaneous contest endogenously emerges. Zhang (2024) applies the mechanism design approach to optimal contest design with convex (or linear) effort cost and identifies a sufficient and necessary condition for the static single-prize contest to be effort maximizing. With linear effort cost, the condition degenerates to Myerson's (1981) classical regularity condition of nondecreasing virtual value, that is, with $a - [1 - F(a)]/f(a)$ being nondecreasing in a in our context. Our Theorem 3, together with Zhang (2024), indicates that a process of decentralized decision on timings of moves leads to a simultaneous contest and generates the maximum amount of expected total effort under the regularity condition.

4.3 *Hybrid payment rule for losers*

Our main results do not rely on the all-pay feature. Specifically, we allow each loser to bear only a portion of his effort cost. The associated payment rule is specified as follows: The winner in the contest is obliged to pay the full cost of his own effort, while a loser pays $\theta \in [0, 1]$ of that.[19,20] To put this formally, fixing a contestant i's ability a^i and the effort profile $\boldsymbol{b} \equiv (b^1, \dots, b^N)$, his ex post payoff is

$$p^i(\boldsymbol{b})\big(1 - b^i/a^i\big) - \big[1 - p^i(\boldsymbol{b})\big]\theta b^i/a^i, \quad \text{for all } i \in \mathcal{N}.$$

The above expression degenerates to (2) in the baseline setting as $\theta = 1$, and the contest game turns into a *first-price auction* with sequential entry as $\theta = 0$.[21] A $\theta \in (0, 1)$ depicts a hybrid payment rule that involves both winner-pay and all-pay elements.

[19] See Amann and Leininger (1996) and Baye, Kovenock, and De Vries (2005, 2012) for similar parameterization.

[20] Note that a bid $b \in (0, \beta_t)$ always leads to a loss and is suboptimal to a period-t player for $\theta > 0$. In contrast, when $\theta = 0$, bidding $b \in (0, \beta_t)$ is strategically equivalent to bidding zero because a bid does not incur a cost to a loser. In this case, we impose the restriction that period-t players bid 0 or weakly above β_t without any loss of generality when characterizing the symmetric PBE.

[21] To the best of our knowledge, the previous studies of first-price auctions have yet to accommodate a setting of unrestricted timing structure like ours.

Fixing a contest architecture $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and $\theta \in [0, 1]$, a sequence of functions $\{Q_t(b; \theta), a_t^*(\beta; \theta), \tilde{\pi}_t(b, a; \theta)\}_{t=1}^T$ in parallel with (3), (4), and (5) can be defined recursively as follows:[22]

$$Q_T(b; \theta) \equiv 1, \qquad Q_{t-1}(b; \theta) := Q_t(b; \theta) F^{n_t}\big(a_t^*(b; \theta)\big), \quad \forall b \in [0, 1], \tag{10}$$

$$a_t^*(\beta; \theta) := \max\big\{0 < a \leq 1 : \tilde{\pi}_t(b, a; \theta) \leq 0, \forall b \geq \beta\big\}, \tag{11}$$

$$\tilde{\pi}_t(b, a; \theta) := Q_t(b; \theta) F^{n_t - 1}(a)\big[1 - (1 - \theta) b/a\big] - \theta b/a. \tag{12}$$

With slight abuse of notation, let $\mathcal{T}_0(\theta) := \{t \in \mathcal{T} : Q_t(b; \theta) \leq \frac{\theta b}{1 - b + \theta b}, \forall b \in [0, 1]\}$ and define $t_0(\theta) := \max \mathcal{T}_0(\theta)$. Again, we can obtain $0 \leq t_0(\theta) \leq T - 1$.

Theorem 4 (Contests With a Generalized Payment Rule for Losers). *Fix $\theta \in [0, 1]$ and consider a generalized contest with sequential entry $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ under a tie-breaking rule as specified in (1). There exists a unique symmetric PBE of the contest game. In the equilibrium, all players in periods 1 through $t_0(\theta)$ choose to stay inactive regardless of their ability and the previous maximum bid; moreover, a player's expected payoff is higher than those of all earlier movers. If players can choose the timing of their move, then all players' choosing to move in the last period constitutes a Nash equilibrium of the first-stage game that uniquely survives IESDS.*

Theorem 4 reinstates the main results of our baseline model under the hybrid payment rule. Our analysis and predictions extend to all the alternative settings with $\theta \in [0, 1]$, such as standard first-price auctions. The main results are not an artifact of the all-pay feature. Instead, the strategic complementarity in these bidding games is the key driver of the results.

5. Concluding remarks

In this paper, we conduct a general analysis of an incomplete-information contest with sequential entry in the form of (first-price) all-pay auctions. Our model allows for a flexible architecture, such that multiple players can be clustered in a single period: They move simultaneously within the period, while observing earlier efforts and anticipating future competitions. Our analysis fully characterizes the unique symmetric equilibrium under a general ability distribution, which adds to the contest literature since a general analysis of contests with sequential entry remains scarce.

Based on our equilibrium analysis, we formally establish a later-mover advantage, in that one secures a higher ex ante expected payoff when he is assigned to a later timing position vis-à-vis an earlier one. We further allow players to choose the timing of their moves in a pre-contest stage. The unique equilibrium that survives iterated elimination of strictly dominated strategies requires that all players choose the last period. Finally, we demonstrate that the all-pay feature is not crucial for our analysis and that all of the results extend to contests with a hybrid payment rule for losers.

[22] We add θ to $\{Q_t(b), a_t^*(\beta), \tilde{\pi}_t(b, a)\}_{t=1}^T$ to highlight the fact that the defined sequence of functions depends on θ.

Large room for extensions remains. For instance, a model of multiple prizes with sequential moves deserves serious scholarly effort. A second-price all-pay auction (e.g., Krishna and Morgan (1997); Bulow and Klemperer (1999); Hafer (2006); Bergemann, Brooks, and Morris (2019)) also deserves serious research effort under a sequential timing architecture, and will be attempted in the future. Further, our analysis assumes ex ante symmetric players. This allows us to identify the effect of timing positions on players' expected payoff. An equilibrium analysis of contests with sequential entry and ex ante heterogeneous players under a general timing architecture is technically challenging, but warrants serious research effort. Also, our paper assumes that each player commits to his effort upon entry. One natural variation is to allow them to add to their bids in future periods as in Yildirim (2005).[23] Such an analysis entails enormous complications in our setting: With incomplete information, players' bidding strategies trigger complicated information updating and give rise to a challenging and subtle signaling game. Finally, we assume that players' timings of moves are well known before they sink their effort. It is intriguing to assume instead that players' timing positions are randomly assigned, so they do not know precisely the timings of future opponents' entries while observing the history of previous bids. This setting also causes technical difficulty: The general and random timing architecture can lead to numerous possibilities for future competitions—which is history-dependent—and, in turn, complexly and reflexively reshape earlier bidding.

Appendix A: Proofs

A.1 *Proof of Lemma 1*

Proof. We prove Lemma 1 along with the following lemma.

Lemma 7 (Equilibrium Winning Probability of a Provisional Winner). *Consider a contest with sequential entry $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and suppose that a symmetric PBE exists. Let $b \geq 0$ be the realized highest effort by the end of period $t \in \mathcal{T}$. Then $Q_t(b)$ gives the probability of effort b's exceeding all subsequent efforts in equilibrium.*

It is useful to prove several intermediate results.

Lemma 8. *The following statements hold:*

(i) $a_t^*(\beta)$ *is continuous, piecewise differentiable, and weakly increasing on* $[0, 1]$, *satisfying* $a_t^*(\beta) \geq \beta$ *for all* $t \in \mathcal{T}$; *and*

(ii) $Q_t(b)$ *is continuous, piecewise differentiable, and strictly increasing on* $[0, 1]$, *with* $Q_t(0) = 0$ *and* $Q_t(1) = 1$ *for all* $t \in \mathcal{T} \setminus \{T\}$.

[23] Relatedly, Quint and Hendricks (2018) let a seller use indicative bids (i.e., nonbinding preliminary bids) before a standard English auction to select a subset of bidders for conducting due diligence and eliciting binding offers. In their model, bidders simultaneously send cheap-talk messages to the seller, who subsequently uses these messages to select participants for the auction. The chosen bidders then partake in the auction.

Proof. We prove the lemma by induction. Note that piecewise analyticity implies piecewise differentiability. Therefore, to show that $a_t^*(\beta)$ and $Q_t(b)$ are piecewise differentiable, it suffices to show that they are piecewise analytic. Denote by $G_n(\cdot)$ the inverse function of $aF^{n-1}(a)$ for an arbitrary positive integer $n \in \mathbb{N}_+$. It can be verified that $G_n(\cdot)$ is strictly increasing, piecewise analytic, and differentiable on $[0, 1]$, with $G_n(0) = 0$ and $G_n(1) = 1$.

Base case: By definition, $Q_T(b) = 1$. Therefore, $\tilde{\pi}_T(b, a) \equiv Q_T(b)F^{n_T-1}(a) - b/a = F^{n_T-1}(a) - b/a$. These facts, together with (4), imply that

$$a_T^*(\beta) := \max\{0 < a \leq 1 : \tilde{\pi}_T(b, a) \leq 0, \forall b \in [\beta, 1]\} = G_{n_T}(\beta)$$

and $Q_{T-1}(b) = F^{n_T}(G_{n_T}(b))$. It is straightforward to verify that $a_T^*(\beta)$ satisfies part (i) of the lemma and $Q_{T-1}(b)$ satisfies part (ii).

Inductive step: Suppose that $Q_t(b)$ satisfies part (ii) of the lemma for some $t \leq T - 1$. It suffices to show that $a_t^*(\beta)$ satisfies part (i) of the lemma and $Q_{t-1}(b)$ satisfies part (ii).

Fixing $b \in (0, 1]$, $\tilde{\pi}_t(b, a)$ strictly increases with $a \in (0, 1)$. Define $\breve{a}_t(b)$ as follows:

$$\breve{a}_t(b) := \begin{cases} G_{n_t}\left(\min\left\{\dfrac{1}{Q_t'(0)}, 1\right\}\right), & \text{if } b = 0, \\ 1, & \text{if } b \in (0, 1] \text{ and } \tilde{\pi}_t(b, 1) < 0, \\ \text{the unique solution to } \tilde{\pi}_t(b, a) = 0, & \text{otherwise.} \end{cases}$$

It can be verified that $a_t^*(\beta) = \min_{b \geq \beta} \breve{a}_t(b)$ and $\breve{a}_t(b)$ is continuous on $[0, 1]$. This in turn implies that $a_t^*(\beta)$ is continuous, piecewise analytic, and weakly increasing on $[0, 1]$. Further, for $b \geq \beta$, we have

$$\tilde{\pi}_t(b, \beta) = F^{n_t-1}(\beta)Q_t(b) - b/\beta \leq 0,$$

which indicates that $\beta \in \{0 < a \leq 1 : \tilde{\pi}_t(b, a) \leq 0, \forall b \geq \beta\}$, and thus $a_t^*(\beta) \geq \beta$. To summarize, $a_t^*(\beta)$ satisfies part (i) of the lemma.

Because $Q_t(b)$ satisfies part (ii) of the lemma by assumption and $a_t^*(\beta)$ satisfies part (i), we can conclude that $Q_{t-1}(b) = Q_t(b)F^{n_t}(a_t^*(b))$ satisfies part (ii). This completes the inductive step.

Conclusion: By the principle of induction, $a_t^*(\beta)$ satisfies part (i) of Lemma 8 for all $t \in \mathcal{T}$ and $Q_t(b)$ satisfies part (ii) for all $t \in \mathcal{T} \setminus \{T\}$. This concludes the proof. □

Lemma 9. *The following statements hold for all $t \in \mathcal{T}$:*

(i) If $a_t^*(\beta) < a < 1$, then there exists $b \in [\beta, 1]$ such that $\tilde{\pi}_t(b, a) > 0$.

(ii) If $0 < a < a_t^*(\beta)$, then $\tilde{\pi}_t(b, a) < 0$ for all $b \in [\beta, 1] \setminus \{0\}$.

Proof. Part (i) of the lemma is obvious and it remains to prove part (ii). Fix $0 < a < a_t^*(\beta)$. Suppose, to the contrary, that $\tilde{\pi}_t(b_0, a) \geq 0$ for some $b_0 \in [\beta, 1] \setminus \{0\}$. It follows immediately that $a \geq b_0$. Further, we have that $\tilde{\pi}_t(b_0, a_t^*(\beta)) > \tilde{\pi}_t(b_0, a) \geq 0$, which contradicts with the fact that $\tilde{\pi}_t(b_0, a_t^*(\beta)) \leq 0$ for all $b \in [\beta, 1]$. This completes the proof. □

Now we can prove Lemmas 1 and 7 by induction.

Base case: Consider the last period, that is, $t = T$. It is evident that the realized highest effort by the end of period T wins the contest with certainty. By definition, $Q_T(b) = 1$. Therefore, Lemma 7 holds for $t = T$ and it remains to show that Lemma 1 holds for the last period. We consider the following two cases:

(a) Suppose $n_T = 1$. Then the optimal bidding strategy of the unique period-T player is to bid β_T if $a > \beta_T$ and bid 0 otherwise. Therefore, Lemma 1(i) and (ii) hold.

(b) Suppose $n_T \geq 2$. We first show that $b_T^*(a; \beta_T)$ is increasing in a. Suppose, to the contrary, that there exists an ability pair (a', a''), with $0 < a' < a'' < 1$, such that $b'' := b_T^*(a''; \beta_T) < b' := b_T^*(a'; \beta_T)$. Denote the equilibrium winning probability of bidding b by $\mathcal{WP}_T^*(b)$. It is obvious that $\mathcal{WP}_T^*(b'') < \mathcal{WP}_T^*(b')$; otherwise, a type-$a'$ player has a strict incentive to bid b''. Moreover, from players' incentive compatibility constraints, we have that

$$\mathcal{WP}_T^*(b')a' - b' \geq \mathcal{WP}_T^*(b'')a' - b'', \quad \text{and} \quad \mathcal{WP}_T^*(b'')a'' - b'' \geq \mathcal{WP}_T^*(b')a'' - b',$$

which is equivalent to

$$a'\left[\mathcal{WP}_T^*(b') - \mathcal{WP}_T^*(b'')\right] \geq b' - b'', \quad \text{and} \quad a''\left[\mathcal{WP}_T^*(b') - \mathcal{WP}_T^*(b'')\right] \leq b' - b''.$$

Combining the above inequalities yield

$$(a' - a'') \times \left[\mathcal{WP}_T^*(b') - \mathcal{WP}_T^*(b'')\right] \geq 0,$$

which is a contradiction given that $\mathcal{WP}_T^*(b'') < \mathcal{WP}_T^*(b')$ and the postulated $a' < a''$.

Let $\bar{a}_T := \inf\{a : b_T^*(a; \beta_T) > 0\}$. We first show that $b_T^*(a; \beta_T)$ strictly increases with a for $a > \bar{a}_T$. Suppose, to the contrary, that $\bar{a}_T < a' < a'' < 1$ and $b' := b_T^*(a'; \beta_T) = b'' := b_T^*(a''; \beta_T)$. It follows immediately that $b_T^*(a; \beta_T) = b'$ for $a \in [a', a'']$. Then a type-a' player has an incentive to deviate from bidding b'. Specifically, he can raise his effort by an infinitesimal amount to substantially increase his winning probability, which leads to an increase in his interim expected payoff. A contradiction.

Further, note that $b_T^*(a; \beta_T) \geq \beta_T$ for $a > \bar{a}_T$ and $b_T^*(a; \beta_T) = 0$ for $a \leq \bar{a}_T$, and it thus remains to prove that $\bar{a}_T = a_T^*(\beta_T) \equiv \max\{0 < a \leq 1 : \tilde{\pi}_T(b, a) \leq 0, \forall b \in [\beta_T, 1]\}$, where $\tilde{\pi}_T(b, a) \equiv Q_T(b)F^{n_T-1}(a) - b/a$. We consider the following two cases:

(i) Suppose that $\bar{a}_T < a_T^*(\beta_T)$. Consider a type-$a'$ player, with $\bar{a}_T < a' < a_T^*(\beta_T)$. Recall that $b_T^*(a; \beta_T)$ strictly increases with a for $a > \bar{a}_T$. Therefore, we have $b_T^*(a'; \beta_T) > 0$. His equilibrium expected payoff is

$$F^{n_T-1}(a') - \frac{b_T^*(a'; \beta_T)}{a'} = \tilde{\pi}_T\left(b_T^*(a'; \beta_T), a'\right) < 0,$$

where the strict inequality follows from $b_T^*(a'; \beta_T) \neq 0$, $b_T^*(a'; \beta_T) \geq \beta_T$, and Lemma 9(ii). However, he can secure a nonnegative expected payoff by bidding zero. A contradiction.

(ii) Suppose that $\bar{a}_T > a_T^*(\beta_T)$. Fix $a' \in (a_T^*(\beta_T), \bar{a}_T)$. It follows immediately from $a' < \bar{a}_T$ that $b_T^*(a'; \beta_T) = 0$. Note that bidding zero must generate zero expected payoff to a type-a' player. Otherwise, we must have $\beta_T = 0$; together with $\bar{a}_T > 0$, we can conclude that a player whose type falls below $\bar{a}_T$ can strictly increase his expected payoff by exerting an infinitesimal amount of effort. A contradiction.

By Lemma 9(i), there exists some $b' \in [\beta_T, 1]$ such that $\tilde{\pi}_T(b', a') > 0$. Then type-$a'$ player's expected payoff of bidding b' is bounded from below by

$$F^{n_T-1}(a') - \frac{b'}{a'} = \tilde{\pi}_T(b', a') > 0.$$

Therefore, a type-a' player has a strict incentive to deviate from exerting zero effort, which is a contradiction.

Inductive step: Suppose that the equilibrium bidding strategy $b_t^*(a; \beta_t)$ satisfies the properties stated in Lemma 1 and $Q_t(b)$ gives the probability of the effort b's exceeding all subsequent efforts in equilibrium, as predicted in Lemma 7, for some $t \leq T$. We show that the same holds for period $t-1$.

Suppose that the realized highest effort by the end of period $t-1$ is b. Then the probability of the effort b's exceeding all subsequent efforts in equilibrium is $Q_t(b)F^{n_t}(a_t^*(b))$, which is exactly $Q_{t-1}(b)$ from (3).

For the case of $n_{t-1} = 1$, note that the problem of the only period-$(t-1)$ player with ability a is $\max_{b \in \{0\} \cup [\beta_{t-1}, 1]} [Q_{t-1}(b) - b/a]$. It is then straightforward to verify that (i) $b_{t-1}^*(a; \beta_{t-1})$ is increasing in a on $(0, 1)$, and (ii) $b_{t-1}^*(a; \beta_{t-1}) = 0$ for $a \leq a_{t-1}^*(\beta_{t-1})$ and $b_{t-1}^*(a; \beta_{t-1}) \geq \beta_{t-1}$ for $a > a_{t-1}^*(\beta_{t-1})$. For the case of $n_{t-1} \geq 2$, by the same argument as in the base case, we can show that $b_{t-1}^*(a, \beta_{t-1})$ satisfies all properties stated in Lemma 1. This completes the inductive step.

Conclusion: By the principle of induction, $b_t^*(a; \beta_t)$ satisfies all properties stated in Lemma 1 for all $t \in \mathcal{T}$. Moreover, $Q_t(b)$ gives the probability of the effort b's exceeding all subsequent efforts in equilibrium for all $t \in \mathcal{T}$, as predicted in Lemma 7. This concludes the proof. □

A.2 *Proof of Lemma 2*

Proof. See the main text. □

A.3 *Proof of Lemma 3*

Proof. It is useful to prove the following intermediate result.

Lemma 10. *Suppose that $a_t^*(\beta_t) < 1$. Then $\tilde{\pi}_t(s_t^1(a_t^*(\beta_t); \beta_t), a_t^*(\beta_t)) = 0$.*

Proof. Evidently, $s_t^1(a_t^*(\beta_t); \beta_t) \geq \beta_t$; together with the definition of $a_t^*(\beta_t)$, we can obtain $\tilde{\pi}_t(s_t^1(a_t^*(\beta_t); \beta_t), a_t^*(\beta_t)) \leq 0$. Suppose, to the contrary, that $\tilde{\pi}_t(s_t^1(a_t^*(\beta_t); \beta_t), a_t^*(\beta_t)) \neq 0$. Then we must have

$$\tilde{\pi}_t\big(s_t^1\big(a_t^*(\beta_t); \beta_t\big), a_t^*(\beta_t)\big) < 0.$$

The above inequality, together with the fact that $s_t^1(a_t^*(\beta_t); \beta_t) \in S_t(a; \beta_t)$, implies that

$$\tilde{\pi}_t\big(b', a_t^*(\beta_t)\big) < \tilde{\pi}_t\big(s_t^1\big(a_t^*(\beta_t); \beta_t\big), a_t^*(\beta_t)\big) < 0, \quad \text{for all } s_t^1\big(a_t^*(\beta_t); \beta_t\big) < b' \leq 1. \tag{13}$$

Next, note that by definition, $s_t^1(a_t^*(\beta_t); \beta_t)$ is the smallest element in the set $S_t(a; \beta_t)$. Therefore, we have that

$$\tilde{\pi}_t\big(b', a_t^*(\beta_t)\big) \leq \tilde{\pi}_t\big(s_t^1\big(a_t^*(\beta_t); \beta_t\big), a_t^*(\beta_t)\big) < 0, \quad \text{for all } \beta_t \leq b' \leq s_t^1\big(a_t^*(\beta_t); \beta_t\big). \tag{14}$$

Combining (13) and (14), $\tilde{\pi}_t(b', a_t^*(\beta_t) + \epsilon) < 0$ for all $b' \in [\beta_t, 1]$ for sufficiently small $\epsilon > 0$, which contradicts the definition of $a_t^*(\beta_t)$ and concludes the proof. □

Now we can prove Lemma 3. Suppose, to the contrary, that $n_t \geq 2$, $a_t^*(\beta_t) < 1$, and $\lim_{a \searrow a_t^*(\beta_t)} b_t^*(a; \beta_t) \neq s_t^1(a_t^*(\beta_t); \beta_t)$. We consider the following two cases:

(a) Suppose that $\lim_{a \searrow a_t^*(\beta_t)} b_t^*(a; \beta_t) < s_t^1(a_t^*(\beta_t); \beta_t)$. Then for sufficiently small $\epsilon > 0$, we have $b_t^*(a; \beta_t) < s_t^1(a_t^*(\beta_t); \beta_t)$ for all $a < a_t^*(\beta_t) + \epsilon$. Consider a type-$a_t^*(\beta_t)$ player. His expected payoff of bidding $s_t^1(a_t^*(\beta_t); \beta_t)$ is at least

$$\begin{aligned}
&Q_t\big(s_t^1\big(a_t^*(\beta_t); \beta_t\big)\big)F^{n_t-1}\big(a_t^*(\beta_t) + \epsilon\big) - \frac{s_t^1\big(a_t^*(\beta_t); \beta_t\big)}{a_t^*(\beta_t)} \\
&> Q_t\big(s_t^1\big(a_t^*(\beta_t); \beta_t\big)\big)F^{n_t-1}\big(a_t^*(\beta_t)\big) - \frac{s_t^1\big(a_t^*(\beta_t); \beta_t\big)}{a_t^*(\beta_t)} \\
&= \tilde{\pi}_t\big(s_t^1\big(a_t^*(\beta_t); \beta_t\big), a_t^*(\beta_t)\big) = 0,
\end{aligned}$$

where the equality follows from Lemma 10. Meanwhile, it follows from Lemma 1(ii) that a type-$a_t^*(\beta_t)$ player would bid 0, and thus earns zero expected payoff in equilibrium. A contradiction.

(b) Suppose that $\lim_{a \searrow a_t^*(\beta_t)} b_t^*(a; \beta_t) > s_t^1(a_t^*(\beta_t); \beta_t)$. Consider a player whose type is $a' = a_t^*(\beta_t) + \epsilon$ for sufficiently small $\epsilon > 0$. His expected payoff of bidding $b' = b_t^*(a'; \beta_t)$ is $\tilde{\pi}_t(b', a')$. It follows from the postulated $\lim_{a \searrow a_t^*(\beta_t)} b_t^*(a; \beta_t) > s_t^1(a_t^*(\beta_t); \beta_t)$ and the definition of $S_t(a; \beta_t)$ that

$$\tilde{\pi}_t\Big(\lim_{a \searrow a_t^*(\beta_t)} b_t^*(a; \beta_t), a_t^*(\beta_t)\Big) < \tilde{\pi}_t\big(s_t^1\big(a_t^*(\beta_t); \beta_t\big), a_t^*(\beta_t)\big) = 0,$$

where the equality again follows from Lemma 10. By continuity, $\tilde{\pi}_t(b', a') < 0$ for sufficiently small $\epsilon > 0$. Therefore, a type-a' player can secure a strictly higher expected payoff by exerting zero effort, which is a contradiction. This concludes the proof. □

A.4 *Proof of Lemma 4*

Proof. Fix some type $\tilde{a} \in (a_t^*(\beta_t), 1]$ such that $b_t^*(\tilde{a} - 0; \beta_t) = e_t^m(\tilde{a}; \beta_t)$ for some $1 \leq m \leq m_t(\tilde{a}; \beta_t) - 1$. Then we have that (see Figure 1)

$$\tilde{\pi}_t\big(e_t^m(\tilde{a}; \beta_t), \tilde{a}\big) = \tilde{\pi}_t\big(s_t^{m+1}(\tilde{a}; \beta_t), \tilde{a}\big). \tag{15}$$

Suppose, to the contrary, that $b_t^*(\tilde{a} + 0; \beta_t) \neq s_t^{m+1}(\tilde{a}; \beta_t)$. We consider the following two cases:

(a) Suppose that $b_t^*(\tilde{a} + 0; \beta_t) < s_t^{m+1}(\tilde{a}; \beta_t)$. Then there exists $\epsilon > 0$ such that $b_t^*(a; \beta_t) < s_t^{m+1}(\tilde{a}; \beta_t)$ for all $a < \tilde{a} + \epsilon$. Consider a player whose ability is $\tilde{a} - \epsilon'$ for sufficiently small $\epsilon' > 0$. His expected payoff of bidding $s_t^{m+1}(\tilde{a}; \beta_t)$ is no less than

$$\begin{aligned} &F^{n_t - 1}(\tilde{a} + \epsilon) Q_t\big(s_t^{m+1}(\tilde{a}; \beta_t)\big) - \frac{s_t^{m+1}(\tilde{a}; \beta_t)}{\tilde{a} - \epsilon'} \\ &> F^{n_t - 1}(\tilde{a} + \epsilon) Q_t\big(s_t^{m+1}(\tilde{a}; \beta_t)\big) - \frac{s_t^{m+1}(\tilde{a}; \beta_t)}{\tilde{a}} \\ &> \tilde{\pi}_t\big(s_t^{m+1}(\tilde{a}; \beta_t), \tilde{a}\big) \\ &= \tilde{\pi}_t\big(e_t^m(\tilde{a}; \beta_t), \tilde{a}\big), \end{aligned}$$

where the second equality follows from (15). Note that $b_t^*(\tilde{a} - 0; \beta_t) = e_t^m(\tilde{a}; \beta_t)$, and thus $\tilde{\pi}_t(e_t^m(\tilde{a}; \beta_t), \tilde{a})$ is the limit of player's equilibrium expected payoff as ϵ' approaches 0. Therefore, the player can obtain a strictly higher payoff by bidding $s_t^{m+1}(\tilde{a}; \beta_t)$, which is a contradiction.

(b) Suppose that $b_t^*(\tilde{a} + 0; \beta_t) > s_t^{m+1}(\tilde{a}; \beta_t)$. Consider a player whose type is $\tilde{a} + \epsilon'$ for sufficiently small $\epsilon' > 0$. Note that his equilibrium expected payoff of bidding $b_t^*(\tilde{a} + \epsilon'; \beta_t)$ can then be bounded from above by

$$\tilde{\pi}_t\big(b_t^*\big(\tilde{a} + \epsilon'; \beta_t\big), \tilde{a} + \epsilon'\big) < \tilde{\pi}_t\big(s_t^{m+1}(\tilde{a}; \beta_t), \tilde{a}\big) = \tilde{\pi}_t\big(e_t^m(\tilde{a}; \beta_t), \tilde{a}\big),$$

where the inequality follows from the definition of $S_t(a; \beta_t)$ and the equality from (15). Meanwhile, his expected payoff of bidding $b_t^*(\tilde{a} - 0; \beta_t)$ is

$$F^{n_t - 1}(\tilde{a}) Q_t\big(e_t^m(\tilde{a}; \beta_t)\big) - \frac{e_t^m(\tilde{a}; \beta_t)}{\tilde{a} + \epsilon'} \geq \tilde{\pi}_t\big(e_t^m(\tilde{a}; \beta_t), \tilde{a}\big).$$

Therefore, the player has a strict incentive to deviate from his equilibrium bid $b_t^*(\tilde{a} + \epsilon'; \beta_t)$, which is a contradiction. □

A.5 *Proof of Theorem 1*

Proof. We consider the following two cases:

(a) Suppose $n_t = 1$. It is evident that $b_t^*(a; \beta_t) = 0$ for $a \leq a_t^*(\beta_t)$ and $b_t^*(a; \beta_t)$ solves

$$\max_{b \geq \beta_t} \big[Q_t(b) - b/a\big]$$

for $a > a_t^*(\beta_t)$. Further, let $a_t^{**}(\beta_t) := \sup_{a_t^*(\beta_t) \le a \le 1}\{a : Q_t(\beta_t) - \beta_t/a > Q_t(b') - b'/a, \forall b' \in (\beta_t, 1]\}$. It follows immediately that

$$\frac{b' - \beta_t}{a} > \frac{b' - \beta_t}{a_t^{**}(\beta_t)} \ge Q_t(b') - Q_t(\beta_t), \quad \text{for all } a_t^*(\beta_t) \le a < a_t^{**}(\beta_t) \text{ and } b' \in (\beta_t, 1],$$

which in turn implies that $b_t^*(a; \beta_t) = \beta_t$ for when the player's ability a lies between $a^*(\beta_t)$ and $a^{**}(\beta_t)$. To summarize, period-t player's equilibrium bidding strategy for the case of $n_t = 1$ is characterized by (7) in part (i) of the theorem.

(b) Suppose that $n_t \ge 2$. For $a \le a_t^*(\beta_t)$, it follows immediately from Lemma 1(ii) that $b_t^*(a; \beta_t) = 0$. For $a > a_t^*(\beta_t)$, we have that

$$b_t^*(a; \beta_t) \in \arg\max_{b > \beta_t}\left[Q_t(b) F^{n_t - 1}\left(\left(b_t^*\right)^{-1}(b; \beta_t)\right) - b/a\right].$$

This implies that

$$a \in \arg\max_{\check{a} > a_t^*(\beta_t)} \check{\pi}_t(\check{a}, a; \beta_t) := Q_t\left(b_t^*(\check{a}; \beta_t)\right) F^{n_t - 1}(\check{a}) - b_t^*(\check{a}; \beta_t)/a.$$

Suppose that $b_t^*(a; \beta_t)$ is continuous in some interval $\mathcal{U}_{\tilde{a}} = (\tilde{a}, \tilde{a} + \epsilon)$. In the equilibrium, the following first-order condition should be satisfied:

$$\left.\frac{\partial \check{\pi}_t(\check{a}, a; \beta_t)}{\partial \check{a}}\right|_{\check{a} = a} = 0, \quad \text{for } a \in \mathcal{U}_{\tilde{a}},$$

which is equivalent to

$$\begin{aligned} &(n_t - 1) Q_t\left(b_t^*(a; \beta_t)\right) F^{n_t - 2}(a) f(a) + \left(b_t^*\right)'(a; \beta_t) \times \left.\frac{\partial \tilde{\pi}_t(b, a)}{\partial b}\right|_{b = b_t^*(a; \beta_t)} \\ &\quad = 0, \quad \text{for } a \in \mathcal{U}_{\tilde{a}}, \end{aligned} \tag{16}$$

and can be further simplified as (8) in the text. Condition (16), together with Lemma 1(ii), Lemma 3, and Lemma 4, indicates that the equilibrium bidding strategy $b_t^*(a; \beta_t)$, if a PBE exists, is fully characterized as in Theorem 1(ii).

It remains to verify that $b_t^*(a; \beta_t)$ as described in Theorem 1(ii) indeed constitutes a PBE of the contest game. We first verify the monotonicity of $b_t^*(a; \beta_t)$. Evidently, the first term on the left-hand side of (16) always remains positive, indicating that $(b_t^*)'(a; \beta_t) \ne 0$. Moreover, suppose that there exists $\tilde{a} \ge a_t^*(\beta_t)$ such that $b_t^*(\tilde{a} + 0; \beta_t) = s_t^m(\tilde{a}; \beta_t)$ for some $1 \le m \le m_t(\tilde{a}; \beta_t)$. From the definition of $s_t^m(\tilde{a}; \beta_t)$, for any sufficiently small $\epsilon > 0$, we have that

$$\left.\frac{\partial \tilde{\pi}_t(b, \tilde{a})}{\partial b}\right|_{b = s_t^m(\tilde{a}; \beta_t) + \epsilon} < 0.$$

Therefore, $(b_t^*)'(a; \beta_t) > 0$ at $a = \tilde{a} + \epsilon$; otherwise, (16) cannot be satisfied. We can thus conclude from these facts that $b_t^*(a; \beta_t)$ strictly increases with a whenever $b_t^*(a; \beta_t)$ is continuous and is governed by (16). It remains to verify the monotonicity of $b_t^*(a; \beta_t)$ at discontinuity points. Suppose that there exists $\tilde{a}$ such that

$b_t^*(\tilde{a} - 0; \beta_t) = e_t^m(\tilde{a}; \beta_t)$ for some $1 \le m \le m_t(\tilde{a}; \beta_t) - 1$. By Lemma 4, we have $b_t^*(\tilde{a} - 0; \beta_t) = e_t^m(\tilde{a}; \beta_t) < s_t^{m+1}(\tilde{a}; \beta_t) = b_t^*(\tilde{a} + 0; \beta_t)$.

Simple algebra would verify that

$$\frac{\partial^2 \breve{\pi}_t(\breve{a}, a; \beta_t)}{\partial \breve{a} \partial a} > 0,$$

which implies that $\frac{\partial \breve{\pi}_t(\breve{a}, a; \beta_t)}{\partial \breve{a}}$ is increasing in a. Therefore, we have

$$\frac{\partial \breve{\pi}_t(\breve{a}, a; \beta_t)}{\partial \breve{a}} > \frac{\partial \breve{\pi}_t(\breve{a}, a; \beta_t)}{\partial \breve{a}}\bigg|_{\breve{a}=a} = 0, \quad \text{for } \breve{a} < a,$$

and

$$\frac{\partial \breve{\pi}_t(\breve{a}, a; \beta_t)}{\partial \breve{a}} < \frac{\partial \breve{\pi}_t(\breve{a}, a; \beta_t)}{\partial \breve{a}}\bigg|_{\breve{a}=a} = 0, \quad \text{for } \breve{a} > a.$$

That is, the necessary first-order condition $\frac{\partial \breve{\pi}_t(\breve{a}, a; \beta_t)}{\partial \breve{a}}|_{\breve{a}=a} = 0$ is also a sufficient condition for global maximizer. This concludes the proof. □

A.6 *Proof of Proposition 1*

Proof. See the main text. □

A.7 *Proof of Corollary 1*

Proof. It is useful to prove the following intermediate result.

Lemma 11. *Suppose that $F(\cdot)$ is continuous, twice differentiable, strictly concave, and satisfies* $\lim_{a \searrow 0}[f(a)a] = 0$. *Then $Q_t(b)$ is continuous, twice differentiable, weakly increasing, strictly concave on* $[0, 1]$, *and satisfies* $\lim_{b \searrow 0}[bq_t(b)] = 0$ *for all $t \le T - 1$.*

Proof. We prove the lemma by induction.

Base case: Consider the penultimate period, that is, $t = T - 1$. Recall from the proof of Lemma 8 that $G_n(\cdot)$ is defined as the inverse function of $aF^{n-1}(a)$ for an arbitrary positive integer $n \in \mathbb{N}_+$. It follows from (4) and (5) that $a_T^*(\beta) = G_{n_T}(\beta)$; together with (3), we have $Q_{T-1}(b) = F^{n_T}(G_{n_T}(b))$. Evidently, $Q_{T-1}(b)$ is continuous, twice differentiable, and weakly increasing, and it remains to show that $Q_{T-1}(b)$ is strictly concave on $[0, 1]$ and satisfies $\lim_{b \searrow 0}[bq_{T-1}(b)] = 0$.

For notational convenience, define $\widehat{G}_t(b) := F^{n_t}(G_{n_t}(b))$, $\forall t \in \{2, \dots, T\}$. It can be verified that $\widehat{G}_t(b)$ is continuous, twice differentiable, and weakly increasing. We first show that $\widehat{G}_t(b)$ is strictly concave. Carrying out the algebra, we have that

$$\widehat{G}_t'(b) = \frac{n_t F\big(G_{n_t}(b)\big) f\big(G_{n_t}(b)\big)}{F\big(G_{n_t}(b)\big) + (n_t - 1) G_{n_t}(b) f\big(G_{n_t}(b)\big)}.$$

Because $G_{n_t}(b)$ is strictly increasing in b, it suffices to show that for all $x \in (0, 1)$,

$$\frac{d}{dx} \frac{F(x) f(x)}{F(x) + (n_t - 1) x f(x)} < 0 \quad \Longleftrightarrow \quad \frac{d}{dx}\left[\frac{1}{f(x)} + (n_t - 1)\frac{x}{F(x)}\right] > 0.$$

The strict concavity of $F(x)$ implies that both $\frac{1}{f(x)}$ and $\frac{x}{F(x)}$ are strictly increasing in x. Therefore, $\widehat{G}_t(b)$ is strictly concave in b.

Next, we show that $\lim_{b\searrow 0}[b\widehat{G}_t'(b)] = 0$. The analysis is straightforward for $n_t = 1$, and it suffices to consider the case of $n_t \geq 2$. Carrying out the algebra, we have that

$$\lim_{b\searrow 0}\frac{b}{G_{n_t}(b)} = \lim_{b\searrow 0}\bigl[F^{n_t-1}\bigl(G_{n_t}(b)\bigr)\bigr] = 0,$$

and

$$\lim_{b\searrow 0}\bigl[G_{n_t}(b)\widehat{G}_t'(b)\bigr] = \lim_{b\searrow 0}\frac{n_t bF(b)f(b)}{F(b)+(n_t-1)bf(b)} = \lim_{b\searrow 0}\frac{n_t}{\dfrac{1}{bf(b)}+(n_t-1)\dfrac{1}{F(b)}} = 0.$$

Therefore,

$$\lim_{b\searrow 0}\bigl[b\widehat{G}_t'(b)\bigr] = \lim_{b\searrow 0}\frac{b}{G_{n_t}(b)} \times \lim_{b\searrow 0}\bigl[G_{n_t}(b)\widehat{G}_t'(b)\bigr] = 0.$$

Note that $Q_{T-1}(b) = \widehat{G}_T(b)$. The above analyses indicate that $Q_{T-1}(b) = \widehat{G}_T(b)$ is strictly concave and $\lim_{b\searrow 0}[bq_{T-1}(b)] = \lim_{b\searrow 0}[b\widehat{G}_T'(b)] = 0$.

Inductive step: Suppose that $Q_t(b)$ is continuous, twice differentiable, weakly increasing, strictly concave on $[0, 1]$, and satisfies $\lim_{b\searrow 0}[bq_t(b)] = 0$ for some $t \leq T-1$. Next, we show that $Q_{t-1}(b)$ has the same properties. Before we proceed, note that $Q_t(0) = 0$ for all $t \leq T-1$.

It is straightforward to verify that $Q_{t-1}(b)$ is continuous and twice differentiable from its definition. Further, it can be verified from the concavity of $Q_t(b)$ that $b/Q_t(b)$ is strictly increasing in b. Because $Q_t(b)$, $\widehat{G}_t(b)$, and $b/Q_t(b)$ are all increasing, $Q_{t-1}(b)$ is an increasing function.

Next, we prove the strict concavity of $Q_{t-1}(b)$. Carrying out the algebra, we can obtain that

$$q_{t-1}(b) = q_t(b)\left[\widehat{G}_t\left(\frac{b}{Q_t(b)}\right) - \widehat{G}_t'\left(\frac{b}{Q_t(b)}\right)\frac{b}{Q_t(b)}\right] + \widehat{G}_t'\left(\frac{b}{Q_t(b)}\right)$$

and

$$\begin{aligned} q_{t-1}'(b) = {} & \underbrace{q_t'(b)}_{<0}\underbrace{\left[\widehat{G}_t\left(\frac{b}{Q_t(b)}\right) - \widehat{G}_t'\left(\frac{b}{Q_t(b)}\right)\frac{b}{Q_t(b)}\right]}_{>0} \\ & + \underbrace{\widehat{G}_t''\left(\frac{b}{Q_t(b)}\right)}_{<0}\underbrace{\left[\frac{Q_t(b)-bq_t(b)}{Q_t(b)}\right]}_{>0}\underbrace{\left[\frac{b}{Q_t(b)}\right]'}_{>0}. \end{aligned}$$

From the previous analysis, $\widehat{G}_t(b)$ is strictly concave and $\lim_{b\searrow 0}[b\widehat{G}_t'(b)] = 0$, which implies that $\widehat{G}_t''(\frac{b}{Q_t(b)}) < 0$ and

$$\widehat{G}_t\left(\frac{b}{Q_t(b)}\right) - \widehat{G}_t'\left(\frac{b}{Q_t(b)}\right)\frac{b}{Q_t(b)} > 0, \quad \forall b \in (0, 1).$$

The monotonicity of $b/Q_t(b)$ implies that $\frac{Q_t(b)-bq_t(b)}{Q_t(b)} > 0$ and $[\frac{b}{Q_t(b)}]' > 0$. Further, the strict concavity of $Q_t(b)$ implies that $q_t'(b) < 0$. Therefore, $q_{t-1}'(b) < 0$, and thus $Q_{t-1}(b)$ is strictly concave.

Finally, we have

$$\lim_{b\searrow 0}\bigl[bq_{t-1}(b)\bigr] = \lim_{b\searrow 0}\left\{bq_t(b)\left[\widehat{G}_t\left(\frac{b}{Q_t(b)}\right) - \widehat{G}_t'\left(\frac{b}{Q_t(b)}\right)\frac{b}{Q_t(b)}\right] + Q_t(b)\frac{b}{Q_t(b)}\widehat{G}_t'\left(\frac{b}{Q_t(b)}\right)\right\}. \tag{17}$$

Note that

$$0 \le \widehat{G}_t\left(\frac{b}{Q_t(b)}\right) - \widehat{G}_t'\left(\frac{b}{Q_t(b)}\right)\frac{b}{Q_t(b)} \le \widehat{G}_t\left(\frac{b}{Q_t(b)}\right) \le 1 \tag{18}$$

and

$$0 \le \frac{b}{Q_t(b)}\widehat{G}_t'\left(\frac{b}{Q_t(b)}\right) \le \widehat{G}_t\left(\frac{b}{Q_t(b)}\right) \le 1. \tag{19}$$

Equations (17) to (19), together with $Q_t(0) = 0$ and the postulated $\lim_{b\searrow 0}[bq_t(b)] = 0$, imply that $\lim_{b\searrow 0}[bq_{t-1}(b)] = 0$. This completes the inductive step.

Conclusion: By the principle of induction, $Q_t(b)$ is continuous, twice differentiable, weakly increasing, strictly concave on $[0, 1]$, and satisfies $\lim_{b\searrow 0}[bq_t(b)] = 0$ for all $t \le T-1$. This concludes the proof. □

Now we can prove Corollary 1. Suppose that $F(\cdot)$ is continuous, twice differentiable, strictly concave, and satisfies $\lim_{a\searrow 0}[f(a)a] = 0$. By Lemma 11, $Q_t(b)$ is strictly concave on $[0, 1]$. Further, by Lemma 8, $Q_t(0) = 0$ and $Q_t(1) = 1$ for all $t \in \mathcal{T} \setminus \{T\}$. It follows immediately that $Q_t(b) > b$ for all $b \in (0, 1)$, which in turn implies that $t_0 = 0$.

Next, suppose that $F(\cdot)$ is continuous, twice differentiable, and weakly convex. Recall from the proof of Lemma 8 that $G_n(\cdot)$ is the inverse function of $aF^{n-1}(a)$—which implies that $b = G_{n_T}(b)F^{n_T-1}(G_{n_T}(b))$—and $F^{n_T}(G_{n_T}(b)) = Q_{T-1}(b)$. Further, the weak convexity of $F(\cdot)$ implies that $a \ge F(a)$. Taken together, we can obtain that

$$b = G_{n_T}(b)F^{n_T-1}\bigl(G_{n_T}(b)\bigr) \ge F^{n_T}\bigl(G_{n_T}(b)\bigr) = Q_{T-1}(b),$$

from which we can conclude that $t_0 = T-1$. This concludes the proof. □

A.8 *Proof of Theorem 2*

Proof. Recall the unique symmetric PBE is denoted by $\boldsymbol{b}^* := \{b_1^*(a; \beta_1), \dots, b_T^*(a; \beta_T)\}$. Fix an arbitrary period $\tau \in \{t_0+1, \dots, T-1\}$ and a player i in period $\tau+1$, that is, $i \in \mathcal{N}_{\tau+1}$. We conduct the following thought experiment. Holding fixed all other players' strategies—including those in period $\tau+1$, if any—we modify player i's equilibrium bidding strategy from $b_{\tau+1}^*(a; \beta_{\tau+1})$ to

$$b_{\tau+1}^{i\dagger}(a; \beta_{\tau+1}) := b_\tau^*(a; \beta_\tau).$$

For ease of exposition, denote the constructed profile of bidding strategies by $\boldsymbol{b}^\dagger$. Further, denote player i's expected payoff and a period-τ player's under $\boldsymbol{b}^\dagger$ by $\Pi^{i\dagger}_{\tau+1}$ and $\Pi^\dagger_\tau$, respectively.

We first show that $\Pi^*_\tau < \Pi^\dagger_\tau$. Consider an indicative period-τ player j, $j \in \mathcal{N}_\tau$, whose ability we denote by a^j. Denote his interim expected payoff under $\boldsymbol{b}^*$ and that under $\boldsymbol{b}^\dagger$ by $\pi^*_\tau(a^j; \beta_\tau)$ and $\pi^\dagger_\tau(a^j; \beta_\tau)$, respectively.

If $b^j := b^*_\tau(a^j; \beta_\tau) = 0$, then the period-$\tau$ player loses under both $\boldsymbol{b}^*$ and $\boldsymbol{b}^\dagger$, indicating $\pi^*_\tau(a^j; \beta_\tau) = \pi^\dagger_\tau(a^j; \beta_\tau) = 0$. If $b^j > 0$, then we have $b^j \geq \beta_\tau$ and

$$\begin{aligned}\pi^*_\tau\big(a^j; \beta_\tau\big) &= F^{n_\tau-1}\big(a^j\big)Q_\tau\big(b^j\big) - b^j/a^j \\ &= F^{n_\tau-1}\big(a^j\big)Q_{\tau+1}\big(b^j\big)F^{n_{\tau+1}}\big(a^*_{\tau+1}\big(b^j\big)\big) - b^j/a^j,\end{aligned}$$

where the second equality follows from (3). Similarly, we have

$$\pi^\dagger_\tau\big(a^j; \beta_\tau\big) = F\big(\acute{a}^j\big)F^{n_\tau-1}\big(a^j\big)Q_{\tau+1}\big(b^j\big)F^{n_{\tau+1}-1}\big(a^*_{\tau+1}\big(b^j\big)\big) - b^j/a^j,$$

where $\acute{a}^j$ is defined as

$$\acute{a}^j := \begin{cases} a^*_\tau(\beta_\tau), & \text{if } n_\tau = 1 \text{ and } a^*_\tau(\beta_\tau) < a^j \leq a^{**}_\tau(\beta_\tau), \\ a^j, & \text{otherwise,}\end{cases}$$

and satisfies $b^*_\tau(\acute{a}^j; \beta_\tau) = b^*_\tau(a^j; \beta_\tau) > 0$. It is straightforward to verify that

$$\pi^*_\tau\big(a^j; \beta_\tau\big) < \pi^\dagger_\tau\big(a^j; \beta_\tau\big) \quad\Longleftrightarrow\quad F\big(a^*_{\tau+1}\big(b^j\big)\big) < F\big(\acute{a}^j\big) \quad\Longleftrightarrow\quad a^*_{\tau+1}\big(b^j\big) < \acute{a}^j. \tag{20}$$

It follows immediately from $b^*_\tau(\acute{a}^j; \beta_\tau) = b^*_\tau(a^j; \beta_\tau) > 0$ that $\pi^*_\tau(\acute{a}^j; \beta_\tau) > 0$, from which we can conclude

$$Q_{\tau+1}\big(b^j\big)F^{n_{\tau+1}-1}\big(a^*_{\tau+1}\big(b^j\big)\big) - b^j/\acute{a}^j > 0. \tag{21}$$

Further, it follows from the definition of $a^*_{\tau+1}(\cdot)$ [see Equation (4)] that

$$\tilde{\pi}_{\tau+1}\big(b^j, a^*_{\tau+1}\big(b^j\big)\big) = Q_{\tau+1}\big(b^j\big)F^{n_{\tau+1}-1}\big(a^*_{\tau+1}\big(b^j\big)\big) - b^j/a^*_{\tau+1}\big(b^j\big) \leq 0. \tag{22}$$

Combining (21) and (22) yields

$$a^*_{\tau+1}\big(b^j\big) \leq \frac{b^j}{Q_{\tau+1}\big(b^j\big)F^{n_{\tau+1}-1}\big(a^*_{\tau+1}\big(b^j\big)\big)} < \acute{a}^j.$$

The above condition, together with (20), implies that $\pi^*_\tau(a^j; \beta_\tau) < \pi^\dagger_\tau(a^j; \beta_\tau)$ and

$$\Pi^*_\tau = \mathbb{E}\big[\pi^*_\tau\big(a^j; \beta_\tau\big)\big] < \mathbb{E}\big[\pi^\dagger_\tau\big(a^j; \beta_\tau\big)\big] = \Pi^\dagger_\tau, \tag{23}$$

where the expectation is taken with respect to both a^j and β_τ.

To complete the proof, first note that $\Pi^*_{\tau+1} \geq \Pi^{i\dagger}_{\tau+1}$ by the definition of PBE. Moreover, it follows immediately from the construction $b^{i\dagger}_{\tau+1}(a; \beta_{\tau+1}) := b^*_\tau(a; \beta_\tau)$ that $\Pi^{i\dagger}_{\tau+1} \geq \Pi^\dagger_\tau$. These inequalities, together with (23), imply that $\Pi^*_{\tau+1} \geq \Pi^{i\dagger}_{\tau+1} \geq \Pi^\dagger_\tau > \Pi^*_\tau$. $\square$

A.9 *Proof of Corollary 2*

Proof. The corollary follows immediately from Theorem 2 and Corollary 1. □

A.10 *Proof of Lemma 5*

Proof. Fixing an arbitrary architecture $\boldsymbol{n} \equiv (n_1, \dots, n_T)$, with $n_t \geq 1$ for all $t \in \{1, \dots, T\}$ and $T \geq 2$, it suffices to show that

$$\mathrm{WP}_1^*(a; \boldsymbol{n}) < F^{N-1}(a) < \mathrm{WP}_T^*(a; \boldsymbol{n}), \quad \text{for almost every } a \in (0, 1).$$

We first prove that $\mathrm{WP}_1^*(a; \boldsymbol{n}) < F^{N-1}(a)$ for all $a \in (0, 1)$. Consider a representative period-1 player $i \in \mathcal{N}_1$. Recall $\beta_1 \equiv 0$. The inequality obviously holds if $b^i := b_1^*(a^i; \beta_1) = 0$, and it remains to consider the case where $b^i > 0$. Player i's expected equilibrium payoff is

$$\pi_1^*\big(a^i; \boldsymbol{n}\big) := \mathrm{WP}_1^*\big(a^i; \boldsymbol{n}\big) - \frac{b^i}{a^i} > 0. \tag{24}$$

Fixing $\ell \in \{2, \dots, T\}$, we have that

$$\mathrm{WP}_1^*\big(a^i; \boldsymbol{n}\big) = F^{n_1-1}\big(a^i\big) \prod_{t=2}^{\ell} F^{n_t}\big(a_t^*\big(b^i\big)\big) Q_\ell\big(b^i\big) \leq F^{n_\ell-1}\big(a_\ell^*\big(b^i\big)\big) Q_\ell\big(b^i\big), \tag{25}$$

where the equality follows from Lemma 1 and Lemma 7. Combining (24) and (25) yields

$$F^{n_\ell-1}\big(a_\ell^*\big(b^i\big)\big) Q_\ell\big(b^i\big) - \frac{b^i}{a^i} > 0. \tag{26}$$

From (3), (4), and (5), we have that $\tilde{\pi}_\ell(b^i, a_\ell^*(b^i)) \leq 0$, which is equivalent to

$$F^{n_\ell-1}\big(a_\ell^*\big(b^i\big)\big) Q_\ell\big(b^i\big) - \frac{b^i}{a_\ell^*\big(b^i\big)} \leq 0. \tag{27}$$

Comparing (26) with (27) yields that $a^i > a_\ell^*(b^i)$ for all $\ell \in \{2, \dots, T\}$, which in turn implies that

$$\mathrm{WP}_1^*\big(a^i; \boldsymbol{n}\big) = F^{n_1-1}\big(a^i\big) \prod_{\ell=2}^{T} F^{n_\ell}\big(a_\ell^*\big(b^i\big)\big) < F^{n_1-1}\big(a^i\big) \prod_{\ell=2}^{T} F^{n_\ell}\big(a^i\big) = F^{N-1}\big(a^i\big).$$

Next, we prove that $F^{N-1}(a) < \mathrm{WP}_T^*(a; \boldsymbol{n})$ for almost every $a \in (0, 1)$. Fix $a \in (0, 1)$, $\beta \in [0, 1]$, and (t, ℓ), with $1 \leq t < \ell \leq T$. Following a similar argument as in the previous analysis, we can show that if $b_t^*(a; \beta) > 0$, then

$$a > a_\ell^*\big(b_t^*(a; \beta)\big). \tag{28}$$

Consider a representative period-T player, $j \in \mathcal{N}_T$, with ability $a^j \in (0, 1)$. By (28), we have $a^j > a_T^*(b_1^*(a^j; 0))$. Note that $a_T^*(b_1^*(a; 0))$ weakly increases with a, and thus is continuous almost everywhere. We can focus on the case where $a_T^*(b_1^*(a; 0))$ is continuous at $a = a^j$. Therefore, there exists $\epsilon > 0$ such that $a^j > a_T^*(b_1^*(a^j + \epsilon; 0))$. Let $\underline{a} := a^j + \epsilon$. It follows immediately that

$$\underline{a} > a^j > a_T^*\big(b_1^*(\underline{a}; 0)\big). \tag{29}$$

It is useful to prove the following intermediate result.

Lemma 12. *Fix an arbitrary architecture $\boldsymbol{n} \equiv (n_1, \dots, n_T)$—with $n_t \geq 1$ for all $t \in \{1, \dots, T\}$ and $T \geq 2$—and consider an indicative period-T player $j \in \mathcal{N}_T$. He wins the contest in the unique PBE if $\underline{a} > a^{j'}$ for all $j' \in \mathcal{N}_1$ and $a^j > a^{j'}$ for all $j' \in \mathcal{N} \setminus (\{j\} \cup \mathcal{N}_1)$.*

Proof. Fix an ability profile $\boldsymbol{a} := (a^1, \dots, a^N)$ such that $\underline{a} > a^{j'}$ for all $j' \in \mathcal{N}_1$ and $a^j > a^{j'}$ for all $j' \in \mathcal{N} \setminus (\{j\} \cup \mathcal{N}_1)$. Let $\imath$ denote the index of the provisional winner by the end of period $T-1$ given that all players use the equilibrium strategy and $\mathfrak{t}$ the period he moves. Then $\beta_T = b_{\mathfrak{t}}^*(a^\imath; \beta_{\mathfrak{t}})$. Evidently, the lemma holds if $b_{\mathfrak{t}}^*(a^\imath; \beta_{\mathfrak{t}}) = 0$ and it remains to consider the situation where $b_{\mathfrak{t}}^*(a^\imath; \beta_{\mathfrak{t}}) > 0$. We consider the following two cases:

(a) Suppose $\mathfrak{t} \geq 2$. Then we have

$$a^j > a^\imath > a_T^*\big(b_{\mathfrak{t}}^*\big(a^\imath; \beta_{\mathfrak{t}}\big)\big) = a_T^*(\beta_T),$$

where the first inequality follows from the postulated $\imath \notin \mathcal{N}_1$ and the second inequality from (28).

(b) Suppose $\mathfrak{t} = 1$. Then we have

$$a^j > a_T^*\big(b_1^*(\underline{a}; 0)\big) \geq a_T^*\big(b_1^*\big(a^\imath; 0\big)\big) = a_T^*(\beta_T),$$

where the first inequality follows from (29).

To summarize, if $b_{\mathfrak{t}}^*(a^\imath; \beta_{\mathfrak{t}}) > 0$, then $a^j > a_T^*(\beta_T)$, indicating that player j places a positive amount of bid in equilibrium. Therefore, he outbids all players up to period $T-1$. Next, note that $a^j > a^{j'}$ for all $j' \in \mathcal{N}_T$ by assumption; together with Lemma 1(iii), player j outbids all of his contemporaneous rivals in period T and wins the contest. □

By Lemma 12, player j's expected winning probability, $\mathrm{WP}_T^*(a^j; \boldsymbol{n})$, can be bounded from below by

$$\mathrm{WP}_T^*\big(a^j; \boldsymbol{n}\big) \geq F^{n_1}(\underline{a}) F^{(\sum_{t=2}^T n_t)-1}\big(a^j\big) > F^{N-1}\big(a^j\big).$$

This concludes the proof. □

A.11 *Proof of Lemma 6*

Proof. Fix an indicative player $i \in \mathcal{N}$ and consider the following two cases:

(a) All other players choose to move in the last period. If player i chooses to move in period 1, the resultant contest architecture is $\hat{\boldsymbol{n}} = (1, N-1)$ and his equilibrium payoff in this subgame is $\Pi_1^*(\hat{\boldsymbol{n}})$. If player i chooses to move in the last period, all players move simultaneously in the second-stage game and his equilibrium payoff amounts to Π^{SIM}. By (9), we have $\Pi_1^*(\hat{\boldsymbol{n}}) < \Pi^{\mathrm{SIM}}$.

(b) At least one of player i' opponents chooses not to move in the last period. Denote the resultant contest architecture when player i chooses to move in period 1 and that when he chooses to move in period L by $\hat{\boldsymbol{n}}'$ and $\hat{\boldsymbol{n}}''$, respectively. Note that $\hat{\boldsymbol{n}}'$ degenerates to a simultaneous-move contest if all other players choose to move in period 1 and a sequential-move one otherwise. By (9), we have $\Pi_1^*(\hat{\boldsymbol{n}}') \leq \Pi^{\text{SIM}}$. Next, note that $\hat{\boldsymbol{n}}''$ is a sequential-move contest. Denote the number of periods with at least one player by $\hat{T}''$. Clearly, player i's equilibrium payoff under $\hat{\boldsymbol{n}}''$ is $\Pi_{\hat{T}''}^*(\hat{\boldsymbol{n}}'')$. Again, we can obtain $\Pi^{\text{SIM}} < \Pi_{\hat{T}''}^*(\hat{\boldsymbol{n}}'')$ from (9). Therefore, we have $\Pi_1^*(\hat{\boldsymbol{n}}') \leq \Pi^{\text{SIM}} < \Pi_{\hat{T}''}^*(\hat{\boldsymbol{n}}'')$.

To summarize, moving in period L yields a strictly higher payoff to player i than moving in period 1. This concludes the proof. □

A.12 *Proof of Theorem 3*

Proof. The theorem follows immediately from Lemma 6. □

A.13 *Proof of Theorem 4*

Proof. Fixing a contest architecture $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and $\theta \in [0, 1]$, denote a symmetric PBE of the contest game, if it exists, by $\{b_t^*(a; \beta_t)\}_{t=1}^T$, with slight abuse of notation. Recall that the sequence of functions $\{Q_t(b; \theta), a_t^*(\beta; \theta), \tilde{\pi}_t(b, a; \theta)\}_{t=1}^T$ is defined by (10), (11), and (12). By arguments similar to the case of $\theta = 1$, we can show that Lemmas 1, 3, 4, and 7 extend to $\theta \in [0, 1]$. The proof of the existence and uniqueness of symmetric PBE resembles that of Theorem 1, except that $\check{\pi}_t(\check{a}, a; \beta)$ is now defined as

$$\check{\pi}_t(\check{a}, a; \beta_t) := Q_t\big(b_t^*(\check{a}; \beta_t); \theta\big) F^{n_t-1}(\check{a})\big[1 - (1-\theta) b_t^*(\check{a}; \beta_t)/a\big] - \theta b_t^*(\check{a}; \beta_t)/a,$$

and the differential equation that governs a period-t player's bidding strategy $b_t^*(a; \beta_t)$—given that $n_t \geq 2$ and $b_t^*(a; \beta_t)$ is continuous in some interval $\mathcal{U}_{\tilde{a}} = (\tilde{a}, \tilde{a} + \epsilon)$—is

$$\begin{aligned} &(n_t - 1) Q_t\big(b_t^*(a; \beta_t); \theta\big) F^{n_t-2}(a) f(a)\big[a - (1-\theta) b_t^*(a; \beta_t)\big] \\ &\quad + \big(b_t^*\big)'(a; \beta_t) a \frac{\partial \tilde{\pi}_t(b, a; \theta)}{\partial b}\bigg|_{b = b_t^*(a; \beta_t)} = 0. \end{aligned}$$

The proofs of the later-mover advantage and the endogenous timing result are similar to those in Theorem 2 and Theorem 3 and omitted for brevity. □

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