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Expected balanced uncertain utility

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We introduce and analyze *expected balanced uncertain utility (EBUU) theory*. A prior and a *balanced outcome-set utility* characterize an EBUU decision maker. Conditional on a reference or "balancing value," the latter assigns a utility to each outcome-set. The decision maker associates with each act, its *envelope*, the minimal measurable mapping from states to outcome-sets that contains the act. She then (implicitly) ranks an act according to the balancing value at which the expected balanced utility of its associated envelope is zero. As a consequence, her risk preferences need only exhibit betweenness allowing for behavior that can accommodate Allais-type paradoxes.

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1. Introduction

In the tradition of the voluminous literature initiated by Ellsberg (1961), consider a decision maker (hereafer, DM) who possesses only partial information about the underlying stochastic process that determines the resolution of the uncertainty she faces. In particular, this means she is not comfortable quantifying with a precise probability the uncertainty she associates with each and every event. There does exist, however, a rich collection of events she deems *measurable* over which is defined her *prior*, a unique probability representing her beliefs over those events.

An object of choice for our DM is an uncertain prospect or *act* that maps each state to an outcome. Using her prior, any act measurable with respect to her prior can be mapped to a corresponding probability distribution or *lottery* over outcomes. Thus the restriction of her preferences to measurable acts may be viewed as inducing a preference relation over lotteries, which we refer to as the DM's *risk preferences*.

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In Gul and Pesendorfer's (2014) expected uncertain utility model, the DM deems measurable precisely those events in which the event and its complement jointly satisfy a version of Savage's (1954) state-separability postulate **P2**.[1] As a consequence, her risk preferences conform to expected utility. The Allais paradoxes, however, clearly illustrate that **P2** is not only challenged when probabilities are unknown. This is certainly the case when it comes to descriptive modeling (see, e.g., Tversky and Shafir (1992) for empirical findings). Furthermore, even on normative grounds **P2** has not gone unchallenged (see, e.g., Heukelom (2015) for an extensive historical account). Hence our goal is to characterize a class of DMs who may perceive ambiguity but whose risk preferences need not conform to expected utility theory.

One approach taken by Grant, Rich, and Stecher (2022) that allows for risk-preferences that can accommodate Allais style violations of expected utility is to assume the set of measurable events are exogeneously specified and then axiomatize a family of preference relations in which the evaluation of an arbitrary act is characterized by a mapping to an equivalent measurable act along with a generalized notion of the certainty equivalent of a lottery.[2]

The approach taken here is to consider an alternative property an event must satisfy in order for it to be deemed measurable by the DM and then explore its implications for the corresponding risk-preferences. The one we propose is simple as it corresponds to Grant, Kajii, and Polak's (2000) (weak) decomposability property (in its strict form). Letting $\succsim$ denote the DM's preferences over acts and writing $f_E g$ for the act that agrees with the act f on the event E and with the act g on its complement, an event R is *decomposable*, if for any pair of acts f and g:

$$[f_R g \succ g \text{ and } g_R f \succ g] \quad \Longrightarrow \quad f \succ g.$$

Grant, Kajii, and Polak (2000) contend that by interpreting a statement like *"f would be preferred to g if the event R were known to obtain"* as *only* entailing $f_R g \succ g$, decomposability may be interpreted as encapsulating the following reasoning:

> *If the DM would prefer f to g knowing R obtains, and she would prefer f to g knowing R does not obtain, then she should prefer f to g even though she currently does not know whether R will or will not obtain.*

That is, similar to the property employed by Gul and Pesendorfer to identify those events deemed measurable by the DM, decomposability provides a way to operationalize Savage's (1954) *extralogical Sure-Thing Principle* (STP). With measurable events classified as those that satisfy decomposability, as Grant, Kajii, and Polak (2000) establish, the corresponding risk-preferences need only exhibit the betweenness property of Chew (1983) and Dekel (1986), and thus can accommodate Allais style violations of expected utility.

[1] Gul and Pesendorfer refer to any such event as *ideal*.

[2] Grant, Rich, and Stecher (2022) also consider endogenizing the set of events the DM views as measurable by utilizing Epstein and Zhang's (2001) preference-based definition for classifying measurable events. Even then, however, in their representation theorem (Theorem 5, p. 14) the structure of the set of measurable events needed for the evaluation of arbitrary acts is *assumed* as part of the hypothesis of the theorem and not derived from the postulates they propose.

Our second point of departure concerns the handling of nonmeasurable acts. To model her valuation of such acts, we adapt Gul and Pesendorfer's construction that assigns to each act a measurable *envelope*. We retain their notion of a *measurable split* of the state space induced by the preimage of the act. Each element of this split corresponds to an outcome-set (with finite cardinality), for which, given her limited information about the underlying stochastic process, renders her incapable of attributing any fraction of the probability her prior assigns to that element of the split to any strict subset of the corresponding outcome-set. The essential difference is that whereas Gul and Pesendorfer define an envelope as a mapping from states to intervals of outcomes, defined in terms of least- and most-preferred outcomes of the corresponding outcome-set, we retain the entire outcome-set. That is, the envelope of the act maps to an outcome-set precisely those states in the element of the measurable split induced by the act's preimage that corresponds to that subset of outcomes. This in turn means the envelope of an act can be characterized as the minimal (with respect to set-inclusion) measurable mapping from states to outcome-sets that contains the act. Indeed from a perceptual perspective, we contend it makes sense to view the DM *as incapable of distinguishing among acts that have a common envelope*.[3] The axioms we adopt guarantee the existence and uniqueness of envelopes.

The interpretation of envelopes in terms of belief and plausibility functions, as described in Gul and Pesendorfer's, becomes even more straightforward: the belief in a particular outcome-set obtaining is the total probability assigned to that outcome-set and all its subsets in the envelope, while its plausibility is the total probability of all subsets containing at least one element of that outcome-set. Moreover, the prior and the envelope of an act induce the outcome-set lottery in which for each outcome-set, the probability assigned to that outcome-set is given by the probability the prior assigns to the set of states that the envelope maps to that particular outcome-set.

Imposing that the DM is indifferent among acts inducing the same outcome-set lottery, we introduce *Expected Balanced Uncertain Utility (EBUU) preferences* corresponding to the family of preferences that admit an *implicit* probability equivalent (utility) representation characterized by a pair $\langle \mu, U \rangle$, where μ is the DM's prior defined over those events she deems measurable and a *balanced outcome-set utility*, $U(Y, p)$, that specifies the utility of an outcome-set Y in a lottery that has a *probability* equivalent of p, by which we mean any lottery the DM views as equally valuable as a binary gamble that yields the best outcome with probability p and the worst outcome with the complementary probability $1 - p$. It exhibits a natural (outcome-set) monotonicity with respect to its first argument.

Thus we obtain a clean separation of the ambiguity she perceives to be present given her knowledge about the random process governing the resolution of the uncertainty she faces from her attitude toward risk (i.e., measurable uncertainty). The former is characterized by those events that lie outside the domain of her prior while the restriction of her balanced outcome-set utility to singleton outcome-sets encodes the latter.

[3] In this regard, the measurable split induced by an act's inverse image is reminiscent of Ghirardato's (2001, p. 249) second scenario of an underspecified state space as one possible way to interpret his model in which preferences are defined over outcome-set acts.

Finally, her attitude toward (general) uncertainty involving both risk and ambiguity is embodied in her (unrestricted) balanced outcome-set utility.

This interpretation clarifies why we do not impose decomposability for non-measurable events. The principle relies on a sharp separation between outcomes of an act on an event and its complement, but this gets blurred when events are nonmeasurable, since outcomes on an event may contribute to the outcome-set outside that event. We refer to Gul and Pesendorfer (2014) (Section 5) for an illustration by way of the Ellsberg paradox of this effect. Hence, their motivation not to impose **P2** for non-measurable events in essence also applies to restricting decomposability to measurable events only.

We develop the formal definition of EBUU preferences in Section 2 with its axiomatic characterization appearing in Section 3. We provide three examples in Section 4. We conclude in Section 5. Proofs appear in the Appendix.

2. The model

Our setting is one in which the purely subjective uncertainty the DM faces is described by a state space Ω. The objects of choice are acts that for each state of nature $\omega \in \Omega$, deliver an outcome x from a set X. Each act f is simple, that is, its image $f(\Omega)$ is a finite subset of X.

We denote the set of all acts by F. We identify any outcome $x \in X$ with the (constant) act f in which $f(\omega) = x$ for all ω. And with further (albeit fairly standard) abuse of notation, X will also refer to the set of constant acts.

For any pair of events $E, B \subseteq \Omega$, $B \setminus E$ shall denote the set of elements that are in B but not in E. For any pair of acts f and g in F and any event $E \subset \Omega$, we write $f_E g$ for the act that agrees with f on E and with g on $\Omega \setminus E$.

The DM is characterized by her preferences over acts, a binary relation $\succsim$ on F, with asymmetric and symmetric parts denoted by $\succ$ and $\sim$, respectively.

We begin our description of expected balanced uncertain utility preferences by first noting the DM possesses *rich coherent beliefs*. This entails the existence of a sufficiently rich collection of (*risky*) events, constituting a σ-algebra of subsets of Ω, over which can be defined a countably-additive and convex-ranged probability measure μ (her "prior") with which the DM *precisely quantifies* the uncertainty she associates with each risky event. We denote the domain of μ by $\mathbf{R}$.

Countable-additivity requires the probability of the union of a countable collection of disjoint measurable events from $\mathbf{R}$ equals the infinite sum of the probabilities of these events. For μ to be convex-ranged requires for any event R in $\mathbf{R}$ and any r in $(0, 1)$ there exists a subset $B \subset R$ that is in $\mathbf{R}$ and for which $\mu(B) = r\mu(R)$. Let $F_\mu \subset F$ denote the set of acts that are measurable with respect to μ.

From this point on, the term *outcome-set* will refer to any nonempty finite subset of X with generic elements denoted by Y, Z, Y', etc. As we alluded to in the Introduction, there is a natural way to use this prior to identify with each act its *outcome-set* envelope. Let $\mathbf{F}_\mu$ be the set of measurable (with respect to μ) functions $\mathbf{f}\colon \Omega \to \{Y \subset X : Y \neq \emptyset, |Y| < \infty\}$. We refer to elements of $\mathbf{F}_\mu$ as outcome-set acts.

DEFINITION 1 (Envelope of an Act). The outcome-set act $\mathbf{f} \in \mathbf{F}_\mu$ is the *envelope* of f if:

(i) $f(\omega) \in \mathbf{f}(\omega)$ for all $\omega \in \Omega$, and

(ii) for any outcome-set act $\mathbf{g} \in \mathbf{F}_\mu$:

$$f(\omega) \in \mathbf{g}(\omega) \quad \text{for all } \omega \in \Omega \quad \Longrightarrow \quad \mu\big(\{\omega \in \Omega : \mathbf{f}(\omega) \subseteq \mathbf{g}(\omega)\}\big) = 1.$$

To construct the envelope of an act, it is useful first to define the *inner measure* of μ, denoted by μ_*, that is derived from the prior by assigning to each event $E \subset \Omega$ the weight $\mu_*(E) \in [0, 1]$ that is the solution to

$$\sup_{R \in \mathbf{R}, R \subseteq E} \mu(R).$$

Since μ is countably additive, the supremum is attained. We shall refer to the measurable event $[E]_* \in \mathbf{R}$ as the *inner-sleeve* of E, if $[E]_* \subseteq E$ and $\mu([E]_*) = \mu_*(E)$.[4]

Following Gul and Pesendorfer, we associate with each act a measurable partition of the state space generated by the act's preimage as follows.

DEFINITION 2 (Measurable Split). The *measurable split* (of the state space) associated with the act $f : \Omega \to X$ and denoted by $\{R_f^Y \in \mathbf{R} : Y \subseteq f(\Omega), Y \neq \emptyset\}$ is inductively defined as follows:

1. For each element $x \in f(\Omega)$, set $R_f^{\{x\}} := [f^{-1}(x)]_*$.
2. For each $Y \subseteq f(\Omega)$ such that $|Y| > 1$, set

$$R_f^Y := \big[f^{-1}(Y)\big]_* \setminus \Bigg(\bigcup_{Z \subset Y, Z \neq \emptyset} R_f^Z\Bigg).$$

We refer to R_f^Y as the *f-marginal inner-sleeve* of the outcome-set Y.

To see how the envelope of an act can be constructed using the measurable split generated by its inverse image, first consider a binary act $x_A y$. The measurable split is the three element partition of the state-space

$$\Big\{\underset{[A]_*}{\underset{\|}{R_{x_A y}^{\{x\}}}}, \quad \underset{[\Omega \setminus A]_*}{\underset{\|}{R_{x_A y}^{\{y\}}}}, \quad \underset{\Omega \setminus ([A]_* \cup [\Omega \setminus A]_*)}{\underset{\|}{R_{x_A y}^{\{x,y\}}}}\Big\}.$$

The first (resp., second) element corresponds to the largest measurable subset in which the binary act $x_A y$ yields the outcome x (resp., y). For the third element, all the DM can discern is that the outcome will be either x or y. However, she is unable to attribute any fraction of the probability her prior assigns to this element of the split to either x or y

[4]Notice that the inner sleeve is unique up to a set of μ-measure 0.

obtaining alone. Thus, it readily follows from Definition 1 that the envelope of $x_A y$ is the outcome-set act $\mathbf{f} \in \mathbf{F}_\mu$ for which

$$\mathbf{f}(\omega) = \begin{cases} \{x\} & \text{if } \omega \in [A]_* \\ \{y\} & \text{if } \omega \in [\Omega \setminus A]_* \\ \{x, y\} & \text{otherwise} \end{cases}$$

This method readily extends to an arbitrary act f in F. Using the measurable split $\{R_f^Y \in \mathbf{R} \colon Y \subseteq f(\Omega), Y \neq \emptyset\}$, its (essentially unique) envelope is the outcome-set act $\mathbf{f} \in \mathbf{F}_\mu$ constructed by setting $\mathbf{f}(\omega) := Y$ whenever $\omega \in R_f^Y$.

An *outcome-set lottery* is a finite ranged function $L : \{Y \subset X \colon Y \neq \emptyset, |Y| < \infty\} \to [0, 1]$ satisfying $\sum_{Y \subset X, |Y| < \infty} L(Y) = 1$. We associate with the act f the outcome-set lottery $\mu \circ \mathbf{f}^{-1}$.

Recalling the approach of Dempster (1967) and Shafer (1976), we shall interpret the outcome-set lottery $\mu \circ \mathbf{f}^{-1}$ as encoding how the DM weights that part of the evidence supporting the belief that the act f leads to an outcome in a given set of outcomes obtaining that is not well specified enough to allow her to distribute any of it across any of the elements of that set or any of the other strict subsets of that set of outcomes. In other words, for each outcome-set $Y \subseteq f(\Omega)$ we shall interpret $\mu \circ \mathbf{f}^{-1}(Y)(= \mu(R_f^Y))$ as the weight assigned by the DM to evidence that directly supports the act f leading to an outcome in Y obtaining that cannot be further refined in terms of any of the strict subsets of Y.

Analogous to Grant's (1995, p. 163) rendition of Machina and Schmeidler's (1992) concept of probabilistic sophistication, we require that no relevant preference information is lost by this association.

Definition 3 (Coherent Beliefs). The prior μ is a *coherent belief* for the preference relation $\succsim$, if for each pair of acts f and $\widehat{f}$, with respective envelopes $\mathbf{f}$ and $\widehat{\mathbf{f}}$, $f \sim \widehat{f}$ whenever $\mu \circ \mathbf{f}^{-1} = \mu \circ \widehat{\mathbf{f}}^{-1}$.

One more element is needed, a *balanced outcome-set utility* that specifies the utility of each outcome-set in a lottery of a given value. Restricted to singletons, we impose the standard properties of utility in betweenness models, but to determine a useful concept of monotonicity for sets turns out to be a more delicate issue. It involves the choice of a dominance relation between sets of outcomes. To impose a complete dominance relation, as for singletons, would overly restrict the model class. However, it is natural to require the DM strictly (resp., weakly) prefer one outcome-set over another if the DM strictly (resp., weakly) prefers all the outcomes in the former to all the outcomes in the latter.

To streamline the exposition, we assume that there exists a best outcome $\bar{x}$ and a worst outcome $\underline{x}$ (i.e., $\bar{x} \succsim x \succsim \underline{x}$ for all x in X), and impose a normalization that for any binary act $\bar{x}_R \underline{x}$ with $\mu(R) = p$, its expected balanced utility is zero.

Definition 4 (Balanced Outcome-Set Utility). A *balanced outcome-set utility* is a function $U \colon \{Y \subset X \colon Y \neq \{\emptyset\}, |Y| < \infty\} \times [0, 1] \to \mathbb{R}$ that:

1. Exhibits *outcome-set monotonicity*, in the sense that for any p in $(0, 1]$ and any pair of outcome-sets Y and Z, $U(Y, p) > (\geq) U(Z, p)$, whenever for all $(y, z) \in Y \times Z$: $U(\{y\}, q) = U(\{z\}, p) = 0 \implies q > (\geq) p$; and,

2. Is normalized with respect to some maximal and minimal outcomes in X, denoted $\bar{x}$ and $\underline{x}$, respectively, in the sense that:

$$pU\big(\{\bar{x}\}, p\big) + (1 - p)U\big(\{\underline{x}\}, p\big) \equiv 0. \tag{1}$$

It is deemed to be *canonical* if, in addition, $U(\{\bar{x}\}, p) - U(\{\underline{x}\}, p) \equiv 1$.

For all the examples presented in Section 4, the balanced outcome-set utility we specify will be canonical. This amounts to setting $U(\{\bar{x}\}, p) := 1 - p$ and $U(\{\underline{x}\}, p) := -p$. That we always can choose U canonical, relies on the fact that rescaling (for a fixed p) $U(\cdot, p)$ by a strictly positive scalar λ_p has no effect in an EBUU representation, as it becomes evident from the next definition.

Definition 5 (EBUU preferences). A preference $\succsim$ is EBUU if there exists a prior μ and a balanced outcome-set utility U such that $\succsim$ admits a *(probability equivalent)* representation $V : \mathcal{F} \to [0, 1]$ defined as the unique solution to

$$\sum_{Y \subseteq f(\Omega)} U\big(Y, V(f)\big)\mu\big(\mathbf{f}^{-1}(Y)\big) = 0, \quad \text{where } \mathbf{f} \text{ is the envelope of } f. \tag{2}$$

The left-hand side of equation (2) functions as a balance scale, in the sense that, for each probability p and any binary act $\bar{x}_R\underline{x}$ with $R \in \mathcal{E}$ and $\mu(R) = p$:

$$\sum_{Y \subseteq f(\Omega)} U(Y, p)\mu\big(\mathbf{f}^{-1}(Y)\big) \geq 0 \quad \iff \quad f \succsim \bar{x}_R\underline{x}.$$

3. Characterization

We begin our characterization of EBUU preferences by first specifying what property an event must satisfy that allows us to infer the DM deems it to be "risky," and thus "measurable." As we noted above in the Introduction, in Gul and Pesendorfer's (2014) expected uncertain utility theory, this requires both it and its complement satisfy a version of Savage's (1954) postulate **P2**.

Definition 6 (Ideal Events). An event $E \subseteq \Omega$ is *ideal* if for any four acts f, g, h and h' in F, $[f_E h \succsim g_E h$ and $h_E f \succsim h_E g] \implies [f_E h' \succsim g_E h'$ and $h'_E f \succsim h'_E g]$.

Instead, we propose the following property of *decomposability*.

Definition 7 (Decomposable Events). An event $R \subseteq \Omega$ is *decomposable* if for every pair of acts f and g in F, $f_R g \succ g$ and $g_R f \succ g \implies f \succ g$.

Our axioms will ensure that decomposability is equivalent to the criterion with $\succ$ replaced by $\prec$, and entails the nonstrict variants with $\precsim$ and $\succsim$ as well.[5] We shall refer to $\mathbf{R}$ as the set of decomposable events. Notice that by construction the set $\mathbf{R}$ (like the set of ideal events) is closed under complements. Also, it readily follows that if $\succsim$ is complete and transitive, then any ideal event is also decomposable.[6] The converse, however, need not hold.

We refer to an act as *decomposable* if it is measurable with respect to $\mathbf{R}$, that is, an act g is decomposable if $g^{-1}(\{x\}) \in \mathbf{R}$ for all $x \in X$. Hence, loosely speaking, a decomposable act coincides with its own envelope. Let $G \subset F$ denote the set of decomposable acts.

A subclass of decomposable events are those for which modifying any act on that event leaves it in the same indifference set. These are known as (Savage-)null events.

Definition 8 (Null Events). An event $N \subseteq \Omega$ is *null* if $f \sim g_N f$ for all $f, g \in F$. Let $\mathbf{N}$ denote the set of null events.

Set $\mathbf{R}^+ := \mathbf{R} \setminus \mathbf{N}$, the class of nonnull decomposable events. And for each non-null decomposable event $R \in \mathbf{R}^+$ and each act $f \in F$, set $f(R)^+ := \{y \in f(R) : f^{-1}(y) \cap R \notin \mathbf{N}\}$. That is, $f(R)^+$ contains each element in the image of $f(R)$ for which its preimage has a nonnull intersection with R.

Analogous to the role played by ideal events in Gul and Pesendorfer (2014), we suppose an EBUU DM uses elements of $\mathbf{R}^+$ to quantify the uncertainty of any event. So, it seems natural to view an event as *maximally ambiguous* if it and its complement contain no element of $\mathbf{R}^+$. Adopting the terminology of Gul and Pesendorfer, we will refer to such an event (as well as its complement) as *diffuse*.

Definition 9 (Diffuse Events). An event $D \subseteq \Omega$ is *diffuse* if, for every nonnull decomposable event $R \in \mathbf{R}^+$, $R \cap D \notin \mathbf{N}$, and $R \cap (\Omega \setminus D) \notin \mathbf{N}$. Let $\mathbf{D}$ denote the set of diffuse events.

We say an act h is *diffuse* if its inverse image generates a diffuse partition of Ω, by which we mean $h^{-1}(x) \in \mathbf{D}$ for all $x \in h(\Omega)^+$. Let $H \subset F$ denote the set of diffuse acts.[7]

With these preliminaries in hand, we can now state the axioms. Our first is the standard ordering axiom.

Axiom 1 (Ordering). *The binary relation $\succsim$ is complete and transitive.*

We next require the collection of events the DM deems unambiguous to be closed under conjunctions. That is, we require for any pair of decomposable events R and $\widehat{R}$ that their intersection is also decomposable.

[5]Where $\precsim$ is the relation derived from $\succsim$ by setting $f \precsim f'$ if $f' \succsim f$, and $\prec$ is the asymmetric part of $\precsim$.

[6]If the event E is ideal, then it follows from Gul and Pesendorfer's (2014) Lemma B0 (p. 25) that it is also left ideal, that is, $g_E f \succsim f$ implies $g \succsim f_E g$ or equivalently, $\neg(g \succsim f_E g)$ implies $\neg(g_E f \succsim f)$. Thus, it follows from the completeness of $\succsim$ that $f_E g \succ g$ implies $f \succ g_E f$. Hence if, in addition, we have $g_E f \succ g$, then $f \succ g$ follows from the transitivity of $\succ$, which in turn follows from the completeness and transitivity of $\succsim$.

[7]It will turn out that the envelope of any diffuse act $h \in H$ will be the constant function $\mathbf{h}(\omega) = h(\Omega)^+$ for all $\omega \in \Omega$.

Axiom 2 (Decomposable Closure Under Intersection). *For any pair of decomposable events $R, \widehat{R} \in \mathbf{R}^+$ and any pair of acts $f, f' \in \mathcal{F}$,*

$$\text{if} \quad f_{R\cap\widehat{R}} f' \succ f' \quad \text{and} \quad f'_{R\cap\widehat{R}} f \succ f' \quad \text{then } f \succ f'.$$

We readily acknowledge this is not without loss of generality. As a number of researchers have demonstrated, various hedging opportunities against ambiguity may be based on events one would expect a DM to deem unambiguous but which are not closed under intersection.[8] However, just as was the case in Gul and Pesendorfer's (2014) model of expected uncertain utility, as our theory builds on the induced order over the (measurable) envelopes the DM associates with acts, we require the set of events the DM deems measurable, constitute a σ-algebra thereby guaranteeing the (measurable) envelope the DM associates with each act is well-defined. Axiom 2 in conjunction with the continuity property imposed by Axiom 5 (see below), plays a key role in establishing that the collection of events $\mathbf{R}$ is a σ-algebra.

Recall from above, a diffuse event is one in which the DM cannot find any nonnull decomposable event that can "fit" inside that event, which would help her quantify the uncertainty she associates with that event. Correspondingly, for a given diffuse act $h \in H$, the DM is unable to estimate the *relative* likelihood of *any strict subset* of the objects in $h(\Omega)^+$ obtaining, either unconditionally or conditionally on any decomposable event R obtaining. Thus the next axiom requires the DM is incapable of expressing a strict preference between any pair of diffuse acts h and $\widehat{h}$ for which $h(\Omega)^+ = \widehat{h}(\Omega)^+$. This indifference extends to any pair of acts that only differ on a decomposable event R, with one having the diffuse act h determine the outcome should R obtain while the other has $\widehat{h}$. Furthermore, her evaluation of a diffuse act h conditional on a decomposable event R obtaining cannot exceed (resp., be no worse) than her conditional evaluation of any outcome in $h(\Omega)^+$.

Axiom 3 (Decomposable Eventwise Monotonicity). *For all diffuse and constant acts h in $H \cup X$, all acts f in F, and all nonnull decomposable events R in $\mathbf{R}^+$:*

(i) $h_R f \sim \widehat{h}_R f$ *for any diffuse act* $\widehat{h}$ *in* H, *in which* $\widehat{h}(\Omega)^+ = h(\Omega)^+$;

(ii) $x_R f \succsim (\succ) h_R f$ *for any* x *such that* $x \succsim (\succ) y$ *for all* $y \in h(\Omega)^+$; *and,*

(iii) $h_R f \succsim (\succ) z_R f$ *for any* z *such that* $y \succsim (\succ) z$ *for all* $y \in h(\Omega)^+$.

This axiom can be interpreted as *recognizable* monotonicity: if the DM can deduce from the envelopes of f and g that f dominates g on Ω, which is the case if and only if f's envelope dominates that of g (according to the strong set-order), then she should prefer f to g.

We do not impose, however, Gul and Pesendorfer's (2014) Axiom 2 of statewise monotonicity.[9] In our interpretation, this would be the axiom of *plausible* monotonicity, requiring that f should be preferred to f' whenever the DM cannot exclude, on the

[8]See, for example, Zhang (2002), Epstein and Zhang (2001), Kopylov (2007), and Nehring (2009).

[9]However, in the next subsection, we do explore the consequences of imposing this axiom.

basis of their envelopes that f may dominate f' on Ω. For diffuse acts h and h', with $h(\Omega)^+ = Y$ and $h'(\Omega)^+ = Y'$, this is already the case when both the worst outcome and best outcome in Y are each strictly preferred to the respectively worst and best in Y'. Moreover, the utility of a set Y would then be completely determined by its worst and best element, that is, we would arrive at interval utilities. For instance, for the case where X is a subset of the real line this would require the outcome-set $Y = \{0.01, 10.01\}$ be assigned a strictly higher utility than the outcome-set $Y' = \{0, 8, 9, 10\}$, not as a matter of taste, but as an axiomatic principle, for which we see no ground.

We follow with an axiom that serves the role played by Savage's comparative probability axiom **P4** in Gul and Pesendorfer's (2014) characterization of expected uncertain utility. However, since the decomposability property does not entail the full separability implied by the definition of an ideal event, it is not enough for us to modify **P4** by substituting decomposable events for ideal events. So, instead we adopt the strong comparative probability axiom **P4*** from Machina and Schmeidler (1992) in indifference form and extend it to apply to diffuse acts as well.

Axiom 4 (Comparative Conditional Probability). *For any pair of disjoint decomposable events R and $\widehat{R}$ in* $\mathbf{R}$, *any pair of outcomes $x^* \succ x$ in X, and any pair of acts f and f' in F,*

$$\left(x^*_R x\right)_{R\cup\widehat{R}} f \sim \left(x^*_{\widehat{R}} x\right)_{R\cup\widehat{R}} f \quad \Longrightarrow \quad \left(h^*_R h\right)_{R\cup\widehat{R}} f' \sim \left(h^*_{\widehat{R}} h\right)_{R\cup\widehat{R}} f',$$

for all h^* and h in $H \cup X$.

To interpret Axiom 4, notice that the first indifference allows us to infer that, conditional on the act f determining the outcome should neither R nor $\widehat{R}$ obtain, the DM views R "as equally as likely" as $\widehat{R}$. The axiom then requires these revealed conditional equal likelihoods should still obtain no matter what pair of *outcome-sets* serve as the "stakes" for bets made on R against $\widehat{R}$ nor what act f' determines the outcome should neither R nor $\widehat{R}$ obtain.

We finish with two continuity axioms. The first helps ensure the collection of events the DM deems unambiguous is closed under countable unions.

Axiom 5 (Cumulative Decomposable-Event Continuity). *Let $f^n = f_{R^n} f^*$ with $\{R^n\} \subset \mathbf{R}$ and $R^n \subset R^{n+1}$, and set $R^\infty := \bigcup_n R^n$. If $f' \succsim f^n \succsim f''$ for all n, then $f' \succsim f_{R^\infty} f^* \succsim f''$.*

Our second continuity axiom plays the same role as Savage's (1954) postulate **P6**, namely requiring the set of decomposable events is sufficiently rich so that the derived comparative likelihood relation over decomposable events is both fine and tight.

Axiom 6 (Small Decomposable-Event Continuity). *For any pair of acts $f, f' \in \mathcal{F}$, if $f \succ f'$, then there exists a finite decomposable partition $\{R_1, \ldots, R_N\}$ of Ω, with $R_n \in \mathbf{R}$, such that $\underline{x}_{R_n} f \succ f'$ and $f \succ \bar{x}_{R_n} f'$ for all $n = 1, \ldots, N$.*

Our main representation result follows.

THEOREM 1. *Fix a binary relation $\succsim$. Suppose there exists a maximal outcome $\bar{x}$ and minimal outcome $\underline{x}$ in X, satisfying $\bar{x} \succ \underline{x}$ and $\bar{x} \succsim x \succsim \underline{x}$ for all x in X.*

Then $\succsim$ satisfies Axioms 1–6 if and only if it is EBUU.

The proof appears in the Appendix but here we provide an outline of how the axioms enable us to obtain an EBUU representation.

Step 1: Deriving the prior We first establish that the set of decomposable events, $\mathbf{R}$, constitutes a σ-algebra of events. Next, we show the restriction of the DM's preferences to binary bets on decomposable events involving only the best and worst outcomes admits a standard expected utility representation, which is characterized by a countably-additive and convex-ranged probability μ defined on $\mathbf{R}$. Since μ is convex-ranged, this means that for any $p \in [0, 1]$, there exists a decomposable event R_p, for which $\mu(R_P) = p$.

Step 2: Variants of decomposability To prepare for the construction of the utility, we formulate a technical result on the variants of the decomposability criterion with $\succ$ replaced by $\prec$, $\precsim$, and $\succsim$. This establishes conditional independence of acts in an indifference class.

Step 3: Constructing the balanced outcome-set utility The value of the outcome-set utility $U(Y, p)$ is determined as follows. Fix $p \in (0, 1)$. Set $U(\{\bar{x}\}, p) := 1 - p$, $U(\{\underline{x}\}, p) := -p$, and find a (decomposable) event R_p in $\mathbf{R}$ such that $\mu(R_p) = p$.

For each singleton $Y = \{y\}$, take R^y in $\mathbf{R}$, to be (i) in case $y \succsim \bar{x}_{R_p}\underline{x}$, the decomposable subset $R^y \subseteq R_p$ for which $y_{R^y}\underline{x} \sim \bar{x}_{R_p}\underline{x}$, otherwise (ii) the decomposable subset $R^y \subset \Omega \setminus R_p$ for which $y_{R^y}\bar{x} \sim \bar{x}_{R_p}\underline{x}$. Set $q := \mu(R^y)$. The only candidate for a solution to the EBUU equation (2) turns out to be, respectively,

$$\text{(i)}: U(\{y\}, p) = p\frac{(1-q)}{q} \quad \text{and} \quad \text{(ii)}: U(\{y\}, p) = -(1-p)\frac{(1-q)}{q}.$$

Do the same for nonsingletons Y, with y replaced by a diffuse act h^Y in H that has associated with it an envelope that maps every state to Y.

Step 4: Verification We show that the prior μ from step 1 and the balanced outcome-set utility $U(\cdot, \cdot)$ from step 3 indeed determine a properly constituted EBUU representation of $\succsim$.

Statewise monotonicity

We conclude this section by considering the implication of strengthening monotonicity by adding Gul and Pesendorfer's (2014) Axiom 2. The simplest way to state it is to adopt their setting with the outcome set $X := [\ell, m]$, for some ℓ and m in $\mathbb{R}$, $\ell < m$.

AXIOM 7 (Gul and Pesendorfer's (2014) Axiom 2). *For any pair of acts $f, \widehat{f} \in F$, $f \succ \widehat{f}$ whenever $f(\omega) > \widehat{f}(\omega)$ for all $\omega \in \Omega$.*

Although this monotonicity property may seem natural in a setting in which outcomes can be viewed as monetary prizes, adding it to the other six axioms reduces outcome-set utilities to interval utilities.

PROPOSITION 1. *Set $X := [\ell, m]$, for some ℓ and m in $\mathbb{R}$, $\ell < m$. Fix a prior μ and balanced outcome-set utility U. Then the following are equivalent:*

1. *The preference relation $\succsim$, characterized by $\langle \mu, U \rangle$, satisfies Axiom 7.*
2. *For every outcome-set Y and every $p \in (0, 1)$,*

$$U(Y, p) = U\left(\left\{\min_{x \in Y} x, \max_{y \in Y} y\right\}, p\right).$$

To get a flavor as to why this should be the case, consider a diffuse act h with outcome set Y with minimum $\underline{y}$ and maximum $\bar{y}$. By replacing all the intermediate outcomes in Y by $\bar{y}$ (resp., $\underline{y}$), we obtain a diffuse act h' (resp., h''). Since both h' and h'' have the same outcome set $\{\underline{y}, \bar{y}\}$, they must have the same value. Furthermore, since $h' \geq h \geq h''$, this in turn means that h must also have the same value when Axiom 7 holds true. By considering the EBUU representation of acts of the form $h_R\underline{y} \sim c$, it then follows that $U(Y, p) = U(\{\underline{y}, \bar{y}\}, p)$ for all $p \in (0, 1)$. Conversely, by comparing the interval envelopes of any pair of acts f and $\widehat{f}$, in which $f(\omega) > \widehat{f}(\omega)$ for all ω in Ω, it readily follows that the subclass of EBUU with interval utilities satisfies statewise monotonicity.

To accommodate outcome-set utilities that cannot be reduced to interval utilities, Axiom 7 clearly cannot be part of the characterization of EBUU. We contend, however, its omission accords with the (implicit) assumption we maintain throughout that the DM's knowledge about those aspects of an act she deems relevant is necessarily restricted to its associated envelope.

4. Examples of EBUU preferences

We present three examples of EBUU preferences. The first is a generalization of Gul and Pesendorfer's (2014) expected uncertain utility model in which the risk preferences need only conform to Gul's (1991) risk preference model of *disappointment aversion*, a parsimoniously parameterized subclass of risk preferences that exhibit the betweenness property of Chew (1983) and Dekel (1986). The second is a generalization of Gul and Pesendorfer's (2015) Hurwicz expected utility model in which the degree of aversion to ambiguity can depend on the indifference class in which an act resides. The third is a subjective extension of a preference model over outcome-set lotteries (or equivalently, *belief functions*) introduced by Eichberger and Pasichnichenko (2021). It allows the decision maker to exhibit the novel phenomenon of aversion to ambiguity about disappointment and elation.

In the first two examples, the balanced utility of the outcome-set Y for a given p cannot depend on any outcome in Y that is neither its worst nor its best element. For the third, however, every element in the outcome-set Y matters.

In what follows, the function $v : X \to [0, 1]$ shall denote a Bernoulli utility with a maximal (resp., minimal) outcome $\bar{x}$ (resp., $\underline{x}$) and normalized so that $v(\bar{x}) := 1$ and $v(\underline{x}) := 0$. For each outcome-set Y, we set $\underline{v}^Y := \min_{z\in Y} v(z)$ and $\bar{v}^Y := \max_{z\in Y} v(z)$.

1. Risk of disappointment averse expected uncertain utility: There exists an interval utility

$$u : \{[\underline{v}, \bar{v}] \in [0, 1]^2 : \underline{v} \le \bar{v}\} \to [0, 1],$$

that is continuous, monotonic in the sense that

$$\underline{v} > \underline{v}' \quad \text{and} \quad \bar{v} > \bar{v}' \quad \Longrightarrow \quad u(\underline{v}, \bar{v}) > u(\underline{v}', \bar{v}'),$$

and is normalized so that $u(v, v) = v$ for all $v \in [0, 1]$. Fixing the *risk of disappointment aversion* parameter $\beta > -1$, the canonical balanced outcome-set utility is defined by setting

$$U(Y, p) := \begin{cases} \dfrac{\big(1 + (1 - p)\beta\big)u\big(\underline{v}^Y, \bar{v}^Y\big) - p}{1 + \beta} & \text{if positive} \\ \big(1 + (1 - p)\beta\big)u\big(\underline{v}^Y, \bar{v}^Y\big) - p & \text{if not positive} \end{cases}$$

To see how this conforms to Gul's (1991) construction, consider the local utility function

$$\phi(\underline{v}, \bar{v}, k) = \begin{cases} \dfrac{u(\underline{v}, \bar{v}) + \beta k}{1 + \beta} & \text{if } u(\underline{v}, \bar{v}) > k \\ u(\underline{v}, \bar{v}) & \text{if } u(\underline{v}, \bar{v}) \le k(p) \end{cases}$$

with k marking the "kink" induced by disappointment aversion.[10]

A subtle point arises concerning the choice of k, not due to the uncertainty of utility, but to the generalization of disappointment aversion to nonmonetary outcomes. For each p, we have to choose k_p so that $U(Y, p) = 0$ precisely when $u(\underline{v}^Y, \bar{v}^Y) = k_p$, to ensure an outcome set Y is deemed an elation if and only if it is better than the reference binary decomposable act $\bar{x}_R\underline{x}$ with $\mu(R) = p$. This yields the rule

$$k_p = \frac{p}{1 + (1 - p)\beta},$$

and the proposed U can then be seen as the affine transformation of ϕ that yields the canonical form.[11]

[10]This is exactly in line with Gul (1991, p. 674), with ordinary utility $u(x)$ replaced by $u(\underline{v}, \bar{v})$, threshold $u(x) > v$ replaced by $u(\underline{v}, \bar{v}) > k$ for some k to be explained, and $\phi(x, v)$ replaced by $\phi(\underline{v}, \bar{v}, k)$.

[11]That is, $U(Y, p) = [1 + (1 - p)\beta][\phi(\underline{v}^Y, \bar{v}^Y, k_p) - k_p]$. Solving equation (2) for a constant act $f = x$ yields $V(x) = M(v(x))$, where $M(u) = (1 + \beta)u/(1 + \beta u)$. If instead we work with the representation $\widehat{V}(f) = T \circ V(f)$, where T is the monotonic transformation defined by setting

$$T(p) := M^{-1}(p) = \frac{p}{1 + \beta(1 - p)},$$

then we have $\widehat{V}(x) = M^{-1} \circ M(v(x)) = v(x)$. Moreover, $\widehat{V}(f)$ is the unique solution to equation (2) in which the outcome-set function $U(Y, p)$ is replaced by the function $\widehat{U} : \{Y \subset X : Y \neq \emptyset, |Y| < \infty\} \times [0, 1]$ defined

So, the risk preferences conform to Gul's (1991) risk preference model of *disappointment aversion*. Expected utility corresponds to $\beta = 0$. The property of aversion to the risk of disappointment, which Gul shows is both necessary and sufficient to generate Allais-style choice patterns, requires $\beta > 0$.

2. Implicit Hurwicz expected utility: Let $\alpha\colon [0, 1] \to [0, 1]$ parametrize ambiguity aversion. We take the canonical outcome-set utility to be

$$U(Y, p) := \alpha(p)\underline{v}^Y + \bigl(1 - \alpha(p)\bigr)\bar{v}^Y - p$$

Imposing α is differentiable in p, with derivative $\alpha'(p) > -1$ for all p, guarantees that $V(f)$ is uniquely defined by the EBUU equation (2).

This example illustrates how decomposability allows for the level of ambiguity aversion to depend on p. For example, consider the two-parameter formula $\alpha(p) = \alpha_0 + \varepsilon p$ with $\varepsilon \in [-\alpha_0, 1 - \alpha_0]$.[12] Preferences exhibiting the comparative static property of *decreasing* (resp., *increasing*) *absolute ambiguity aversion* then corresponds to $\varepsilon < 0$ (resp., $\varepsilon > 0$).

3. Ambiguity of disappointment averse quasi-arithmetic mean uncertain utility: There exists: *(i)* a *second-order* utility $\phi\colon [0, 1] \to \mathbb{R}$ that is twice continuously differentiable with $\phi' > 0$ and $\phi'' \leq 0$, and normalized with $\phi(1) := 1$ and $\phi(0) := 0$ and *(ii)* an *ambiguity of disappointment aversion* parameter, $\gamma > -1$. The canonical balanced outcome-set utility is defined by setting

$$U(Y, p) := M_p\bigl(v(Y)\bigr) - p,$$

where for any $W \subseteq v(X)$ such that $|W| < \infty$, $M_p(W)$ is the *disappointment (relative to p) adjusted quasi-arithmetic mean* of the set W given by

$$M_p(W) := \phi_p^{-1}\left(\frac{1}{|W|}\sum_{v \in W} \phi_p(v)\right),$$

where for each $p \in [0, 1]$:

$$\phi_p(v) = \begin{cases} \dfrac{\phi(v) + \gamma p}{1 + \gamma} & \text{if } v > p \\ \phi(v) & \text{if } v \leq p \end{cases}$$

Notice that

$$M_0(W) = M_1(W) = \phi^{-1}\left(\frac{1}{|W|}\sum_{v \in W} \phi(v)\right),$$

by setting $\widehat{U}(Y, u) := U(Y, M(u))$. Although, properly speaking $\widehat{U}(Y, u)$ is not an outcome-set function as it does not satisfy the normalization equation (1) of Definition 4, it is still normalized with respect to the maximal $\bar{x}$ and minimal $\underline{x}$ in the sense that

$$p\widehat{U}\bigl(\{\bar{x}\}, T(p)\bigr) + (1 - p)\widehat{U}\bigl(\{\underline{x}\}, T(p)\bigr) \equiv 0.$$

[12] We thank the coeditor for suggesting we consider such a possibility.

the standard (i.e., *unadjusted*) quasi-arithmetic mean. Moreover, if *every* Bernoulli utility in the set $v(Y)$ is greater than (resp., less than or equal to) p, then the balanced utility is equal to the difference between the quasi-arithmetic mean of the Bernoulli utilities in the set $v(Y)$ and p.

However, if $\gamma > 1$ and p resides in the open interval $(\underline{v}^Y, \bar{v}^Y)$, then

$$M_p\big(v(Y)\big) < \phi^{-1}\left(\frac{1}{|W|}\sum_{v\in v(Y)}\phi(v)\right).$$

That is, the quasi-arithmetic mean is adjusted downward, reflecting the decision-maker's *aversion* to ambiguity about whether the outcome she will receive from the set Y will prove disappointing because its Bernoulli utility is less than p, or will be a cause for elation because its Bernoulli utility is greater than p.

The case for each example in which the balanced outcome-set utility is quasi-linear with respect to p (thus allowing for a straightforward rearrangement of the implicit representation to obtain an explicit representation) corresponds to $\beta = 0$, $\alpha'(p) \equiv 0$, and $\gamma = 0$, respectively. And, for each of the three, standard definitions of aversion to ambiguity from the literature respectively correspond to $u(\underline{v}, \bar{v}) \le (\underline{v} + \bar{v})/2$; $\alpha(p) \ge 1/2$ for all p; ϕ is concave and $\gamma \ge 0$, respectively.

5. Concluding comments

Like Gul and Pesendorfer's (2014) expected uncertain utility, EBUU affords the outside observer the ability to infer those events the DM deems measurable, *solely* from her preferences. And just as it was the case for ideal events in expected uncertain utility, decomposable events may be viewed as ones for which Savage's (1954) *sure-thing principle* applies.

Unlike Gul and Pesendorfer, however, we have not "operationalized" this principle by imposing Savage's postulate **P2**. Instead, following Grant, Kajii, and Polak (2000), we have interpreted statements like *"f would be preferred to g if the event R were known to obtain"* as only entailing $f_R g \succ g$. As a consequence and following as an immediate corollary to Theorem 1, for a DM who exhibits rich coherent beliefs, her risk preferences (over outcome lotteries) need only satisfy the betweenness property of Chew (1983) and Dekel (1986), rather than (full) independence. Moreover, the third example from Section 4 provides us with a parsimoniously parameterized model that not only can accommodate Ellsberg-style choice patterns but allows the DM to exhibit the novel phenomenon of aversion to ambiguity about disappointment.

Although arguably they have received a reasonable amount of attention in the risk literature, preferences that exhibit betweenness properties are almost completely absent in the ambiguity literature. However, as decomposability provides us with a natural way to operationalize the sure-thing principle, we contend EBUU theory provides us with a normatively attractive approach for modeling choice under uncertainty.

Appendix

Proof of Theorem 1

To set the stage for the proof, we first describe how every act can be expressed in terms of constants acts or (and) diffuse acts on its measurable split, as explained in Gul and Pesendorfer (2014). Without loss of generality, we take $f(\Omega) = f(\Omega)^+$.

Lemma 1. *Fix a act $f \in F$ with measurable split $\{R_f^Y \in \mathbf{R}\colon Y \subseteq f(\Omega), Y \neq \emptyset\}$. For each nonnull element R_f^Y in the measurable split of f, if $|Y| > 1$, there exists a diffuse act $h^Y \in H$ such that $h^Y_{R_f^Y} f = f$, and if $Y = \{y\}$, $f = y_{R_f^Y} f$.*

Proof. Given a nondecomposable $f \in F$, choose a nonnull R_f^Y with $Y = \{y_1, \dots, y_n\}$, $n > 1$ (the case $n = 1$ is obvious). There exists a sequence of disjoint events $\{B_n\}$ such that $B_i = f^{-1}(y_i) \cap R_f^Y$ for all i. Lemma A2 of GP show that there exists a diffuse partition $\{D_1, \dots, D_n\} \in \mathbf{D}$ of Ω. Now define

$$D_1^* = \big(D_1 \cap \big(\Omega \setminus R_f^Y\big)\big) \cup B_1,$$

$$\dots\dots$$

$$D_n^* = \big(D_n \cap \big(\Omega \setminus R_f^Y\big)\big) \cup B_n.$$

We next show that $\{D_1^*, \dots, D_n^*\}$ is a diffuse partition of Ω. Assume by way of contradiction that D_i^* is not a diffuse event for some i. Then there is $R \in \mathbf{R}^+$ such that $R \in D_i^*$. Since $\mathbf{R}$ is a σ-algebra, $(\Omega \setminus (R_f^Y))\backslash R \in \mathbf{R}$. Moreover, $(\Omega \setminus (R_f^Y))\backslash R \in B_i$, which contradicts B_i containing no decomposable event. Thus, D_i^* is a diffuse event. It is easy to check that D_i^*s are all disjoint and their union is Ω. Therefore, $\{D_1^*, \dots, D_n^*\}$ is also a diffuse partition of Ω. Set $h^Y = (D_1^* : y_1, \dots, D_n^* : y_n)$. Then $h^Y_{R_f^Y} f = f$. □

Sufficiency of the axioms

Step 1: Deriving the prior First, we show that the set of decomposable events constitutes a σ-algebra: That is, (1) it contains both the universal set Ω and the empty set, $\emptyset$; (2) it is closed under complements; (3) it is closed under intersection; and (4) it is closed under countable unions.

Lemma 2. *The set of decomposable events* $\mathbf{R}$ *is a σ-algebra.*

Proof. (1) From the definition of a decomposable event, it is immediate that $\emptyset \in \mathbf{R}$ and $\Omega \in \mathbf{R}$. (2) If $R \in \mathbf{R}$, then also by definition we have $\Omega \setminus R \in \mathbf{R}$. (3) Axiom 2 ensures $\mathbf{R}$ is closed under intersection.

(4) Finally, let $\{R^n\}$ be a set of (increasing) decomposable events with $R^n \subset R^{n+1}$. Assume by way of contradiction that $R^\infty := \bigcup R^n$ is not a decomposable event. That is, there exist $f, g \in \mathcal{F}$ such that $f_{R^\infty} g \succ g$ and $g_{R^\infty} f \succ g$ but $g \succsim f$. But since each R^n is decomposable, this means for every n either $g \succsim f_{R^n} g$ or $g \succsim g_{R^n} f$ (or both). Hence, we can find an infinite subsequence $\{\widehat{R}^n\}$ of $\{R^n\}$ with $\bigcup \widehat{R}^n = R^\infty$, and for which:

(i) either $g \succsim f_{\widehat{R}^n} g$ ($\succsim \underline{x}$) for all n,

(ii) or $g \succsim g_{\widehat{R}^n} f$ ($\succsim \underline{x}$) for all n.

If (i) (resp., (ii)) holds, axiom 5 implies $g \succsim f_{R^\infty} g$ (resp., $g \succsim g_{R^\infty} f$) contradicting $f_{R^\infty} g \succ g$ (resp., $g_{R^\infty} f \succ g$). Thus we have established there must exist at least one n for which $f_{R^n} g \succ g$ and $g_{R^n} f \succ g$, which since R^n is decomposable, in turn implies $f \succ g$—a contradiction. Thus, $\bigcup R^n$ is a decomposable event. Therefore, $\mathbf{R}$ is a σ-algebra. □

The following auxiliary result is fundamental and used both here and in subsequent steps.

Lemma 3. *For all $x^* \succ x$, all $f, f' \in F$, and $R \in \mathbf{R}^+$, if $x^*{}_R f \succsim f' \succsim x_R f$, then there is a $R' \in \mathbf{R}$ for which $(x^*{}_{R'} x)_R f \sim f'$.*

Proof. If either $x^*{}_R f \sim f'$ or $f' \sim x_R f$, set $R' := \Omega$ or set $R' := \emptyset$. We now only need to find a decomposable event R' when $x^*{}_R f \succ f' \succ x_R f$.

Since $x^*{}_R f \succ f'$, Axiom 6 implies there is a decomposable event $R_1 \subset R$ such that $(x_{R_1} x^*)_R f \succ f'$. Applying Axiom 6 again implies there is a decomposable event $R_2 \subset R \setminus R_1$ such that $(x_{R_1 \cup R_2} x^*)_R f \succ f'$. By repeating the argument, this yields a series $\{R_n\} \subset \mathbf{R}^+$ of disjoint subsets of R that satisfies

$$\left(x_{\bigcup_{n=1}^k R_n} x^*\right)_R f \succ f' \quad \text{for all } k. \tag{3}$$

Let A denote the collection of all such series, and define $B := \{\bigcup_{n=1}^\infty R_n : \{R_n\} \in A\}$. Notice that $B \subset \mathbf{R}$, and that $(x_{\widehat{R}} x^*)_R f \succsim f'$ for each $\widehat{R} \in B$, by Axiom 5. We show that we can take $R' = R \setminus M$ with M a maximal element of B if it exists, and invoke Zorn's lemma to establish that otherwise we can take M as the upper bound outside B of a chain in B.

Lemma (Zorn). *Let $\mathcal{P}$ be a partially ordered set in which each chain C has an upper bound. Then $\mathcal{P}$ has at least one maximal element.*

This applies to B as a partially ordered set, by set-inclusion. First, assume B has a maximal element M. We can exclude that $(x_M x^*)_R f \succ f'$, since otherwise the procedure above would determine $\tilde{R} \in \mathbf{R}^+$ for which still $((x_{M \cup \tilde{R}} x^*)_E f \succ f'$, implying that also $M \cup \tilde{R} \in B$, as limit of the series $(M \cup \tilde{R}, \emptyset, \ldots) \in A$, contradicting that M is maximal of B with $M \subset M \cup \tilde{R}$. So, $(x^*{}_{R \setminus M} x)_f \sim f'$.

Next, suppose B has no maximal element. By Zorn's lemma, B must contain a chain C with upper bound $\bigcup_{\tilde{R} \in C} \tilde{R} \notin B$. Now we can take M as this upper bound, we can again exclude that $(x_{M \cup \tilde{R}} x^*)_R f \succ f'$.

To conclude, in both cases, we can take R' as the decomposable event $R \setminus M$ such that $(x_M x^*)_R f \sim f'$. □

Next, we establish Machina and Schmeidler's (1992) Axiom **P4*** holds on the restriction of $\succsim$ to G (the set of decomposable acts).

LEMMA 4. *For any decomposable events R, $\widehat{R}$, $T \in \mathbf{R}$, $R \cup \widehat{R} \subseteq T$, any four outcomes $x^* \succ x$ and $y^* \succ y$ in X, and any pair of acts $f, f' \in F$:*

$$(x^*_R x)_T f \succsim (x^*_{\widehat{R}} x)_T f \quad \Longrightarrow \quad (y^*_R y)_T f' \succsim (y^*_{\widehat{R}} y)_T f'.$$

PROOF. From $(x^*_R x)_T f \succsim (x^*_{\widehat{R}} x)_T f \succ x_T f$, by applying Lemma 3 we can find a decomposable event $R' \subseteq R$ such that $(x^*_{R'} x)_T f \sim (x^*_{\widehat{R}} x)_T f$. Thus by Axiom 4 it follows $(y^*_{R'} y)_T f' \sim (y^*_{\widehat{R}} y)_T f'$. And since $R' \subseteq R$ it follows from Axiom 3 that $(y^*_R y)_T f' \succsim (y^*_{R'} y)_T f'$. The desired implication then follows from the transitivity of $\succsim$. □

Now we consider the restriction of $\succsim$ to binary bets on decomposable events involving the best $\bar{x}$ and worst $\underline{x}$ outcomes, denoted as $\mathcal{F}^{\{\underline{x},\bar{x}\}}$, and establish it admits an SEU representation. Our structural assumption on X that $\bar{x} \succ \underline{x}$ implies (Savage's) **P5**. Axiom 1 is **P1**. Axiom 3 directly implies **P3**. **P4** is redundant, and Axiom 6 is **P6**. Finally, in a setting with exactly two outcomes $\bar{x} \succ \underline{x}$, **P4*** (as derived above from Axiom 4) is equivalent to **P2**; cf. Machina and Schmeidler (1992, p. 764).

Thus, there is a finitely additive, convex-ranged μ on Σ and a function $v : \{\underline{x}, \bar{x}\} \to \mathbb{R}$ such that $V : \mathcal{F}^{\{\underline{x},\bar{x}\}} \to \mathbb{R}$, defined by $V(\bar{x}_R \underline{x}) = \mu(R)v(\bar{x}) + (1 - \mu(R))v(\underline{x})$, represents $\succsim$ on $\mathcal{F}^{\{\underline{x},\bar{x}\}}$. Without loss of generality, we can set $v(\underline{x}) := 0$ and $v(\bar{x}) := 1$. Hence

$$V(\bar{x}_R \underline{x}) = \mu(R). \tag{4}$$

Axiom 5 implies that $V(\bar{x}_{R^1 \cup \cdots \cup R^n} \underline{x})$ converges to $V(\bar{x}_{\bigcup_{n=1}^{\infty} R^n} \underline{x})$ for any disjoint sequence of decomposable events $\{R_n\}$, which yields

$$\lim_{n \to \infty} \sum_{n=1}^{n} \mu(R_i) = \mu\left(\bigcup_{n=1}^{\infty} R^n\right),$$

that is, μ is countably additive.

Step 2: Variants of decomposability To prepare for the construction and validation of the EBUU representation, we address some variants of decomposability conditions.

LEMMA 5. *Let $\succsim$ be a relation that satisfies Axiom 1 and Axiom 3. For any decomposable event $E \in \mathbf{R}$ and any pair of acts $f, g \in F$:*

1. *$f \succ f_E g$ and $f \succ g_E f$ implies $f \succ g$;*

and furthermore, if $\succsim$ also satisfies Axioms 2 and 6, then

2. *$f \succsim f_E g$ and $f \succsim g_E f$ implies $f \succsim g$; and,*
3. *$f_E g \succsim g$ and $g_E f \succsim g$ implies $f \succsim g$.*

PROOF. Fix $E \in \mathbf{R}$. We show that statement 1 holds by the same techniques in Grant, Kajii, and Polak (2000). Assume by way of contradiction that there exist two acts $f, g \in F$, such that $f \succ f_E g$, $f \succ g_E f$, and $g \succsim f$. We consider two cases:

(a) Suppose $f_E g \succsim g_E f$. Then we have

$$g \succsim f \succ f_E g \succsim g_E f.$$

Set $\widehat{f} := f_E g$ and $\widehat{g} := g_E f$. Notice $\widehat{f}_E \widehat{g} = f$ and $\widehat{g}_E \widehat{f} = g$. Thus

$$\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{f} \succsim \widehat{g}.$$

Since E is decomposable, $\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{f}$ implies that $\widehat{g} \succ \widehat{f}$, which contradicts $\widehat{f} \succsim \widehat{g}$.

(b) Now suppose, $g_E f \succ f_E g$. Then we have

$$g \succsim f \succ g_E f \succ f_E g.$$

So again, set $\widehat{f} := f_E g$ and $\widehat{g} := g_E f$, and again notice that $\widehat{f}_E \widehat{g} = f$ and $\widehat{g}_E \widehat{f} = g$. Thus

$$\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{g} \succ \widehat{f}.$$

Since E is decomposable, $\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{g}$ implies that $\widehat{f} \succ \widehat{g}$, which contradicts $\widehat{g} \succ \widehat{f}$.

Therefore, we have established that statement 1 holds.

For statement 2, we first show that for all (arbitrary) acts $f \in F$ and decomposable acts $g \in G$,

$$f \sim f_E g \sim g_E f \quad \text{implies} \quad f \sim g \tag{5}$$

Assume by way of contradiction that $f \sim f_E g \sim g_E f$ and $[f \succ g$ or $g \succ f]$:

(a) $f \sim f_E g \sim g_E f$ and $f \succ g$.

(i) In case, $E \subset g^{-1}(\bar{x})$ or $\Omega \setminus E \subset g^{-1}(\bar{x})$. If $E \subset g^{-1}(\bar{x})$, then

$$f \sim f_E g \sim \bar{x}_E f \succ g = \bar{x}_E g, \quad \text{that is,} \quad f_E g \succ \bar{x}_E g$$

and Axiom 3 is violated. Similarly, if $\Omega \setminus E \subset g^{-1}(\bar{x})$, then

$$f \sim f_E \bar{x} \sim g_E f \succ g = g_E \bar{x}, \quad \text{that is,} \quad g_E f \succ g_E \bar{x}.$$

Again, Axiom 3 is violated. The same argument applies when $g(E)$ or $g(\Omega \setminus E)$ only has outcomes indifferent to $\bar{x}$.

(ii) Otherwise, Axiom 6 implies there is a partition $\{R_i\}$ such that $f \succ \bar{x}_{R_i} g$ for all R_i. Since E is nonnull, there exist R_j such that $R_i \cap E$ is nonnull. Applying Axiom 6 again, there is a partition $\{R'_i\}$ such that $f \succ \bar{x}_{R'_i}(\bar{x}_{R_j} g)$ for all R'_i. Then there is R'_j such that $R'_j \cap E^c$ is non-null. Let $R = R_j \cup R'_j$ and so $f \succ \bar{x}_R g$ with both $R \cap E$ and $R \cap E^c$ nonnull. Together with Axiom 3, $f_E(\bar{x}_R g) \succ f$ and $(\bar{x}_R g)_E f \succ f$, and so $\bar{x}_R g \succ f$ and we reach a contraction.

(b) $f \sim f_E g \sim g_E f$ and $g \succ f$. This case can be proved in the same way as what we have done in part (a) by applying Axiom 6 on $\underline{x}$.

Next, we show statement 2 holds. Assume for contradiction that there are f, g such that any of the following holds:

(i) $f \sim f_E g \sim g_E f$ and $g \succ f$.

(ii) $f \succ f_E g$, $f \sim g_E f$, and $g \succ f$.

(iii) $f \sim f_E g$, $f \succ g_E f$ and $g \succ f$.

We only need to show case (i) since the other two are in favor of the direction to get contradiction. Applying Lemma 3, there exist decomposable events R_g^1 and R_g^2 such that

$$f \sim f_E(\underline{x}_{R_g^1}\bar{x}) \sim (\underline{x}_{R_g^2}\bar{x})_E f$$

and so $f \sim \underline{x}_{R_g^1 \cup R_g^2}\bar{x}$ by Eq. (5). Similarly, there there exist decomposable events R_f^1 and R_f^2 such that

$$f \sim g_E(\underline{x}_{R_f^1}\bar{x}) \sim (\underline{x}_{R_f^2}\bar{x})_E g \tag{6}$$

Now we let $f^* = \underline{x}_{R_g^1 \cup R_g^2}\bar{x}$ and $g^* = \underline{x}_{R_f^1 \cup R_f^2}\bar{x}$, the above preferences are reduced to

$$f \sim f^* \sim f_E f^* \sim f^*_E f \sim g_E g^* \sim g^*_E g \sim f_E g \sim g_E f$$

Notice that $f^*_E f = (f^*_E g^*)_E(g_E f)$ and $g_E g^* = (g_E f)_E(f^*_E g^*)$ and $f^*_E g^* \in G$ and so $f^*_E g^* \sim f^*$ by expression (5). Similarly, $g^*_E f^* \sim f^*$, so $f^* \sim g^*$. From (4), it follows that $\mu(R_f^1) = \mu(R_g^1)$ and $\mu(R_f^2) = \mu(R_g^2)$. Expression (6) becomes

$$f \sim g_E(\underline{x}_{R_g^1}\bar{x}) \sim (\underline{x}_{R_g^2}\bar{x})_E g \sim \underline{x}_{R_g^1 \cup R_g^2}\bar{x}$$

and so $g \sim \underline{x}_{R_g^1 \cup R_g^2}\bar{x} \sim f$, which gives us the contradiction.

Statement 3 can be proved in the same way as statement 2. □

Lemma 5 implies the conditional independence of acts on an indifference class.

Proposition 2. *For any event $R \in \mathbf{R}$ and any four acts $f, f', f'', f^* \in F$:*

$$f_R f' \sim f_R f'' \sim f^*_R f' \quad \Longrightarrow \quad f^*_R f'' \sim f^*_R f' \sim f_R f' \sim f_R f''.$$

Proof. By the nonstrict criterion in Lemma 5, $f_R f' \succsim f_R f''$ and $f_R f' \succsim f^*_R f'$ implies $f_R f' \succsim f^*_R f''$. Similarly, $f_R f' \precsim f_R f''$ and $f_R f' \precsim f^*_R f'$ implies $f_R f' \precsim f^*_R f''$. Thus, $f^*_R f' \sim f_R f'$ implies $f^*_R f'' \sim f^*_R f' \sim f_R f' \sim f_R f''$. □

Step 3: Constructing the balanced outcome-set utility The preliminary results above (particularly, Lemmas 3 and 5) enable us to define the balanced outcome-set utility $U(\cdot, \cdot)$, as follows. Set $U(\{\bar{x}\}, p) := 1 - p$ and $U(\{\underline{x}\}, p) := -p$, for all $p \in [0, 1]$.

Fix an outcome set Y and $p \in [0,1]$. We employ a shorthand notation $\bar{x}_p\underline{x}$ for an act of the form $\bar{x}_R\underline{x}$ with $\mu(R) = p$. When Y is not a singleton, choose a diffuse act $h^Y \in H$, for which its envelope $\mathbf{h}^Y$ is the constant function $\mathbf{h}^Y(\omega) = Y$ for all $\omega \in \Omega$. When $Y = \{y\}$, set $h^Y := y$.

(a) For the case $h^Y \succ \bar{x}_p\underline{x}$, determine a decomposable event R such that $h^Y_R\underline{x} \sim \bar{x}_p\underline{x}$. Such an R exists, since the balance probability of $h^Y_{R'}$ with $R' \in \mathbf{R}$ only depends on $\mu(R')$, again by Axiom 4, and we take any $R \in \mathbf{R}$ with $\mu(R) = q$, for q the maximum probability of R' such that $h^Y_{R'} \succsim \bar{x}_p\underline{x}$. Notice that q does not depend on the choice of h^Y, by Axiom 3(i). In order for $\succsim$ to admit an EBUU representation requires

$$\begin{aligned} &qU(Y,p) + (1-q)U\big(\{\underline{x}\},p\big) \\ &\quad = pU\big(\{\bar{x}\},p\big) + (1-p)U\big(\{\underline{x}\},p\big) = 0. \end{aligned}$$

Solving for $U(Y,p)$ yields

$$U(Y,p) := \frac{1-q}{q} \times p.$$

(b) Otherwise, determine a decomposable event R such that $h^Y_R\bar{x} \sim \bar{x}_p\underline{x}$, and set $q := \mu(R)$. Again, q does not depend on the choice of R and h^Y. In order for $\succsim$ to admit a EBUU representation requires

$$qU(Y,p) + (1-q)U\big(\{\bar{x}\},p\big) = 0,$$

which yields

$$U(Y,p) := -\frac{1-q}{q} \times (1-p).$$

By construction, this function satisfies the two properties required for a balanced outcome-set utility.

Step 4: Establishing the EBUU representation It remains to verify that μ and U, as specified above, constitute a proper EBUU representation that represents the given ordering $\succsim$ on F.

Fix an arbitrary act f in F that has associated with it the measurable split $\{R^Y_f : Y \subseteq f(\Omega)^+\}$, and the diffuse acts h^Y_f for which $(h^Y_f)_{R^Y_f} f = f$.

There exists a $p \in [0,1]$, such that $f \sim \bar{x}_p\underline{x}$ (again by Lemma 3, taking $R = \Omega$). For each $Y \subseteq f(\Omega)^+$, we can find a decomposable subevent $\bar{R}^Y_f \subseteq R^Y_f$ for which $[\bar{x}_{\bar{R}^Y_f}\underline{x}]_{R^Y_f} f \sim f$. Also, there is $A \subset \Omega \setminus R^Y_f$ in $\mathbf{R}$ such that $f \sim f_{R^Y_f}[\bar{x}_A\underline{x}]$, and Proposition 2 guarantees that also $[\bar{x}_{\bar{R}^Y_f}\underline{x}]_{R^Y_f}[\bar{x}_A\underline{x}] \sim f$.

To analyze these expressions, notice that the value of an act of the form $(h^Y_R\underline{x})_E\bar{x}$ for measurable events $R \subset E$, only depends on $\mu(R)$ and $\mu(E)$, again by Axiom 4. Since we can split R and E in k subsets R_i, E_i with equal probabilities, respectively $\mu(R)/k$ and $\mu(E)/k$, we have $(h^Y_{R_i}\underline{x})_{E_i}\bar{x} \sim (\bar{x}_{A_i}\underline{x})_{E_i}\bar{x}$ for events $A_i \subset E_i$, with also $\mu(A_i)$ independent of i. From Proposition 2, it follows now that $(h^Y_R \underline{x})_E \bar{x} \sim (\bar{x}_A \underline{x})_E \bar{x}$. This means that this indifference in fact only prescribes the ratio $\mu(R)/\mu(A)$ for any $E \in \mathbf{R}$.

More generally, we know from Axiom 4 that the ratio $\mu(\bar{R}^Y_f)/\mu(R^Y_f)$ must be the same in all acts of the form $h^Y_{R^Y_p} f'$ that are on the same indifference curve as f, being all indifferent to $h^Y_{R^Y_p}[\bar{x}_A \underline{x}]$ for some $A \in \mathbf{R}$. Since we defined $U(Y, p)$ from the rule $h^Y_{R^Y_p} \underline{x} \sim \bar{x}_{R_p} \underline{x}$ (when $h^Y \succsim f$) and $h^Y_{R^Y_p} \bar{x} \sim \bar{x}_{R_p} \underline{x}$ (when $f \succ h^Y$), we can determine this ratio as

$$\frac{\mu(\bar{R}^Y_f)}{\mu(R^Y_f)} = \begin{cases} \dfrac{p}{q} = U(Y, p) + p & \text{if } h^Y \succsim f \\ \dfrac{q - 1 + p}{q} = U(Y, p) + p & \text{otherwise.} \end{cases}$$

So, the contribution of each outcome set Y to the EBUU equation is

$$\mu(R^Y_f)\left[\frac{\mu(\bar{R}^Y_f)}{\mu(R^Y_f)}(1-p) + \left(1 - \frac{\mu(\bar{R}^Y_f)}{\mu(R^Y_f)}\right)(-p)\right] = \mu(R^Y_f) U(Y, p),$$

as desired.

Necessity of the axioms Axiom 1 follows from the fact that for any $f \in F$, there exists a unique $p \in [0, 1]$ such that (2) holds true for $V(f) = p$.

Let $\mathbf{R}$ denote the domain of μ, which is a σ-algebra. To show that the events in $\mathbf{R}$ are decomposable, consider an arbitrary $R \in \mathbf{R}$, and fix a pair of acts $f, g \in F$, with $f \sim \underline{x}_R \bar{x} \in X$ and $\mu(R) = p$. If $g_R f \succ f$, then

$$\sum_{Y \in \mathcal{X}^{(\Omega)}} U(Y, p) \mu\big(\mathbf{g}^{-1}(Y) \cap R\big) > \sum_{Y \in \mathcal{X}^{f(\Omega)}} U(Y, p) \mu\big(\mathbf{f}^{-1}(Y) \cap R\big). \tag{7}$$

And $f_R g \succ f$ implies that

$$\sum_{Y \in \mathcal{X}^{g(\Omega)}} U(Y, p) \mu\big(\mathbf{g}^{-1}(Y) \cap \Omega \setminus R\big) > \sum_{Y \in \mathcal{X}^{f(\Omega)}} U(Y, p) \mu\big(\mathbf{f}^{-1}(Y) \cap \Omega \setminus R\big). \tag{8}$$

Adding inequalities (7) and (8), we get

$$\sum_{Y \in \mathcal{X}^{g(\Omega)}} U(Y, p) \mu\big(\mathbf{g}^{-1}(Y)\big) > 0.$$

That is, $g \succ f$. So, $\mathbf{R}$ only contains decomposable events.

Next, we show that any event outside $\mathbf{R}$ is not decomposable. Consider an event $E \subset \Omega$ but $E \notin \mathbf{R}_\mu$. Let $[E]_*$ (resp., $[\Omega \setminus E]_*$) denote the inner sleeve of E (resp., $\Omega \setminus E$), and define $\tilde{E} := \Omega \setminus (E_* \cup [\Omega \setminus E]_*)$. Since $E \notin \mathbf{R}$, $r := \mu(\tilde{E}) > 0$. By Lemma A2 (p. 22) and the proof of B11 (p. 31) in GP, we can partition $E \setminus [E]_*$ (resp., $(\Omega \setminus E) \setminus [\Omega \setminus E]_*$) into two nonnull events B^{11} and B^{12} (resp., B^{21} and B^{22}). Notice by construction none of the four events B^{11}, B^{12}, B^{21}, and B^{22} contain any nonnull measurable event.

To show E is not decomposable, observe that $U(\{\bar{x}\}, 0) > U(\{\underline{x}\}, 0) = 0$ and $U(\{\bar{x}\}, 0) \geq U(\{\underline{x}, \bar{x}\}, 0) \geq U(\{\underline{x}\}, 0) = 0$. Hence at least one of following two inequalities: (i) $U(\{\bar{x}, \underline{x}\}, 0) > 0$ and (ii) $U(\{\bar{x}\}, 0) > U(\{\bar{x}, \underline{x}\}, 0)$ must hold.

Consider first, the case $U(\{\bar{x},\underline{x}\},0) > U(\{\underline{x}\},0)$ and the act $f=\bar{x}_{B^{11}\cup B^{21}}\underline{x}$. Since $\underline{x}_E f=\bar{x}_{B^{21}}\underline{x}$ and $f_E\underline{x}=\bar{x}_{B^{11}}\underline{x}$, it follows that $\mathbf{f}=\{\underline{x}\}_E\mathbf{f}=\mathbf{f}_E\{\underline{x}\}$ (i.e., the envelope of each of those three acts are all the same). Since by construction, the measure μ is a coherent belief for the preferences generated by the (implicitly defined) EBUU functional, this means $f\sim\underline{x}_E f\sim f_E\underline{x}$. However, since

$$\begin{aligned}&\big[1-\mu\big([E]_*\big)-\mu\big([\Omega\setminus E]_*\big)\big]u\big(\{\bar{x},\underline{x}\},0\big)+\big[\mu\big([E]_*\big)+\mu\big([\Omega\setminus E]_*\big)\big]u\big(\{\underline{x}\},0\big)\\&\quad>\big[1-\mu\big([E]_*\big)-\mu\big([\Omega\setminus E]_*\big)\big]u\big(\{\underline{x}\},0\big)+\big[\mu\big([E]_*\big)+\mu\big([\Omega\setminus E]_*\big)\big]u\big(\{\underline{x}\},0\big)(=0),\end{aligned}$$

it follows that $f\succ\underline{x}$, say $f\sim\bar{x}_{R_p}\underline{x}$ for some R_p with $\mu(R_p)=p>0$. To arrive at a violation of the decomposability criterion, choose a measurable event $R\subset\tilde{E}$ with $0<\mu(R)<p$, so that $f\succ f':=\underline{x}_R f\succ\underline{x}$. Since the envelopes of $f'_E\underline{x}$ and $\underline{x}_E f'$ are the same as those of respectively $f_E\underline{x}$ and $\underline{x}_E f$, we have $f'_E\underline{x}\succ f'$ and $\underline{x}_E f'\succ f'$, yet $f'\succ\underline{x}$. So, E is not decomposable.

So, now consider the case $U(\{\bar{x}\},0)>U(\{\bar{x},\underline{x}\},0)$ and the pair of acts $f=\bar{x}_{B^{11}\cup B^{21}}\underline{x}$ and $f'=\underline{x}_{[E]_*\cup[\Omega\setminus E]_*}\bar{x}$. Since $f_E f'=\underline{x}_{B^{11}\cup B^{21}\cup B^{22}}\bar{x}$ and $f'_E f=\underline{x}_{B^{11}\cup B^{12}\cup B^{22}}\bar{x}$, $\mathbf{f}=\mathbf{f}_\mathbf{E}\mathbf{f}'=\mathbf{f}'_\mathbf{E}\mathbf{f}$, that is, all three acts come from the same indifference set, and hence $f\sim f_E f'$ and $f\sim f'_E f$. However, since

$$\begin{aligned}&\big[1-\mu\big([E]_*\big)-\mu\big([\Omega\setminus E]_*\big)\big]u\big(\{\bar{x}\},0\big)+\big[\mu\big([E]_*\big)+\mu\big([\Omega\setminus E]_*\big)\big]u\big(\{\underline{x}\},0\big)\\&\quad>\big[1-\mu\big([E]_*\big)-\mu\big([\Omega\setminus E]_*\big)\big]u\big(\{\bar{x},\underline{x}\},0\big)+\big[\mu\big([E]_*\big)+\mu\big([\Omega\setminus E]_*\big)\big]u\big(\{\underline{x}\},0\big)(=0),\end{aligned}$$

it follows $f'\succ f$. A violation of the decomposability criterion for E can be established as above. So, also in this case, E is not decomposable, and hence $\mathbf{R}$ is the set of all decomposable events.

Axiom 2 now follows directly from the assumption that $\mathbf{R}$ is a σ-algebra. The necessity of the rest of the axioms follows straightforwardly from the EBUU representation combined with the fact that for any pair of acts f and g with respective envelopes $\mathbf{f}$ and $\mathbf{g}$, and any decomposable event R in $\mathbf{R}$, the envelope of $g_R f$ is $\mathbf{g}_R\mathbf{f}$. In particular, the envelope of $h_R f$ has outcome-set $h(\Omega)$ on R, and equals $\mathbf{f}$ outside R. The necessity of Axiom 4 is now obvious. Axioms 3 and 5 follow directly from the corresponding properties of U. Axiom 6 follows from the fact that μ is convex-ranged. □

Proof of Proposition 1

Let $\succsim$ be characterized by $\langle\mu,U\rangle$, and satisfy Axiom 7. Given $h,h'\in H$ with $h(\Omega)^+$, $h'(\Omega)^+\subset(\ell,m)$, $h\geq h'$. Let $\widehat{h}_n=h+\varepsilon_n\in H$ with $\varepsilon_n>0$ and $\lim_{n\to\infty}\varepsilon_n=0$. Axiom 7 implies $\widehat{h}_n\succ h$ for all n. Since $\widehat{h}_n$ converges to h uniformly with $|\widehat{h}_n(\Omega)=h(\Omega)|$ for all n, Axiom 5.1 implies $\widehat{h}_n$ converges to h in preference, that is, $h\succsim h'$. Choose $h=x_D y\in H$ with $D\in\mathbf{D}$ and $x,y\in(\ell,m)$ with $x>y$. Lemma A2 of Gul and Pesendorfer (2014) implies there are disjoint $D_1,D_2\in\mathbf{D}$ with $D_1\cup D_2=D$. Similarly, there are disjoint $D'_1,D'_2\in\mathbf{D}$ with $D'_1\cup D'_2=D^c$. For any z with $x>z>y$, define $h'=x_{D_1}z_{D_2}y$ and $h''=x_D z_{D'_1}y$. We have $h''\geq h\geq h'$ and so $h''\succsim h\succsim h'$. Since $h'(\Omega)=h''(\Omega)$, $h''\sim h'$, that is, $h''\sim h\sim h'$. The same argument gives that for all $h,h'\in H$, $h\sim h'$ when $\max h(\Omega)=\max h(\Omega)$, $\min h(\Omega)=\min h(\Omega)$ and $h(\Omega),h'(\Omega)\subset(\ell,m)$.

Let $h, h' \in H$ with $h(\Omega) = \{m, \ell\}$ and $h'(\Omega) = \{m, x_1, \dots, x_n, \ell\}$ with $m > x_1 >, \dots, > x_n > \ell$. Given a positive decreasing sequence $\{\varepsilon_n\}$ such that $m - x_1 > \varepsilon_n$, $x_n - \ell > \varepsilon_n$, and for all n and ε_n goes to 0 as $n \to \infty$. Define h_n, h'_n as follows:

$$
\begin{aligned}
h_n(\omega) &= h(\omega) - \varepsilon_n \quad \text{if } \omega \in h^{-1}(m) & h'_n(\omega) &= h'(\omega) - \varepsilon_n \quad \text{if } \omega \in h'^{-1}(m) \\
h_n(\omega) &= h(\omega) + \varepsilon_n \quad \text{if } \omega \in h^{-1}(l) & h'_n(\omega) &= h'(\omega) + \varepsilon_n \quad \text{if } \omega \in h'^{-1}(l) \\
h_n(\omega) &= h(\omega) \quad \text{otherwise} & h'_n(\omega) &= h'(\omega) \quad \text{otherwise}
\end{aligned}
$$

and so $h_n \sim h'_n$ for all n. Since h_n, h'_n converges uniformly to h, h' respectively with $|h_n(\Omega)| = |h(\Omega)|$ and $|h'_n(\Omega)| = |h'(\Omega)|$ for all n. Axiom 5 implies $h \sim h'$. Therefore, for all $h, h' \in H$, $h \sim h'$ if $\max h(\Omega) = \max h(\Omega)$ and $\min h(\Omega) = \min h(\Omega)$. And so for all outcome-sets Y and all $p \in (0, 1)$, $U(Y, p) = U(\{\min_{x \in Y} x, \max_{y \in Y} y\}, p)$. □

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