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Bayesian social aggregation with almost-objective uncertainty

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We consider collective decisions under uncertainty, when agents have *generalized Hurwicz* preferences, a broad class allowing many different ambiguity attitudes, including subjective expected utility preferences. We consider sequences of acts that are "almost-objectively uncertain" in the sense that asymptotically, all agents almost agree about the probabilities of the underlying events. We introduce a Pareto axiom, which applies *only* to asymptotic preferences along such almost-objective sequences. This axiom implies that the social welfare function is utilitarian, but it does not impose any constraint on collective beliefs. Next, we show that a Pareto axiom restricted to two-valued acts implies that collective beliefs are contained in the closed, convex hull of individual beliefs, but imposes no constraints on the social welfare function. Neither axiom entails any link between individual and collective ambiguity attitudes.

Keywords. Almost-objective uncertainty, Bayesian social aggregation, Bewley preferences, generalized Hurwicz, utilitarian.

JEL classification. D70, D81.

Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin.
—John von Neumann

1. Introduction

From a democratic point of view, collective decisions should be made by aggregating the preferences or opinions of the affected individuals. But almost all nontrivial decisions involve uncertainty. Normative decision theory considers the question of how rational agents should cope with such uncertainty. Bayesian social aggregation combines these

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two ingredients: it aims for collective decisions that are both rational and democratic. The foundational result is Harsanyi's (1955) social aggregation theorem. Harsanyi considered a society in which all agents are von Neumann–Morgenstern (vNM) expected utility maximizers. He showed that if the vNM preferences of the social planner satisfy an ex ante Pareto axiom relative to the vNM preferences of the individuals, then the social welfare function—that is, the vNM utility function of the social planner—must be a weighted average of the individual vNM utility functions. Harsanyi interpreted this as a strong argument for utilitarianism.

Harsanyi's result is highly influential in social choice theory, but its dependence on the vNM framework curtails its applicability. The vNM framework assumes that all risks can be quantified with known, objective probabilities. But in many complex decision problems (e.g., macroeconomics, climate change, pandemics), it is not clear how to assign precise probabilities to the relevant contingencies. Indeed, when considering *sui generis* events in the future (e.g., hypothetical wars or financial crises in 2060), it is not clear that "objective" probabilities even exist. This led Savage (1954) to propose an approach to decision-making based on the maximization of subjective expected utility (SEU), that is, expected utility computed using the agent's own "subjective" probabilistic beliefs.

A central tenet of the Savagean framework is that different rational agents may reasonably hold different subjective beliefs. But Mongin (1995) showed that Harsanyi's theorem breaks down in settings with heterogeneous beliefs. Mongin (1997) diagnosed the root of the problem as "spurious unanimity": different individuals might have different utility functions *and* different beliefs, but these differences might "cancel out" to yield a unanimous ex ante preference among them for one act over another, thereby entailing (via the ex ante Pareto axiom) a corresponding ex ante social preference.

This suggests that to avoid Mongin's impossibility theorem, one should weaken the ex ante Pareto axiom to avoid cases of spurious unanimity. This strategy was realized in a landmark paper by Gilboa, Samet, and Schmeidler (2004), who proposed a "restricted" ex ante Pareto axiom that only applied to acts for which all individuals have the *same* probabilistic beliefs about the underlying events. Gilboa, Samet, and Schmeidler showed that this Restricted Pareto axiom has two consequences: (1) the social welfare function (SWF) must be a weighted sum of individual utility functions, and (2) the social beliefs must be a weighted average of individual beliefs.[1]

However, while it escapes from the spurious unanimity diagnosed by Mongin, the Restricted Pareto axiom of Gilboa, Samet, and Schmeidler is still susceptible to another form of spurious unanimity, which Mongin and Pivato (2020, Section 6) call "complementary ignorance." Agents might "agree" about the probabilities of certain events—and unanimously prefer one act over another—only because they have different private information. Restricted Pareto will then require the social preferences to agree with these unanimous individual preferences, even when this contradicts the preference that

[1]See Section 6 for a more detailed discussion of Gilboa, Samet, and Schmeidler (2004). Recently, Brandl (2021) has obtained a similar result, but in his case, the SWF is *relative* utilitarian: it is a sum of the utility functions of individuals rescaled to range from 0 to 1. See also Billot and Qu (2021).

all agents would have if they had adequately pooled their private information (see Section 6 for details).

Importantly, this private information is already identifiable from the *support* of the individuals' beliefs. So, a social planner who knew enough about the individuals' beliefs to even apply Restricted Pareto would already know enough to pool their private information. This brings us to another objection to Gilboa, Samet, and Schmeidler's result: it is not always appropriate to construct social beliefs as an arithmetic average of individual beliefs. In particular, arithmetic averaging obfuscates precisely the private information just mentioned. But it can even malfunction when all agents receive the *same* information, because it does not interact well with Bayesian updating.[2]

In response, Dietrich (2021) has recently obtained a result similar to that of Gilboa, Samet, and Schmeidler (2004), in which social beliefs are a weighted *geometric* average of individual beliefs. This ensures compatibility with Bayesian updating. But it does not address a broader issue. Different belief-aggregation rules are suitable in different contexts, and the criteria that determine the appropriate belief-aggregation rule are not necessarily the criteria that determine the correct social welfare function. The specification of collective beliefs is an *epistemic* problem, whereas the specification of the SWF is an *ethical* problem; there is no reason that these two problems should be solved by the same theorem.[3] For this reason, Mongin and Pivato (2020) and Pivato (2022) have recently introduced weak Pareto axioms, which entail a utilitarian SWF, but do not impose any constraints on collective beliefs. They thus concentrate on the ethical problem, leaving the epistemic problem to be solved later by other methods.

The present paper takes up this challenge: it addresses both problems, but deals with them independently of one another. We assume an uncountably infinite state space, on which beliefs are represented by finitely additive, nonatomic probability measures. This enables us to exploit the phenomenon of "almost-objective uncertainty" (due to Poincaré (1912) and Machina (2004, 2005)), which involves a sequence of partitions $\mathfrak{G}^1, \mathfrak{G}^2, \mathfrak{G}^3, \ldots$ such that even agents with very different beliefs will assign increasingly similar probabilities to the cells of $\mathfrak{G}^n$ as $n \to \infty$. We propose a weak Pareto axiom, which only applies to asymptotic preferences for sequences of acts measurable with respect to these partitions. Our first main result says that this axiom is both necessary and sufficient for the SWF to be a weighted sum of individual utility functions (Theorem 1). But unlike results in the aforementioned literature, it does not impose any relationship between individual and collective beliefs.

We then turn to belief aggregation. We consider a second weak Pareto axiom, which only applies to preferences between two-valued acts for which all agents have the same preferences over the outcomes. Our second main result (Theorem 2) connects this axiom to the social aggregation of individual beliefs. But it does not impose any constraint on the SWF. Thus, the two theorems decouple the ethical problem from the epistemic problem, and deal with them separately.

[2]To be precise: the Bayesian update of the arithmetic average of the individuals' prior beliefs is generally not the arithmetic average of the Bayesian updates of these beliefs.

[3]See Section 4.7 of Pivato (2022) for further elaboration of these points.

Our last result (Theorem 3) is a variant of the theorem of Gilboa, Samet, and Schmeidler (2004), and yields the same conclusion as Theorem 1: a characterization of utilitarian social welfare *without* linear aggregation of beliefs. The Pareto axiom invoked by Theorem 3 is simpler than the one invoked by Theorem 1. But like the axiom of Gilboa, Samet, and Schmeidler (2004), it is susceptible to complementary ignorance, as we explain in Section 6.

The earlier discussion was vague about the belief aggregation in Theorem 2. If all agents have SEU preferences, then Theorem 2 says the social beliefs are a weighted average of individual beliefs, as in Gilboa, Samet, and Schmeidler (2004). But to fully explain this result, we must broaden our perspective. All of the aforementioned literature assumes that all agents have SEU preferences. But in ambiguous decision environments, this might be inappropriate; it might be difficult to specify any single probability measure over contingencies as an adequate description of the uncertainty faced by an agent. This objection is both normative and descriptive. At a descriptive level, many agents might simply be unable to condense their uncertainty into a single probability measure. At a normative level, it is perhaps not even rational for an agent to resort to such a probabilistic description. These concerns have inspired a variety of "non-SEU" models of decision making. Typically, such models represent an agent's beliefs not with a single probability measure but with an ensemble of probability measures, and in addition to her utility function, they often involve other parameters. For succinctness, we shall describe this entire package (i.e., a non-SEU decision model and its associated parameters) as the agent's "ambiguity attitude."

This raises the question of whether non-SEU ambiguity attitudes can be incorporated into collective decisions. But just as different agents can reasonably hold different probabilistic beliefs, different agents can reasonably adopt different ambiguity attitudes. Such heterogeneity leads once again to impossibility theorems (Chambers and Hayashi (2006), Gajdos, Tallon, and Vergnaud (2008), Mongin and Pivato (2015), Zuber (2016)). In general, to satisfy the ex ante Pareto axiom, all agents must not only have the same beliefs, but the same ambiguity attitudes—indeed, they must be SEU maximizers.[4] Once again, to escape this undesirable conclusion, one must weaken the ex ante Pareto axiom; this strategy has been explored in a series of elegant papers by Alon and Gayer (2016), Danan, Gajdos, Hill, and Tallon (2016), Qu (2017), and Hayashi and Lombardi (2019).[5] Like the foundational result of Gilboa, Samet, and Schmeidler (2004), these more recent papers axiomatically characterize not only a SWF, but a procedure for aggregating individual beliefs into a collective belief. As already noted, non-SEU models generally represent agents' beliefs by ensembles of probability measures, so these procedures aggregate these ensembles. Thus, they are vulnerable to the same objections earlier raised against Gilboa, Samet, and Schmeidler (2004) and Dietrich (2021): different belief-aggregation rules are appropriate in different environments, and in any case, collective beliefs should not necessarily be determined at the same time as the social

[4] In fact, when all agents have maximin SEU preferences, or all have Hurwicz preferences, Hayashi (2021) has shown that ex ante Pareto implies dictatorship, even if all agents have the same beliefs.

[5] See Mongin and Pivato (2016) or Fleurbaey (2018) for reviews of this literature.

welfare function. Furthermore, these theorems generally impose a particular ambiguity attitude on society (either in their hypotheses or in their conclusions).

The results of the present paper are compatible with both heterogeneity of beliefs and heterogeneity of ambiguity attitudes. Theorems 1 and 3 are formulated for "generalized Hurwicz" preferences, a broad class that includes SEU preferences, maximin SEU preferences, Hurwicz preferences, and second-order SEU preferences, among others. Theorem 2 is formulated for "Bewley preferences." In both preference classes, each agent's beliefs are described by a *set* of probability measures. The precise statement of Theorem 2 is that the belief set underlying collective preferences must be contained in the closed, convex hull of the union of the belief sets underlying the individual preferences. Importantly, none of Theorems 1–3 impose any relationship between individual ambiguity attitudes and collective ambiguity attitudes. We see this as an advantage. Just as the specification of the SWF is an ethical problem, and the specification of collective beliefs is an epistemic problem, the specification of collective ambiguity attitudes is a problem of *prudential rationality*. It is better to disentangle these three problems. This paper focuses on the first two problems, leaving the prudential problem for future work.

The rest of this paper is organized as follows. Section 2 introduces generalized Hurwicz representations. Section 3 introduces almost-objective uncertainty, and provides several sufficient conditions for the existence of almost-objective uncertainty. Section 4 turns to social welfare; it introduces a concept of "asymptotic preferences" based on almost-objective uncertainty and a corresponding Pareto axiom, along with the statement of Theorem 1 and several corollaries. Section 5 turns to belief aggregation, and contains our second Pareto axiom and Theorem 2. Section 6 contains Theorem 3, and compares our results to some prior literature. Appendices A, B, C, and F contain proofs of results stated in the main text, while Appendices D and E contains additional results, which may be of interest to some readers.

2. Generalized Hurwicz representations

Let $\mathcal{S}$ and $\mathcal{X}$ be measurable spaces, that is, sets equipped with sigma algebras.[6] We shall refer to $\mathcal{S}$ as the *state space* and $\mathcal{X}$ as the *outcome space*. Let $\Delta(\mathcal{S})$ be the set of all finitely additive probability measures on $\mathcal{S}$. An *act* is a measurable function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ that takes only finitely many values. Let $\mathcal{A}$ be the set of all acts. Let $\succcurlyeq$ be a preference order on $\mathcal{A}$. In the Savage model of uncertainty, $\mathcal{X}$ is a set of "outcomes," while $\mathcal{S}$ is a set of possible "states of nature"; the true state is unknown. The order $\succcurlyeq$ describes an agent's ex ante preferences. A *representation* of $\succcurlyeq$ is a function $V : \mathcal{A} \longrightarrow \mathbb{R}$ such that

$$\text{for all } \alpha, \beta \in \mathcal{A}, \quad (\alpha \succcurlyeq \beta) \quad \iff \quad \big(V(\alpha) \geq V(\beta)\big). \tag{1}$$

In particular, V is a *subjective expected utility* (SEU) *representation* if there is some $\rho \in \Delta(\mathcal{S})$ and a bounded measurable function $u : \mathcal{X} \longrightarrow \mathbb{R}$ such that

$$V(\alpha) = \int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\rho, \quad \text{for all } \alpha \in \mathcal{A}. \tag{2}$$

[6] For simplicity, we shall not make these sigma algebras explicit in our notation. A set will never be equipped with more than one sigma algebra in this paper.

Here, ρ is interpreted as the agent's subjective beliefs about the unknown state of nature, while u describes the utility she would obtain from each outcome. But as noted in Section 1, in situations of ambiguity, it might be inappropriate to represent an agent's beliefs as a single probability measure over $\mathcal{S}$. This has led to classes of preferences that use an *ensemble* of probability measures. This paper will focus on a broad class of such preferences: those admitting a "generalized Hurwicz" representation.

A representation V is *generalized Hurwicz* (GH) if there is a closed, convex subset $\mathcal{P} \subseteq \Delta(\mathcal{S})$ and a bounded measurable function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that

$$\text{for all } \alpha \in \mathcal{A}, \quad \underline{V}(\alpha) \leq V(\alpha) \leq \overline{V}(\alpha),$$
$$\text{where } \underline{V}(\alpha) := \inf_{\rho \in \mathcal{P}} \int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\rho \text{ and } \overline{V}(\alpha) := \sup_{\rho \in \mathcal{P}} \int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\rho. \tag{3}$$

The idea here is that the agent is not only unsure of the true state of nature, but also unsure about the correct probability distribution to put on $\mathcal{S}$; the *belief set* $\mathcal{P}$ contains all probabilities that she considers "possible." The GH representation (3) encompasses a wide gamut of preferences. It reduces to the SEU representation (2) if $\mathcal{P}$ is a singleton. It obviously includes the class of "maximin SEU" (or "multiple priors") preferences characterized by Gilboa and Schmeidler (1989) (for which $V(\alpha) = \underline{V}(\alpha)$, for all $\alpha \in \mathcal{A}$), and also the classical "Hurwicz" (or "α-maximin") preferences introduced by Hurwicz (1951) and recently characterized by Chateauneuf, Ventura, and Vergopoulos (2020) and Hartmann (2023) (for which $V(\alpha) = q\underline{V}(\alpha) + (1-q)\overline{V}(\alpha)$, for all $\alpha \in \mathcal{A}$, for some constant $q \in [0, 1]$). It also includes the class of "second-order SEU" (or "smooth ambiguity") preferences characterized by Klibanoff, Marinacci, and Mukerji (2005) and the "Choquet expected utility" preferences of Schmeidler (1989). More generally, Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011, Proposition 5) show that any "Monotone, Bernoullian, Archimedean" (MBA) preference admits a GH representation like (3), generalizing an earlier result of Ghirardato, Maccheroni, and Marinacci (2004, Proposition 7) for "invariant biseparable" preferences.

Let $\mathrm{ba}(\mathcal{S})$ be the Banach space of all finitely additive signed measures ("charges") on $\mathcal{S}$, which have finite *total variation norm*

$$\|\mu\|_{\mathrm{VT}} := \sup_{\substack{\mathcal{H}_1, \ldots, \mathcal{H}_N \subseteq \mathcal{S} \\ \text{disjoint measurable}}} \sum_{n=1}^{N} \left|\mu[\mathcal{H}_n]\right|. \tag{4}$$

We will say that a GH representation (3) is *compact* if $\mathcal{P}$ is compact in this norm. We shall say it is *nonatomic* if all elements of $\mathcal{P}$ are nonatomic measures. (A measure ρ is *nonatomic* if, for any $\epsilon > 0$, there is a measurable partition $\{\mathcal{G}_1, \ldots, \mathcal{G}_N\}$ of $\mathcal{S}$ such that $\rho(\mathcal{G}_n) < \epsilon$ for all $n \in [1 \ldots N]$.) We shall say that a representation V is *contiguous* if its image $V(\mathcal{A})$ is a dense subset of an interval in $\mathbb{R}$. For example, if $\mathcal{X}$ is a connected topological space and $u : \mathcal{X} \longrightarrow \mathbb{R}$ is continuous, then any GH representation (3) with u as its utility function is contiguous.[7]

[7] To see this, let α range over all constant-valued acts, to deduce that $V(\mathcal{A}) = u(\mathcal{X})$.

The goal of this paper is not to axiomatically characterize GH representations. We shall simply assume that the agents' preference have such representations; in light of the generality of this class, this is a reasonable assumption. But different agents might have *different* representations, with different u and $\mathcal{P}$. Thus, our framework allows great diversity in the beliefs and ambiguity attitudes of the agents.

The utility function u that appears in the GH representation (3) of a preference order $\succcurlyeq$ is unique up to positive affine transformations. But the belief set $\mathcal{P}$ is not unique. There are certain "natural" choices for $\mathcal{P}$; for example, in the frameworks of Ghirardato, Maccheroni, and Marinacci (2004) and Cerreia-Vioglio et al. (2011), there is a unique belief set that yields a Bewley representation for the "unambiguous" part of $\succcurlyeq$; we will discuss this further in Section 5. Alternatively, one could use an inclusion-minimal belief set (see Lemma E.1 in Appendix E). The characterizations of utilitarianism in this paper apply to *any* GH representations for the preferences of the agents. But the smaller the corresponding belief sets are, the easier it will be to satisfy our hypotheses.

3. Almost-objective uncertainty

A *measurable partition* of $\mathcal{S}$ is a finite collection $\mathfrak{G} = \{\mathcal{G}_n\}_{n=1}^N$ of disjoint measurable subsets such that $\mathcal{S} = \bigsqcup_{n=1}^N \mathcal{G}_n$. For any $K \in \mathbb{N}$, let $\Delta^K := \{\mathbf{q} = (q_1, \ldots, q_K) \in \mathbb{R}_+^K; \sum_{k=1}^K q_k = 1\}$, the set of K-dimensional probability vectors.

Let $\mathcal{R}$ be a collection of probability measures on $\mathcal{S}$. Let $K \in \mathbb{N}$ and let $\mathbf{q} \in \Delta^K$. For all $n \in \mathbb{N}$, let $\mathfrak{G}^n := \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$ be a K-cell measurable partition of $\mathcal{S}$. We shall say that the sequence of partitions $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{R}$-*almost-objectively uncertain* and *subordinate to* $\mathbf{q}$ if, for all $\rho \in \mathcal{R}$, we have

$$\lim_{n\to\infty} \rho\big(\mathcal{G}_k^n\big) = q_k, \quad \text{for all } k \in [1 \ldots K]. \tag{5}$$

For example, let $\mathcal{S} = [0, 1]$, and let $\mathcal{R}$ be the set of all probability measures that are absolutely continuous with respect to the Lebesgue measure, with continuous density functions. Suppose $\mathbf{q} = (0.1, 0.2, 0.3, 0.4)$. For any number $s \in [0, 1]$ and $n \in \mathbb{N}$, let $s_{(n)}$ be the nth digit in the decimal expansion of s.[8] For all $n \in \mathbb{N}$, let $\mathfrak{G}^n := \{\mathcal{G}_1^n, \mathcal{G}_2^n, \mathcal{G}_3^n, \mathcal{G}_4^n\}$, where $\mathcal{G}_1^n := \{s \in [0, 1]; s_{(n)} = 0\}$, $\mathcal{G}_2^n := \{s \in [0, 1]; s_{(n)} \in \{1, 2\}\}$, $\mathcal{G}_3^n := \{s \in [0, 1]; s_{(n)} \in \{3, 4, 5\}\}$, and $\mathcal{G}_4^n := \{s \in [0, 1]; s_{(n)} \in \{6, 7, 8, 9\}\}$. It is easily seen that $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{R}$-almost-objectively uncertain and subordinate to $\mathbf{q}$.

Almost-objective uncertainty was first introduced by Poincaré (1912) to explain why it is reasonable to hold particular epistemic probabilities regarding a physical randomization device such as a roulette wheel, even if we do not have an exact understanding of how this apparent randomness is generated. Its first application to decision-making under ambiguity was due to Machina (2004, 2005), who also coined the term "almost-objective uncertainty." Poincaré and Machina considered almost-objective uncertainty on the unit interval $[0, 1]$, as in the above example. We will now generalize this concept to a much broader collection of state spaces and probability measures. Let $\mathcal{S}$ be a

[8] There is a countable subset of $[0, 1]$ of numbers with nonunique decimal expansions, for whom $s_{(n)}$ is not well-defined. But it has Lebesgue measure zero, so it is irrelevant to this construction.

measurable space, and let $\mathcal{R} \subseteq \Delta(\mathcal{S})$. We shall say that $\mathcal{R}$ is *consilient* if, for any $K \in \mathbb{N}$ and $\mathbf{q} \in \Delta^K$, there is an $\mathcal{R}$-almost-objectively uncertain sequence of partitions $(\mathfrak{G}^n)_{n=1}^{\infty}$ subordinate to $\mathbf{q}$. The results in this section give sufficient conditions for consilience. We need some terminology. A subset $\mathcal{R} \subseteq \Delta(\mathcal{S})$ is *nonatomic* if all elements of $\mathcal{R}$ are nonatomic. It is *separable* if it has a countable dense subset in the topology of the total variation norm (4).

Proposition 1. *If $\mathcal{R}$ is nonatomic and separable, then $\mathcal{R}$ is consilient.*

It is sometimes convenient to have a consilient set that is closed under Bayesian updating. For any $\mu, \rho \in \Delta(\mathcal{S})$, we shall write "$\mu \lll \rho$" if there is a measurable function $\phi : \mathcal{S} \longrightarrow \mathbb{R}$ such that $\mu(\mathcal{B}) = \int_{\mathcal{B}} \phi \, \mathrm{d}\rho$ for all measurable $\mathcal{B} \subseteq \mathcal{S}$; in this case, we define $\frac{\mathrm{d}\mu}{\mathrm{d}\rho} := \phi$.[9] For any subset $\mathcal{R} \subseteq \Delta(\mathcal{S})$, let $\langle\mathcal{R}\rangle := \{\mu \in \Delta(\mathcal{S}); \mu \lll \rho$ for some $\rho \in \mathcal{R}$, and $\frac{\mathrm{d}\mu}{\mathrm{d}\rho}$ is bounded$\}$. In particular, $\langle\mathcal{R}\rangle$ includes all measures that arise from a Bayesian update of some element of $\mathcal{R}$. Let us say that $\mathcal{R}$ is *strongly consilient* if $\langle\mathcal{R}\rangle$ is consilient.

The next result gives two sufficient conditions for strong consilience. First, we need some terminology. A probability measure $\mu \in \Delta(\mathcal{S})$ is *separable* if there is a countable set of events $\{\mathcal{E}_n\}_{n=1}^{\infty}$ that is *dense*: for any measurable $\mathcal{B} \subseteq \mathcal{S}$, and any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\mathcal{B}$ is "ϵ-approximated" by $\mathcal{E}_n$ in the sense that $\mu[\mathcal{B} \setminus \mathcal{E}_n] < \epsilon$ and $\mu[\mathcal{E}_n \setminus \mathcal{B}] < \epsilon$. (Equivalently, μ is separable if the normed vector space $\mathcal{L}^1(\mathcal{S}, \mu)$ is separable.)[10] For example, the Lebesgue measure on $[0, 1]$ is separable. Most probability spaces that arise in practical applications are separable.

A *standard Borel space* is a measurable space $\mathcal{S}$ that is measurably isomorphic to a complete, separable metric space $\mathcal{S}'$ (e.g., a closed subset of $\mathbb{R}^N$), endowed with its Borel sigma algebra (i.e., there is a measurable bijection from $\mathcal{S}$ to $\mathcal{S}'$ whose inverse is also measurable). Every Polish space is a standard Borel space. But a standard Borel space need not have a Polish topology (or indeed, any topology at all). Almost every measurable space encountered in applications is standard Borel.[11] Let $\Delta_\sigma(\mathcal{S})$ be the space of *countably* additive probability measures on $\mathcal{S}$.

Proposition 2. *Suppose that $\mathcal{R} \subseteq \Delta(\mathcal{S})$ is nonatomic and separable, and suppose that either* (a) *every element of $\mathcal{R}$ is separable; or* (b) *$\mathcal{S}$ is a standard Borel space and $\mathcal{R} \subseteq \Delta_\sigma(\mathcal{S})$. Then $\mathcal{R}$ is strongly consilient.*

Further sufficient conditions under which a collection $\mathcal{R} \subseteq \Delta(\mathcal{S})$ is (strongly) consilient can be found in Appendix D.

[9] If μ and ρ are *countably* additive, then the Radon–Nikodym theorem says that $\mu \lll \rho$ if and only if μ is absolutely continuous relative to ρ ("$\mu \ll \rho$"). But we only assume that μ and ρ are *finitely* additive. Given supplementary technical conditions, finitely additive versions of the Radon–Nikodym theorem have been obtained by Berti, Regazzini, and Rigo (1992, Corollaries 3.1 and 3.2), Basile and Bhaskara Rao (2000, Theorem 7.5), and others; see Candeloro and Volčič (2002, Section 3) for a summary.

[10] If μ is only finitely additive, then $\mathcal{L}^1(\mathcal{S}, \mu)$ might not be a Banach space, unless certain technical conditions are satisfied (Basile and Bhaskara Rao (2000)). But this is irrelevant for our purposes.

[11] For a good introduction to standard Borel spaces, see Section 424, p. 158 of Fremlin (2006a).

4. Social aggregation of utility

As noted in Section 1, a central problem in Bayesian social aggregation is that different agents might have different probabilistic beliefs and different attitudes toward ambiguity. We shall now use almost-objective uncertainty to obviate these problems.

Almost-objective acts Let $\mathcal{R}$ be a consilient collection of probability measures on a measurable space $\mathcal{S}$. Let $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$ be a sequence of acts. We shall say that $\boldsymbol{\alpha}$ is an *$\mathcal{R}$-almost-objective act* if there is a K-tuple of outcomes $\mathbf{x} \in \mathcal{X}^K$ (for some $K \in \mathbb{N}$), and an $\mathcal{R}$-almost-objectively uncertain sequence of K-cell partitions $\boldsymbol{\mathcal{G}} = (\mathfrak{G}^n)_{n=1}^{\infty}$, with $\mathfrak{G}^n := \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$ for all $n \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ and $k \in [1 \ldots K]$ we have $\alpha^n(s) = x_k$ for all $s \in \mathcal{G}_k^n$. If $\boldsymbol{\mathcal{G}}$ is subordinate to the probability vector $\mathbf{q} \in \Delta^K$, then we shall say that $\boldsymbol{\alpha}$ is *subordinate to* $(\mathbf{q}, \mathbf{x})$.

Let $\boldsymbol{\beta} = (\beta^n)_{n=1}^{\infty}$ be another $\mathcal{R}$-almost-objective act. We shall say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are *compatible* if β^n is also measurable with respect to $\mathfrak{G}^n$ for all $n \in \mathbb{N}$.

Asymptotic preferences Let $\succcurlyeq$ be a preference order on $\mathcal{A}$. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be $\mathcal{R}$-almost-objective acts. We shall say $\succcurlyeq$ *asymptotically prefers* $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$, and write $\boldsymbol{\alpha} \succ^{\infty} \boldsymbol{\beta}$ if there exist $\alpha', \beta' \in \mathcal{A}$, and $N \in \mathbb{N}$ such that $\alpha^n \succ \alpha' \succ \beta' \succ \beta^n$ for all $n \geq N$.

Almost-objective Pareto Let $\mathcal{I}$ be a set of individuals. Let o be another agent, representing a social planner or social observer. Let $\mathcal{J} = \mathcal{I} \sqcup \{o\}$. For all $j \in \mathcal{J}$, let $\succcurlyeq_j$ be a preference order on $\mathcal{A}$. We shall require $\succcurlyeq_o$ to satisfy the following axiom, relative to $\{\succcurlyeq_i\}_{i\in\mathcal{I}}$ and $\mathcal{R}$:

$\mathcal{R}$-Almost-Objective Pareto. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are compatible $\mathcal{R}$-almost-objective acts, and $\boldsymbol{\alpha} \succ_i^{\infty} \boldsymbol{\beta}$ for all $i \in \mathcal{I}$, then $\boldsymbol{\alpha} \not\prec_o^{\infty} \boldsymbol{\beta}$.

This axiom does *not* require $\boldsymbol{\alpha} \succ_o^{\infty} \boldsymbol{\beta}$; it simply requires the social planner not to form the *opposite* asymptotic preference to that of the individuals.

Minimal Agreement Suppose that each of the preference orders $\{\succcurlyeq_j\}_{j\in\mathcal{J}}$ has a GH representation (3) with an associated utility function $u_j : \mathcal{X} \longrightarrow \mathbb{R}$. We shall say that the utility functions $\{u_i\}_{i\in\mathcal{I}}$ satisfy *Minimal Agreement* if there exist probability measures μ_1 and μ_2 on $\mathcal{X}$ such that $\int_{\mathcal{X}} u_i \, \mathrm{d}\mu_1 > \int_{\mathcal{X}} u_i \, \mathrm{d}\mu_2$ for all $i \in \mathcal{I}$. In other words, there exist two "objective lotteries" over outcomes, for which all individuals have the same strict preference. Versions of this condition are widespread in the literature on Bayesian social aggregation; see, for example, Mongin (1995, 1998), Alon and Gayer (2016), or Danan et al. (2016).

Utilitarianism and weak utilitarianism Recall that u_o is the ex post utility function associated to the social preference order $\succcurlyeq_o$. We shall say that u_o is *weakly utilitarian* if there exist constants $c_i \geq 0$ for all $i \in \mathcal{I}$ and $b \in \mathbb{R}$ such that

$$u_o = b + \sum_{i\in\mathcal{I}} c_i u_i. \tag{6}$$

It is possible that $c_i = 0$ for some $i \in \mathcal{I}$; thus, the preferences of some individuals might be ignored. If $c_i > 0$ for all $i \in \mathcal{I}$, then u_o is *utilitarian*. Under mild conditions, weak utilitarianism is equivalent to utilitarianism (see Proposition F.1 in Appendix F). So, we focus on establishing *weak* utilitarianism. We now come to our main result.

THEOREM 1. *Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$ be consilient. For all $j \in \mathcal{J}$, suppose $\succeq_j$ has a compact, contiguous GH representation* (3) *with $\mathcal{P}_j \subseteq \mathcal{R}$. Assume that $\{u_i\}_{i \in \mathcal{I}}$ satisfy* Minimal Agreement. *Then $\succeq_o$ satisfies $\mathcal{R}$*-Almost-Objective Pareto *if and only if u_o is weakly utilitarian.*

The next result applies this to the original problem of Bayesian social aggregation.

COROLLARY 1. *Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$ be consilient. For all $j \in \mathcal{J}$, suppose $\succeq_j$ has a contiguous SEU representation* (2) *with $\rho_j \in \mathcal{R}$. Suppose $\{u_i\}_{i \in \mathcal{I}}$ satisfy* Minimal Agreement. *Then $\succeq_o$ satisfies $\mathcal{R}$*-Almost-Objective Pareto *if and only if u_o is weakly utilitarian.*

Intrinsic consilience A possible criticism of Theorem 1 and Corollary 1 is that $\mathcal{R}$-Almost-Objective Pareto involves an exogenous set $\mathcal{R}$ of probability measures. The next axiom endogenizes $\mathcal{R}$.

Almost-Objective Pareto*. For all $j \in \mathcal{J}$, let $\succeq_j$ be a preference order on $\mathcal{A}$ with a GH representation (3) given by some set $\mathcal{P}_j \subseteq \Delta(\mathcal{S})$. Let $\mathcal{R} := \bigcup_{j \in \mathcal{J}} \mathcal{P}_j$.
If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are compatible $\mathcal{R}$-almost-objective acts, and $\boldsymbol{\alpha} \succ_i^{\infty} \boldsymbol{\beta}$ for all $i \in \mathcal{I}$, then $\boldsymbol{\alpha} \not\prec_o^{\infty} \boldsymbol{\beta}$.

Combining Proposition 1 with Theorem 1 yields the following result.

COROLLARY 2. *For all $j \in \mathcal{J}$, suppose $\succeq_j$ has a compact, contiguous, nonatomic GH representation* (3), *and suppose $\{u_i\}_{i \in \mathcal{I}}$ satisfy* Minimal Agreement. *Then $\succeq_o$ satisfies* Almost-Objective Pareto* *if and only if u_o is weakly utilitarian.*

One can likewise obtain versions of Corollary 1 using Almost-Objective Pareto*. These results follows from Proposition 1 because $\bigcup_{j \in \mathcal{J}} \mathcal{P}_j$ is compact, and hence separable (see the end of Appendix B for details). The advantage of Corollary 2 over Theorem 1 is that the relevant Pareto axiom is defined "by the agents themselves," via their belief sets $\{\mathcal{P}_j\}_{j \in \mathcal{J}}$. The disadvantage is that, to verify Almost-Objective Pareto*, one must exactly identify the sets $\{\mathcal{P}_j\}_{j \in \mathcal{J}}$. In contrast, to apply Theorem 1, one need only know that these sets are all contained in *some* consilient set $\mathcal{R}$.

Proof sketch Recall that Harsanyi's (1955) original result involved expected-utility preferences over objective lotteries. In that setting, if $\succeq_o$ is not weakly utilitarian, then the separating hyperplane theorem can be used to construct a pair of lotteries that violate the *ex ante Pareto* axiom. By restricting the Pareto axiom to asymptotic preferences between almost-objective acts, we have restricted it to a domain where agents' preferences are "almost" described by such objective expected utilities. This is expressed precisely by the next result, which is also of independent interest.

PROPOSITION 3. *Let $\mathcal{R}$ be a consilient set of probability measures on $\mathcal{S}$. Let $K \in \mathbb{N}$, let $\mathbf{q} \in \Delta^K$, let $\mathbf{x} \in \mathcal{X}^K$, and let $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$ be an $\mathcal{R}$-almost-objective act subordinate to $(\mathbf{q}, \mathbf{x})$. Let V be a compact GH representation* (3) *with $\mathcal{P} \subseteq \mathcal{R}$. Then*

$$\lim_{n \to \infty} V(\alpha^n) = \sum_{k=1}^{K} q_k u(x_k). \tag{7}$$

By virtue of Proposition 3, a separating hyperplane argument can be applied to prove Theorem 1. Proposition 3 also has another important consequence: when considering an agent's asymptotic preferences over almost-objective acts, *all information about that agent's beliefs is effaced*. This explains why $\mathcal{R}$-Almost-Objective Pareto *cannot* entail any link between individual beliefs and collective beliefs. We will turn to this question in the next section.

5. Collective beliefs

In this section, we shall assume *Minimal Agreement on Outcomes* (MAO): there exist $x, y \in \mathcal{X}$ such that $x \succ_j y$ for all $j \in \mathcal{J}$. Let us call the pair (x, y) a *dichotomy*. Let $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ be an act. Say that α is a *dichotomous act* if there is a dichotomy (x, y) such that $\alpha(s) \in \{x, y\}$ for all $s \in \mathcal{S}$. Two dichotomous acts α and β are *congruent* if they range over the same dichotomy $\{x, y\}$. Consider the following axiom.

Dichotomous Pareto. For any congruent dichotomous acts $\alpha, \beta \in \mathcal{A}$, if $\alpha \succcurlyeq_i \beta$ for all $i \in \mathcal{I}$, then $\alpha \succcurlyeq_o \beta$.

The next result is derived from a result of Mongin (1995).

Proposition 4. *Suppose the preferences $\{\succcurlyeq_j\}_{j\in\mathcal{J}}$ all have SEU representations with nonatomic beliefs $\{\rho_j\}_{j\in\mathcal{J}}$, and they satisfy* MAO. *Then $\succcurlyeq_o$ satisfies* Dichotomous Pareto *if and only if ρ_o is a convex combination of $\{\rho_i\}_{i\in\mathcal{I}}$.*

Consistent with the philosophy of this paper, Proposition 4 decouples the problem of belief aggregation from that of utility aggregation: it determines the collective beliefs but says nothing about social welfare. But it only applies when all agents are SEU maximizers. Are there similar results for other ambiguity attitudes? In uncertain decision environments where all agents have the *same* utility function, the social aggregation of beliefs has been studied by Crès, Gilboa, and Vieille (2011), Nascimento (2012), Gajdos and Vergnaud (2013), and Stanca (2021) for various ambiguity attitudes including maximin expected utility and second-order subjective expected utility. By restricting to dichotomous acts, Dichotomous Pareto simulates a world where all agents have the same utility function, so Proposition 4 is comparable to this literature. This raises the question of whether there is a version of Proposition 4 for GH preferences.

Unfortunately, the class of GH preferences does not admit a result analogous to Proposition 4. In the representation (3), the function V conflates the agent's *beliefs* (the set $\mathcal{P}$) with her *ambiguity attitudes*. This conflation remains even if we restrict to congruent dichotomous acts. To forge a link between the social belief set $\mathcal{P}_o$ and the individual belief sets $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$, we must isolate the part of the agents' preferences that is determined solely by their beliefs, and is independent of their ambiguity attitudes.

Unambiguous preferences Suppose temporarily that $\mathcal{X}$ is a convex space, as in the Anscombe–Aumann framework. For any $\alpha, \beta \in \mathcal{A}$, and $q \in [0, 1]$, define $\alpha \oplus_q \beta \in \mathcal{A}$ by setting $(\alpha \oplus_q \beta)(s) := q\alpha(s) + (1 - q)\beta(s)$ for all $s \in \mathcal{S}$. Let $\succcurlyeq$ be a preference order on $\mathcal{A}$.

The *unambiguous part* of $\succcurlyeq$ is the binary relation $\trianglerighteq$ on $\mathcal{A}$ defined

$$(\alpha \trianglerighteq \beta) \quad \Longleftrightarrow \quad \big(\alpha \oplus_q \gamma \succcurlyeq \beta \oplus_q \gamma, \text{ for all } \gamma \in \mathcal{A} \text{ and all } q \in (0, 1]\big).$$

This is the largest subrelation of $\succeq$ that satisfies the vNM *Independence* axiom (Ghirardato, Maccheroni, and Marinacci (2004), Proposition 4, part 7). Under certain conditions, there is a unique weak* compact, convex set $\mathcal{P} \subseteq \Delta(\mathcal{S})$ and a utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$ that yield both a generalized Hurwicz representation (3) for $\succcurlyeq$, and a *Bewley representation* for $\trianglerighteq$, meaning that[12]

$$\text{for all } \alpha, \beta \in \mathcal{A}, \quad (\alpha \trianglerighteq \beta) \quad \Longleftrightarrow \quad \left(\int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\rho \geq \int_{\mathcal{S}} u \circ \beta \, \mathrm{d}\rho \text{ for all } \rho \in \mathcal{P}\right). \tag{8}$$

In fact, a convex $\mathcal{X}$ is not necessary to obtain these results. Recently, working in the Savage framework, and generalizing the work of Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003), Ghirardato and Pennesi (2020) have shown that if $\succcurlyeq$ has even one "locally biseparable event," then one can define a "subjective mixture" operation on $\mathcal{X}$ for $\succeq$. The aforementioned representation results can then be extended to any monotone, locally biseparable preference using this subjective mixture operation, yielding combined GH/Bewley representations for $\succcurlyeq$ and $\trianglerighteq$.[13]

More generally, let $\trianglerighteq$ be any preorder on $\mathcal{A}$, that is, a transitive, reflexive (but possibly incomplete) binary relation. A *Bewley representation* for $\trianglerighteq$ is a pair $(\mathcal{P}, u)$, where $\mathcal{P} \subset \Delta(\mathcal{S})$ and $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that statement (8) holds. If $\trianglerighteq$ has such a representation, then we shall call it a *Bewley preference*. When restricted to constant acts, a Bewley preference defines a complete order on $\mathcal{X}$. So, the property of Minimal Agreement on Outcomes, the definition of dichotomous acts, and the Dichotomous Pareto axiom are all meaningful for Bewley preferences.

Theorem 2. *Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$ be strongly consilient. For all $j \in \mathcal{J}$, suppose $\trianglerighteq_j$ has a Bewley representation* (8) *given by a compact subset $\mathcal{P}_j \subseteq \mathcal{R}$, and suppose these preferences satisfy* MAO. *Let $\overline{\mathcal{P}}$ be the closed, convex hull of $\bigcup_{i \in \mathcal{I}} \mathcal{P}_i$. Then $\trianglerighteq_o$ satisfies* Dichotomous Pareto *if and only if $\mathcal{P}_o \subseteq \overline{\mathcal{P}}$.*

In the special case when all agents have SEU preferences, we have $\mathcal{P}_j = \{\rho_j\}$ for all $j \in \mathcal{J}$, so that Theorem 2 reduces to Proposition 4. As explained earlier, if $\succcurlyeq$ is a preference order on $\mathcal{A}$ with unambiguous part $\trianglerighteq$, then in many cases $\trianglerighteq$ has a Bewley representation (8) with a set $\mathcal{P}$ that also appears in a GH representation (3) of $\succcurlyeq$.[14] In this case,

[12] Ghirardato, Maccheroni, and Marinacci (2004, Propositions 5 and 7) showed this in the case when $\succcurlyeq$ is an "invariant biseparable" preference. This result was then extended to Monotone, Bernoullian, Archimedean (MBA) preferences by Cerreia-Vioglio et al. (2011, Propositions 2 and Corollary 3). The original reference is Bewley (2002).

[13] See Remark 1 of Ghirardato and Pennesi (2020) for details. However, different agents generally have *different* subjective mixture operations. So unlike almost-objective uncertainty, subjective mixtures cannot be used for Bayesian social aggregation.

[14] More generally, given *any* preference $\trianglerighteq$ with a Bewley representation (8), Danan et al. (2016, Proposition 2) show that any transitive, Archimedean completion of $\trianglerighteq$ has a GH representation (3) using the same set $\mathcal{P}$ of beliefs. (They refer to GH preferences as *variable caution rules*.)

Theorem 2 establishes a relationship between the belief set underlying the GH representation of the social preferences and the belief sets of the individuals' GH representations. But Theorem 2 applies to *any* collection of Bewley preferences.

The proof strategy is as follows. Suppose (x, y) is a dichotomy. Renormalize all agents' utilities such that $u(x) = 1$ and $u(y) = 0$. For any $p \in [0, 1]$, one can create an almost-objective act such that x and y appear with probabilities p and $1 - p$ according to every element of $\mathcal{P}_j$, for all $j \in \mathcal{J}$. So, all agents assign this act an almost-objective expected utility of p. For any measurable partition $\mathfrak{H}$ of $\mathcal{S}$, we can then "stitch together" such almost-objective acts across the cells of $\mathfrak{H}$ to create a "piecewise almost-objective act" such that for all agents, the conditional expected utility in each cell of $\mathfrak{H}$ takes some specified value (Lemma C.2 in Appendix C). Each agent's asymptotic preferences over these gadgets then entail inequalities between linear functionals, which must hold for all elements of her belief set. We can thus use the separating hyperplane theorem to derive $\mathcal{P}_o \subseteq \overline{\mathcal{P}}$ from Dichotomous Pareto.

Bayesian social aggregation of Bewley preferences has previously been analyzed by Danan et al. (2016). In particular, Danan et al.'s Theorem 2 shows that a certain Pareto axiom implies that $\mathcal{P}_o \subseteq \overline{\mathcal{P}}$. However, like Gilboa, Samet, and Schmeidler (2004), the results of Danan et al. simultaneously characterize belief aggregation *and* utility aggregation, whereas we separate these problems. By combining $\mathcal{R}$-Almost-Objective Pareto and Dichotomous Pareto, we can characterize both the social welfare function and social belief set using Theorems 1 and 2. But we can also choose to impose only one or the other of these axioms, thereby constraining *either* the social welfare function *or* the social belief set, while leaving the other unconstrained.

6. Discussion

We have considered a decision environment of radical uncertainty, in which the ex ante preferences of each agent admit generalized Hurwicz representation. We have introduced a very weak Pareto axiom, which applies only to asymptotic preferences along a sequence of acts for which all possible probabilistic beliefs entertained by all agents converge to the same limit. We have shown that social preferences satisfy this weak Pareto axiom if and only if the ex post social welfare function is a weighted sum of the ex post utility functions of the individuals. In other words, social preferences must be ex post utilitarian. A different Pareto axiom characterizes the formation of collective beliefs. Importantly, these results separate utility aggregation from belief aggregation, and they do not impose any relationship between collective ambiguity attitudes and individual ambiguity attitudes. As explained in Section 1, we see this as an advantage. We will now relate our results to some prior literature.

Restricted Pareto For all $i \in \mathcal{I}$, suppose $\succsim_i$ has a GH representation (3) with belief set $\mathcal{P}_i$. Let $\mathfrak{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_K\}$ be a partition of $\mathcal{S}$. Let us say that $\mathfrak{G}$ is a *consensus partition* if there is some $\mathbf{q} \in \Delta^K$ such that $\rho(\mathcal{G}_k) = q_k$ for all $k \in [1 \ldots K]$, all $\rho \in \mathcal{P}_i$, and all $i \in \mathcal{I}$—in other words, all individuals *exactly agree* on the probabilities of all cells of $\mathfrak{G}$. In a watershed paper, Gilboa, Samet, and Schmeidler (2004) proposed a version of the following axiom.

Restricted Pareto. Let $\alpha, \beta \in \mathcal{A}$ be measurable with respect to a consensus partition $\mathfrak{G}$. If $\alpha \succcurlyeq_i \beta$ for all $i \in \mathcal{I}$, then $\alpha \succcurlyeq_o \beta$.[15]

Let us say that a GH representation (3) is *polytopic* if the set $\mathcal{P}$ is a polytope, that is, the convex hull of a finite subset of $\Delta(\mathcal{S})$. Gilboa, Samet, and Schmeidler (2004) worked with SEU preferences based on countably additive probability measures. But their result has the following generalization to (finitely additive) GH representations.

PROPOSITION 5. *Suppose that $\succcurlyeq_o$ has an SEU representation given by some $u_o : \mathcal{X} \longrightarrow \mathbb{R}$ and $\rho_o \in \Delta(\mathcal{S})$. For all $i \in \mathcal{I}$, suppose that $\succcurlyeq_i$ has a nonatomic, polytopic GH representation* (3). *Then $\succcurlyeq_o$ satisfies* Restricted Pareto *if and only if u_o is weakly utilitarian and ρ_o is in the span of $\bigcup_{i \in \mathcal{I}} \mathcal{P}_i$.*

Unlike Theorem 1, this result only applies if the social preference order $\succcurlyeq_o$ has an SEU representation. Also, as we have already argued, a simultaneous characterization of utilitarianism and linear belief aggregation is a mixed blessing. But a slight weakening of the Restricted Pareto axiom addresses both of these concerns. For all $j \in \mathcal{J}$ (including o), suppose that $\succcurlyeq_j$ has a GH representation (3) with belief set $\mathcal{P}_j$. Let us say that a partition $\mathfrak{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_K\}$ is a *strong consensus partition* if there is some $\mathbf{q} \in \Delta^K$ such that $\rho(\mathcal{G}_k) = q_k$ for all $k \in [1 \dots K]$, all $\rho \in \mathcal{P}_j$, and all $j \in \mathcal{J}$ (including o). We shall weaken the axiom of Gilboa, Samet, and Schmeidler (2004) as follows.

Restricted Pareto*. Let $\alpha, \beta \in \mathcal{A}$ be measurable with respect to a strong consensus partition $\mathfrak{G}$. If $\alpha \succcurlyeq_i \beta$ for all $i \in \mathcal{I}$, then $\alpha \succcurlyeq_o \beta$.

This axiom seems quite similar to Almost-Objective Pareto. Indeed, if $\mathfrak{G}$ is a strong consensus partition, and we define $\mathfrak{G}^n := \mathfrak{G}$ for all $n \in \mathbb{N}$, then the sequence $(\mathfrak{G}^n)_{n=1}^{\infty}$ is trivially an "almost-objective" sequence with respect to the family $\mathcal{R} := \bigcup_{j \in \mathcal{J}} \mathcal{P}_j$. Thus, if α and β are measurable with respect to $\mathfrak{G}$, and we define $\alpha^n := \alpha$ and $\beta^n := \beta$ for all $n \in \mathbb{N}$, then the sequences $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$ and $\boldsymbol{\beta} := (\beta^n)_{n=1}^{\infty}$ are compatible almost-objective acts. Thus, any unanimous preference, which is admissible as input to Restricted Pareto*, is also admissible to Almost-Objective Pareto, except that Almost-Objective Pareto accepts a larger variety of inputs, and yields a weaker conclusion.

THEOREM 3. *For all $j \in \mathcal{J}$, suppose that $\succcurlyeq_j$ has a nonatomic, polytopic GH representation (3). Then $\succcurlyeq_o$ satisfies* Restricted Pareto* *if and only if u_o is weakly utilitarian.*

In comparison with Theorem 1, the main advantage of Theorem 3 is that Restricted Pareto* is a simpler and more natural axiom than $\mathcal{R}$-Almost-Objective Pareto. But there are three major disadvantages. First, Theorem 3 only applies to *polytopic* GH representations. Second, Restricted Pareto* suffers from the same weakness as Almost-Objective Pareto*, as remarked after Corollary 2: to apply Restricted Pareto* in a particular situation, we must be able to recognize strong consensus partitions, which requires precise

[15] Gilboa, Samet, and Schmeidler used only the "indifference" part of this axiom, and assumed SEU representations, so $\mathcal{P}_i$ was a singleton for all $i \in \mathcal{I}$. So, their definition of "consensus partition" is simpler than ours.

knowledge of the sets $\{\mathcal{P}_j\}_{j\in\mathcal{J}}$—something which may be difficult to achieve in practice. In contrast, to apply $\mathcal{R}$-Almost-Objective Pareto, we need only know that $\{\mathcal{P}_j\}_{j\in\mathcal{J}}$ are contained in $\mathcal{R}$, a broad family of probability measures. It is possible to determine whether a partition sequence is $\mathcal{R}$-almost-objectively uncertain without knowing anything about $\{\mathcal{P}_j\}_{j\in\mathcal{J}}$, and also possible to construct such partition sequences on demand (e.g., using the methods of Appendix A).

But the third and most serious disadvantage is that the Restricted Pareto axiom (in either form) is vulnerable to a form of "spurious unanimity," as we now explain.

Complementary ignorance In real decision environments, new information arrives all the time. This creates a potential problem: as agents acquire more information and Bayes update their beliefs, *different* partitions of $\mathcal{S}$ will become consensus partitions. Thus, the scope of application of Restricted Pareto* will shift as the information available to the agents changes. As noted by Mongin and Pivato (2020, Section 6, p. 649), different agents might "spuriously" assign the same probabilities to the cells of a partition because they receive different information. This can lead Restricted Pareto* to make recommendations that are obviously incorrect in light of the aggregate information of the entire group.

For a simple illustration, suppose that there are two individuals $\mathcal{I} = \{i, j\}$, along with the social planner o. The state space $\mathcal{S}$ is the triangle shown in Figure 1. Divide $\mathcal{S}$ into four triangular regions, and suppose that the three agents have prior beliefs ρ_o, ρ_i, and ρ_j, which assign probabilities to these regions as shown in Figure 1(a). (We do not care how these probabilities are distributed within each region—indeed, it does not

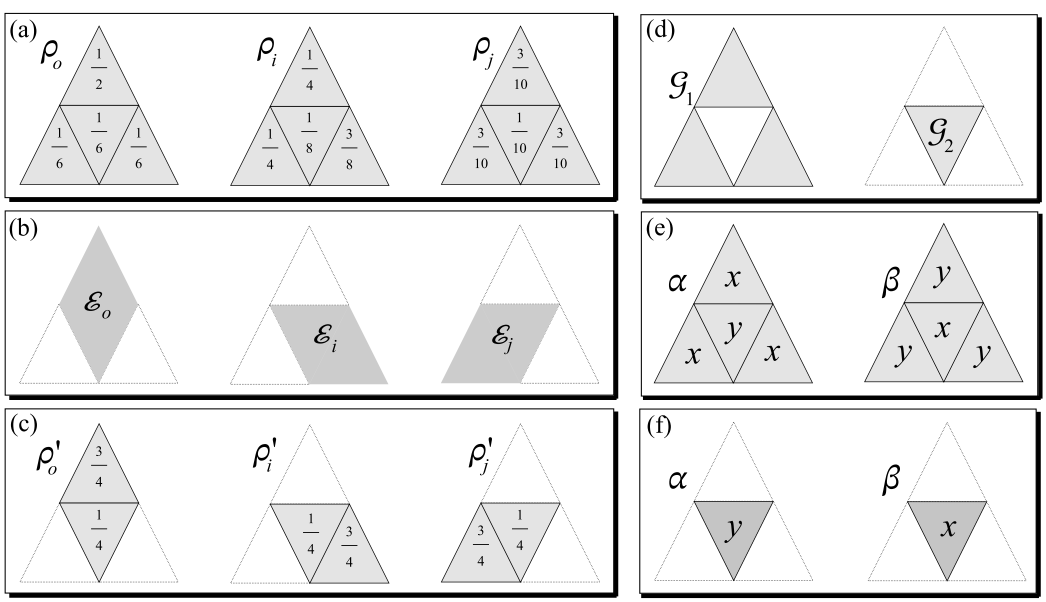

FIGURE 1. An example of complementary ignorance: (a) the prior beliefs of the three agents, (b) the events they observe, (c) their posterior beliefs, (d) the partition $\mathfrak{G} = \{\mathcal{G}_1, \mathcal{G}_2\}$, (e) the acts α and β, *ex ante*, and (f) the acts α and β in light of the combined information of all agents.

even matter whether they are atomic or nonatomic.) Each agent then receives private information. The social planner observes the event $\mathcal{E}_o$ (Figure 1(b)) and updates her beliefs to ρ'_o (Figure 1(c)). Meanwhile, individual i observes $\mathcal{E}_i$, and updates her beliefs to ρ'_i, while j observes $\mathcal{E}_j$, and updates her beliefs to ρ'_j.

Consider the partition $\mathfrak{G} = \{\mathcal{G}_1, \mathcal{G}_2\}$ shown in Figure 1(d). This is a *strong consensus* partition, because $\rho'_o(\mathcal{G}_1) = \rho'_i(\mathcal{G}_1) = \rho'_j(\mathcal{G}_1) = \frac{3}{4}$ and $\rho'_o(\mathcal{G}_2) = \rho'_i(\mathcal{G}_2) = \rho'_j(\mathcal{G}_2) = \frac{1}{4}$. Let $x, y \in \mathcal{X}$. Define the acts $\alpha, \beta : \mathcal{S} \longrightarrow \mathcal{X}$ as shown in Figure 1(e). That is, $\alpha(s) = x$ for all $s \in \mathcal{G}_1$ and $\alpha(s) = y$ for all $s \in \mathcal{G}_2$, whereas $\beta(s) = y$ for all $s \in \mathcal{G}_1$ and $\beta(s) = x$ for all $s \in \mathcal{G}_2$. So, α and β are both measurable with respect to $\mathfrak{G}$. Suppose that $x \succ_i y$ and $x \succ_j y$. Then $\alpha \succ_i \beta$ and $\alpha \succ_j \beta$, because α yields the better outcome x with probability $\frac{3}{4}$ (according to either individual's beliefs), whereas β only yields it with probability $\frac{1}{4}$. Thus, Restricted Pareto* forces $\alpha \succ_o \beta$.

However, combining the private information of any two of the three agents yields the event $\mathcal{G}_2$. And for all $s \in \mathcal{G}_2$, we have $\alpha(s) = y \prec x = \beta(s)$ for both individuals, as shown in Figure 1(f). So, Restricted Pareto* leads society to the wrong answer.[16] Mongin and Pivato refer to this phenomenon as "complementary ignorance."[17]

$\mathcal{R}$-Almost-Objective Pareto is much less vulnerable to complementary ignorance. To see this, suppose $\mathcal{R}$ is strongly consilient, and $\succcurlyeq$ has a GH representation V with utility function u and belief set $\mathcal{P} \subseteq \mathcal{R}$. Let $\mathcal{E} \subseteq \mathcal{S}$ be an event, which gets positive probability from all elements of $\mathcal{P}$, and let $\mathcal{P}'$ be obtained by Bayes updating every element of $\mathcal{P}$ by $\mathcal{E}$. Suppose $\succcurlyeq'$ is another preference, having a GH representation V' with the utility function u and belief set $\mathcal{P}'$; this could be the updated preferences of the $\succcurlyeq$-agent upon learning $\mathcal{E}$.[18] If $\boldsymbol{\alpha}$ is any $\mathcal{R}$-almost-objective act, then Proposition 3 says $\lim_{n\to\infty} V(\alpha^n) = \lim_{n\to\infty} V'(\alpha^n)$. Thus, $\succcurlyeq$ and $\succcurlyeq'$ have *exactly the same asymptotic preferences* over $\mathcal{R}$-almost-objective acts.

Now, suppose we have a collection $\{\succcurlyeq_j\}_{j\in\mathcal{J}}$ of GH preferences and a collection $\{\mathcal{E}_j\}_{j\in\mathcal{J}}$ of events. For all $j \in \mathcal{J}$, let $\succcurlyeq'_j$ be a GH preference obtained by Bayes updating $\succcurlyeq_j$ with $\mathcal{E}_j$, as in the previous paragraph. Since the asymptotic preferences of each agent are unchanged by these updates, it follows that $\mathcal{R}$-Almost-Objective Pareto will apply to $\{\succcurlyeq'_j\}_{j\in\mathcal{J}}$ in exactly the same situations as it applies to $\{\succcurlyeq_j\}_{j\in\mathcal{J}}$. In other words, unlike Restricted Pareto, it is impossible to induce "spurious" instances of Almost-Objective Pareto by exposing different agents to different information.

Sources of uncertainty The distinction between Gilboa, Samet, and Schmeidler (2004) and the present paper is analogous to the distinction between universal and existential quantifiers.[19] The Restricted Pareto axioms say that for any source of uncertainty, if all agents happen to share the same beliefs about that source (for whatever reason), then the ex ante Pareto axiom should apply to preferences over acts contingent on that

[16] For the same reason, the original Restricted Pareto axiom also yields the wrong answer here.

[17] Note that $\mathfrak{G}$ is *not* a consensus partition for the agents' prior beliefs, because $\rho_o(\mathcal{G}_2) = \frac{1}{6}$, $\rho_i(\mathcal{G}_2) = \frac{1}{8}$ and $\rho_j(\mathcal{G}_2) = \frac{1}{10}$. This shows how the "consensus" status of a partition depends on what information agents have received. But heterogeneity of priors is not required for complementary ignorance; it is easy to construct similar examples where all agents have the *same* prior beliefs.

[18] Note that we do not impose any other relationship between V and V'.

[19] We thank a referee for this apt comparison.

source. But to achieve utilitarian aggregation *à la* Harsanyi, we do not need to quantify over *every* source of such "common-belief uncertainty." It suffices to apply the ex ante Pareto axiom to *certain* sources of common-belief uncertainty.

In the models of Mongin and Pivato (2020) and Pivato (2022), these sources of common-belief uncertainty were either exogenous, or the asymptotic outcome of a learning process. In the first paper, there is an exogenous distinction between two sources of uncertainty: one "subjective" and one "objective." If social preferences satisfy ex ante Pareto only for the objective source, then the social utility function is utilitarian and all agents have SEU preferences with the same beliefs about the objective source, but there is no relationship between their beliefs regarding the subjective source. In the second paper, all agents have SEU preferences, and there is an infinite stream of information arriving over time, from which all agents update their beliefs, and hence their preferences over acts. If social preferences satisfy ex ante Pareto only for unanimous preferences, which persist in the long term under this learning process, then the social utility function must be utilitarian, but no relationship is required between the original beliefs of the agents, except for a weak condition called "concordance" (roughly, the supports of their beliefs must have a common overlap).

In the present paper, the source of common-belief uncertainty is the almost-objective uncertainty introduced in Section 3. Unlike Mongin and Pivato (2020), this source is not exogenous. Unlike Pivato (2022), it does not arise from a dynamical process, and does not require any compatibility between the beliefs of different agents (their beliefs could even have pairwise disjoint support). But like these two papers, and unlike Gilboa, Samet, and Schmeidler (2004), this focus on carefully selected sources of common-belief uncertainty not only allows us to cleanly separate utility-aggregation from belief-aggregation, but also precludes complementary ignorance.

Appendix A: Proofs from Section 3

The following result will play a crucial role in many of our proofs.

Lemma A.1. (Dubins–Spanier Theorem) *Let $\mathcal{S}$ be a measurable space. Let $\mu_1, \dots, \mu_N \in \Delta(\mathcal{S})$ be finitely additive, nonatomic measures. For any $K \in \mathbb{N}$ and probability vector $\mathbf{q} \in \Delta^K$, there exists a measurable partition $\mathfrak{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_K\}$ such that $\mu_n(\mathcal{G}_k) = q_k$ for all $k \in [1 \dots K]$ and all $n \in [1 \dots N]$.*

This is a straightforward corollary of Lyapunov's Convexity Theorem; see, for example, Theorem 13.34 of Aliprantis and Border (2006). Lyapunov's theorem was originally stated for *countably* additive measures, but was generalized to *finitely* additive measures by Armstrong and Prikry (1981). The proof of Dubins–Spanier in the finitely additive case is much the same, but for logical completeness we repeat it here.

Proof of Lemma A.1. Let $\mathfrak{B}$ be the sigma algebra on $\mathcal{S}$. Define $\boldsymbol{\mu} : \mathfrak{B} \longrightarrow \mathbb{R}^N$ by setting $\boldsymbol{\mu}(\mathcal{B}) := (\mu_1(\mathcal{B}), \dots, \mu_N(\mathcal{B}))$ for all $\mathcal{B} \in \mathfrak{B}$. Then $\boldsymbol{\mu}$ is a finitely additive, nonatomic, bounded, $\mathbb{R}^N$-valued measure. Thus, its range $\boldsymbol{\mu}(\mathfrak{B})$ is a convex subset of $\mathbb{R}^N$ (Theorem 2-2, Armstrong and Prikry (1981); for another proof, see Theorem 1 of Khan and Rath (2013)).[20]

Let $\mathbf{0}$ be the all-0 vector in $\mathbb{R}^N$, and let $\mathbf{1}$ be the all-1 vector. Then $\boldsymbol{\mu}(\emptyset) = \mathbf{0}$ and $\boldsymbol{\mu}(\mathcal{S}) = \mathbf{1}$, so the image of $\boldsymbol{\mu}$ contains all vectors on the line segment between $\mathbf{0}$ and $\mathbf{1}$. In particular, there is some measurable $\mathcal{G}_1 \in \mathfrak{B}$ with $\boldsymbol{\mu}[\mathcal{G}_1] = q_1\mathbf{1}$. Now, consider the restriction of $\boldsymbol{\mu}$ to the subspace $\mathcal{G}_1^{\complement}$. This is again a nonatomic, bounded, $\mathbb{R}^N$-valued measure, so its image is again convex, and contains all vectors on the line segment between $\mathbf{0}$ and $(1-q_1)\mathbf{1}$. So, there is some measurable $\mathcal{G}_2 \subseteq \mathcal{G}_1^{\complement}$ with $\boldsymbol{\mu}[\mathcal{G}_2] = q_2\mathbf{1}$.

Inductively, for all $k \in [3 \dots K-1]$, restrict $\boldsymbol{\mu}$ to $(\mathcal{G}_1 \sqcup \dots \sqcup \mathcal{G}_{k-1})^{\complement}$ and apply Lyapunov convexity to get a measurable subset $\mathcal{G}_k \subseteq (\mathcal{G}_1 \sqcup \dots \sqcup \mathcal{G}_{k-1})^{\complement}$ with $\boldsymbol{\mu}[\mathcal{G}_k] = q_k\mathbf{1}$. Finally, let $\mathcal{G}_K := (\mathcal{G}_1 \sqcup \dots \sqcup \mathcal{G}_{K-1})^{\complement}$. Then $\mathcal{G}_K$ is measurable, and

$$\boldsymbol{\mu}[\mathcal{G}_K] = \boldsymbol{\mu}(\mathcal{S}) - \boldsymbol{\mu}(\mathcal{G}_1) - \dots - \boldsymbol{\mu}(\mathcal{G}_{K-1}) = \mathbf{1} - q_1\mathbf{1} - \dots - q_{K-1}\mathbf{1} = q_K\mathbf{1}.$$

Thus, $\mathfrak{G} := \{\mathcal{G}_k\}_{k=1}^K$ is a measurable partition of $\mathcal{S}$, and for all $k \in [1 \dots K]$ and all $n \in [1 \dots N]$, we have $\mu_n(\mathcal{G}_k) = q_k$, as desired. □

Proof of Proposition 1. Let $\{\mu_n\}_{n=1}^\infty$ be a countable dense subset of $\mathcal{R}$. Let $\mathbf{q} \in \Delta^K$. For all $n \in \mathbb{N}$, Lemma A.1 yields a partition $\mathfrak{G}^n = \{\mathcal{G}_1^n, \dots, \mathcal{G}_K^n\}$ such that $\mu_m(\mathcal{G}_k^n) = q_k$ for all $k \in [1 \dots K]$ and all $m \in [1 \dots n]$ (because $\mu_1, \dots, \mu_n$ are all nonatomic). We claim that the sequence $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{R}$-almost-objectively uncertain and subordinate to $\mathbf{q}$.

To see this, let $\rho \in \mathcal{R}$ and let $\epsilon > 0$. Since $\{\mu_n\}_{n=1}^\infty$ is dense in the norm topology, there exists $N \in \mathbb{N}$ such that $\|\mu_N - \rho\| < \epsilon$. Now, let $k \in [1 \dots K]$. For any $n \geq N$, we have $\mu_N(\mathcal{G}_k^n) = q_k$, by the definition of $\mathfrak{G}^n$, while $|\rho(\mathcal{G}_k^n) - \mu_N(\mathcal{G}_k^n)| < \epsilon$ because $\|\mu_N - \rho\|_{\mathrm{vr}} < \epsilon$. Thus, $|\rho(\mathcal{G}_k^n) - q_k| < \epsilon$, for all $n \geq N$. This works for any $\epsilon > 0$; thus, $\lim_{n\to\infty} \rho(\mathcal{G}_k^n) = q_k$. This works for all $k \in [1 \dots K]$, and all $\rho \in \mathcal{R}$. □

Remark. Although we have assumed $\mathcal{S}$ is equipped with a sigma algebra, Proposition 1 can be extended to the case when $\mathcal{S}$ is only equipped with a *Boolean* algebra of sets, by using Lemma 1-1 of Armstrong and Prikry (1981) to obtain an "approximate" version of Lemma A.1. But we do not need this level of generality here.

Proposition 2(a) follows immediately from Proposition 1 and the next lemma.

Lemma A.2. *Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$, If $\mathcal{R}$ is separable in the norm topology, and every element of $\mathcal{R}$ is separable, then $\langle\mathcal{R}\rangle$ is separable in the norm topology.*

Proof. Let $\{\nu_n\}_{n=1}^\infty$ be a countable dense subset of $\mathcal{R}$. For all $n \in \mathbb{N}$, let $\Delta(\mathcal{S}, \nu_n) := \{\phi \in \mathcal{L}^1(\mathcal{S}, \nu_n); \phi \geq 0 \text{ and } \int_{\mathcal{S}} \phi \, d\nu_n = 1\}$; in other words, $\Delta(\mathcal{S}, \nu_n) = \{\frac{d\rho}{d\nu_n}; \rho \in \Delta(\mathcal{S}) \text{ and } \rho \lll \nu_n\}$. Recall that the normed vector space $\mathcal{L}^1(\mathcal{S}, \nu_n)$ is separable (because all elements

[20] Armstrong and Prikry (1981) formulate their theorem the case when $\mathfrak{B}$ is an *F-algebra*. But any sigma algebra is an F-algebra. Khan and Rath (2013) use the term "strongly continuous" to mean what we and Armstrong and Prikry (1981) mean by "nonatomic.".

of $\mathcal{R}$ are separable probability measures). Thus, the subset $\Delta(\mathcal{S}, \nu_n)$ is also separable in the L^1 norm. So, let $\{\psi_n^m\}_{m=1}^\infty$ be a countable dense subset of $\Delta(\mathcal{S}, \nu_n)$. For all $m \in \mathbb{N}$, let $\lambda_n^m \in \Delta(\mathcal{S})$ be the probability measure such that $\frac{\mathrm{d}\lambda_n^m}{\mathrm{d}\nu_n} = \psi_n^m$. Note that λ_n^m is nonatomic because ν_n is nonatomic.

We claim that the countable set $\{\lambda_n^m\}_{m,n=1}^\infty$ is dense in $\langle\mathcal{R}\rangle$ in the total variation norm. To see this, let $\mu \in \langle\mathcal{R}\rangle$. Then there exists $\rho \in \mathcal{R}$ such that $\mu \lll \rho$, and if $\phi := \frac{\mathrm{d}\mu}{\mathrm{d}\rho}$, then there exists $C > 0$ such that $0 \leq \phi(s) < C$ for all $s \in \mathcal{S}$. Let $\epsilon > 0$. Since $\{\nu_n\}_{n=1}^\infty$ is dense in $\mathcal{R}$, there exists $n \in \mathbb{N}$ such that $\|\nu_n - \rho\|_{\mathrm{vr}} < \epsilon/2C$. Automatically, $\phi \in \mathcal{L}^1(\mathcal{S}, \nu_n)$, because ϕ is bounded. Thus, there exists $m \in \mathbb{N}$ such that $\|\phi - \psi_n^m\|_{1,\nu_n} < \epsilon/2$, where this refers to the L^1 norm on $\mathcal{L}^1(\mathcal{S}, \nu_n)$. We will show that $\|\lambda_n^m - \mu\|_{\mathrm{vr}} < \epsilon$.

To see this, let $\mathcal{H}_1, \ldots, \mathcal{H}_J \subseteq \mathcal{S}$ be disjoint and measurable. For all $j \in [1 \ldots J]$,

$$\begin{aligned}
\left|\lambda_n^m(\mathcal{H}_j) - \mu(\mathcal{H}_j)\right| &\underset{(*)}{=} \left|\int_{\mathcal{H}_j} \psi_n^m \,\mathrm{d}\nu_n - \int_{\mathcal{H}_j} \phi \,\mathrm{d}\rho\right| \\
&\leq \left|\int_{\mathcal{H}_j} \psi_n^m \,\mathrm{d}\nu_n - \int_{\mathcal{H}_j} \phi \,\mathrm{d}\nu_n\right| + \left|\int_{\mathcal{H}_j} \phi \,\mathrm{d}\nu_n - \int_{\mathcal{H}_j} \phi \,\mathrm{d}\rho\right| \\
&= \left|\int_{\mathcal{H}_j} \left(\psi_n^m - \phi\right) \mathrm{d}\nu_n\right| + \left|\int_{\mathcal{H}_j} \phi \,\mathrm{d}(\nu_n - \rho)\right| \\
&\leq \int_{\mathcal{H}_j} \left|\psi_n^m - \phi\right| \mathrm{d}\nu_n + \int_{\mathcal{H}_j} |\phi| \,\mathrm{d}|\nu_n - \rho|,
\end{aligned} \tag{A1}$$

where $(*)$ is because $\psi_n^m = \frac{\mathrm{d}\lambda_n^m}{\mathrm{d}\nu_n}$ and $\phi = \frac{\mathrm{d}\mu}{\mathrm{d}\rho}$. Thus, if $\mathcal{H} := \bigsqcup_{j=1}^J \mathcal{H}_j$, then

$$\begin{aligned}
\sum_{j=1}^J \left|\lambda_n^m(\mathcal{H}_j) - \mu[\mathcal{H}_j]\right| &\underset{(*)}{\leq} \sum_{j=1}^J \int_{\mathcal{H}_j} \left|\psi_n^m - \phi\right| \mathrm{d}\nu_n + \sum_{j=1}^J \int_{\mathcal{H}_j} |\phi| \,\mathrm{d}|\nu_n - \rho| \\
&= \int_{\mathcal{H}} \left|\psi_n^m - \phi\right| \mathrm{d}\nu_n + \int_{\mathcal{H}} |\phi| \,\mathrm{d}|\nu_n - \rho| \\
&\leq \int_{\mathcal{S}} \left|\psi_n^m - \phi\right| \mathrm{d}\nu_n + \int_{\mathcal{S}} |\phi| \,\mathrm{d}|\nu_n - \rho| \\
&\leq \left\|\psi_n^m - \phi\right\|_{1,\nu_n} + C \cdot \|\nu_n - \rho\|_{\mathrm{vr}} < \frac{\epsilon}{2} + C \cdot \frac{\epsilon}{2C} = \epsilon,
\end{aligned}$$

where $(*)$ is by inequality (A1). This works for any disjoint collection $\mathcal{H}_1, \ldots, \mathcal{H}_J \subseteq \mathcal{S}$, so from definition (4) we conclude that $\|\lambda_n^m - \mu\|_{\mathrm{vr}} \leq \epsilon$. This argument works for any $\epsilon > 0$, and any $\mu \in \langle\mathcal{R}\rangle$. Thus, $\{\lambda_n^m\}_{m,n=1}^\infty$ is dense in $\langle\mathcal{R}\rangle$. □

The proof of Proposition 2(b) is somewhat more involved, and requires an auxiliary concept and four preliminary lemmas. Recall that in Proposition 2(b), $\mathcal{S}$ was assumed to be a standard Borel space, that is, it is measurably isomorphic to a complete separable metric space endowed with its Borel sigma algebra. Therefore, without loss of generality, we will sometimes assume in the following material that $\mathcal{S}$ is endowed with a metric d that makes it a complete separable metric space, and the sigma algebra on $\mathcal{S}$ is the resulting Borel sigma algebra.

For any $\mathcal{Y} \subseteq \mathcal{S}$, the *diameter* of $\mathcal{Y}$ is defined: $\operatorname{diam}(\mathcal{Y}) := \sup_{s,t\in\mathcal{Y}} d(s,t)$. For any $\epsilon > 0$, an ϵ*-partition* is a collection $\mathfrak{Y} = \{\mathcal{Y}_n\}_{n=1}^N$ of disjoint measurable subsets of $\mathcal{S}$ (for some $N \in \mathbb{N} \cup \{\infty\}$) such that $\bigsqcup_{n=1}^N \mathcal{Y}_n = \mathcal{S}$, and $\operatorname{diam}(\mathcal{Y}_n) \leq \epsilon$ for all $n \in [1\dots N]$.[21]

Lemma A.3. *Let $(\mathcal{S}, d)$ be any metric space. Then $(\mathcal{S}, d)$ is separable if and only if it admits an ϵ-partition for all $\epsilon > 0$.*

Proof. "$\Longrightarrow$" Let $\{s_n\}_{n=1}^\infty$ be a countable dense subset of $\mathcal{S}$. Let $\epsilon > 0$. For all $s \in \mathcal{S}$, let $\mathcal{B}(s, \epsilon)$ be the open ball of radius $\frac{\epsilon}{2}$ around s. For all $N \in \mathbb{N}$, let $\mathcal{Y}_N := \mathcal{B}(s_N, \epsilon) \setminus \bigcup_{n=1}^{N-1} \mathcal{B}(s_n, \epsilon)$; then $\operatorname{diam}(\mathcal{Y}_N) \leq \epsilon$. Thus, $\{\mathcal{Y}_n\}_{n=1}^\infty$ is an ϵ-partition of $\mathcal{S}$.

"$\Longleftarrow$" For all $m \in \mathbb{N}$, let $\mathfrak{Y}^m = \{\mathcal{Y}_n^m\}_{n=1}^\infty$ be a $(\frac{1}{m})$-partition. For all $(n, m) \in \mathbb{N}^2$, let $s_{n,m} \in \mathcal{Y}_n^m$. Then $\{s_{n,m}\}_{n,m=1}^\infty$ is a countable dense subset of $\mathcal{S}$. □

Let $\mathcal{P}$ be a collection of Borel probability measures on $\mathcal{S}$, let $K \in \mathbb{N}$, and let $\mathbf{q} = (q_1, \dots, q_K) \in \Delta^K$. A $\mathbf{q}$*-Poincaré sequence* for $\mathcal{P}$ is a sequence $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^\infty$, where for all $n \in \mathbb{N}$, $\mathfrak{G}^n = \{\mathcal{G}_1^n, \dots, \mathcal{G}_K^n\}$ is a K-cell measurable partition of $\mathcal{S}$, $\epsilon_n > 0$ and $\mathfrak{Y}^n$ is an ϵ_n-partition, such that $\lim_{n\to\infty} \epsilon_n = 0$, and such that for all $\rho \in \mathcal{P}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, all $k \in [1\dots K]$, and all $\mathcal{Y} \in \mathfrak{Y}^n$, $\rho[\mathcal{G}_k^n \cap \mathcal{Y}] = q_k \rho[\mathcal{Y}]$ (and thus, $\rho[\mathcal{G}_k^n] = q_k$).

Example. Let $\mathcal{S} := [0, 1)$. Let $\mathcal{P} := \{\lambda\}$ where λ is the Lebesgue measure. Let $\mathbf{q} = (\frac{1}{2}, \frac{1}{2})$. For all $n \in \mathbb{N}$, let $\epsilon := 1/2^n$ and let $\mathfrak{Y}^n := \{\mathcal{Y}_1^n, \dots, \mathcal{Y}_{2^n}^n\}$ where $\mathcal{Y}_k^n := [\frac{k-1}{2^n}, \frac{k}{2^n})$ for all $k \in [1\dots 2^n]$. Finally, let $\mathfrak{G}^n := \{\mathcal{G}_1^n, \mathcal{G}_2^n\}$, where

$$\mathcal{G}_1^n := \bigcup_{\substack{k=1\\ k \text{ odd}}}^{2^{n+1}-1} \mathcal{Y}_k^{n+1} \quad \text{and} \quad \mathcal{G}_2^n := \bigcup_{\substack{k=2\\ k \text{ even}}}^{2^{n+1}} \mathcal{Y}_k^{n+1}.$$

Then $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^\infty$ is a $(\frac{1}{2}, \frac{1}{2})$-Poincaré sequence for $\{\lambda\}$. ♢

Lemma A.4. *Let $(\mathcal{S}, d)$ be any separable metric space. Let $\mathcal{H} \subseteq \operatorname{ba}(\mathcal{S})$ be a countable collection of nonatomic signed measures on $\mathcal{S}$. Let $\mathcal{F}$ be the linear subspace of $\operatorname{ba}(\mathcal{S})$ consisting of all finite linear combinations of elements of $\mathcal{H}$. Let $\mathcal{P} \subseteq \mathcal{F}$ be the set of all probability measures in $\mathcal{F}$. Then for all $K \in \mathbb{N}$ and all $\mathbf{q} \in \Delta^K$, $\mathcal{P}$ has a $\mathbf{q}$-Poincaré sequence.*

Proof. Suppose $\mathcal{H} = \{\eta_n\}_{n=1}^\infty$. For all $n \in \mathbb{N}$, the Jordan decomposition theorem says $\eta_n = \eta_n^+ - \eta_n^-$, where $\eta_n^+, \eta_n^- \in \operatorname{ba}(\mathcal{S})$ are either zero or positive measures (Bhaskara Rao and Bhaskara Rao (1983), Theorem 2.5.3). They are nonatomic because η_n is nonatomic. By replacing $\{\eta_n\}_{n=1}^\infty$ with $\{\eta_n^\pm\}_{n=1}^\infty$ if necessary, we can assume without loss of generality that all elements of $\mathcal{H}$ are positive, nonatomic measures.

Let $\{\epsilon_n\}_{n=1}^\infty$ be a positive sequence with $\lim_{n\to\infty} \epsilon_n = 0$. For all $N \in \mathbb{N}$, Lemma A.3 says $\mathcal{S}$ has an ϵ_N-partition $\mathfrak{Y}^N$.

[21] Note that we allow these partitions to have a countably infinite number of elements. This is necessary because $\mathcal{S}$ is not necessarily compact.

CLAIM 1. *For all $N \in \mathbb{N}$, and all $\mathcal{Y} \in \mathfrak{Y}^N$, there is a measurable partition $\{\mathcal{G}_1^{\mathcal{Y}}, \ldots, \mathcal{G}_K^{\mathcal{Y}}\}$ of $\mathcal{Y}$ such that $n \in [1 \ldots N]$, we have*

$$\eta_n\big(\mathcal{G}_k^{\mathcal{Y}}\big) = q_k \cdot \eta_n(\mathcal{Y}), \quad \textit{for all } k \in [1 \ldots K]. \tag{A2}$$

PROOF. Let $n \in [1 \ldots N]$. If $\eta_n(\mathcal{Y}) = 0$, then the equations (A2) are trivially satisfied for *any* partition $\{\mathcal{G}_{\mathcal{Y}}^1, \ldots, \mathcal{G}_{\mathcal{Y}}^K\}$. So, let $\mathcal{N} := \{n \in [1 \ldots N];\ \eta_n(\mathcal{Y}) > 0\}$; it suffices to construct a partition satisfying the equations (A2) for all $n \in \mathcal{N}$. For all $n \in \mathcal{N}$, let $\widetilde{\eta}_n$ be the nonatomic probability measure on $\mathcal{Y}$ defined by setting $\widetilde{\eta}_n(\mathcal{U}) := \eta_n(\mathcal{U})/\eta_n(\mathcal{Y})$ for all measurable $\mathcal{U} \subseteq \mathcal{Y}$. Thus, $\{\widetilde{\eta}_n\}_{n \in \mathcal{N}}$ is a finite collection of nonatomic probability measures, so Lemma A.1 yields a partition $\{\mathcal{G}_1^{\mathcal{Y}}, \ldots, \mathcal{G}_K^{\mathcal{Y}}\}$ of $\mathcal{Y}$ such that

$$\widetilde{\eta}_n\big(\mathcal{G}_k^{\mathcal{Y}}\big) = q_k \quad \text{for all } k \in [1 \ldots K] \text{ and } n \in \mathcal{N}. \tag{A3}$$

For all $n \in \mathcal{N}$, multiply both sides of (A3) by $\eta_n(\mathcal{Y})$ to obtain (A2). $\diamondsuit$[Claim 1]

Fix $N \in \mathbb{N}$, and apply Claim 1 to all $\mathcal{Y} \in \mathfrak{Y}^N$. Observe that the sets in the family $\{\mathcal{G}_k^{\mathcal{Y}};\ \mathcal{Y} \in \mathfrak{Y}^N \text{ and } k \in [1 \ldots K]\}$ are all disjoint. For all $k \in [1 \ldots K]$, define

$$\mathcal{G}_k^N \quad := \quad \bigsqcup_{\mathcal{Y} \in \mathfrak{Y}^N} \mathcal{G}_k^{\mathcal{Y}}. \tag{A4}$$

Then $\{\mathcal{G}_1^N, \ldots, \mathcal{G}_K^N\}$ is a measurable partition of $\mathcal{S}$: these sets are are disjoint, and

$$\bigsqcup_{k=1}^{K} \mathcal{G}_k^N = \bigsqcup_{k=1}^{K}\Big(\bigsqcup_{\mathcal{Y} \in \mathfrak{Y}^N} \mathcal{G}_k^{\mathcal{Y}}\Big) = \bigsqcup_{\mathcal{Y} \in \mathfrak{Y}^N}\left(\bigsqcup_{k=1}^{K} \mathcal{G}_k^{\mathcal{Y}}\right) = \bigsqcup_{\mathcal{Y} \in \mathfrak{Y}^N} \mathcal{Y} = \mathcal{S}.$$

Furthermore, for all $\mathcal{Y} \in \mathfrak{Y}^N$, we have $\mathcal{G}_k^N \cap \mathcal{Y} = \mathcal{G}_k^{\mathcal{Y}}$ for all $k \in [1 \ldots K]$; thus, for all $n \in [1 \ldots N]$,

$$\eta_n\big(\mathcal{G}_k^N \cap \mathcal{Y}\big) = \eta_n\big(\mathcal{G}_k^{\mathcal{Y}}\big) \underset{(*)}{=} q_k \eta_n(\mathcal{Y}), \tag{A5}$$

where $(*)$ is by equation (A2).

Now, let $\rho \in \mathcal{P}$. Then there exists some $N \in \mathbb{N}$ such that ρ is a linear combination of $\eta_1, \ldots, \eta_N$. Thus, for any $n \geq N$, ρ is also a linear combination of $\eta_1, \ldots, \eta_n$ (with zero coefficients for $\eta_{N+1}, \ldots, \eta_n$). Thus, for all $\mathcal{Y} \in \mathfrak{Y}^n$ and all $k \in [1 \ldots K]$, equation (A5) yields $\rho[\mathcal{G}_k^n \cap \mathcal{Y}] = q_k \rho[\mathcal{Y}]$, as desired. $\square$

LEMMA A.5. *Suppose $(\mathcal{S}, d)$ is a complete, separable metric space. Let $K \in \mathbb{N}$, let $\mathbf{q} \in \Delta^K$, let $\mathcal{P} \subseteq \Delta_\sigma(\mathcal{S})$ be a collection of countably additive Borel probability measures on $\mathcal{S}$, and let $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^{\infty}$ be a $\mathbf{q}$-Poincaré sequence for $\mathcal{P}$. Let $\mathcal{L} = \langle \mathcal{P} \rangle$. Then $(\mathfrak{G}^n)_{n=1}^{\infty}$ is $\mathcal{L}$-almost-objectively uncertain and subordinate to $\mathbf{q}$.*

PROOF. Let $\lambda \in \mathcal{L}$ and let $k \in [1 \ldots K]$. We will show that

$$\lim_{n \to \infty} \lambda\big(\mathcal{G}_k^n\big) = q_k. \tag{A6}$$

There exists $\rho \in \mathcal{P}$ such that $\lambda \lll \rho$. Let $\phi := \frac{d\lambda}{d\rho}$ and $C := \sup_{s \in \mathcal{S}} \phi(s)$. Then $C < \infty$. Fix $\epsilon > 0$. By hypothesis, $\mathcal{S}$ is a Polish space, so Lusin's theorem yields a compact subset $\mathcal{K} \subseteq \mathcal{S}$ such that $\phi_{\upharpoonright \mathcal{K}}$ is uniformly continuous on $\mathcal{K}$ and

$$\rho(\mathcal{K}^{\complement}) < \frac{\epsilon}{8C}. \tag{A7}$$

(Aliprantis and Border (2006), Theorem 12.8, p. 438).[22] It follows that

$$\lambda[\mathcal{K}^{\complement}] = \int_{\mathcal{K}^{\complement}} \phi \, d\rho \underset{(*)}{\leq} C \cdot \rho[\mathcal{K}^{\complement}] \underset{(\dagger)}{\leq} C \cdot \frac{\epsilon}{8C} = \frac{\epsilon}{8}, \tag{A8}$$

where $(*)$ is because $0 \leq \phi(s) \leq C$ for all $s \in \mathcal{S}$, and $(\dagger)$ is by inequality (A7). Since $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^{\infty}$ is a Poincaré sequence for $\mathcal{P}$, there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ and all $\mathcal{Y} \in \mathfrak{Y}^n$,

$$\rho[\mathcal{G}_k^n \cap \mathcal{Y}] = q_k \rho[\mathcal{Y}]. \tag{A9}$$

Claim 1. *For all $n \geq N_1$, $\sum_{\mathcal{Y} \in \mathfrak{Y}^n} |\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - q_k \rho[\mathcal{Y} \cap \mathcal{K}]| \leq \frac{\epsilon}{4C}$.*

Proof. Let $n \geq N$. For all $\mathcal{Y} \in \mathfrak{Y}^n$,

$$\begin{aligned}
&\big|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - q_k \rho[\mathcal{Y} \cap \mathcal{K}]\big| \\
&\quad = \big|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - \rho[\mathcal{G}_k^n \cap \mathcal{Y}] + \rho[\mathcal{G}_k^n \cap \mathcal{Y}] - q_k \rho[\mathcal{Y} \cap \mathcal{K}]\big| \\
&\quad \underset{(*)}{=} \big|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - \rho[\mathcal{G}_k^n \cap \mathcal{Y}] + q_k \rho[\mathcal{Y}] - q_k \rho[\mathcal{Y} \cap \mathcal{K}]\big| \\
&\quad = \big|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - \rho[\mathcal{G}_k^n \cap \mathcal{Y}] + q_k \big(\rho[\mathcal{Y}] - \rho[\mathcal{Y} \cap \mathcal{K}]\big)\big| \\
&\quad \leq \big|\rho[\mathcal{G}_k^n \cap \mathcal{Y}] - \rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}]\big| + q_k \big|\rho[\mathcal{Y}] - \rho[\mathcal{Y} \cap \mathcal{K}]\big| \\
&\quad = \rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}^{\complement}] + q_k \rho[\mathcal{Y} \cap \mathcal{K}^{\complement}].
\end{aligned} \tag{A10}$$

Here, $(*)$ is by equation (A9). Thus,

$$\begin{aligned}
\sum_{\mathcal{Y} \in \mathfrak{Y}^n} |\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - q_k \rho[\mathcal{Y} \cap \mathcal{K}]| &\underset{(\dagger)}{\leq} \sum_{\mathcal{Y} \in \mathfrak{Y}^n} \big(\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}^{\complement}] + q_k \rho[\mathcal{Y} \cap \mathcal{K}^{\complement}]\big) \\
&= \rho\Big[\bigsqcup_{\mathcal{Y} \in \mathfrak{Y}^n} (\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}^{\complement})\Big] + q_k \rho\Big[\bigsqcup_{\mathcal{Y} \in \mathfrak{Y}^n} (\mathcal{Y} \cap \mathcal{K}^{\complement})\Big] \\
&= \rho\Big[\mathcal{G}_k^n \cap \mathcal{K}^{\complement} \cap \bigsqcup_{\mathcal{Y} \in \mathfrak{Y}^n} \mathcal{Y}\Big] + q_k \rho\Big[\mathcal{K}^{\complement} \cap \bigsqcup_{\mathcal{Y} \in \mathfrak{Y}^n} \mathcal{Y}\Big] \\
&\underset{(*)}{=} \rho[\mathcal{G}_k^n \cap \mathcal{K}^{\complement}] + q_k \rho[\mathcal{K}^{\complement}] \underset{(\diamond)}{\leq} \frac{\epsilon}{8C} + \frac{\epsilon}{8C} = \frac{\epsilon}{4C},
\end{aligned}$$

as claimed. Here, $(\dagger)$ is by applying inequality (A10) to each $\mathcal{Y} \in \mathfrak{Y}^n$, $(*)$ is because $\bigsqcup_{\mathcal{Y} \in \mathfrak{Y}^n} \mathcal{Y} = \mathcal{S}$, and $(\diamond)$ is by inequality (A7). ◇[Claim 1]

[22] This is the one place in the proof of Proposition 2(b) that requires countably additive measures.

Recall that $\phi_{\upharpoonright\mathcal{K}}$ is uniformly continuous on $\mathcal{K}$. Thus, there exists some $\delta > 0$ such that, for all $s_1, s_2 \in \mathcal{K}$, if $d(s_1, s_2) \le \delta$, then $|\phi(s_1) - \phi(s_2)| < \frac{\epsilon}{4}$. Find $N_2 \in \mathbb{N}$ such that $\epsilon_n \le \delta$ for all $n \ge N_2$. Thus, if $n \ge N_2$ and $\mathcal{Y} \in \mathfrak{Y}^n$, then $\operatorname{diam}(\mathcal{Y}) \le \epsilon_n \le \delta$, so that for all $y_1, y_2 \in \mathcal{Y} \cap \mathcal{K}$ we have $|\phi(y_1) - \phi(y_2)| < \frac{\epsilon}{4}$. Thus, there is some $c_{\mathcal{Y}} \in \mathbb{R}_+$ such that $|\phi(y) - c_{\mathcal{Y}}| < \frac{\epsilon}{4}$ for all $y \in \mathcal{Y} \cap \mathcal{K}$. Thus, for all $n \ge N_2$,

$$\begin{aligned}
\big|\lambda\big[\mathcal{Y} \cap \mathcal{K} \cap \mathcal{G}_k^n\big] - c_{\mathcal{Y}} \cdot \rho\big[\mathcal{Y} \cap \mathcal{K} \cap \mathcal{G}_k^n\big]\big| &\underset{(*)}{=} \left|\int_{\mathcal{Y}\cap\mathcal{K}\cap\mathcal{G}_k^n} (\phi - c_{\mathcal{Y}})\,\mathrm{d}\rho\right| \\
&\le \int_{\mathcal{Y}\cap\mathcal{K}\cap\mathcal{G}_k^n} |\phi - c_{\mathcal{Y}}|\,\mathrm{d}\rho \le \int_{\mathcal{Y}\cap\mathcal{K}\cap\mathcal{G}_k^n} \frac{\epsilon}{4}\,\mathrm{d}\rho \\
&= \frac{\epsilon}{4} \cdot \rho\big[\mathcal{Y} \cap \mathcal{K} \cap \mathcal{G}_k^n\big], \qquad \text{(A11)}
\end{aligned}$$

where $(*)$ is because $\phi = \frac{\mathrm{d}\lambda}{\mathrm{d}\rho}$. By a very similar argument,

$$\big|\lambda[\mathcal{Y} \cap \mathcal{K}] - c_{\mathcal{Y}}\rho[\mathcal{Y} \cap \mathcal{K}]\big| \le \frac{\epsilon}{4} \cdot \rho[\mathcal{Y} \cap \mathcal{K}], \quad \text{for all } n \ge N_2. \qquad \text{(A12)}$$

Now, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\lambda\big[\mathcal{G}_k^n \cap \mathcal{K}\big] - q_k\lambda[\mathcal{K}] &\underset{(*)}{=} \sum_{\mathcal{Y}\in\mathfrak{Y}} \lambda\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big] - q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} \lambda[\mathcal{K} \cap \mathcal{Y}] \\
&= \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\rho\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big] - \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\rho\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big] + \sum_{\mathcal{Y}\in\mathfrak{Y}} \lambda\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big] \\
&\quad - q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} \lambda[\mathcal{K} \cap \mathcal{Y}] + q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\rho[\mathcal{K} \cap \mathcal{Y}] - q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\rho[\mathcal{K} \cap \mathcal{Y}] \\
&= \sum_{\mathcal{Y}\in\mathfrak{Y}} \big(c_{\mathcal{Y}}\rho\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big] - q_k c_{\mathcal{Y}}\rho[\mathcal{K} \cap \mathcal{Y}]\big) \\
&\quad + \sum_{\mathcal{Y}\in\mathfrak{Y}} \big(\lambda\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big] - c_{\mathcal{Y}}\rho\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big]\big) \\
&\quad - q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} \big(\lambda[\mathcal{K} \cap \mathcal{Y}] - c_{\mathcal{Y}}\rho[\mathcal{K} \cap \mathcal{Y}]\big), \qquad \text{(A13)}
\end{aligned}$$

where $(*)$ is because $\bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^n} \mathcal{Y} = \mathcal{S}$. Now, let $N_\epsilon := \max\{N_1, N_2\}$. Then for all $n \ge N_\epsilon$,

$$\begin{aligned}
&\big|\lambda\big[\mathcal{G}_k^n \cap \mathcal{K}\big] - q_k\lambda[\mathcal{K}]\big| \\
&\quad \underset{(\diamond)}{\le} \left|\sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\big(\rho\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big] - q_k\rho[\mathcal{K} \cap \mathcal{Y}]\big)\right| \\
&\qquad + \left|\sum_{\mathcal{Y}\in\mathfrak{Y}} \big(\lambda\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big] - c_{\mathcal{Y}}\rho\big[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}\big]\big)\right| \\
&\qquad + q_k\left|\sum_{\mathcal{Y}\in\mathfrak{Y}} \big(\lambda[\mathcal{K} \cap \mathcal{Y}] - c_{\mathcal{Y}}\rho[\mathcal{K} \cap \mathcal{Y}]\big)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\left|\rho\left[\mathcal{G}_k^n\cap\mathcal{K}\cap\mathcal{Y}\right]-q_k\rho[\mathcal{K}\cap\mathcal{Y}]\right|+\sum_{\mathcal{Y}\in\mathfrak{Y}}\left|\lambda\left[\mathcal{G}_k^n\cap\mathcal{K}\cap\mathcal{Y}\right]-c_{\mathcal{Y}}\rho\left[\mathcal{G}_k^n\cap\mathcal{K}\cap\mathcal{Y}\right]\right| \\
&\quad + q_k\sum_{\mathcal{Y}\in\mathfrak{Y}}\left|\lambda[\mathcal{K}\cap\mathcal{Y}]-c_{\mathcal{Y}}\rho[\mathcal{K}\cap\mathcal{Y}]\right| \\
&\underset{(*)}{\leq} C\sum_{\mathcal{Y}\in\mathfrak{Y}}\left|\rho\left[\mathcal{G}_k^n\cap\mathcal{K}\cap\mathcal{Y}\right]-q_k\rho[\mathcal{K}\cap\mathcal{Y}]\right|+\sum_{\mathcal{Y}\in\mathfrak{Y}}\frac{\epsilon}{4}\rho\left[\mathcal{G}_k^n\cap\mathcal{K}\cap\mathcal{Y}\right]+q_k\sum_{\mathcal{Y}\in\mathfrak{Y}}\frac{\epsilon}{4}\rho[\mathcal{K}\cap\mathcal{Y}] \\
&\underset{(\dagger)}{\leq} C\frac{\epsilon}{4C}+\frac{\epsilon}{4}\sum_{\mathcal{Y}\in\mathfrak{Y}}\rho\left[\mathcal{G}_k^n\cap\mathcal{K}\cap\mathcal{Y}\right]+\frac{\epsilon}{4}\sum_{\mathcal{Y}\in\mathfrak{Y}}\rho[\mathcal{K}\cap\mathcal{Y}] \\
&\leq \frac{\epsilon}{4}+\frac{\epsilon}{4}\rho\left[\mathcal{G}_k^n\cap\mathcal{K}\right]+\frac{\epsilon}{4}\rho[\mathcal{K}]\leq\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}=\frac{3\epsilon}{4}. \qquad \text{(A14)}
\end{aligned}$$

Here, $(\diamond)$ is by equation (A13), while $(*)$ is by inequalities (A11) and (A12). Finally, $(\dagger)$ is by Claim 1, and also uses the fact that $q_k \leq 1$. Thus, for all $n \geq N_\epsilon$,

$$\begin{aligned}
\left|\lambda\left[\mathcal{G}_k^n\right]-q_k\right| &= \left|\lambda\left[\mathcal{G}_k^n\cap\mathcal{K}^{\complement}\right]+\lambda\left[\mathcal{G}_k^n\cap\mathcal{K}\right]-q_k\left(\lambda[\mathcal{K}]+\lambda\left[\mathcal{K}^{\complement}\right]\right)\right| \\
&\leq \left|\lambda\left[\mathcal{G}_k^n\cap\mathcal{K}^{\complement}\right]\right|+\left|\lambda\left[\mathcal{G}_k^n\cap\mathcal{K}\right]-q_k\lambda[\mathcal{K}]\right|+\left|\lambda\left[\mathcal{K}^{\complement}\right]\right| \\
&\underset{(*)}{\leq} \frac{\epsilon}{8}+\left|\lambda\left[\mathcal{G}_k^n\cap\mathcal{K}\right]-q_k\lambda[\mathcal{K}]\right|+\frac{\epsilon}{8} \\
&\underset{(\dagger)}{\leq} \frac{\epsilon}{8}+\frac{3\epsilon}{4}+\frac{\epsilon}{8}=\epsilon
\end{aligned}$$

where $(*)$ is by two applications of inequality (A8), while $(\dagger)$ is by inequality (A14).

We can construct such an N_ϵ for any $\epsilon > 0$. This proves the limit (A6). □

Lemma A.6. *Let $\mathcal{S}$ be any measurable space, and let $\mathcal{L} \subseteq \Delta(\mathcal{S})$ be a set of probability measures on $\mathcal{S}$. Let $\mathcal{R}$ be the convex closure of $\mathcal{L}$ in the total variation norm. Let $\mathbf{q} \in \Delta^K$. If a partition sequence $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{L}$-almost-objectively uncertain and subordinate to $\mathbf{q}$, then $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{R}$-almost-objectively uncertain and subordinate to $\mathbf{q}$.*

Proof. Let $\mathcal{R}_0$ be the convex hull of $\mathcal{L}$. If $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{L}$-almost-objectively uncertain and subordinate to $\mathbf{q}$, then it is easily shown that $(\mathfrak{G}^n)_{n=1}^\infty$ is also $\mathcal{R}_0$-almost-objectively uncertain subordinate to $\mathbf{q}$.

For all $n \in \mathbb{N}$, suppose $\mathfrak{G}^n = \{\mathcal{G}_1^n, \dots, \mathcal{G}_K^n\}$. Let $\rho \in \mathcal{R}$. Then there is a sequence $\{\rho_m\}_{m=1}^\infty$ in $\mathcal{R}_0$ such that $\lim_{m\to\infty} \|\rho_m - \rho\|_{\mathrm{vr}} = 0$. For all $k \in [1 \dots K]$, we must show that the limit (5) holds for ρ.

Let $\epsilon > 0$. There exists $m \in \mathbb{N}$, with $\|\rho_m - \rho\|_{\mathrm{vr}} < \frac{\epsilon}{2}$. This means that $|\rho_m(\mathcal{G}) - \rho(\mathcal{G})| < \epsilon/2$ for all measurable $\mathcal{G} \subseteq \mathcal{S}$. In particular,

$$\left|\rho\left(\mathcal{G}_k^n\right)-\rho_m\left(\mathcal{G}_k^n\right)\right| < \frac{\epsilon}{2}, \quad \text{for all } n \in \mathbb{N},\ \text{all } k \in [1 \dots K]. \qquad \text{(A15)}$$

The limit (5) holds for ρ_m, so there exists some $N_\epsilon \in \mathbb{N}$ such that

$$\left|\rho_m\left(\mathcal{G}_k^n\right)-q_k\right| < \frac{\epsilon}{2} \quad \text{for all } k \in [1 \dots K] \text{ and all } n \geq N_\epsilon. \qquad \text{(A16)}$$

Combining inequalities (A15) and (A16) yield $|\rho(\mathcal{G}_k^n) - q_k| < \epsilon$ for all $n \geq N_\epsilon$. We can obtain such an N_ϵ for any $\epsilon > 0$. Therefore, the limit (5) holds for ρ. □

Proof of Proposition 2(b). Suppose $\mathcal{S}$ is a standard Borel space. We can assume without loss of generality that there is a metric d making $(\mathcal{S}, d)$ a complete separable metric space, and the sigma algebra on $\mathcal{S}$ is the Borel sigma algebra. Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$ be separable and nonatomic; we must show that $\langle \mathcal{R} \rangle$ is consilient.

Let $\mathcal{N}$ be the closed subspace of $\mathrm{ba}(\mathcal{S})$ spanned by $\mathcal{R}$. Then $\mathcal{N}$ is separable because $\mathcal{R}$ is separable. Thus, $\mathcal{N}$ it is spanned by a countable subset $\mathcal{H}$.[23] Since $\mathcal{R}$ (and hence $\mathcal{N}$) is nonatomic, all elements of $\mathcal{H}$ are nonatomic. Let $\mathcal{F}$ be the linear subspace of $\mathrm{ba}(\mathcal{S})$ consisting of all finite linear combinations of elements from $\mathcal{H}$. Then $\mathcal{N}$ is the norm closure of $\mathcal{F}$. Let $\mathcal{P} := \mathcal{F} \cap \Delta(\mathcal{S})$, and then let $\mathcal{L} := \langle \mathcal{P} \rangle$.

Claim 1. *$\langle \mathcal{R} \rangle$ is contained in the norm closure of $\mathcal{L}$.*

Proof. Let $\mu \in \langle \mathcal{R} \rangle$. Find $\rho \in \mathcal{R}$ such that $\mu \lll \rho$ and $\phi := \frac{d\mu}{d\rho}$ is bounded. Since $\mathcal{R} \subset \mathcal{N}$, and $\mathcal{N}$ is the norm closure of $\mathcal{F}$, there exists a sequence $(\nu_n)_{n=1}^\infty$ in $\mathcal{F}$ converging to ρ in norm. For all $n \in \mathbb{N}$, let $\widetilde{\lambda}_n \in \mathrm{ba}(\mathcal{S})$ be the measure such that $\widetilde{\lambda}_n \lll \nu_n$ and $\frac{d\widetilde{\lambda}_n}{d\nu_n} = \phi$. Next, let $\lambda_n := \widetilde{\lambda}_n / \ell_n$, where $\ell_n := \widetilde{\lambda}_n(\mathcal{S})$. Then $\lambda_n \in \mathcal{L}$. (*Proof*: By construction, λ_n is a probability measure, and $\lambda_n \lll \nu_n$. Let $\pi_n := \nu_n / \nu_n(\mathcal{S})$; then $\pi_n \in \mathcal{P}$, $\lambda_n \lll \pi_n$, and $\frac{d\lambda_n}{d\pi_n}$ is a multiple of ϕ, hence bounded.) To prove the claim, it suffices to show that the sequence $\{\lambda_n\}_{n=1}^\infty$ converges to μ in norm. For any $n \in \mathbb{N}$,

$$\|\mu - \lambda_n\|_{\mathrm{vr}} \leq \|\mu - \widetilde{\lambda}_n\|_{\mathrm{vr}} + \|\widetilde{\lambda}_n - \lambda_n\|_{\mathrm{vr}}. \tag{A17}$$

Now, for any measurable $\mathcal{U} \subseteq \mathcal{S}$,

$$\begin{aligned} \left|\mu(\mathcal{U}) - \widetilde{\lambda}_n(\mathcal{U})\right| &\underset{(*)}{=} \left|\int_{\mathcal{U}} \phi \, d\rho - \int_{\mathcal{U}} \phi \, d\nu_n\right| = \left|\int_{\mathcal{U}} \phi \, d(\rho - \nu_n)\right| \\ &\leq \|\phi\|_\infty \cdot \left|\rho(\mathcal{U}) - \nu_n(\mathcal{U})\right|, \end{aligned}$$

where $(*)$ is because $\frac{d\mu}{d\rho} = \phi = \frac{d\widetilde{\lambda}_n}{d\nu_n}$. Combining this inequality with defining formula (4), we deduce that $\|\mu - \widetilde{\lambda}_n\|_{\mathrm{vr}} \leq \|\phi\|_\infty \cdot \|\rho - \nu_n\|_{\mathrm{vr}} \xrightarrow[n\to\infty]{(\dagger)} 0$, where $(\dagger)$ is because ν_n converges to ρ in norm by hypothesis. Thus,

$$\lim_{n\to\infty} \|\mu - \widetilde{\lambda}_n\|_{\mathrm{vr}} = 0. \tag{A18}$$

Meanwhile,

$$\begin{aligned} \|\widetilde{\lambda}_n - \lambda_n\|_{\mathrm{vr}} &= \|\ell_n \lambda_n - \lambda_n\|_{\mathrm{vr}} = |1 - \ell_n| \cdot \|\lambda_n\|_{\mathrm{vr}} = |1 - \ell_n| \\ &= \left|\mu(\mathcal{S}) - \widetilde{\lambda}_n(\mathcal{S})\right| \underset{(*)}{=} \left|\int_{\mathcal{S}} \phi \, d\rho - \int_{\mathcal{S}} \phi \, d\nu_n\right| \\ &= \left|\int_{\mathcal{S}} \phi \, d(\rho - \nu_n)\right| \leq \|\phi\|_\infty \cdot \|\rho - \nu_n\|_{\mathrm{vr}} \xrightarrow[n\to\infty]{(\dagger)} 0, \end{aligned}$$

[23] That is, $\mathcal{N}$ is the norm closure of the vector space of all finite linear combinations of elements of $\mathcal{H}$.

where again, $(*)$ is because $\frac{d\mu}{d\rho} = \phi = \frac{d\widetilde{\lambda}_n}{d\nu_n}$ and $(\dagger)$ is because ν_n converges to ρ in norm. Thus,

$$\lim_{n\to\infty} \|\widetilde{\lambda}_n - \lambda_n\|_{\mathrm{vr}} = 0. \tag{A19}$$

Equations (A17), (A18), and (A19) yield $\lim_{n\to\infty} \|\mu - \lambda_n\|_{\mathrm{vr}} = 0$, as desired. ◇[Claim 1]

Let $\mathbf{q} \in \Delta^K$. Since $\mathcal{S}$ is separable, Lemma A.4 says that $\mathcal{P}$ has a $\mathbf{q}$-Poincaré sequence $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^\infty$. Then Lemma A.5 says that $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{L}$-almost-objectively uncertain, subordinate to $\mathbf{q}$. Then Lemma A.6 and Claim 1 says that $(\mathfrak{G}^n)_{n=1}^\infty$ is $\langle\mathcal{R}\rangle$-almost-objectively uncertain, subordinate to $\mathbf{q}$. □

Appendix B: Proofs from Section 4

The proof of Theorem 1 uses Proposition 3, so we will prove that first. The proof of Proposition 3, in turn, uses the following result, which can be seen as the special case of Proposition 3 for SEU representations.

Lemma B.1. *Let $\mathcal{R}$, $\mathbf{q} \in \Delta^K$, $\mathbf{x} \in \mathcal{X}^K$, and $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^\infty$ be as in Proposition 3. For any $\rho \in \mathcal{R}$, and any measurable $u : \mathcal{X} \longrightarrow \mathbb{R}$,*

$$\lim_{n\to\infty} \int_{\mathcal{S}} u \circ \alpha^n \, d\rho = \sum_{k=1}^K q_k u(x_k).$$

Proof. By hypothesis, there is an $\mathcal{R}$-almost-objectively uncertain partition sequence $\boldsymbol{\mathcal{G}} = (\mathfrak{G}^n)_{n=1}^\infty$ subordinate to the probability vector $\mathbf{q}$, and for all $n \in \mathbb{N}$, the act α^n is $\mathfrak{G}^n$-measurable. Suppose $\mathbf{q} = (q_1, \ldots, q_K) \in \Delta^K$. For all $n \in \mathbb{N}$, write $\mathfrak{G}^n := \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$, such that the limit equations (5) hold. By hypothesis, there is a K-tuple $\mathbf{x} \in \mathcal{X}^K$ such that for all $n \in \mathbb{N}$, all $k \in [1 \ldots K]$, and all $s \in \mathcal{G}_k^n$, we have $\alpha^n(s) = x_k$. Thus, for any $\rho \in \mathcal{R}$,

$$\int_{\mathcal{S}} u \circ \alpha^n \, d\rho = \sum_{k=1}^K u(x_k) \rho(\mathcal{G}_k^n).$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathcal{S}} u \circ \alpha^n \, d\rho &= \lim_{n\to\infty} \sum_{k=1}^K u(x_k)\rho(\mathcal{G}_k^n) = \sum_{k=1}^K u(x_k) \lim_{n\to\infty} \rho(\mathcal{G}_k^n) \\ &\underset{(*)}{=} \sum_{k=1}^K u(x_k) q_k, \end{aligned}$$

where $(*)$ is by the limit equations (5). □

Proof of Proposition 3. Recall the notation of equation (3). We will first show that the limit equation (7) holds for $\underline{V}$ and $\overline{V}$, and then show that it holds for V itself.

Claim 1. $\lim_{n\to\infty} \underline{V}(\alpha^n) = \sum_{k=1}^K q_k u(x_k)$.

Proof. Let $B := \|u\|_\infty$. Then $B < \infty$, and the sequence $\{\underline{V}(\alpha^n)\}_{n=1}^\infty$ is bounded in the interval $[-B, B]$, so it has convergent subsequences. To prove the claim, it suffices to show that *every* convergent subsequence of $\{\underline{V}(\alpha^n)\}_{n=1}^\infty$ converges to $\sum_{k=1}^K q_k u(x_k)$.

So, let $\{n(\ell)\}_{\ell=1}^\infty$ be an increasing sequence in $\mathbb{N}$ such that the subsequence $\{\underline{V}(\alpha^{n(\ell)})\}_{\ell=1}^\infty$ converges to some limit V^*. We must show that $V^* = \sum_{k=1}^K q_k u(x_k)$. For all $\ell \in \mathbb{N}$, define the linear function $v_\ell : \Delta(\mathcal{S}) \longrightarrow \mathbb{R}$ by

$$v_\ell(\rho) := \int_{\mathcal{S}} u \circ \alpha^{n(\ell)} \, \mathrm{d}\rho, \quad \text{for all } \rho \in \Delta(\mathcal{S}). \tag{B1}$$

This function is continuous in the norm topology, while $\mathcal{P}$ is closed in this topology. Thus,

$$\underline{V}\big(\alpha^{n(\ell)}\big) = \min_{\rho \in \mathcal{P}} v_\ell(\rho) = v_\ell(\rho_\ell), \tag{B2}$$

for some $\rho_\ell \in \mathcal{P}$. Furthermore, $\mathcal{P}$ is norm compact. Thus, the sequence $\{\rho_\ell\}_{\ell=1}^\infty$ has a subsequence $\{\rho_{\ell_m}\}_{m=1}^\infty$ that converges to some limit point $\rho_* \in \mathcal{P}$ in the norm topology.

Let $\epsilon > 0$. There exists $M_1 \in \mathbb{N}$ such that, for all $m \geq M_1$, $\|\rho_{\ell_m} - \rho_*\|_{\mathrm{vr}} < \frac{\epsilon}{3B}$. Thus, for all $n \in \mathbb{N}$ and all $m \geq M_1$,

$$\begin{aligned}\left| \int_{\mathcal{S}} u \circ \alpha^n \, \mathrm{d}\rho_{\ell_m} - \int_{\mathcal{S}} u \circ \alpha^n \, \mathrm{d}\rho_* \right| &= \left| \int_{\mathcal{S}} u \circ \alpha^n \, \mathrm{d}(\rho_{\ell_m} - \rho_*) \right| \\ &\leq \left\| u \circ \alpha^n \right\|_\infty \cdot \|\rho_{\ell_m} - \rho_*\|_{\mathrm{vr}} < B \cdot \frac{\epsilon}{3B} = \frac{\epsilon}{3}. \end{aligned} \tag{B3}$$

In particular, setting $n := n(\ell_m)$ in (B3) and invoking equation (B1) yields

$$\left| v_{\ell_m}(\rho_{\ell_m}) - v_{\ell_m}(\rho_*) \right| < \frac{\epsilon}{3}. \tag{B4}$$

Next, substituting equation (B2) into inequality (B4) yields

$$\left| \underline{V}\big(\alpha^{n(\ell_m)}\big) - v_{\ell_m}(\rho_*) \right| < \frac{\epsilon}{3}. \tag{B5}$$

Meanwhile, $\rho_* \in \mathcal{R}$, so Lemma B.1 yields some $N \in \mathbb{N}$ such that

$$\left| \int_{\mathcal{S}} u \circ \alpha^n \, \mathrm{d}\rho_* - \sum_{k=1}^K q_k u(x_k) \right| < \frac{\epsilon}{3} \quad \text{for all } n \geq N. \tag{B6}$$

Since the sequence $\{n(\ell_m)\}_{m=1}^\infty$ is strictly increasing, there is some $M_2 \in \mathbb{N}$ such that $n(\ell_m) > N$ for all $m \geq M_2$. From this and inequality (B6), it follows that

$$\left| \int_{\mathcal{S}} u \circ \alpha^{n(\ell_m)} \, \mathrm{d}\rho_* - \sum_{k=1}^K q_k u(x_k) \right| < \frac{\epsilon}{3}, \quad \text{for all } m \geq M_2. \tag{B7}$$

Using the defining equation (B1), we can rewrite inequality (B7) as follows:

$$\left| v_{\ell_m}(\rho_*) - \sum_{k=1}^K q_k u(x_k) \right| < \frac{\epsilon}{3}, \quad \text{for all } m \geq M_2. \tag{B8}$$

Finally, by hypothesis, $\lim_{\ell \to \infty} \underline{V}(\alpha^{n(\ell)}) = V^*$. So, there is some $L \in \mathbb{N}$ such that

$$\left|V^* - \underline{V}\left(\alpha^{n(\ell)}\right)\right| < \frac{\epsilon}{3}, \quad \text{for all } \ell \geq L. \tag{B9}$$

Since the sequence $\{\ell_m\}_{m=1}^{\infty}$ is strictly increasing, there is some $M_3 \in \mathbb{N}$ such that $\ell_m > L$ for all $m \geq M_3$. From this and inequality (B9), it follows that

$$\left|V^* - \underline{V}\left(\alpha^{n(\ell_m)}\right)\right| < \frac{\epsilon}{3}, \quad \text{for all } m \geq M_3. \tag{B10}$$

Now, let $M_\epsilon := \max\{M_1, M_2, M_3\}$. Then for all $m \geq M_\epsilon$, we have

$$\begin{aligned}
&\left|V^* - \sum_{k=1}^{K} q_k u(x_k)\right| \\
&\leq \left|V^* - \underline{V}\left(\alpha^{n(\ell_m)}\right)\right| + \left|\underline{V}\left(\alpha^{n(\ell_m)}\right) - v_{\ell_m}(\rho_*)\right| + \left|v_{\ell_m}(\rho_*) - \sum_{k=1}^{K} q_k u(x_k)\right| \\
&\underset{(*)}{<} \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

where $(*)$ is by inequalities (B5), (B8), and (B10).

This argument works for any $\epsilon > 0$. Thus, $V^* = \sum_{k=1}^{K} q_k u(x_k)$. ◇[Claim 1]

By an argument similar to Claim 1 (replacing min with max), we can show that

$$\lim_{n \to \infty} \overline{V}\left(\alpha^n\right) = \sum_{k=1}^{K} q_k u(x_k). \tag{B11}$$

Combining inequality (3) with Claim 1 and equation (B11) yields equation (7), proving the theorem. □

Proposition 3 yields a convenient condition for asymptotic preferences.

Lemma B.2. *Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$ be consilient. Suppose $\succcurlyeq$ has a compact, contiguous GH representation (3) V with $\mathcal{P} \subseteq \mathcal{R}$. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be almost-objective acts. Then $\boldsymbol{\alpha} \succ^\infty \boldsymbol{\beta}$ if and only if there exist $N \in \mathbb{N}$ and $\epsilon > 0$ such that $V(\alpha^n) > V(\beta^n) + \epsilon$ for all $n \geq N$.*

Proof. "$\Longrightarrow$" If $\boldsymbol{\alpha} \succ^\infty \boldsymbol{\beta}$, then there exist $\alpha', \beta' \in \mathcal{A}$, and $N \in \mathbb{N}$ such that for all $n \geq N$, we have $V(\alpha^n) > V(\alpha') > V(\beta') > V(\beta^n)$, and thus $V(\alpha^n) - V(\beta^n) > V(\alpha') - V(\beta') > 0$. So, let $\epsilon := V(\alpha') - V(\beta')$. Then $\epsilon > 0$, and $V(\alpha^n) > V(\beta^n) + \epsilon$ for all $n \geq N$.

"$\Longleftarrow$" Let $\mathbf{q} \in \Delta^K$ and $\mathbf{x} \in \mathcal{X}^K$ (for some $K \in \mathbb{N}$) and suppose that $\boldsymbol{\alpha}$ is subordinate to the lottery $(\mathbf{q}, \mathbf{x})$. Let $\mathbf{p} \in \Delta^L$ and $\mathbf{y} \in \mathcal{X}^L$ (for some $L \in \mathbb{N}$) and suppose that $\boldsymbol{\beta}$ is subordinate to the lottery $(\mathbf{p}, \mathbf{y})$. Let $A := \sum_{k=1}^{K} q_k u(x_k)$ and $B := \sum_{\ell=1}^{L} p_\ell u(y_\ell)$. Then Proposition 3 says that

$$\lim_{n \to \infty} V\left(\alpha^n\right) = A \quad \text{and} \quad \lim_{n \to \infty} V\left(\beta^n\right) = B. \tag{B12}$$

If $V(\alpha^n) > V(\beta^n) + \epsilon$ for all $n \geq N$, then the limits (B12) imply that $A \geq B + \epsilon$. Thus, $A - \frac{\epsilon}{3} > B + \frac{\epsilon}{3}$. The limits (B12) yield $M \in \mathbb{N}$ such that $V(\alpha^m) > A - \frac{\epsilon}{3}$ and $V(\beta^m) < B + \frac{\epsilon}{3}$ for all $m \geq M$. Since V is contiguous, its image $V(\mathcal{A})$ is a dense subset of an interval in $\mathbb{R}$. By prior observations, this interval must contain the subinterval $[B + \frac{\epsilon}{3}, A - \frac{\epsilon}{3}]$. So, there exist $a, b \in V(\mathcal{A})$ such that $A - \frac{\epsilon}{3} > a > b > B + \frac{\epsilon}{3}$. Then for all $m \geq M$,

$$V(\alpha^m) > A - \frac{\epsilon}{3} > a > b > B + \frac{\epsilon}{3} > V(\beta^m). \tag{B13}$$

Let $\alpha', \beta' \in \mathcal{A}$ be such that $V(\alpha') = a$ and $V(\beta') = b$. Then for all $m \geq M$, the inequalities (B13) imply that $\alpha^m \succ \alpha' \succ \beta' \succ \beta^m$, as desired. □

Let $\mathcal{U}$ be the Banach space of bounded, measurable, real-valued functions on $\mathcal{X}$, endowed with the norm $\|\cdot\|_\infty$ defined by $\|u\|_\infty := \sup_{x\in\mathcal{X}} |u(x)|$ for all $u \in \mathcal{U}$. We shall use the following straightforward consequence of the separating hyperplane theorem.

Lemma B.3. *Let $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{U}$, and suppose $\{u_i\}_{i\in\mathcal{I}}$ satisfy* Minimal Agreement. *Suppose there exists $z \in \mathcal{X}$ such that $u_j(z) = 0$ for all $j \in \mathcal{J}$. Let $\mathcal{C}$ be the convex cone in $\mathcal{U}$ spanned by $\{u_i\}_{i\in\mathcal{I}}$ and* 0. *If $u_o \notin \mathcal{C}$, then there exist finitely additive probability measures ν_1 and ν_2 on $\mathcal{X}$ such that*

$$\int_{\mathcal{X}} u_o \,\mathrm{d}\nu_1 < \int_{\mathcal{X}} u_o \,\mathrm{d}\nu_2, \quad \text{while} \int_{\mathcal{X}} u_i \,\mathrm{d}\nu_1 > \int_{\mathcal{X}} u_i \,\mathrm{d}\nu_2 \quad \text{for all } i \in \mathcal{I}. \tag{B14}$$

Proof. (Pivato (2022), Lemma A.2). □

Proof of Theorem 1. "$\Longrightarrow$" (by contradiction) Suppose $\succcurlyeq_o$ satisfies $\mathcal{R}$-Almost-Objective Pareto, but u_o is *not* weakly utilitarian. Let $z \in \mathcal{X}$. We can assume without loss of generality that $u_j(z) = 0$ for all $j \in \mathcal{J}$. To see this, let $c_j := u_j(z)$, and then define $\widetilde{u}_j(x) := u_j(x) - c_j$ for all $x \in \mathcal{X}$. If $\succcurlyeq_j$ has a GH representation (3), then $\succcurlyeq_j$ also admits a GH representation where u_j is replaced by $\widetilde{u}_j$.

Now, let $\mathcal{C}$ be the closed, convex cone in $\mathcal{U}$ spanned by $\{u_i\}_{i\in\mathcal{I}}$ and 0. Then u_o is weakly utilitarian if and only if $u_o \in \mathcal{C}$. Thus, if u_o is *not* weakly utilitarian, then $u_o \notin \mathcal{C}$, in which case Lemma B.3 yields finitely additive probability measures ν_1 and ν_2 on $\mathcal{X}$ satisfying the inequalities (B14). For all $j \in \mathcal{J}$, let $\epsilon_j := |\int_{\mathcal{X}} u_j \,\mathrm{d}\nu_1 - \int_{\mathcal{X}} u_j \,\mathrm{d}\nu_2|$. Let

$$\epsilon := \frac{1}{5} \min_{j\in\mathcal{J}} \epsilon_j. \tag{B15}$$

Then $\epsilon > 0$. Inequalities (B14) and definition (B15) yield

$$\int_{\mathcal{X}} u_o \,\mathrm{d}\nu_2 - \int_{\mathcal{X}} u_o \,\mathrm{d}\nu_1 > 5\epsilon, \tag{B16}$$

$$\text{while} \int_{\mathcal{X}} u_i \,\mathrm{d}\nu_1 - \int_{\mathcal{X}} u_i \,\mathrm{d}\nu_2 > 5\epsilon, \text{ for all } i \in \mathcal{I}. \tag{B17}$$

Let $R := \max\{\|u_j\|_\infty\}_{j\in\mathcal{J}}$; this value is finite because $\{u_j\}_{j\in\mathcal{J}}$ are bounded. Let $N := \lceil R/\epsilon \rceil + 1$; then $N\epsilon > R$, so the interval $[-N\epsilon, N\epsilon)$ contains the ranges of $\{u_j\}_{j\in\mathcal{J}}$. For all $j \in \mathcal{J}$ and all $n \in [-N \dots N]$, let $\mathcal{Y}_n^j := (u_j)^{-1}[n\epsilon, (n+1)\epsilon)$. Then $\mathfrak{Y}^j := \{\mathcal{Y}_n^j\}_{n=-N}^N$ is a measurable partition of $\mathcal{X}$. Let $\mathfrak{Y}$ be the common refining partition of $\{\mathfrak{Y}^j\}_{j \in \mathcal{J}}$. This is a measurable partition of $\mathcal{X}$. Suppose it has K cells, and write $\mathfrak{Y} = \{\mathcal{Y}_k\}_{k=1}^K$. For all $k \in [1 \dots K]$, let $p_k^1 := \nu_1(\mathcal{Y}_k)$ and $p_k^2 := \nu_2(\mathcal{Y}_k)$. Then $\mathbf{p}^1 := (p_k^1)_{k=1}^K$ and $\mathbf{p}^2 := (p_k^2)_{k=1}^K$ are K-dimensional probability vectors. For all $k \in [1 \dots K]$, let $x_k \in \mathcal{Y}_k$.

Claim 1. *For all* $j \in \mathcal{J}$,

$$\left|\sum_{k=1}^K p_k^1 u_j(x_k) - \int_{\mathcal{X}} u_j \, d\nu_1\right| < \epsilon \quad \textit{and} \quad \left|\sum_{k=1}^K p_k^2 u_j(x_k) - \int_{\mathcal{X}} u_j \, d\nu_2\right| < \epsilon.$$

Proof. To prove the first inequality, note that

$$\begin{aligned}
\left|\sum_{k=1}^K p_k^1 u_j(x_k) - \int_{\mathcal{X}} u_j \, d\nu_1\right| &= \left|\sum_{k=1}^K \nu_1(\mathcal{Y}_k) u_j(x_k) - \sum_{k=1}^K \int_{\mathcal{Y}_k} u_j \, d\nu_1\right| \\
&= \left|\sum_{k=1}^K \left(\int_{\mathcal{Y}_k} u_j(x_k) \, d\nu_1 - \int_{\mathcal{Y}_k} u_j \, d\nu_1\right)\right| \\
&= \left|\sum_{k=1}^K \left(\int_{\mathcal{Y}_k} u_j(x_k) - u_j(y) \, d\nu_1[y]\right)\right| \\
&\leq \sum_{k=1}^K \int_{\mathcal{Y}_k} \left|u_j(x_k) - u_j(y)\right| d\nu_1[y] \underset{(*)}{<} \sum_{k=1}^K \int_{\mathcal{Y}_k} \epsilon \, d\nu_1 \\
&= \sum_{k=1}^K \epsilon \nu_1(\mathcal{Y}_k) = \epsilon,
\end{aligned}$$

as claimed. Here, $(*)$ is because for all $k \in [1 \dots K]$, we have $x_k \in \mathcal{Y}_k$ while $n\epsilon \leq u_j(y) < (n+1)\epsilon)$ for all $y \in \mathcal{Y}_k$, so that $|u_j(x_k) - u_j(y)| < \epsilon$ for all $y \in \mathcal{Y}_k$. The proof of the second inequality is similar. ◇[Claim 1]

Combining inequalities (B16) and (B17) with Claim 1 yield

$$\sum_{k=1}^K p_k^2 u_o(x_k) - \sum_{k=1}^K p_k^1 u_o(x_k) > 3\epsilon, \tag{B18}$$

$$\text{while } \sum_{k=1}^K p_k^1 u_i(x_k) - \sum_{k=1}^K p_k^2 u_i(x_k) > 3\epsilon, \text{ for all } i \in \mathcal{I}. \tag{B19}$$

Let $\mathbf{q} \in \Delta^{K \times K}$ be the probability vector defined by $q_{k,\ell} := p_k^1 p_\ell^2$ for all $k, \ell \in [1 \dots K]$. Since $\mathcal{R}$ is consilient, there is an $\mathcal{R}$-almost-objectively uncertain partition sequence $(\mathfrak{G}^n)_{n=1}^\infty$ subordinate to $\mathbf{q}$. For all $n \in \mathbb{N}$, write $\mathfrak{G}^n = \{\mathcal{G}_{k,\ell}^n\}_{k,\ell=1}^K$, with

$$\lim_{n \to \infty} \rho\left(\mathcal{G}_{k,\ell}^n\right) = q_{k,\ell}, \quad \text{for all } \rho \in \mathcal{R} \text{ and } k, \ell \in [1 \dots K]. \tag{B20}$$

For all $n \in \mathbb{N}$, and $\ell, k \in [1 \dots K]$, define $\mathcal{G}^n_{k,*} := \mathcal{G}^n_{k,1} \cup \mathcal{G}^n_{k,2} \cup \cdots \cup \mathcal{G}^n_{k,K}$ and $\mathcal{G}^n_{*,\ell} := \mathcal{G}^n_{1,\ell} \cup \mathcal{G}^n_{2,\ell} \cup \cdots \cup \mathcal{G}^n_{K,\ell}$. Then the equation (B20) yields

$$\lim_{n\to\infty} \rho\big(\mathcal{G}^n_{k,*}\big) = p^1_k \quad \text{and} \quad \lim_{n\to\infty} \rho\big(\mathcal{G}^n_{*,\ell}\big) = p^2_\ell, \quad \text{for all } \rho \in \mathcal{R}. \tag{B21}$$

For all $n \in \mathbb{N}$, define acts $\alpha^n, \beta^n : \mathcal{S} \longrightarrow \mathcal{X}$ as follows:

- For all $k \in [1 \dots K]$, let $\alpha^n(s) := x_k$ for all $s \in \mathcal{G}^n_{k,*}$.
- For all $\ell \in [1 \dots K]$, let $\beta^n(s) := x_\ell$ for all $s \in \mathcal{G}^n_{*,\ell}$.

Thus, $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^\infty$ and $\boldsymbol{\beta} = (\beta^n)_{n=1}^\infty$ are $\mathcal{R}$-almost-objectively uncertain acts. They are compatible because for all $n \in \mathbb{N}$, α^n and β^n are both $\mathfrak{G}^n$-measurable. By construction and equations (B21), $\boldsymbol{\alpha}$ is subordinate to $(\mathbf{p}^1, \mathbf{x})$, while $\boldsymbol{\beta}$ is subordinate to $(\mathbf{p}^2, \mathbf{x})$.

Claim 2. *$\boldsymbol{\alpha} \succ^\infty_i \boldsymbol{\beta}$ for all $i \in \mathcal{I}$.*

Proof. For all $i \in \mathcal{I}$, let $V_i : \mathcal{A} \longrightarrow \mathbb{R}$ be a GH representation for $\succcurlyeq_i$ in which $\mathcal{P}_i \subseteq \mathcal{R}$ is norm compact. Proposition 3 says that

$$\lim_{n\to\infty} V_i\big(\alpha^n\big) = \sum_{k=1}^K p^1_k u_i(x_k) \quad \text{and} \quad \lim_{n\to\infty} V_i\big(\beta^n\big) = \sum_{k=1}^K p^2_k u_i(x_k).$$

Thus, there exists $N \in \mathbb{N}$ such that

$$\left| V_i\big(\alpha^n\big) - \sum_{k=1}^K p^1_k u_i(x_k) \right| < \epsilon \quad \text{and} \quad \left| V_i\big(\beta^n\big) - \sum_{k=1}^K p^2_k u_i(x_k) \right| < \epsilon, \quad \text{for all } n \geq N. \tag{B22}$$

Combining inequalities (B19) and (B22), we obtain $V_i(\alpha^n) - V_i(\beta^n) > \epsilon$, for all $n \geq N$. Thus, $\boldsymbol{\alpha} \succ^\infty_i \boldsymbol{\beta}$ by Lemma B.2. ◇[Claim 2]

By an argument identical to Claim 2, but using inequality (B18) rather than (B19), it is easy to prove that $\boldsymbol{\alpha} \prec^\infty_o \boldsymbol{\beta}$. This, together with Claim 2, is a violation of $\mathcal{R}$-Almost-Objective Pareto. Contradiction. To avoid this contradiction, u_o must be weakly utilitarian.

"$\Longleftarrow$" (by contradiction) Suppose u_o is weakly utilitarian; thus, $u_o = \sum_{i\in\mathcal{I}} c_i u_i$ for some constants $c_i \geq 0$. Suppose $\mathcal{R}$-Almost-Objective Pareto is violated. Then there exist compatible almost-objective acts $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} \succ^\infty_i \boldsymbol{\beta}$ for all $i \in \mathcal{I}$, while $\boldsymbol{\alpha} \prec^\infty_o \boldsymbol{\beta}$. Thus, for all $i \in \mathcal{I}$, Lemma B.2 yields $\epsilon_i > 0$ and $N_i \in \mathbb{N}$ such that

$$V_i\big(\alpha^n\big) - V_i\big(\beta^n\big) > 2\epsilon_i, \quad \text{for all } n \geq N_i, \tag{B23}$$

whereas there is some $\epsilon_o > 0$ and some $N_o \in \mathbb{N}$ such that

$$V_o\big(\beta^n\big) - V_o\big(\alpha^n\big) > 2\epsilon_o, \quad \text{for all } n \geq N_o. \tag{B24}$$

There exist $K \in \mathbb{N}$, $\mathbf{p} \in \Delta^K$, and $\mathbf{x} \in \mathcal{X}^K$ such that $\boldsymbol{\alpha}$ is subordinate to $(\mathbf{p}, \mathbf{x})$. Likewise, there exist $L \in \mathbb{N}$, $\mathbf{q} \in \Delta^L$, and $\mathbf{y} \in \mathcal{X}^L$ such that $\boldsymbol{\beta}$ is subordinate to $(\mathbf{q}, \mathbf{y})$.

CLAIM 3. *For all* $i \in \mathcal{I}$, $\sum_{k=1}^{K} p_k u_i(x_k) - \sum_{\ell=1}^{L} q_\ell u_i(y_\ell) > 0$.

PROOF. For all $i \in \mathcal{I}$, let $V_i : \mathcal{A} \longrightarrow \mathbb{R}$ be a GH representation for $\succcurlyeq_i$ in which $\mathcal{P}_i \subseteq \mathcal{R}$ is norm compact. Now, follow the argument from the proof of Claim 2 to obtain $M_i \in \mathbb{N}$ such that

$$\left|V_i(\alpha^m) - \sum_{k=1}^{K} p_k u_i(x_k)\right| < \epsilon_i \quad \text{and} \quad \left|V_i(\beta^m) - \sum_{\ell=1}^{L} q_\ell u_i(y_\ell)\right| < \epsilon_i, \quad \text{for all } m \geq M_i. \tag{B25}$$

Now, let $n \geq \max\{N_i, M_i\}$, and combine (B23) and (B25) to get the claimed inequality.

◇[Claim 3]

By an argument similar to Claim 3, but using inequality (B24) rather than (B23), one can show that

$$\sum_{k=1}^{K} p_k u_o(x_k) - \sum_{\ell=1}^{L} q_\ell u_o(y_\ell) < 0. \tag{B26}$$

Now, $u_o = \sum_{i\in\mathcal{I}} c_i u_i$. Thus,

$$\begin{aligned}\sum_{k=1}^{K} p_k u_o(x_k) - \sum_{\ell=1}^{L} q_\ell u_o(y_\ell) &= \sum_{k=1}^{K} p_k \sum_{i\in\mathcal{I}} c_i u_i(x_k) - \sum_{\ell=1}^{L} q_\ell \sum_{i\in\mathcal{I}} c_i u_i(y_\ell) \\ &= \sum_{i\in\mathcal{I}} c_i \left(\sum_{k=1}^{K} p_k u_i(x_k) - \sum_{\ell=1}^{L} q_\ell u_i(y_\ell)\right).\end{aligned} \tag{B27}$$

But $c_i \geq 0$ for all $i \in \mathcal{I}$, so equation (B27), inequality (B26), and Claim 3 are logically inconsistent. To avoid this contradiction, $\mathcal{R}$-Almost-Objective Pareto must be satisfied. □

PROOF OF COROLLARY 2. For all $j \in \mathcal{J}$, the preference $\succcurlyeq_j$ has a GH representation induced by a compact set $\mathcal{P}_j \subseteq \Delta(\mathcal{S})$ of nonatomic probability measures. Let $\mathcal{R} := \bigcup_{j\in\mathcal{J}} \mathcal{P}_j$. Then $\mathcal{R}$ is compact (because $\mathcal{J}$ is finite), hence a separable subset of $\Delta(\mathcal{S})$. Thus, Proposition 1 says that $\mathcal{R}$ is consilient. By definition, $\succcurlyeq_o$ satisfies Almost-Objective Pareto* if and only if it satisfies $\mathcal{R}$-Almost-Objective Pareto, which (by Theorem 1) is the case if and only if u_o is weakly utilitarian. □

APPENDIX C: PROOF OF RESULTS FROM SECTION 5

PROOF OF PROPOSITION 4. The following axiom about beliefs is due to Mongin (1995):

C1. For all $\mathcal{B}, \mathcal{C} \subseteq \mathcal{S}$, if $\rho_i(\mathcal{B}) \geq \rho_i(\mathcal{C})$ for all $i \in \mathcal{I}$, then $\rho_o(\mathcal{B}) \geq \rho_o(\mathcal{C})$.

CLAIM 1. $\succeq_o$ *satisfies* Dichotomous Pareto *if and only if* ρ_o *satisfies Axiom C1.*

Proof. Let β, γ be congruent dichotomous acts, ranging over a dichotomy $\{x, y\}$. Then there exist measurable subsets $\mathcal{B}, \mathcal{C} \subseteq \mathcal{S}$ such that for all $s \in \mathcal{S}$, we have

$$\beta(s) = \begin{cases} x & \text{if } s \in \mathcal{B}; \\ y & \text{otherwise.} \end{cases} \qquad \gamma(s) = \begin{cases} x & \text{if } s \in \mathcal{C}; \\ y & \text{otherwise.} \end{cases} \tag{C1}$$

Thus, for all $j \in \mathcal{J}$, we have $\beta \succeq_j \gamma$ if and only if $\rho_j(\mathcal{B}) \geq \rho_j(\mathcal{C})$. It follows that Dichotomous Pareto (for β vs. γ) is equivalent to Axiom C1 (for $\mathcal{B}$ vs. $\mathcal{C}$). We can make this argument for any congruent pair of dichotomous acts β and γ. Conversely, for any measurable $\mathcal{B}, \mathcal{C} \subseteq \mathcal{S}$, we can construct congruent dichotomous acts β and γ satisfying statement (C1). ◇[Claim 1]

"$\Longrightarrow$" If $\succeq_o$ satisfies Dichotomous Pareto, then Claim 1 says that ρ_o satisfies C1. Thus, ρ_o is a weighted average of $\{\rho_i\}_{i \in \mathcal{I}}$ by Proposition 2 of Mongin (1995).[24]

"$\Longleftarrow$" If ρ_o is a weighted average of $\{\rho_i\}_{i \in \mathcal{I}}$, then it clearly satisfies C1. Thus, $\succeq_o$ satisfies Dichotomous Pareto, by Claim 1. □

Theorem 2 is a consequence of a more general result. Let $\trianglerighteq$ be a preorder on $\mathcal{A}$ (e.g., a Bewley preference). We will write $\boldsymbol{\alpha} \trianglerighteq^{\omega} \boldsymbol{\beta}$ if there exists $N \in \mathbb{N}$ such that $\alpha_n \trianglerighteq \beta_n$ for all $n \geq N$.

Now, let $\{\trianglerighteq_j\}_{j \in \mathcal{J}}$ be a family of Bewley preferences satisfying MAO, and consider a sequence of acts $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$. We shall say that $\boldsymbol{\alpha}$ is *dichotomous* if there is some dichotomy (x, y) such that α^n ranges over $\{x, y\}$ for all $n \in \mathbb{N}$. Suppose that $\mathcal{R} \subseteq \Delta(\mathcal{S})$ is consilient. We shall say that $\boldsymbol{\alpha}$ is $\mathcal{R}$*-piecewise almost-objective* if there is a measurable partition $\mathfrak{H} = \{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_J\}$ of $\mathcal{S}$ and a family of $\mathcal{R}$-almost-objective acts $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_J$ such that for all $n \in \mathbb{N}$, and all $j \in [1 \dots J]$, we have

$$\alpha^n(s) = \alpha_j^n(s) \quad \text{for all } s \in \mathcal{H}_j. \tag{C2}$$

In other words, $\boldsymbol{\alpha}$ is achieved by "patching together" $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_J$ according to the partition $\mathfrak{H}$. Any almost-objective act is piecewise almost-objective (via the trivial partition). Consider the following axiom.

$\mathcal{R}$-Dichotomous Piecewise Almost-Objective Pareto. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two dichotomous $\mathcal{R}$-piecewise almost-objective acts. If $\boldsymbol{\alpha} \trianglerighteq_i^{\omega} \boldsymbol{\beta}$ for all $i \in \mathcal{I}$, then $\boldsymbol{\alpha} \trianglerighteq_o^{\omega} \boldsymbol{\beta}$.

Compared to $\mathcal{R}$-Almost-Objective Pareto, this new axiom is broader in one way (it applies to *piecewise* almost-objective acts), but narrower in another way (it applies only to *dichotomous* almost-objective acts). It also differs from $\mathcal{R}$-Almost-Objective Pareto in that it involves the (possibly incomplete) Bewley preferences $\{\trianglerighteq_j\}_{j \in \mathcal{J}}$ instead of the weak orders $\{\succcurlyeq_j\}_{j \in \mathcal{J}}$, and it requires the planner's asymptotic preferences to actually *agree* with those of the individuals, rather than simply not disagree. Theorem 2 is an immediate consequence of the following more general result.

[24] Mongin assumes *countably* additive measures. But he does this only so that he can invoke the Lyapunov convexity theorem, which was extended to *finitely* additive measures by Armstrong and Prikry (1981, Theorem 2-2).

Theorem C.1. *Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$ be strongly consilient. For all $j \in \mathcal{J}$, let $\trianglerighteq_j$ be a Bewley preferences induced by a compact subset $\mathcal{P}_j \subseteq \mathcal{R}$ and utility function $u_j : \mathcal{X} \longrightarrow \mathbb{R}$. Suppose $\{\trianglerighteq_j\}_{j\in\mathcal{J}}$ satisfy* MAO. *Let $\overline{\mathcal{P}}$ be the $\mathfrak{T}$-closed, convex hull of $\bigcup_{i\in\mathcal{I}} \mathcal{P}_i$. The following are equivalent:*

(a) $\trianglerighteq_o$ *satisfies* $\mathcal{R}$-Dichotomous Piecewise Almost-Objective Pareto.

(b) $\trianglerighteq_o$ *satisfies* Dichotomous Pareto.

(c) $\mathcal{P}_o \subseteq \overline{\mathcal{P}}$.

The proof of Theorem C.1 requires some preliminaries. A measurable function $\phi : \mathcal{S} \longrightarrow \mathbb{R}$ is *simple* if it takes only a finite number of values. For any simple function ϕ, define $\phi^* : \text{ba}(\mathcal{S}) \longrightarrow \mathbb{R}$ by setting by $\phi^*(\mu) := \int_{\mathcal{S}} \phi \, d\mu$ for all $\mu \in \text{ba}(\mathcal{S})$. Then ϕ^* is a linear functional and continuous in the norm topology.

Lemma C.2. *Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$ be strongly consilient. Let $\{\trianglerighteq_j\}_{j\in\mathcal{J}}$ be Bewley preferences on $\mathcal{A}$ that satisfy* MAO, *and let (x, y) be a dichotomy for $\{\trianglerighteq_j\}_{j\in\mathcal{J}}$. Suppose their Bewley representations (8) have belief sets contained in $\mathcal{R}$, and utility functions $\{u_j\}_{j\in\mathcal{J}}$ that are renormalized such that $u_j(x) = 1$ and $u_j(y) = 0$ for all $j \in \mathcal{J}$. Let $\phi : \mathcal{S} \longrightarrow \mathbb{R}$ be a simple function, and consider the functional $\phi^* : \text{ba}(\mathcal{S}) \longrightarrow \mathbb{R}$. There exists a dichotomous piecewise $\mathcal{R}$-almost-objective act $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} \int_{\mathcal{S}} u_j \circ \alpha^n \, d\rho_j = \phi^*(\rho_j), \quad \textit{for all } j \in \mathcal{J} \textit{ and all } \rho_j \in \mathcal{P}_j.$$

Proof. By hypothesis, there exists a measurable partition $\{\mathcal{G}_1, \ldots, \mathcal{G}_L\}$ of $\mathcal{S}$ and some $r_1, \ldots, r_L \in \mathbb{R}$ such that $\phi = \sum_{\ell=1}^{L} r_\ell \mathbf{1}_{\mathcal{G}_\ell}$, where $\mathbf{1}_{\mathcal{G}_\ell}$ is the indicator function of $\mathcal{G}_\ell$.

Claim 1. *For all $\ell \in [1 \ldots L]$, there exists a sequence $(\mathcal{F}_\ell^n)_{n=1}^{\infty}$ of subsets of $\mathcal{G}^\ell$ such that $\lim_{n\to\infty} \rho(\mathcal{F}_\ell^n) = r_\ell \cdot \rho(\mathcal{G}_\ell)$ for all $\rho \in \mathcal{R}$.*

Proof. For all $\rho \in \mathcal{R}$, let $\rho_{\mathcal{G}_\ell} \in \Delta(\mathcal{S})$ be the measure obtained by Bayes updating ρ on $\mathcal{G}_\ell$. Then $\rho_{\mathcal{G}_\ell} \in \langle\mathcal{R}\rangle$, because $\rho_{\mathcal{G}_\ell} \lll \rho$ and $\frac{d\rho_{\mathcal{G}_\ell}}{d\rho} = \mathbf{1}_{\mathcal{G}_\ell}/\rho(\mathcal{G}_\ell)$ are bounded.

By strong consilience, there is a sequence of measurable subsets $(\mathcal{E}^n)_{n=1}^{\infty}$ in $\mathcal{S}$ such that $\lim_{n\to\infty} \mu(\mathcal{E}^n) = r_\ell$ for all $\mu \in \langle\mathcal{R}\rangle$. Thus, $\lim_{n\to\infty} \rho_{\mathcal{G}_\ell}(\mathcal{E}^n) = r_\ell$ for all $\rho \in \mathcal{R}$, by the previous paragraph. For all $n \in \mathbb{N}$, let $\mathcal{F}_\ell^n := \mathcal{E}^n \cap \mathcal{G}_\ell$. Then $\mathcal{F}_\ell^n \subseteq \mathcal{G}_\ell$. For all $\rho \in \mathcal{R}$, we have $\rho(\mathcal{F}_\ell^n) = \rho_{\mathcal{G}_\ell}(\mathcal{F}_\ell^n) \cdot \rho(\mathcal{G}_\ell)$ and $\rho_{\mathcal{G}_\ell}(\mathcal{F}_\ell^n) = \rho_{\mathcal{G}_\ell}(\mathcal{E}^n)$. Thus, $\lim_{n\to\infty} \rho_{\mathcal{G}_\ell}(\mathcal{F}_\ell^n) = r_\ell$, and hence $\lim_{n\to\infty} \rho(\mathcal{F}_\ell^n) = r_\ell \cdot \rho(\mathcal{G}_\ell)$. ◇[Claim 1]

Now, for all $n \in \mathbb{N}$, let $\mathcal{F}_1^n, \mathcal{F}_2^n, \ldots, \mathcal{F}_L^n$ be as in Claim 1; these sets are disjoint because $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_L$ are disjoint. Let $\mathcal{F}^n := \bigsqcup_{\ell=1}^{L} \mathcal{F}_\ell^N$, and then define $\alpha^n \in \mathcal{A}$ by

$$\text{for all } s \in \mathcal{S}, \quad \alpha^n(s) := \begin{cases} x & \text{if } s \in \mathcal{F}^n; \\ y & \text{otherwise.} \end{cases}$$

The sequence $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$ is clearly dichotomous, and is piecewise $\mathcal{R}$-almost objective (with respect to the original partition $\mathfrak{G}$). For all $j \in \mathcal{J}$, we have $u_j \circ \alpha^n = \mathbf{1}_{\mathcal{F}^n}$. Thus, for any $\rho \in \mathcal{R}$,

$$\int_{\mathcal{S}} u_j \circ \alpha^n \, \mathrm{d}\rho = \int_{\mathcal{S}} \mathbf{1}_{\mathcal{F}^n} \, \mathrm{d}\rho = \rho\big[\mathcal{F}^n\big] = \rho\left[\bigsqcup_{\ell=1}^{L} \mathcal{F}_\ell^N\right] = \sum_{\ell=1}^{L} \rho\big[\mathcal{F}_\ell^n\big]. \tag{C3}$$

Thus,

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathcal{S}} u_j \circ \alpha^n \, \mathrm{d}\rho &\underset{(*)}{=} \lim_{n\to\infty} \sum_{\ell=1}^{L} \rho\big[\mathcal{F}_\ell^n\big] = \sum_{\ell=1}^{L} \lim_{n\to\infty} \rho\big[\mathcal{F}_\ell^n\big] \\ &\underset{(\dagger)}{=} \sum_{\ell=1}^{L} r_\ell \cdot \rho(\mathcal{G}_\ell) = \int_{\mathcal{S}} \sum_{\ell=1}^{L} r_\ell \mathbf{1}_{\mathcal{G}_\ell} \, \mathrm{d}\rho = \int_{\mathcal{S}} \phi \, \mathrm{d}\rho = \phi^*(\rho),\end{aligned}$$

as desired. Here, $(*)$ is by equation (C3), and $(\dagger)$ is by Claim 1. □

Let $\mathfrak{T}$ be the weak topology on $\mathrm{ba}(\mathcal{S})$ induced by the family $\{\phi^*; \phi : \mathcal{S} \longrightarrow \mathcal{R}$ a simple function$\}$. The total variation norm topology on $\mathrm{ba}(\mathcal{S})$ is finer than $\mathfrak{T}$. Thus, if a subset $\mathcal{P} \subset \mathrm{ba}(\mathcal{S})$ is compact in the total variation norm topology, then $\mathcal{P}$ is compact in $\mathfrak{T}$. For any measurable $\mathcal{B} \subseteq \mathcal{S}$, define $\eta_{\mathcal{B}} : \mathrm{ba}(\mathcal{S}) \longrightarrow \mathbb{R}$ by setting $\eta_{\mathcal{B}}(\mu) := \mu[\mathcal{B}]$ for all $\mu \in \mathrm{ba}(\mathcal{S})$. For any simple function $\phi : \mathcal{S} \longrightarrow \mathbb{R}$, with corresponding linear functional $\phi^* : \mathrm{ba}(\mathcal{S}) \longrightarrow \mathbb{R}$, if $\phi = \sum_{\ell=1}^{L} r_\ell \mathbf{1}_{\mathcal{G}_\ell}$, then $\phi^* := \sum_{\ell=1}^{L} r_\ell \eta_{\mathcal{G}_\ell}$.

Proof of Theorem C.1. "(b) $\Longrightarrow$ (a)" Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be dichotomous $\mathcal{R}$-piecewise almost-objective acts, and suppose that $\boldsymbol{\alpha} \trianglerighteq_i^{\omega} \boldsymbol{\beta}$ for all $i \in \mathcal{I}$. Thus, for all $i \in \mathcal{I}$ there is some $N_i \in \mathbb{N}$ such that $\alpha^n \trianglerighteq_i \beta^n$ for all $n \geq N_i$. Let $N := \max\{N_i\}_{i\in\mathcal{I}}$. Then $\alpha^n \trianglerighteq_o \beta^n$ for all $n \geq N$, by Dichotomous Pareto. Thus, $\boldsymbol{\alpha} \trianglerighteq_o^{\omega} \boldsymbol{\beta}$, as desired.

"(c) $\Longrightarrow$ (b)" Let (x, y) be a dichotomy for $\{\trianglerighteq_j\}_{j\in\mathcal{J}}$. Define $v : \{x, y\} \longrightarrow \mathbb{R}$ by $v(x) = 1$ and $v(y) = 0$. For all $j \in \mathcal{J}$, suppose $\trianglerighteq_j$ has a Bewley representation $(u_j, \mathcal{P}_j)$ for some $u_j : \mathcal{X} \longrightarrow \mathbb{R}$. By applying positive affine transformations to $\{u_j\}_{j\in\mathcal{J}}$ if necessary, we can assume without loss of generality that u_j agrees with v on $\{x, y\}$, for all $j \in \mathcal{J}$.

Let $\alpha, \beta \in \mathcal{A}$ be congruent dichotomous acts ranging over $\{x, y\}$. Then $u_j \circ \alpha = v \circ \alpha$ and $u_j \circ \beta = v \circ \beta$ for all $j \in \mathcal{J}$. Suppose $\alpha \trianglerighteq_i \beta$ for all $i \in \mathcal{I}$. Then for all $i \in \mathcal{I}$, we have $\int_{\mathcal{S}} u_i \circ \alpha \, \mathrm{d}\rho \geq \int_{\mathcal{S}} u_i \circ \beta \, \mathrm{d}\rho$ for all $\rho \in \mathcal{P}_i$. Using the above identities, we can rewrite this $\int_{\mathcal{S}} v \circ \alpha \, \mathrm{d}\rho \geq \int_{\mathcal{S}} v \circ \beta \, \mathrm{d}\rho$ for all $\rho \in \mathcal{P}_i$ and all $i \in \mathcal{I}$. Convex combinations of probability measures preserve weak inequalities of expected values, so this inequality also holds for all ρ in the convex hull of $\bigcup_{i\in\mathcal{I}} \mathcal{P}_i$. Furthermore, $v \circ \alpha$ and $v \circ \beta$ are simple functions, and $\mathfrak{T}$-limits preserve weak inequalities of expected values for simple functions (because $\mathfrak{T}$ is the weak topology generated by simple functions). Thus, we deduce that $\int_{\mathcal{S}} v \circ \alpha \, \mathrm{d}\rho \geq \int_{\mathcal{S}} v \circ \beta \, \mathrm{d}\rho$ for all $\rho \in \overline{\mathcal{P}}$. Since $\mathcal{P}_o \subseteq \overline{\mathcal{P}}$, this implies that $\int_{\mathcal{S}} v \circ \alpha \, \mathrm{d}\rho \geq \int_{\mathcal{S}} v \circ \beta \, \mathrm{d}\rho$ for all $\rho \in \mathcal{P}_o$. In other words, $\int_{\mathcal{S}} u_o \circ \alpha \, \mathrm{d}\rho \geq \int_{\mathcal{S}} u_o \circ \beta \, \mathrm{d}\rho$ for all $\rho \in \mathcal{P}_o$. Thus, $\alpha \trianglerighteq_o \beta$, as desired.[25]

[25]This proof does not use consilience. So, in fact, it works for any $\mathcal{R} \subseteq \Delta(\mathcal{S})$.

"(a) $\Longrightarrow$ (c)" (by contrapositive) Suppose $\mathcal{P}_o \not\subseteq \overline{\mathcal{P}}$. Let $\mathcal{P}_*$ be a nonempty, norm compact, convex subset of $\mathcal{P}_o$ that is disjoint from $\overline{\mathcal{P}}$. (For example, let $\mathcal{P}_* := \{\rho_o\}$, for any $\rho_o \in \mathcal{P}_o \setminus \overline{\mathcal{P}}$.) Then $\mathcal{P}_*$ is also $\mathfrak{T}$-compact, as explained above. In the $\mathfrak{T}$ topology, $\mathrm{ba}(\mathcal{S})$ is a locally convex topological vector space, and $\mathcal{P}_*$ and $\overline{\mathcal{P}}$ are disjoint, closed, convex subsets, one of which is compact. So, the strong separating hyperplane theorem (Aliprantis and Border, Theorem 5.79, p. 207) yields a $\mathfrak{T}$-continuous linear functional $\varphi : \mathrm{ba}(\mathcal{S}) \longrightarrow \mathbb{R}$ and $r_1 < r_2 \in \mathbb{R}$ such that

$$\varphi(\mu) < r_1 < r_2 < \varphi(\rho), \quad \text{for all } \mu \in \mathcal{P}_* \text{ and } \rho \in \overline{\mathcal{P}}. \tag{C4}$$

Let $r := (r_1 + r_2)/2$ and let $\epsilon := (r_1 - r_2)/6$; then $r_1 = r - 3\epsilon$ and $r_2 = r + 3\epsilon$. Consider the $\mathfrak{T}$-continuous linear functional $\eta_{\mathcal{S}} : \mathrm{ba}(\mathcal{S}) \longrightarrow \mathbb{R}$ defined by $\eta_{\mathcal{S}}(\mu) := \mu[\mathcal{S}]$ for all $\mu \in \mathrm{ba}(\mathcal{S})$. Let $\varphi' := \varphi - r \cdot \eta_{\mathcal{S}}$. Then φ' is also a $\mathfrak{T}$-continuous linear functional, and inequality (C4) yields:

$$\varphi'(\mu) < -3\epsilon < 0 < 3\epsilon < \varphi'(\rho), \quad \text{for all } \mu \in \mathcal{P}_* \text{ and } \rho \in \overline{\mathcal{P}}. \tag{C5}$$

Any $\mathfrak{T}$-linear functional on $\mathrm{ba}(\mathcal{S})$ has the form ϕ^* for some simple function $\phi : \mathcal{S} \longrightarrow \mathbb{R}$, because $\mathfrak{T}$ is the weak topology on $\mathrm{ba}(\mathcal{S})$ generated by the vector space of simple functions (Aliprantis and Border (2006), Theorem 5.93, p. 212). Thus, $\varphi' = \sum_{\ell=1}^{L} r_\ell \eta_{\mathcal{G}_\ell}$ for some disjoint measurable subsets $\mathcal{G}_1, \ldots, \mathcal{G}_L \subseteq \mathcal{S}$ and some $r_1, \ldots, r_L \in \mathbb{R}$. By rearranging $\mathcal{G}_1, \ldots, \mathcal{G}_L$ if necessary, we can assume that $r_1, \ldots, r_J < 0$ and $r_{J+1}, \ldots, r_L > 0$ for some $J \in \mathbb{N}$. Let $\varphi_- := -\sum_{j=1}^{J} r_j \eta_{\mathcal{G}_j}$ and $\varphi_+ := \sum_{\ell=J+1}^{L} r_\ell \eta_{\mathcal{G}_\ell}$. Then $\varphi' = \varphi_+ - \varphi_-$, so we can rewrite inequality (C5) as

$$\varphi_+(\mu) - \varphi_-(\mu) < -3\epsilon < 0 < 3\epsilon < \varphi_+(\rho) - \varphi_-(\rho) \quad \text{for all } \mu \in \mathcal{P}_* \text{ and } \rho \in \overline{\mathcal{P}}.$$

In other words,

$$\varphi_+(\mu) < \varphi_-(\mu) - 3\epsilon \quad \text{for all } \mu \in \mathcal{P}_*, \text{ whereas } \varphi_+(\rho) > \varphi_-(\rho) + 3\epsilon \text{ for all } \rho \in \overline{\mathcal{P}}. \tag{C6}$$

Now, let $\{x, y\}$ be a dichotomy, and assume without loss of generality that $u_j(x) = 1$ and $u_j(y) = 0$ for all $j \in \mathcal{J}$, as in the proof of "(c) $\Longrightarrow$ (b)." Lemma C.2 yields piecewise $\mathcal{R}$-almost-objective dichotomous acts $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^\infty$ and $\boldsymbol{\beta} = (\beta^n)_{n=1}^\infty$ such that for all $j \in \mathcal{J}$, and all $\rho_j \in \mathcal{P}_j$,

$$\lim_{n\to\infty} \int_{\mathcal{S}} u_j \circ \alpha^n \, \mathrm{d}\rho_j = \varphi_+(\rho_j) \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathcal{S}} u_j \circ \beta^n \, \mathrm{d}\rho_j = \varphi_-(\rho_j). \tag{C7}$$

Now, let $i \in \mathcal{I}$. Since $\mathcal{P}_i$ is compact in the total variation norm, there is a finite subset $\{\lambda_i^\ell\}_{\ell=1}^{L_i} \subset \mathcal{P}_i$ that is ϵ-*dense* in $\mathcal{P}_i$, in the sense that for any $\rho \in \mathcal{P}_i$, we have $\|\rho - \lambda_i^\ell\|_{\mathrm{var}} < \epsilon$ for some $\ell \in [1 \ldots L_i]$. For all $\ell \in [1 \ldots L_i]$, the right inequality in statement (C6) applies to λ_i^ℓ, because $\mathcal{P}_i \subseteq \overline{\mathcal{P}}$. Combining this inequality with the limit equations (C7) yield some $N_i^\ell \in \mathbb{N}$ such that

$$\int_{\mathcal{S}} u_i \circ \alpha^n \, \mathrm{d}\lambda_i^\ell > 2\epsilon + \int_{\mathcal{S}} u_i \circ \beta^n \, \mathrm{d}\lambda_i^\ell, \quad \text{for all } n \geq N_i^\ell. \tag{C8}$$

Let $N_i := \max\{N_i^\ell\}_{\ell=1}^{L_i}$. For any $n \geq N_i$, the inequality (C8) holds for all $\ell \in [1 \ldots L_i]$. Now, let $\rho \in \mathcal{P}_i$ be arbitrary. By construction, there is some $\ell \in [1 \ldots L_i]$ such that $\|\rho - \lambda_i^\ell\|_{\mathrm{var}} <$

ϵ. Thus, for any $n \geq N_i$,

$$\left|\int_{\mathcal{S}} u_i \circ \alpha^n \, \mathrm{d}\rho - \int_{\mathcal{S}} u_i \circ \alpha^n \, \mathrm{d}\lambda_i^\ell\right| \leq \|u_i \circ \alpha_n\|_\infty \cdot \left\|\rho - \lambda_i^\ell\right\|_{\text{var}} < \epsilon,$$

$$\text{and likewise,} \quad \left|\int_{\mathcal{S}} u_i \circ \beta^n \, \mathrm{d}\rho - \int_{\mathcal{S}} u_i \circ \beta^n \, \mathrm{d}\lambda_i^\ell\right| < \epsilon, \tag{C9}$$

where we use the fact that $\|u_i \circ \alpha_n\|_\infty = \|u_i \circ \beta_n\|_\infty = 1$ because $\alpha_n(\mathcal{S}) = \beta_n(\mathcal{S}) = \{x, y\}$ and $u_i(\{x, y\}) = \{0, 1\}$. Combining inequalities (C8) and (C9), we get

$$\int_{\mathcal{S}} u_i \circ \alpha^n \, \mathrm{d}\rho > \int_{\mathcal{S}} u_i \circ \beta^n \, \mathrm{d}\rho, \quad \text{for all } \rho \in \mathcal{P}_i, \tag{C10}$$

and thus $\alpha^n \rhd_i \beta^n$. This holds for all $n \geq N_i$, so $\boldsymbol{\alpha} \rhd_i^\omega \boldsymbol{\beta}$. This holds for all $i \in \mathcal{I}$.

Now, let $\rho_o \in \mathcal{P}_*$ be arbitrary. The limit equations (C7) and the left inequality in statement (C6) yield some $N \in \mathbb{N}$ such that

$$\int_{\mathcal{S}} u_o \circ \alpha^n \, \mathrm{d}\rho_o < \int_{\mathcal{S}} u_o \circ \beta^n \, \mathrm{d}\rho_o \quad \text{for all } \rho_o \in \mathcal{P}_* \text{ and } n \geq N. \tag{C11}$$

Since $\mathcal{P}_* \subseteq \mathcal{P}_o$, this means it is impossible that $\alpha_n \trianglerighteq \beta_n$. This holds for all $n \geq N$; thus, it is not the case that $\boldsymbol{\alpha} \trianglerighteq_o^\omega \boldsymbol{\beta}$. This contradicts $\mathcal{R}$-Dichotomous Piecewise Almost-Objective Pareto. □

Appendix D: Further examples of consilience

Although the scopes of Propositions 1 and 2 are already very broad, there are many other examples of consilient collections of measures. To illustrate this, let $\widehat{\mathcal{S}}$ and $\mathcal{S}$ be two measurable spaces, and let $\phi : \widehat{\mathcal{S}} \longrightarrow \mathcal{S}$ be any measurable function. This induces a function $\phi_* : \text{ba}(\widehat{\mathcal{S}}) \longrightarrow \text{ba}(\mathcal{S})$ where, for any $\widehat{\mu} \in \text{ba}(\widehat{\mathcal{S}})$ and any measurable $\mathcal{B} \subseteq \mathcal{S}$, we define $\phi_*(\widehat{\mu})[\mathcal{B}] := \widehat{\mu}[\phi^{-1}(\mathcal{B})]$.

Proposition D.1. *Let $\widehat{\mathcal{S}}$ and $\mathcal{S}$ be measurable spaces, and let $\phi : \widehat{\mathcal{S}} \longrightarrow \mathcal{S}$ be measurable. Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$, and let $\widehat{\mathcal{R}} := (\phi_*)^{-1}(\mathcal{R}) \subseteq \Delta(\widehat{\mathcal{S}})$. If $\mathcal{R}$ is (strongly) consilient, then $\widehat{\mathcal{R}}$ is (strongly) consilient.*

Proof. Let $K \in \mathbb{N}$ and let $\mathbf{q} \in \Delta^K$. By hypothesis, there is an $\mathcal{R}$-almost-objectively uncertain sequence of partitions $(\mathfrak{G}^n)_{n=1}^\infty$ of $\mathcal{S}$ that is subordinate to $\mathbf{q}$. For all $n \in \mathbb{N}$, suppose $\mathfrak{G}^n = \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$. For all $k \in [1 \ldots K]$, let $\widehat{\mathcal{G}}_1^n := \phi^{-1}(\mathcal{G}_1^n)$. Then $\widehat{\mathfrak{G}}^n := \{\widehat{\mathcal{G}}_1^n, \ldots, \widehat{\mathcal{G}}_K^n\}$ is a measurable partition of $\widehat{\mathcal{S}}$ (because ϕ is measurable). This yields a partition sequence $(\widehat{\mathfrak{G}}^n)_{n=1}^\infty$ of $\widehat{\mathcal{S}}$. We will show that it is $\widehat{\mathcal{R}}$-almost-objectively uncertain and subordinate to $\mathbf{q}$.

To see this, let $\widehat{\rho} \in \widehat{\mathcal{R}}$. Let $\rho := \phi_*(\widehat{\rho})$. Then $\rho \in \mathcal{R}$. For all $k \in [1 \ldots K]$, we have $\widehat{\rho}(\widehat{\mathcal{G}}_k^n) = \rho(\mathcal{G}_k^n)$ for all $n \in \mathbb{N}$, so $\lim_{n\to\infty} \widehat{\rho}(\widehat{\mathcal{G}}_k^n) = \lim_{n\to\infty} \rho(\mathcal{G}_k^n) = q_k$, as desired.

To prove the claim for *strong* consilience, it suffices to show that $\langle\widehat{\mathcal{R}}\rangle \subseteq (\phi_*)^{-1}(\langle\mathcal{R}\rangle)$. To see this, let $\widehat{\mu} \in \langle\widehat{\mathcal{R}}\rangle$. Then there exists $\widehat{\rho} \in \widehat{\mathcal{R}}$ such that $\widehat{\mu} \lll \widehat{\rho}$ and such that $\widehat{\psi} := \frac{\mathrm{d}\widehat{\mu}}{\mathrm{d}\widehat{\rho}}$ is bounded. Let $\mu := \phi_*(\widehat{\mu})$ and $\rho := \phi_*(\widehat{\rho})$. Then $\mu \lll \rho$ and $\rho \in \mathcal{R}$. Furthermore, if $\psi := \frac{\mathrm{d}\mu}{\mathrm{d}\rho}$, then $\psi \circ \phi = \widehat{\psi}$. Thus, ψ is also bounded. Thus, $\mu \in \langle\mathcal{R}\rangle$. Thus, $\widehat{\mu} \in (\phi_*)^{-1}(\langle\mathcal{R}\rangle)$. □

Dynamical systems are mathematical models of systems evolving deterministically in time. They arise in the study of ordinary differential equations, difference equations, and all parts of applied mathematics. Formally, a *(measurable) dynamical system* is a pair $(\mathcal{S}, \phi)$, where $\mathcal{S}$ is a measurable space and $\phi : \mathcal{S} \longrightarrow \mathcal{S}$ is a measurable function. A (countably additive) probability measure μ on $\mathcal{S}$ is ϕ*-invariant* if $\phi_*(\mu) = \mu$. The triple $(\mathcal{S}, \mu, \phi)$ is called a *measure-preserving dynamical system* (MPDS). A wide variety of dynamical systems admit invariant measures, and hence can be treated as MPDS. For example, if $\mathcal{S}$ is a compact metric space and $\phi : \mathcal{S} \longrightarrow \mathcal{S}$ is continuous, then the Krylov–Bogolyubov theorem yields a ϕ-invariant measure (Walters (1982), Section 6.2).

An MPDS $(\mathcal{S}, \mu, \phi)$ is *mixing* if, for all measurable subsets $\mathcal{B}, \mathcal{C} \subseteq \mathcal{S}$, we have $\lim_{t\to\infty} \mu[\mathcal{B} \cap \phi^{-t}(\mathcal{C})] = \mu(\mathcal{B}) \cdot \mu(\mathcal{C})$. Many MPDS are mixing—in particular, ones which exhibit so-called "chaotic" behavior. For example, let $\mathcal{S} = [0, 1]$. The *tent map* $\phi : [0, 1] \longrightarrow [0, 1]$ is defined

$$\phi(s) = \begin{cases} 2s & \text{if } s \le \dfrac{1}{2}; \\ 1 - 2s & \text{if } s > \dfrac{1}{2}. \end{cases}$$

The Lebesgue measure on $[0, 1]$ is ϕ-invariant, and the resulting MPDS is mixing.

Proposition D.2. *Let $(\mathcal{S}, \mu, \phi)$ be any mixing MPDS, where μ is countably additive. Let $\mathcal{R} := \{\rho \in \Delta_\sigma(\mathcal{S}); \rho \ll \mu$ and $\frac{\mathrm{d}\rho}{\mathrm{d}\mu} \in \mathcal{L}^2(\mathcal{S}, \mu)\}$. Then $\mathcal{R}$ is strongly consilient.*

This result addresses a possible concern about Propositions 1 and 2. Whereas the almost-objectively uncertain partition sequences constructed in Propositions 1 and 2 might seem somewhat exotic, the sequences constructed in Proposition D.2 are extremely natural: they take a single partition of $\mathcal{S}$ and shift it into the far future via ϕ. Many standard examples of "effectively random" questions have this form, such as "What will the temperature in Times Square be at 12:00 PM on April 1, 2062?"[26] It is not implausible that such questions could arise in collective decisions. This provides further motivation for the Almost-Objective Pareto axiom of Section 4.

Proof of Proposition D.2. If $(\mathcal{S}, \mu, \phi)$ is mixing, then it is ergodic, and hence μ is nonatomic. Let $\mathfrak{G} = (\mathcal{G}_1, \dots, \mathcal{G}_K)$ be a measurable partition such that $\mu[\mathcal{G}_k] = q_k$ for all $k \in [1 \dots K]$; this exists because μ is nonatomic. Now, for all $n \in \mathbb{N}$, let $\mathfrak{G}^n := (\mathcal{G}_1^n, \dots, \mathcal{G}_K^n)$, where $\mathcal{G}_k^n := \phi^{-n}(\mathcal{G}_k)$ for all $k \in [1 \dots K]$. We shall show that the sequence $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{R}$-almost-objectively uncertain and subordinate to $\mathbf{q}$.

Let $\rho \in \mathcal{R}$; then $\rho \ll \mu$. Let $\psi := \frac{\mathrm{d}\rho}{\mathrm{d}\mu}$, then $\psi \in \mathcal{L}^2(\mathcal{S}, \mu)$ by hypothesis. For any measurable $\mathcal{G} \subseteq \mathcal{S}$, let $\mathbf{1}_\mathcal{G}$ be its indicator function. Then $\mathbf{1}_\mathcal{G} \in \mathcal{L}^2(\mathcal{S}, \mu)$, and

$$\begin{aligned} \lim_{n\to\infty} \int_\mathcal{G} \psi \circ \phi^n \, \mathrm{d}\mu &= \lim_{n\to\infty} \langle \mathbf{1}_\mathcal{G}, \psi \circ \phi^n \rangle \qquad \text{(D1)} \\ &\underset{(*)}{=} \int_\mathcal{S} \mathbf{1}_\mathcal{G} \, \mathrm{d}\mu \cdot \int_\mathcal{S} \psi \, \mathrm{d}\mu = \mu[\mathcal{G}] \cdot \rho[\mathcal{S}] = \mu[\mathcal{G}], \end{aligned}$$

[26]Here, we assume that global weather patterns can be described as a chaotic dynamical system.

where $(*)$ is a standard property of mixing MPDS (Walters (1982, Theorem 1.23(iii.2) on p. 45 of Section 1.7); Fremlin (2006b, Proposition 372Q(iv), p. 195)). By applying change of variables, (D1) becomes

$$\lim_{n\to\infty}\int_{\phi^{-n}(\mathcal{G})}\psi\,\mathrm{d}\mu=\mu[\mathcal{G}]. \tag{D2}$$

In particular, we can apply (D2) to all $\mathcal{G}_k$ for all $k\in[1\dots K]$ to conclude that

$$\lim_{n\to\infty}\rho\big[\mathcal{G}_k^n\big]=\lim_{n\to\infty}\int_{\mathcal{G}_k^n}\psi\,\mathrm{d}\mu=\lim_{n\to\infty}\int_{\phi^{-n}(\mathcal{G}_k)}\psi\,\mathrm{d}\mu\underset{(*)}{=}\mu[\mathcal{G}_k]=q_k,$$

as desired. Here, $(*)$ is by (D2).

This proves that $\mathcal{R}$ is consilient. It is *strongly* consilient because $\langle\mathcal{R}\rangle=\mathcal{R}$. To see this, suppose $\nu\in\langle\mathcal{R}\rangle$. Then $\nu\ll\rho$ for some $\rho\in\mathcal{R}$, and $\phi:=\frac{\mathrm{d}\nu}{\mathrm{d}\rho}$ is bounded. By the definition of $\mathcal{R}$, $\rho\ll\mu$ and $\psi:=\frac{\mathrm{d}\rho}{\mathrm{d}\mu}\in\mathcal{L}^2(\mathcal{S},\mu)$. Thus, $\nu\ll\mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}=\phi\cdot\psi$ are also in $\mathcal{L}^2(\mathcal{S},\mu)$ (because $\|\phi\cdot\psi\|_2\leq\|\phi\|_\infty\cdot\|\psi\|_2$). Thus, $\nu\in\mathcal{R}$. □

Appendix E: Uniqueness of GH representations[27]

A preference order $\succcurlyeq$ might have many GH representations. How much do they have in common? First, note that if $V:\mathcal{A}\longrightarrow\mathbb{R}$ is a GH representation for $\succcurlyeq$, then the utility function u in expression (3) is entirely determined by V: for any $x\in\mathcal{X}$, we have $u(x)=V(\kappa_x)$, where $\kappa_x\in\mathcal{A}$ is just the constant act with value x. Conversely, suppose that $\succcurlyeq$ satisfies the following mild condition:

Certainty Equivalents For any $\alpha\in\mathcal{A}$, there is some $x\in\mathcal{X}$ such that $\alpha\approx\kappa_x$.

(For example, if $\mathcal{X}$ is connected and $u:\mathcal{X}\longrightarrow\mathbb{R}$ is continuous, then $\succcurlyeq$ satisfies Certainty Equivalents.) In this case, V is also entirely determined by u, because for any $\alpha\in\mathcal{A}$ we have $V(\alpha)=u(x)$, where $x\in\mathcal{X}$ is any outcome such that $\alpha\approx\kappa_x$. Thus, for preferences satisfying Certainty Equivalents, V and u codetermine each other.

Unfortunately, the set $\mathcal{P}$ in a GH representation is far from unique. Indeed, let $\mathfrak{P}$ be the set of all *belief sets* for V, that is, all subsets $\mathcal{P}\subseteq\Delta(\mathcal{S})$ that satisfy (3). This set is closed under upwards inclusion: if $\mathcal{P}\subseteq\mathcal{P}'\subseteq\Delta(\mathcal{S})$ and $\mathcal{P}\in\mathfrak{P}$, then $\mathcal{P}'\in\mathfrak{P}$ also. But the next lemma allows us to isolate some "natural" elements of $\mathfrak{P}$.

Lemma E.1. *If GH is compact, then $\mathfrak{P}$ has inclusion-minimal elements.*

Proof. Recall that a *chain* in $\mathfrak{P}$ is a subset $\mathfrak{Q}\subseteq\mathfrak{P}$ that is linearly ordered under inclusion. We first show that every chain has a lower bound in $\mathfrak{P}$. Let $\mathfrak{Q}\subseteq\mathfrak{P}$ be a chain, and let $\mathcal{Q}^*:=\bigcap_{\mathcal{Q}\in\mathfrak{Q}}\mathcal{Q}$. Then $\mathcal{Q}^*\in\mathfrak{P}$ also. To see this, let $\alpha\in\mathcal{A}$. For all $\mathcal{Q}\in\mathfrak{Q}$, the inequalities (3) imply that there exist $\underline{\rho}_{\mathcal{Q}},\overline{\rho}_{\mathcal{Q}}\in\mathcal{Q}$ such that

$$\int_{\mathcal{S}}u\circ\alpha\,\mathrm{d}\underline{\rho}_{\mathcal{Q}}\leq V(\alpha)\leq\int_{\mathcal{S}}u\circ\alpha\,\mathrm{d}\overline{\rho}_{\mathcal{Q}}. \tag{E1}$$

[27]The proofs in this Appendix rely on Propositions 1 and 3, but are independent of the other results in the paper

If we order $\mathfrak{Q}$ by reverse inclusion (so that $\mathcal{Q} < \mathcal{Q}'$ if $\mathcal{Q} \supset \mathcal{Q}'$), then $\mathfrak{Q}$ is a directed set, and $\{\underline{\rho}_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathfrak{Q}}$ and $\{\overline{\rho}_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathfrak{Q}}$ are nets in $\Delta(\mathcal{S})$. For any $\mathcal{Q}_0 \in \mathfrak{Q}$, we have $\underline{\rho}_{\mathcal{Q}}, \overline{\rho}_{\mathcal{Q}} \in \mathcal{Q}_0$ for all $\mathcal{Q} \in \mathfrak{Q}$ with $\mathcal{Q} \subseteq \mathcal{Q}_0$. Thus, the tails of these nets are contained in $\mathcal{Q}_0$, which is compact. Thus, they have convergent subnets. Let $\underline{\rho}$ be a limit of a subnet of $\{\underline{\rho}_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathfrak{Q}}$, and let $\overline{\rho}$ be a limit of a subnet of $\{\overline{\rho}_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathfrak{Q}}$. Then the inequalities (E1) imply that

$$\int_{\mathcal{S}} u \circ \alpha \,\mathrm{d}\,\mathrm{d}\underline{\rho} \leq V(\alpha) \leq \int_{\mathcal{S}} u \circ \alpha \,\mathrm{d}\overline{\rho}, \tag{E2}$$

because the functional $\Delta(\mathcal{S}) \ni \rho \mapsto \int_{\mathcal{S}} u \circ \alpha \,\mathrm{d}\rho \in \mathbb{R}$ is continuous.

It remains to show that $\underline{\rho}, \overline{\rho} \in \mathcal{Q}^*$. To see this, note that for any $\mathcal{Q}' \in \mathfrak{Q}$, the tails of $\{\underline{\rho}_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathfrak{Q}}$ and $\{\overline{\rho}_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathfrak{Q}}$ are contained in $\mathcal{Q}'$. So, any limit points of subnets of $\{\underline{\rho}_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathfrak{Q}}$ and $\{\overline{\rho}_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathfrak{Q}}$ must also be contained in $\mathcal{Q}'$ (because $\mathcal{Q}'$ is closed). Thus, $\underline{\rho}, \overline{\rho} \in \mathcal{Q}'$. We conclude that $\underline{\rho}, \overline{\rho} \in \mathcal{Q}'$ for all $\mathcal{Q}' \in \mathfrak{Q}$, and thus $\underline{\rho}, \overline{\rho} \in \mathcal{Q}^*$.

Inequality (E2) implies that inequality (3) holds for $\mathcal{Q}^*$ and α. This argument works for all $\alpha \in \mathcal{A}$. Thus, $\mathcal{Q}^* \in \mathfrak{P}$.

Thus, any chain in $\mathfrak{P}$ has a lower bound in $\mathfrak{P}$. Thus, Zorn's lemma implies that $\mathfrak{P}$ contains inclusion-minimal elements. □

Unfortunately, Lemma E.1 does not say that $\mathfrak{P}$ has a *unique* inclusion-minimal element; it may have more than one. But this question of uniqueness is beyond the scope of this paper. Furthermore, an inclusion-minimal element of $\mathfrak{P}$ might not be the most natural choice for other purposes. For example, as explained in Section 5, under certain conditions, there is a unique weak* compact, convex set $\mathcal{P} \subseteq \Delta(\mathcal{S})$ and a utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$ that yield both a generalized Hurwicz representation (3) for $\succcurlyeq$, and a Bewley representation (8) for the unambiguous part of $\succcurlyeq$ (Ghirardato, Maccheroni, and Marinacci (2004), Propositions 5 and 7; Cerreia-Vioglio et al. (2011), Proposition 5). But this $\mathcal{P}$ is not necessarily minimal in $\mathfrak{P}$.

Nevertheless, whether we wish to work with an inclusion-minimal belief set in $\mathfrak{P}$, or with the unique belief set in $\mathfrak{P}$ that is suitable for the Bewley representation of the unambiguous part of $\succcurlyeq$, this discussion shows that there are a relatively small number of "natural" belief sets for V. And we have already seen that the utility function u is determined by V. Could not $\succcurlyeq$ have two *different* representations V_1 and V_2, described by two different utility functions u_1 and u_2 and two different collections of minimal belief sets $\mathfrak{P}_1$ and $\mathfrak{P}_2$? The next result addresses this question.

Proposition E.2. *Suppose $\succcurlyeq$ satisfies* Certainty Equivalents. *If V_1 and V_2 are compact, nonatomic GH representations for $\succcurlyeq$, then they have the same inclusion-minimal belief sets, and there are constants $a > 0$ and $b \in \mathbb{R}$ such that $V_1 = aV_2 + b$.*

This is actually a consequence of a more general result.

Proposition E.3. *Let $\mathcal{R} \subseteq \Delta(\mathcal{S})$ be consilient. Let $\succcurlyeq$ be a preference order on $\mathcal{A}$, and let $V_1, V_2 : \mathcal{A} \longrightarrow \mathbb{R}$ be two compact GH representations of $\succcurlyeq$ with utility functions $u_1, u_2 : \mathcal{X} \longrightarrow \mathbb{R}$ and belief sets $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{R}$. Then:*

(a) *There exist constants $a > 0$ and $b \in \mathbb{R}$ such that $u_1 = au_2 + b$.*

(b) *If $\succcurlyeq$ satisfies* Certainty Equivalents, *then also $V_1 = aV_2 + b$.*

(c) *If $\mathfrak{P}_1$ is the set of all belief sets for V_1, and $\mathfrak{P}_2$ is the set of all belief sets for V_2, then $\mathfrak{P}_1 = \mathfrak{P}_2$. Hence, they have the same inclusion-minimal elements.*

Proof. Part (c) follows from (a) and (b). To prove part (a), recall that for all $\alpha \in \mathcal{A}$,

$$\inf_{\rho\in\mathcal{P}_1} \int_{\mathcal{S}} u_1 \circ \alpha \, \mathrm{d}\rho \leq V_1(\alpha) \leq \sup_{\rho\in\mathcal{P}_1} \int_{\mathcal{S}} u_1 \circ \alpha \, \mathrm{d}\rho, \quad \text{and} \tag{E3}$$

$$\inf_{\rho\in\mathcal{P}_2} \int_{\mathcal{S}} u_2 \circ \alpha \, \mathrm{d}\rho \leq V_2(\alpha) \leq \sup_{\rho\in\mathcal{P}_2} \int_{\mathcal{S}} u_2 \circ \alpha \, \mathrm{d}\rho. \tag{E4}$$

Let $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$ and $\boldsymbol{\beta} = (\beta^n)_{n=1}^{\infty}$ be compatible $\mathcal{R}$-almost-objective acts, and suppose that $\boldsymbol{\alpha} \succ^{\infty} \boldsymbol{\beta}$. Then Lemma B.2 yields $\epsilon_1, \epsilon_2 > 0$ such that for all sufficiently large $n \in \mathbb{N}$, we have $V_1(\alpha^n) > V_1(\beta^n) + \epsilon_1$ and $V_2(\alpha^n) > V_2(\beta^n) + \epsilon_2$.

Suppose $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are measurable with respect to the almost-objectively uncertain partition sequence $\boldsymbol{\mathcal{G}} = (\mathfrak{G}^n)_{n=1}^{\infty}$, where $\mathfrak{G}^n = \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$ for all $n \in \mathbb{N}$, and suppose $\boldsymbol{\mathcal{G}}$ is subordinate to the probability vector $\mathbf{q} = (q_1, \ldots, q_K)$. Suppose $\boldsymbol{\alpha}$ is subordinate to the K-tuple $(x_1, \ldots, x_K) \in \mathcal{X}^K$, while $\boldsymbol{\beta}$ is subordinate to the K-tuple $(y_1, \ldots, y_K)$. Then Proposition 3 says that $\lim_{n\to\infty} V_1(\beta^n) = \sum_{k=1}^K q_k u_1(y_k)$ and $\lim_{n\to\infty} V_2(\beta^n) = \sum_{k=1}^K q_k u_2(y_k)$.

Thus, since $V_1(\alpha^n) > V_1(\beta^n) + \epsilon_1$ and also $V_2(\alpha^n) > V_2(\beta^n) + \epsilon_2$ for all sufficiently large $n \in \mathbb{N}$, we conclude that

$$\sum_{k=1}^K q_k u_1(x_k) \geq \sum_{k=1}^K q_k u_1(y_k) + \epsilon_1 \quad \text{and} \quad \sum_{k=1}^K q_k u_2(x_k) \geq \sum_{k=1}^K q_k u_2(y_k) + \epsilon_2. \tag{E5}$$

Now, by a suitable choice of almost-objective acts $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we can achieve versions of (E5) for any $\epsilon_1, \epsilon_2 > 0$ and $K \in \mathbb{N}$, any probability vector $\mathbf{q} \in \Delta^K$, and any K-tuples of outcomes $(x_1, \ldots, x_K)$ and $(y_1, \ldots, y_K)$. We conclude that for all $K \in \mathbb{N}$, all $\mathbf{q} \in \Delta^K$, and all $(x_1, \ldots, x_K)$ and $(y_1, \ldots, y_K)$ in $\mathcal{X}^K$,

$$\left(\sum_{k=1}^K q_k u_1(x_k) > \sum_{k=1}^K q_k u_1(y_k)\right) \iff \left(\sum_{k=1}^K q_k u_2(x_k) > \sum_{k=1}^K q_k u_2(y_k)\right). \tag{E6}$$

By standard uniqueness theorems for SEU representations, it follows from (E6) that u_1 is a positive affine transformation of u_2—in other words, there exist $a > 0$ and $b \in \mathbb{R}$ such that $u_1 = au_2 + b$. This proves (a).

To prove (b), suppose that $\succcurlyeq$ satisfies Certainty Equivalents. Let $\mathcal{V} := V_2(\mathcal{A}) \subseteq \mathbb{R}$. V_1 and V_2 both represent $\succcurlyeq$, so there is an increasing function $\phi : \mathcal{V} \longrightarrow \mathbb{R}$ such that $V_1 = \phi \circ V_2$. We must show that $\phi(x) = av + b$ for all $v \in \mathcal{V}$.

For any $v \in \mathcal{V}$, there is some $\alpha \in \mathcal{A}$ such that $v = V_2(\alpha)$. By Certainty Equivalents, there is some constant act κ such that $\alpha \approx \kappa$. Thus, $V_2(\kappa) = V_2(\alpha)$. If κ has the constant value x, then the inequalities (E4) force $V_2(\kappa) = u_2(x)$. Thus, $u_2(x) = v$.

By a similar argument, $V_1(\alpha) = u_1(x) = au_2(x) + b = av + b$. But we also have $\phi \circ V_2(\alpha) = V_1(\alpha)$. Thus, we get $\phi(v) = av + b$, as desired. This argument works for any $v \in \mathcal{V}$. We conclude that $V_1 = aV_2 + b$. □

Proof of Proposition E.2. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any compact, nonatomic belief sets for the GH representations V_1 and V_2. Let $\mathcal{R} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{R}$ is nonatomic and separable, so Proposition 1 says that $\mathcal{R}$ is consilient. Thus, Proposition E.3 says that $\mathfrak{P}_1 = \mathfrak{P}_2$ and $V' = aV + b$ (because $\succcurlyeq$ satisfies Certainty Equivalents). □

Since Propositions E.2 and E.3 rely on consilience, they require nonatomic beliefs.

Appendix F: Proofs of other results

This Appendix contains proofs of additional statements made in the text, regarding the relationship between utilitarianism and weak utilitarianism, and observations made in Section 6. These proofs are logically independent from the rest of the paper.

To explain the logical relationship between utilitarianism and weak utilitarianism, we need two hypotheses: *Ex post Pareto* and *Independent Prospects*. The social preference $\succcurlyeq_o$ satisfies the *Ex post Pareto* axiom with respect to $\{\succcurlyeq_i\}_{i \in \mathcal{I}}$ if, for any constant acts $\alpha, \beta \in \mathcal{A}$,

- If $\alpha \succcurlyeq_i \beta$ for all $i \in \mathcal{I}$, then $\alpha \succcurlyeq_o \beta$.
- If, in addition, $\alpha \succ_i \beta$ for some $i \in \mathcal{I}$, then $\alpha \succ_o \beta$.

Now, suppose that each of the preference orders $\{\succcurlyeq_j\}_{j \in \mathcal{J}}$ has a GH representation (3) with an associated utility function $u_j : \mathcal{X} \longrightarrow \mathbb{R}$. We shall say that the collection $\{u_i\}_{i \in \mathcal{I}}$ satisfies *Independent Prospects* if, for all $j \in \mathcal{J}$, there exist outcomes $x, y \in \mathcal{X}$ such that $u_j(x) > u_j(y)$ whereas $u_i(x) = u_i(y)$ for all $i \in \mathcal{I} \setminus \{j\}$.

Proposition F.1. *Suppose* $\{u_i\}_{i \in \mathcal{I}}$ *satisfy* Independent Prospects. *Then* u_o *is utilitarian if and only if it is weakly utilitarian and* $\succcurlyeq_o$ *satisfies* Ex post Pareto *for* $\{\succcurlyeq_i\}_{i \in \mathcal{I}}$.

Proof. By definition, if u_o is utilitarian, then it is weakly utilitarian. We will just show that Ex post Pareto is satisfied. Let α and β be two constant acts such that $\alpha \succcurlyeq_i \beta$ for all i. Assume that $\alpha(s) = x$ and $\beta(s) = y$ for all states $s \in \mathcal{S}$. We will have $V_i(\alpha) = u_i(x)$ and $V_i(\beta) = u_i(y)$, for all $i \in \mathcal{J}$. Thus, with $u_i(x) \geq u_i(y)$ for all $i \in \mathcal{I}$ and $u_o = b + \sum_{i \in \mathcal{I}} c_i u_i$ we have $u_o(x) \geq u_o(y)$. Furthermore, if there is $i \in \mathcal{I}$ such that $u_i(x) > u_i(y)$, since $c_i > 0$, we will obviously have $u_o(x) > u_o(y)$.

Conversely, if u_o is weakly utilitarian, then for all $i \in \mathcal{I}$, there is $c_i \geq 0$ such that $u_o = b + \sum_{i \in \mathcal{I}} c_i u_i$. Let $i \in \mathcal{I}$. To show that $c_i > 0$, let $x_i, y_i \in \mathcal{X}$ such that $u_i(x_i) > u_i(y_i)$ and $u_j(x_i) = u_j(y_i)$ for $j \neq i$; this exists by the hypothesis of Independent Prospects. Considering the constant acts $\alpha_i(s) = x_i$ and $\beta_i(s) = y_i$, we have $V_j(\alpha_i) \geq V_j(\beta_i)$ for all $j \in \mathcal{I}$ and $V_i(\alpha_i) > V_i(\beta_i)$. By Ex post Pareto, we have $V_o(\alpha_i) > V_o(\beta_i)$. Thus, $u_o(x_i) - u_o(y_i) = c_i(u_i(x_i) - u_i(y_i)) > 0$. But since $(u_i(x_i) - u_i(y_i)) > 0$, we get $c_i > 0$. □

It is more efficient to prove Theorem 3 before proving Proposition 5.

Proof of Theorem 3. Repurposing the terminology of Gilboa, Samet, and Schmeidler (2004), let us say that an act is a *strong lottery* if it is measurable with respect to a strong consensus partition.[28] In this case, there is a probability vector $\mathbf{p} = (p_1, \ldots, p_M)$ and a set of outcomes $\mathcal{Y} = \{y_1, \ldots, y_M\} \subseteq \mathcal{X}$ (for some $M \in \mathbb{N}$) such that, for all $j \in \mathcal{J}$ and all $\rho \in \mathcal{P}_j$, we have $\rho\{s \in \mathcal{S}; \alpha(s) = y_m\} = p_m$ for all $m \in [1 \ldots M]$. We will indicate this by writing "$\alpha \sim (\mathbf{p}, \mathcal{Y})$."

For all $j \in \mathcal{J}$, suppose $\mathcal{P}_j$ is the convex hull of some finite collection $\mathcal{R}_j := \{\rho_j^1, \ldots, \rho_j^{N_j}\}$ of nonatomic probability measures. If $\alpha \in \mathcal{A}$ is a strong lottery and $\alpha \sim (\mathbf{p}, \mathcal{Y})$, then for all $j \in \mathcal{J}$, it is easily checked that

$$V_j(\alpha) = \int_{\mathcal{S}} u_j \circ \alpha \, \mathrm{d}\rho_j^1 = \cdots = \int_{\mathcal{S}} u_j \circ \alpha \, \mathrm{d}\rho_j^{N_j} = \sum_{m=1}^{M} p_m u_j(y_m). \tag{F1}$$

Thus, for any $\mathcal{Y} = \{y_1, \ldots, y_M\} \subseteq \mathcal{X}$, and any probability vectors $\mathbf{p} = (p_1, \ldots, p_M)$ and $\mathbf{q} = (q_1, \ldots, q_M)$, and any strong lotteries $\alpha \sim (\mathbf{p}, \mathcal{Y})$ and $\beta \sim (\mathbf{q}, \mathcal{Y})$,

$$(\alpha \succcurlyeq_i \beta \text{ for all } i \in \mathcal{I}) \quad \Longleftrightarrow \quad \left(\sum_{m=1}^{M} p_m u_i(y_m) \geq \sum_{m=1}^{M} q_m u_i(y_m) \text{ for all } i \in \mathcal{I}\right). \tag{F2}$$

Likewise

$$(\alpha \succcurlyeq_o \beta) \quad \Longleftrightarrow \quad \left(\sum_{m=1}^{M} p_m u_o(y_m) \geq \sum_{m=1}^{M} q_m u_o(y_m)\right). \tag{F3}$$

Let $\Delta(\mathcal{Y})$ be the M-dimensional simplex of all probability measures on $\mathcal{Y}$. For all $j \in \mathcal{J}$, let $u_{j|\mathcal{Y}}$ be the restriction of u_j to $\mathcal{Y}$, and let $\succcurlyeq_{j|\mathcal{Y}}$ be the vNM expected utility preferences on $\Delta(\mathcal{Y})$ induced by $u_{j|\mathcal{Y}}$. For any $\mathbf{p} \in \Delta(\mathcal{Y})$, Lemma A.1 yields a strong lottery $\alpha \sim (\mathbf{p}, \mathcal{Y})$. (This uses the fact that $\mathcal{P}_j$ is the convex hull of a finite set $\mathcal{R}_j$ of nonatomic measures, for all $j \in \mathcal{J}$.) Thus, the axiom Restricted Pareto*, combined with statements (F2) and (F3), implies that $\succcurlyeq_{o|\mathcal{Y}}$ satisfies the following Pareto axiom with respect to $\{\succcurlyeq_{i|\mathcal{Y}}\}_{i \in \mathcal{I}}$:

vNM Pareto For all $\mathbf{p}, \mathbf{q} \in \Delta(\mathcal{Y})$, if $\mathbf{p} \succcurlyeq_{i|\mathcal{Y}} \mathbf{q}$ for all $i \in \mathcal{I}$, then $\mathbf{p} \succcurlyeq_{o|\mathcal{Y}} \mathbf{q}$.

Thus, a variant of Harsanyi's social aggregation theorem implies that $u_{o|\mathcal{Y}}$ is a constant plus a nonnegative linear combination of $\{u_{j|\mathcal{Y}}\}_{i \in \mathcal{I}}$; see Domotor (1979) or Weymark (1993, 1994).[29]

[28] Gilboa, Samet, and Schmeidler use *lottery* for an act measurable with respect to a (nonstrong) consensus partition.

[29] Harsanyi's original (1955) result used the weaker axiom of *Pareto Indifference*, and concluded only that social utility is an affine combination of individual utilities, with possibly negative coefficients. But Domotor and Weymark show that vNM Pareto (which Weymark calls *Semistrong Pareto*) implies that these coefficients must be nonnegative.

Now, fix distinct $x, y \in \mathcal{X}$, and let $\mathfrak{Y}$ be the set of all finite subsets of $\mathcal{X}$ containing $\{x, y\}$. For all $\mathcal{Y} \in \mathfrak{Y}$, define

$$\mathcal{C}_{\mathcal{Y}} := \left\{(c, w_1, \ldots, w_I) \in \mathbb{R} \times \mathbb{R}_+^{\mathcal{I}};\, u_{o|\mathcal{Y}} = c + \sum_{i \in \mathcal{I}} w_i u_{i|\mathcal{Y}}\right\}.$$

By the above argument, $\mathcal{C}_{\mathcal{Y}} \neq \emptyset$, and it is clearly a convex, compact subset of $\mathbb{R} \times \mathbb{R}_+^{\mathcal{I}}$. Furthermore, if $\mathcal{Y} \subseteq \mathcal{Y}'$, then $\mathcal{C}_{\mathcal{Y}'} \subseteq \mathcal{C}_{\mathcal{Y}}$. Since $\mathfrak{Y}$ is a directed set under inclusion, it follows that the set $\mathcal{C} := \bigcap_{\mathcal{Y} \in \mathfrak{Y}} \mathcal{C}_{\mathcal{Y}}$ is not empty. Now, let $(c, w_1, \ldots, w_I) \in \mathcal{C}$. Then $u_o = c + \sum_{i \in \mathcal{I}} w_i u_i$, as claimed. □

Remark. Theorem 3 requires the GH representations of all agents to be "polytopic" because Lemma A.1 only applies to *finite*-dimensional vector-valued measures, since it relies on a version of Lyapunov's convexity theorem for $\mathbb{R}^N$-valued measures. There are versions of Lyapunov's theorem for $\mathcal{V}$-valued measures where $\mathcal{V}$ is an infinite-dimensional locally convex vector space (Khan and Sagara (2013, 2015), Greinecker and Podczeck (2013), Urbinati (2019)). These yield corresponding versions of the Dubins–Spanier theorem (by the same proof as our Lemma A.1). This yields versions of Theorem 3 for nonpolytopic GH representations. But these results impose strong "largeness" conditions on the state space $\mathcal{S}$ and the measures it supports (in terms of Maharam type), which exclude standard Borel spaces like $[0, 1]^n$. This limits their applicability to the state spaces usually encountered in decision theory.

Proof of Proposition 5. We use the same notation as in the proof of Theorem 3. For all $i \in \mathcal{I}$, and all $n \in [1 \ldots N_i]$, let $\succsim_i^n$ be the preference order on $\mathcal{A}$ defined by the SEU representation with utility function u_i and probability measure ρ_i^n. Clearly, a partition of $\mathcal{S}$ is a consensus partition for the original individuals $\{\succsim_i\}_{i \in \mathcal{I}}$ if and only if it is a consensus partition for the new "individuals" $\{\succsim_i^n\}_{i \in \mathcal{I}}^{n \in [1 \ldots N_i]}$; thus, an act is a lottery for the former group of individuals if and only if it is a lottery for the latter group. Thus, the scope of the Restricted Pareto axiom for the former group is exactly the same as the scope of this axiom for the latter group.

For all $i \in \mathcal{I}$ and any lottery α, statement (F1) is still true. Thus, the "individuals" $\{\succsim_i^1, \ldots, \succsim_i^{N_i}\}$ all have the same preferences over lotteries as the individual $\succsim_i$. It follows that for any lotteries α and β,

$$(\alpha \succsim_i \beta \text{ for all } i \in \mathcal{I}) \quad \Longleftrightarrow \quad \big(\alpha \succsim_i^n \beta \text{ for all } n \in [1 \ldots N_i] \text{ and } i \in \mathcal{I}\big).$$

Thus, $\succsim_o$ satisfies Restricted Pareto with respect to $\{\succsim_i\}_{i \in \mathcal{I}}$ if and only if $\succsim_o$ satisfies Restricted Pareto with respect to $\{\succsim_i\}_{i \in \mathcal{I}}^{n \in [1 \ldots N_i]}$. Thus, the theorem of Gilboa, Samet, and Schmeidler (2004) says that $\succsim_o$ satisfies Restricted Pareto (the "indifference" part) with respect to $\{\succsim_i\}_{i \in \mathcal{I}}$ if and only if (i) u_o is an affine combination of $\{u_i\}_{i \in \mathcal{I}}$; and (ii) ρ_o is a linear combination of the elements of $\bigcup_{i \in \mathcal{I}} \mathcal{R}_i$.[30]

[30] Gilboa, Samet, and Schmeidler assume *countably* additive measures. But like Mongin (1995), they do this only to invoke the Lyapunov Convexity Theorem, which was extended to *finitely* additive measures by Armstrong and Prikry (1981), as noted in Footnote 24. (In fact, their Claim 4 is our Lemma A.1.).

As explained in the proof of Theorem 3, our "semistrong" Pareto axiom implies that the coefficients in the affine combination (i) are nonnegative, so that u_o is weakly utilitarian (Domotor (1979), Weymark (1993, 1994)). □

References

Aliprantis, Charalambos D. and Kim C. Border (2006), *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third edition. Springer, Berlin. [1367, 1372, 1386]

Alon, Shiri and Gabrielle Gayer (2016), "Utilitarian preferences with multiple priors." *Econometrica*, 84, 1181–1201. [1354, 1359]

Armstrong, Thomas E. and Karel Prikry (1981), "Liapounoff's theorem for nonatomic, finitely-additive, bounded, finite-dimensional, vector-valued measures." *Transactions of the American Mathematical Society*, 266, 499–514. [1367, 1368, 1383, 1394]

Basile, Achille and Kopparty P. S. Bhaskara Rao (2000), "Completeness of L_p-spaces in the finitely additive setting and related stories." *Journal of Mathematical Analysis and Applications*, 248, 588–624. [1358]

Berti, Patrizia, Eugenio Regazzini, and Pietro Rigo (1992), "Finitely additive Radon–Nikodỳm theorem and concentration function of a probability with respect to a probability." *Proceedings of the American Mathematical Society*, 114, 1069–1078. [1358]

Bewley, Truman F. (2002), "Knightian decision theory. Part I." *Decisions in economics and finance*, 25, 79–110. [1362]

Bhaskara Rao, Kopparty P. S. and M. Bhaskara Rao (1983), *Theory of Charges: A Study of Finitely Additive Measures*. Academic Press. [1370]

Billot, Antoine and Xiangyu Qu (2021), "Utilitarian aggregation with heterogeneous beliefs." *American Economic Journal: Microeconomics*, 13, 112–123. [1352]

Brandl, Florian (2021), "Belief-averaging and relative utilitarianism." *Journal of Economic Theory*, 198, 105368. [1352]

Candeloro, Domenico and Aljosa Volčič (2002), "Radon–Nikodym theorems." In *Handbook of Measure Theory: in Two Volumes* (Endre Pap, ed.), Elsevier. [1358]

Cerreia-Vioglio, Simone, Paolo Ghirardato, Fabio Maccheroni, Massimo Marinacci, and Marciano Siniscalchi (2011), "Rational preferences under ambiguity." *Economic Theory*, 48, 341–375. [1356, 1357, 1362, 1390]

Chambers, Christopher and Takashi Hayashi (2006), "Preference aggregation under uncertainty: Savage vs. Pareto." *Games Econom. Behav.*, 54, 430–440. [1354]

Chateauneuf, Alain, Caroline Ventura, and Vassili Vergopoulos (2020), "A simple characterization of the Hurwicz criterium under uncertainty." *Revue economique*, 71, 331–336. [1356]

Crès, Hervé, Itzhak Gilboa, and Nicolas Vieille (2011), "Aggregation of multiple prior opinions." *J. Econom. Theory*, 146, 2563–2582. [1361]

Danan, Eric, Thibault Gajdos, Brian Hill, and Jean-Marc Tallon (2016), "Robust social decisions." *Am. Econ. Rev.*, 106, 2407–2425. [1354, 1359, 1362, 1363]

Dietrich, Franz (2021), "Fully Bayesian aggregation." *Journal of Economic Theory*, 194, 105255. [1353, 1354]

Domotor, Zoltan (1979), "Ordered sum and tensor product of linear utility structures." *Theory and Decision*, 11, 375–399. [1393, 1395]

Fleurbaey, Marc (2018), "Welfare economics, risk and uncertainty." *Canadian Journal of Economics*, 51, 5–40. [1354]

Fremlin, David H. (2006a), *Measure Theory, Volume 4: Topological Measure Spaces (Part I)*. Torres Fremlin, Colchester. [1358]

Fremlin, David H. (2006b), *Measure Theory, Volume 4: Topological Measure Spaces (Part II)*. Torres Fremlin, Colchester. [1389]

Gajdos, Thibault, Jean-Marc Tallon, and Jean-Christophe Vergnaud (2008), "Representation and aggregation of preferences under uncertainty." *J. Econom. Theory*, 141, 68–99. [1354]

Gajdos, Thibault and Jean-Christophe Vergnaud (2013), "Decisions with conflicting and imprecise information." *Social Choice and Welfare*, 41, 427–452. [1361]

Ghirardato, Paolo, Fabio Maccheroni, Massimo Marinacci, and Marciano Siniscalchi (2003), "A subjective spin on roulette wheels." *Econometrica*, 71, 1897–1908. [1362]

Ghirardato, Paolo, Fabio Maccheroni, and Massimo Marinacci (2004), "Differentiating ambiguity and ambiguity attitude." *Journal of Economic Theory*, 118, 133–173. [1356, 1357, 1362, 1390]

Ghirardato, Paolo and Daniele Pennesi (2020), "A general theory of subjective mixtures." *Journal of Economic Theory*, 188, 105056. [1362]

Gilboa, Itzhak, Dov Samet, and David Schmeidler (2004), "Utilitarian aggregation of beliefs and tastes." *Journal of Political Economy*, 112, 932–938. [1352, 1353, 1354, 1363, 1364, 1366, 1367, 1393, 1394]

Gilboa, Itzhak and David Schmeidler (1989), "Maxmin expected utility with non-unique prior." *Journal of Mathematical Economics*, 18, 141–153. [1356]

Greinecker, Michael and Konrad Podczeck (2013), "Liapounoff's vector measure theorem in Banach spaces and applications to general equilibrium theory." *Economic Theory Bulletin*, 1, 157–173. [1394]

Harsanyi, John C. (1955), "Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility." *Journal of Political Economy*, 63, 309–321. [1352, 1360, 1393]

Hartmann, Lorenz (2023), "Strength of preference over complementary pairs axiomatizes alpha-MEU preferences." *Journal of Economic Theory*, 213, 105719. [1356]

Hayashi, Takashi (2021), "Collective decision under ignorance." *Social Choice and Welfare*, 57, 347–359. [1354]

Hayashi, Takashi and Michele Lombardi (2019), "Fair social decision under uncertainty and responsibility for beliefs." *Econ. Theory*, 67, 775–816. [1354]

Hurwicz, Leonid (1951), "Optimality criteria for decision making under ignorance." Cowles commission papers, 370. [1356]

Khan, M. Ali and Kali P. Rath (2013), "The Shapley–Folkman theorem and the range of a bounded measure: An elementary and unified treatment." *Positivity*, 17, 381–394. [1368]

Khan, M. Ali and Nobusumi Sagara (2013), "Maharam-types and Lyapunov's theorem for vector measures on Banach spaces." *Illinois Journal of Mathematics*, 57, 145–169. [1394]

Khan, M. Ali and Nobusumi Sagara (2015), "Maharam-types and Lyapunov's theorem for vector measures on locally convex spaces with control measures." *Journal of Convex Analysis*, 22, 647–672. [1394]

Klibanoff, Peter, Massimo Marinacci, and Sujoy Mukerji (2005), "A smooth model of decision making under ambiguity." *Econometrica*, 73, 1849–1892. [1356]

Machina, Mark J. (2004), "Almost-objective uncertainty." *Economic Theory*, 24, 1–54. [1353, 1357]

Machina, Mark J. (2005), "'Expected utility/subjective probability' analysis without the sure-thing principle or probabilistic sophistication." *Economic Theory*, 26, 1–62. [1353, 1357]

Mongin, Philippe (1995), "Consistent Bayesian aggregation." *Journal of Economic Theory*, 66, 313–351. [1352, 1359, 1361, 1382, 1383, 1394]

Mongin, Philippe (1997), "Spurious unanimity and the Pareto principle." Technical report, THEMA, Université de Cergy-Pontoise. Published as Mongin (2016). [1352, 1397]

Mongin, Philippe (1998), "The paradox of the Bayesian experts and state-dependent utility theory." *Journal of Mathematical Economics*, 29, 331–361. [1359]

Mongin, Philippe (2016), "Spurious unanimity and the Pareto principle." *Economics and Philosophy*, 32, 511–532. (earlier circulated as Mongin (1997)). [1397]

Mongin, Philippe and Marcus Pivato (2015), "Ranking multidimensional alternatives and uncertain prospects." *Journal of Economic Theory*, 157, 146–171. [1354]

Mongin, Philippe and Marcus Pivato (2016), "Social evaluation under risk and uncertainty." In *Handbook of Well-Being and Public Policy* (Matthew D. Adler and Marc Fleurbaey, eds.), 711–745, Oxford University Press. Chapter 24. [1354]

Mongin, Philippe and Marcus Pivato (2020), "Social preference under twofold uncertainty." *Economic Theory*, 70, 633–663. [1352, 1353, 1365, 1366, 1367]

Nascimento, Leandro (2012), "The ex-ante aggregation of opinions under uncertainty." *Theoretical Economics*, 7, 535–570. [1361]

Pivato, Marcus (2022), "Bayesian social aggregation with accumulating evidence." *Journal of Economic Theory*, 200, 105399. [1353, 1367, 1379]

Poincaré, Henri (1912), *Calcul des probabilités*, second edition. Gauthier-Villars, Paris. [1353, 1357]

Qu, Xiangyu (2017), "Separate aggregation of beliefs and values under ambiguity." *Economic Theory*, 63, 503–519. [1354]

Savage, Leonard J. (1954), *The Foundations of Statistics*. John Wiley & Sons, New York. [1352]

Schmeidler, David (1989), "Subjective probability and expected utility without additivity." *Econometrica*, 571–587. [1356]

Stanca, Lorenzo (2021), "Smooth aggregation of Bayesian experts." *Journal of Economic Theory*, 196, 105308. [1361]

Urbinati, Niccolò (2019), "A convexity result for the range of vector measures with applications to large economies." *Journal of Mathematical Analysis and Applications*, 470, 16–35. [1394]

Walters, Peter (1982), *An Introduction to Ergodic Theory*. Springer-Verlag, New York. [1388, 1389]

Weymark, John (1993), "Harsanyi's social aggregation theorem and the weak Pareto principle." *Social choice and welfare*, 10, 209–221. [1393, 1395]

Weymark, John (1994), "Harsanyi's social aggregation theorem with alternative Pareto principles." In *Models and Measurement of Welfare and Inequality* (Wolfgang Eichhorn, ed.), Springer, Heidelberg. [1393, 1395]

Zuber, Stéphane (2016), "Harsanyi's theorem without the sure-thing principle: On the consistent aggregation of monotonic Bernoullian and Archimedean preferences." *Journal of Mathematical Economics*, 63, 78–83. [1354]

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