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Extensive measurement in social choice

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Extensive measurement is the standard measurement-theoretic approach for constructing a ratio scale. It involves the comparison of objects that can be concatenated in an additively representable way. This paper studies the implications of extensively measurable welfare for social choice theory. We do this in two frameworks: an Arrovian framework with a fixed population and no interpersonal comparisons, and a generalized framework with variable populations and full interpersonal comparability. In each framework we use extensive measurement to introduce novel domain restrictions, independence conditions, and constraints on social evaluation. We prove a welfarism theorem for these domains and characterize the social welfare functions that satisfy the axioms of extensive measurement at both individual and social levels. The main results are simple axiomatizations of strong dictatorship in the Arrovian framework and classical utilitarianism in the generalized framework.

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JEL classification. D63, D71.

1. Introduction

Kenneth Arrow once called himself "a kind of utilitarian manqué":

> I'd like to be utilitarian but…I have nowhere those utilities come from.… What are those objects we are adding up? I have no objection to adding them up if there's something to add. (Kelly and Arrow 1987, p. 59)

The content of Arrow's complaint is not entirely transparent. In the orthodox economic sense of "utility," anyone who takes individuals to have numerically representable preferences certainly has somewhere "those utilities come from": a person's utility is just the numerical value of a function that represents her preferences. There is no mystery about how such utilities can be added together: they are just numbers, and we can add whatever numbers we like.

Arrow's complaint cannot be that he lacks a foundation for the numerical representation of preferences. A different complaint, which is at least inspired by Arrow's remarks, is this. A classical utilitarian believes that we should maximize *the sum of well-being*, where a person's well-being is how good things are for her. But what does it mean

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to "add up" people's well-beings? A person's well-being is not a number, any more than her height or weight is a number. Some properties can, intuitively, be added together: we can add together two heights or two masses. But we cannot add heights to masses. And it is unclear what would be meant by the "sum" of one person's intelligence and another's or of the hardness of two minerals. The complaint is that the classical utilitarian has not shown well-being to be the kind of thing that, like height or mass, can be added up, as opposed to the kind of thing, like intelligence or hardness, that cannot.

Extensive measurement offers a way to make this contrast precise. The idea of extensive measurement is to compare objects that can be concatenated, or combined, to yield new objects. If the comparison of concatenated objects satisfies certain axioms (stated in Section 2), it can be represented by a real-valued function with concatenation represented by the arithmetic operation of addition (Suppes 1951, Krantz, Luce, Suppes, and Tversky 1971). A classic example is the measurement of length by adjoining rods from end to end or of mass by stacking together objects in a weight pan.

There are various ways of trying to apply extensive measurement to well-being, which differ based on what kinds of objects are evaluated and how they are concatenated (Nebel 2023b). Each of these methods depends on controversial assumptions about well-being. It is therefore, in my view, an open question whether well-being is susceptible to extensive measurement. In this paper, I want to assume that it is, and thus that well-being can be meaningfully added up, so as to study the implications of extensive measurement for social choice and welfare theory. In particular, I want to understand what further commitments are necessary and sufficient to characterize classical utilitarianism, once it is granted that well-being is extensively measurable.

We explore the social-choice–theoretic implications of extensive measurement in two frameworks. In both frameworks, the set of alternatives is equipped with a concatenation operation. (When alternatives belong to a vector space, for example, this operation can simply be vector addition.) In Section 3, we consider an Arrovian framework in which each profile is an n-tuple of individual orderings on the set of alternatives. We restrict the domain to profiles in which each individual's ordering satisfies the axioms of extensive measurement. We provide a characterization of welfarism on this domain (Theorem 1), using Pareto indifference and a suitable weakening of Arrow's Independence of Irrelevant Alternatives (IIA). This IIA condition allows for nondictatorial social welfare functions that satisfy the weak and even strong Pareto principles on our domain. However, such social welfare functions cannot be anonymous (Theorem 2), and their social preference relations are not extensively measurable (Theorem 3). These negative results motivate the use of interpersonal comparisons in our second framework, based on Hammond (1976), which is explored in Section 4.

In Hammond's framework, each profile is a single relation over alternative–individual pairs. The pair (x, i) stands in this relation to (y, j) if and only if alternative x is at least as good for person i as y is for person j. Interpersonal comparisons of this form are utilized and defended by Suppes (1966), Sen (1970), Arrow (1977), Harsanyi (1977), Kolm (1998), and Adler (2014). A *generalized social welfare function*, as defined by Hammond, assigns a social ordering of alternatives to each ordering of alternative–individual pairs.

We modify Hammond's framework in two ways. First, we do not assume that the population is fixed. Instead, different alternatives have potentially different populations. This generalization is crucial for evaluating choices that affect the size or composition of the population, such as responses to climate change (Scovronick et al. 2017) or allocations of fertility-affecting resources (Pérez-Nievas, Conde-Ruiz, and Giménez 2019, Córdoba and Liu 2022). Indeed, Parfit (1984, Chapter 16) argues that almost all social and economic policy choices have far-reaching effects on which people will exist in the future. Variable-population comparisons are also needed to distinguish classical (i.e., total) utilitarianism from other varieties of utilitarianism (e.g., average utilitarianism) that coincide with it in fixed-population cases. Second, we restrict the domain, as in the Arrovian case, to orderings of alternative–individual pairs that are extensively measurable. Our characterization of welfarism, in terms of Pareto indifference and an appropriate IIA condition, continues to hold on this domain (Theorem 4).

Our main result is an axiomatization of classical utilitarianism in this generalized framework. Theorem 5 shows that classical utilitarianism is the only social welfare function on our domain that satisfies the weak Pareto principle, our IIA condition, a fixed-population anonymity requirement, and the axioms of extensive measurement imposed on social preferences.

1.1 *Background*

Issues of measurement have played a central role in social choice theory since Sen (1970). In Sen's framework of social welfare functionals, a social preference ordering of alternatives is assigned to each profile of real-valued utility functions in some domain. Different views about the measurability and interpersonal comparability of welfare are captured by imposing *informational invariance* conditions on the social welfare functional. These conditions require the social ranking of alternatives to be preserved under certain classes of transformations of utilities, namely, those transformations up to which the utility representation is assumed to be unique.

The social welfare functional framework is extremely flexible. It has been used to provide axiomatic characterizations of many important theories of welfare aggregation (Roemer 1998, d'Aspremont and Gevers 2002, Bossert and Weymark 2004). The informational invariance conditions lie at the core of these results. These conditions have recently been criticized, however, on the grounds that they do not really follow from the underlying measurability and comparability assumptions with which they are associated. As Sen (1977a, p. 1542) observes, the invariance conditions fail to distinguish between real changes in well-being (e.g., everyone becoming twice as well off) and merely representational changes in the scale on which well-being is measured (e.g., halving the unit of measurement). It is not obvious why invariance with respect to the latter kind of transformation should require invariance with respect to the former. This criticism has been further developed by Morreau and Weymark (2016) and Nebel (2021, 2022, 2024).

Sen's framework takes numerical scales of welfare for granted but provides no way to specify what structures they are supposed to represent—only the class of transformations up to which they are unique. This makes it difficult to defend the invariance

conditions against the criticism just mentioned. An alternative approach is to formulate our social choice problem and principles in terms of the relational structure of individual welfare, rather than (at the outset, at least) a numerical representation thereof. Let me explain.

In measurement theory, a *qualitative relational structure* is a set of objects together with one or more relations on that set (Heilmann 2015). An example is a set X of alternatives together with an ordering $\succcurlyeq$ on that set. Another is an ordered set L of lotteries closed under an operation $\otimes : [0, 1] \times L \times L \to L$, which takes any probability $\lambda \in [0, 1]$ and lotteries $p, q \in L$, and returns their convex combination $\lambda p + (1 - \lambda)q$. The role of this "natural operation" is explicit in von Neumann and Morgenstern (1953, 24) (see also Fishburn 1989, Weymark 2005).

A central business of measurement theory is to provide conditions under which qualitative relational structures can be represented by certain *numerical* relational structures. There are familiar conditions that are necessary and sufficient for an ordered set $(X, \succcurlyeq)$ to be be mapped into the numerical structure $(\mathbb{R}, \geq)$, via an order-preserving function $U : X \to \mathbb{R}$. Such a representation is unique up to strictly increasing transformation. And there are familiar conditions that are necessary and sufficient for $(L, \succcurlyeq, \otimes)$ to be mapped into the numerical structure $(\mathbb{R}, \geq, \otimes^*)$, where $\otimes^* : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ takes any $\lambda \in [0, 1]$ and $a, b \in \mathbb{R}$, and returns their convex combination $\lambda a + (1 - \lambda)b$. Such a representation is unique up to positive affine transformation.

Qualitative relational structures of these kinds are the primitive ingredients of the Arrow (1951) and Harsanyi (1955) approaches to social choice. This is in contrast to the later framework of Sen, where the primitive ingredients are numerical utility functions. Other work in the relational tradition includes Hammond (1976), Dhillon and Mertens (1999), Harvey (1999), Pivato (2015), Marchant (2019), Brandl and Brandt (2020), and Raschka (2024), among others. None of these works, however, considers the social-choice–theoretic implications of extensive measurement.

Extensive structures are formally quite similar to the examples mentioned above. The qualitative relational structure is of the form $(X, \succcurlyeq, \circ)$, where $\circ$ concatenates each pair of alternatives in X. There are natural axioms that are necessary and sufficient for this structure to be mapped into $(\mathbb{R}, \geq, +)$, via a function $U : X \to \mathbb{R}$, which is both order-preserving and *additive*, in the sense that $U(x \circ y) = U(x) + U(y)$ for all $x, y \in X$. This representation is unique up to similarity transformation (multiplication by a positive constant). This is the characteristic uniqueness condition of a ratio scale, such as the gram scale of mass or the meter scale of length.

Our project is therefore intimately related to the study of social welfare functionals with ratio-scale measurable welfare, typically spelled out in terms of invariance to similarity transformations of utilities (Roberts 1980, Blackorby and Donaldson 1982, Tsui and Weymark 1997, Nebel 2023a). Many social welfare functionals, especially in variable-population contexts (Blackorby, Bossert, and Donaldson 1999), appear to require a ratio scale, because they violate invariance conditions associated with weaker scale types. But no one in this literature has explained how such a scale can be derived. For that, we need to identify a relational structure whose numerical representation is unique up to similarity transformation. Our approach does this without requiring us to

assume an invariance condition to capture the intended scale type. Rather, the desired invariance properties will be *derived* from conditions formulated in entirely relational terms.

1.2 *Related literature*

There appears to be no canonical axiomatic characterization of classical utilitarianism—as distinguished from other varieties of utilitarianism—in the literature. This is a striking gap, given the historical importance of this doctrine in social ethics and welfare economics. There are, of course, several axiomatizations of fixed-population utilitarian social welfare functionals (d'Aspremont and Gevers 1977, Deschamps and Gevers 1978, Maskin 1978), but these do not discriminate between classical utilitarianism and its variants. Indeed, they rest on informational invariance conditions that, when extended to a variable-population setting, rule out classical utilitarianism (Blackorby, Bossert, and Donaldson 1999). The most systematic treatment of variable-population social choice is Blackorby, Bossert, and Donaldson (2005). They characterize various kinds of utilitarianism, but none that singles out classical utilitarianism in particular. Hammond (1988) derives a principle that formally resembles classical utilitarianism, but in later work he is careful to acknowledge the resemblance as "only formal" (Fleurbaey and Hammond 2004, 1268). Xu (1990) provides axiomatic characterizations of classical and average utilitarianism that differ, quite usefully, over a single pair of axioms; his article appears to have been universally overlooked in the literature.

Our characterization of classical utilitarianism is in many ways analogous to the fixed-population aggregation theorem of Harsanyi (1955). Whereas Harsanyi applies expected utility theory at both the individual and social levels, we appeal to extensive measurement. Harsanyi's aggregation theorem has been extended to variable-population cases by Blackorby, Bossert, and Donaldson (1998), Broome (2004), and McCarthy, Mikkola, and Thomas (2020). Our contribution is especially related to that of Mongin (1994), who adapts Harsanyi's theorem to a multi-profile setting via a domain restriction and an IIA condition: he requires each individual to have a mixture-preserving utility function on some convex subset of a vector space, and requires the social ordering of two alternatives to coincide on profiles that assign the same utility vectors to those alternatives. He also considers the implications of informational invariance conditions in this setting and concludes that the interpersonal comparability of welfare is a "surprise effect" of Harsanyi's theorem (Mongin 1994, p. 349). The invariance conditions derived in our framework are much weaker than their counterparts considered by Mongin. In the interpersonally noncomparable case, for example, our invariance condition avoids dictatorship where his corresponding condition implies it. Additionally, none of the invariance conditions considered by Mongin is compatible with classical utilitarianism when extended to variable-population comparisons.

This paper is also related to the study of Arrovian social welfare functions on restricted domains. Le Breton and Weymark (2011) survey the consistency of Arrow's axioms on various domains of economic interest. Our Arrovian domain in Section 3 is an example of what they (following Kalai, Muller, and Satterthwaite 1979) call *saturating*

preference domains. Arrow's axioms are inconsistent on these domains (Le Breton and Weymark 2011, Theorem 6), but our IIA condition is much weaker than Arrow's, in a way that makes it compatible with nondictatorial, Paretian social choice on our domain. Our Arrovian domain is also quite different than those studied by Brandl and Brandt (2020). They characterize the domains on which Arrow's axioms are consistent with one another and with an anonymity requirement; they show that such anonymous Arrovian aggregation must take a certain utilitarian form. Their domains require preferences to be continuous and convex, but not transitive. Our Arrovian domain plainly does not meet their conditions; nonetheless, our impossibility result involving anonymity (Theorem 2) does not follow from their characterization of anonymously Arrow-consistent domains, again, because of our weaker IIA condition.

2. Extensive measurement

An extensive structure has three ingredients. There is a set X of objects to be measured: for example, rods of differing lengths. There is a binary relation $\succcurlyeq$ on that set (e.g., the "at least as long as" relation). As usual $\succ$ denotes the asymmetric part of $\succcurlyeq$; $\sim$ denotes its symmetric part. There is a binary concatenation operation $\circ : X \times X \to X$ that, in some sense, combines the objects together (e.g., by adjoining rods from end to end). Our set of objects is assumed to be closed under this operation, so that we can concatenate any two elements of X to form a new element of X. For any object $a \in X$, define $1a := a$ and, for any natural number $n > 1$, let $na := (n-1)a \circ a$, so that na is the concatenation of n copies of a.

The triple $(X, \succcurlyeq, \circ)$ is called an *extensive structure* if and only if the following five axioms are satisfied.

Transitivity. For all $a, b, c \in X$, if $a \succcurlyeq b$ and $b \succcurlyeq c$, then $a \succcurlyeq c$.
Completeness. For all $a, b \in X$, either $a \succcurlyeq b$ or $b \succcurlyeq a$.
Weak Associativity. For all $a, b, c \in X$, $a \circ (b \circ c) \sim (a \circ b) \circ c$.
Monotonicity. For all $a, b, c \in X$, $a \succcurlyeq b$ if and only if $a \circ c \succcurlyeq b \circ c$ if and only if $c \circ a \succcurlyeq c \circ b$.
Archimedean Axiom. For all $a, b, c, d \in X$, if $a \succ b$, then there is some natural number n such that $na \circ c \succcurlyeq nb \circ d$.

These conditions are necessary and sufficient for a numerical representation of $\succcurlyeq$ that is additive with respect to concatenation.

Proposition 1 (Krantz et al. 1971, Theorem 3.1). *The set $(X, \succcurlyeq, \circ)$ is an extensive structure if and only if there is a function $U : X \to \mathbb{R}$ such that, for all $a, b \in X$,*

(i) $a \succcurlyeq b$ if and only if $U(a) \geq U(b)$;

(ii) $U(a \circ b) = U(a) + U(b)$.

Another function U' satisfies (i) and (ii) if and only if $U' = kU$ for some real number $k > 0$.

We call U an *additive representation* of $\succcurlyeq$.

Here is an example, based on Kahneman, Wakker, and Sarin (1997), of how extensive measurement might be applied to well-being. Consider a set of hedonic episodes. Each episode is individuated by its duration and by its felt quality—pleasure or pain—at each moment. The concatenation of two episodes is simply an episode that starts with the first and ends with the second. These concatenable episodes are ordered by their desirability to some agent. Narens and Skyrms (2020, Chapter 12) defend the axioms of extensive measurement for this sort of structure. Other structures, which carry no commitment to hedonism about well-being, are explored by Nebel (2023b).

An important difference between length and well-being is that, in the case of length, all values of an additive representation are positive. Formally, this is captured by an additional positivity axiom, which requires that $a \circ b \succ a$ for all $a, b \in X$. That axiom would make $(X, \succcurlyeq, \circ)$ a *positive* extensive structure. Though this restriction is of formal interest, it is not imposed here, because I find it hard to think of a conception of well-being on which it seems reasonable in full generality.

3. Arrovian social welfare functions

Let X be a set of alternatives that is closed under a concatenation operation $\circ : X \times X \to X$. The operation is assumed to be associative: $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in X$.[1] An alternative $a \in X$ is *atomic* if and only if it is not identical to the concatenation of any alternatives (i.e., there are no $x, y \in X$ such that $x \circ y = a$). We assume there to be a subset $A \subset X$ of at least three atomic alternatives, and that every alternative $x \in X$ decomposes into finitely many atomic alternatives: that is, $x = a_1 \circ \cdots \circ a_k$ for some $a_1, \ldots, a_k \in A$ $(k \geq 1)$. This decomposition is assumed to be unique up to the order in which the atoms are concatenated: that is, if $a_1 \circ \cdots \circ a_k = b_1 \circ \cdots \circ b_l$, with a_i and b_j atoms, then $k = l$ and there exists a permutation σ on $\{1, \ldots, k\}$, such that $a_i = b_{\sigma(i)}$ for all $i = 1, \ldots, k$.

We assume a fixed population $N = \{1, 2, \ldots, n\}$ of at least two individuals. An *Arrovian profile* $R = (R_1, \ldots, R_n)$ is an n-tuple of orderings on X, one for each individual in N. Our interpretation of these orderings is that xR_iy if and only if (according to profile R) x is at least as good for i as y. As usual, I_i denotes the symmetric part of R_i and P_i denotes its asymmetric part. The set of all orderings on X is $\mathcal{R}$. An *Arrovian social welfare function* is a function $f : \mathcal{D} \subseteq \mathcal{R}^n \to \mathcal{R}$, which assigns an overall "betterness" or social preference ordering to some set $\mathcal{D}$ of Arrovian profiles. For any profile $R \in \mathcal{D}$, let $\succcurlyeq_R$ denote the ordering $f(R)$.

We adopt the following domain assumption.

Extensive Domain. An Arrovian profile R is in $\mathcal{D}$ if and only if $(X, R_i, \circ)$ is an extensive structure for all $i \in N$.

[1] That is, $(X, \circ)$ is a semigroup. Ordered semigroups play a central role in the proof of Proposition 1 (Krantz et al. 1971, Chapter 2), and certain kinds of semigroups have been used in social choice theory (Pivato 2013, Pivato and Tchouante 2024). The associativity of $\circ$ is not strictly necessary for our results, since we will soon be imposing Weak Associativity anyway, but it simplifies notation by allowing us to omit parentheses without ambiguity.

By Proposition 1, every profile R in an extensive domain can be represented by a utility profile $U = (U_1, \dots, U_n)$, where each U_i additively represents R_i (that is, $U_i(x) \geq U_i(y)$ if and only if xR_iy, and $U_i(x \circ y) = U_i(x) + U_i(y)$) in which case we say that U itself additively represents R. For any Arrovian profile R, let $\mathcal{U}_R$ denote the set of all utility profiles that additively represent R, and let $\mathcal{U}_{\mathcal{D}} \coloneqq \bigcup_{R \in \mathcal{D}} \mathcal{U}_R$.

Here is a simple example. Suppose there are $m \geq 3$ public goods. Let A be the set of standard unit vectors in $\mathbb{R}^m_+$.[2] Each atomic alternative represents an arbitrarily small increment of a distinct public good. The concatenation operation is vector addition. Then $X = \mathbb{Z}^m_+ \setminus \{\mathbf{0}\}$ represents all possible bundles of those public goods in those increments, excluding the null bundle.[3] On this interpretation, Extensive Domain amounts to the assumption that each individual's preferences can be additively represented by a linear utility function (this of course implies nothing about their attitudes toward risk). Domains like this are considered by Kalai, Muller, and Satterthwaite (1979) and Le Breton and Weymark (2011, Example 9).

Here is another example, based on Weymark (1981). Suppose there are $m \geq 3$ sources of income. Let A be the set of all vectors whose first k components are 1 and all others are 0 for all $k \in \{1, \dots, m\}$. The unit of income can be as small as we like. The concatenation operation is again vector addition. Then $X = \{x \in \mathbb{Z}^m_+ \setminus \{\mathbf{0}\} \mid x_1 \geq x_2 \geq \cdots \geq x_m\}$ represents all income distributions, in the chosen unit, ordered from greatest to least. Each R_i, on this interpretation, represents individual i's ethical ranking of income distributions. The Monotonicity axiom of extensive measurement here corresponds to the "comonotonic independence" axiom for ranking such distributions (Weymark 1981, Axiom 4). Extensive Domain amounts to the requirement that each individual's ethical ranking is of the generalized Gini form. An Arrovian social welfare function aggregates these generalized Gini rankings into a collective ranking.

Here is a third, more abstract example. Social welfare theorists are often interested in alternatives that are much richer than income distributions or bundles of goods. It is often supposed that an alternative is a possible history of the world or of some society over some period of time (Gibbard 1982, 1984, Hylland 1989, Broome 2004, Blackorby, Bossert, and Donaldson 2005, Dasgupta 1995, 2007, 2009, Adler 2019). Let A be a set of at least three such histories. We might imagine that histories can be concatenated into successive epochs of a single history: things proceed according to the first history, and then according to the second (much like Kahneman et al.'s concatenation of hedonic episodes). Then X is simply the closure of A under this concatenation operation. Our domain assumption then requires each individual's well-being in any history to be representable by the sum of that person's well-being across the epochs that make it up (see Nebel 2023b for discussion).

We now turn to the characterization of welfarism on our domain.

3.1 *Welfarism*

We will be interested in three standard Pareto principles:

[2] I use $\mathbb{R}^m_+$ to denote the nonnegative orthant of $\mathbb{R}^m$, $\mathbb{R}^m_{++}$ the strictly positive orthant.

[3] The null bundle is excluded because the unique decomposition property rules out identity elements.

Weak Pareto. For any $x, y \in X$ and any $R \in \mathcal{D}$, if xP_iy for every $i \in N$, then $x \succ_R y$.
Pareto Indifference. For any $x, y \in X$ and any $R \in \mathcal{D}$, if xI_iy for every $i \in N$, then $x \sim_R y$.
Strong Pareto. For any $x, y \in X$ and any $R \in \mathcal{D}$, if xR_iy for every $i \in N$, then xRy; if, in addition, xP_iy for some $i \in N$, then $x \succ_R y$.

For any binary relation R on X and any $S \subseteq X$, let $R|_S$ denote the restriction of R to S. Arrow required, via his IIA condition, that, for any alternatives $x, y \in X$ and profiles $R, R' \in \mathcal{D}$, if $R_i|_{\{x,y\}} = R'_i|_{\{x,y\}}$ for every $i \in N$, then $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$. We will instead use a weaker principle, which allows the social comparison of alternatives to depend not just on individuals' rankings of *those* alternatives, but also on their rankings of concatenations involving them. For any $S \subseteq X$, let S° denote the closure of S under $\circ$. Our weaker IIA principle is as follows.

Ratio IIA. For any alternatives $x, y \in X$ and any Arrovian profiles $R, R' \in \mathcal{D}$, if $R_i|_{\{x,y\}^\circ} = R'_i|_{\{x,y\}^\circ}$ for every $i \in N$, then $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$.

The motivation for weakening Arrow's condition to Ratio IIA (and for its name) is that each $R_i|_{\{x,y\}^\circ}$ fully determines the ratio of $U_i(x)$ to $U_i(y)$ for any U_i that additively represents R_i. In a setting where such information is well defined, there is no reason to exclude it as "irrelevant" to the comparison of alternatives.

For any utility profile $U \in \mathcal{U}_\mathcal{D}$ and alternative $x \in X$, the utility vector assigned by U to x is $U(x) = (U_1(x), \ldots, U_n(x))$. A social welfare function is *welfarist* if and only if the ordering it assigns to any profile is determined by a single *social welfare ordering* $\succcurlyeq^*$ on the set of attainable utility vectors.

Welfarism. There is a unique ordering $\succcurlyeq^*$ on $\mathbb{R}^n$ such that, for any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $x \succcurlyeq_R y$ if and only if $U(x) \succcurlyeq^* U(y)$.

When f and $\succcurlyeq^*$ are so related, we say that $\succcurlyeq^*$ is *associated* with f.

The standard welfarism theorem in the framework of social welfare functionals appeals to Pareto Indifference and an IIA condition formulated in terms of numerical utilities (Bossert and Weymark 2004, Theorem 2.2). We show (Proposition 2 in Appendix A) that Ratio IIA is equivalent to this utility-theoretic condition, given Extensive Domain. The standard welfarism theorem, however, assumes an unrestricted domain of utility profiles. It does not apply to the present setting because we have restricted the domain; neither do analogous results for restricted domains due to Mongin (1994) and Weymark (1998).[4] Fortunately, we can still characterize Welfarism in terms of Pareto Indifference and Ratio IIA.

THEOREM 1. *If an Arrovian social welfare function f satisfies Extensive Domain, then f satisfies Pareto Indifference and Ratio IIA if and only if f satisfies Welfarism.*

[4]Mongin is concerned with profiles of mixture-preserving utility functions on a convex subset of a vector space. Weymark characterizes welfarism on "saturating" and "hypersaturating" *utility* domains. Our $\mathcal{U}_\mathcal{D}$ is not saturating because some pairs of nonatomic alternatives are, in Weymark's terminology, nontrivial but also not free and thus not connected. (This is compatible with $\mathcal{D}$ being a saturating *preference* domain in the sense of Kalai, Muller, and Satterthwaite 1979 and Le Breton and Weymark 2011.)

The basic insight behind the proof of Theorem 1 is that the set of utility vectors attainable by the atomic alternatives is unrestricted (Lemma 1). We are therefore able to define a social welfare ordering using only atomic alternatives, and then show how it determines the social ordering over all alternatives.

Not just any social welfare ordering is compatible with Extensive Domain, however—only those that are invariant to individual-specific similarity transformations of utilities.

Intrapersonal Ratio-Scale Invariance. For any utility vectors $u, v, u', v' \in \mathbb{R}^n$, if for every $i \in N$ there is some $k_i > 0$ such that $u'_i = k_i u_i$ and $v'_i = k_i v_i$, then $u \succcurlyeq^* v$ if and only if $u' \succcurlyeq^* v'$.

(See Proposition 3 in Appendix A.) Intrapersonal Ratio-Scale Invariance plays a key role in the results of Section 3.2.

The various Pareto principles have obvious analogues in terms of the social welfare ordering. We do not state them separately. When we say that $\succcurlyeq^*$ violates or satisfies one of the Pareto principles, we mean that it violates or satisfies the obvious translation of that principle for $\succcurlyeq^*$.

3.2 *Possibilities and impossibilities*

Arrow (1951) showed that if a social welfare function defined on an unrestricted domain satisfies Weak Pareto and his IIA condition, then it must be *dictatorial*: there must be some $i \in N$ such that, for any profile $R \in \mathcal{D}$ and alternatives $x, y \in X$, $x \succ_R y$ whenever $xP_i y$. If we weaken Arrow's domain and independence axioms to Extensive Domain and Ratio IIA, this implication is avoided, and even Strong Pareto can be satisfied. For there are nondictatorial social welfare orderings on $\mathbb{R}^n$ that satisfy Intrapersonal Ratio-Scale Invariance and Strong Pareto. Here is a two-person example, based on a class of social welfare orderings axiomatized by Naumova and Yanovskaya (2001); it is easily generalized to larger populations.

Example 1. Take any $u, v \in \mathbb{R}^2$. Suppose $\operatorname{sgn}(u_i) = \operatorname{sgn}(v_i)$ for both $i \in \{1, 2\}$, where $\operatorname{sgn}(0) = 0$. Then letting $0^0 = 1$,

$$u \succcurlyeq^* v \quad \text{if and only if} \quad |u_1|^{\operatorname{sgn}(u_1)} |u_2|^{\operatorname{sgn}(u_2)} \geq |v_1|^{\operatorname{sgn}(v_1)} |v_2|^{\operatorname{sgn}(v_2)}.$$

If $\operatorname{sgn}(u_i) \neq \operatorname{sgn}(v_i)$ for some $i \in \{1, 2\}$, then u and v are ranked according to the following linear ordering of the quadrants and their boundaries (plus the origin):

$$(+,+) \succ (+,0) \succ (0,+) \succ (0,0) \succ (-,+) \succ (+,-) \succ (0,-) \succ (-,0) \succ (-,-).$$

This social welfare ordering satisfies Intrapersonal Ratio-Scale Invariance and Strong Pareto, but is not dictatorial (see Naumova and Yanovskaya 2001). ◊

Orderings of this kind satisfy a number of further properties. They are, *within* each quadrant or boundary, continuous and impartial between individuals. They are also representable by a real-valued social utility function (Naumova and Yanovskaya 2001,

Corollary 4.1). However, they are not fully impartial between individuals, in the following sense. A social welfare ordering $\succcurlyeq^*$ is *anonymous* if and only if, for every $u, v \in \mathbb{R}^n$, $u \sim^* v$ whenever there is a permutation $\sigma : N \to N$ such that $u_i = v_{\sigma(i)}$ for every $i \in N$. Given Welfarism, this condition is equivalent to the following property of social welfare functions (see Proposition 4 in Appendix B).

Anonymity. For all profiles $R, R' \in \mathcal{D}$, if there is a permutation $\sigma : N \to N$ such that $R_i = R'_{\sigma(i)}$ for every individual $i \in N$, then $f(R) = f(R')$.

The social welfare function associated with Example 1 violates Anonymity (for example, $(1, 0) \succ^* (0, 1)$). Indeed, the failure of Anonymity is quite general.

THEOREM 2. *There is no Arrovian social welfare function that satisfies Extensive Domain, Anonymity, Ratio IIA, and Strong Pareto (or, when n is even, Weak Pareto and Pareto Indifference).*[5]

Nonpositive utilities play an essential role in the proof of Theorem 2. There do exist anonymous social welfare orderings on $\mathbb{R}^n_{++}$ that satisfy Weak Pareto and Intrapersonal Ratio-Scale Invariance. These conditions uniquely characterize the symmetric Cobb–Douglas ordering, which compares vectors by the unweighted product of utilities (Moulin 1988, 38). As Example 1 illustrates, this ordering can be generalized to each orthant $\mathbb{R}^n$ taken separately, but the resulting orderings cannot be pieced together in a way that satisfies Weak Pareto, Intrapersonal Ratio-Scale Invariance, and Anonymity.

Theorem 2 may suggest that Anonymity is too much to ask of a social welfare function in the present environment. However, the Arrovian axioms can be strengthened in a way that requires the social welfare function to be *strongly* dictatorial: there must be some $i \in N$ such that, for any $R \in \mathcal{D}$ and $x, y \in X$, $x \succcurlyeq_R y$ if and only if xR_iy. One way to do this is to require the social welfare ordering to be continuous. Tsui and Weymark (1997) show that a continuous social welfare ordering that satisfies Weak Pareto and Intrapersonal Ratio-Scale Invariance must be strongly dictatorial (see also Nebel 2023a). In my view, however, the ethical content of and motivation for continuity is not obvious. It is standardly motivated by considerations regarding slight measurement errors (e.g., by d'Aspremont and Gevers 2002, 496). But while sensitivity to such errors may be unfortunate, it is far from obvious that the ethical ordering of utility vectors should not be sensitive to such errors. To figure out which alternatives are better or worse, why should we not have to identify the correct profile (as opposed to one that is merely arbitrarily close to the correct profile)? Especially given the distinguished role of neutral elements in an extensive structure, discontinuities when some utilities are zero in particular do not seem unreasonable. We therefore consider a different requirement that does not by itself entail continuity.

Extensive Social Preference. For each profile $R \in \mathcal{D}$, the triple $(X, \succcurlyeq_R, \circ)$ is an extensive structure.

[5]Recall that we are assuming $n \geq 2$. (Anonymity would be trivially satisfied if $n = 1$.)

The axioms of extensive measurement may of course be questioned in this context. But *if* we take individual welfare to be extensively measurable, we might reasonably take the social ordering to be extensively measurable as well. For example, on the successive-epochs interpretation of $\circ$, Monotonicity can be motivated by the thought that a choice between histories $c \circ a$ and $c \circ b$ is relevantly like choosing between futures a and b after a past epoch c; what happened in previous epochs, we might think, should not matter for future evaluation except insofar as it affects people today or in the future, in which case those effects should be considered in the valuation of a and b. As in the case of individual welfare, my view is that the applicability of extensive measurement to social evaluation should be regarded as an open question, which depends on the nature of the alternatives and the interpretation of $\circ$, as well as our general ethical commitments.

Our second negative result for Arrovian social welfare functions is as follows.

THEOREM 3. *If an Arrovian social welfare function f satisfies Extensive Domain, then f satisfies Ratio IIA, Weak Pareto, and Extensive Social Preference if and only if f is strongly dictatorial.*

The proof goes as follows. First, we show that Extensive Domain, Ratio IIA, Weak Pareto, and Extensive Social Preference together yield a "semistrong" Pareto principle that entails Pareto Indifference (Lemma 4). These axioms therefore entail Welfarism, by Theorem 1. Next, given Welfarism, Extensive Social Preference is equivalent to $(\mathbb{R}^n, \succcurlyeq^*, +)$ being an extensive structure (Lemma 5). Thus, by Proposition 1, $\succcurlyeq^*$ must be additively representable by a social utility function $W : \mathbb{R}^n \to \mathbb{R}$. Semistrong Pareto forces this function to be of the weighted utilitarian form—i.e., a linear combination of utilities—with nonnegative weights (Lemma 6). Finally, Weak Pareto and Intrapersonal Ratio-Scale Invariance together require exactly one person's weight to be positive; this proves the theorem. An obvious corollary of this result is that there is no Arrovian social welfare function that satisfies Extensive Domain, Ratio IIA, Strong Pareto, and Extensive Social Preference.

Sen (1977b, p. 80) influentially claims that "n-tuples of individual orderings," as in Arrow's framework, "are informationally inadequate for representing conflicts of interests." The lesson I am inclined to draw from Theorems 2 and 3 is that—at least within the confines of welfarism—Arrow's framework is still inadequate for this task even when the individual orderings are supplemented by an extensive concatenation operation. For I take Anonymity to be a fundamental requirement of impartiality. And it seems reasonable to want social preferences to have the same structure as individual welfare. We will see in Section 4 that both of these desiderata can be satisfied in an informationally richer framework that allows for interpersonal comparisons of well-being.

4. GENERALIZED VARIABLE-POPULATION SOCIAL WELFARE FUNCTIONS

We now generalize the framework of Section 3 in two ways.

First, so as to distinguish classical utilitarianism from its variants, we need the population to vary between alternatives. Following Blackorby, Bossert, and Donaldson (2005),

let $\mathbb{N} = \{1, 2, \dots\}$ represent the set of all possible individuals. Let $\mathcal{P}$ denote the set of all finite, nonempty subsets of $\mathbb{N}$. Let X once again denote the set of alternatives. Each alternative x has a population, $N(x) \in \mathcal{P}$, the set of individuals who would exist (or be members of some particular society) if x obtained. For any individual $i \in \mathbb{N}$, $X_i := \{x \in X : i \in N(x)\}$ denotes the set of alternatives in which i exists. For any $N \in \mathcal{P}$, $X^N := \{x \in X : N(x) = N\}$ denotes the set of alternatives whose populations are N.

We assume, as before, that X is closed under an associative concatenation operation $\circ : X \times X \to X$. For each $i \in \mathbb{N}$, there is a set $A^{\{i\}} \subset X^{\{i\}}$ of at least three atomic alternatives in which only i exists.[6] (Recall that an alternative a is atomic if and only if there are no $x, y \in X$ such that $x \circ y = a$.) We assume for simplicity that these are the only atomic alternatives. And we assume, as before, that all nonatomic alternatives decompose uniquely (up to rearrangement) into finitely many atoms. We also require that, for all $x, y \in X$, $N(x \circ y) = N(x) \cup N(y)$. These assumptions together imply that, for each population $N \in \mathcal{P}$, there are infinitely many alternatives in X^N.

Suppose, for example, that there are $m \geq 3$ private goods. We can regard each alternative $x \in X$ as a mapping that assigns a vector in $\mathbb{Z}_+^m \setminus \{\mathbf{0}\}$ to each individual $i \in N(x)$, with the atomic alternatives mapping just a single individual to a unit vector. These alternatives can be concatenated by letting $(x \circ y)_i = (x + y)_i$ for all $i \in N(x) \cap N(y)$, $(x \circ y)_i = x_i$ for all $i \in N(x) \setminus N(y)$, and $(x \circ y)_i = y_i$ for all $i \in N(y) \setminus N(x)$. Or suppose that the alternatives are possible histories of the world over some duration of time. Each history has a unique population: the set of individuals who exist at some point in that history. The atomic alternatives are histories in which only a single person exists. We imagine again that histories can be concatenated into successive epochs of a single history, with the population of the larger history being the union of the populations of its subhistories.

Second, to avoid the impossibilities that arose in our Arrovian framework, we include interpersonal comparisons of well-being. Such comparisons are ubiquitous in variable-population frameworks (see, e.g., Xu 1990, Blackorby, Bossert, and Donaldson 1995, Broome 2004, Asheim and Zuber 2014, Pivato 2020). Indeed, such frameworks typically assume not only that interpersonal comparisons are meaningful, but also that there is a meaningful "neutral level" of welfare that divides lives that are "worth living" from those that are not. Unlike these authors, we incorporate interpersonal comparisons in entirely relational terms, and claims about positive, neutral, or negative well-being will be derived rather than assumed.

As mentioned in Section 1, interpersonal comparisons can be formalized as a relation over alternative–individual pairs. For any $x \in X$ and $i \in N(x)$, I call the pair (x, i) a *life*. (By a "life," I just mean a pair of this form; the definition is simply meant to exclude pairs of the form (y, j) where $j \notin N(y)$.) Let $\mathcal{L} := \{(x, i) \in X \times \mathbb{N} \mid i \in N(x)\}$ denote the set of all lives. An *interpersonal profile* R is an ordering on $\mathcal{L}$. The intended interpretation is

[6] Compare with Blackorby, Bossert, and Donaldson (2005), who assume $|X^N| \geq 3$ for all $N \in \mathcal{P}$. It is unrealistic, of course, to suppose that any individual could exist without her parents ever existing. Weymark (2019) has raised this concern for the intertemporal framework of Blackorby, Bossert, and Donaldson (1995). But it also applies to standard variable-population frameworks that, like ours, lack an explicit time dimension.

that $(x, i)R(y, j)$ if and only if x is at least as good for i (according to profile R) as y is for j, or, equivalently, that i is at least as well off in x as j is in y. Such comparisons are often understood in terms of the "extended preferences" of a social observer—preferring, for one's own sake, to be one person (or to be in their "position" in some sense, having all of their tastes, values, and so on) in one alternative rather than another person in another alternative (Suppes 1966, Sen 1970, 1997, Arrow 1977, Harsanyi 1977, Suzumura 1996, Kolm 1998, Adler 2014). But I do not insist on this or any other particular way of making interpersonal comparisons. My own view is that, clearly, some people are better off than others, and any plausible theory of well-being must be able to accommodate such comparisons (Scanlon 1991, Hausman 1995, Broome 1999, Greaves and Lederman 2018). (This is not to say, of course, that it is easy to explain what makes such comparisons true or to discover which ones are true.)

Let $\mathcal{R}_{\mathcal{L}}$ denote the set of all orderings on $\mathcal{L}$, and let $\mathcal{R}_X$ denote the set of all orderings on X. Adapting terminology from Hammond (1976), a *generalized social welfare function* is a mapping $f : \mathcal{D} \subseteq \mathcal{R}_{\mathcal{L}} \to \mathcal{R}_X$. For any interpersonal profile $R \in \mathcal{D}$, we write $\succcurlyeq_R$ for $f(R)$.

For our interpersonal profiles to be extensively measurable, we need a concatenation operation on the set of lives. Instead of taking such an operation as primitive, we define it here in terms of the alternative-concatenation operation $\circ$ that we already have, at the cost of two additional assumptions. The first says that for any individuals i and j and alternatives x and y in which they (respectively) exist, there is an individual k and alternatives x' and y' such that x' and y' are just as good for k as x and y are for i and j (respectively); and in the special case where $i = j$, $x' \circ y'$ must be just as good for k as $x \circ y$ is for i.

Matching. For any interpersonal profile $R \in \mathcal{D} \subseteq \mathcal{R}_{\mathcal{L}}$ and any lives $(x, i), (y, j) \in \mathcal{L}$, there is some individual $k \in \mathbb{N}$ and alternatives $x', y' \in X_k$ such that $(x', k)I(x, i)$ and $(y', k)I(y, j)$; and for any such k, x', y', if $i = j$, then $(x \circ y, i)I(x' \circ y', k)$.

Matching lets us, for any $R \in \mathcal{D}$, define an operation $\oplus^R : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ as follows: for any $(x, i), (y, j) \in \mathcal{L}$, let $(x, i) \oplus^R (y, j) = (x' \circ y', k)$ for some k, x', y' such that $(x', k)I(x, i)$ and $(y', k)I(y, j)$. (When there are multiple such k, x', y', the choice can be arbitrary, since Matching requires all such choices to be equally good according to R.) Thus any number of lives led by distinct individuals can, in any profile, be concatenated into a single life. This is compatible, however, with a social preference for the existence of the many rather than the single "utility monster" (Nozick 1974). Further axioms are needed to rule out such a preference.[7]

Consider an individual i who exists in both x and y, and therefore also in $x \circ y$ (recall that $N(x \circ y) = N(x) \cup N(y)$). The axioms of extensive measurement will tell us how to

[7]Note also that what allows us to concatenate any number of lives is not just Matching, but also the assumption that X is closed under $\circ$. Those who wish to avoid this implication might therefore prefer a version of extensive measurement in which the concatenation operation is restricted (Krantz et al. 1971, Section 3.4). Analogous implications also hold in the standard utility-theoretic framework, where for any number of individuals and any utilities they might attain, there is some individual who can attain, in some outcome and some profile, the sum of those utilities.

value i's life in $x \circ y$ in terms of her life in x and her life in y: $(x \circ y, i) I ((x, i) \oplus^R (y, i))$. But what if i exists in x but not y? A natural hypothesis is that, since i does not even exist in y, concatenating y to x should not affect i's well-being. This is our second assumption.

Irrelevance of Nonexistence. For any interpersonal profile $R \in \mathcal{D}$, alternatives $x, y \in X$, and individual $i \in N(x) \setminus N(y)$, $(x \circ y, i) I (x, i)$.

This seems a plausible extension of the orthodox view that nothing can be better or worse for a person who does not exist (Broome 2004, Blackorby, Bossert, and Donaldson 2005).

We can now state our domain condition.

Interpersonal Extensive Domain. An interpersonal profile R is in $\mathcal{D}$ if and only if R satisfies Matching and Irrelevance of Nonexistence, and $(\mathcal{L}, R, \oplus^R)$ is an extensive structure.

Given Interpersonal Extensive Domain, each profile $R \in \mathcal{D}$ can be additively represented by a real-valued utility function. The function $U : \mathcal{L} \to \mathbb{R}$ additively represents a profile $R \in \mathcal{D}$ if and only if, for all $(x, i), (y, j) \in \mathcal{L}$,

(i) $U(x, i) \geq U(y, j)$ if and only if $(x, i) R (y, j)$;

(ii) $U((x, i) \oplus^R (y, j)) = U(x, i) + U(y, j)$.

As before, let $\mathcal{U}_R$ denote the set of all utility functions that additively represent R, and $\mathcal{U}_{\mathcal{D}} := \bigcup_{R \in \mathcal{D}} \mathcal{U}_R$.

4.1 *Variable-population welfarism*

The various Pareto conditions have the same interpretation as in Section 3, so we do not state them separately here; see Appendix C.

The reformulation of Ratio IIA in this framework requires some care because our life-concatenation operation is profile-dependent. For any subset of alternatives $S \subseteq X$, let $L(S) := \{(x, i) \in \mathcal{L} \mid x \in S\}$ denote the set of all lives led among the alternatives in S. For any such S and any profile R, let $L(S)^{\oplus^R}$ denote the closure of $L(S)$ under $\oplus^R$. For any $S, T \subseteq X$ and any profiles R, R', we define a *weak homomorphism* from $(L(S)^{\oplus^R}, R, \oplus^R)$ to $(L(T)^{\oplus^{R'}}, R', \oplus^{R'})$ as a mapping induced by a function $\phi : L(S)^{\oplus^R} \to L(T)^{\oplus^{R'}}$ such that, for all $(x, i), (y, j) \in L(S)^{\oplus^R}$,

(i) $(x, i) R (y, j)$ if and only if $\phi(x, i) R' \phi(y, j)$;

(ii) $\phi((x, i) \oplus^R (y, j)) I' (\phi(x, i) \oplus^{R'} \phi(y, j))$.

(For ease of exposition, we call $\phi : L(S)^{\oplus^R} \to L(T)^{\oplus^{R'}}$ itself a weak homomorphism. It is "weak" because condition (ii) only requires $\phi((x, i) \oplus^R (y, j))$ and $\phi(x, i) \oplus^{R'} \phi(y, j)$ to be indifferent according to R', rather than identical.)

Our Independence of Irrelevant Alternatives condition is as follows.

Interpersonal Ratio IIA. For all interpersonal profiles $R, R' \in \mathcal{D}$ and alternatives $x, y \in X$, if there is a weak homomorphism $\phi : L(\{x, y\})^{\oplus^R} \to L(\{x, y\})^{\oplus^{R'}}$ such that $\phi(x, i) =$

(x, i) and $\phi(y, j) = (y, j)$ for all individuals $i \in N(x)$ and $j \in N(y)$, then $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$.

As with Ratio IIA, this principle is equivalent to a more familiar condition formulated in terms of utility functions (see Proposition 5 in Appendix C).

For any utility profile $U : \mathcal{L} \to \mathbb{R}$, let $U(x, \cdot) : N(x) \to \mathbb{R}$ denote x's utility distribution in profile U. For any population $N \in \mathcal{P}$, $\mathbb{R}^N$ denotes the set of all utility distributions with domain N. The set of all utility distributions is $\Omega := \bigcup_{N \in \mathcal{P}} \mathbb{R}^N$. We call these "distributions" rather than "vectors" because Ω is not a vector space: we cannot add together utility distributions with different populations. The variable-population analogue of Welfarism is as follows.

Variable-Population Welfarism. There is a unique social welfare ordering $\succcurlyeq^*$ on Ω such that, for any $R \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $x, y \in X$, $x \succcurlyeq_R y$ if and only if $U(x, \cdot) \succcurlyeq^* U(y, \cdot)$.

As in Section 3, the key to our welfarism theorem in this setting is that the set of attainable utility distributions for the atomic alternatives is unrestricted. We have not assumed the existence of atomic alternatives for each population, however—only for each *singleton* population. But, for any population, we can find an atomic alternative for each member of the population and concatenate them to form an alternative in which all of those individuals exist. This is the strategy behind the proof of Theorem 4 in Appendix C.

THEOREM 4 (Variable-Population Welfarism Theorem). *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Pareto Indifference and Interpersonal Ratio IIA if and only if f satisfies Variable-Population Welfarism.*

As in the fixed-population setting, Interpersonal Extensive Domain requires the social welfare ordering to be invariant to similarity transformations of individual utilities. However, the same transformation must be applied to all individuals so as to preserve interpersonal comparisons.

Interpersonal Ratio-Scale Invariance. For every $u, v \in \Omega$ and positive real number k, $u \succcurlyeq^* v$ if and only if $ku \succcurlyeq^* kv$.

(See Proposition 6 in Appendix C.) This weaker invariance condition is what allows us to avoid the negative results of our Arrovian setting.

4.2 *A qualitative axiomatization of classical utilitarianism*

In the present framework, classical utilitarianism has a natural qualitative formulation. For any alternative $x \in X$ and profile $R \in \mathcal{D}$, let $\bigoplus^R_{i \in N(x)} (x, i)$ denote the concatenation of all the individuals' lives in x in arbitrary order.

Classical Utilitarianism. For any $x, y \in X$ and $R \in \mathcal{D}$, $x \succcurlyeq_R y$ if and only if

$$\bigoplus^R_{i \in N(x)} (x, i) \; R \bigoplus^R_{i \in N(y)} (y, i).$$

Given Interpersonal Extensive Domain, Classical Utilitarianism is equivalent to the claim that, for any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $x \succcurlyeq_R y$ if and only if $\sum_{i\in N(x)} U(x,i) \geq \sum_{i\in N(y)} U(y,i)$. For each $U \in \mathcal{U}_R$ additively represents R, so $U(\bigoplus^R_{i\in N(x)}(x,i)) = \sum_{i\in N(x)} U(x,i)$ and $U(\bigoplus^R_{i\in N(y)}(y,i)) = \sum_{i\in N(y)} U(y,i)$.

Our axiomatization of Classical Utilitarianism appeals to Weak Pareto, Interpersonal Ratio IIA, and two further conditions. First, we require the restriction of the social ordering to the alternatives facing a *fixed* population to be invariant to permutations on that fixed set of individuals.

Fixed-Population Anonymity. For any $N \in \mathcal{P}$ and $R, R' \in \mathcal{D}$, if there is a permutation $\sigma : N \to N$ and a weak homomorphism $\phi : L(X^N)^{\oplus R} \to L(X^N)^{\oplus R'}$ such that $\phi(x,i) = (x, \sigma(i))$ for all $(x,i) \in L(X^N)$, then for all $x, y \in X^N$, $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$.

(See Appendix D for the utility-theoretic analogue of this condition.) Our second principle has much the same interpretation as in Section 3.2.

Extensive Social Preference. For all $R \in \mathcal{D}$, $(X, \succcurlyeq_R, \circ)$ is an extensive structure.

Analogues of these two principles led to our negative results for Arrovian social welfare functions in Section 3. It is therefore noteworthy that they are not just compatible with our other axioms in the generalized framework; they lead, in conjunction with the other axioms, to Classical Utilitarianism.

THEOREM 5. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Interpersonal Ratio IIA, Weak Pareto, Fixed-Population Anonymity, and Extensive Social Preference if and only if f satisfies Classical Utilitarianism.*

The strategy behind the proof is this. We first show that, given our axioms, adding an individual with zero utility to a population is always a matter of indifference (Proposition 8). We are therefore able to strengthen Variable-Population Welfarism by constructing an "extended" social welfare ordering on the space $\mathbb{R}^\infty$ of all infinite sequences with finite support (Lemma 11). Fixed-Population Anonymity then requires this extended social welfare ordering to be fully anonymous (Lemma 12). By Extensive Social Preference and Proposition 1, the extended ordering can be additively represented by a real-valued social utility function. The proof of Theorem 5 then amounts to showing that this additive representation is of the weighted utilitarian form and that all weights must be equal.

The reason why our axioms lead to such a different result in this framework is the presence of interpersonal comparisons. The richer informational basis provided by interpersonal comparability leads to a considerably weaker invariance condition, which avoids the impossibilities that arose in the Arrovian framework.

5. CONCLUSION

Extensive measurement gives rise to natural weakenings of Arrow's conditions that are jointly consistent even when welfare is not interpersonally comparable. But, while there

are nondictatorial Arrovian social welfare functions that satisfy Extensive Domain, Ratio IIA, and Strong Pareto, there are none that also satisfy Anonymity or Extensive Social Preference. In the generalized framework, by contrast, analogues of these conditions are not only consistent; together, they uniquely characterize Classical Utilitarianism.

Extensive measurement, as we have seen, does much more for the classical utilitarian than just giving "meaning to the utilities to be added" (Arrow 1973, p. 255). It is, when applied at the social level, what distinguishes classical utilitarianism from other anonymously welfarist approaches to social choice, including other versions of utilitarianism.

Appendix A: Proofs for Section 3.1

We first show that Ratio IIA is equivalent, on our domain, to the following familiar condition.

Utility IIA. For any $x, y \in X$, $R, R' \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$, if $U_i(x) = U'_i(x)$ and $U_i(y) = U'_i(y)$ for every $i \in N$, then $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$.

Proposition 2. *If an Arrovian social welfare function f satisfies Extensive Domain, then f satisfies Ratio IIA if and only if f satisfies Utility IIA.*

Proof. Suppose that f satisfies Extensive Domain and Ratio IIA. Take some $x, y \in X$, $R, R' \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$ such that $U_i(x) = U'_i(x)$ and $U_i(y) = U'_i(y)$ for every $i \in N$. For each $z \in \{x, y\}^\circ$, there must be nonnegative integers n and m such that $U_i(z) = nU_i(x) + mU_i(y)$ and $U'_i(z) = nU'_i(x) + mU'_i(y)$ for every $i \in N$. Thus $U_i(z) = U'_i(z)$ for all $z \in \{x, y\}^\circ$. We must therefore have $R_i|_{\{x,y\}^\circ} = R'_i|_{\{x,y\}^\circ}$ for every $i \in N$, so $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$ by Ratio IIA, and Utility IIA is therefore satisfied.

For the other direction, suppose that f satisfies Extensive Domain and Utility IIA. Take some $x, y \in X$ and $R, R' \in \mathcal{D}$ such that $R_i|_{\{x,y\}^\circ} = R'_i|_{\{x,y\}^\circ}$ for every $i \in N$. Take some $U \in \mathcal{U}_R$ and $V \in \mathcal{U}_{R'}$. For any $w, z \in \{x, y\}^\circ$ and $i \in N$, we have wR_iz if and only if wR'_iz if and only if $V_i(w) \geq V_i(z)$, and $V_i(w \circ z) = V_i(w) + V_i(z)$. It follows that each $V_i|_{\{x,y\}^\circ}$ additively represents $R_i|_{\{x,y\}^\circ}$. Since $U \in \mathcal{U}_R$, $U_i|_{\{x,y\}^\circ}$ also additively represents $R_i|_{\{x,y\}^\circ}$. Thus, by the uniqueness component of Proposition 1, for each $i \in N$ there must be some $k_i > 0$ such that $V_i = k_iU_i$. Now let $U'_i = (1/k_i)V_i$ for every $i \in N$, so that $U' = (U'_1, \ldots, U'_n) \in \mathcal{U}_{R'}$ and $U'_i|_{\{x,y\}^\circ} = U_i|_{\{x,y\}^\circ}$. We have $U_i(x) = U'_i(x)$ and $U_i(y) = U'_i(y)$ for every $i \in N$, so $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$ by Utility IIA, and Ratio IIA is therefore satisfied. (Indeed, since $U_i|_{\{x,y\}^\circ} = U'_i|_{\{x,y\}^\circ}$, we also have the stronger consequence that $\succcurlyeq_R|_{\{x,y\}^\circ} = \succcurlyeq_{R'}|_{\{x,y\}^\circ}$.) □

We now show that our domain imposes no restriction on the assignment of utility vectors to the set A of atomic alternatives. This plays an important role in deriving our welfarism theorem, and appeals crucially to our assumption that each alternative has a unique decomposition into atoms, up to the order of concatenation.

Lemma 1. *Given Extensive Domain, for any function $g : A \to \mathbb{R}^n$, there is a utility profile $U \in \mathcal{U}_\mathcal{D}$ such that $U(a) = g(a)$ for all $a \in A$.*

PROOF. Assume Extensive Domain. Recall that every alternative is uniquely decomposable into finitely many atoms, up to the order of concatenation. Thus, for each $x \in X$, there is a unique function $\mu_x : A \to \mathbb{Z}_+$ (where $\mathbb{Z}_+$ denotes the set of nonnegative integers) such that each atom $a \in A$ occurs exactly $\mu_x(a)$ times in the decomposition of x. Let $A_x := \{a \in A \mid \mu_x(a) > 0\}$ denote the (finite, nonempty) set of atoms that occur at least once in the decomposition of x. Observe that, for all $x, y \in X$, $\mu_{x \circ y}(a) = \mu_x(a) + \mu_y(a)$ for all $a \in A$, and so $A_{x \circ y} = A_x \cup A_y$.

Take any function $g : A \to \mathbb{R}^n$. We define $U : X \to \mathbb{R}^n$ as follows: $U(x) = \sum_{a \in A_x} \mu_x(a) g(a)$ for all $x \in X$. This is well defined because μ_x is unique, and A_x is unique, finite, and nonempty, for each $x \in X$.

Clearly $U(a) = g(a)$ for every atom $a \in A$, since $A_a = \{a\}$ with $\mu_a(a) = 1$. To show that U is in $\mathcal{U}_\mathcal{D}$, define the profile R as follows: for all $x, y \in X$ and $i \in N$, $xR_i y$ if and only if $U_i(x) \geq U_i(y)$. For all $x, y \in X$ and $i \in N$, we have

$$\begin{aligned} U_i(x \circ y) &= \sum_{a \in A_{x \circ y}} \mu_{x \circ y}(a) U_i(a) \\ &= \sum_{a \in A_x \cup A_y} \big[\mu_x(a) + \mu_y(a)\big] U_i(a) \\ &= \sum_{a \in A_x} \mu_x(a) U_i(a) + \sum_{a \in A_y} \mu_y(a) U_i(a) \\ &= U_i(x) + U_i(y). \end{aligned}$$

It therefore follows from Proposition 1 that $(X, R_i, \circ)$ is an extensive structure for every $i \in N$, and thus, from Extensive Domain that $R \in \mathcal{D}$, so $U \in \mathcal{U}_\mathcal{D}$. □

We now use Lemma 1 to derive two technical results that are used in proving Theorem 1. Both appeal crucially to our assumption that there are at least three atomic alternatives, in addition to the unique decomposability of each alternative into atoms.

LEMMA 2. *If an Arrovian social welfare function f satisfies Extensive Domain, then for any alternatives $x, y \in X$, utility profile $U \in \mathcal{U}_\mathcal{D}$, and any utility vector $w \in \mathbb{R}^n$, there is an atomic alternative $a \in A$ and some profile $V \in \mathcal{U}_\mathcal{D}$ such that $V(x) = U(x)$, $V(y) = U(y)$, and $V(a) = w$.*

PROOF. As in the proof of Lemma 1, for each alternative $x \in X$, let $\mu_x(a)$ denote the number of occurrences of atom $a \in A$ in the decomposition of x, and let A_x denote the set of all atoms that occur at least once in the decomposition of x.

Take any alternatives x and y. Since A_x and A_y are finite, we can enumerate their elements: let $A_x \cup A_y = \{a_1, \ldots, a_k\}$, with $k \geq 1$. Extensive Domain implies that $U(x) = \sum_{i=1}^k \mu_x(a_i) U(a_i)$ and $U(y) = \sum_{i=1}^k \mu_y(a_i) U(a_i)$ for all $U \in \mathcal{U}_\mathcal{D}$.

Fix a particular profile U. If $k < 3$, the proof is trivial: since there are at least three atomic alternatives in A, simply let $V(a_i) = w$ for some $a_i \in A \setminus \{a_1, a_2\}$ and $V(a_j) = U(a_j)$ for all $j \neq i$ (such a V exists by Lemma 1). This preserves $V(x) = U(x)$ and $V(y) = U(y)$. Suppose instead, then, that $k \geq 3$.

We know that the system

$$\begin{pmatrix} \mu_x(a_1) & \cdots & \mu_x(a_k) \\ \mu_y(a_1) & \cdots & \mu_y(a_k) \end{pmatrix} \begin{pmatrix} U_1(a_1) & U_2(a_1) & \cdots & U_n(a_1) \\ \vdots & \vdots & \ddots & \vdots \\ U_1(a_k) & U_2(a_k) & \cdots & U_n(a_k) \end{pmatrix} = \begin{pmatrix} U_1(x) & \cdots & U_n(x) \\ U_1(y) & \cdots & U_n(y) \end{pmatrix}.$$

is satisfied. Write the above system as $\mathbf{MA} = \mathbf{U}$ and pick any vector $w \in \mathbb{R}^n$.

Since we have $\mathbf{MA} = \mathbf{U}$, we know (by the Rouché–Capelli theorem) that $\operatorname{rank}(\mathbf{M}) = \operatorname{rank}(\mathbf{M} \mid \mathbf{U})$, where $\operatorname{rank}(\mathbf{M} \mid \mathbf{U})$ denotes the rank of the augmented matrix $\mathbf{M} \mid \mathbf{U}$, formed by appending the columns of $\mathbf{U}$ to the right of $\mathbf{M}$. In addition, since $k \geq 3 > \operatorname{rank}(\mathbf{M})$, there must be some $2 \times (k-1)$ submatrix $\hat{\mathbf{M}}$ of $\mathbf{M}$ such that $\operatorname{rank}(\hat{\mathbf{M}}) = \operatorname{rank}(\mathbf{M})$. (Just find some 2×2 submatrix of $\mathbf{M}$ with $\operatorname{rank}(\mathbf{M})$-many linearly independent columns—there must be at least one—and delete a column not in that submatrix.)

Without loss of generality let

$$\hat{\mathbf{M}} = \begin{pmatrix} \mu_x(a_1) & \cdots & \mu_x(a_{k-1}) \\ \mu_y(a_1) & \cdots & \mu_y(a_{k-1}) \end{pmatrix}, \hat{\mathbf{U}} = \begin{pmatrix} U_1(x) - \mu_x(a_k)w_1 & \cdots & U_n(x) - \mu_x(a_k)w_n \\ U_1(y) - \mu_y(a_k)w_1 & \cdots & U_n(y) - \mu_y(a_k)w_n \end{pmatrix}.$$

It is not difficult to see that $\operatorname{rank}(\hat{\mathbf{M}} \mid \hat{\mathbf{U}}) = \operatorname{rank}(\mathbf{M} \mid \mathbf{U})$, since $\hat{\mathbf{U}} = \mathbf{U} - \binom{\mu_x(a_k)}{\mu_y(a_k)} w$ and $\operatorname{rank}(\mathbf{M} \mid \mathbf{U}) = \operatorname{rank}(\mathbf{M}) = \operatorname{rank}(\hat{\mathbf{M}})$. Thus $\operatorname{rank}(\hat{\mathbf{M}} \mid \hat{\mathbf{U}}) = \operatorname{rank}(\hat{\mathbf{M}})$. So (by Rouché–Capelli again) there is a $(k-1) \times n$ matrix $\mathbf{B}$ such that $\hat{\mathbf{M}}\mathbf{B} = \hat{\mathbf{U}}$. Using Lemma 1, we simply let $V(a_j)$ equal the jth row of $\mathbf{B}$ for all $j \in \{1, \ldots, k-1\}$, and $V(a_k) = w$, so that $V(x) = U(x)$ and $V(y) = U(y)$, as desired. □

Next we derive a multi-profile "neutrality" property for the atomic alternatives.

LEMMA 3. *If an Arrovian social welfare function f satisfies Extensive Domain, Pareto Indifference, and Utility IIA, then for any $a, b, a', b' \in A$, $R, R' \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$, if $U'(a') = U(a)$ and $U'(b') = U(b)$, then $a \succcurlyeq_R b$ if and only if $a' \succcurlyeq_{R'} b'$.*

PROOF. Take any $a, b, a', b' \in A$, $R, R' \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$. Suppose $U'(a') = U(a) = u$ and $U'(b') = U(b) = v$.

Since $|A| \geq 3$, we can use Lemma 1 to find some $c \in A \setminus \{b, b'\}$, $R^1, R^2, R^3 \in \mathcal{D}$, and $U^1 \in \mathcal{U}_{R^1}$, $U^2 \in \mathcal{U}_{R^2}$, $U^3 \in \mathcal{U}_{R^3}$ such that

(i) $U^1(a) = U^1(c) = u$ and $U^1(b) = v$

(ii) $U^2(c) = u$ and $U^2(b) = U^2(b') = v$

(iii) $U^3(a') = U^3(c) = u$ and $U^3(b') = v$.

These profile assignments are displayed in Table 1.

Utility IIA and Pareto Indifference (plus transitivity) imply, in alternating order, that $a \succcurlyeq_R b$ if and only if $a \succcurlyeq_{R^1} b$ if and only if $c \succcurlyeq_{R^1} b$ if and only if $c \succcurlyeq_{R^2} b$ if and only if $c \succcurlyeq_{R^2} b'$ if and only if $c \succcurlyeq_{R^3} b'$ if and only if $a' \succcurlyeq_{R^3} b'$ if and only if $a' \succcurlyeq_{R'} b'$. Thus $a \succcurlyeq_R b$ if and only if $a' \succcurlyeq_{R'} b'$. □

TABLE 1. $a \succcurlyeq_R b$ if and only if $a' \succcurlyeq_{R'} b'$.

	a	a'	b	b'	c
U	u		v		
U^1	u		v		u
U^2			v	v	u
U^3		u		v	u
U'		u		v	

PROOF OF THEOREM 1. Suppose that f satisfies Extensive Domain, Pareto Indifference, and Ratio IIA. By Proposition 2, f also satisfies Utility IIA. Define the following relation $\succcurlyeq^*$ on $\mathbb{R}^n$: for any $u, v \in \mathbb{R}^n$, $u \succcurlyeq^* v$ if and only if $a \succcurlyeq_R b$ for some $a, b \in A$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$ such that $U(a) = u$ and $U(b) = v$. We show that this relation is the unique social welfare ordering associated with f, by establishing that (i) it determines the social preference ordering assigned to each profile, (ii) it is transitive, (iii) it is complete, and (iv) it is unique.

(i) Take any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$. We show that $x \succcurlyeq_R y$ if and only if $U(x) \succcurlyeq^* U(y)$. Suppose without loss of generality that $U(x) = u$ and $U(y) = v$.

Use Lemma 2 to find an $R^1 \in \mathcal{D}$, $U^1 \in \mathcal{U}_{R^1}$, and $a \in A$ such that $U^1(a) = U^1(x) = u$ and $U^1(y) = v$, and then another $R^2 \in \mathcal{D}$, $U^2 \in \mathcal{U}_{R^2}$, and $b \in A$ such that $U^2(a) = u$ and $U^2(b) = U^2(y) = v$. (See Table 2.) Utility IIA and Pareto Indifference (given transitivity) imply, in alternating order, that $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R^1} y$ if and only if $a \succcurlyeq_{R^1} y$ if and only if $a \succcurlyeq_{R^2} y$ if and only if $a \succcurlyeq_{R^2} b$. Lemma 3 then implies that for any $a', b' \in A$, $R' \in \mathcal{D}$, and $U' \in \mathcal{U}_{R'}$ such that $U'(a') = u$ and $U'(b') = v$, $a \succcurlyeq_{R^2} b$ if and only if $a' \succcurlyeq_{R'} b'$. It follows that $x \succcurlyeq_R y$ if and only if $U(x) \succcurlyeq^* U(y)$, as desired.

(ii) To show that $\succcurlyeq^*$ is transitive, suppose that $u \succcurlyeq^* v$ and $v \succcurlyeq^* w$ for some $u, v, w \in \mathbb{R}^n$. Since there are at least three atoms in A, we can use Lemma 1 to find some $R \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $a, b, c \in A$ such that $U(a) = u$, $U(b) = v$, and $U(c) = w$. By the result of step (i) above, we must have $a \succcurlyeq_R b \succcurlyeq_R c$ and thus $a \succcurlyeq_R c$ by the transitivity of $\succcurlyeq_R$. Thus $u \succcurlyeq^* w$.

(iii) To show that $\succcurlyeq^*$ is complete, take any $u, v \in \mathbb{R}^n$. Since there are at least three atoms in A, we can use Lemma 1 to find some $a, b \in A$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$ such

TABLE 2. $x \succcurlyeq_R y$ if and only if $a \succcurlyeq_{R^2} b$ if and only if $a' \succcurlyeq_{R'} b'$.

	x	y	a	b	a'	b'
U	u	v				
U^1	u	v	u			
U^2		v	u	v		
U'					u	v

that $U(a) = u$ and $U(b) = v$. So, by the completeness of $\succcurlyeq_R$, either $u \succcurlyeq^* v$ or $v \succcurlyeq^* u$.

(iv) To see that $\succcurlyeq^*$ is unique, take any ordering $\succcurlyeq^{**}$ on $\mathbb{R}^n$ such that, for all $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $U(x) \succcurlyeq^{**} U(y)$ if and only if $x \succcurlyeq_R y$. Take any $u, v \in \mathbb{R}^n$. By Lemma 1, there must be some $a, b \in A$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$ such that $U(a) = u$ and $U(b) = v$. If $u \succcurlyeq^{**} v$, then $a \succcurlyeq_R b$ and thus $u \succcurlyeq^* v$, and if $u \succcurlyeq^* v$, then $x \succcurlyeq_R y$ for any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$ such that $U(x) = u$ and $U(y) = v$, by step (i) above; thus $u \succcurlyeq^{**} v$. Therefore $\succcurlyeq^*$ and $\succcurlyeq^{**}$ are identical.

It is easy to see that Welfarism implies Pareto Indifference and Utility IIA, and thus, given Extensive Domain and Proposition 2, Ratio IIA. □

PROPOSITION 3. *If an Arrovian social welfare function f satisfies Extensive Domain and Welfarism, then the social welfare ordering associated with f must satisfy Intrapersonal Ratio-Scale Invariance.*

PROOF. Suppose that f satisfies Extensive Domain and Welfarism. Take any utility vectors $u, v, u', v' \in \mathbb{R}^n$ for which, for every $i \in N$, there is some $k_i > 0$ such that $u'_i = k_i u_i$ and $v'_i = k_i v_i$. Suppose that $u \succcurlyeq^* v$, where $\succcurlyeq^*$ is the social welfare ordering associated with f. Then for any $R \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $x, y \in X$ such that $U(x) = u$ and $U(y) = v$, $x \succcurlyeq_R y$. For any such R and U, the profile $U' = (k_1 U_1, \ldots, k_n U_n)$ additively represents R as well, by the uniqueness component of Proposition 1. So by Welfarism, $u' \succcurlyeq^* v'$ as well. □

APPENDIX B: PROOFS FOR SECTION 3.2

The following condition is equivalent to Anonymity on our domain.

Utility Anonymity. For all $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $U' \in \mathcal{U}_{R'}$, if there is a permutation $\sigma : N \to N$ such that $U_i = U'_{\sigma(i)}$ for every $i \in N$, then $f(R) = f(R')$.

PROPOSITION 4. *If an Arrovian social welfare function f satisfies Extensive Domain, then f satisfies Anonymity if and only if f satisfies Utility Anonymity. If, in addition, f satisfies Welfarism, then f satisfies Anonymity or Utility Anonymity if and only if $\succcurlyeq^*$ is anonymous.*

PROOF. Suppose that f satisfies Extensive Domain and Anonymity. Take any $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $U' \in \mathcal{U}_{R'}$, and permutation $\sigma : N \to N$ such that $U_i = U'_{\sigma(i)}$ for every $i \in N$. Since U_i and $U'_{\sigma(i)}$ additively represent R_i and $R'_{\sigma(i)}$ respectively, this implies $R_i = R'_{\sigma(i)}$ for all $i \in N$. So $f(R) = f(R')$ by Anonymity, and Utility Anonymity is satisfied.

Suppose next that f satisfies Extensive Domain and Utility Anonymity. Take any $R, R' \in \mathcal{D}$ and $\sigma : N \to N$ such that $R_i = R'_{\sigma(i)}$ for every $i \in N$. Fix a profile $U \in \mathcal{U}_R$. Let $U' = (U_{\sigma(1)}, \ldots, U_{\sigma(n)})$. Clearly $U' \in \mathcal{U}_{R'}$. So $f(R) = f(R')$ by Utility Anonymity, and Anonymity is satisfied.

Now suppose that f satisfies Extensive Domain, Welfarism, and Anonymity and therefore Utility Anonymity. The anonymity of $\succcurlyeq^*$ follows from the proofs of d'Aspremont and Gevers (1977, Lemmas 4 and 5). It is easy to see that if $\succcurlyeq^*$ is anonymous, then f must satisfy Utility Anonymity and therefore Anonymity. □

PROOF OF THEOREM 2. Take an Arrovian social welfare function f that satisfies Extensive Domain, Ratio IIA, and either Strong Pareto or the conjunction of Pareto Indifference and Weak Pareto. By Theorem 1 and Proposition 3, f satisfies Welfarism and its associated social welfare ordering $\succcurlyeq^*$ satisfies Intrapersonal Ratio-Scale Invariance. By Proposition 4, f satisfies Anonymity if and only if its associated social welfare ordering is anonymous. We show that $\succcurlyeq^*$ cannot be anonymous given Strong Pareto or, when n is even, Weak Pareto.

First assume Strong Pareto. Let $a > b > 0$. By the anonymity of $\succcurlyeq^*$ and Strong Pareto, $(a, 0, \ldots, 0) \sim (0, \ldots, 0, a) \succ (0, \ldots, 0, b)$, so $(a, 0, \ldots, 0) \succ (0, \ldots, 0, b)$. By the same reasoning, $(0, \ldots, 0, a) \succ (b, 0, \ldots, 0)$. But Intrapersonal Ratio-Scale Invariance implies that $(a, 0, \ldots, 0) \succ (0, \ldots, 0, b)$ if and only if $(b, 0, \ldots, 0) \succ (0, \ldots, 0, a)$, by multiplying person 1's utilities in both vectors by b/a and person n's by a/b.

Next assume Weak Pareto and suppose that n is even. For any $x, y \in \mathbb{R}$, let $(\mathbf{x}, \mathbf{y})$ denote the vector in $\mathbb{R}^n$ the first half of whose components equal x and whose second half equals y. As before, assume that $a > b > 0$. By the anonymity of $\succcurlyeq^*$ and Weak Pareto, $(\mathbf{a}, -\mathbf{b}) \sim (-\mathbf{b}, \mathbf{a}) \succ (-\mathbf{a}, \mathbf{b})$, so $(\mathbf{a}, -\mathbf{b}) \succ (-\mathbf{a}, \mathbf{b})$. By the same reasoning, $(\mathbf{b}, -\mathbf{a}) \sim (-\mathbf{a}, \mathbf{b}) \prec (-\mathbf{b}, \mathbf{a})$, so $(\mathbf{b}, -\mathbf{a}) \prec (-\mathbf{b}, \mathbf{a})$. But these are inconsistent with Intrapersonal Ratio-Scale Invariance, which implies that $(\mathbf{a}, -\mathbf{b}) \succ (-\mathbf{a}, \mathbf{b})$ if and only if $(\mathbf{b}, -\mathbf{a}) \succ (-\mathbf{b}, \mathbf{a})$. □

We now lay out three results concerning Extensive Social Preference; these lead to the proof of Theorem 3.

First, we derive the following Pareto condition from Extensive Domain, Ratio IIA, Weak Pareto, and Extensive Social Preference.

Semistrong Pareto. For any $x, y \in X$ and any Arrovian profile $R \in \mathcal{D}$, if xR_iy for every $i \in N$, then $x \succcurlyeq_R y$.

Semistrong Pareto is, like Strong Pareto and unlike Weak Pareto, a strengthening of Pareto Indifference; it was named and distinguished by Weymark (1991, 1993).

LEMMA 4. *If an Arrovian social welfare function f satisfies Extensive Domain, Ratio IIA, Weak Pareto, and Extensive Social Preference, then it must also satisfy Semistrong Pareto.*

PROOF. Suppose that f satisfies Extensive Domain, Ratio IIA, Weak Pareto, and Extensive Social Preference. Suppose for reductio that, for some $x, y \in X$ and $R \in \mathcal{D}$, xR_iy for all $i \in N$, but $y \succ_R x$. Take some $U \in \mathcal{U}_R$ and use Lemma 2 to find an $R' \in \mathcal{D}$, $V \in \mathcal{U}_{R'}$, and $z \in X$ such that $V(x) = U(x)$, $V(y) = U(y)$, and $V(z) = U(y) - (1, \ldots, 1)$. By Ratio IIA and Proposition 2, $y \succ_{R'} x$. This implies, by the Archimedean property, that for some natural number n, $ny \circ z \succcurlyeq_{R'} nx \circ x$. By Extensive Domain, $V(ny \circ z) = V(ny) + V(z) = (n+1)V(y) - (1, \ldots, 1)$, and $V(nx \circ x) = V(nx) + V(x) = (n+1)V(x)$. But since $V_i(x) \geq V_i(y)$ for every $i \in N$, $(n+1)V_i(x) > (n+1)V_i(y) - 1$ for every $i \in N$ and natural number n. Thus we cannot have $ny \circ z \succcurlyeq_{R'} nx \circ x$ by Weak Pareto. □

LEMMA 5. *If an Arrovian social welfare function f satisfies Extensive Domain and Welfarism, then f satisfies Extensive Social Preference if and only if its associated social welfare ordering (SWO) $\succcurlyeq^*$ satisfies Extensive SWO.*

Extensive SWO. *The triple* $(\mathbb{R}^n, \succcurlyeq^*, +)$ *is an extensive structure.*

Proof of Lemma 5. Suppose that f satisfies Extensive Domain and Welfarism. Transitivity and Completeness are built into the definitions of $\succcurlyeq_R$ and $\succcurlyeq^*$. Weak Associativity follows from the associativity of $\circ$ and $+$, and the reflexivity of $\sim_R$ and $\sim^*$, respectively. So it remains to show that $(X, \succcurlyeq_R, \circ)$ satisfies Monotonicity and Archimedean if and only if $(\mathbb{R}^n, \succcurlyeq^*, +)$ does.

For Monotonicity , take any $u, v, w \in \mathbb{R}^n$, and any $R \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $x, y, z \in X$ such that $U(x) = u$, $U(y) = v$, and $U(z) = w$. Welfarism implies that $u \succcurlyeq^* v$ if and only if $x \succcurlyeq_R y$, and $x \circ z \succcurlyeq_R y \circ z$ if and only if $u + w \succcurlyeq^* v + w$. Extensive Social Preference implies that $x \succcurlyeq_R y$ if and only if $x \circ z \succcurlyeq_R y \circ z$; Extensive SWO implies $u \succcurlyeq^* v$ if and only if $u + w \succcurlyeq^* v + w$. Whichever we assume, the other follows. The proof for the Archimedean axiom is analogous. □

Lemma 6. *If a social welfare ordering* $\succcurlyeq^*$ *satisfies Extensive SWO and Semistrong Pareto, then it is additively represented by a social utility function* $W : \mathbb{R}^n \to \mathbb{R}$ *of the form, for some* $c_1, \dots, c_n \geq 0$,

$$W(u) = \sum_{i \in N} c_i u_i \quad \textit{for all } u \in \mathbb{R}^n. \tag{1}$$

Proof. By Extensive SWO and Proposition 1, $\succcurlyeq^*$ is representable by some $W : \mathbb{R}^n \to \mathbb{R}$ that satisfies Cauchy's functional equation

$$W(u + v) = W(u) + W(v) \quad \text{for all } u, v \in \mathbb{R}^n. \tag{2}$$

The general solution to such an equation is of the form (Aczél and Dhombres 1989, p. 35)

$$W(u) = \sum_{i=1}^{n} W_i(u_i), \tag{3}$$

where each $W_i : \mathbb{R} \to \mathbb{R}$ satisfies

$$W_i(x + y) = W_i(x) + W_i(y) \quad \text{for all } x, y \in \mathbb{R}. \tag{4}$$

So as to satisfy Semistrong Pareto, each W_i must be nondecreasing. Thus, by Aczél and Dhombres (1989, Corollary 2.5, p. 15), for each W_i there must be a constant $c_i \geq 0$ such that

$$W_i(x) = c_i x \quad \text{for all } x \in \mathbb{R}. \tag{5}$$

Putting (3) and (5) together, we get (1). □

Proof of Theorem 3. Suppose that f satisfies Extensive Domain, Ratio IIA, Weak Pareto, and Extensive Social Preference. By Lemma 4, f also satisfies Semistrong Pareto and thus Pareto Indifference. So, by Theorem 1, Proposition 3, and Lemma 5, f satisfies Welfarism, and the associated social welfare ordering $\succcurlyeq^*$ satisfies Intrapersonal Ratio-Scale Invariance and Extensive SWO. Lemma 6 then implies that $\succcurlyeq^*$ must be additively representable by a $W : \mathbb{R}^n \to \mathbb{R}$ that satisfies (1) with nonnegative weights.

So as to satisfy Weak Pareto, there must be some $i \in N$ such that $c_i > 0$. We then show that, for any $j \in N \setminus \{i\}$, $c_j = 0$. Suppose for reductio that, for some distinct $i, j \in N$, $c_i > 0$ and $c_j > 0$. Consider the unit vectors $\mathbf{e}_i, \mathbf{e}_j \in \mathbb{R}^n$ with all components equal to 0 except the ith (respectively, jth), which equals 1. We have $W(\mathbf{e}_i) = c_i$ and $W(\mathbf{e}_j) = c_j$ by (1). If c_i and c_j are both positive, then there must be some natural numbers n and m such that $nc_i > c_j$ and $mc_j > c_i$ by the Archimedean property of the real numbers. Since $W(n\mathbf{e}_i) = nc_i$ and $W(m\mathbf{e}_j) = mc_j$, this implies that $n\mathbf{e}_i \succ^* \mathbf{e}_j$ and $m\mathbf{e}_j \succ^* \mathbf{e}_i$. But, by Intrapersonal Ratio-Scale Invariance, $n\mathbf{e}_i \succ^* \mathbf{e}_j$ implies $\mathbf{e}_i \succ^* m\mathbf{e}_j$.

We have shown there to be exactly one $i \in N$ such that $c_i > 0$; for all other $j \in N$, $c_j = 0$. Thus $W(u) = c_i u_i$ for all $u \in \mathbb{R}^n$, so the social welfare function must be strongly dictatorial. It is easy to see that if f satisfies Extensive Domain and is strongly dictatorial, it must also satisfy Ratio IIA, Weak Pareto, and Extensive Social Preference. □

Appendix C: Proofs for Section 4.1

We first reformulate our Pareto and utility-theoretic IIA conditions in the generalized framework.

Weak Pareto. For any $N \in \mathcal{P}$, $x, y \in X^N$, and $R \in \mathcal{D}$, if $(x, i)P(y, i)$ for every $i \in N$, then $x \succ_R y$.

Pareto Indifference. For any $N \in \mathcal{P}$, $x, y \in X^N$, and $R \in \mathcal{D}$, if $(x, i)I(y, i)$ for every $i \in N$, then $x \sim_R y$.

Semistrong Pareto. For any $N \in \mathcal{P}$, $x, y \in X^N$, and $R \in \mathcal{D}$, if $(x, i)R(y, i)$ for every $i \in N$, then $x \succcurlyeq_R y$.

Strong Pareto. For any $N \in \mathcal{P}$, $x, y \in X^N$, and $R \in \mathcal{D}$, if $(x, i)R(y, i)$ for every $i \in N$, then $x \succcurlyeq_R y$; if, in addition, $(x, i)P(y, i)$ for some $i \in N$, then $x \succ_R y$.

Generalized Utility IIA. For any $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$ and $x, y \in X$, if for all $i \in N(x)$ and $j \in N(y)$, $U(x, i) = U'(x, i)$ and $U(y, j) = U'(y, j)$, then $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$.

Proposition 5. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Interpersonal Ratio IIA if and only if f satisfies Generalized Utility IIA.*

Proof. Suppose first that f satisfies Interpersonal Extensive Domain and Interpersonal Ratio IIA, and that for some $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$ and $x, y \in X$, $U(x, i) = U'(x, i)$ and $U(y, j) = U'(y, j)$ for all $i \in N(x)$, $j \in N(y)$. Define $\phi : L(\{x, y\})^{\oplus^R} \to L(\{x, y\})^{\oplus^{R'}}$ as follows: for all $s \in L(\{x, y\})$, let $\phi(s) = s$; for all $s \in L(\{x, y\})^{\oplus^R} \setminus L(\{x, y\})$, let $\phi(s) = s_1 \oplus^{R'} \cdots \oplus^{R'} s_k$ for some $s_1, \ldots, s_k \in L(\{x, y\})$ such that $s = s_1 \oplus^R \cdots \oplus^R s_k$.

Clearly $U'(\phi(s)) = U(s)$ for all $s \in L(\{x, y\})$. For all $s \in L(\{x, y\})^{\oplus^R} \setminus L(\{x, y\})$, there are some $s_1, \dots, s_k \in L(\{x, y\})$ such that

$$\begin{aligned} U'\big(\phi(s)\big) &= U'\big(s_1 \oplus^{R'} \cdots \oplus^{R'} s_k\big) \\ &= U'(s_1) + \cdots + U'(s_k) \\ &= U(s_1) + \cdots + U(s_k) \\ &= U\big(s_1 \oplus^{R} \cdots \oplus^{R} s_k\big) \\ &= U(s). \end{aligned}$$

Therefore, for any $s, t \in L(\{x, y\})^{\oplus^R}$, $U(s) \geq U(t)$ if and only if $U'(\phi(s)) \geq U'(\phi(t))$, so sRt if and only if $\phi(s)R'\phi(t)$; and, by construction, $U'(\phi(s \oplus^R t)) = U'(\phi(s) \oplus^{R'} \phi(t))$, so $\phi(s \oplus^R t)I'(\phi(s) \oplus^{R'} \phi(t))$. Thus ϕ is a weak homomorphism, so by Interpersonal Ratio IIA, $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$ and Generalized Utility IIA is satisfied.

Suppose next that f satisfies Interpersonal Extensive Domain and Generalized Utility IIA, and that for some $R, R' \in \mathcal{D}$ and $x, y \in X$, there is a weak homomorphism $\phi : L(\{x, y\})^{\oplus^R} \to L(\{x, y\})^{\oplus^{R'}}$ such that $\phi(x, i) = (x, i)$ and $\phi(y, j) = (y, j)$ for all $i \in N(x)$ and $j \in N(y)$. Pick a $U \in \mathcal{U}_R$ and $U' \in \mathcal{U}_{R'}$. For any $s, t \in L(\{x, y\})^{\oplus^R}$, we have sRt if and only if $\phi(s)R'\phi(t)$ if and only if $U'(\phi(s)) \geq U'(\phi(t))$, and $U'(\phi(s \oplus^R t)) = U'(\phi(s) \oplus^{R'} \phi(t)) = U'(\phi(s)) + U'(\phi(t))$. Let $V : L(\{x, y\})^{\oplus^R} \to \mathbb{R}$ denote the composition of $U'|_{L(\{x,y\})^{\oplus^{R'}}}$ with ϕ. We have just seen that V additively represents $R|_{L(\{x,y\})^{\oplus^R}}$: for any $s, t \in L(\{x, y\})^{\oplus^R}$, sRt if and only if $V(s) \geq V(t)$ if and only if $U'(\phi(s)) \geq U'(\phi(t))$, and $V(s \oplus^R t) = V(s) + V(t) = U'(\phi(s)) + U'(\phi(t))$. Since $U \in \mathcal{U}_R$, $U|_{L(\{x,y\})^{\oplus R}}$ also additively represents $R|_{L(\{x,y\})^{\oplus R}}$. Thus, by the uniqueness component of Proposition 1, there must be some $k > 0$ such that $V(s) = kU(s)$ for all $s \in L(\{x, y\})^{\oplus^R}$. Now let $V' = (1/k)U'$, so that $V' \in \mathcal{U}_{R'}$ and $V'(\phi(s)) = U(s)$ for all $s \in L(\{x, y\})^{\oplus^R}$. Remember that $\phi(s) = s$ for all $s \in L(\{x, y\})$. So $V'(x, i) = U(x, i)$ and $V'(y, j) = U(y, j)$ for all $i \in N(x)$ and $j \in N(y)$. Therefore, by Generalized Utility IIA, $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$, and Interpersonal Ratio IIA is satisfied. □

Lemma 7. *Given Interpersonal Extensive Domain, for any finite, nonempty set of atoms $B \subseteq A$ and any function $g : B \to \mathbb{R}$, there is a utility profile $U \in \mathcal{U}_{\mathcal{D}}$ such that, for every $a \in B$ and $i \in N(a)$, $U(a, i) = g(a)$.*

Proof. As in the proof of Lemma 1, let $\mu_x(a)$ denote the number of occurrences of atom $a \in A$ in the decomposition of alternative $x \in X$. Let $A_x := \{a \in A \mid \mu_x(a) > 0\}$ denote the atoms of x, as before. For each $x \in X$ and $i \in N(x)$, let $A_x^{\{i\}} := A_x \cap A^{\{i\}}$ denote the set of atoms with population $\{i\}$ that occur at least once in the decomposition of x. Let $N = \bigcup_{a \in B} N(a)$ denote the (finite) set of individuals who exist in some element of B.

We define $U : \mathcal{L} \to \mathbb{R}$ as follows. First, let $U(a, i) = g(a)$ for all $a \in B$ and $i \in N(a)$. For all $i \in N$ and $a \in A^{\{i\}} \setminus B$, let $U(a, i) = 0$.

Next, to ensure that Matching is satisfied, we need to assign utilities to lives led by individuals not in N in a way that "matches" the utilities of lives in $L(B^\circ)$. Since B is

finite, its closure B° under $\circ$ is countably infinite. So $L(B^\circ)$ is also countably infinite. It follows that the set $\binom{L(B^\circ)}{2}$ of all two-element subsets of $L(B^\circ)$ is countably infinite as well. Thus there is a bijection from $\binom{L(B^\circ)}{2}$ to $\mathbb{N} \setminus N$ that assigns an individual $k \in \mathbb{N} \setminus N$ to each unordered pair $\{(x, i), (y, j)\} \subseteq L(B^\circ)$. For each such k, choose two atoms $b, c \in A^{\{k\}}$, and let $U(b, k) = \sum_{a \in A_x^{\{i\}}} \mu_x(a) g(a)$ and $U(c, k) = \sum_{a \in A_y^{\{j\}}} \mu_y(a) g(a)$, where (x, i) and (y, j) are the elements of the pair to which k is assigned; let $U(d, k) = 0$ for all other atoms $d \in A^{\{k\}} \setminus \{b, c\}$ (of which there is at least one).

We have now defined $U(a, i)$ for all atoms $a \in A$ and $i \in N(a)$. Finally, for all $x \in X \setminus A$ and $i \in N(x)$, let $U(x, i) = \sum_{a \in A_x^{\{i\}}} \mu_x(a) U(a, i)$. Thus, more generally, $U(x, i) = \sum_{a \in A_x^{\{i\}}} \mu_x(a) U(a, i)$ for all $(x, i) \in \mathcal{L}$, since $A_a^{\{i\}} = \{a\}$ and $\mu_a(a) = 1$ for all $a \in A$.

Consider the profile R defined by $(x, i) R (y, j)$ if and only if $U(x, i) \geq U(y, j)$. Clearly R satisfies Irrelevance of Nonexistence: for all $x, y \in X$ and $i \in N(x) \setminus N(y)$, $U(x \circ y, i) = U(x, i)$, since $A_{x \circ y}^{\{i\}} = A_x^{\{i\}}$ and $\mu_{x \circ y}|_{A^{\{i\}}} = \mu_x|_{A^{\{i\}}}$ for all such x, y, i. And it is easy to see that R satisfies Matching, by the construction of U. So there is an operation $\oplus^R : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that, for all $(x, i), (y, j) \in \mathcal{L}$, $(x, i) \oplus^R (y, j) = (x' \circ y', k)$ for some $k \in \mathbb{N}$ and $x', y' \in X_k$ such that $(x, i) I (x', k)$ and $(y, j) I (y', k)$. For any such (x, i), (y, j), and $(x' \circ y', k)$, we have

$$\begin{aligned}
U\big((x, i) \oplus^R (y, j)\big) &= U\big(x' \circ y', k\big) \\
&= \sum_{a \in A_{x' \circ y'}^{\{k\}}} \mu_{x' \circ y'}(a) U(a, k) \\
&= \sum_{a \in A_{x'}^{\{k\}} \cup A_{y'}^{\{k\}}} \big[\mu_{x'}(a) + \mu_{y'}(a)\big] U(a, k) \\
&= \sum_{a \in A_{x'}^{\{k\}}} \mu_{x'}(a) U(a, k) + \sum_{a \in A_{y'}^{\{k\}}} \mu_{y'}(a) U(a, k) \\
&= U\big(x', k\big) + U\big(y', k\big) \\
&= U(x, i) + U(y, j).
\end{aligned}$$

It therefore follows from Proposition 1 that $(\mathcal{L}, R, \oplus^R)$ is an extensive structure, and thus from Interpersonal Extensive Domain that $R \in \mathcal{D}$, so $U \in \mathcal{U}_\mathcal{D}$. □

For any population $N \in \mathcal{P}$, let $\{a_i\}_{i \in N}$ be a set of atomic alternatives with $N(a_i) = \{i\}$ for each a_i. Let $\bigcirc_{i \in N} a_i$ denote the concatenation of each of these alternatives (exactly once) in arbitrary order, so that $N(\bigcirc_{i \in N} a_i) = N$. Let A^N denote the set of all such concatenations of one-person alternatives involving the members of N. For any populations $M, N \in \mathcal{P}$, and $x \in A^M$ and $y \in A^N$, where $x = \bigcirc_{i \in M} a_i$ and $y = \bigcirc_{i \in N} b_i$, say that x and y are *nonoverlapping* if and only if $\{a_i\}_{i \in M} \cap \{b_i\}_{i \in N} = \emptyset$. We have the following lemma.

Lemma 8. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then for any populations $M, N, O \in \mathcal{P}$, there are nonoverlapping alternatives*

$x \in A^M$, $y \in A^N$, and $z \in A^O$. And for any such x, y, z, and any utility distributions $u \in \mathbb{R}^M$, $v \in \mathbb{R}^N$, $w \in \mathbb{R}^O$, there is a utility profile $U \in \mathcal{U}_{\mathcal{D}}$ such that $U(x,\cdot) = u$, $U(y,\cdot) = v$, $U(z,\cdot) = w$.

Proof. Recall that we have assumed, for each individual, the existence of at least three atomic alternatives in which only that individual exists. So we can find disjoint sets of atomic alternatives $\{a_i\}_{i\in M}$, $\{b_j\}_{j\in N}$, and $\{c_k\}_{k\in O}$. Let $x = \bigcirc_{i\in M} a_i$, $y = \bigcirc_{j\in N} b_j$, and $z = \bigcirc_{k\in O} c_k$, so that $x \in A^M$, $y \in A^N$, and $z \in A^O$ are nonoverlapping concatenations of single-person atomic alternatives.

For any $u \in \mathbb{R}^M$, $v \in \mathbb{R}^N$, and $w \in \mathbb{R}^O$, we can use Lemma 7 to find some $U \in \mathcal{U}_{\mathcal{D}}$ such that $U(a_i, i) = u_i$, $U(b_j, j) = v_j$, and $U(c_k, k) = w_k$ for all $i \in M$, $j \in N$, and $k \in O$. By the Irrelevance of Nonexistence condition of Interpersonal Extensive Domain, $(x, i)I(a_i, i)$, $(y, j)I(b_j, j)$, and $(z, k)I(c_k, k)$ for all $i \in M$, $j \in N$, and $k \in O$. So $U(x, i) = u_i$, $U(y, j) = v_j$, and $U(z, k) = w_k$ for every $i \in M$, $j \in N$, and $k \in O$. Thus $U(x,\cdot) = u$, $U(y,\cdot) = v$, and $U(z,\cdot) = w$, as desired. □

Lemma 8 provides us with a set of free triples in the sense of Weymark (1998): that is, a set of three alternatives for which the domain of attainable utility distributions is unrestricted. We also have the following analogues of Lemmas 2 and 3.

Lemma 9. *If f satisfies Interpersonal Extensive Domain, then for any populations $M, N, O \in \mathcal{P}$, alternatives $x \in X^M$ and $y \in X^N$, any utility profile $U \in \mathcal{U}_{\mathcal{D}}$, and utility distribution $w \in \mathbb{R}^O$, there is some $z \in A^O$ and $V \in \mathcal{U}_{\mathcal{D}}$ such that $V(x,\cdot) = U(x,\cdot)$, $V(y,\cdot) = U(y,\cdot)$, and $V(z,\cdot) = w$.*

Proof. The proof is analogous to that of Lemma 2, except that we choose an atomic alternative $a_i \in A^{\{i\}}$ for each $i \in O$ and let z be the concatenation of all these alternatives. This is trivial for $i \in O \setminus (M \cap N)$. For $i \in O \cap M \cap N$, we can use the same strategy as the one used in the proof of Lemma 2 to find an $a_i \in A^{\{i\}}$ and a $V \in \mathcal{U}_{\mathcal{D}}$ such that $V(a_i, i) = w_i$ while preserving $V(x, i) = U(x, i)$ and $V(y, i) = U(y, i)$. (This relies on Lemma 7.) We then let z be the concatenation of all these atomic, one-person alternatives, so that $V(z,\cdot) = w_i$ for every $i \in O$ while preserving $V(x,\cdot) = U(x,\cdot)$ and $V(y,\cdot) = U(y,\cdot)$, as desired. □

Lemma 10. *If f satisfies Interpersonal Extensive Domain, Pareto Indifference, and Generalized Utility IIA, then for any $M, N \in \mathcal{P}$, $a, a' \in A^M$, $b, b' \in A^N$, $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $U' \in \mathcal{U}_{R'}$, if $U'(a', i) = U(a, i)$ for all $i \in M$ and $U'(b', j) = U(b, j)$ for all $j \in N$, then $a \succcurlyeq_R b$ if and only if $a' \succcurlyeq_{R'} b'$.*

(The proof of Lemma 10 is omitted; it is analogous to that of Lemma 3.)

We can now define our social welfare ordering on Ω.

Proof of Theorem 4. Suppose that f satisfies Interpersonal Extensive Domain, Pareto Indifference, and Interpersonal Ratio IIA. Define the social welfare ordering as follows:

for any $M, N \in \mathcal{P}$, $u \in \mathbb{R}^M$, and $v \in \mathbb{R}^N$, $u \succcurlyeq^* v$ if and only if $a \succcurlyeq_R b$ for some nonoverlapping $a \in A^M$ and $b \in A^N$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$ such that $U(a, \cdot) = u$ and $U(b, \cdot) = v$.

It suffices to show (i) that this relation determines the social preference ordering assigned to each profile, (ii) that this relation is transitive, (iii) that it is complete, and (iv) that it is unique. The demonstrations of these claims are exactly analogous to those of the corresponding claims in the proof of Theorem 1 and are therefore omitted.

It is easy to see that Variable-Population Welfarism implies Pareto Indifference and Generalized Utility IIA and therefore Interpersonal Ratio IIA. □

PROPOSITION 6. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain and Variable-Population Welfarism, then the social welfare ordering associated with f must satisfy Interpersonal Ratio-Scale Invariance.*

The proof of Proposition 6 is exactly similar to that of Proposition 3 and is therefore omitted.

APPENDIX D: PROOFS FOR SECTION 4.2

Fixed-Population Anonymity is equivalent to the following condition formulated in terms of numerical utilities.

Fixed-Population Utility Anonymity. For any $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$, and $N \in \mathcal{P}$, if there is a permutation $\sigma : N \to N$ such that $U(x, i) = U'(x, \sigma(i))$ for all $\mathcal{L} \in X^N \times N$, then for all $x, y \in X^N$, $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$.

PROPOSITION 7. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Fixed-Population Anonymity if and only if f satisfies Fixed-Population Utility Anonymity.*

PROOF. Suppose that f satisfies Interpersonal Extensive Domain and Fixed-Population Anonymity. Take some $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$, and $N \in \mathcal{P}$, and permutation $\sigma : N \to N$ such that $U(x, i) = U'(x, \sigma(i))$ for all $\mathcal{L} \in X^N \times N$. Define $\phi : L(X^N)^{\oplus^R} \to L(X^N)^{\oplus^{R'}}$ as follows: for all $(x, i) \in L(X^N)$, let $\phi(x, i) = (x, \sigma(i))$; for all $s \in L(X^N)^{\oplus^R} \setminus L(X^N)$, let $\phi(s) = \phi(s_1) \oplus^{R'} \cdots \oplus^{R'} \phi(s_k)$ for some $s_1, \ldots, s_k \in L(X^N)$ such that $s = s_1 \oplus^R \cdots \oplus^R s_k$. By reasoning analogous to that in the second paragraph of the proof of Proposition 5, ϕ is a weak homomorphism. Therefore, for all $x, y \in X^N$, $x \succcurlyeq_R y$ if and only if $x \succcurlyeq_{R'} y$, so Fixed-Population Utility Anonymity is satisfied.

Suppose next that f satisfies Interpersonal Extensive Domain and Fixed-Population Utility Anonymity. Take some $R, R' \in \mathcal{D}$ and $N \in \mathcal{P}$, permutation $\sigma : N \to N$, and weak homomorphism $\phi : L(X^N)^{\oplus^R} \to L(X^N)^{\oplus^{R'}}$ such that $\phi(x, i) = (x, \sigma(i))$ for all $(x, i) \in X^N \times N$. By reasoning analogous to that in the third paragraph of the proof of Proposition 5, there exist $U \in \mathcal{U}_R$ and $U' \in \mathcal{U}_{R'}$ such that $U(x, i) = U'(x, \sigma(i))$ for all $(x, i) \in X^N \times N$. So, by Fixed-Population Utility Anonymity, $\succcurlyeq_R |_{X^N} = \succcurlyeq_{R'} |_{X^N}$, and Fixed-Population Anonymity is satisfied. □

One especially powerful implication of Extensive Social Preference in the variable-population setting is that, in the presence of Interpersonal Extensive Domain and Pareto Indifference, it implies that the addition of "null" lives to a population is always a matter of social indifference. An alternative $z \in X$ is *null for individual* $i \in N(z)$, relative to a profile R, if and only if $((z,i) \oplus^R (z,i))I(z,i)$. An alternative z is *universally null*, relative to R, if and only if z is null for all $i \in N(z)$. In any given profile, there may or may not be universally null alternatives. But if there are, the following condition says that their concatenation to an alternative is always a matter of indifference.

Null Critical Levels. For any $R \in \mathcal{D}$ and any $z \in X$ that is universally null in R, $x \circ z \sim_R x$ for all $x \in X$.

Proposition 8. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, Pareto Indifference, and Extensive Social Preference, then f satisfies Null Critical Levels.*

Proof. Take any profile R and $z \in X$ such that $((z,i) \oplus^R (z,i))I(z,i)$ for all $i \in N(z)$. By the Matching condition of Interpersonal Extensive Domain, $((z,i) \oplus^R (z,i))I(z \circ z, i)$ for all $i \in N(z)$. Thus $(z \circ z, i)I(z,i)$ for all $i \in N(z)$, so $z \circ z \sim_R z$ by Pareto Indifference. So, by the Monotonicity condition of Extensive Social Preference, $x \circ (z \circ z) \sim_R x \circ z$; by Weak Associativity, $x \circ (z \circ z) \sim_R (x \circ z) \circ z$, so $(x \circ z) \circ z \sim_R x \circ z$ by Transitivity; by Monotonicity again, $x \circ z \sim_R x$. □

As mentioned in Section 4, the field Ω of the social welfare ordering $\succcurlyeq^*$ is not a vector space: we cannot add together utility distributions with different domains. This can be rectified by strengthening Variable-Population Welfarism in the following way. Let $\mathbb{R}^\infty$ denote the set of all infinite sequences with finite support: that is, $\mathbb{R}^\infty := \{u : \mathbb{N} \to \mathbb{R} \mid u_i \neq 0 \text{ for finitely many } i \in \mathbb{N}\}$. Unlike Ω, $\mathbb{R}^\infty$ is a vector space: for any $u, v \in \mathbb{R}^\infty$, $(u+v)_i = u_i + v_i$ for every $i \in \mathbb{N}$. For any population $N \in \mathcal{P}$, let $\iota_N : \mathbb{R}^N \hookrightarrow \mathbb{R}^\infty$ denote the canonical inclusion such that for each $u \in \mathbb{R}^N$, $\iota_N(u)_i = u_i$ for all $i \in N$ and $\iota_N(u)_j = 0$ for all $j \in \mathbb{N} \setminus N$. Let $\iota : \Omega \hookrightarrow \mathbb{R}^\infty$ (no subscript) denote the union of all these inclusions. We call an ordering $\succcurlyeq^\infty$ on $\mathbb{R}^\infty$ an *extended* social welfare ordering.

Extended Welfarism. There is a unique ordering $\succcurlyeq^\infty$ on $\mathbb{R}^\infty$ such that, for any profile $R \in \mathcal{D}$, any $U \in \mathcal{U}_R$, and any alternatives $x, y \in X$, $x \succcurlyeq_R y$ if and only if $\iota(U(x,\cdot)) \succcurlyeq^\infty \iota(U(y,\cdot))$.

Lemma 11. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Variable-Population Welfarism and Null Critical Levels if and only if f satisfies Extended Welfarism.*

Proof. Take any $M, N \in \mathcal{P}$, $u \in \mathbb{R}^M$, and $v \in \mathbb{R}^N$. Suppose $\iota_M(u) = \iota_N(v)$. We show that $u \sim^* v$. This is obvious if $M = N$, since then $u = v$. So suppose $M \neq N$. Let $u \frown v$ denote the utility distribution in $\mathbb{R}^{M \cup N}$ such that, for all $i \in M \cup N$, $(u \frown v)_i = u_i = v_i$ if $i \in M \cap N$ and $(u \frown v)_i = 0$ otherwise. We show that $u \sim^* (u \frown v) \sim^* v$.

By Lemma 8, there must be some $x \in X^M$, $z \in X^N$, $R \in \mathcal{D}$, and $U : \mathcal{L} \to \mathbb{R}$ that additively represents R such that $U(x, \cdot) = u$ and $U(z, i) = 0$ for all $i \in N$. It follows from Proposition 1 that z is universally null. So by Proposition 8, $x \circ z \sim_R x$. Notice, however, that $U(x \circ z, \cdot) = u \frown v$, so by Variable-Population Welfarism, $u \sim^* (u \frown v)$. An exactly similar argument shows $v \sim^* (u \frown v)$. Thus $u \sim^* v$.

We now define $\succcurlyeq^\infty$ as follows: for all $u, v \in \mathbb{R}^\infty$, $u \succcurlyeq^\infty v$ if and only if, for some $M, N \in \mathcal{P}$, and $u' \in \mathbb{R}^M$ and $v' \in \mathbb{R}^N$ such that $\iota(u') = u$ and $\iota(v') = v$, $u' \succcurlyeq^* v'$. For any such $u, v \in \mathbb{R}^\infty$, there exist $M, N \in \mathcal{P}$, and $u' \in \mathbb{R}^M$ and $v' \in \mathbb{R}^N$ such that $\iota(u') = u$ and $\iota(v') = v$, so $\succcurlyeq^\infty$ inherits completeness from $\succcurlyeq^*$. In addition, we have just seen that for any $M', N' \in \mathcal{P}$, and $u^* \in \mathbb{R}^{M'}$ and $v^* \in \mathbb{R}^{N'}$ such that $\iota(u^*) = \iota(u') = u$ and $\iota(v^*) = \iota(v') = v$, $u' \sim^* u^*$ and $v' \sim^* v^*$, so $u^* \succcurlyeq^* v^*$ if and only if $u \succcurlyeq^\infty v$. It is easy to see that $\succcurlyeq^\infty$ must also be transitive and unique.

For the other direction, suppose that f satisfies Extended Welfarism. Then we define the social welfare ordering $\succcurlyeq^*$ as follows: for all $u, v \in \Omega$, $u \succcurlyeq^* v$ if and only if $\iota(u) \succcurlyeq^\infty \iota(v)$. It is clear that $\succcurlyeq^*$ is an ordering and that, by Extended Welfarism, for any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $x \succcurlyeq_R y$ if and only if $U(x, \cdot) \succcurlyeq^* U(y, \cdot)$. Finally, to see that Extended Welfarism implies Null Critical Levels, suppose that z is universally null in a profile R. Then for any $U \in \mathcal{U}_R$, $U(z, i) = 0$ for all $i \in N(z)$. For any $x \in X$, $\iota(U(x \circ z, \cdot)) = \iota(U(x, \cdot))$, so by Extended Welfarism, $x \circ z \sim_R x$. □

An extended social welfare ordering is *fully anonymous* if and only if, for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and $u, v \in \mathbb{R}^\infty$ such that $u_i = v_{\sigma(i)}$ for every $i \in \mathbb{N}$, $u \sim^\infty v$.

Lemma 12. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain and Extended Welfarism, then f satisfies Fixed-Population Anonymity if and only if its associated extended social welfare ordering $\succcurlyeq^\infty$ is fully anonymous.*

Proof. Suppose f satisfies Interpersonal Extensive Domain and Extended Welfarism. Clearly, if $\succcurlyeq^\infty$ is fully anonymous, then Fixed-Population Anonymity must be satisfied. For the other direction, suppose that f satisfies Fixed-Population Anonymity and thus Fixed-Population Utility Anonymity (by Proposition 7). Take any $u, v \in \mathbb{R}^\infty$ such that, for some permutation $\sigma : \mathbb{N} \to \mathbb{N}$, $u_i = v_{\sigma(i)}$ for every $i \in \mathbb{N}$. Let $N = \{i \in \mathbb{N} \mid u_i \neq v_i\}$. Since u and v have finite support, N must be finite even if σ itself has infinite support. Consider the distributions $u^*, v^* \in \mathbb{R}^N$ such that $\iota(u^*) = u$ and $\iota(v^*) = v$. There is a permutation $\sigma^* : N \to N$ such that $u^*_i = v^*_{\sigma^*(i)}$ for every $i \in N$. By Fixed-Population Utility Anonymity and Proposition 7, $u^* \sim^* v^*$. Thus, by Extended Welfarism, $u \sim^\infty v$, as desired. □

Proof of Theorem 5. Suppose that f satisfies Interpersonal Extensive Domain, Interpersonal Ratio IIA, Weak Pareto, Fixed-Population Anonymity, and Extensive Social Preference. By Lemma 4, f must also satisfy Semistrong Pareto and thus Pareto Indifference. So by Theorem 4 and Proposition 8, f satisfies Variable-Population Welfarism and Null Critical Levels, and thus, by Lemma 11, Extended Welfarism. By Fixed-Population Anonymity and Lemma 12, the extended social welfare ordering $\succcurlyeq^\infty$ is fully anonymous.

The proof of Lemma 5 can be easily adapted to show that $(\mathbb{R}^\infty, \succcurlyeq^\infty, +)$ is an extensive structure. So $\succcurlyeq^\infty$ is additively representable by a social utility function $W : \mathbb{R}^\infty \to \mathbb{R}$ that satisfies Cauchy's functional equation:

$$W(u+v) = W(u) + W(v) \quad \text{for all } u, v \in \mathbb{R}^\infty. \tag{6}$$

For each $i \in \mathbb{N}$, define $W_i : \mathbb{R} \to \mathbb{R}$ so that $W_i(x) = W(\iota_{\{i\}}(i \mapsto x))$ for all $x \in \mathbb{R}$.

For any $u \in \mathbb{R}^\infty$, there must be some $k \in \mathbb{N}$ such that $u_i = 0$ for all $i > k$. (Otherwise u would have infinite support.) Thus, for any $u \in \mathbb{R}^\infty$, there is a $k \in \mathbb{N}$ such that

$$u = (u_1, 0, 0, \dots) + (0, u_2, 0, 0, \dots) + \cdots + (0, \dots, 0, u_k, 0, 0, \dots) + (0, 0, \dots). \tag{7}$$

This implies, by (6),

$$\begin{aligned} W(u) = {} & W(u_1, 0, 0, \dots) + W(0, u_2, 0, 0, \dots) + \cdots \\ & + W(0, \dots, 0, u_k, 0, 0, \dots) + W(0, 0, \dots) \end{aligned} \tag{8}$$

Since $W(0, 0, \dots) = 0$, this simplifies to

$$W(u) = W(u_1, 0, 0, \dots) + W(0, u_2, 0, 0, \dots) + \cdots + W(0, \dots, 0, u_k, 0, 0, \dots). \tag{9}$$

So, by (6) and the definition of W_i, we have that for any $u \in \mathbb{R}^\infty$, there is a $k \in \mathbb{N}$ such that

$$W(u) = \sum_{i=1}^{k} W_i(u_i). \tag{10}$$

For each $i \in \mathbb{N}$, we must have

$$W_i(x+y) = W_i(x) + W_i(y) \quad \text{for all } x, y \in \mathbb{R}. \tag{11}$$

Thus $W_i(0) = 0$ for all $i \in \mathbb{N}$, so

$$W(u) = \sum_{i=1}^{\infty} W_i(u_i) \quad \text{for all } u \in \mathbb{R}^\infty. \tag{12}$$

Each W_i must be nondecreasing so as to satisfy Semistrong Pareto. So by Aczél and Dhombres (1989, Corollary 2.5, p. 15), for each W_i there must be a constant $c_i \geq 0$ such that

$$W_i(x) = c_i x \quad \text{for all } x \in \mathbb{R}. \tag{13}$$

To satisfy Weak Pareto and the full anonymity of $\succcurlyeq_\infty$, there must be some $c > 0$ such that $c_i = c$ for all $i \in \mathbb{N}$. So

$$W(u) = \sum_{i=1}^{\infty} c(u_i) = c \sum_{i=1}^{\infty} u_i \quad \text{for all } u \in \mathbb{R}^\infty. \tag{14}$$

For any such c, and any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $W(\iota(U(x, \cdot))) \geq W(\iota(U(y, \cdot)))$ if and only if $\sum_{i \in N(x)} U(x, i) \geq \sum_{i \in N(y)} U(y, i)$, which is equivalent to Classical Utilitarianism.

It is straightforward to verify that, given Interpersonal Extensive Domain, Classical Utilitarianism satisfies Interpersonal Ratio IIA, Weak Pareto, Fixed-Population Anonymity, and Extensive Social Preference. □

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