Dynamic economics with quantile preferences

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This paper studies a dynamic quantile model for intertemporal decisions under uncertainty, in which the decision maker maximizes the τ-quantile of the stream of future utilities, for $\tau \in (0, 1)$. We present two sets of contributions. First, we generalize existing results in directions that are important for applications. In particular, the sets of choices and random shocks are general metric spaces, either connected or finite. Moreover, the future state is not exclusively determined by the agent's choice, but can also be influenced by shocks. Under these generalizations, we establish the principle of optimality, show that the corresponding dynamic problem yields a value function, and under suitable assumptions, this value function is concave and differentiable. Additionally, we derive the corresponding Euler equation. Second, we illustrate the usefulness of this approach by studying two prominent dynamic economics models. The first deals with intertemporal consumption with one asset. We obtain closed-form expressions for the value function, the optimal asset allocation and consumption, as well as for the consumption path. These closed-form solutions allow us to obtain useful comparative statics that shed light on how consumption and savings respond to increase in risk aversion, impatience, and interest rates. For the second model, we discuss a quantile-based version of the job-search model with uncertainty.

Keywords. Quantile preferences, dynamic programming, recursive model, intertemporal consumption, job search with unemployment.

JEL classification. C61, D1, E2.

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1. Introduction

Dynamic programming is a basic tool for intertemporal economic analysis that allows economists to examine a wide variety of problems. This framework has been extensively used because it is sufficiently rich to model problems involving sequential decision making over time and under uncertainty. See, among others, Stokey, Lucas, and Prescott (1989), Rust (1996), Ljungqvist and Sargent (2012), and Sargent and Stachurski (2023).

Many applications of intertemporal maximization use the standard recursive expected utility (EU). These models have been workhorses in several economic fields. EU is simple and amenable to theoretical modeling. The assumption of maximization of average utility, the average being a simple measure of centrality, has intuitive appeal as a behavioral postulate. Nevertheless, the usual EU framework has been subjected to a number of criticisms, including in its dynamic version.[1] An expanding literature considers alternative recursive models. We refer the reader to Epstein and Zin (1989, 1991), Weil (1990), Grant, Kajii, and Polak (2000), Epstein and Schneider (2003), Hansen and Sargent (2004), Maccheroni, Marinacci, and Rustichini (2006), Klibanoff, Marinacci, and Mukerji (2009), Marinacci and Montrucchio (2010), Strzalecki (2013), Bommier, Kochov, and Le Grand (2017), Sarver (2018), and Dejarnette, Dillenberger, Gottlieb, and Ortoleva (2020) among others.

Recently, de Castro and Galvao (2019) suggested a new alternative to the EU recursive model. In their model, the economic agent chooses the alternative that leads to the highest τ-quantile of the stream of future utilities for a fixed $\tau \in (0, 1)$. The dynamic quantile preferences for intertemporal decisions are represented by an additively separable quantile model with standard discounting. The associated recursive equation is characterized by the sum of the current period utility function and the discounted value of the certainty equivalent, which is obtained from a quantile operator. This intertemporal model is tractable and simple to interpret, since the value function and Euler equation are transparent, and easy to calculate (analytically or numerically). This framework allows the separation of the risk attitude from the intertemporal substitution, which is not possible with EU, while maintaining important features of the standard model, such as dynamic consistency and monotonicity.[2] Static quantile preferences were first studied by Manski (1988) and axiomatized by Chambers (2009), Rostek (2010), and de Castro and Galvao (2022). There are several recent applications of quantile preferences models; see, e.g., Bhattacharya (2009), Giovannetti (2013), Baruník and Čech (2021), Long, Sethuraman, and Xue (2021), and Chen, Dolado, and Gonzalo (2021), de Castro, Galvao, Montes-Rojas, and Olmo (2022b), Baruník and Nevrla (forthcoming). From an experimental point of view, de Castro, Galvao, Noussair, and Qiao (2022c) find that the behavior of between 30% and 50% of the individuals can be better described with quantile

[1]For example, it has been well documented in the literature that it is not possible to separate the intertemporal substitution from the risk attitude parameters when using standard dynamic models based on the EU (see, e.g., Hall (1988)). The framework proposed by Kreps and Porteus (1978) to study temporal resolution of uncertainty was one of the first efforts to go beyond EU in the dynamic setting.

[2]In the quantile model, the risk attitude is captured by τ. Therefore, the model allows a separation of risk attitude (governed by τ) and the elasticity of intertemporal substitution, which is exclusively determined by the utility function; see Section 2 for details.

preferences rather than the standard EU. Moreover, de Castro, Galvao, Kim, Montes-Rojas, and Olmo (2022a) provide experimental evidence that when individuals selecting a portfolio are able to clearly assess the differences in the lotteries' payoff distributions, their portfolio choices are closer to the optimal decision of a quantile maximizer than of a mean-variance maximizer.

The first main contribution of this paper is to generalize the quantile dynamic programming model. We extend existing results in important directions that are useful for practical applications. First, the sets of choices and random shocks are now general metric spaces, either connected or finite. This generalization substantially broadens the scope of economic applications. Moreover, we relax the assumption that the future state variable is exclusively determined by the agent's choice. Now the future state can also be influenced by shocks, and the choice variable is completely separate from the state variable, with the agent choosing a contingent action plan, which could also be influenced by the shock. This allows, for instance, to study the case in which the wealth in the current state is influenced by the random returns and not directly chosen from a previous investment decision.

Under these generalizations, we show that theoretical properties of the dynamic quantile model remain valid. In particular, we first establish the validity of the principle of optimality. Second, we show that the optimization problem leads to a contraction, which therefore has a unique fixed point. This fixed point is the value function of the problem and satisfies the Bellman equation. Third, under suitable assumptions, we prove that the value function is concave and differentiable, thus establishing the quantile analog of the envelope theorem. Fourth, using these results, we derive the corresponding Euler equation for the infinite horizon problem. These extensions are nontrivial.[3]

The second main contribution of this paper is to provide examples to illustrate the usefulness of the recursive quantile framework, exploring its economic and empirical implications. In particular, we revisit two important models. First, we illustrate the methods with a simple intertemporal consumption model with a single asset (see, e.g., Ljungqvist and Sargent (2012)), where the economic agent decides on how much to consume and save by maximizing a quantile recursive function subject to a linear budget constraint. Following a large body of literature, we specify an isoelastic utility function and derive several properties of the model. The quantile model is characterized by three parameters: the discount factor, the risk attitude, and the elasticity of intertemporal substitution. We solve the dynamic problem and obtain the Euler equation. Interestingly, we are able to obtain closed-form expressions for the fixed-point value function, and the optimal consumption and asset allocation. These closed-form solutions allow us to do comparative statics with respect to the parameters of the model and establish how consumption and savings decisions are influenced by changes in the risk attitude, impatience, or intertemporal substitution of decision makers, or by interest rate changes.

[3]The main difficult in establishing our results is related to the continuity of the quantile operator. This continuity is specially delicate when the variables are not required to have continuous densities. See discussion in Section 3.4.

In the second example, we discuss a quantile-based version of the job-search model discussed in McCall (1970). In a labor market characterized by uncertainty and costly information, both employers and employees will be searching. The analysis is directed to the employee's job-searching strategy. This model illustrates the use of the quantile framework when the decision variable is discrete and one of the shocks—namely keeping or losing the job—is also discrete.[4] We establish a characterization of the value function as a function of the wage, as well as the optimal wage.

The remaining of the paper is organized as follows. Section 2 describes the dynamic economic model and introduces the dynamic programming approach for determining the optimal solution of the recursive quantile model. We begin the discussion motivating the quantile model with a review of a dynamic model of intertemporal consumption without uncertainty. Section 3 presents the main theoretical results. Section 4 illustrates the empirical usefulness of the the new approach by providing different examples of the dynamic quantile model. Finally, Section 5 concludes. We relegate all proofs to the Appendix.

2. An introduction to quantile preferences

This section introduces the dynamic programming approach for determining the optimal solution of the recursive quantile model, which was introduced by de Castro and Galvao (2019). The objective is to write a recursive problem and solve the infinite horizon sequence problem, subject to a given constraint.

We begin by briefly revisiting the definition of quantiles. Given two random variables, W and Z, let $F(w|Z=z)=F_{W|Z=z}(w)=\Pr(W\leq w|Z=z)$ denote the conditional cumulative distribution function (c.d.f.) of W given Z. If the function $w\mapsto F_{W|Z=z}(w)$ is strictly increasing and continuous in its support, its inverse is the quantile of W given Z, i.e., $\mathrm{Q}_\tau[W|Z=z]=F^{-1}_{W|Z=z}(\tau)$, for $\tau\in(0,1)$.[5] This case is illustrated in Figure 1(a). If $w\mapsto F_{W|Z=z}(w)$ is not invertible, we can still define the quantile as one of its generalized inverses. Following the standard practice, we define the quantile as the left-continuous version of the generalized inverse:

$$\mathrm{Q}_\tau[W|Z=z]\equiv\inf\big\{w\in\mathbb{R}:\Pr[W\leq w|Z=z]\geq\tau\big\}. \tag{1}$$

For simplicity, in the rest of the paper we will denote $\mathrm{Q}_\tau[W|Z=z]$ by $\mathrm{Q}_\tau[W|z]$ or $\mathrm{Q}_\tau[w|z]$.

Before we define quantile preferences both in the static and dynamic settings, it is useful to consider a simple investment problem *without uncertainty*: at date t, a consumer that had invested x_t in the previous period, receives interests R, risk-free. The consumer then needs to decide how much to consume in period t, c_t, enjoying utility $U(c_t)$ and how much to invest for future period, x_{t+1}. Thus, $x_{t+1}=x_tR-c_t$ or $c_t=x_tR-x_{t+1}$. The consumer's problem is

$$\max_{\{x_t\}_{t=0}^{\infty},x_t\geq 0}\sum_{t=0}^{\infty}\beta^t U(x_tR-x_{t+1}),$$

[4]The model also contemplates a continuous shock, determining the distribution of new wages.

[5]In this paper, we will not consider the cases in which $\tau\in\{0,1\}$.

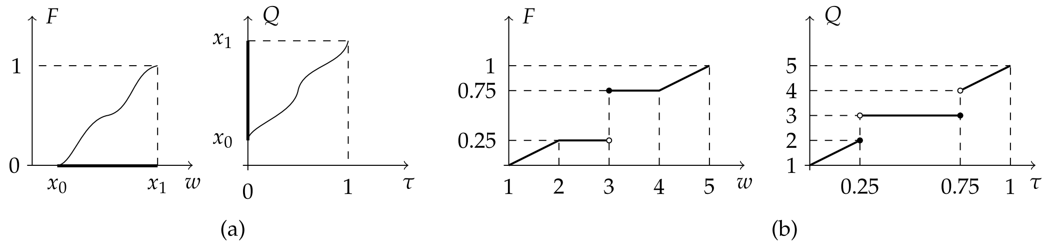

FIGURE 1. c.d.f. (F) and quantile (Q) functions when (a) F is continuous and strictly increasing in its support $[x_0, x_1]$; and (b) F is not invertible.

where $\beta \in (0, 1)$ is the discount factor. This model can be conveniently written in recursive form. For this, let $v(x_t)$ denote the present value of all future optimal consumption, given that the initial wealth is x_t. The recursive problem is

$$v(x_t) = \max_{x_{t+1} \in [0, x_t R]} \{U(x_t R - x_{t+1}) + \beta v(x_{t+1})\}. \tag{2}$$

It is easy to see that the concavity of U determines the consumption and investment decision by the consumer. In fact, if U is the isoelastic utility $U(c) = c^{1-\gamma}/(1-\gamma)$ for $\gamma > 0$, $\gamma \neq 1$, the elasticity of intertemporal substitution (EIS) is equal to $1/\gamma$.

Now, we would like to consider uncertainty, where the interest rate is represented by the random shock z_t. It is convenient to adapt the recursive form of the risk-free problem (2) by considering a certainty equivalent of the continuation utility, represented by value function $v(x_t, z_t)$ that depends in how much was invested in the previous period, x_t, and the current shock z_t. If, as usual, we adopt expectation as the certainty equivalent, then the recursive problem becomes

$$v(x_t, z_t) = \max_{x_{t+1} \in [0, z_t x_t]} \{U(x_t z_t - x_{t+1}) + \beta \mathrm{E}[v(x_{t+1}, z_{t+1} | z_t]\}, \tag{3}$$

where E_t is the conditional expectation with respect to the information at time t. Note that now the utility function $U(\cdot)$ in (3) determines both the risk attitude and the EIS. For the isoelastic function mentioned above, a single parameter, γ, determines both the EIS and the coefficient of relative risk aversion (CRRA). This creates a conceptual problem, since risk attitude and intertemporal substitution are distinct economic concepts that should be mutually independent.

This problem has been recognized a long time ago; see, for instance, Hall (1978, 1988). The preferred approach to deal with it has been to consider Epstein and Zin (1989)'s preferences. In this paper, we take a different route, by considering dynamic quantile preferences, that are defined by substituting the certainty equivalent expectation $\mathrm{E}[\cdot]$ in (3) by a quantile operator $\mathrm{Q}_\tau[\cdot]$. That is, we consider the following recursive problem:

$$v(x_t, z_t) = \max_{x_{t+1} \in [0, z_t x_t]} \{U(x_t z_t - x_{t+1}) + \beta \mathrm{Q}_\tau[v(x_{t+1}, z_{t+1} | z_t]\}, \tag{4}$$

With this change, U determines exclusively the intertemporal substitution, exactly as it does in the case without uncertainty. The risk attitude in the quantile model is not influenced by U. To understand this claim, let us consider quantile preferences in the static case.[6]

Recall that an expected utility maximizer with utility $U : \mathbb{R} \to \mathbb{R}$ prefers lottery X to Y if $\mathrm{E}[U(X)] \geq \mathrm{E}[U(Y)]$. Thus, it seems natural to define quantile preferences by simply substituting the expectation by the quantile operator in this comparison, i.e.,

$$X \succeq Y \quad \Longleftrightarrow \quad \mathrm{Q}_\tau\big[u(X)\big] \geq \mathrm{Q}_\tau\big[u(Y)\big]. \tag{5}$$

However, quantiles enjoy the following property: for any continuous and increasing function $f : \mathbb{R} \to \mathbb{R}$, $f(\mathrm{Q}_\tau[X]) = \mathrm{Q}_\tau[f(X)]$.[7] If $U : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous, as usual, then we can take its inverse and apply to (5), to obtain

$$X \succeq Y \quad \Longleftrightarrow \quad U^{-1}\big(\mathrm{Q}_\tau\big[U(X)\big]\big) \geq U^{-1}\big(\mathrm{Q}_\tau\big[U(Y)\big]\big) \quad \Longleftrightarrow \quad \mathrm{Q}_\tau[X] \geq \mathrm{Q}_\tau[Y].$$

Therefore, the utility function plays absolutely no role in the static quantile preference "defined" by (5). In particular, we could change a concave U by a convex U and obtain the same preference, that depends only on the quantile of the random variables themselves. However, this does not mean that U does not play a role in dynamic quantile preferences. On the contrary, in a dynamic setting U has exactly the same role that it had in the risk-free model: to define the intertemporal substitution. In fact, once uncertainty is resolved, the dynamic quantile preference model reduces to the risk-free model (2).

The discussion so far leads to an important question: if the concavity of the utility function does not play a role in the risk attitude for quantile preferences, what does?[8] The answer, first observed by Manski (1988), is quite simple: τ itself. To see this, consider Mendelson (1987)'s concept of "quantile-preserving spreads," that is an adaptation of the famous Rothschild and Stiglitz (1970)'s mean-preserving spreads. The idea is that Y is a quantile preserving spread of X if it is more likely to have both worse and better outcomes than X. Formally, Mendelson (1987) defines the following.

Definition 2.1 (Quantile-preserving spread). We say that Y is a τ-quantile-preserving spread of X if $\mathrm{Q}_\tau[Y] = \mathrm{Q}_\tau[X] = q$ and the following holds: (i) $t < q \implies F_Y(t) \geq F_X(t)$; and (ii) $t > q \implies F_Y(t) \leq F_X(t)$. Y is a quantile-preserving spread of X if it is a τ-quantile-preserving spread of X for some $\tau \in (0, 1)$.

[6] Quantile preferences were first introduced by Manski (1988). Rostek (2010) and Chambers (2009) provide axioms for the static case, and de Castro and Galvao (2022) formally axiomatize both the static and dynamic quantile preferences. Giovannetti (2013) studies a two-period economy for an intertemporal consumption model under quantile utility maximization. de Castro and Galvao (2019) establish the properties of a general dynamically consistent quantile preferences model.

[7] This property holds for expectation only if f is linear. For quantiles, it is sufficient that f is nondecreasing and left-continuous; see de Castro and Galvao (2019, Lemma A.2, p. 1927).

[8] Appendix B discusses the relationship of risk attitudes in EU and quantile preferences.

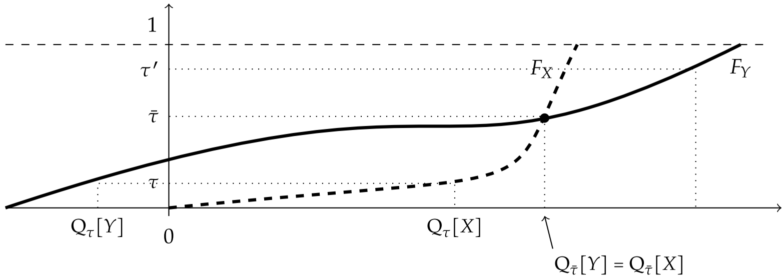

FIGURE 2. Y is a $\bar{\tau}$-quantile-preserving spread of X.

Figure 2 illustrates the c.d.f.'s of random variables Y and X when Y is a $\bar{\tau}$-quantile-preserving spread of X.[9] Notice that this definition captures the notion that Y is riskier than X, since it puts weight in more extreme values than X. Manski (1988) uses a different terminology for the same concept referring to the property of "single crossing from below": F_X crosses F_Y from below when Y is a quantile-preserving spread of X.

Note that if $Q_\tau[Y] = q$ and X is equal to q with probability 1, then Y is a τ-quantile-preserving spread of X. In other words, any risk asset Y with τ-quantile q is a quantile-preserving spread of any riskless asset X with value q.

Figure 2 suggests that the choice of a τ-quantile maximizer or τ-decision maker (τ-DM) depends on whether τ is below or above the quantile $\bar{\tau}$ where the two c.d.f.'s cross. That is, when $\tau < \bar{\tau}$ as in Figure 2, a τ-DM prefers the safer asset X: $Q_\tau[X] \geq Q_\tau[Y]$. On the other hand, if $\tau' > \bar{\tau}$, a τ-DM prefers the riskier asset Y: $Q_\tau[X] \leq Q_\tau[Y]$.

The following result formalizes the relationship between risk and the quantile τ for the simple static case.

PROPOSITION 2.2 (Manski, 1988). *Let Y be a $\bar{\tau}$-quantile-preserving spread of X for $\bar{\tau} \in (0, 1)$. Then: (i) $\tau \leq \bar{\tau} \implies Q_\tau[X] \geq Q_\tau[Y]$, i.e., a τ-DM prefers the asset X if τ is low; and (ii) $\tau \geq \bar{\tau} \implies Q_\tau[X] \leq Q_\tau[Y]$, i.e., a τ-DM prefers asset Y if τ is high.*

A relevant question is whether it is possible to reduce quantile preferences to expected utility with special subjective beliefs. If we restrict the set of alternatives (random prospects) from which the decision maker has to choose, and focus on a parametrized class of utility functions, it is possible to define a map of risk attitude between quantile preferences and expected utility. For example, suppose that we restrict our attention to log-normal variables X, i.e., $\ln(X) \sim \mathcal{N}(\mu, \sigma)$, the certainty equivalent of an expected utility maximizer, with isoelastic utility function $U(x) = x^{1-\gamma}/(1-\gamma)$, $\gamma \neq 1$, will be a function of μ, σ, and γ, while the certainty equivalent of a τ-quantile maximizer will be a function of μ, σ, and τ. By equating these certainty equivalents, we obtain a map between the downside risk aversion parameter τ and the risk aversion parameter γ for those class of assets. Appendix B shows that this map depends only on σ, i.e., by fixing

[9] Mendelson (1987) formalizes other four conditions and shows that they are all equivalent to the above definition; see the paper for further discussion and intuition.

the standard deviation σ, there is a one-to-one map between the risk attitude parameters τ and γ. However, the pairs of corresponding τ and γ will change if we change σ. Moreover, a completely new map may be obtained for different classes of random variables. Besides exploring this construction, Appendix B shows that quantile preferences in general cannot be reduced to expected utility preferences even with special priors, and also that quantile preferences do not belong to the general class of preferences considered by Epstein and Zin (1989).

3. Theoretical results

This section generalizes existing results for dynamic quantile models and provides theoretical foundations for the applications discussed in Section 4 below. Such generalizations are important for potential applications of dynamic economic models, thus substantially enlarging the scope of applicability of the recursive quantile model.

We begin by establishing the principle of optimality, and then the existence of the value function associated to the dynamic programming problem for the quantile preferences. We also present results on monotonicity, concavity, and differentiability of the value function. Finally, we derive the Euler equation. Derivations for dynamic consistency are similar to those contained in de Castro and Galvao (2019), and are omitted.

3.1 *States, decisions, shocks, and notation*

Let $\mathcal{X}$ denote the state space, $\mathcal{Y}$ be the set of possible actions the decision maker (DM) may take, and $\mathcal{Z}$, the range of the shocks (random variables) in the model. We require these sets to be metric spaces. Let $x_t \in \mathcal{X}$ denote the state in period t, and $z_t \in \mathcal{Z}$ the shock after the end of period $t-1$, both of which are known by the DM at the beginning of period t. In each period t, the DM chooses a feasible action y_t from a constraint subset $\Gamma(x_t, z_t) \subset \mathcal{Y}$.

In the model above, the resolution of uncertainty at period t occurs after the DM chooses an action so the next period's state x_{t+1} may be affected by the shock z_{t+1}, as discussed in Stokey, Lucas, and Prescott (1989, p. 240). This influence is described by a law of motion function ϕ from $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ to $\mathcal{X}$ that determines the next period state variable x_{t+1} as function of the current state x_t, the choice y_t, and the shock z_{t+1} realized at the beginning of period $t+1$, i.e.,

$$x_{t+1} = \phi(x_t, y_t, z_{t+1}). \tag{6}$$

It is common in the literature to write the law of motion in equation (6) as simply a function of x_t, y_t, and z_{t+1}; see, e.g., Stokey, Lucas, and Prescott (1989, p. 256). In most models, this is even simpler and we could write $\phi(x_t, y_t, z_{t+1}) = y_t$.

Let $\mathcal{Z}^t = \mathcal{Z} \times \cdots \times \mathcal{Z}$ (t-times, for $t \in \mathbb{N}$), $\mathcal{Z}^\infty = \mathcal{Z} \times \mathcal{Z} \times \cdots$ and $\mathbb{N}^0 \equiv \mathbb{N} \cup \{0\}$. Given $z \in \mathcal{Z}^\infty$, $z = (z_1, z_2, \ldots)$, we denote $(z_t, z_{t+1}, \ldots)$ by ${}_tz$ and $(z_t, z_{t+1}, \ldots, z_{t'})$ by ${}_tz_{t'}$. A similar notation can be used for $x \in \mathcal{X}^\infty$ and $y \in \mathcal{Y}^\infty$.

The random shocks will follow a time-invariant (stationary) Markov process. The set of random shocks $\mathcal{Z}$ is a (subset of a) metric space, assumed to be either connected or finite. Some results require $\mathcal{Z}$ to be Euclidean, i.e., $\mathcal{Z} \subseteq \mathbb{R}^k$. Stationary Markov processes are modeled by a Markov kernel $K : \mathcal{Z} \times \Sigma \to [0, 1]$, where Σ is the Borel σ-algebra of the metric space $\mathcal{Z}$.[10] This means that the probability that $Z' \in A \subset \mathcal{Z}$ given $Z = z$ is $\Pr(Z' \in A|Z = z) = K(z, A)$. The expectation of a function $h : \mathcal{Z} \to \mathbb{R}$ is $\mathrm{E}[h(w)|z] = \int_{\mathcal{Z}} h(z')k(z, dz')$.

We now introduce the concept of the *quantile martingale process*. This class of processes will be especially useful later to investigate particular examples of the model with closed-form solutions. Recall that for a standard martingale process, the best predictor of the expectation of the future value of the shock is its expectation. We adapt this notion for quantiles as follows.

Definition 3.1. We say that Z is a τ-quantile martingale if

$$\mathrm{Q}_\tau[Z_{t+1}|Z_t = z_t] = z_t. \tag{7}$$

This means that the best τth conditional quantile predictor of the random variable Z_{t+1} is the current value z_t. A simple and useful example of quantile martingale process is given by the following.

Example 3.2. Let $Z_{t+1} = Z_t + e_t$, where e_t satisfies $\mathrm{Q}_\tau[e_t|Z_t = z_t] = 0$. Then (7) holds, since $\mathrm{Q}_\tau[Z_{t+1}|Z_t = z_t] = \mathrm{Q}_\tau[Z_t + e_t|Z_t = z_t] = z_t + \mathrm{Q}_\tau[e_t|Z_t = z_t] = z_t + 0 = z_t$. ◊

3.2 *The recursive problem*

Given the current state x_t and current shock z_t, $\Gamma(x_t, z_t)$ denotes the set of possible choices y_t, i.e., the feasibility constraint set. Given x_t, z_t, and $y_t \in \Gamma(x_t, z_t)$, $u(x_t, y_t, z_t)$ denotes the instantaneous utility obtained in period t. The next period x_{t+1} is defined by a function $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ of the current state x_t, the choice y_t and the next period shock z_{t+1}, i.e.,

$$x_{t+1} = \phi(x_t, y_t, z_{t+1}).$$

In our model, the uncertainty with respect to the future realizations of z_t are evaluated by a quantile. In the dynamic quantile model, the intertemporal choices can be represented by the maximization of a value function $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ that satisfies the recursive equation:

$$v(x, z) = \sup_{y \in \Gamma(x,z)} \{u(x, y, z) + \beta \mathrm{Q}_\tau[v(\phi(x, y, z'), z')|z]\}, \tag{8}$$

where z' indicates the next period shock.

Note that this is similar to the usual dynamic programming problem, in which the expectation operator $\mathrm{E}[\cdot]$ is in place of $\mathrm{Q}_\tau[\cdot]$. See de Castro and Galvao (2019) for a

[10]Recall that a mapping $K : \mathcal{Z} \times \Sigma \to [0, 1]$ is a Markov kernel if for each $z \in \mathcal{Z}$, the set function $K(z, \cdot) : \Sigma \to [0, 1]$ is a probability measure and, for each $S \in \Sigma$, the mapping $K(\cdot, S) \to [0, 1]$ is Σ-measurable. See Aliprantis and Border (2006, Definition 19.11, p. 630).

construction of this recursive model from dated preferences. Section 3.5 below proves uniqueness of the solution to problem (8), under assumptions that will be discussed there and in Section 3.4. However, before we establish these results, it is useful to study the infinite horizon problem, which deals with a sequence of plans. We introduce the relevant notation and definitions in Section 3.3. A reader that is content to focus only on the recursive problem (8) will be able to skip it, since the rest of the paper is independent of these developments.

3.3 *Infinite horizon problem and the principle of optimality*

In this section, we define the infinite horizon problem and establish the principle of optimality, analogous to Stokey, Lucas, and Prescott (1989, Section 9.1) and that generalizes de Castro and Galvao (2019, Proposition 3.17). That is, we show that optimizing period by period, as in the recursive problem (8), yields the same result as choosing the best plan for the infinite horizon problem. This requires to formally define plans and the value function evaluated at those plans. Hence, we need to introduce some notation that is specific to this section and will not be used in the rest of the paper. There will be no loss to a reader that decides to skip this subsection.

At period t, the DM has learned the realization of the finite sequence of shocks $z^t = (z_1, \dots, z_t) \in \mathcal{Z}^t$ and can make a choice based upon this knowledge. This leads us to the following.

Definition 3.3. A plan h is a profile $h = (h_t)_{t\in\mathbb{N}}$ where, for each $t \in \mathbb{N}$, h_t is a measurable function from $\mathcal{X} \times \mathcal{Z}^t$ to $\mathcal{Y}$. The set of plans is denoted by H.

The interpretation of the above definition is that a plan $h_t(x_t, z^t)$ represents the choice that the individual makes at time t upon observing the current state x_t and the sequence of previous shocks z^t. The following notation will simplify statements below.

Definition 3.4. Given a plan $h = (h_t)_{t\in\mathbb{N}} \in H$, $x \in \mathcal{X}$ and realization $z^\infty = (z_1, \dots) \in \mathcal{Z}^\infty$, its associated sequence of states and choices is the sequence $(x_t^h, y_t^h)_{t\in\mathbb{N}} \in \mathcal{X}^\infty \times \mathcal{Y}^\infty$ defined recursively by $x_1^h = x$ and, for $t \geq 1$, by

$$y_t^h = h_t(x_t^h, z^t), \tag{9}$$

$$x_{t+1}^h = \phi(x_t^h, y_t^h, z_{t+1}). \tag{10}$$

Similarly, given $h \in H$, $(x, z^t) \in \mathcal{X} \times Z^t$, the t-sequence associated to (x, z^t) is $(x_t^h, y_t^h)_{l=1}^t \in \mathcal{X}^t \times \mathcal{Y}^t$ defined recursively by (9) and (10).

Since the elements of the sequence depend on x and z^∞, we may write them as $x_t^h(x, z^\infty)$ and $y_t^h(x, z^\infty)$. However, for simplicity and whenever convenient, we will write only x_t^h, $x_t^h(\cdot)$, $x_t^h(x, \cdot)$, or even $x_t^h(x, z^t)$ to emphasize that x_t^h depends on the initial state x and on the sequence of shocks z^∞, up to time t. Notice that $(x_t^h)_{t=1}^n$ is a random variable (function of z^∞) for each (h, x_0, z_0) and it is not known (realized) before time $t = n$.

Definition 3.5. A plan h is feasible from $(x, z) \in \mathcal{X} \times \mathcal{Z}$ if $h_t(x_t^h, z^t) \in \Gamma(x_t^h, z_t)$ for every $t \in \mathbb{N}$ and $z^\infty \in \mathcal{Z}^\infty$ such that $x_1^h = x$ and $z_1 = z$.

We denote by $H(x, z)$ the set of feasible plans from $(x, z) \in \mathcal{X} \times \mathcal{Z}$. Let H denote the set of all feasible plans from some point, i.e., $H \equiv \bigcup_{(x,z)\in\mathcal{X}\times\mathcal{Z}} H(x, z)$. We will give sufficient conditions for $H(x, z) \neq \emptyset$. Before that, we need to introduce some additional notation. For each $h \in H$ and $n \in \mathbb{N} \cup \{0\}$, define the function $S^{h,n} : \mathcal{X} \times \mathcal{Z}^{n+1} \to \mathbb{R}$ by

$$S^{h,n}(x, z^{n+1}) \equiv \sum_{t=0}^{n} \beta^t u(x_{t+1}^h, y_{t+1}^h, z_{t+1}). \tag{11}$$

It is sometimes convenient to abuse notation and write $S^{h,n}$ as a function of $\mathcal{X} \times \mathcal{Z}^\infty$ instead of $\mathcal{X} \times \mathcal{Z}^{n+1}$, i.e., $S^{h,n}(x, z^\infty)$ instead of $S^{h,n}(x, z^{n+1})$.

For a measurable $S : \mathcal{Z}^\infty \to \mathbb{R}$, let $\mathrm{Q}_\tau[S|z^t]$ denote the conditional quantile given z^t. Define $\mathrm{Q}_\tau^1[S|z]$ as $\mathrm{Q}_\tau[S|z]$ and, recursively, $\mathrm{Q}_\tau^{n+1}[S|z] = \mathrm{Q}_\tau^n[\mathrm{Q}_\tau[S|z^{n+1}]|z]$, i.e.,[11]

$$\mathrm{Q}_\tau^n[S|z] = \mathrm{Q}_\tau\big[\cdots \mathrm{Q}_\tau\big[\mathrm{Q}_\tau\big[S|z^n\big]|z^{n-1}\big]\cdots|z\big].$$

The Appendix discusses some properties of this operator that are essentially the same of the standard quantile operator.[12] For $n \in \mathbb{N}$ and $h \in H$, define

$$V^n(h, x, z) \equiv \mathrm{Q}_\tau^n\big[S^{h,n}(x, \cdot)|z\big].$$

The following assumption adapts Stokey, Lucas, and Prescott (1989, Assumption 9.2) to our setting.

Assumption 0. *The feasibility correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$ is nonempty-valued, its graph is measurable, and it has at least a measurable selection. Moreover, for all $x \in \mathcal{X}$, $z \in \mathcal{Z}$, and $h \in H$, there exists the limit*

$$V(h, x, z) \equiv \lim_{n\to\infty} V^n(h, x, z). \tag{12}$$

Assumption 0 is implied by our other assumptions introduced below. It is a very weak requirement, that allows us to define the value function, but will not be used outside this subsection. This assumption enables us to state our first result, which establishes that the set of feasible plans departing from $(x, z) \in \mathcal{X} \times \mathcal{Z}$ at time t is nonempty.

Lemma 3.6. *Let Assumption 0 hold. For any $x \in \mathcal{X}$ and $z \in \mathcal{Z}$, $H(x, z) \neq \emptyset$.*

[11] Since the "law of iterated expectations" does not have an analogue for quantiles, the iterative quantiles defining $\mathrm{Q}_\tau^n[\cdot|\cdot]$ do not collapse to a single quantile as they would do for expectations; see the further discussion in de Castro and Galvao (2019).

[12] Proposition 6.3 of de Castro, Costa, Galvao, and Zubelli (2023) gives sufficient conditions for the existence of the limit $\mathrm{Q}_\tau^\infty[S|z] \equiv \lim_{n\to\infty} \mathrm{Q}_\tau^n[S|z]$. In particular, if S is in $\mathcal{L}^\infty$, $\mathrm{Q}_\tau^\infty[S|z]$ exists. However, we are interested in another type of limit that we discuss next.

Now, we can define $v^* : \mathcal{X} \times \mathcal{Z} \to \overline{\mathbb{R}}$, the value function of infinite horizon problem:

$$v^*(x, z) \equiv \sup_{h \in H(x,z)} V(h, x, z). \tag{13}$$

Our objective is to establish the principle of optimality, that roughly states that solving the infinite horizon problem choosing plans as in (13) is essentially equivalent to solving the functional equation (8), in which the problem step by step, in a recursive fashion. To formalize this result, we need a few more definitions. The most important of those is the transversality condition. Recall that the standard transversality condition requires that the product of β^n and the integral of v up to time n converges to zero for all plans and initial states; cf. equation (7) in Stokey, Lucas, and Prescott (1989, p. 246). Our analogue is that $\lim_{n\to\infty} \beta^n \mathrm{Q}_\tau[v(\cdot)|z^n] = 0$. This is formalized as follows.

Definition 3.7. We say that a function $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ satisfies the conditional quantile transversality condition (CQTC) if for any $h \in H$, $(x, z) \in \mathcal{X} \times \mathcal{Z}$, and $\epsilon > 0$, there exists n_ϵ such that $n \geq n_\epsilon$ implies that for all $z^n = (z_1, \dots, z_n) \in \mathcal{Z}^n$, with $z_1 = z$,

$$-\epsilon < \beta^n \mathrm{Q}_\tau\big[v\big(\phi\big(x_n^h, y_n^h, z_{n+1}\big), z_{n+1}\big)|z^n\big] < \epsilon. \tag{14}$$

Notice that the above condition does not require v to be integrable, as the standard transversality condition does. Indeed, the quantile can even be uniformly bounded, which would imply (14), for a nonintegrable v, for which the transversality condition would not hold. When we assume additional structure, we can offer an alternative transversality condition. See Definition A.16 in Appendix A.8.1.

For a function $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$, let $G_v : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$ be the correspondence defined by

$$G_v(x, z) \equiv \big\{y \in \Gamma(x, z) : v(x, z) = u(x, y, z) + \beta \mathrm{Q}_\tau\big[v\big(\phi\big(x, y, z'\big), z'\big)|z\big]\big\}. \tag{15}$$

Of course, this correspondence may have empty values in general. We say that a plan $h \in H$ is obtained from G_v if there exists a sequence of selections $g_t : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$ such that for all $t \in \mathbb{N}$ and all $(x, z) \in \mathcal{X} \times \mathcal{Z}$, $g_t(x, z) \in G_v(x, z)$ and $h_t(x, z^t) = g_t(h_{t-1}(x, z^{t-1}), z_t)$.

We are now ready to state our principle of optimality.[13]

Theorem 3.8 (Principle of optimality). *Let Assumption 0 hold. Suppose that a function $v : \mathcal{X} \times \mathcal{Z} \to \overline{\mathbb{R}}$ satisfies (8) and (14). Suppose that G_v is nonempty and has a measurable selection. Then $v = v^*$ and any plan h obtained from G_v attains the supremum in (13).*

In the rest of the paper, we will introduce stronger assumptions that would imply not only Assumption 0, but also the existence of functions satisfying the functional equation (8) and (14). The significance of the above result is that it suggests a partial *uniqueness* result for the value function: even if our stronger assumptions introduced below do not hold but only the much weaker Assumption 0, there will be just one function that satisfies both the functional equation (8) and the CQTC (14).

[13]See Stokey, Lucas, and Prescott (1989, Theorem 9.2).

3.4 *Basic assumption on random shocks*

Now we state the main assumptions concerning the shocks, used for establishing the results. The following basic assumption is assumed throughout the paper.

ASSUMPTION 1 (Markov). *$\mathcal{Z}$ is a metric space, which is either connected or finite, and the process is Markov, with transition function $K : \mathcal{Z} \times \Sigma \to [0, 1]$ satisfying the following:*

(i) for each $z \in \mathcal{Z}$ and $\eta \in (0, 1)$, there exists compact $\mathcal{Z}' \subset \mathcal{Z}$ such that $K(z, \mathcal{Z}') > 1 - \eta$;

(ii) for each compact $A \subset \mathcal{Z}$, the function $z \in \mathcal{Z} \mapsto K(z, A) \in [0, 1]$ is continuous;

(iii) for each $A \in \Sigma$ open and nonempty, $K(z, A) > 0$ for all $z \in \mathcal{Z}$.

Assumption 1 is adopted in all results of this paper, even if it is not explicitly mentioned. Note that Assumption 1 allows an unbounded multidimensional Markov process. Condition (i) is equivalent to the requirement that, for each $z \in \mathcal{Z}$, $K(z, \cdot)$ is a tight measure, i.e., $K(z, A) = \sup\{K(z, C) : C \text{ is compact}, C \subset A\}$ for all $A \in \Sigma$. When $\mathcal{Z}$ is compact, this condition is trivially satisfied by choosing $\mathcal{Z}' = \mathcal{Z}$. Condition (ii) is a continuity property for Markov kernels that is satisfied for the most familiar processes. Condition (iii) is just the requirement that open subsets of $\mathcal{Z}$ have positive measure. We need to impose this condition to rule out discontinuities in the quantile. The property that quantiles are continuous is necessary to establish the continuity of the value function and, therefore, fundamental to many of the results in this paper.[14] Obviously, the continuity of the value function is also a desirable feature in itself.

Establishing the continuity of the quantile operator is the most delicate step in the proof. This is complicated by the fact that we allow discrete random shocks. Some of the problems that may arise when allowing discrete and continuous variables are illustrated by de Castro and Galvao (2022); see their Example 3.11 and Remark 3.12.

The following example shows that Assumption 1(iii) is necessary for the continuity of quantiles.

EXAMPLE 3.9. Let $\mathcal{Z} = [0, 1]$. Define

$$f(w, z) = \begin{cases} 2(1 - z) & \text{if } w \in \left[0, \dfrac{1}{4}\right] \cup \left[\dfrac{3}{4}, 1\right], \\ 2z & \text{if } w \in \left(\dfrac{1}{4}, \dfrac{3}{4}\right) \end{cases}$$

Thus, for all $z \in [0, 1]$, $f(z) = \int_0^1 f(w, z)\, dw = \frac{2(1-z)}{4} + \frac{2z}{2} + \frac{2(1-z)}{4} = 1$, and we obtain $f(w|z) = \frac{f(w,z)}{f(z)} = f(w, z)$. Consider $K : \mathcal{Z} \times \Sigma \to [0, 1]$ defined by $K(z, A) = \int_A f(w|z)\, dw$. This Markov kernel satisfies Assumption 1(i) and (ii), but not (iii). Indeed,

[14]The continuity of the value function requires, as an intermediary step, that the map $(x, y, z) \mapsto Q_\tau[v(\phi(x, y, w), w)|z]$ is continuous for continuous and bounded v; see Proposition A.4 in the Appendix. This result obviously requires that quantiles are continuous, i.e., $z \mapsto Q_\tau[w|z]$ is continuous.

the interval $A \equiv (\frac{1}{4}, \frac{3}{4})$ is open, but $K(0, A) = 0$. We will show that this leads to a failure of continuity of the quantile at $z = 0$ for $\tau = \frac{1}{2}$. We have

$$\Pr[w \leq \alpha | z] = K\big(z, \{w \in \mathcal{Z} : w \leq \alpha\}\big) = \begin{cases} 2(1-z)\alpha & \text{if } \alpha \in \left[0, \dfrac{1}{4}\right], \\ \dfrac{1-z}{2} + 2z(\alpha - \dfrac{1}{4}) & \text{if } \alpha \in \left(\dfrac{1}{4}, \dfrac{3}{4}\right], \\ \dfrac{1+z}{2} + 2(1-z)(\alpha - \dfrac{3}{4}) & \text{if } \alpha \in \left(\dfrac{3}{4}, 1\right]. \end{cases}$$

Thus, for $\tau = \frac{1}{2}$, we have $\mathrm{Q}_\tau[w|z] = \frac{1}{2}$ if $z \neq 0$, but $\mathrm{Q}_\tau[w|0] = \frac{1}{4}$. ◊

Example 3.9 can be modified to justify also the requirement of Assumption 1 that $\mathcal{Z}$ is either connected or finite. Indeed, all conditions of Assumption 1 are satisfied in the following example, but for the fact that $\mathcal{Z}$ is not connected.

Example 3.10. Let $\mathcal{Z} = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, and $\tau = \frac{1}{2}$. Define

$$f(w, z) = \begin{cases} 4 - 2z & \text{if } (w, z) \in \left[0, \dfrac{1}{4}\right] \times \left[0, \dfrac{1}{4}\right], \\ 4 + 2z & \text{if } (w, z) \in \left[\dfrac{3}{4}, 1\right] \times \left[0, \dfrac{1}{4}\right], \\ 4 & \text{otherwise.} \end{cases}$$

Then $f(w, z) > 0$ for all $(w, z) \in \mathcal{Z} \times \mathcal{Z}$ and $f(z) = 2, \forall z \in \mathcal{Z}$, which implies that $f(w|z) = \frac{1}{2} f(w, z)$. Defining the Markov kernel as before, Assumption 1 is satisfied, but for the fact that $\mathcal{Z}$ is not connected. We have

$$\Pr[w \leq \alpha | z] = K\big(z, \{w \in \mathcal{Z} : w \leq \alpha\}\big) = \begin{cases} (2-z)\alpha & \text{if } \alpha \in \left[0, \dfrac{1}{4}\right], \\ \dfrac{2-z}{4} & \text{if } \alpha \in \left(\dfrac{1}{4}, \dfrac{3}{4}\right], \\ \dfrac{2-z}{4} + (2+z)\left(\alpha - \dfrac{3}{4}\right) & \text{if } \alpha \in \left(\dfrac{3}{4}, 1\right]. \end{cases}$$

Consider a sequence $z_n \equiv \frac{1}{n} \to z^* = 0$. Since $\tau = \frac{1}{2}$, $\mathrm{Q}_\tau[w|z^*] = \frac{1}{4}$, while

$$\mathrm{Q}_\tau[w|z_n] = \frac{3}{4} + \frac{z_n}{2 + z_n} \to \frac{3}{4} \quad \text{when } n \to \infty.$$

Thus, $z \mapsto \mathrm{Q}_\tau[w|z]$ is not continuous at $z^* = 0$. ◊

It is worth noting that Assumption 1 extends the setting in de Castro and Galvao (2019) by allowing the set of random shocks $\mathcal{Z}$ to be a (connected or finite) metric space, instead of a convex subset of an Euclidean space. We also impose some less stringent assumptions on the distribution of the shocks, which generalize the setting of de Castro and Galvao (2019). These extensions are some of the major theoretical contributions of the current paper.

3.5 *Existence of the value function*

We prove the existence of the value function through a contraction fixed-point theorem. The first step is to the define the contraction operator. Let $\mathcal{C}$ denote the space of bounded and continuous functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$. For $\tau \in (0, 1)$, and $v \in \mathcal{C}$, define $\mathbb{M}(v)$ by

$$\mathbb{M}(v)(x, z) = \sup_{y \in \Gamma(x,z)} u(x, y, z) + \beta \mathrm{Q}_\tau\big[v\big(\phi(x, y, w), w\big)|z\big], \tag{16}$$

where $\beta \in (0, 1)$ and $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$ is the feasibility correspondence. The functional in (16) is similar to the usual dynamic programming problem with the expectation operator $\mathrm{E}[\cdot]$ instead of $\mathrm{Q}_\tau[\cdot]$. We show below that $\mathbb{M}^\tau$ has a fixed point, which is the value function of the dynamic programming problem and that the supremum is achieved, i.e., the the policy correspondence $\Upsilon : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$ defined by

$$\Upsilon(x, z) \equiv \big\{y \in \Gamma(x, z) : y \text{ achieves the supremum in (16)}\big\} \tag{17}$$

has nonempty values. For this, we need the following.

Assumption 2 (Continuity). *The discount rate* $\beta \in (0, 1)$ *and the following hold:*

(i) $\mathcal{X}$ *and* $\mathcal{Y}$ *are metric spaces;*

(ii) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is continuous and bounded;*

(iii) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ *is continuous;*

(iv) The correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{Y}$ *is continuous, with nonempty, compact values.*[15]

Note that in Assumption 2(i), the state space $\mathcal{X}$ is not required to be Euclidean nor convex, as in de Castro and Galvao (2019). This allows $\mathcal{X}$ to be infinite-dimensional or finite. The same is true for the action space $\mathcal{Y}$. In fact, since now we do not assume that the choice is the next period state as de Castro and Galvao (2019) do, it is possible that $\mathcal{Y} \neq \mathcal{X}$ and the richness on $\mathcal{Y}$ does matter. Property (ii) is the standard continuity assumption of the utility function, which is extended to the transition function ϕ in (iii) and to the feasibility correspondence in (iv). Conditions (ii), (iii), and (iv) guarantee that an optimal choice always exist.

Together, Assumptions 1 and 2 generalize the existing setting in the literature. The following result establishes that under those assumptions, $\mathbb{M}$ is a β-contraction in $\mathcal{C}$, i.e., $\mathbb{M}(v) \in \mathcal{C}$ for any $v \in \mathcal{C}$, and $\|\mathbb{M}(v) - \mathbb{M}(v')\| \leq \beta\|v - v'\|$ for any $v, v' \in \mathcal{C}$.

Theorem 3.11. *Under Assumptions 1 and 2,* $\mathbb{M}$ *is a* β*-contraction in* $\mathcal{C}$*. Thus, it has a unique fixed-point* $\bar{v} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R} \in \mathcal{C}$*. Moreover, the policy correspondence* $\Upsilon : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$ *is upper semicontinuous with nonempty and compact values.*

[15]Since at this point convexity is not required, we may have Γ finite-valued, representing the case where only finitely many options are available to the DM at each period.

The unique fixed point of the problem will be the value function of the problem. The proof of this result is not a routine application of similar theorems from the expected utility case, since continuity of quantiles is not immediate. We explore the Markov transition properties required in Assumption 1 to establish that the quantile functional is continuous; see Proposition A.4 in the Appendix for details. As we have shown with counterexamples after Assumption 1, this result may fail if the requirements in that assumption are not met.

Remark 3.12. Theorem 3.11 and all the other results of this paper still hold if we relax Assumption 1 by allowing $\mathcal{Z}$ to be disconnected, provided that we work with a subset of continuous functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ such that $w \mapsto v(x, w)$ has a connected image, for all $x \in \mathcal{X}$. Notice that the image of this function is connected if v is continuous and $\mathcal{Z}$ is connected.[16] Section 4.2 illustrates this approach. See also comments before Lemma A.9 in the Appendix.

Below we derive some sharper properties of the value function, namely monotonicity, concavity and differentiability, as well as single-valuedness of the policy correspondence.

3.6 *Monotonicity*

In this section, we establish monotonicity of the value function with respect to the x and z variables. This section imposes only that the metric spaces $\mathcal{X}$ and $\mathcal{Y}$ are Euclidean, so monotonicity has a natural meaning. We start with an assumption necessary to prove strict increasingness of the value function with respect to the state variable x.[17]

Assumption 3 (Monotonicity in x). *The following holds: (i) $\mathcal{X} \subset \mathbb{R}^p$; (ii) $\mathcal{Y} \subset \mathbb{R}^m$; (iii) u and ϕ are nondecreasing in x; and (iv) for every $z \in \mathcal{Z}$ and $x \le x'$, $\Gamma(x, z) \subseteq \Gamma(x', z)$.*

From the next result and all that follow, by $\bar{v}$ we mean the unique fixed point of $\mathbb{M}$ in $\mathcal{C}$, guaranteed to exist by Theorem 3.11.

Theorem 3.13. *Under Assumptions 1, 2, and 3, $\bar{v}$ is nondecreasing in x. If u is also strictly increasing in x, so is $\bar{v}$.*

It is also possible to establish increasingness of the value function also with respect to the shocks z. For this, we need to require monotonicity of u, ϕ, and Γ with respect to z. This is content of the following.

[16]Example 3.10 obtains discontinuity for the quantile by using the function $v(x, w) = w$, which has the disconnected image $\mathcal{Z} = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$.

[17]As it is well known, there are two related notions of strict increasingness for a function $h : \mathbb{R}^p \to \mathbb{R}$: (i) $x^1 \ge x^0$, but $x^1 \ne x^0$ implies $h(x^1) > h(x^0)$; and the weaker notion that (ii) $x^1_i > x^0_i$ for $i = 1, \dots, p$ implies $h(x^1) > h(x^0)$. Theorem 3.13 holds with any definition, provided that they are applied consistently in the assumption and in the result.

Assumption 4 (Monotonicity in z). *Both u and ϕ are nondecreasing in z and, for every $x \in \mathcal{X}$, and $z \leq z'$, $\Gamma(x, z) \subseteq \Gamma(x, z')$.*

For establishing monotonicity with respect to shocks, we also need the following assumption. It assumes that $\mathcal{Z}$ is a subset of an Euclidean space $\mathbb{R}^k$, for which that $z = (z_1, \dots, z_k) \leq z' = (z'_1, \dots, z'_k)$ means $z_i \leq z'_i$ for all $i = 1, \dots, k$.

Assumption 5. *$\mathcal{Z} \subset \mathbb{R}^k$ and for any weakly increasing function $h : \mathcal{Z} \to \mathbb{R}$ and $z, z' \in \mathcal{Z}$ such that $z \leq z'$, $\mathrm{E}[h(w)|z] \leq \mathrm{E}[h(w)|z']$.*

Assumption 5 is just a requirement that the conditional distribution $K(z', \cdot)$ first-order stochastically dominates $K(z, \cdot)$ whenever $z' \geq z$. It implies, in particular, an analogous inequality for quantiles, i.e., under the above conditions we also have $\mathrm{Q}_\tau[h(w)|z] \leq \mathrm{Q}_\tau[h(w)|z']$; see Lemma A.12 in the Appendix. We have the following result.

Theorem 3.14. *Under Assumptions 1, 2, 3, 4, and 5, $\bar{v}$ is nondecreasing in x and z. If u is also strictly increasing in z, so is $\bar{v}$.*

It should be noted that Theorem 3.14 holds not only for very general Euclidean $\mathcal{X}$ and $\mathcal{Y}$, which may be even discrete, but also for any $\mathcal{Z} \subset \mathbb{R}^k$ satisfying Assumptions 1 and 5, which allow multidimensional shocks. In the next section, where we establish concavity, more restrictions will be imposed over the sets $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$.

3.7 *Concavity*

In this section, we establish concavity of the value function. Moreover, we show that the policy correspondence is convex-valued. For establishing this, we naturally need to require the spaces to be convex and the functions to be concave. Moreover, the feasibility constraint set also needs to satisfy a convexity requirement. This is the content of the following.

Assumption 6 (Convexity and concavity). *The following holds: (i) $\mathcal{X}$ and $\mathcal{Y}$ are convex; (ii) u and ϕ are concave in (x, y); and (iii) for all $z \in \mathcal{Z}$ and all $x, x' \in \mathcal{X}$, $y \in \Gamma(x, z)$, and $y' \in \Gamma(x', z)$ imply*

$$\theta y + (1-\theta) y' \in \Gamma\big[\theta x + (1-\theta) x', z\big] \quad \textit{for all } \theta \in [0, 1].$$

Notice that Assumption 6(iii) implies that $\Gamma(x, z)$ is a convex set for each $(x, z) \in \mathcal{X} \times \mathcal{Z}$. In addition to the standard convexity and concavity requirements of Assumption 6, to deal with the continuous shock scenario, we need to work with unidimensional shocks, as required by the following.

Assumption 7. $\mathcal{Z} \subseteq \mathbb{R}$.

We need to restrict the dimension of the Markov process to $k=1$ for using comonotonicity arguments, that guarantee that the quantile of sums of random variables is the sum of quantiles. This property fails in general. The next result establishes concavity of the value function.

THEOREM 3.15. *Let Assumptions 1–7 hold. Then $\bar{v}$ is concave in x and $\Upsilon : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$ is convex-valued. If, additionally, u is strictly concave in (x, y), then $\bar{v}$ is strictly concave in x and Υ has convex values. Whenever $\bar{v}$ is strictly concave, Υ is single-valued and continuous.*

3.8 *Differentiability*

This section presents results for differentiability of the value function with respect to the state variable x. In this case, two different approaches are needed depending on whether the choice space, $\mathcal{Y}$, is convex or discrete. Nevertheless, both cases rely on the following common basic assumption.

ASSUMPTION 8. *The function u is C^1 in x and ϕ does not depend on x.*

The second part of Assumption 8 imposes that the next period state can depend on the choice y and the observed shock z, but not on the current state x. The set of actions $\Gamma(x, z)$ available to the DM may depend on x. The requirement is that in no other way the current state x can affect the next period state after an action y is picked and a shock z is realized. It is important to note that Assumption 8 is also required in the expected utility context; see Stokey, Lucas, and Prescott (1989, p. 270, item f).[18] In any case, this condition is satisfied in many practical applications.

Now, we present a result on the differentiability of $\bar{v}$ for convex $\mathcal{Y}$,[19] which follows the classical Benveniste and Scheinkman (1979)'s argument.

THEOREM 3.16. *Let Assumptions 1–8 hold and assume that $x \in \mathcal{X} \subset \mathbb{R}^p$ is interior. Then $\bar{v}$ is differentiable in x and for $i = 1, \dots, p$,*

$$\frac{\partial \bar{v}}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}\big(x, y^*, z\big), \tag{18}$$

where $y^* \in \Upsilon(x, z)$ is a maximizer of (16) for $\bar{v}$.

Although the standard result presented above requires convexity of $\mathcal{Y}$, we are able develop different arguments (not based on concavity) to establish differentiability of the value function even if $\mathcal{Y}$ is finite, and hence, not convex. This is the content of the following.

[18] Blume, Easley, and O'Hara (1982) assume that the shock z_t is an argument of the law of motion ϕ, but z_t is not in Γ or the instantaneous utility function. Nevertheless, they apply different techniques to show that optimal plans can be obtained by an application of the implicit function theorem to first-order conditions.

[19] Notice that convexity of $\mathcal{Y}$ is required by Assumption 6.

THEOREM 3.17. *Assume that the choice set $\mathcal{Y}$ is finite. Let Assumptions 1, 2, and 8 hold. Fix $x \in \mathcal{X} \subset \mathbb{R}^p$, $z \in \mathcal{Z}$. Assume that $x \in \mathcal{X}$ is an interior point where the optimal correspondence $\Upsilon(x, z) \subset \Gamma(x, z)$ is lower hemicontinuous.*[20] *Then $\bar{v}$ is differentiable in x, and*

$$\frac{\partial \bar{v}}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*, z),$$

where $y^* \in \Upsilon(x, z)$ is a maximizer of (16) for $\bar{v}$.

Notice that Theorem 3.17 requires less assumptions than Theorem 3.16, but it restricts to finite choice sets and requires lower hemicontinuity of the optimal correspondence.

3.9 *Euler equation*

The final step is to characterize the solutions of the quantile recursive problem through the Euler equation. As before, let $\bar{v}$ be the unique fixed point of $\mathbb{M}$ in $\mathcal{C}$, guaranteed to exist by Theorem 3.11. By Theorem 3.16, if ϕ does not depend on x, $\bar{v}$ is differentiable in its first coordinate, satisfying $\frac{\partial \bar{v}}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*, z)$. Given that we have shown the differentiability of value function, we are able to apply the standard technique to obtain the Euler equation, as formalized in the following theorem.

Below, we will assume that Assumption 8 holds, so that ϕ does not depend on x. Using again $\bar{v}$ as the fixed point of $\mathbb{M}$, we can define $\tilde{v} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ by $\tilde{v}(y, z) \equiv \bar{v}(\phi(y, z), z)$.

THEOREM 3.18 (Euler equation). *Let Assumptions 1–8 hold. Let $(x_t, y_t, z_t)_{t\in\mathbb{N}}$ be a sequence of states, optimal decisions, and shocks, such that (x_t, y_t) are interior for all t. If $z_t \mapsto \frac{\partial u}{\partial x}(x_t, y_t, z_t) \cdot \frac{\partial \phi}{\partial y_i}(y_{t-1}, z_t)$ is strictly increasing, then $\forall n \in \mathbb{N}$ and $i = 1, \ldots, m$:*

$$\frac{\partial u}{\partial y_i}(x_t, y_t, z_t) + \beta \mathrm{Q}_\tau\left[\frac{\partial u}{\partial x}(x_{t+1}, y_{t+1}, z_{t+1}) \cdot \frac{\partial \phi}{\partial y_i}(y_t, z_{t+1}) \middle| z_t\right] = 0. \tag{19}$$

In (19), $\frac{\partial u}{\partial y_i}$ represents the derivative of u with respect to the ith coordinate of its second variable (y) (i.e., an unidimensional value) and $\frac{\partial u}{\partial x}$ represents the derivative of u with respect to its first variable (x) (i.e., a p-dimensional vector). Since ϕ takes value on $\mathcal{X} \subset \mathbb{R}^p$, $\frac{\partial \phi}{\partial y_i}$ stands for the p-dimensional derivative vector of ϕ with respect to the ith coordinate of y. We could also rewrite (19) as follows:

$$\frac{\partial u}{\partial y_i}(x_t, y_t, z_t) + \beta \mathrm{Q}_\tau\left[\sum_{j=1}^{p} \frac{\partial u}{\partial x_j}(x_{t+1}, y_{t+1}, z_{t+1}) \frac{\partial \phi_j}{\partial y_i}(y_t, z_{t+1}) \middle| z_t\right] = 0, \tag{20}$$

where ϕ_j stands for the jth component of ϕ.

[20]Recall that this means that for every sequence $x_n \to x$, and every $y^* \in \Upsilon(x, z)$, there exists some sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \in \Upsilon(x_n, z)$ for every $n \in \mathbb{N}$ and $y_n \to y^*$.

Theorem 3.18 provides the Euler equation, that is the optimality conditions for the quantile dynamic programming problem. This result is the generalization of the traditional expected utility to the quantile preferences. The Euler equation in (19) is displayed as an implicit function, nevertheless for any particular application, and given utility function, one is able to solve it explicitly as a conditional quantile function.

When $\phi(y, z) = y$ and we identify $\mathcal{X} \equiv \mathcal{Y}$, as in the model where the shock occurs before the DM chooses his action, so in practice it is the same as considering his choice being directly the next period state, (19) simplifies to

$$\frac{\partial u}{\partial y_i}(x_t, y_t, z_t) + \beta \mathrm{Q}_\tau\left[\frac{\partial u}{\partial x_i}(x_{t+1}, y_{t+1}, z_{t+1})\bigg| z_t\right] = 0.$$

The proof of Theorem 3.18 relies on a result about the differentiability inside the quantile function. Indeed, if h is differentiable and the derivative $\frac{\partial h}{\partial y_i}(y, Z)$ is integrable, then

$$\frac{\partial}{\partial y_i}\mathrm{E}\big[h(y, Z)\big] = \mathrm{E}\left[\frac{\partial h}{\partial y_i}(y, Z)\right], \quad \text{but } \frac{\partial}{\partial y_i}\mathrm{Q}_\tau\big[h(y, Z)\big] \neq \mathrm{Q}_\tau\left[\frac{\partial h}{\partial y_i}(y, Z)\right],$$

in general. However, de Castro and Galvao (2019) establish conditions under which the commutability of the two operations holds. See their paper for details.

4. Applications

In this section, we discuss two well-known economic models that can be adapted to quantile preferences. The analysis of these canonical models are useful to illustrate the recursive quantile model, as well as the new theoretical results in this paper.

4.1 *Intertemporal consumption*

In a seminal work, Modigliani and Brumberg (1954) investigated intertemporal consumption and life cycle. This framework has been used as a standard economic approach to the study of consumption behavior and served as basis for a very large literature and subsequent models of intertemporal consumption (see, e.g., Deaton (1992)).

This first example uses a consumption-based model to illustrate the dynamic quantile preferences methods. We establish results as an explicit formula for the value function, the optimal consumption and asset hold, as well as their corresponding paths. We also compare the results with the case without uncertainty, and make a parallel with the permanent income hypothesis.

Consider the following economy. At the beginning of period t, the DM has $x_t \in \mathcal{X} \subset \mathbb{R}_+$ units of the risky asset, with return $z_t \in \mathcal{Z} \subseteq \mathbb{R}_{++}$. With wealth $x_t z_t$ at the beginning of period t, the DM decides $y_t = (c_t, x_{t+1})$, which includes the amount consumed in period t, c_t, and next period's state, x_{t+1}. Therefore, the next period units of the risky asset x_{t+1} is given by the law of motion $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ introduced in equation (6), as follows:

$$x_{t+1} = \phi(x_t, y_t, z_{t+1}) = \phi\big(x_t, (c_t, x_{t+1}), z_{t+1}\big).$$

The transformation that defines the recursive equation is the following:

$$\mathbb{M}(\bar{v})(x_t, z_t) \equiv \max_{(c_t, x_{t+1}) \in \Gamma(x_t, z_t)} \{U(c_t) + \beta \mathrm{Q}_\tau[\bar{v}(x_{t+1}, z_{t+1})|z_t]\}, \tag{21}$$

where $\beta \in (0, 1)$ is the discount factor, $\tau \in (0, 1)$ is the risk attitude, $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} = \mathcal{X} \times \mathcal{X}$ is the feasibility correspondence and $U : \mathbb{R}_+ \to \mathbb{R}$ defines the utility function, that is related to the function $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ of Section 3 by the following:

$$u(x_t, y_t, z_t) = u(x_t, (c_t, x_{t+1}), z_t) = U(c_t). \tag{22}$$

We impose the following assumption.

Assumption 9. *The following holds: (i) $\mathcal{X} = [0, \overline{x}]$ for some $\overline{x} > 0$; (ii) $\mathcal{Z} = [\underline{z}, \overline{z}]$, with $\overline{z} > \underline{z} > 0$; (iii) $U : \mathcal{X} \to \mathbb{R}$ is C^2, $U' > 0$, $U'' < 0$; (iv) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ is defined by $\phi(x, y, z) = \phi(x, (c, x'), z) = x'$; and (v) $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} = \mathcal{X} \times \mathcal{X}$ is given by $\Gamma(x, z) \equiv \{(c, x') \in \mathcal{X} \times \mathcal{X} : c + x' \leq xz\}$.*

Assumption 9 encompass many useful specifications for applications. Although it restricts the domain to be $[0, \overline{x}]$ instead of the usual $\mathbb{R}_+$ or $\mathbb{R}_{++}$, this limitation does not create significant issues; see discussion after Corollary 4.2 below. It should be noted that Assumption 9 allows all commonly used utility functions, such as the isoelastic and exponential that are explicitly discussed below.

Now, we use the results from Section 3 above to show that the transformation defined by (21) possesses a fixed point, which is a value function satisfying the recursive equation and previous properties.

Theorem 4.1. *Let Assumptions 1 and 9 hold. There exists a unique continuous and bounded function $\bar{v} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ satisfying the recursive equation (21). This function is increasing in x.*

If Assumption 5 also holds and if the optimal point in (21) is interior to $\mathcal{Y}$, then $\bar{v}$ is differentiable in x, strictly increasing in x and z, strictly concave in x and satisfies, for an optimal path $\{(c_t, x_t)\}_{t=1}^{\infty}$ that is interior, with $c_t = x_t z_t - x_{t+1}$,

$$\frac{\partial \bar{v}}{\partial x}(x_t, z_t) = U'(x_t z_t - x_{t+1}) z_t = (x_t z_t - x_{t+1})^{-\gamma} z_t. \tag{23}$$

Moreover, for this optimal interior path, the following Euler equation holds:[21]

$$-U'(c_t) + \mathrm{Q}_\tau[\beta U'(c_{t+1}) z_{t+1}|z_t] = 0. \tag{24}$$

Theorem 4.1 follows from results in Section 3, although some of the previous assumptions are not strictly satisfied. See the proof in the Appendix for details.

The Euler equation for the intertemporal consumption model (24) has a very simple intertemporal substitution interpretation. The marginal rate of substitution between

[21] To obtain this Euler equation, we change the setup above. See details in the proof of this theorem in the Appendix.

consumption in two periods must be equal to the marginal rate of transformation. Suppose the DM decreases the consumption by dc_t at time t, invests dc_t in the asset, and consumes the proceeds at time $t+1$. The decrease in utility at time t is $U'(c_t)$. The increase in utility at time $t+1$ is uncertain because of the shock, but viewed at t, it is evaluated as the τ-quantile $\mathrm{Q}_\tau[\beta U'(c_{t+1})z_{t+1}|z_t]$. The future uncertainty is solved using the τ-quantile.

Next, we specialize the utility function to the isoelastic and exponential utility cases. We will see that in those specific cases, we can obtain closed-form solutions for the value function.

4.1.1 *Isoelastic utility function* In this section, we specify $U:\mathcal{X}\to\mathbb{R}$ to be the isoelastic utility function, for $\gamma\in(0,1)$,[22]

$$U(c)=\frac{c^{1-\gamma}}{1-\gamma},\quad \gamma\in(0,1). \tag{25}$$

In the Appendix, we discuss the cases of $\gamma\geq 1$. Now, we specialize the conclusions of Theorem 4.1.

Corollary 4.2. *Let Assumptions 1, 5, and 9 hold, with U given by (25). Then there exists a unique continuous and bounded function $\bar{v}:\mathcal{X}\times\mathcal{Z}\to\mathbb{R}$ satisfying the recursive equation (21), i.e., $\bar{v}=\mathbb{M}(\bar{v})$. Moreover, if the optimal choice is interior, $\bar{v}$ is differentiable in x, strictly increasing in z, strictly concave in x, satisfies (23) and the following Euler equation holds:*

$$\mathrm{Q}_\tau\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}z_{t+1}\middle|z_t\right]=1. \tag{26}$$

The Euler equation (26) provides an equilibrium condition for consumption. Empirically, together with instrumental variables quantile regression methods—as in, for instance, de Castro, Galvao, Kaplan, and Liu (2019)—it could be used to estimate the parameters characterizing the preferences for intertemporal substitution in the model.

Although Corollary 4.2 restricts the domain of the utility function in (25) to $[0,\bar{x}]$, we develop below results for the usual unbounded domain $\mathbb{R}_+$. Indeed, Theorem 4.3 offers a closed-form expression for the value function $\bar{v}$ in $\mathbb{R}_+$. This theorem does not state uniqueness for $\bar{v}$ because involved functions are not bounded and the contraction argument does not apply. However, Theorem 4.4 establishes that this value function $\bar{v}$ is indeed the unique fixed point among all functions satisfying the functional equation (21) that also satisfy the conditional quantile transversality condition introduced in Section 3.3.

We focus here in the case $\gamma\in(0,1)$, which is simpler to state. The results for $\gamma=1$ and $\gamma>1$ are developed separately in the Appendix; see, respectively, Theorems A.24 and A.30.

[22]The case $\gamma\geq 1$ is studied in the Appendix.

Assumption 10. *The following holds: (i) U is given by (25), for $\gamma \in (0,1)$; (ii) $\mathcal{X} = \mathbb{R}_+$; (iii) $\mathcal{Z} \subset \mathbb{R}_+$ is a closed interval; (iv) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ is defined by $\phi(x, y, z) = \phi(x, (c, x'), z) = x'$; (v) $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} = \mathcal{X} \times \mathcal{X}$ is given by $\Gamma(x, z) \equiv \{(c, x') \in \mathcal{X} \times \mathcal{X} : c + x' \leq xz\}$; (vi) there exists $\tilde{z} > 0$ such that $0 < \mathrm{Q}_\tau[w|z] \leq \tilde{z}$, for for all $z \in \mathcal{Z}$;*[23] *and (vii) $\beta \tilde{z}^{1-\gamma} < 1$.*

The following functions are useful in the statement below. Let $r_{\tau,s}(z)$ be defined recursively by $r_{\tau,0}(z) = 1$, and

$$r_{\tau,s}(z) = r_{\tau,s-1}\big(\mathrm{Q}_\tau[w|z]\big) \cdot \mathrm{Q}_\tau[w|z] \quad \text{for } s \geq 1. \tag{27}$$

Given this, define the functions:

$$R(z) \equiv \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} \big[r_{\tau,s}(z)\big]^{\frac{1-\gamma}{\gamma}} \quad \text{and} \quad S(z) \equiv \frac{R(z)}{1 + R(z)}. \tag{28}$$

Assumption 10 guarantees that R is well-defined.[24] Observe that both functions $R(z)$ and $S(z)$ depend on all three parameters β, τ, and γ. We have the following.

Theorem 4.3. *Let Assumptions 1, 5, and 10 hold. Let $\bar{v} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ be given by*

$$\bar{v}(x, z) = \frac{1}{1-\gamma} \cdot (xz)^{1-\gamma} \cdot \big[1 + R(z)\big]^{\gamma}. \tag{29}$$

Then $\bar{v}$ is a fixed point of the transformation $\mathbb{M}$ defined in (21). Moreover, the optimal policy function $y^* : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} = \mathcal{X} \times \mathcal{X}$ is given by

$$y^*(x, z) = (c, x') = \big(\big[1 - S(z)\big] \cdot xz, S(z) \cdot xz\big), \tag{30}$$

and for an optimal consumption path $\{c_t\}_{t=1}^{\infty}$ associated with shocks $\{z_t\}_{t=1}^{\infty}$,

$$\frac{c_{t+1}}{c_t} = z_{t+1} \cdot R(z_t) \cdot \big[1 - S(z_{t+1})\big]. \tag{31}$$

Notice that the value function in (29) is characterized by three parameters: the discount factor (β), the risk attitude (τ), and the parameter in the utility function (γ). The discount factor characterizes consumer's impatience. It is used to discount future payments of intertemporal utility functions, and allows to obtain the present value of future consumption. The risk attitude parameter—given by the quantile τ, as discussed in Section 2—describes consumer's reluctance to substitute consumption across states of the world under uncertainty and is meaningful even in an atemporal setting. The elasticity of intertemporal substitution (EIS), i.e., the elasticity of consumption growth with

[23] Notice that $\mathcal{Z}$ does not need to be bounded and may include zero. This condition only requires that the *quantile* $\mathrm{Q}_\tau[w|z]$ is strictly positive and bounded by $\tilde{z}$ for all $z \in \mathcal{Z}$.

[24] By Assumption 10(vi), $r_{\tau,s}(z) \leq \tilde{z} r_{\tau,s-1}(z)$. Therefore, $r_{\tau,s}(z) \leq \tilde{z}^s$. This implies that $\beta^{\frac{s}{\gamma}} [r_{\tau,s}(z)]^{\frac{1-\gamma}{\gamma}} \leq (\beta^{\frac{1}{\gamma}} \tilde{z}^{\frac{1-\gamma}{\gamma}})^s$. By Assumption 10(vii), $\beta^{\frac{1}{\gamma}} \tilde{z}^{\frac{1-\gamma}{\gamma}} < 1$. This implies that the infinite sum defining $R(z)$ converges.

respect to marginal utility growth, is just $1/\gamma$ in this dynamic quantile model.[25] As discussed in Section 2, an important feature of the recursive quantile model is that it allows for the complete separation of the risk and EIS parameters, while maintaining important properties as dynamic consistency and monotonicity.

The main contribution of Theorem 4.3 is to provide explicit solutions for the value function and the optimal savings (investment) and consumption for each given state and random shock, in equations (29) and (30). In addition, equation (31) derives a recursive equation for the optimal path for the consumption, that allows to obtain explicit expressions for the growth of consumption as function of random shocks. These expressions may be very useful in theoretical and empirical analysis.

We observe that, in contrast with the results shown in Theorem 4.3 for the quantile model, it is difficult to obtain closed-form expressions in the standard recursive EU case for general Markov shock processes.[26] For this reason, it has been standard in the literature (see, e.g., Adda and Cooper (2003)) to use numerical methods to solve dynamic programming problems under the EU model.

Notice that in Theorem 4.3 we refrain from stating uniqueness of $\bar{v}$ because the transformation may fail to be a contraction.[27] The following theorem establishes the desired uniqueness result using another argument, namely the principle of optimality (Theorem 3.8) that uses the conditional quantile transversality condition (CQTC); see Definition 3.7.

Theorem 4.4 (Uniqueness of the value function). *Let Assumptions 1, 5, and 10 hold. Let $\bar{v}: \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}$ be the function defined by (29). Suppose that $v: \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ is a fixed point of $\mathbb{M}$ defined by (21) that satisfies the CQTC. Then $v = \bar{v}$.*

In fact, in Appendix A.8.1, we state a more general result than Theorem 4.4, by introducing another transversality condition that may be easier to verify. There, we also show how a mild condition would imply CQTC.[28] In any case, Theorem 4.4 shows that function $\bar{v}$ defined by (29) is essentially the only function that satisfies the functional equation for this isoelastic model, even with unbounded domain. Notice we have reached this conclusion without using the standard contractions arguments that are usually restricted to bounded functions.

In the rest of this section, we explore particularizations of this model, to obtain simpler expressions for the value function $\bar{v}$. We will consider conditions that are stronger

[25]Under time separable utility, the EIS is also the percent change in consumption growth per percent increase in the net interest rate.

[26]In the EU case, the solution is also separable in the form $v(x, z) = \frac{x^{1-\gamma}}{1-\gamma} L(z)$, where $L(z)$ is the fixed point of the operator $T(L(z)) = z^{1-\gamma}\{1 + \beta^{\frac{1}{\gamma}}(E[L(w)|z])^{\frac{1}{\gamma}}\}^{\gamma}$. However, the fact that $E[\cdot]$ does not commute with increasing functions makes it hard to find a simple closed form for L. Thus, a numerical approach seems unavoidable for the EU version of this model. In contrast, for quantiles we have $(\mathrm{Q}_\tau[L(w)|z])^{\frac{1}{\gamma}} = \mathrm{Q}_\tau[(L(w))^{\frac{1}{\gamma}}|z]$ and the exponent $\frac{1}{\gamma}$ will cancel with the exponent γ coming from the iteration.

[27]Remember, in particular, that the usual argument for establishing that $\mathbb{M}$ is a contraction requires that this transformation is restricted to bounded functions.

[28]See Remark A.23.

than the general Markov assumption. For instance, if we assume that the shocks are independent and identically distributed (i.i.d.), we can specialize the above results as follows.

Example 4.5 (The i.i.d. case). If the shocks are independent, then $Q_\tau[w|z]$ becomes a constant, $Q_\tau[w]$, such that (27) reduces to $r_{\tau,s}(z) = r_{\tau,s} = (Q_\tau[w])^s$. Similarly, let $a_{\beta,\tau,\gamma} = \beta^{\frac{1}{\gamma}}(Q_\tau[w])^{\frac{1-\gamma}{\gamma}}$. Then

$$R(z) = \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} \big[r_{\tau,s}(z)\big]^{\frac{1-\gamma}{\gamma}} = \sum_{s=1}^{\infty} a^s_{\beta,\tau,\gamma} = \frac{a_{\beta,\tau,\gamma}}{1 - a_{\beta,\tau,\gamma}},$$

since $0 < a_{\beta,\tau,\gamma} < 1$ by Assumption 10(vi, vii). Therefore, $S(z) = a_{\beta,\tau,\gamma}$. With this, the above results simplify to $x_{t+1} = a_{\beta,\tau,\gamma} x_t z_t$, $c_t = (1 - a_{\beta,\tau,\gamma}) x_t z_t$, $c_{t+1} = a_{\beta,\tau,\gamma} c_t z_{t+1}$, and $\bar{v}(x_t, z_t) = \frac{(1-a_{\beta,\tau,\gamma})^{-\gamma}}{1-\gamma}(x_t z_t)^{1-\gamma}$. ◊

If we interpret the shocks z_t as coming from productivity determined by human capital, the last expressions in Example 4.5 above capture the essence of the standard permanent income hypothesis (PIH) discussed in Hall (1988): current consumption is determined by a combination of current nonhuman wealth x_t and human capital wealth z_t. The fraction of total wealth consumed today further depends on $a_{\beta,\tau,\gamma}$, which is a function of all parameters of the model. Also, notice that the uncertainty is resolved using the quantile operator, which is inside $a_{\beta,\tau,\gamma}$. An increase in the risk attitude has the same effect as increase in the discount factor making the current consumption decrease. When $\gamma < 1$, an increase in γ (decrease in EIS) also decreases current consumption.

Another case of interest is when the shocks are τ-quantile martingales (see Definition 3.1).

Example 4.6 (τ-quantile martingales). Assume that z follows a τ-quantile martingale process; see Definition 3.1 and equation (7). Then $Q_\tau[w|z] = z$ for all z, and

$$r_{\tau,s}(z) = z^s \quad \text{for all } s \geq 1.$$

Therefore, Theorem 4.3 implies that the value function is explicitly given by

$$\bar{v}(x, z) = \frac{1}{1-\gamma}(xz)^{1-\gamma}\left[\sum_{s=0}^{\infty}\big(\beta^{\frac{1}{\gamma}} z^{\frac{1-\gamma}{\gamma}}\big)^s\right]^{\gamma} = \frac{1}{1-\gamma}(xz)^{1-\gamma}\big(1 - \beta^{\frac{1}{\gamma}} z^{\frac{1-\gamma}{\gamma}}\big)^{-\gamma}, \tag{32}$$

with optimal consumption $c^*(x, z) = (1 - \beta^{\frac{1}{\gamma}} z^{\frac{1-\gamma}{\gamma}})xz$ and optimal savings $(\beta z)^{\frac{1}{\gamma}} x$. Notice that the general formulas for the value function and the optimal assets and consumption depend on all parameters of the model. These expressions are explicitly dependent on β and γ, but they are functions of τ implicitly, because we assumed that the process is a τ-quantile martingale process, which means that for a given risk attitude τ, the uncertainty is solved as $Q_\tau[w|z] = z$. ◊

These expressions give us the opportunity to compare them with with those for the model without uncertainty, where it is also possible to obtain closed-form solutions. This is the subject of the next section.

4.1.2 *Comparison with the riskless case* We will see now how the closed-form expressions obtained in Theorem 4.3 for the quantile model generalize similar expressions for a model without risk. To see this, it is sufficient to consider the case in equation (21), where the set of shocks reduces to a singleton, i.e., $\mathcal{Z} = \{R\}$ so that the budget constraint $c_t = x_t z_t - x_{t+1}$ becomes $c_t = x_t R - x_{t+1}$. Notice that the familiar "cake eating problem" is a special case, in which $R = 1$. If we consider $R < 1$, in general this problem is called "ice cream eating problem" since a fraction $1 - R > 0$ of the "ice cream" "melts" each period and is no longer available for consumption.

Consider a recursive model without uncertainty as

$$\bar{v}(x_t) = \max_{x_{t+1} \in [0, x_t R]} U(x_t R - x_{t+1}) + \beta \bar{v}(x_{t+1}). \tag{33}$$

The first-order condition (Euler equation) for this problem leads to $U'(c_t) = \beta R U'(c_{t+1})$. Assume that U is the isoelastic utility function in (25), and let $a_{\beta,\gamma} = \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}}$. If we denote the optimal savings (next period assets) by x^* and optimal consumption by c^*, we have the following closed-form expressions:

$$\bar{v}(x) = \frac{(1 - a_{\beta,\gamma})^{-\gamma}}{1 - \gamma} (xR)^{1-\gamma}, \tag{34}$$

$$x^* = a_{\beta,\gamma} \cdot xR, \tag{35}$$

$$c^* = (1 - a_{\beta,\gamma}) \cdot xR, \tag{36}$$

$$c_{t+1} = a_{\beta,\gamma} c_t R. \tag{37}$$

Equations (34)–(37) are parallel to those in Theorem 4.3 and Examples 4.5 and 4.6. We observe that the equations for both cases, with and without uncertainty, have similar functional forms. However, in the case with uncertainty, the quantile operator appears in the expressions to account for the uncertainty. This is interesting since it shows that the quantile model is able to capture important features of the model without risk, but at the same time allows for studying risk and risk attitudes. Notice also that the optimal savings (x^*) and consumption (c^*) are expressed as shares of the available resources, namely xz in the uncertain case and xR in the risk-free case. Nevertheless, in the presence of uncertainty, these values are influenced by the risk attitude parameter τ and the quantile operator. Thus, a small increment in x impacts y^* and c^* similarly in both models, but differences between the models depend on the quantile τ. Another interesting point regards the consumption path, where consumption at time $t + 1$ is a share of the previous consumption in both models. Notice that $a_{\beta,\gamma} = \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}}$ above corresponds to $a_{\beta,\tau,\gamma} = \beta^{\frac{1}{\gamma}} (\mathrm{Q}_\tau[w])^{\frac{1-\gamma}{\gamma}}$ of Example 4.5, where τ plays an explicit role. However, the two models will not be equivalent even if $\mathrm{Q}_\tau[w] = R$, which implies that $a_{\beta,\gamma} = a_{\beta,\tau,\gamma}$. Indeed, in the model with uncertainty of Example 4.5, we have $c_{t+1} = a_{\beta,\tau,\gamma} c_t z_{t+1}$, while $c_{t+1} = a_{\beta,\gamma} c_t R$ in the model without risk. Notice that z_{t+1} is a random element and may take values different from $\mathrm{Q}_\tau[w]$ or R.

It is also interesting to see that equation (37) shows that the parameter characterizing the utility function is the EIS. Recall that the EIS is defined as $EIS = d \ln(c_{t+1}/c_t)/d \ln(R)$.

Taking the logarithm of (37), we obtain

$$\ln\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\gamma}\ln(\beta) + \frac{1}{\gamma}\ln(R).$$

Taking the derivative with respect to $\ln(R)$, we obtain $EIS = 1/\gamma$.

4.1.3 *Comparative statics* As previously observed, expressions (29), (30), and (31) show that the value function and the optimal saving and consumption decisions are functions of the three parameters characterizing the model, the discount factor β, the EIS $1/\gamma$, and the risk attitude (quantile) τ. These closed-form solutions allow us to obtain comparative statics results as established by the following.

THEOREM 4.7. *Let the assumptions of Theorem 4.3 hold, with the appropriate modifications to allow $\gamma \geq 1$.*[29] *Then we have the following:*

1. *If the DM becomes more impatient, i.e., the discount factor β decreases, then the DM consumes more (and saves less).*[30]
2. *If the elasticity of intertemporal substitution (EIS$= \frac{1}{\gamma}$) increases and $\beta\tilde{z} < 1$, then the DM consumes more (and saves less).*[31]
3. *If the DM becomes more risk averse, i.e., the risk attitude parameter τ decreases, then the DM consumes more (and saves less) if $\gamma \in (0, 1)$; and if $\gamma > 1$, then the DM consumes less (and saves more). Moreover, if $\gamma = 1$, consumption and savings decisions are not affected by the risk attitude.*
4. *If the distribution of returns increases, i.e., the τ-quantile $Q_\tau[w|z]$ of future returns increases for all $z \in \mathcal{Z}$, then the DM consumes less (and saves more) if $\gamma \in (0, 1)$; and if $\gamma > 1$, the DM consumes more (and saves less). Moreover, if $\gamma = 1$, consumption and savings decisions are not affected by these changes.*

Theorem 4.7 sheds light on the impact of changes of the parameters of the model on intertemporal consumption and savings decisions. First, it confirms the intuitive result that an increase in impatience makes the DM to consume more and save less. Second, it clarifies the impact of changes in EIS. Recall that the EIS measures the sensitivity of consumption growth to changes in the interest rate (the return of investment opportunities). As the EIS increases, the DM becomes more sensitive to changes in investment opportunities, and hence, consumes more and saves less, provided that the gains from investment are not too high. Third, we note the very interesting result concerning changes in the risk attitude. When the DM becomes more risk averse (τ decreases), changes in consumption and savings depend on the EIS $(1/\gamma)$. When the EIS is larger

[29]The precise conditions for the cases $\gamma = 1$ and $\gamma > 1$ are given, respectively, in the statements of Theorems A.24 and A.30 in the Appendix.

[30]We refer to consumption and savings as fractions of available assets xz.

[31]See Assumption 10 for a definition of $\tilde{z}$ for the case $\gamma \in (0, 1)$ and Theorem A.30 in the Appendix, for $\gamma > 1$. Since item 2 deals with changes in γ, it is not meaningful to consider $\gamma = 1$.

than 1, the DM is sensitive to investment opportunities, and an increase in risk aversion leads to larger consumption and smaller investment. Finally, Theorem 4.7 shows that the consumption and savings decisions react differently to changes in the rate of return, depending on the EIS: for high EIS—$\gamma \in (0, 1)$—an increase in interest rates leads to less consumption and more savings. If the EIS is low—$\gamma > 1$—the same change leads to opposite behavior. These implications are empirically testable and may shed light, among other things, on the debate whether the EIS is larger or smaller than 1; see, for instance, Thimme (2017).

4.1.4 *Exponential utility and the permanent income hypothesis* Now, we can consider the exponential utility, $U(c) = -\frac{1}{\gamma}\exp(-\gamma c)$, for $\gamma > 0$.[32] Under the conditions of Theorem 4.1, we can obtain the following Euler equation for an interior optimal path $\{c_t\}_{t=1}^{\infty}$ in this case:

$$Q_\tau[c_{t+1}|z_t] = \frac{1}{\gamma}\ln\big(Q_\tau[z_{t+1}|z_t]\big) + c_t + \frac{1}{\gamma}\ln\beta. \tag{38}$$

The model in equation (38) is very similar to the well-known permanent income hypothesis (PIH) model in Hall (1978, 1988) and Flavin (1981) for the conditional expectations. Indeed, Hall (1988, equation (1), p. 341) writes the following equation resulting from an EU model and lognormal returns:

$$E[c_{t+1}|z_t] = \frac{1}{\gamma}\ln\big(E[z_{t+1}|z_t]\big) + c_t + k, \tag{39}$$

adapting his notation to ours.

Generally the PIH predicts that consumption depends on permanent income, which is the annuity value of lifetime resources. If rational expectations are also assumed, together with a constant rate of return, the PIH implies that consumption follows a random walk, so that only consumption in the previous period contains information which can predict current consumption. Therefore, the DM adjusts current consumption immediately to the point where consumption is not expected to change, smoothing the consumption path.

If one assumes that $z_t = z$ for all t, as it is common the literature (see, e.g., Flavin (1981)), then equation (38) becomes a quantile (regression) version of the unit root model for the conditional average widely analyzed in the literature. Thus, the quantile model predicts a τ-quantile martingale (see Definition 3.1). This conjecture could be empirically tested by using quantile regression unit root tests, as for example in Koenker and Xiao (2004), using data on consumption.

We notice similarities and differences between equations (38) and (39). First, conditional expectations in (39) are substituted by conditional quantiles in (38). Second, notice that these two models produce different empirical implications. For the quantile case, the model implies existence of an unit root for the τ-quantile of the conditional

[32]This function is also usually known as the Constant Absolute Risk Aversion (CARA) function, as the isoelastic function discussed in Sections 4.1.1 and A.8.2 is known as Constant Relative Risk Aversion (CRRA) function. Both terms are not appropriate for quantile preferences since, as we have explained in Section 2, the utility functions do not have any implications for risk attitude, but for intertemporal substitution.

quantile function of c_{t+1}. On the other hand, the EU implies a unit root for the conditional average. While both models predict unit root behavior of the time series of consumption, these predictions are not nested, since the τ-quantile does not need to coincide with the conditional mean.

4.2 *Search with unemployment*

We now present a quantile-based version of the job-search model discussed in McCall (1970); see also Lippman and McCall (1976a, 1976b). In a labor market characterized by uncertainty and costly information, both employers and employees will be searching. The analysis presented here is directed to the employee's job-searching strategy.[33]

The worker begins each period t with a wage offer w_t and has to decide if she accepts the offer and works at that wage ($y_t = 1$) or refuses the offer ($y_t = 0$) and searches for a new one. Hence, the decision variable y_t takes discrete values in $\{0, 1\}$. If she decides to search, she earns nothing during the period t, and a new wage offer $w_{t+1} \in [0, \bar{w}]$ will be her best option for the next period, when she will be making another choice between searching or working. This new wage offer w_{t+1} is modeled as a continuous shock. If the worker chooses to work at period t, there is a chance that she looses her job ($e_{t+1} = 0$) in the next period $t + 1$, or keeps it ($e_{t+1} = 1$), and thus maintains the same wage x_t as in the previous period, where x_t denotes the effectively earned wage at period t. Hence, the random variable determining whether the worker keeps or looses the job (e_t), which can be interpreted as employer's decision, is a discrete shock. The next period state x_{t+1} given by (6) satisfies the following law of motion:[34]

$$x_{t+1} = \phi(x_t, y_t, e_{t+1}, w_{t+1}) = e_{t+1}x_t y_t + (1 - y_t)w_{t+1}. \tag{40}$$

We assume that the worker cannot lend nor borrow, so consumption will equal earnings x_t at each period t. The variable z is a vector $z_t = (e_t, w_t)$ representing the shocks concerning the employer's decision e_t of keeping the worker and the wage offer w_t resulting from the search. Thus, the DM's problem can be represented by

$$\bar{v}(x_t, z_t) = \sup_{y_t \in \{0,1\}} \{y_t U(x_t) + \beta \mathrm{Q}_\tau[\bar{v}(\phi(x_t, y_t, z_{t+1}), z_{t+1})|z_t]\}, \tag{41}$$

where $U : [0, \bar{w}] \to \mathbb{R}$ denotes the utility over consumption, satisfying $U(0) = 0$.

We impose the following.

Assumption 11 (Independence and i.i.d.). *The sequences w_t and e_t are i.i.d., independent of each other, and w_t has a continuous distribution with support $[0, \bar{w}]$, with $\mathrm{Q}_\tau[w] > 0$.*

Observe that if $y_t = 1$, then $x_{t+1} = e_{t+1}x_t$ and if $y_t = 0$, $x_{t+1} = w_{t+1}$. Since the shocks are independent by Assumption 11, then the future state does not depend on the current

[33] An interesting extension of this model would encompass unemployment benefit. We leave it for future research.

[34] Observe that this ϕ does not satisfy Assumption 8 since it depends on x_t.

value of the shocks $z_t = (e_t, w_t)$. Moreover, for any of the choices $y_t \in \{0, 1\}$, the value function does not depend on the current value of shocks. Therefore, we can write the value function (41) as a function only of x_t, i.e.,

$$\bar{v}(x_t) = \max\{\underbrace{\beta \mathrm{Q}_\tau[\bar{v}(w_{t+1})]}_{\text{value for } y_t=0}, \underbrace{U(x_t) + \beta \mathrm{Q}_\tau[\bar{v}(e_{t+1}x_t)]}_{\text{value for } y_t=1}\}. \tag{42}$$

The characterization of the value function requires a few definitions. Observe that $\mathrm{Q}_\tau[e] \in \{0, 1\}$ and $\mathrm{Q}_\tau[w] \in [0, \bar{w}]$ are values determined by the primitives of the model. Let us define the following constant:

$$A = \frac{\beta(1 + \beta \mathrm{Q}_\tau[e])}{1 - \beta^2} U(\mathrm{Q}_\tau[w]). \tag{43}$$

It is easy to see that $A(1 - \beta) < U(\mathrm{Q}_\tau[w])$. Thus, if U is continuous and strictly increasing, we can define uniquely x^* by

$$U(x^*) = (1 - \beta)A. \tag{44}$$

We have the following.

Theorem 4.8. *Let Assumption 11 hold, $\beta \in (0, 1)$, and assume that $U : [0, \bar{w}] \to \mathbb{R}$ is strictly increasing and continuous, with $U(0) = 0$. Then there exists a unique continuous and bounded value function $\bar{v}$ satisfying (41), and this $\bar{v}$ is strictly increasing in x_t, does not depend on z_t and is given by*

$$\bar{v}(x) = \begin{cases} A & \text{if } x \leq x^*, \\ \left(1 + \dfrac{\beta}{1 - \beta} \mathrm{Q}_\tau[e]\right) U(x) + (1 - \mathrm{Q}_\tau[e])\beta A & \text{if } x > x^*. \end{cases} \tag{45}$$

Moreover, it is optimal to accept the offer $(y^* = 1)$ if $x \geq x^*$ and it is optimal to search $(y^* = 0)$ if $x \leq x^*$.

The solution in Theorem 4.8 is interesting and intuitive, being similar to the one obtainable for the expected utility model.[35] Indeed, equation (44) is the same as equation (3) in Stokey, Lucas, and Prescott (1989, p. 306). However, the expressions for A and $v(x)$ are different. Compare equations (2) and (4) in Stokey, Lucas, and Prescott (1989, p. 306) with (43) and (45), respectively.

The DM has an optimal benchmark salary x^* given by (44), which depends on both $\mathrm{Q}_\tau[e]$ and $\mathrm{Q}_\tau[w]$. Whenever a wage offer is below this level, the worker rejects the offer and searches for a new one. If, on the contrary, the DM receives an offer greater than x^*, the offer is accepted. It is worth noticing that this critical wage x^* is increasing in τ, as shown in expression (43). Since the parameter τ captures the risk attitude of the DM, larger values of τ—meaning that the agent is more risk lover—are associated with larger

[35] See Stokey, Lucas, and Prescott (1989, Section 10.7).

wages x^*. Therefore, a risk loving DM will have a relatively higher benchmark wage level x^*, and hence, will be more likely to engage in searching for a better salary, whereas a more risk averse DM is more likely to accept a given wage offer, since the benchmark x^* is lower. We note that the quantile search model generates different implications from that for the EU, since the solution depends on both $\mathrm{Q}_\tau[e]$ and $\mathrm{Q}_\tau[w]$. These quantile values are, in general, different from the expectation of their corresponding distributions. Potential coincidence depends on the skewness of these distributions, and the particular risk attitude parameter τ.

5. Conclusion

This paper studies dynamic quantile preferences introduced by de Castro and Galvao (2019). In this model, an agent maximizes the stream of future τ-quantile utilities. We are able to generalize and sharpen their results in many directions that are relevant for economic applications. In particular, we allow the shocks to be in a finite set or in a general connected metric space. Also, the future state is not directly chosen, but can be affected by shocks. These features allow to deal with applications that were not covered by de Castro and Galvao (2019)'s results. We show that the recursive quantile preferences model yields a value function, using a fixed-point argument. We also obtain desirable properties of the value function. In addition, we derive the corresponding Euler equation. These results are illustrated for two models: an intertemporal consumption and savings problem, and a search with unemployment model.

Appendix A: Appendix

This Appendix collects all the formal proofs of the results in the main text. Before we proceed to the proofs, we review a few useful properties of quantiles.[36]

A.1 *Preliminaries*

In this Appendix, we state and prove a number of results about quantiles, most of which are well known. Quantiles are monotonic in the following sense: if X first-order stochastically dominates Y then $\mathrm{Q}_\tau[X] \geq \mathrm{Q}_\tau[Y]$. If X is risk-free, i.e., $X = x$ with probability one for some x, then $\mathrm{Q}_\tau[X] = x$. Quantiles are also translation-invariant, i.e., $\mathrm{Q}_\tau[\alpha + X] = \alpha + \mathrm{Q}_\tau[X]$, $\forall \alpha \in \mathbb{R}$; and scale-invariant, i.e., $\mathrm{Q}_\tau[\alpha X] = \alpha \mathrm{Q}_\tau[X]$, $\forall \alpha \in \mathbb{R}_+$. On the other hand, quantiles do not share many of the convenient properties of expectations. For instance, $\mathrm{Q}_\tau[-X] = -\mathrm{Q}_{1-\tau}[X]$, provided the c.d.f. of X is invertible.[37] We highlight three other important properties that fail for quantiles and would be important for our results. First, in general, quantiles are not linear: $\mathrm{Q}_\tau[X + Y] \neq \mathrm{Q}_\tau[X] + \mathrm{Q}_\tau[Y]$ in general; but see Proposition A.2 below. Second, quantiles do not satisfy an analogue of the law of iterated expectations: if $\Sigma_0 \subset \Sigma_1$ are two σ-algebras, then, in general, $\mathrm{Q}_\tau[\mathrm{Q}_\tau[X|\Sigma_1]|\Sigma_0] \neq \mathrm{Q}_\tau[X|\Sigma_0]$. Third, in general, it is not possible to interchange

[36] See de Castro and Galvao (2019) for proofs of the stated properties and other results.
[37] See Lemma A.27 below for a more general statement.

a differentiation and a quantile operator, as it is for expectations, i.e., $\frac{\partial Q_\tau}{\partial x}[h(x, Z)] \neq Q_\tau[\frac{\partial h}{\partial x}(x, Z)]$. Despite these shortcomings, we are able to overcome the difficulties to obtain our results.

We proceed to state some properties that will be used repeatedly below, beginning with the following:[38]

$$\Pr\big[X < Q_\tau[X]\big] \leq \tau \leq \Pr\big[X \leq Q_\tau[X]\big] = F_X\big(Q_\tau[X]\big). \tag{46}$$

Let Θ be a set (of parameters) and $g : \Theta \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ be a measurable function. We denote by $Q_\tau[g(\theta, \cdot)|z]$ the quantile function associated with g, i.e.,

$$Q_\tau\big[g(\theta, \cdot)|z\big] \equiv \inf\big\{\alpha \in \mathbb{R} : \Pr\big[\big(g(\theta, W) \leq \alpha\big)|Z = z\big] \geq \tau\big\}. \tag{47}$$

The following result is a generalization of de Castro and Galvao (2019, Lemma A.2) to the case in which $\mathcal{Z}$ can be discrete, not only finite but also countable. Since the proof is identical, we omit it.

Lemma A.1. *Assume that $\mathcal{Z} \subset \mathbb{R}$ is closed and $g : \Theta \times \mathcal{Z} \to \mathbb{R}$ is nondecreasing and left-continuous in $\mathcal{Z}$, where closedness and left-continuity are relative to the usual topology on $\mathbb{R}$. Then*

$$Q_\tau\big[g(\theta, \cdot)|z\big] = g\big(\theta, Q_\tau[w|z]\big). \tag{48}$$

Although this paper focus attention on $\mathcal{Z}$ connected or finite, our results can be extended for countable $\mathcal{Z}$. In this setup, it is usual to endow $\mathcal{Z}$ with the discrete topology. Since this topology is trivial, every function is continuous with respect to it. But for the purpose of this lemma, more structure is needed for the case in which $\mathcal{Z}$ is countable, by requiring continuity with respect to the usual topology of $\mathbb{R}$. To see this assumption is needed, we provide the following counterexample. Let $\mathcal{Z} = \{1 - 1/n; n \in \mathbb{N}\} \cup \{1, 2\}$. Then $\mathcal{Z}$ is discrete and closed in the usual $\mathbb{R}$-topology. Consider the probabilities

$$\Pr\left[Z = 1 - \frac{1}{n}\right] = \frac{1}{2^{n+1}}; \quad n \in \mathbb{N} \quad \text{and} \quad \Pr[Z = 1] = \Pr[Z = 2] = \frac{1}{4}.$$

Instead of considering functions continuous with respect to the usual topology, assume only continuity with respect to the discrete topology on $\mathcal{Z}$. Let g be given by

$$g(1 - 1/n) = (1 - 1/n); \quad g(1) = 2 \text{ and } g(2) = 3.$$

For $\tau = 1/2$, one has $Q_\tau[g(Z)] = 1$ while $g(Q_\tau[Z]) = g(1) = 2$.

The next result is just a generalization of de Castro and Galvao (2019, Proposition A.4).

Proposition A.2. *Given random variables X and Y, assume that there are continuous and increasing functions h and g such that $X = h(Z)$ and $Y = g(Z)$. Then*

$$Q_\tau[X + Y] = Q_\tau[X] + Q_\tau[Y] \tag{49}$$

[38]See de Castro and Galvao (2019, Lemma A.1, p. 1926).

and if $X, Y > 0$,

$$Q_\tau[X \cdot Y] = Q_\tau[X] \cdot Q_\tau[Y]. \tag{50}$$

Proof. The property (49) is well known; a proof can be found in de Castro and Galvao (2019, Proposition A.4). To see that (50) also holds, observe that we can apply the monotonic function ln to the positive random variables X and Y, and use (48) and (49) for the comonotonic variables $\ln(X)$ and $\ln(Y)$, to obtain

$$\begin{aligned}\ln\bigl(Q_\tau[X \cdot Y]\bigr) &= Q_\tau\bigl[\ln(X \cdot Y)\bigr] = Q_\tau\bigl[\ln(X) + \ln(Y)\bigr] = Q_\tau\bigl[\ln(X)\bigr] + Q_\tau\bigl[\ln(Y)\bigr] \\ &= \ln\bigl(Q_\tau[X]\bigr) + \ln\bigl(Q_\tau[Y]\bigr) = \ln\bigl(Q_\tau[X] \cdot Q_\tau[Y]\bigr).\end{aligned}$$

Applying the exponential function to the terms on the left and on the right of the above equation, we obtain (50). □

We conclude with a useful property of convergence.

Lemma A.3. *Left* $f_n : \mathcal{X} \subset \mathbb{R}^p \to \mathbb{R}$ *be a sequence of functions converging uniformly to a function* $f : \mathcal{X} \to \mathbb{R}$. *Then*

$$\lim_{n\to\infty} Q_\tau\bigl[f_n(X)\bigr] = Q_\tau\bigl[f(X)\bigr].$$

Proof of Lemma A.3. Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists some $N \in \mathbb{N}$ such that

$$-\frac{\epsilon}{2} + f(x) < f_n(x) < f(x) + \frac{\epsilon}{2}$$

for all $x \in \mathcal{X}$ whenever $n \geq N$. Taking quantiles imply

$$\begin{aligned}-\epsilon + Q_\tau\bigl[f(X)\bigr] &< -\frac{\epsilon}{2} + Q_\tau\bigl[f(X)\bigr] = Q_\tau\left[-\frac{\epsilon}{2} + f(X)\right] \leq Q_\tau\bigl[f_n(X)\bigr] \\ &\leq Q_\tau\left[f(X) + \frac{\epsilon}{2}\right] = Q_\tau\bigl[f(X)\bigr] + \frac{\epsilon}{2} \\ &< Q_\tau\bigl[f(X)\bigr] + \epsilon,\end{aligned}$$

so

$$\bigl|Q_\tau\bigl[f_n(X)\bigr] - Q_\tau\bigl[f(X)\bigr]\bigr| < \epsilon,$$

if $n \geq N$. Thus, the result follows. □

A.2 *Proofs of Section* 3.3

Proof of Lemma 3.6. By Stokey, Lucas, and Prescott (1989, Theorem 7.6), Γ has a measurable selection. Therefore, the argument in Stokey, Lucas, and Prescott (1989, Lemma 9.1) establishes the result. □

Proof of Theorem 3.8. Under Assumption 0, v^* is well-defined by (13). Let $v : \mathcal{X} \times \mathcal{Z} \to \overline{\mathbb{R}}$ be a function satisfying the assumptions of Theorem 3.8. We need to show that:

(A) $v(x, z) \geq V(h, x, z)$, for all $(x, z) \in \mathcal{X} \times \mathcal{Z}$ and $h \in H(x, z)$; and

(B) $v(x, z) = \lim_{n\to\infty} V(h^n, x, z)$, for a sequence of plans $\{h^n\}$ obtained from G_v.

To see (A), assume that it is false, i.e., there exists $(x_1, z_1) \in \mathcal{X} \times \mathcal{Z}$, $h \in H$, and $\epsilon > 0$ such that $v(x_1, z_1) + 2\epsilon < V(h, x_1, z_1)$. From (12), there exists n_1 such that $n \geq n_1$ implies that

$$V^{n-1}(h, x_1, z_1) - \epsilon > v(x_1, z_1). \tag{51}$$

From (8), and using (9) and (10),

$$\begin{aligned} v(x_1, z_1) &= \sup_{y_1 \in \Gamma(x_1, z_1)} \{u(x_1, y_1, z_1) + \beta Q_\tau[v(\phi(x_1, y_1, z_2), z_2)|z_1]\} \\ &\geq Q_\tau[u(x_1, y_1^h, z_1) + \beta v(\phi(x_1, y_1^h, z_2), z_2)|z_1] \end{aligned}$$

We can use again (8) to obtain that $v(x_1, z_1)$ is not smaller than

$$\begin{aligned} &Q_\tau\Big[S^{h,0}(x_1, z_1) + \beta \sup_{y_2 \in \Gamma(x_2^h, z_2)} \{u(x_2^h, y_2, z_2) + \beta Q_\tau[v(\phi(x_2^h, y_2, z_3), z_3)|z^2]\}|z_1\Big] \\ &\geq Q_\tau[Q_\tau[S^{h,0}(x_1, z_1) + \beta u(x_2^h, y_2^h, z_2) + \beta^2 v(\phi(x_2^h, y_2^h, z_3), z_3)|z^2]|z_1] \\ &= Q_\tau^2[S^{h,1}(x_1, z^2) + \beta^2 v(\phi(x_2^h, y_2^h, z_3), z_3)|z_1]. \end{aligned}$$

Proceeding in this fashion, we obtain

$$\begin{aligned} v(x_1, z_1) &\geq Q_\tau^n[S^{h,n-1}(x_1, z^n) + \beta^n v(\phi(x_n^h, y_n^h, z_{n+1}), z_{n+1})|z_1] \\ &= Q_\tau^{n-1}[Q_\tau[S^{h,n-1}(x_1, z^n) + \beta^n v(\phi(x_n^h, y_n^h, z_{n+1}), z_{n+1})|z^n]|z_1] \\ &= Q_\tau^{n-1}[S^{h,n-1}(x_1, z^n) + \beta^n Q_\tau[v(\phi(x_n^h, y_n^h, z_{n+1}), z_{n+1})|z^n]|z_1]. \end{aligned}$$

From the transversality condition, there exists $n_2 \geq n_1$ such that $n \geq n_2$ implies (14). Therefore,

$$\begin{aligned} v(x_1, z_1) &\geq Q_\tau^{n-1}[S^{h,n-1}(x_1, z^n) + \beta^n Q_\tau[v(\phi(x_n^h, y_n^h, z_{n+1}), z_{n+1})|z^n]|z_1] \qquad (52) \\ &\geq Q_\tau^{n-1}[S^{h,n-1}(x_1, z^n) - \epsilon|z_1] \\ &= V^{n-1}(h, x, z) - \epsilon, \end{aligned}$$

but this contradicts (51). The contradiction establishes (A).

From (A), we conclude that $v(x, z) \geq \sup_{h\in H(x,z)} V(h, x, z)$. For a contradiction, assume that (B) is false, i.e., there exists $\epsilon > 0$ and $(x_1, z_1) \in \mathcal{X} \times \mathcal{Z}$, such that $v(x_1, z_1) - 2\epsilon > \sup_{h\in H(x_1,z_1)} V(h, x_1, z_1)$. Let h be any plan obtained from G_v, which exists because of the stated assumptions on G_v. Then $v(x_1, z_1) - 2\epsilon > V(h, x_1, z_1)$. From (12), there exists n_1 such that $n \geq n_1$ implies that

$$v(x_1, z_1) > V^{n-1}(h, x_1, z_1) + \epsilon. \tag{53}$$

We can repeat the developments above after (51), where all inequalities hold with equality since h is obtained from G. In this fashion, we obtain

$$v(x_1, z_1) = \mathrm{Q}_\tau^{n-1}\big[S^{h,n-1}(x_1, z^n) + \beta^n \mathrm{Q}_\tau\big[v\big(\phi\big(x_n^h, y_n^h, z_{n+1}\big), z_{n+1}\big)|z^n\big]|z_1\big], \tag{54}$$

From CQTC, there exists $n_2 \geq n_1$ such that $n \geq n_2$ implies (14). Therefore,

$$v(x_1, z_1) \leq \mathrm{Q}_\tau^{n-1}\big[S^{h,n-1}(x_1, z^n) + \epsilon|z_1\big] = V^{n-1}(h, x, z) + \epsilon,$$

but this contradicts (53). The contradiction establishes (B) and concludes the proof.

Notice that this also establishes the last claim. Indeed, if there exists a plan h obtained from G_v such that $v(x_1, z_1) > V(h, x_1, z_1)$, the above arguments would lead to the same contradiction. This concludes the proof. □

A.3 *Proofs of Section 3.5*

Proof of Theorem 3.11. This will be established through a series of lemmas and the next proposition. In the following proofs, we denote by w the next period shock, given that the current shock is z.

Proposition A.4. *If $v \in \mathcal{C}$, the map $(x, y, z) \mapsto \mathrm{Q}_\tau[v(\phi(x, y, w), w)|z]$ is continuous.*

The proof of Proposition A.4 is divided in a series of lemmas. Before presenting them, we need to make a simple observation and introduce some notation. Observe that, since ϕ is continuous, by setting $y' = (x, y)$ and $v'(y', w) = v(\phi(x, y, w), w)$, it suffices to prove that $(y', z) \mapsto \mathrm{Q}_\tau[v'(y', w)|z]$ is continuous. We proceed in this direction, simply writing y and v instead of y' and v', respectively.

Now consider a sequence $(y^n, z^n) \to (y^*, z^*)$. Let $K : \mathcal{Z} \times \Sigma \to [0, 1]$ be the transition function representing the Markov process of the shocks $\mathcal{Z}$, where Σ is the Borel σ-algebra. Let

$$m^n(\alpha) \equiv \Pr\big(\{w \in \mathcal{Z} : v(y^n, w) \leq \alpha\}|z^n\big) = K\big(z^n, \{w \in \mathcal{Z} : v(y^n, w) \leq \alpha\}\big)$$

and

$$m^*(\alpha) \equiv \Pr\big(\{w \in \mathcal{Z} : v(y^*, w) \leq \alpha\}|z^*\big) = K\big(z^*, \{w \in \mathcal{Z} : v(y^*, w) \leq \alpha\}\big).$$

Let $\alpha^n \equiv \inf\{\alpha \in \mathbb{R} : m^n(\alpha) \geq \tau\} = \mathrm{Q}_\tau[v(y^n, \cdot)|z^n]$ and $\alpha^* \equiv \inf\{\alpha \in \mathbb{R} : m^*(\alpha) \geq \tau\} = \mathrm{Q}_\tau[v(y^*, \cdot)|z^*]$. We want to show that $\alpha^n \to \alpha^*$.[39] We will proceed in two main steps, first showing that $\underline{\alpha} \equiv \liminf_n \alpha^n \geq \alpha^*$ and then showing that $\bar{\alpha} \equiv \limsup_n \alpha^n \leq \alpha^*$.[40] This will establish the result.

[39]Notice that we do not claim that $m^n(\alpha) \to m^*(\alpha)$, as in equation (50) of de Castro and Galvao (2019), and the proof does not depend on this convergence.

[40]Since v is bounded, it is not possible that $\alpha^n \to \infty$ or $-\infty$, i.e., $\underline{\alpha}, \bar{\alpha} \in \mathbb{R}$ are well-defined.

Lemma A.5. *Suppose that* $\alpha^n \to \hat{\alpha}$. *Given* $\epsilon, \delta > 0$, *there exists* $n_{\epsilon,\delta} \in \mathbb{N}$ *and a compact* $\mathcal{Z}' \subset \mathcal{Z}$ *such that* $n \geq n_{\epsilon,\delta}$ *implies that*

$$K\big(z^*, \{w \in \mathcal{Z}' : v(y^*, w) \leq \hat{\alpha} + \epsilon\}\big) + \delta > \tau; \quad \text{and} \tag{55}$$

$$K\big(z^n, \{w \in \mathcal{Z}' : v(y^*, w) \leq \hat{\alpha} - \epsilon\}\big) + \delta > K\big(z^*, \{w \in \mathcal{Z} : v(y^*, w) \leq \hat{\alpha} - \epsilon\}\big). \tag{56}$$

Proof. By Assumption 1(i), given $\epsilon > 0$, there exists $\mathcal{Z}' \subset \mathcal{Z}$ compact such that

$$K\big(z^*, \mathcal{Z}\backslash\mathcal{Z}'\big) < \frac{\delta}{4}. \tag{57}$$

Using (57) and Assumption 1(ii), there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$,[41]

$$\begin{aligned} &\big|K\big(z^n, \mathcal{Z}\backslash\mathcal{Z}'\big) - K\big(z^*, \mathcal{Z}\backslash\mathcal{Z}'\big)\big| < \frac{\delta}{4} \\ &\Rightarrow \quad K\big(z^n, \mathcal{Z}\backslash\mathcal{Z}'\big) < K\big(z^*, \mathcal{Z}\backslash\mathcal{Z}'\big) + \frac{\delta}{4} < \frac{\delta}{2}. \end{aligned} \tag{58}$$

Let D be a compact set containing the sequence $\{y^n\}_{n\in\mathbb{N}}$ and, of course, its limit y^*. Then, since v is continuous, it is uniformly continuous in the compact $D \times \mathcal{Z}'$. Hence, there exists $n_2 \geq n_1$ such that if $n \geq n_2$ then

$$\big|v\big(y^n, w\big) - v\big(y^*, w\big)\big| < \frac{\epsilon}{2}, \quad \forall w \in \mathcal{Z}' \quad \text{and} \quad \big|\hat{\alpha} - \alpha^n\big| < \frac{\epsilon}{2}. \tag{59}$$

Now, let $w \in \mathcal{Z}'$ be such that $v(y^n, w) \leq \alpha^n$. By (59), $v(y^*, w) \leq v(y^n, w) + \frac{\epsilon}{2} \leq \alpha^n + \frac{\epsilon}{2} < \hat{\alpha} + \epsilon$. Thus, if $n \geq n_2$, $\{w \in \mathcal{Z}' : v(y^n, w) \leq \alpha^n\} \subset \{w \in \mathcal{Z}' : v(y^*, w) \leq \hat{\alpha} + \epsilon\}$. Defining $E \equiv \{w \in \mathcal{Z}' : v(y^*, w) \leq \hat{\alpha} + \epsilon\}$, which is compact, we conclude that

$$K\big(z^n, \{w \in \mathcal{Z} : v\big(y^n, w\big) \leq \alpha^n\}\big) \leq K\big(z^n, E\big) + K\big(z^n, \mathcal{Z}\backslash\mathcal{Z}'\big). \tag{60}$$

Since the expression on the left above is greater than or equal to τ by (46), we can use (58) to conclude that $\tau < K(z^n, E) + \frac{\delta}{2}$. Again by Assumption 1(ii), there exists $n_3 \geq n_2$ such that $n \geq n_3$ implies $K(z^n, E) < K(z^*, E) + \frac{\delta}{2}$. Therefore, we have proved (55).

To see (56), define $F \equiv \{w \in \mathcal{Z}' : v(y^*, w) \leq \hat{\alpha} - \epsilon\}$. This set is compact. Observe that

$$K\big(z^*, \{w \in \mathcal{Z} : v\big(y^*, w\big) \leq \hat{\alpha} - \epsilon\}\big) \leq K\big(z^*, F\big) + K\big(z^*, \mathcal{Z}\backslash\mathcal{Z}'\big). \tag{61}$$

Again by Assumption 1(ii), there exists $n_4 \geq n_3$ such that $n \geq n_4$ implies

$$K\big(z^*, F\big) < K\big(z^n, F\big) + \frac{\delta}{2}. \tag{62}$$

Combining (61) with (57) and (62), we obtain (56). Finally, let $n_{\epsilon,\delta} \equiv n_4$. □

Lemma A.6. *Let Assumption 1 hold. Then* $\underline{\alpha} \equiv \liminf_n \alpha^n \geq \alpha^*$.

[41]Since $K(z, \mathcal{Z}\backslash\mathcal{Z}') = 1 - K(z, \mathcal{Z}')$ for any $z \in \mathcal{Z}$, Assumption 1(ii) implies $K(z^n, \mathcal{Z}\backslash\mathcal{Z}') \to K(z^*, \mathcal{Z}\backslash\mathcal{Z}')$.

PROOF. We will show that $\underline{\alpha} \geq \alpha^*$ by contradiction. So, assume that $\underline{\alpha} < \alpha^*$. This means that there exists $\epsilon > 0$ and a subsequence n_j such that $\alpha^{n_j} \to \underline{\alpha}$, with $\underline{\alpha} + \epsilon < \alpha^*$. Since $\alpha^* \equiv \inf\{\alpha \in \mathbb{R} : m^*(\alpha) \geq \tau\}$, $m^*(\underline{\alpha} + \epsilon) < \tau$. Choose $\eta > 0$ such that

$$\tau - \eta > m^*(\underline{\alpha} + \epsilon) = K\big(z^*, \{w \in \mathcal{Z} : v(y^*, w) \leq \underline{\alpha} + \epsilon\}\big).$$

Given this $\eta > 0$, Lemma A.5 implies that there exists a compact $\mathcal{Z}' \subset \mathcal{Z}$ such that

$$m^*(\underline{\alpha} + \epsilon) \geq K\big(z^*, \{w \in \mathcal{Z}' : v(y^*, w) \leq \underline{\alpha} + \epsilon\}\big) > \tau - \eta,$$

i.e., $\tau - \eta > m^*(\underline{\alpha} + \epsilon) > \tau - \eta$, which is a contradiction that establishes the result. □

Let us denote by E^* the set $\{w \in \mathcal{Z} : v(y^*, w) \leq \alpha^*\}$. Since v is continuous, E^* is closed.

LEMMA A.7. *Let $\bar{\alpha} \equiv \limsup_n \alpha^n$. If $K(z^*, E^*) > \tau$, then $\bar{\alpha} \leq \alpha^*$.*

PROOF. For a contradiction, assume that there exists $\epsilon > 0$ such that $\bar{\alpha} - \epsilon > \alpha^\star$. Let $\{\alpha^{n_j}\}_{j\in\mathbb{N}}$ be a subsequence converging to $\bar{\alpha}$. Let $\delta > 0$ be such that $K(z^*, E^*) > \tau + 2\delta$. By Lemma A.5, there exists $j_1 \in \mathbb{N}$ and a compact $\mathcal{Z}' \subset \mathcal{Z}$ such that $j \geq j_1$ implies that

$$K\big(z^{n_j}, \{w \in \mathcal{Z}' : v(y^*, w) \leq \bar{\alpha} - \epsilon\}\big) > K\big(z^*, \{w \in \mathcal{Z} : v(y^*, w) \leq \bar{\alpha} - \epsilon\}\big) - \delta.$$

Since $\bar{\alpha} - \epsilon > \alpha^*$, $K(z^*, \{w \in \mathcal{Z} : v(y^*, w) \leq \bar{\alpha} - \epsilon\}) \geq K(z^*, E^*) > \tau + 2\delta$. Thus, if $j \geq j_1$,

$$K\big(z^{n_j}, \{w \in \mathcal{Z} : v(y^*, w) \leq \bar{\alpha} - \epsilon\}\big) > \tau + \delta.$$

Since $\alpha^{n_j} = \inf\{\alpha \in \mathbb{R} : K(z^{n_j}, \{w \in \mathcal{Z} : v(y^{n_j}, w) \leq \alpha\}) \geq \tau\}$, then $\alpha^{n_j} \leq \bar{\alpha} - \epsilon$ for all $j \geq j_1$. However, this contradicts $\alpha^{n_j} \to \bar{\alpha}$, and concludes the proof. □

Now, we have to deal with the case in which

$$K\big(z^*, E^*\big) = K\big(z^*, \{w \in \mathcal{Z} : v(y^*, w) \leq \alpha^*\}\big) = \tau. \tag{63}$$

We deal with this case by considering first the case in which $\mathcal{Z}$ is finite and then the case in which $\mathcal{Z}$ is connected.

LEMMA A.8. *Let Assumption 1 hold, in the specific case in which $\mathcal{Z}$ is finite. If (63) holds, then $\bar{\alpha} \equiv \limsup_n \alpha^n \leq \alpha^*$.*

PROOF. For a contradiction, assume that for some $\epsilon > 0$, $\bar{\alpha} - \epsilon > \alpha^*$. Given that $\mathcal{Z}$ is a finite metric space, it is endowed with the discrete topology. Since $z^n \to z^*$, we may assume, without loss of generality, that there exists a subsequence such that $z^{n_j} = z^*$ for all $j \in \mathbb{N}$ and $\alpha^{n_j} \to \bar{\alpha}$. There exists $j_1 \in \mathbb{N}$ such that for all $j \geq j_1$, $|\alpha^{n_j} - \bar{\alpha}| < \frac{\epsilon}{2}$. Since $\bar{\alpha} - \epsilon > \alpha^*$, for all $j \geq j_1$,

$$\alpha^* < \alpha^* + \frac{\epsilon}{2} < \bar{\alpha} - \frac{\epsilon}{2} < \alpha^{n_j}. \tag{64}$$

Since $\alpha^{n_j} = \inf\{\alpha \in \mathbb{R} : K(z^{n_j}, \{w \in \mathcal{Z} : v(y^{n_j}, w) \leq \alpha\}) \geq \tau\}$, for all $j \geq j_1$,

$$K\left(z^*, \left\{w \in \mathcal{Z} : v(y^{n_j}, w) \leq \bar{\alpha} - \frac{\epsilon}{2}\right\}\right) < \tau. \tag{65}$$

Fix a compact D containing $\{y^n\}_{n\in\mathbb{N}}$. Again, since v continuous and $\mathcal{Z}$ is finite, v is uniformly continuous on $D \times \mathcal{Z}$. Hence, there exists $j_2 \geq j_1$ such that $\forall w \in \mathcal{Z}$ and $j \geq j_2$,

$$\left|v(y^*, w) - v(y^{n_j}, w)\right| < \frac{\epsilon}{2}.$$

Thus, if $j \geq j_2$ and $w \in \mathcal{Z}$ is such that $v(y^*, w) \leq \alpha^*$, we have

$$v(y^{n_j}, w) - \frac{\epsilon}{2} < v(y^*, w) \leq \alpha^*,$$

so, by (64), for all $j \geq j_2$,

$$v(y^{n_j}, w) < \alpha^* + \frac{\epsilon}{2} < \bar{\alpha} - \frac{\epsilon}{2} < \alpha^{n_j}.$$

Therefore, for all $j \geq j_2$,

$$E^* = \{w \in \mathcal{Z} : v(y^*, w) \leq \alpha^*\} \subset \left\{w \in \mathcal{Z} : v(y^{n_j}, w) \leq \bar{\alpha} - \frac{\epsilon}{2}\right\}.$$

Using this, (63) and (65), we have t, for $j \geq j_2$,

$$\tau = K(z^*, E^*) \leq K\left(z^*, \left\{w \in \mathcal{Z} : v(y^{n_j}, w) \leq \bar{\alpha} - \frac{\epsilon}{2}\right\}\right) < \tau,$$

a contradiction. Thus, the result is established. □

Now, we consider (63) for the case in which $\mathcal{Z}$ is connected. In fact, we will establish the result for a condition that is implied but it is slightly more general than the requirement that $\mathcal{Z}$ is connected, namely the assumption that the image of $w \mapsto v(y^*, w)$ is connected, i.e., it is an interval. Since v is continuous, the condition that $\mathcal{Z}$ is connected implies this property. Lemma A.9 below establishes the relevant result, assuming only this condition on the v.

Lemma A.9. *Let Assumption 1 hold, $v \in \mathcal{C}$, and assume that $v(y^*, \mathcal{Z})$ is an interval. If (63) holds, then $\bar{\alpha} \equiv \limsup_n \alpha^n \leq \alpha^*$.*

Proof. For a contradiction, assume that $\bar{\alpha} > \alpha^*$. Fix $\delta > 0$ such that $\alpha^* + 2\delta < \bar{\alpha}$. Let n_j be a subsequence such that $\alpha^{n_j} \to \bar{\alpha}$. Thus, there exists $j_1 \in \mathbb{N}$ such that $j \geq j_1$ implies $\alpha^{n_j} > \bar{\alpha} - \delta > \alpha^* + \delta$. We claim that the set $B \equiv \{w \in \mathcal{Z} : \alpha^* < v(y^*, w) < \bar{\alpha} - \delta\}$ is nonempty. To see this, suppose that $B = \emptyset$. Since $\tau \in (0, 1)$, (63) implies that

$$0 < 1 - \tau = K(z^*, \{w \in \mathcal{Z} : v(y^*, w) > \alpha^*\}) = K(z^*, \{w \in \mathcal{Z} : v(y^*, w) \geq \bar{\alpha} - \delta\}).$$

Therefore, there exists $w_1 \in \{w \in \mathcal{Z} : v(y^*, w) \geq \bar{\alpha} - \delta\}$. Again by (63), there exists $w_0 \in \{w \in \mathcal{Z} : v(y^*, w) \leq \alpha^*\}$. Therefore, $v(y^*, w_0) \leq \alpha^* < \bar{\alpha} - \delta \leq v(y^*, w_1)$. Since $v(y^*, \mathcal{Z})$ is an interval, $[\alpha^*, \bar{\alpha} - \delta] \subset v(y^*, \mathcal{Z})$, which contradicts $B = \emptyset$. Thus, there exists w_2 such that $\alpha^* < v^*(w_2) < \bar{\alpha} - \delta$. Let $\epsilon \in (0, \delta]$ be such that $v^*(w_2) \in (\alpha^* + \epsilon, \bar{\alpha} - \epsilon)$.

Since $w_2 \in \{w \in \mathcal{Z} : \alpha^* + \epsilon < v(y^*, w) < \bar{\alpha} - \epsilon\}$, and v is continuous, this set is nonempty and open. Assumption 1(iii) guarantees that there exists $\eta > 0$ such that $5\eta < \tau$ and

$$K\left(z^*, \left\{w \in \mathcal{Z} : \alpha^* + \epsilon < v\left(y^*, w\right) < \bar{\alpha} - \epsilon\right\}\right) > 5\eta. \tag{66}$$

Assumption 1(i) enables us to find a compact $\mathcal{Z}' \subset \mathcal{Z}$ such that

$$K\left(z^*, \mathcal{Z} \backslash \mathcal{Z}'\right) < \eta. \tag{67}$$

Let $C \equiv \{w \in \mathcal{Z}' : \alpha^* + \epsilon \leq v(y^*, w) \leq \bar{\alpha} - \epsilon\}$. Notice that $C \subset \mathcal{Z}'$ is compact, since v is continuous. From (66) and (67),

$$K\left(z^*, C\right) \geq K\left(z^*, \left\{w \in \mathcal{Z} : \alpha^* + \epsilon \leq v\left(y^*, w\right) \leq \bar{\alpha} - \epsilon\right\}\right) - K\left(z^*, \mathcal{Z} \backslash \mathcal{Z}'\right) > 4\eta. \tag{68}$$

Let $D \subset \mathcal{X}$ be a compact containing the sequence $\{y^n\}_{n \in \mathbb{N}}$ and, naturally, its limit y^*. Thus, v is uniformly continuous in $D \times \mathcal{Z}'$ and there exists $j_2 \geq j_1$, such that for all $w \in \mathcal{Z}'$ and $j \geq j_2$,

$$\left|v\left(y^{n_j}, w\right) - v\left(y^*, w\right)\right| < \frac{\epsilon}{2} \quad \text{and} \tag{69}$$

$$\left|\bar{\alpha} - \alpha^{n_j}\right| < \frac{\epsilon}{2}. \tag{70}$$

Now notice that, if $w \in \mathcal{Z}'$ and $\alpha^* + \epsilon \leq v(y^*, w)$, (69) implies that for all $j \geq j_2$,

$$v\left(y^{n_j}, w\right) > v\left(y^*, w\right) - \frac{\epsilon}{2} \geq \alpha^* + \epsilon - \frac{\epsilon}{2} = \alpha^* + \frac{\epsilon}{2}. \tag{71}$$

Notice also that, if $w \in \mathcal{Z}'$ and $v(y^*, w) \leq \bar{\alpha} - \epsilon$, (69) implies that

$$v\left(y^{n_j}, w\right) < v\left(y^*, w\right) + \frac{\epsilon}{2} \leq \bar{\alpha} - \epsilon + \frac{\epsilon}{2} = \bar{\alpha} - \frac{\epsilon}{2}. \tag{72}$$

Hence, (71), (72), and the fact that $\bar{\alpha} - \epsilon/2 < \alpha^{n_j}$, from (70), imply that for $j \geq j_2$,

$$C = \left\{w \in \mathcal{Z}' : \alpha^* + \epsilon \leq v\left(y^*, w\right) \leq \bar{\alpha} - \epsilon\right\} \subset \left\{w \in \mathcal{Z} : \alpha^* + \frac{\epsilon}{2} < v\left(y^{n_j}, w\right) < \alpha^{n_j}\right\}. \tag{73}$$

Since $K(z^*, C) > 4\eta$ by (68) and $K(z^{n_j}, C) \to K(z^*, C)$, there exists $j_3 \geq j_2$ such that for all $j \geq j_3$,

$$K\left(z^{n_j}, C\right) > 2\eta.$$

Therefore, from the definition of m^{n_j}, for all $j \geq j_3$,

$$\begin{aligned} m^{n_j}\left(\alpha^* + \frac{\epsilon}{2}\right) + 2\eta &< K\left(z^{n_j}, \left\{w \in \mathcal{Z} : v\left(y^{n_j}, w\right) \leq \alpha^* + \frac{\epsilon}{2}\right\}\right) + K\left(z^{n_j}, C\right) \\ &\leq K\left(z^{n_j}, \left\{w \in \mathcal{Z} : v\left(y^{n_j}, w\right) < \alpha^{n_j}\right\}\right) \leq \tau, \end{aligned}$$

where we have used (73) to obtain the penultimate inequality, while the last inequality holds from the definition of α^{n_j} and (46). Thus, we have established that for all $j \geq j_3$,

$$m^{n_j}\left(\alpha^* + \frac{\epsilon}{2}\right) < \tau - 2\eta. \tag{74}$$

On the other hand, if $j \geq j_3$ and $w \in \mathcal{Z}'$ is such that $v(y^*, w) \leq \alpha^*$, then from (69),

$$v(y^{n_j}, w) < v(y^*, w) + \frac{\epsilon}{2} \leq \alpha^* + \frac{\epsilon}{2},$$

i.e.,

$$\{w \in \mathcal{Z}' : v(y^*, w) \leq \alpha^*\} \subset \left\{w \in \mathcal{Z}' : v(y^{n_j}, w) \leq \alpha^* + \frac{\epsilon}{2}\right\}. \tag{75}$$

From the definition of α^*, (46), and (67), we have for all $j \geq j_3$,

$$\tau \leq K(z^*, E^*) \leq K(z^*, E^* \cap \mathcal{Z}') + K(z^*, \mathcal{Z} \backslash \mathcal{Z}') < K(z^*, E^* \cap \mathcal{Z}') + \eta. \tag{76}$$

By Assumption 1(ii), $K(z^{n_j}, E^* \cap \mathcal{Z}') \to K(z^*, E^* \cap \mathcal{Z}')$. Therefore, there exists $j_4 \geq j_3$ such that $j \geq j_4$ implies, using (76), that

$$K(z^{n_j}, E^* \cap \mathcal{Z}') > K(z^*, E^* \cap \mathcal{Z}') - \eta > \tau - 2\eta. \tag{77}$$

Hence, for each $j \geq j_4$,

$$\begin{aligned}
m^{n_j}\left(\alpha^* + \frac{\epsilon}{2}\right) &= K\left(z^{n_j}, \left\{z \in \mathcal{Z} : v(y^{n_j}, w) \leq \alpha^* + \frac{\epsilon}{2}\right\}\right) \quad \text{by definition} \\
&\geq K\left(z^{n_j}, \left\{z \in \mathcal{Z}' : v(y^{n_j}, w) \leq \alpha^* + \frac{\epsilon}{2}\right\}\right) \quad \text{by set inclusion} \\
&\geq K\left(z^{n_j}, \{z \in \mathcal{Z}' : v(y^*, w) \leq \alpha^*\}\right) \quad \text{by (75)} \\
&= K(z^{n_j}, E^* \cap \mathcal{Z}') \quad \text{by definition} \\
&> \tau - 2\eta \quad \text{by (77)}.
\end{aligned}$$

However, this contradicts (74), concluding the proof. □

Proof of Proposition A.4. Observe that $K(z^*, \{z \in \mathcal{Z} : v(y^*, w) \leq \alpha^*\}) \in [\tau, 1]$, by (46) and the fact that $K(z^*, \cdot)$ is a probability. The case $\mathcal{Z}$ finite is established by Lemmas A.6, A.7, and A.8 and the case $\mathcal{Z}$ connected is proved by Lemmas A.6, A.7, and A.9. □

Lemma A.10. *For each $v \in \mathcal{C}$ the supremum in (16) is attained and $\mathbb{M}(v) \in \mathcal{C}$. Moreover, the optimal correspondence $\Upsilon : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{Y}$ defined by*

$$\Upsilon(x, z) \equiv \arg \max_{y \in \Gamma(x,z)} \mathrm{Q}_\tau\left[u(x, y, z) + \beta \bar{v}(\phi(x, y, w), w) | z\right] \tag{78}$$

is upper semicontinuous with nonempty compact values.

Proof. The proof repeats the proof of de Castro and Galvao (2019, Lemma A.6). □

We conclude the proof of Theorem 3.11 by showing that $\mathbb{M}$ satisfies Blackwell's sufficient conditions for a contraction.

Lemma A.11. *$\mathbb{M}$ satisfies the following conditions:*

(a) For any $v, v' \in \mathcal{C}$, $v \le v'$ implies $\mathbb{M}(v) \le \mathbb{M}(v')$.

(b) For any $a \ge 0$ and $x \in X$, $\mathbb{M}(v + a)(x) \le \mathbb{M}(v)(x) + \beta a$, with $\beta \in (0, 1)$.

Then $\|\mathbb{M}(v) - \mathbb{M}(v')\| \le \beta\|v - v'\|$, i.e., $\mathbb{M}$ is a contraction with modulus β. Therefore, $\mathbb{M}$ has a unique fixed-point $\bar{v} \in \mathcal{C}$.

Proof. To see (a), let $v, v' \in \mathcal{C}$, $v \le v'$ and define g as

$$g(x, y, z, w) = u(x, y, z) + \beta v\big(\phi(x, y, w), w\big) \tag{79}$$

and analogously for g'. It is clear that $g \le g'$. Then, by de Castro and Galvao (2019, Lemma A.1(vi)), $\mathrm{Q}_\tau[g(\cdot)|z] \le \mathrm{Q}_\tau[g'(\cdot)|z]$, which implies (a).

To verify (b), since a is a constant,

$$\mathrm{Q}_\tau\big[v\big(\phi(x, y, w), w\big) + a|z\big] = \mathrm{Q}_\tau\big[v\big(\phi(x, y, w), w\big)|z\big] + a.$$

Thus, $\mathbb{M}(v + a) = \mathbb{M}(v) + \beta a$, i.e., (b) is satisfied with equality. By a standard argument, (a) and (b) imply that $\mathbb{M}$ is a contraction and the result follows. □

A.4 *Proofs of Section 3.6*

Proof of Theorem 3.13. We will present the proof for the case with strict increasingness. For nondecreasing functions, the same argument works with weak inequalities where strict inequalities appear below. Let $\mathcal{C}' \subset \mathcal{C}$ be the set of the bounded and continuous functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$, which are nondecreasing in x. It is easy to see that $\mathcal{C}'$ is a closed subset of $\mathcal{C}$. Let $\mathcal{C}'' \subset \mathcal{C}'$ be the set of strictly increasing functions x. If we show that $\mathbb{M}(\mathcal{C}') \subset \mathcal{C}''$, then the fixed point of $\mathbb{M}$ will be strictly increasing in x.

Let $v \in \mathcal{C}'$ and consider $x_0, x_1 \in \mathcal{X}$, $x_0 < x_1$. For $i = 0, 1$, let $y_i \in \Gamma(x_i, z)$ attain the maximum, i.e.,

$$\mathbb{M}(v)(x_i, z) = u(x_i, y_i, z) + \beta\mathrm{Q}_\tau\big[v\big(\phi(x_i, y_i, w), w\big)|z\big],$$

By Assumption 3, $\Gamma(x_0, z) \subset \Gamma(x_1, z)$, so $y_0 \in \Gamma(x_1, z)$. Therefore,

$$\begin{aligned}\mathbb{M}v(x_0, z) &= u(x_0, y_0, z) + \beta\mathrm{Q}_\tau\big[v\big(\phi(x_0, y_0, w), w\big)|z\big] \\ &< u(x_1, y_0, z) + \beta\mathrm{Q}_\tau\big[v\big(\phi(x_1, y_0, w), w\big)|z\big] \\ &\le \mathbb{M}v(x_1, z),\end{aligned}$$

where in the first inequality we used that u is strictly increasing in x, both v and ϕ are weakly increasing in x, and the fact that quantiles is a monotonic operator, i.e., for any

g, h functions such that $g \le h$, $\mathrm{Q}_\tau[g(Z)] \le \mathrm{Q}_\tau[h(Z)]$; see de Castro and Galvao (2019, Lemma A.1(vi)). In the last inequality, we have used the optimality of $\mathbb{M}$. This shows that $\mathbb{M}v$ is strictly increasing in x when $v \in \mathcal{C}'$, i.e., $\mathbb{M}^\tau(\mathcal{C}') \subset \mathcal{C}''$, since $v \in \mathcal{C}'$ was arbitrary. □

To establish Theorem 3.14, we need the following two lemmas, which rely on its assumptions.

Lemma A.12. *If $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \le z'$, then $\mathrm{Q}_\tau[h(w)|z] \le \mathrm{Q}_\tau[h(w)|z']$.*

Proof. From Assumption 5, if $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \le z'$,

$$\mathrm{E}[-1_{\{W\in\mathcal{Z}:h(W)\le\alpha\}}|z] \le \mathrm{E}\big[-1_{\{W\in\mathcal{Z}:h(W)\le\alpha\}}|z'\big].$$

Thus,

$$\Pr\big(\big[h(W)\le\alpha\big]|z\big) = \mathrm{E}[1_{\{W\in\mathcal{Z}:h(W)\le\alpha\}}|z] \ge \mathrm{E}\big[1_{\{W\in\mathcal{Z}:h(W)\le\alpha\}}|z'\big] = \Pr\big(\big[h(W)\le w\big]|z'\big). \quad (80)$$

If we define $H(w|z) = \Pr([h(W) \le w]|Z = z)$, then (80) can be written as

$$H(w|z) \ge H\big(w|z'\big).$$

Observe that $\mathrm{Q}_\tau[h(w)|z] = \inf\{\alpha \in \mathbb{R} : H(\alpha|z) \ge \tau\}$, and whenever $z \le z'$, $H(w|z') \le H(w|z)$, for all w. Therefore, if $z \le z'$, then

$$\big\{\alpha\in\mathbb{R} : H(\alpha|z)\ge\tau\big\} \supset \big\{\alpha\in\mathbb{R} : H\big(\alpha|z'\big)\ge\tau\big\},$$

which implies that

$$\mathrm{Q}_\tau\big[h(w)|z\big] = \inf\big\{\alpha\in\mathbb{R} : H(\alpha|z)\ge\tau\big\} \le \inf\big\{\alpha\in\mathbb{R} : H\big(\alpha|z'\big)\ge\tau\big\} = \mathrm{Q}_\tau\big[h(w)|z'\big],$$

as we wanted to show. □

Lemma A.13. *If u and v are weakly increasing in x and z, then $\mathbb{M}(v)$ is weakly increasing in z. If u is strictly increasing in z, so is $\mathbb{M}(v)$.*

Proof. Let $z_1, z_2 \in \mathcal{Z}$, with $z_1 < z_2$. For $i = 1, 2$, let $y_i \in \Gamma(x, z_i)$ realize the maximum, i.e.,

$$\mathbb{M}(v)(x, z_i) = u(x, y_i, z_i) + \beta\mathrm{Q}_\tau\big[v\big(\phi(x, y_i, w), w\big)|z_i\big].$$

If u is strictly increasing in z, we have

$$\begin{aligned}\mathbb{M}(v)(x, z_1) &= u(x, y_1, z_1) + \beta\mathrm{Q}_\tau\big[v\big(\phi(x, y_1, w), w\big)|z_1\big]\\ &< u(x, y_1, z_2) + \beta\mathrm{Q}_\tau\big[v\big(\phi(x, y_1, w), w\big)|z_1\big].\end{aligned}$$

If u is just weakly increasing, the above remains true with weak inequality (and the same is true below; for simplicity, we focus only in the case of strict inequality). By Assumption 4, ϕ is weakly increasing in z. Since v is weakly increasing in x and z, the function $h(w) \equiv v(\phi(x, y_1, w), w)$ is weakly increasing in w. Lemma A.12 implies that

$$\mathrm{Q}_\tau\big[v\big(\phi(x, y_1, w), w\big)|z_1\big] \le \mathrm{Q}_\tau\big[v\big(\phi(x, y_1, w), w\big)|z_2\big],$$

which gives

$$\mathbb{M}(v)(x, z_1) < u(x, y_1, z_2) + \beta Q_\tau\big[v\big(\phi(x, y_1, w), w\big)|z_2\big].$$

From Assumption 4, $\Gamma(x, z_1) \subseteq \Gamma(x, z_2)$, i.e., $y_1 \in \Gamma(x, z_2)$. Optimality thus implies that

$$\begin{aligned} u(x, y_1, z_2) + \beta Q_\tau\big[v\big(\phi(x, y_1, w), w\big)|z_2\big] &\le u(x, y_2, z_2) + \beta Q_\tau\big[v\big(\phi(x, y_2, w), w\big)|z_2\big] \\ &= \mathbb{M}(v)(x, z_2). \end{aligned}$$

Therefore, $\mathbb{M}(v)(x, z_1) < \mathbb{M}(v)(x, z_2)$, which shows strict increasingness in z. □

Proof of Theorem 3.14. That $\bar{v}$ is increasing with respect to x was already proved in Theorem 3.13. The argument for the similar property with respect to z is analogous: Let $\mathcal{C}' \subset \mathcal{C}$ be the set of the bounded and continuous functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$, which are nondecreasing in z; $\mathcal{C}'$ is again a closed subset of $\mathcal{C}$. Let $\mathcal{C}'' \subset \mathcal{C}'$ be the set of strictly increasing functions z. Lemma A.13 shows that $\mathbb{M}(\mathcal{C}') \subset \mathcal{C}''$ if u is strictly increasing in z and $\mathbb{M}(\mathcal{C}') \subset \mathcal{C}'$ if u is only weakly increasing. Thus, the fixed-point of $\mathbb{M}$ has the stated properties, which concludes the proof. □

A.5 *Proofs of Section 3.7*

Proof of Theorem 3.15. As we have done in the proof of Theorem 3.13, we will present the arguments just for the strict concavity case. For weak concavity, the same argument works with weak inequalities where strict inequalities appear below. We organize the proof in a series of lemmas. It is convenient to introduce the following notation. Let $\mathcal{C}' \subset \mathcal{C}$ be the set of the bounded and continuous functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$, which are concave in x and nondecreasing in both x and z. It is easy to see that $\mathcal{C}'$ is a closed subset of $\mathcal{C}$. Let $\mathcal{C}'' \subset \mathcal{C}'$ be the set of strictly concave functions in x and strictly increasing in both x and z. If we show that $\mathbb{M}(\mathcal{C}') \subset \mathcal{C}''$, then the fixed point of $\mathbb{M}$ will be strictly concave in x, as well as strictly increasing in both x and z (see, for instance, Stokey, Lucas, and Prescott (1989, Corollary 1, p. 52)).

Lemma A.14. *Under the assumptions of Theorem 3.15,* $\mathbb{M}(\mathcal{C}') \subseteq \mathcal{C}'$. *If u is strictly concave,* $\mathbb{M}(\mathcal{C}')$ *is also strictly concave.*

Proof. As commented above, we will prove the result just with strict conditions; the weak conditions follow by using weak inequalities. Let $\alpha \in (0, 1)$, $v \in \mathcal{C}'$ and consider $x_0, x_1 \in \mathcal{X}$, $x_0 \neq x_1$. For $i = 0, 1$, let $y_i \in \Gamma(x_i, z)$ attain the maximum, i.e.,

$$\mathbb{M}(v)(x_i, z) = u(x_i, y_i, z) + \beta Q_\tau\big[v\big(\phi(x_i, y_i, w), w\big)|z\big].$$

Let $x_\alpha \equiv \alpha x_0 + (1-\alpha)x_1$ and $y_\alpha \equiv \alpha y_0 + (1-\alpha)y_1$. Hence,

$$\begin{aligned} \alpha\mathbb{M}v(x_0, z) + (1-\alpha)\mathbb{M}v(x_1, z) &= \alpha\big\{u(x_0, y_0, z) + \beta Q_\tau\big[v\big(\phi(x_0, y_0), w\big), w)|z\big]\big\} \\ &\quad + (1-\alpha)\big\{u(x_1, y_1, z) + \beta Q_\tau\big[v\big(\phi(x_1, y_1, w), w\big)|z\big]\big\} \end{aligned}$$

$$
\begin{aligned}
&< u(x_\alpha, y_\alpha, z) + \beta\big\{\mathrm{Q}_\tau\big[\alpha v\big(\phi(x_0, y_0, w), w\big)|z\big] \\
&\quad + \mathrm{Q}_\tau\big[(1-\alpha)v\big(\phi(x_1, y_1, w), w\big)|z\big]\big\} \\
&= u(x_\alpha, y_\alpha, z) + \beta\mathrm{Q}_\tau\big[\alpha v\big(\phi(x_0, y_0, w), w\big) \\
&\quad + (1-\alpha)v\big(\phi(x_1, y_1, w), w\big)|z\big] \\
&\leq u(x_\alpha, y_\alpha, z) + \beta\mathrm{Q}_\tau\big[v\big(\phi(x_\alpha, y_\alpha, w), w\big)|z\big] \\
&\leq \mathbb{M}v(x_\alpha, z),
\end{aligned} \tag{81}
$$

where the first inequality is due to the strict concavity of u in the first two variables. The equality in (81) is justified by Proposition A.2; since v is increasing in both variables and ϕ is increasing in the last variable, $v(\phi(x, y, w), w)$ is both increasing and continuous on w, so comonotonicity applies. The second inequality follows from concavity in x of v and in (x, y) of ϕ, as well as the fact that quantiles preserve order; see de Castro and Galvao (2019, Lemma A.1(vi)). The last inequality follows from Assumption 6 and the definition of $\mathbb{M}(v)$. This proves that $\mathbb{M}v$ is strictly concave in x when $v \in \mathcal{C}'$. □

We conclude the proof of Theorem 3.15 by showing that the policy correspondence (78) is single-valued and continuous.[42]

Lemma A.15. *If u is strictly concave on y and $\bar{v}$ is concave in x or $\bar{v}$ is strictly concave in x, then the optimal correspondence $\Upsilon(x, z)$ is single-valued.*

Proof. For an absurd, assume that there were $y \neq y'$ in $\Upsilon(x, z)$, i.e.,

$$
\bar{v}(x, z) = u(x, y, z) + \beta\mathrm{Q}_\tau\big[\bar{v}\big(\phi(x, y, w), w\big)|z\big] = u\big(x, y', z\big) + \beta\mathrm{Q}_\tau\big[\bar{v}\big(\phi\big(x, y', w\big), w\big)|z\big].
$$

Let $y_\alpha \equiv \alpha y + (1-\alpha)y'$. By Assumption 6, $y_\alpha \in \Gamma(x, z)$. Hence,

$$
\begin{aligned}
\bar{v}(x, z) &= \alpha\bar{v}(x, z) + (1-\alpha)\bar{v}(x, z) \\
&= \alpha\big\{u(x, y, z) + \beta\mathrm{Q}_\tau\big[\bar{v}\big(\phi(x, y, w), w\big)|z\big]\big\} \\
&\quad + (1-\alpha)\big\{u\big(x, y', z\big) + \beta\mathrm{Q}_\tau\big[\bar{v}\big(\phi\big(x, y', w\big), w\big)|z\big]\big\} \\
&< u(x, y_\alpha, z) + \beta\big\{\mathrm{Q}_\tau\big[\alpha\bar{v}\big(\phi(x, y, w), w\big)|z\big] + \mathrm{Q}_\tau\big[(1-\alpha)\bar{v}\big(\phi\big(x, y', w\big), w\big)|z\big]\big\} \\
&= u(x, y_\alpha, z) + \beta\mathrm{Q}_\tau\big[\alpha\bar{v}\big(\phi(x, y, w), w\big) + (1-\alpha)\bar{v}\big(\phi\big(x, y', w\big), w\big)|z\big] \\
&\leq u(x, y_\alpha, z) + \beta\mathrm{Q}_\tau\big[\bar{v}\big(\phi(x, y_\alpha, w), w\big)|z\big] \\
&\leq \bar{v}(x, z),
\end{aligned} \tag{82}
$$

where the first inequality is due to the strict concavity of u in the first two variables. The following equality in (82) is justified by the same argument that established (81) in Lemma A.14. The second inequality follows from concavity in x of $\bar{v}$ (by Lemma A.14) and in (x, y) of ϕ, as well as in de Castro and Galvao (2019, Lemma A.1(vi)). The last

[42] With weak concavity, the inequalities are weak and the proof establishes that Υ is convex.

inequality follows from Assumption 6 and the definition of $\bar{v}$. This contradiction proves that the policy correspondence given by (78) is single-valued. Lemma A.10 shows that the correspondence is upper semicontinuous. Since it is single-valued, it is continuous as a function. □

A.6 *Proofs of Section 3.8*

Proof of Theorem 3.16. The proof follows from an easy adaptation of Benveniste and Scheinkman (1979)'s argument, as developed by de Castro and Galvao (2019). We will reproduce the argument here for readers' convenience.

Let v be the fixed point of $\mathbb{M}:\mathcal{C}\to\mathcal{C}$, which exists by Theorem 3.11. Let $y^*:\mathcal{X}\times\mathcal{Z}\to\mathcal{Y}$ be a measurable selection of $\Upsilon:\mathcal{X}\times\mathcal{Z}\to\mathcal{Y}$, i.e., $y^*(x,z)\in\Upsilon(x,z)$. A measurable selection exists because of Aliprantis and Border (2006, Theorem 18.19, p. 605). Thus, for all (x,z),

$$v(x,z)=u\big(x,y^*(x,z),z\big)+\beta \mathrm{Q}_\tau\big[v\big(\phi\big(x,y^*(x,z),w\big),w\big)|z\big].$$

Recall that v is concave by Theorem 3.15. Fix x_0 in the interior of X and define

$$\bar{w}(x,z)\equiv u\big(x,y^*(x_0,z),z\big)+\beta \mathrm{Q}_\tau\big[v\big(\phi\big(x_0,y^*(x_0,z),w\big),w\big)|z\big].$$

From the optimality of v, for a neighborhood of x_0, we have $\bar{w}(x,z)\le v(x,z)$, with equality at $x=x_0$, which implies $\bar{w}(x,z)-\bar{w}(x_0,z)\le v(x,z)-v(x_0,z)$. Note that $\bar{w}$ is concave and differentiable in x because u is. Thus, any subgradient p of v at x_0 must satisfy

$$p\cdot(x-x_0)\ge v(x,z)-v(x_0,z)\ge \bar{w}(x,z)-\bar{w}(x_0,z).$$

Thus, p is also a subgradient of $\bar{w}$. But since $\bar{w}$ is differentiable, p is unique. Therefore, v is a concave function with a unique subgradient. Therefore, it is differentiable at x_0 (cf. Rockafellar (1970, Theorem 25.1, p. 242)) and its derivative with respect to x is the same as that of $\bar{w}$, i.e., for each $i=1,\dots,p$,[43]

$$\frac{\partial v}{\partial x_i}(x,z)=\frac{\partial \bar{w}}{\partial x_i}(x,z)=\frac{\partial u}{\partial x_i}\big(x,y^*(x,z),z\big),$$

as we wanted to show. □

Proof of Theorem 3.17. Let $\{h_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ be a sequence of real numbers converging to 0. Let $e_i=(0,\dots,0,1,0,\dots,0)\subset\mathbb{R}^p$ be the ith canonical basis vector. Since $x\in\mathcal{X}\subset\mathbb{R}^p$ is assumed to be interior, we can suppose that the h_n are small enough so that $x_n\equiv x+h_ne_i\in\mathcal{X}$ for all $n\in\mathbb{N}$. Clearly, we have $x_n\to x$. Let $y^*\in\Upsilon(x,z)$. Since Υ is lower hemicontinuous at (x,z), there exists $y_n\in\Upsilon(x_n,z)$ such that $y_n\to y^*$. Since $\mathcal{Y}$ is discrete, this means that $y_n=y^*$ for sufficiently high n. Therefore, without loss of generality, we can assume that $y^*\in\Upsilon(x,z)\cap\Upsilon(x_n,z)$ for all $n\in\mathbb{N}$. From (16),

$$v(x+h_ne_i,z)=u\big(x+h_ne_i,y^*,z\big)+\beta \mathrm{Q}_\tau\big[v\big(\phi\big(y^*,z'\big),z'\big)|z\big].$$

[43] Recall that $x\in\mathcal{X}\subset\mathbb{R}^p$ from Assumption 3.

Since $v(x, z) = u(x, y^*, z) + \beta Q_\tau[v(\phi(y^*, z'), z')|z]$, we obtain

$$\lim_{n\to\infty} \frac{v(x + h_n e_i, z) - v(x, z)}{h_n} = \lim_{n\to\infty} \frac{u(x + h_n e_i, y^*, z) - u(x, y^*, z)}{h_n}$$
$$= \frac{\partial u}{\partial x_i}(x, y^*, z).$$

Since $\{h_n\}$ is an arbitrary sequence converging to 0, the proof is complete. □

A.7 *Proofs of Section 3.9*

Proof of Theorem 3.18. Let $g(x, y, z) \equiv u(x, y, z) + \beta Q_\tau[\bar{v}(\phi(y, w), w)|z]$ and $y^*(x, z)$ be an interior solution of the problem (16). Let $\tilde{v}(y, w) = \bar{v}(\phi(y, w), w)$. Observe that $\tilde{v}$ is weakly increasing in w, differentiable in its first variable and for $0 < y_i' - y_i < \epsilon$, for some small $\epsilon > 0$,

$$\begin{aligned}
&\tilde{v}(y_i', y_{-i}, w) - \tilde{v}(y_i, y_{-i}, w) \\
&\quad = \int_{y_i}^{y_i'} \frac{\partial \tilde{v}}{\partial y_i}(\alpha, y_{-i}, w)\, d\alpha \\
&\quad = \int_{y_i}^{y_i'} \frac{\partial \bar{v}}{\partial x}\big(\phi(\alpha, y_{-i}, w), w\big) \cdot \frac{\partial \phi}{\partial y_i}(\alpha, y_{-i}, w)\, d\alpha \\
&\quad = \int_{y_i}^{y_i'} \frac{\partial u}{\partial x}\big(\phi(\alpha, y_{-i}, w), y^*\big(\phi(\alpha, y_{-i}, w), w\big), w\big) \cdot \frac{\partial \phi}{\partial y_i}(\alpha, y_{-i}, w)\, d\alpha,
\end{aligned}$$

where we have applied the chain rule in the second equality, and Theorem 3.16 in the third. Thus, the difference in the first line is weakly increasing in w, because by hypothesis, the integrand in the last line is. Therefore, the assumptions of Proposition 3.19 from de Castro and Galvao (2019) are satisfied and we conclude that $\frac{\partial Q_\tau}{\partial y_i}[\tilde{v}(y, w)] = Q_\tau[\frac{\partial \tilde{v}}{\partial y_i}(y, w)]$. Since u is differentiable in y, so is g. Since $y^*(x, z)$ is interior, the following first-order condition holds:

$$\frac{\partial g}{\partial y_i}\big(x, y^*(x, z), z\big) = \frac{\partial u}{\partial y_i}\big(x, y^*(x, z), z\big) + \beta Q_\tau\left[\frac{\partial \tilde{v}}{\partial y_i}\big(y^*(x, z), w\big)|z\right] = 0.$$

Now we apply Theorem 3.16 and its expression: $\frac{\partial \bar{v}}{\partial x}(x, z) = \frac{\partial u}{\partial x}(x, y^*(x, z), z)$, together with the chain rule, to conclude that

$$\begin{aligned}
&\frac{\partial u}{\partial y_i}\big(x, y^*(x, z), z\big) \\
&\quad + \beta Q_\tau\left[\frac{\partial u}{\partial x}\big(\phi\big(y^*(x, z), w\big), y^*\big(\phi\big(y^*(x, z), w\big), w\big), w\big) \cdot \frac{\partial \phi}{\partial y_i}\big(y^*(x, z), w\big)|z\right] = 0.
\end{aligned}$$

Now, we have just to put the notation of a sequence. For this, let $h = (x_t)$ denote an optimal path beginning at (x_0, z_0). Then the above equation can be rewritten, substituting x for x_t^h, $y^*(x, z)$ for y_t^h, $\phi(y^*(x, z), w)$ for x_{t+1}, $y^*(\phi(y^*(x, z), w), w)$ for y_{t+1}^h, z for z_t

and w for z_{t+1}, as

$$\frac{\partial u}{\partial y_i}(x_t^h, y_t^h, z_t) + \beta Q_\tau\left[\frac{\partial u}{\partial x}(x_{t+1}^h, y_{t+1}^h, z_{t+1}) \cdot \frac{\partial \phi}{\partial y_i}(y_t^h, z_{t+1})|z_t\right] = 0, \tag{83}$$

which we wanted to establish. □

A.8 *Proofs of Section 4.1*

We first generalize Assumption 9 to the following.

Assumption 12. *The following holds: (i) $\mathcal{X} = [\underline{x}, \overline{x}]$ for some $\overline{x} > \underline{x} \geq 0$; (ii) $\mathcal{Z} = [\underline{z}, \overline{z}]$, with $\overline{z} > \underline{z} > 0$; (iii) $U : \mathcal{X} \to \mathbb{R}$ is C^2, $U' > 0$, $U'' < 0$; (iv) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ is defined by $\phi(x, y, z) = \phi(x, (c, x'), z) = x'$; and (v) $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} = \mathcal{X} \times \mathcal{X}$ is given by $\Gamma(x, z) \equiv \{(c, x') \in \mathcal{X} \times \mathcal{X} : c + x' \leq \max\{xz, 2\underline{x}\}\}$.*[44]

Proof of Theorem 4.1 with Assumption 12 in place of Assumption 9. For the existence of the value function, it is sufficient to check that Assumption 2 holds and apply Theorem 3.11. Since $y = (c, x') \in \mathcal{Y} = \mathcal{X} \times \mathcal{X}$, $U : \mathcal{X} \to \mathbb{R}$ is C^2, $\mathcal{X}$ is compact, then $u(x, y, z) = u(x, (c, x'), z) = U(c)$ is continuous and bounded. The function $\phi(x, y, z) = \phi(x, (c, x'), z) = x'$ is also continuous and bounded. The correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$ defined by $\Gamma(x, z) \equiv \{(c, x') \in \mathcal{X} \times \mathcal{X} : c + x' \leq \max\{xz, 2\underline{x}\}\}$ is continuous, with nonempty, compact values. Then Assumption 2 holds and there exists a value function that satisfies the functional equation (21).

Assumption 6(i) is satisfied because $\mathcal{X}$ and $\mathcal{Y} = \mathcal{X} \times \mathcal{X}$ are convex. The function $u(x, y, z) = u(x, (c, x'), z) = U(c)$ is increasing, but not strictly increasing, in the first and last variables and it is concave, but not strictly concave in the first two variables. Also, $\varphi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ defined by $\phi(x, y, z) = \phi(x, (c, x'), z) = x'$ is nondecreasing and concave in all variables. Thus, Assumption 6(ii) is satisfied. Assumption 6(iii) is also satisfied.

Assumption 7 is trivially satisfied since $\mathcal{Z} = [\underline{z}, \overline{z}] \subset \mathbb{R}_{++}$. Assumption 8(i) holds since $u(x, y, z) = u(x, (c, x'), z) = U(c)$ is constant with x, and thus, C^1. Assumption 8(ii) holds since $\varphi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ defined by $\phi(x, y, z) = \phi(x, (c, x'), z) = x'$ does not depend on x.

Thus, the other claims in Theorem 4.1 follow from Theorems 3.15, 3.16, and 3.18, although the Euler equation needs some further work. In the above context, the Euler equation cannot be applied because it refers to interior points, and the optimal choices of c and the next period x' will not be interior to $\Gamma(x, z)$, but in its boundary. We can change the definition of $\mathcal{Y}$ and $\Gamma(x, z)$ to remedy this. Let $\mathcal{Y} = \mathcal{X}$ and $y = x'$, i.e., the DM chooses directly the next state. To keep the boundary limits, consider the function $l : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $l(\alpha) = \min\{\max\{\alpha, 2\underline{x}\}, 2\overline{x}\}$. Thus, if $xz \in (2\underline{x}, 2\overline{x})$, $l(xz) = xz$, but it is equal to $2\underline{x}$ if $xz \leq 2\underline{x}$ and to $2\overline{x}$ if $xz \geq 2\overline{x}$. By the strict monotonicity of U, $c = l(xz) - x' = l(xz) - y$ and we can redefine $u(x, y, z) = U(l(xz) - y)$, $\phi(x, y, z) = y$

[44] The requirement $c + x' \leq \max\{xz, 2\underline{x}\}$ implies that $(c, x') = (\underline{x}, \underline{x})$ is still in the budget set even if $xz < 2\underline{x}$. This guarantees that $\Gamma(x, z)$ is never empty.

and $\Gamma(x,z) \equiv \{y \in \mathcal{X} : l(xz) - y \in \mathcal{X}\}$. For interior points, $xz \in (2\underline{x}, 2\overline{x})$, and the Euler equation (24) is easily obtained. □

Proof of Corollary 4.2. It follows immediately from Theorem 4.1. □

Proof of Theorem 4.3. We want to show that $\bar{v} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ given by (29) is a fixed point of $\mathbb{M} : \mathcal{C} \to \mathcal{C}$:

$$\mathbb{M}(\bar{v})(x,z) = \max_{(c,x')\in\Gamma(x,z)} \left\{\frac{c^{1-\gamma}}{1-\gamma} + \beta \mathrm{Q}_\tau\big[\bar{v}(x',w)|z_t\big]\right\},$$

where $\mathcal{X} = [0,+\infty)$ and $\Gamma(x,z) = \{(c,x') \in \mathcal{X}^2 : c + x' \le xz\}\}$.

Let us analyze this maximization problem. If $xz = 0$, then $c = x' = \underline{x} = 0$ is the only choice possible, i.e., $\Gamma(x,z) = \{(0,0)\}$. Assume from now on that $xz > 0$. Since the utility is strictly increasing, it is not possible that $c + x' < xz$, since in this case we could increase consumption to obtain a higher utility. Therefore, $c + x' = xz$. Thus, we can define $t = \frac{x'}{x}$ so that $c = x(z-t)$ and $t \in [0,z]$. Substituting (29) into the above expression of $\mathbb{M}$, we obtain

$$\begin{aligned}\mathbb{M}(\bar{v})(x,z) &= \sup_{t\in[0,z]}\left\{\frac{x^{1-\gamma}(z-t)^{1-\gamma}}{1-\gamma} + \beta \mathrm{Q}_\tau\left[\frac{x^{1-\gamma}t^{1-\gamma}}{1-\gamma} w^{1-\gamma}\big(1+R(w)\big)^\gamma|z\right]\right\}\\ &= x^{1-\gamma}\cdot \sup_{t\in[0,z]}\left\{\frac{(z-t)^{1-\gamma}}{1-\gamma} + \beta\frac{t^{1-\gamma}}{1-\gamma}\mathrm{Q}_\tau\big[w^{1-\gamma}\big(1+R(w)\big)^\gamma|z\big]\right\}.\end{aligned}$$

Let us define $q : \mathcal{Z} \to \mathbb{R}$ by

$$q(z) \equiv \mathrm{Q}_\tau\big[w^{1-\gamma}\big(1+R(w)\big)^\gamma|z\big]. \tag{84}$$

We are interested in the maximization problem $\max_{t\in[0,z]} v(t)$ where $v : [0,z] \to \mathbb{R}$ is

$$v(t) \equiv \frac{(z-t)^{1-\gamma}}{1-\gamma} + \beta\frac{t^{1-\gamma}}{1-\gamma}q(z). \tag{85}$$

The first-order condition is $v'(t) = -(z-t)^{-\gamma} + t^{-\gamma}\beta q(z) = 0$, which leads to the optimal,

$$t^* = \frac{z\big[\beta q(z)\big]^{\frac{1}{\gamma}}}{1+\big[\beta q(z)\big]^{\frac{1}{\gamma}}}. \tag{86}$$

Notice that $t^* \in (0,z)$ if $z, q(z) > 0$. If $t \in [0,z)$, then $v''(t) = -\gamma[(z-t)^{-\gamma-1} + t^{-\gamma-1}\beta q(z)] < 0$, which implies that t^* given by (86) is optimal. Substituting (86) into (85), we obtain

$$\begin{aligned}v(t^*) &= \frac{z^{1-\gamma}}{(1-\gamma)\big\{1+\big[\beta q(z)\big]^{\frac{1}{\gamma}}\big\}^{1-\gamma}} + \frac{\beta q(z)}{(1-\gamma)}\cdot\frac{z^{1-\gamma}\big[\beta q(z)\big]^{\frac{1-\gamma}{\gamma}}}{\big\{1+\big[\beta q(z)\big]^{\frac{1}{\gamma}}\big\}^{1-\gamma}}\\ &= \frac{z^{1-\gamma}}{(1-\gamma)}\big\{1+\big[\beta q(z)\big]^{\frac{1}{\gamma}}\big\}^{\gamma}.\end{aligned}$$

Therefore,

$$\mathbb{M}(\bar{v})(x,z)=\frac{(xz)^{1-\gamma}}{1-\gamma}\cdot\left\{1+\left[\beta q(z)\right]^{\frac{1}{\gamma}}\right\}^{\gamma}.$$

Thus, we conclude that $\bar{v}$ is a fixed point of $\mathbb{M}$, i.e., $\bar{v}=\mathbb{M}(\bar{v})$, if we establish that $\{1+[\beta q(z)]^{\frac{1}{\gamma}}\}^{\gamma}=[1+R(z)]^{\gamma}$, i.e.,

$$R(z)=\beta^{\frac{1}{\gamma}}\left[q(z)\right]^{\frac{1}{\gamma}}=\beta^{\frac{1}{\gamma}}\left\{\mathrm{Q}_{\tau}\left[w^{1-\gamma}\left(1+R(w)\right)^{\gamma}|z\right]\right\}^{\frac{1}{\gamma}}. \tag{87}$$

Using the commutability with monotonic function (48), this is equivalent to

$$R(z)=\beta^{\frac{1}{\gamma}}\mathrm{Q}_{\tau}\left[w^{\frac{1-\gamma}{\gamma}}\left(1+R(w)\right)|z\right].$$

Recall that $R(z)=\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}[r_{\tau,s}(z)]^{\frac{1-\gamma}{\gamma}}$, where $r_{\tau,s}(z)$ is defined recursively by (27), i.e., $r_{\tau,0}(z)=1$, and $r_{\tau,s}(z)=r_{\tau,s-1}(\mathrm{Q}_{\tau}[w|z])\cdot\mathrm{Q}_{\tau}[w|z]$ for $s\geq 1$. Notice that all $r_{\tau,s}(z)$ are nondecreasing and continuous in z. Therefore, they are all comonotonic. Moreover, they are all strictly positive by Assumption 10(vi). From Proposition A.2, we have

$$\mathrm{Q}_{\tau}\left[wr_{\tau,s-1}(w)|z\right]=\mathrm{Q}_{\tau}[w|z]r_{\tau,s-1}\left(\mathrm{Q}_{\tau}[w|z]\right)=r_{\tau,s}(z).$$

Using these properties and expressions, we obtain

$$\begin{aligned}
\beta^{\frac{1}{\gamma}}\mathrm{Q}_{\tau}\left[w^{\frac{1-\gamma}{\gamma}}\left(1+R(w)\right)|z\right]&=\beta^{\frac{1}{\gamma}}\mathrm{Q}_{\tau}\left[w^{\frac{1-\gamma}{\gamma}}\left(1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left[r_{\tau,s}(w)\right]^{\frac{1-\gamma}{\gamma}}\right)\middle|z\right]\\
&=\mathrm{Q}_{\tau}\left[\beta^{\frac{1}{\gamma}}w^{\frac{1-\gamma}{\gamma}}+\beta^{\frac{1}{\gamma}}w^{\frac{1-\gamma}{\gamma}}\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left[r_{\tau,s}(w)\right]^{\frac{1-\gamma}{\gamma}}|z\right]\\
&=\mathrm{Q}_{\tau}\left[\beta^{\frac{1}{\gamma}}w^{\frac{1-\gamma}{\gamma}}|z\right]+\mathrm{Q}_{\tau}\left[\sum_{s=1}^{\infty}\beta^{\frac{s+1}{\gamma}}\left[wr_{\tau,s}(w)\right]^{\frac{1-\gamma}{\gamma}}|z\right]\\
&=\beta^{\frac{1}{\gamma}}r_{\tau,1}(z)^{\frac{1-\gamma}{\gamma}}+\sum_{s=1}^{\infty}\beta^{\frac{s+1}{\gamma}}\left\{\mathrm{Q}_{\tau}\left[wr_{\tau,s}(w)|z\right]\right\}^{\frac{1-\gamma}{\gamma}}\\
&=\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left[r_{\tau,s}(z)\right]^{\frac{1-\gamma}{\gamma}},
\end{aligned}$$

which is just $R(z)$, as we wanted to verify.

Finally, we observe that the optimal savings and consumption decisions are determined by t^* from (86), i.e.,

$$x'=xt^*=\frac{xz\left[\beta q(z)\right]^{\frac{1}{\gamma}}}{1+\left[\beta q(z)\right]^{\frac{1}{\gamma}}}=\frac{xzR(z)}{1+R(z)}=xzS(z),$$

where we used using (87) and the definition (28) of $S(z)$. Since $c=xz-x'$, $c=[1-S(z)]xz$, we obtain (30).

Finally, given a sequence of shocks $\{z_t\}_{t=1}^{\infty}$, since $c_t = c(x_t, z_t) = [1 - S(z_t)]x_t z_t$, $x_{t+1} = S(z_t)x_t z_t$ and $c_{t+1} = c(x_{t+1}, z_{t+1}) = [1 + S(z_{t+1})]x_{t+1}z_{t+1}$, we have

$$\frac{c_{t+1}}{c_t} = \frac{[1 + S(z_{t+1})]S(z_t)x_t z_t z_{t+1}}{[1 - S(z_t)]x_t z_t} = z_{t+1}[1 + S(z_{t+1})]\frac{S(z_t)}{1 - S(z_t)},$$

which proves (31), since $\frac{S(z_t)}{1-S(z_t)} = R(z_t)$ by simple manipulations. □

A.8.1 *Uniqueness of the value function* In this section, we will prove a stronger version of Theorem 4.4. For this, consider the following variant of the transversality condition.

DEFINITION A.16. We say that a function $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ satisfies the quantile transversality condition (QTC) if for any optimal plan $h \in H$ starting at any $(x, z) \in \mathcal{X} \times \mathcal{Z}$, and $\epsilon > 0$, there exists n_ϵ such that $n \geq n_\epsilon$ implies that

$$-\epsilon < \beta^n \mathrm{Q}_\tau^n\big[v\big(\phi\big(x_n^h, y_n^h, z_{n+1}\big), z_{n+1}\big)|z_1\big] < \epsilon. \tag{88}$$

We have the following.

THEOREM A.17 (Uniqueness of the value function). *Let Assumptions 1, 5, and 10 hold. Let $\bar{v} : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}$ be the function defined by (29). Suppose that $v : \mathcal{X} \times \mathcal{Z} \to \overline{\mathbb{R}}$ is a fixed point of $\mathbb{M}$ defined by (21) and either: (i) the CQTC; see (14) or (ii) v and all optimal plans are weakly increasing and continuous in all its arguments, and v satisfies QTC. Then $v = \bar{v}$.*

The proof of Theorem A.17 is organized in many steps. The first one is to argue that Assumption 0 is satisfied, as Lemma A.18 establishes.

LEMMA A.18. *Let Assumptions 1, 5, and 10 hold. Then Assumption 0 is satisfied.*

PROOF. From Assumption 10, $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} = \mathcal{X} \times \mathcal{X}$ is given by $\Gamma(x, z) = \{(c, x') \in \mathcal{X} \times \mathcal{X} : c + x' \leq xz\}$. Therefore, $\Gamma(x, z) \neq \emptyset$ and Γ has measurable selections. Recall that

$$\begin{aligned}
&V^n(h, x, z)\\
&= \mathrm{Q}_\tau^n\big[S^{h,n}(x, \cdot)|z\big] = \mathrm{Q}_\tau^n\left[\sum_{t=0}^{n} \beta^t u\big(x_{t+1}^h, y_{t+1}^h, z_{t+1}\big)\middle| z\right]\\
&= \mathrm{Q}_\tau\left[\cdots \mathrm{Q}_\tau\left[\mathrm{Q}_\tau\left[\sum_{t=0}^{n-1} \beta^t u\big(x_{t+1}^h, y_{t+1}^h, z_{t+1}\big) + \beta^n u\big(x_{n+1}^h, y_{n+1}^h, z_{n+1}\big)\middle| z^n\right]\middle| z^{n-1}\right]\cdots\middle| z\right]\\
&= \mathrm{Q}_\tau\left[\cdots \mathrm{Q}_\tau\left[\sum_{t=0}^{n-1} \beta^t u\big(x_{t+1}^h, y_{t+1}^h, z_{t+1}\big) + \beta^n \mathrm{Q}_\tau\big[u\big(x_{n+1}^h, y_{n+1}^h, z_{n+1}\big)|z^n\big]\middle| z^{n-1}\right]\cdots\middle| z\right].
\end{aligned}$$

From (22) and Assumption 10(i), we have, for $\gamma \in (0, 1)$,

$$0 \leq u\big(x_{n+1}^h, y_{n+1}^h, z_{n+1}\big) = U(c_{n+1}) = \frac{c_{n+1}^{1-\gamma}}{1-\gamma}. \tag{89}$$

Since $\Gamma(x,z)=\{(c,x')\in\mathcal{X}\times\mathcal{X}:c+x'\le xz\}$, $c_{n+1}\le x^h_{n+1}z_{n+1}\le x^h_n z_n z_{n+1}$, and continuing in this manner, we obtain $c_{n+1}\le x^h_1 z_1\cdots z_n\cdot z_{n+1}$. Moreover,

$$\beta^n \mathrm{Q}_\tau\big[u\big(x^h_{n+1},y^h_{n+1},z_{n+1}\big)|z^n\big]\le \frac{\beta^n\big(x^h_1 z_1\cdots z_n\big)^{1-\gamma}}{1-\gamma}\big(\mathrm{Q}_\tau[z_{n+1}|z_n]\big)^{1-\gamma}.$$

From Assumption 10(vi) and (vii), there exists $\tilde{z}>0$ such that $0<\mathrm{Q}_\tau[w|z]\le\tilde{z}$, for all $z\in\mathcal{Z}$ and $\ell\equiv\beta\tilde{z}^{1-\gamma}<1$. Substituting ℓ in the above inequality, we obtain

$$V^n(h,x,z)\le \mathrm{Q}_\tau\left[\cdots\mathrm{Q}_\tau\left[\sum_{t=0}^{n-1}\beta^t u\big(x^h_{t+1},y^h_{t+1},z_{t+1}\big)+\frac{\beta^{n-1}\big(x^h_1 z_1\cdots z_n\big)^{1-\gamma}}{1-\gamma}l|z^{n-1}\right]\cdots\Bigg|z\right].$$

By repeating the same reasoning repeatedly, we conclude that

$$V^n(h,x,z)\le\frac{(xz)^{1-\gamma}}{1-\gamma}\sum_{t=0}^{n}l^t=\frac{(xz)^{1-\gamma}}{(1-\gamma)(1-l)}.$$

Notice that, from the fact that each term in the sum of $V^n(h,x,z)$ is nonnegative, as observed by (89), the sequence $V^n(h,x,z)$ is nondecreasing and bounded. Therefore, it is convergent. This concludes the proof of Assumption 0. □

The following result will be useful below.

Lemma A.19. *Let Assumptions 1, 5, and 10 hold. Assume that $f:\mathcal{Z}^n\to\mathbb{R}_+$ and $g:\mathcal{Z}^{n+1}\to\mathbb{R}_+$ are weakly increasing and continuous in all arguments. Then*

$$\mathrm{Q}^{n-1}_\tau\big[f\big(z^n\big)+\beta^n\mathrm{Q}_\tau\big[g\big(z^{n+1}\big)|z_n\big]|z_1\big]=\mathrm{Q}^{n-1}_\tau\big[f\big(z^n\big)|z_1\big]+\beta^n\mathrm{Q}^n_\tau\big[g\big(z^{n+1}\big)|z_1\big].\tag{90}$$

Proof. The function $z_n\mapsto f(z^n)$ is weakly increasing. By Assumption 5 and Lemma A.12, $z_n\mapsto \mathrm{Q}_\tau[g(z^{n+1})|z_n]$ is also weakly increasing. Therefore, these two functions are comonotonic and Proposition A.2 implies that

$$\begin{aligned}&\mathrm{Q}_\tau\big[f\big(z^n\big)+\beta^n\mathrm{Q}_\tau\big[g\big(z^{n+1}\big)|z_n\big]|z_{n-1}\big]\\&\quad=\mathrm{Q}_\tau\big[f\big(z^n\big)|z_{n-1}\big]+\beta^n\mathrm{Q}_\tau\big[\mathrm{Q}_\tau\big[g\big(z^{n+1}\big)|z_n\big]|z_{n-1}\big].\end{aligned}$$

Again by Assumption 5 and Lemma A.12, the functions $z_{n-1}\mapsto\mathrm{Q}_\tau[f(z^n)|z_{n-1}]$ and $z_{n-1}\mapsto\mathrm{Q}^2_\tau[g(z^{n+1})|z_{n-1}]$ are weakly increasing, and hence, comonotonic. As before,

$$\begin{aligned}&\mathrm{Q}_\tau\big[\mathrm{Q}_\tau\big[f\big(z^n\big)+\beta^n\mathrm{Q}_\tau\big[g\big(z^{n+1}\big)|z_n\big]|z_{n-1}\big]|z_{n-2}\big]\\&\quad=\mathrm{Q}_\tau\big[\mathrm{Q}_\tau\big[f\big(z^n\big)|z_{n-1}\big]|z_{n-2}\big]+\beta^n\mathrm{Q}_\tau\big[\mathrm{Q}_\tau\big[\mathrm{Q}_\tau\big[g\big(z^{n+1}\big)|z_n\big]|z_{n-1}\big]|z_{n-2}\big].\end{aligned}$$

Proceeding in this way, we obtain (90). □

Given $v:\mathcal{X}\times\mathcal{Z}\to\mathbb{R}$, recall from (15) that $G_v:\mathcal{X}\times\mathcal{Z}\to\mathcal{Y}$ denotes the optimal correspondence and that a plan obtained from G_v if there exists a sequence of selections $g_t:\mathcal{X}\times\mathcal{Z}\to\mathcal{Y}$ such that for all $t\in\mathbb{N}$ and all $(x,z)\in\mathcal{X}\times\mathcal{Z}$, $g_t(x,z)\in G_v(x,z)$ and

$h_t(x, z^t) = g_t(h_{t-1}(x, z^{t-1}), z_t)$. Moreover, a plan is optimal if it is obtained from G_v. Recall from (9) and (10) that $y_t^h = h_t(x_t^h, z^t)$ and $x_{t+1}^h = \phi(x_t^h, y_t^h, z_{t+1})$. Therefore, we may define the sequence of optimal states by functions $h_t^x(x, z^{t+1}) = \phi(x_t^h, y_t^h, z_{t+1}) = x_{t+1}^h$.

Lemma A.20. *Let Assumptions 1, 5, and 10 hold. Fix $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}_+$ and optimal plan h. If $z^{n+1} \mapsto v(h_t^x(x, z^{n+1}), z_{n+1})$ is weakly increasing and continuous, then*

$$\begin{aligned} &\mathrm{Q}_\tau^{n-1}\big[S^{h,n-1}(x_1, z^n) + \beta^n \mathrm{Q}_\tau\big[v\big(\phi\big(x_n^h, y_n^h, z_{n+1}\big), z_{n+1}\big)|z_n\big]|z_1\big] \\ &\quad = \mathrm{Q}_\tau^{n-1}\big[S^{h,n-1}(x_1, z^n)|z_1\big] + \beta^n \mathrm{Q}_\tau^n\big[v\big(\phi\big(x_n^h, y_n^h, z_{n+1}\big), z_{n+1}\big)|z_1\big], \end{aligned} \tag{91}$$

where $S^{h,n}(x, z^{n+1})$ is defined by (11).

Proof. The assumptions imply that the functions $f(z^n) \equiv S^{h,n-1}(x, z^n)$ and $g(z^{n+1}) \equiv v(h_t^x(x, z^{n+1}), z_{n+1})$ satisfy the assumptions of Lemma A.19. The conclusion follows from that lemma. □

Lemma A.21. *Let Assumptions 1, 5, and 10 hold. Let $v^* : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}$ be defined by (13) and let $v : \mathcal{X} \times \mathcal{Z} \to \overline{\mathbb{R}}$ satisfy QTC and $v = \mathbb{M}(v)$ for $\mathbb{M}$ defined by (21). Assume further that $z^{n+1} \mapsto v(h_t^x(x, z^{n+1}), z_{n+1})$ is weakly increasing and continuous for all n. Then $v = v^*$.*

Proof. It is sufficient to adapt the proof of Theorem 3.8. We can repeat everything in that proof up to (52), i.e., there exists n_1 such that $n \geq n_1$ implies (51) and (52), i.e.,

$$\begin{aligned} v(x_1, z_1) &\geq \mathrm{Q}_\tau^{n-1}\big[S^{h,n-1}(x_1, z^n) + \beta^n \mathrm{Q}_\tau\big[v\big(\phi\big(x_n^h, y_n^h, z_{n+1}\big), z_{n+1}\big)|z^n\big]|z_1\big] \\ &= \mathrm{Q}_\tau^{n-1}\big[S^{h,n-1}(x_1, z^n)|z_1\big] + \beta^n \mathrm{Q}_\tau^n\big[v\big(\phi\big(x_n^h, y_n^h, z_{n+1}\big), z_{n+1}\big)|z_1\big], \end{aligned}$$

where the equality comes from Lemma A.20. By the QTC, we conclude that there exists $n_2 \geq n_1$ such that $n \geq n_1$, $v(x_1, z_1) > V^n(h, x_1, z_1) - \epsilon$, which contradicts (51). The contradiction establishes (A) in the proof of Theorem 3.8.

The proof of (B) is the same up to equation (54), for an optimal plan $h \in H$, i.e.,

$$\begin{aligned} v(x_1, z_1) &= \mathrm{Q}_\tau^{n-1}\big[S^{h,n-1}(x_1, z^n) + \beta^n \mathrm{Q}_\tau\big[v\big(\phi\big(x_n^h, y_n^h, z_{n+1}\big), z_{n+1}\big)|z^n\big]|z_1\big] \\ &= \mathrm{Q}_\tau^{n-1}\big[S^{h,n-1}(x_1, z^n)|z_1\big] + \beta^n \mathrm{Q}_\tau^n\big[v\big(\phi\big(x_n^h, y_n^h, z_{n+1}\big), z_{n+1}\big)|z_1\big], \end{aligned}$$

where again the equality comes from Lemma A.20. By QTC, we conclude that there exists $n_2 \geq n_1$ such that $n \geq n_1$, $v(x_1, z_1) < V^n(h, x_1, z_1) + \epsilon$, but this contradicts (53). The contradiction establishes (B) and concludes the proof. □

Lemma A.22. *The function $\bar{v} : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}$, defined by (29), satisfies QTC (88).*

Proof. Footnote 24 argues that $R(z)$ converges. Indeed, if we denote $(\beta^{\frac{1}{\gamma}} \tilde{z}^{\frac{1-\gamma}{\gamma}})$ by $B \in (0, 1)$, that footnote shows that $R(z) \leq \sum_{s=1}^{\infty} B^s = \frac{B}{1-B}$. Therefore, for $(x, z) \in D$,

$$\begin{aligned} \bar{v}(x, z) &= \frac{1}{1-\gamma} \cdot (xz)^{1-\gamma} \cdot \big[1 + R(z)\big]^\gamma \leq \frac{1}{1-\gamma} \cdot (xz)^{1-\gamma} \cdot \left[1 + \frac{B}{1-B}\right]^\gamma \\ &= \frac{(xz)^{1-\gamma}}{(1-\gamma)(1-B)^\gamma}. \end{aligned} \tag{92}$$

Notice that $\bar{v} \geq 0$. Thus, to establish QTC (88), it is sufficient to show that there exists n_ϵ such that $n \geq n_\epsilon$ and

$$\beta^n \, \mathrm{Q}_\tau^n \Big[\, \bar{v} \Big(\phi(x_n^h, y_n^h, z_{n+1}), \; z_{n+1} \Big) \; \Big| \; z_1 \Big] < \epsilon. \tag{93}$$

By the definition of Γ, we have $x_{n+1}^h z_{n+1} \leq x_n^h z_n z_{n+1}$, and finally,

$$x_{n+1}^h \cdot z_{n+1} \leq x_1 \cdot z_1 \cdots z_n \cdot z_{n+1}. \tag{94}$$

Thus, the left hand side of (93) is smaller than

$$\begin{aligned} &\beta^n \frac{(x)^{1-\gamma}}{(1-\gamma)(1-B)^\gamma} \Big(z_1 \mathrm{Q}_\tau \Big[z_2 \cdots \; \mathrm{Q}_\tau \Big[z_n \mathrm{Q}_\tau \big[z_{n+1} \,|\, z_n \big] \; \Big| \, z_{n-1} \Big] \cdots \, \big| z_1 \Big] \Big)^{1-\gamma} \\ &\leq \beta^n \frac{(x_1)^{1-\gamma}}{(1-\gamma)(1-B)^\gamma} \Big(z_1 \mathrm{Q}_\tau \Big[z_2 \cdots \; \mathrm{Q}_\tau \Big[z_n \tilde{z} \, \Big| \, z_{n-1} \Big] \cdots \, \big| z_1 \Big] \Big)^{1-\gamma}, \end{aligned}$$

where we have used Assumption 10-(vi): there exists $\tilde{z} > 0$ such that $0 < \mathrm{Q}_\tau[w|z] \leq \tilde{z}$, for all $z \in \mathcal{Z}$. Proceeding this way, and using Assumption 10-(vii), i.e., $\ell = \beta \tilde{z}^{1-\gamma} < 1$, we obtain

$$\beta^n \mathrm{Q}_\tau^n \big[\bar{v} \big(\phi \big(x_n^h, y_n^h, z_{n+1} \big), z_{n+1} \big) | z_1 \big] \leq \frac{(x_1 z_1)^{1-\gamma}}{(1-\gamma)(1-B)^\gamma} (\ell)^n.$$

From this inequality, it is clear that we can find n_ϵ such that for all $n \geq n_\epsilon$, (93) holds. □

Proof of Theorem A.17. By Lemma A.22, $\bar{v}$ satisfies QTC. By Theorem 4.3, $\bar{v}$ is continuous and increasing in all variables and so is the optimal plan. By Lemma A.21, $\bar{v} = v^*$. Let v be a fixed point of $\mathbb{M}$. If v satisfies CQTC, $v = v^*$ by Theorem 3.8. On the other hand, if v satisfies (ii) of Theorem A.17, then the conditions of Lemma A.21 are met, and $v = v^*$. In any case, we conclude that $v = \bar{v}$, as we wanted to show. □

Remark A.23. Although we did not prove directly that $\bar{v}$ satisfies CQTC, a mild addition to Assumption 10 allows us to establish it; namely it is sufficient to add the requirement that $\mathcal{Z} \subset [0, \tilde{z}]$ for $\tilde{z}$ satisfying Assumption 10(vii). In this case, using (92), (94), and $\ell = \beta \tilde{z}^{1-\gamma} < 1$, we obtain

$$\beta^n \mathrm{Q}_\tau \big[v \big(\phi \big(x_n^h, y_n^h, z_{n+1} \big), z_{n+1} \big) | z^n \big] \leq \beta^n \frac{(x_1 \cdot z_1 \cdot z_2 \cdots z_n \tilde{z})^{1-\gamma}}{(1-\gamma)(1-B)^\gamma} \leq \frac{(x_1 z_1)^{1-\gamma}}{(1-\gamma)(1-B)^\gamma} \ell^n$$

Since $\ell < 1$ and $\bar{v} \geq 0$, this establishes (14).

A.8.2 *Log utility function* Our results can also be adapted to the case of an isoelastic utility function with $\gamma = 1$. Indeed, Corollary 4.2 can be adapted to yield exactly the same Euler equation (26), with $\gamma = 1$. Instead of repeating these results, we obtain directly the closed-form solutions for this case, parallel to Theorem 4.3.

THEOREM A.24. *Let Assumptions 1, 5, and 10 hold, with the following modifications:* $\gamma = 1$, $\mathcal{X} = (0, \infty)$; $\mathcal{Z} \subset \mathbb{R}_{++}$, $U(x) = \ln(x)$, *there exists* $\tilde{z} > 0$ *such that* $0 < \mathrm{Q}_\tau[w|z] \leq \tilde{z}$, $\forall z \in \mathcal{Z}$. *Let* $\bar{v} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *be defined by*

$$\bar{v}(x, z) \equiv \sum_{s=0}^{\infty} \frac{\beta^s}{1-\beta} \ln q_{\tau,s}(z) + \frac{\ln x}{1-\beta} + \frac{\ln\left[\beta^\beta (1-\beta)^{1-\beta}\right]}{(1-\beta)^2}, \tag{95}$$

where $q_{\tau,s}(z)$ *is given recursively by* $q_{\tau,0}(z) = z$ *and*

$$q_{\tau,s}(z) = q_{\tau,s-1}\big(\mathrm{Q}_\tau[w|z]\big) \quad \text{for } s \geq 1. \tag{96}$$

Then $\bar{v}$ *is a fixed point of* $\mathbb{M}$ *defined by (21). Moreover, the optimal policy is*

$$y^*(x, z) = \big(c, x'\big) = \big((1-\beta)xz, \beta xz\big). \tag{97}$$

PROOF OF THEOREM A.24. First, observe that since $\mathrm{Q}_\tau[w|z] \leq \tilde{z}$, $q_{\tau,0}(z) \leq \tilde{z}$ for all $z \in \mathcal{Z}$. By induction, $q_{\tau,s}(z) \leq \tilde{z}$, for all $s \in \mathbb{N}$ and $z \in \mathcal{Z}$. This shows that the infinite sum in (95) converges. Let $\bar{v}$ be given by (95). For conciseness, denote $\frac{\ln[\beta^\beta(1-\beta)^{1-\beta}]}{(1-\beta)^2}$ by C. Then

$$\begin{aligned}
\mathbb{M}\bar{v}(x, z) &= \sup_{y \in [0, xz]} \left\{ \ln(xz - y) + \beta \mathrm{Q}_\tau\left[\sum_{s=0}^{\infty} \frac{\beta^s}{1-\beta} \ln q_{\tau,s}\big(z'\big) + \frac{\ln y}{1-\beta} + C \Big| z \right] \right\} \\
&= \sup_{y \in [0, xz]} \left\{ \ln(xz - y) + \beta \left[\sum_{s=0}^{\infty} \frac{\beta^s}{1-\beta} \ln q_{\tau,s}\big(\mathrm{Q}_\tau\big[z'|z\big]\big) + \frac{\ln y}{1-\beta} + C \right] \right\} \\
&= \sum_{s=1}^{\infty} \frac{\beta^s}{1-\beta} \ln q_{\tau,s}(z) + \beta C + \sup_{y \in [0, xz]} \left\{ \ln(xz - y) + \frac{\beta}{1-\beta} \ln y \right\},
\end{aligned} \tag{98}$$

where we used Lemma A.1 in the second equality, since the $q_{\tau,s}(z)$ are increasing by a successive application of Lemma A.12; and in the third equality, we used the recursive relation (96). The first-order condition for the expression in brackets from (98) for optimal y is

$$\frac{1}{xz - y} = \frac{\beta}{1-\beta} \frac{1}{y},$$

which leads to the optimal savings policy $y = \beta xz$. Thus, consumption is $c = xz - x' = (1-\beta)xz$. Substituting these expressions into (98),

$$\begin{aligned}
\mathbb{M}\bar{v}(x, z) &= \sum_{s=1}^{\infty} \frac{\beta^s}{1-\beta} \ln q_{\tau,s}(z) + \beta C + \ln\big[(xz)(1-\beta)\big] + \frac{\beta}{1-\beta} \ln\big[(xz)\beta\big] \\
&= \sum_{s=1}^{\infty} \frac{\beta^s}{1-\beta} \ln q_{\tau,s}(z) + \beta C + \left(1 + \frac{\beta}{1-\beta}\right) \ln(xz) + \frac{(1-\beta)\ln(1-\beta) + \beta \ln(\beta)}{1-\beta} \\
&= \sum_{s=1}^{\infty} \frac{\beta^s}{1-\beta} \ln q_{\tau,s}(z) + \frac{\ln(xz)}{1-\beta} + \beta C + \frac{\ln\big[(\beta)^\beta (1-\beta)^{1-\beta}\big]}{(1-\beta)}
\end{aligned}$$

$$= \left[\sum_{s=1}^{\infty} \frac{\beta^s}{1-\beta} \ln q_{\tau,s}(z) + \frac{\ln(z)}{1-\beta}\right] + \frac{\ln(x)}{1-\beta} + \beta C + (1-\beta)C$$

$$= \sum_{s=0}^{\infty} \frac{\beta^s}{1-\beta} \ln q_{\tau,s}(z) + \frac{\ln(x)}{1-\beta} + C$$

$$= \bar{v}(x, z).$$

This concludes the proof. □

Now we note that equation (95) simplifies further when the shocks z are i.i.d.

Example A.25 (i.i.d.). Assume that the shocks are i.i.d. In this case, $q_{\tau,s}(z) = \mathrm{Q}_\tau[Z]$ for all $s \geq 1$. Hence, (95) can be written as

$$\bar{v}(x, z) = \sum_{s=1}^{\infty} \frac{\beta^s}{1-\beta} \ln \mathrm{Q}_\tau[Z] + \frac{\ln z}{1-\beta} + \frac{\ln x}{1-\beta} + \frac{\ln\left[(\beta)^\beta (1-\beta)^{1-\beta}\right]}{(1-\beta)^2} = \frac{\ln xz}{1-\beta} + \bar{\kappa},$$

where

$$\bar{\kappa} = \frac{\ln\left[\left(\mathrm{Q}_\tau[Z]\right)^\beta \beta^\beta (1-\beta)^{1-\beta}\right]}{(1-\beta)^2}$$

is a constant. ◊

Analogously, we can treat the case of τ-quantile martingale process.

Example A.26 (τ-quantile martingale process). When Z follows a τ-quantile martingale process (see equation (7)), the recursive functions from (96) are $q_{\tau,s}(z) = z$ for all s, so (95) takes the form

$$\bar{v}(x, z) = \sum_{s=0}^{\infty} \frac{\beta^s}{1-\beta} \ln z + \frac{\ln x}{1-\beta} + \frac{\ln\left[\beta^\beta (1-\beta)^{1-\beta}\right]}{(1-\beta)^2}$$

$$= \frac{\ln z}{(1-\beta)^2} + \frac{(1-\beta)\ln x}{(1-\beta)^2} + \frac{\ln\left[\beta^\beta (1-\beta)^{1-\beta}\right]}{(1-\beta)^2}$$

$$= \frac{\ln\left\{z x^{(1-\beta)} \beta^\beta (1-\beta)^{1-\beta}\right\}}{(1-\beta)^2}.$$

◊

A.8.3 *The right-continuous quantile* To deal with the case in which $\gamma > 1$, we need some additional definitions and properties. In particular, we need to define the τ-quantile* (or right-continuous quantile) as

$$\mathrm{Q}_\tau^*[X] = \sup\left\{\alpha \in \mathbb{R} : \Pr[X \leq \alpha] \leq \tau\right\}.$$

This definition allows to study $\mathrm{Q}_\tau[\alpha X]$ for $\alpha < 0$. We have the following.

Lemma A.27. *Let X be a random variable and $\tau \in (0, 1)$. Then*

$$\mathrm{Q}_\tau[X] = -\mathrm{Q}_{1-\tau}^*[-X] \tag{99}$$

Proof of Lemma A.27. Recall that whenever $A \subset \mathbb{R}$, $\inf A = -\sup(-A)$. Hence,

$$\begin{aligned}-\mathrm{Q}^*_{1-\tau}[-X] &= -\sup\{\alpha \in \mathbb{R}; P[-X \le \alpha] \le 1-\tau\} = \inf\{-\alpha \in \mathbb{R}; P[X \ge -\alpha] \le 1-\tau\}\\ &= \inf\{\alpha \in \mathbb{R}; P[X \ge \alpha] \le 1-\tau\} = \inf\{\alpha \in \mathbb{R}; 1-P[X \ge \alpha] \ge \tau\}\\ &= \inf\{\alpha \in \mathbb{R}; P[X < \alpha] \ge \tau\}.\end{aligned}$$

So, it suffices to prove that

$$\inf\{\alpha \in \mathbb{R}; P[X < \alpha] \ge \tau\} = \inf\{\alpha \in \mathbb{R}; P[X \le \alpha] \ge \tau\}, \tag{100}$$

since the right-hand side equals $\mathrm{Q}_\tau[X]$ by definition.

Let $A = \{\alpha \in \mathbb{R}; P[X < \alpha] \ge \tau\}$, $B = \{\alpha \in \mathbb{R}; P[X \le \alpha] \ge \tau\}$. Since $A \subset B$, we have $\inf B \le \inf A$. For a contradiction, suppose that $\inf B < \inf A$. Then there would be some $b \in B$ and $y \in \mathbb{R}$ such that $\inf B < b < y < \inf A$. Therefore,

$$\tau \le P[X \le b] \le P[X < y]. \tag{101}$$

On the other hand, $y < \inf A$ implies that $y \notin A$, so $P[X < y] < \tau$, which contradicts (101). This establishes (100), thus completing the proof. □

We have the following result concerning interchangeability between quantiles and monotone functions.

Lemma A.28. *Let* $\tau \in [0, 1]$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be increasing. Then*

$$\mathrm{Q}_\tau[g(X)] = g(\mathrm{Q}_\tau[X]) \quad \text{if } g \text{ is left-continuous} \tag{102}$$

and

$$\mathrm{Q}^*_\tau[g(X)] = g(\mathrm{Q}^*_\tau[X]) \quad \text{if } g \text{ is right-continuous.} \tag{103}$$

If, instead, $g : \mathbb{R} \to \mathbb{R}$ *is decreasing, then*

$$\mathrm{Q}_\tau[g(X)] = g(\mathrm{Q}^*_{1-\tau}[X]) \quad \text{if } g \text{ is right-continuous} \tag{104}$$

and

$$\mathrm{Q}^*_{1-\tau}[g(X)] = g(\mathrm{Q}_\tau[X]) \quad \text{if } g \text{ is left-continuous.} \tag{105}$$

Proof of Lemma A.28. Equation (102) is exactly Lemma A.2 from de Castro and Galvao (2019). Now assume that g is increasing and right-continuous. To prove (103), we show that $g(\mathrm{Q}^*_\tau[X])$ is the supremum of $\{\alpha \in \mathbb{R}; P[g(X) \le \alpha] \ge \tau\}$. For this, let $y < g(\mathrm{Q}^*_\tau[X])$. Then

$$P[g(X) \le y] \le P[g(X) < g(\mathrm{Q}^*_\tau[X])] \le P[X < \mathrm{Q}^*_\tau[X]] \le \tau,$$

i.e.,

$$y < g(\mathrm{Q}^*_\tau[X]) \quad \text{implies} \quad P[g(X) \le y] \le \tau. \tag{106}$$

Now, let $y > g(Q^*_\tau[X])$. We want to show that $Q^*_\tau[X] < \inf\{x; g(x) \geq y\} = \hat{\alpha}$, since it implies that $P[g(X) \leq y] \geq P[X \leq \hat{\alpha}] > \tau$, i.e., it proves that

$$y > g\big(Q^*_\tau[X]\big) \quad \text{implies} \quad P\big[g(X) \leq y\big] > \tau. \tag{107}$$

Let x_n be a strictly decreasing sequence converging to $\hat{\alpha}$. Since $x_n > \hat{\alpha}$, then $g(x_n) \geq y$. Hence, $g(Q^*_\tau[X]) < y \leq \lim_{n\to\infty} g(x_n) = g(\hat{\alpha})$, since g is right-continuous. As g is increasing, this implies that $Q^*_\tau[X] < \hat{\alpha}$, thus establishing (107). Since (106) and (107) together characterize the supremum of $\{\alpha \in \mathbb{R}; P[g(X) \leq \alpha] \geq \tau\}$, this proves (103). Now, if g is decreasing and right-continuous, then $Q_\tau[g(X)] = -Q^*_{1-\tau}[-g(X)] = g(Q^*_{1-\tau}[X])$, where we used Lemma A.27 in the first equality and (103) in the second, since $-g$ is increasing and right-continuous. This proves (104). Finally, if g is decreasing and left-continuous, then $Q^*_{1-\tau}[g(X)] = -Q_\tau[-g(X)] = g(Q_\tau[X])$, where we used Lemma A.27 in the first equality and (102) in the second, since $-g$ is increasing and left-continuous. This proves (105) and concludes the proof. □

Remark A.29. We conclude this subsection by observing that Proposition A.2 and Lemma A.12 are also valid with $Q^*_{1-\tau}$ in place of Q_τ. The proof is similar to the proofs of those results, substituting Q_τ by $Q^*_{1-\tau}$ and using Lemma A.28.

A.8.4 *Closed-form solution for $\gamma > 1$* Now, we can consider the case $\gamma > 1$. The following functions are parallel to the ones defined by (27) and (28). Let $r^*_{\tau,s}(z)$ be defined recursively by $r^*_{\tau,0}(z) = 1$, and

$$r^*_{\tau,s}(z) = r^*_{\tau,s-1}\big(Q^*_{1-\tau}[w|z]\big) \cdot Q^*_{1-\tau}[w|z] \quad \text{for } s \geq 1. \tag{108}$$

Given this, define the functions:

$$R^*(z) \equiv \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} \big[r^*_{\tau,s}(z)\big]^{\frac{1-\gamma}{\gamma}} \quad \text{and} \quad S^*(z) \equiv \frac{R^*(z)}{1+R^*(z)}. \tag{109}$$

Convergence of the sum defining $R^*(z)$ by Assumption 10(vii). We have the following.

Theorem A.30. *Let Assumptions 1, 5, and 10 hold, with the following modifications: $\mathcal{X} = \mathbb{R}_{++}$, $\mathcal{Z} \subset \mathbb{R}_{++}$ $\gamma > 1$, and there exists $\tilde{z} > 0$ such that $0 < Q^*_{1-\tau}[w|z] \leq \tilde{z}$, for all $z \in \mathcal{Z}$, and $\beta\tilde{z}^{1-\gamma} < 1$. Let $\bar{v} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ be given by*

$$\bar{v}(x, z) = \frac{1}{1-\gamma} \cdot (xz)^{1-\gamma} \cdot \big[1 + R^*(z)\big]^\gamma. \tag{110}$$

Then $\bar{v}$ is a fixed point of the transformation $\mathbb{M}$ defined in (21). Moreover, the optimal policy function $y^* : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} = \mathcal{X} \times \mathcal{X}$ is

$$y^*(x, z) = \big(c, x'\big) = \big(\big[1 - S^*(z)\big] \cdot xz, S^*(z) \cdot xz\big), \tag{111}$$

and for an optimal consumption path $\{c_t\}_{t=1}^\infty$ associated with shocks $\{z_t\}_{t=1}^\infty$,

$$\frac{c_{t+1}}{c_t} = z_{t+1} \cdot R^*(z_t) \cdot \big[1 - S^*(z_{t+1})\big]. \tag{112}$$

Proof. The proof is very similar to the proof of Theorem 4.3. For the reader's convenience, we will repeat here the relevant details and modifications. First, observe that $r^*_{\tau,s}(z) \leq \tilde{z} r^*_{\tau,s-1}(z)$ and $r^*_{\tau,s}(z) \leq \tilde{z}^s$. This implies that $\beta^{\frac{s}{\gamma}}[r^*_{\tau,s}(z)]^{\frac{1-\gamma}{\gamma}} \leq (\beta^{\frac{1}{\gamma}} \tilde{z}^{\frac{1-\gamma}{\gamma}})^s$. Since $\beta \tilde{z}^{1-\gamma} < 1$, we have $\beta^{\frac{1}{\gamma}} \tilde{z}^{\frac{1-\gamma}{\gamma}} < 1$. This implies that the infinite sum defining $R^*(z)$ converges.

Now, we want to show that $\bar{v} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ given by (110) is a fixed point of $\mathbb{M} : \mathcal{C} \to \mathcal{C}$:

$$\mathbb{M}(v)(x, z) = \max_{(c,x') \in \Gamma(x,z)} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \mathrm{Q}_\tau \big[v\big(x', w\big) | z_t \big] \right\},$$

where $\mathcal{X} = (0, +\infty)$ and $\Gamma(x, z) = \{(c, x') \in \mathcal{X}^2 : c + x' \leq xz\}\}$.

Let us analyze this maximization problem. Since $0 \notin \mathcal{X} \cup \mathcal{Z}$, it is not possible that $xz = 0$ for $(x, z) \in \mathcal{X} \times \mathcal{Z}$. Since the utility is strictly increasing, it is not possible that $c + x' < xz$, since in this case we could increase consumption to obtain a higher utility. Therefore, $c + x' = xz$. Thus, we can define $t = \frac{x'}{x}$ so that $z - t = \frac{c}{x}$, for $t \in [0, z]$. Thus,

$$\begin{aligned}\mathbb{M}(\bar{v})(x, z) &= \sup_{t \in [0,z]} \left\{ \frac{x^{1-\gamma}(z-t)^{1-\gamma}}{1-\gamma} + \beta \mathrm{Q}_\tau \left[\frac{x^{1-\gamma} t^{1-\gamma}}{1-\gamma} w^{1-\gamma} \big(1 + R(w)\big)^\gamma | z \right] \right\} \\ &= x^{1-\gamma} \cdot \sup_{t \in [0,z]} \left\{ \frac{(z-t)^{1-\gamma}}{1-\gamma} + \beta \frac{t^{1-\gamma}}{1-\gamma} \mathrm{Q}^*_{1-\tau} \big[w^{1-\gamma} \big(1 + R(w)\big)^\gamma | z \big] \right\},\end{aligned}$$

where we have used Lemma A.28 for the decreasing map $x \mapsto \frac{x}{1-\gamma}$. Let us define

$$q^*(z) \equiv \mathrm{Q}^*_{1-\tau} \big[w^{1-\gamma} \big(1 + R(w)\big)^\gamma | z \big].$$

We are interested in the maximization problem $\max_{t \in [0,z]} v(t)$, where $v : [0, z] \to \mathbb{R}$ is

$$v(t) \equiv \frac{(z-t)^{1-\gamma}}{1-\gamma} + \beta \frac{t^{1-\gamma}}{1-\gamma} q^*(z). \tag{113}$$

The first-order condition is $v'(t) = -(z-t)^{-\gamma} + t^{-\gamma} \beta q^*(z) = 0$, which leads to the optimal,

$$t^* = \frac{z \big[\beta q^*(z)\big]^{\frac{1}{\gamma}}}{1 + \big[\beta q^*(z)\big]^{\frac{1}{\gamma}}}. \tag{114}$$

Notice that $t^* \in (0, z)$. Then $v''(t) = -\gamma[(z-t)^{-\gamma-1} + t^{-\gamma-1} \beta q^*(z)] < 0$, which implies that t^* given by (114) is optimal. Substituting (114) into (113), we obtain

$$\begin{aligned}v\big(t^*\big) &= \frac{z^{1-\gamma}}{(1-\gamma)\big\{1 + \big[\beta q^*(z)\big]^{\frac{1}{\gamma}}\big\}^{1-\gamma}} + \frac{\beta q^*(z)}{(1-\gamma)} \cdot \frac{z^{1-\gamma} \big[\beta q^*(z)\big]^{\frac{1-\gamma}{\gamma}}}{\big\{1 + \big[\beta q^*(z)\big]^{\frac{1}{\gamma}}\big\}^{1-\gamma}} \\ &= \frac{z^{1-\gamma}}{(1-\gamma)} \big\{1 + \big[\beta q^*(z)\big]^{\frac{1}{\gamma}}\big\}^\gamma.\end{aligned}$$

Therefore, $\mathbb{M}(\bar{v})(x,z) = \frac{(xz)^{1-\gamma}}{1-\gamma} \cdot \{1 + [\beta q^*(z)]^{\frac{1}{\gamma}}\}^{\gamma}$ and $\bar{v}$ will be a fixed point of $\mathbb{M}$ if we establish that $\{1 + [\beta q^*(z)]^{\frac{1}{\gamma}}\}^{\gamma} = [1 + R^*(z)]^{\gamma}$, i.e.,

$$R^*(z) = \beta^{\frac{1}{\gamma}}\big[q^*(z)\big]^{\frac{1}{\gamma}} = \beta^{\frac{1}{\gamma}}\big\{\mathrm{Q}^*_{1-\tau}\big[w^{1-\gamma}\big(1 + R^*(w)\big)^{\gamma}|z\big]\big\}^{\frac{1}{\gamma}}.$$

Using again Lemma A.28, (103), for the increasing function $x \mapsto x^{\frac{1}{\gamma}}$, this is equivalent to

$$R^*(z) = \beta^{\frac{1}{\gamma}}\mathrm{Q}^*_{1-\tau}\big[w^{\frac{1-\gamma}{\gamma}}\big(1 + R^*(w)\big)|z\big].$$

Recall that $R^*(z) = \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}}[r^*_{\tau,s}(z)]^{\frac{1-\gamma}{\gamma}}$, where $r^*_{\tau,s}(z)$ is defined recursively by $r^*_{\tau,0}(z) = 1$ and $r^*_{\tau,s}(z) = r^*_{\tau,s-1}(\mathrm{Q}^*_{1-\tau}[w|z]) \cdot \mathrm{Q}^*_{1-\tau}[w|z]$ for $s \geq 1$. Notice that all $r^*_{\tau,s}(z)$ are nondecreasing and continuous in z, by Remark A.29 and an adaptation of Lemma A.12. By Proposition A.2, the product of those terms can commute with the $\mathrm{Q}^*_{1-\tau}$ operator. Thus, $\mathrm{Q}^*_{1-\tau}[wr^*_{\tau,s-1}(w)|z] = \mathrm{Q}^*_{1-\tau}[w|z]r^*_{\tau,s-1}(\mathrm{Q}_{\tau}[w|z]) = r^*_{\tau,s}(z)$. Using these properties and expressions, we obtain

$$\begin{aligned}
\beta^{\frac{1}{\gamma}}\mathrm{Q}^*_{1-\tau}\big[w^{\frac{1-\gamma}{\gamma}}\big(1 + R^*(w)\big)|z\big] &= \beta^{\frac{1}{\gamma}}\mathrm{Q}^*_{1-\tau}\left[w^{\frac{1-\gamma}{\gamma}}\left(1 + \sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\big[r^*_{\tau,s}(w)\big]^{\frac{1-\gamma}{\gamma}}\right)\middle| z\right] \\
&= \mathrm{Q}^*_{1-\tau}\left[\beta^{\frac{1}{\gamma}}w^{\frac{1-\gamma}{\gamma}} + \beta^{\frac{1}{\gamma}}w^{\frac{1-\gamma}{\gamma}}\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\big[r^*_{\tau,s}(w)\big]^{\frac{1-\gamma}{\gamma}}|z\right] \\
&= \mathrm{Q}^*_{1-\tau}\big[\beta^{\frac{1}{\gamma}}w^{\frac{1-\gamma}{\gamma}}|z\big] + \mathrm{Q}^*_{1-\tau}\left[\sum_{s=1}^{\infty}\beta^{\frac{s+1}{\gamma}}\big[wr^*_{\tau,s}(w)\big]^{\frac{1-\gamma}{\gamma}}|z\right] \\
&= \beta^{\frac{1}{\gamma}}r^*_{\tau,1}(z)^{\frac{1-\gamma}{\gamma}} + \sum_{s=1}^{\infty}\beta^{\frac{s+1}{\gamma}}\big\{\mathrm{Q}^*_{1-\tau}\big[wr^*_{\tau,s}(w)|z\big]\big\}^{\frac{1-\gamma}{\gamma}} \\
&= \sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\big[r^*_{\tau,s}(z)\big]^{\frac{1-\gamma}{\gamma}},
\end{aligned}$$

which is just $R^*(z)$, as we wanted to verify. The expressions for the optimal savings, consumption and consumption growth are obtained in the same fashion as in Theorem 4.3. This concludes the proof. □

A.8.5 *Proof of Theorem 4.7* It is useful to collect the expressions for consumption and savings for all cases ($\gamma \in (0,1)$, $\gamma = 1$, $\gamma > 1$). From (30), (97), and (111), the optimal consumption is given by

$$c^*(x,z) = \begin{cases} \left\{1 + \sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\big[r_{\tau,s}(z)\big]^{\frac{1-\gamma}{\gamma}}\right\}^{-1} \cdot xz & \text{if } \gamma \in (0,1), \\ (1-\beta)\cdot xz & \text{if } \gamma = 1, \\ \left\{1 + \sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\big[r^*_{\tau,s}(z)\big]^{\frac{1-\gamma}{\gamma}}\right\}^{-1} \cdot xz & \text{if } \gamma > 1. \end{cases}$$

Since the optimal savings is given by $xz - c^*(x, z)$, it is sufficient to study the claims for consumption, since the ones for savings follow from those. We restate below the claims in Theorem 4.7, with the respective proofs.

1. If the DM becomes more impatient, i.e., the discount factor β decreases, then the DM consumes more (and saves less).

   If we take the derivative of c^* with respect to β, we obtain

$$\frac{\partial c^*}{\partial \beta}(x, z) = \begin{cases} (-1)\dfrac{\sum_{s=1}^{\infty} \dfrac{s}{\gamma}\beta^{\frac{s}{\gamma}-1}\left[r_{\tau,s}(z)\right]^{\frac{1-\gamma}{\gamma}}}{\left\{1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}}\left[r_{\tau,s}(z)\right]^{\frac{1-\gamma}{\gamma}}\right\}^2} \cdot xz & \text{if } \gamma \in (0, 1), \\ -xz & \text{if } \gamma = 1, \\ (-1)\dfrac{\sum_{s=1}^{\infty} \dfrac{s}{\gamma}\beta^{\frac{s}{\gamma}-1}\left[r^*_{\tau,s}(z)\right]^{\frac{1-\gamma}{\gamma}}}{\left\{1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}}\left[r^*_{\tau,s}(z)\right]^{\frac{1-\gamma}{\gamma}}\right\}^2} \cdot xz & \text{if } \gamma > 1. \end{cases}$$

   Thus, $\frac{\partial c^*}{\partial \beta} < 0$ in all cases, which establishes the claim.

2. If the elasticity of intertemporal substitution (EIS$= \frac{1}{\gamma}$) increases and $\beta\tilde{z} < 1$, then the DM consumes more (and saves less).

   Taking the derivative of c^* with respect to γ, we obtain[45]

$$\frac{\partial c^*}{\partial \gamma}(x, z) = \begin{cases} (-1)\dfrac{\sum_{s=1}^{\infty} \dfrac{\left[\beta^s r_{\tau,s}(z)\right]^{\frac{1}{\gamma}}}{r_{\tau,s}(z)} \cdot \ln\left[\beta^s r_{\tau,s}(z)\right] \cdot (-1)\dfrac{1}{\gamma^2}}{\left\{1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}}\left[r_{\tau,s}(z)\right]^{\frac{1-\gamma}{\gamma}}\right\}^2} \cdot xz & \text{if } \gamma \in (0, 1), \\ (-1)\dfrac{\sum_{s=1}^{\infty} \dfrac{\left[\beta^s r^*_{\tau,s}(z)\right]^{\frac{1}{\gamma}}}{r^*_{\tau,s}(z)} \cdot \ln\left[\beta^s r^*_{\tau,s}(z)\right] \cdot (-1)\dfrac{1}{\gamma^2}}{\left\{1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}}\left[r^*_{\tau,s}(z)\right]^{\frac{1-\gamma}{\gamma}}\right\}^2} \cdot xz & \text{if } \gamma > 1. \end{cases}$$

   If $\gamma \in (0, 1)$, $\frac{\partial c^*}{\partial \gamma}$ has the same signal of $\ln[\beta^s r_{\tau,s}(z)]$ (if it is the same for all s). By Assumption 10(vi), $r_{\tau,s}(z) \le \tilde{z} r_{\tau,s-1}(z)$. Therefore, $r_{\tau,s}(z) \le \tilde{z}^s$. This implies that $\beta^s r_{\tau,s}(z) \le (\beta\tilde{z})^s$. If $\beta\tilde{z} < 1$, then $\beta^s r_{\tau,s}(z) < 1$, which implies that $\frac{\partial c^*}{\partial \gamma} < 0$ if $\gamma \in (0, 1)$. If $\frac{1}{\gamma}$ increases, γ decreases and the result follows.

[45] We omit the case $\gamma = 1$ because it does not make sense to take the derivative with respect to γ in this case.

If $\gamma > 1$, $\frac{\partial c^*}{\partial \gamma}$ has the same signal of $\ln[\beta^s r^*_{\tau,s}(z)]$ (if it is the same for all s). Since $Q^*_{1-\tau}[w|z] \le \tilde{z}$, $r^*_{\tau,s}(z) \le \tilde{z} r^*_{\tau,s-1}(z)$. Therefore, $r^*_{\tau,s}(z) \le \tilde{z}^s$. This implies that $\beta^s r^*_{\tau,s}(z) \le (\beta\tilde{z})^s$. The rest of the argument is the same as above.

3. If the DM becomes more risk averse, i.e., the risk attitude parameter τ decreases, then the DM consumes more (and saves less) if $\gamma \in (0, 1)$ and consumes less (and saves more) if $\gamma > 1$. Moreover, if $\gamma = 1$, consumption and savings decisions are not affected by the risk attitude.

   Taking the derivative of c^* with respect to γ, we obtain

$$\frac{\partial c^*}{\partial \tau}(x, z) = \begin{cases} (-1)\dfrac{\displaystyle\sum_{s=1}^{\infty} \frac{1-\gamma}{\gamma} \beta^{\frac{s}{\gamma}} \big[r_{\tau,s}(z)\big]^{\frac{1-2\gamma}{\gamma}} \frac{\partial r_{\tau,s}(z)}{\partial \tau}}{\left\{1 + \displaystyle\sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} \big[r_{\tau,s}(z)\big]^{\frac{1-\gamma}{\gamma}}\right\}^2} \cdot xz & \text{if } \gamma \in (0, 1), \\ 0 & \text{if } \gamma = 1, \\ (-1)\dfrac{\displaystyle\sum_{s=1}^{\infty} \frac{1-\gamma}{\gamma} \beta^{\frac{s}{\gamma}} \big[r^*_{\tau,s}(z)\big]^{\frac{1-2\gamma}{\gamma}} \frac{\partial r^*_{\tau,s}(z)}{\partial \tau}}{\left\{1 + \displaystyle\sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} \big[r^*_{\tau,s}(z)\big]^{\frac{1-\gamma}{\gamma}}\right\}^2} \cdot xz & \text{if } \gamma > 1. \end{cases}$$

   Since $\frac{\partial Q_\tau[w|z]}{\partial \tau} > 0$, we have $\frac{\partial r_{\tau,s}(z)}{\partial \tau} > 0$. Therefore, if $\gamma \in (0, 1)$, $\frac{\partial c^*}{\partial \tau} < 0$. On the other hand, since $\frac{\partial Q^*_{1-\tau}[w|z]}{\partial \tau} < 0$, we have $\frac{\partial r^*_{\tau,s}(z)}{\partial \tau} < 0$. Thus, if $\gamma > 1$, $\frac{\partial c^*}{\partial \tau} > 0$. This establishes the claims.

4. If the distribution of returns increases, or more specifically, the τ-quantile $Q_\tau[w|z]$ of future interest rates increases for a fixed quantile τ for all $z \in \mathcal{Z}$, then the DM consumes less (and saves more) if $\gamma \in (0, 1)$ and consumes more (and saves less) if $\gamma > 1$. Moreover, if $\gamma = 1$, consumption and savings decisions are not affected by these changes.

   This comes from the expressions obtained in the previous item.

A.9 *Proofs of Section 4.2*

Proof of Theorem 4.8. Assumption 11 implies Assumption 1, but for the fact that $\mathcal{Z} = [0, \bar{w}] \times \{0, 1\}$ is not connected. However, since $z_t \mapsto \bar{v}(x_t, z_t)$ is constant (it does not depend on z_t), its image is connected and the conditions described in Remark 3.12 are met. If we define $u(x, y, z) = U(x)$, this is continuous and bounded; ϕ given by (40) is continuous and $\Gamma(x, z) = \{0, 1\}$ is continuous, with nonempty, compact values. Therefore, Assumption 2 holds. Thus, existence and uniqueness of the fixed-point $\bar{v}$ follows from Theorem 3.11. The claim that $\bar{v}$ is strictly increasing in x_t follows from Theorem 3.13, since Assumption 3 is also satisfied: u and ϕ are strictly increasing in their first variable and Γ is constant.

Since $\bar{v}$ is a function of x_t only and only next period shocks matter, we can simplify notation by letting e and w denote the future realizations of the shocks. With this notation, we can rewrite (42) as

$$\bar{v}(x) = \max\{\beta v(\mathrm{Q}_\tau[w]), U(x) + \beta\bar{v}(\mathrm{Q}_\tau[e]x)\}. \tag{115}$$

Since $e \in \{0, 1\}$, then $\mathrm{Q}_\tau[e] \in \{0, 1\}$. Let us consider separately these two cases.

First case: $\mathrm{Q}_\tau[e] = 0$.

In this case, we have $\bar{v}(x) = \max\{\beta\bar{v}(\mathrm{Q}_\tau[w]), U(x) + \beta\bar{v}(0)\}$. In the particular case in which $x = 0$, this becomes $\bar{v}(0) = \max\{\beta\bar{v}(\mathrm{Q}_\tau[w]), \beta\bar{v}(0)\}$, since $U(0) = 0$. By Assumption 11, $\mathrm{Q}_\tau[w] > 0$ and from the fact that $\bar{v}$ is strictly increasing, $\beta\bar{v}(\mathrm{Q}_\tau[w]) > \beta\bar{v}(0)$, which implies that $\bar{v}(0) = \beta\bar{v}(\mathrm{Q}_\tau[w]) > 0$. For simplicity, denote $\beta\bar{v}(\mathrm{Q}_\tau[w])$ by $\tilde{A} > 0$. Then we have established that $\bar{v}(x) = \max\{\tilde{A}, U(x) + \beta\tilde{A}\}$. Since U is strictly increasing, if there exists $\tilde{x}$ such that $U(\tilde{x}) = (1-\beta)\tilde{A}$, then

$$\bar{v}(x) = \begin{cases} \tilde{A} & \text{if } x \le \tilde{x}, \\ U(x) + \beta\tilde{A} & \text{if } x > \tilde{x}. \end{cases}$$

Since $\beta < 1$, $\tilde{A} = \beta\bar{v}(\mathrm{Q}_\tau[w]) < \bar{v}(\mathrm{Q}_\tau[w])$. Therefore, $\mathrm{Q}_\tau[w] > \tilde{x}$. This implies that $\bar{v}(\mathrm{Q}_\tau[w]) = U(\mathrm{Q}_\tau[w]) + \beta\tilde{A} = U(\mathrm{Q}_\tau[w]) + \beta^2\bar{v}(\mathrm{Q}_\tau[w])$. Therefore,

$$\bar{v}(\mathrm{Q}_\tau[w]) = \frac{1}{1-\beta^2}U(\mathrm{Q}_\tau[w]) \quad \text{and} \quad \tilde{A} = \frac{\beta}{1-\beta^2}U(\mathrm{Q}_\tau[w])$$

Since $\mathrm{Q}_\tau[e] = 0$, this is exactly the expression of A given by (43), i.e., $\tilde{A} = A$. With this equality, then the definition of $\tilde{x}$ as the value such that $U(\tilde{x}) = (1-\beta)\tilde{A} = (1-\beta)A$ becomes exactly the definition of x^* in (44). Moreover, using $\mathrm{Q}_\tau[e] = 0$, we see that the expression of $\bar{v}(X)$ in (45) for $x > x^*$ is equal to $U(x) + \beta A$, exactly as above. This concludes the proof for this case.

Second case: $\mathrm{Q}_\tau[e] = 1$.

In this case, we have $\bar{v}(x) = \max\{\beta\bar{v}(\mathrm{Q}_\tau[w]), U(x) + \beta\bar{v}(x)\}$. Repeating the same arguments given above, we conclude that $\bar{v}(0) = \max\{\beta\bar{v}(\mathrm{Q}_\tau[w]), \beta\bar{v}(0)\} = \beta\bar{v}(\mathrm{Q}_\tau[w])$. Again, denote $\beta\bar{v}(\mathrm{Q}_\tau[w])$ by $\tilde{A} > 0$. Then we have established that $\bar{v}(x) = \max\{\tilde{A}, U(x) + \beta\bar{v}(x)\}$. Since both U and $\bar{v}$ are strictly increasing, if there exists $\tilde{x}$ such that $U(\tilde{x}) + \beta\bar{v}(\tilde{x}) = \tilde{A}$,

$$\bar{v}(x) = \begin{cases} \tilde{A} & \text{if } x \le \tilde{x}, \\ U(x) + \beta\bar{v}(x) & \text{if } x > \tilde{x}. \end{cases}$$

Since $\beta < 1$, $\tilde{A} = \beta\bar{v}(\mathrm{Q}_\tau[w]) < \bar{v}(\mathrm{Q}_\tau[w])$. Therefore, $\mathrm{Q}_\tau[w] > \tilde{x}$. This implies that $\bar{v}(\mathrm{Q}_\tau[w]) = U(\mathrm{Q}_\tau[w]) + \beta\bar{v}(\mathrm{Q}_\tau[w])$. Therefore,

$$\bar{v}(\mathrm{Q}_\tau[w]) = \frac{1}{1-\beta}U(\mathrm{Q}_\tau[w]) \quad \text{and} \quad \tilde{A} = \frac{\beta}{1-\beta}U(\mathrm{Q}_\tau[w]).$$

Since $\mathrm{Q}_\tau[e] = 1$, from (43) we have again $\tilde{A} = A$. From this, the definition of $\tilde{x}$ as the value such that $U(\tilde{x}) + \beta\bar{v}(\tilde{x}) = A$ becomes $U(\tilde{x})(1 + \frac{\beta}{1-\beta}) = A$ leading to the definition of x^* in (44). Now, it remains to observe that if $x > \tilde{x} = x^*$, the expression $\bar{v}(x) = U(x) + \beta\bar{v}(x)$ implies $\bar{v}(x) = \frac{1}{1-\beta}U(x)$. Since $Q_\tau[e] = 1$, this is exactly the expression that we find in (45). This concludes the proof. □

Appendix B: Relating QP and other preferences

In this Appendix, we further discuss the relationship between the quantile preferences and alternative models, specially subjective expected utility (EU) and Epstein–Zin.

B.1 *Quantile preferences are not expected utility*

In this subsection, we show that it is not possible, in general, to reduce a given quantile preference to EU.[46] More formally, we show that given a quantile preference $\succcurlyeq_\tau$, in general it is not possible to find subjective beliefs π and utility function $u : \mathbb{R} \to \mathbb{R}$ such that the expected utility $\succcurlyeq_{(\pi,u)}$ defined by π and u is equivalent to $\succcurlyeq_\tau$, in the sense that

$$\begin{aligned} Q_\tau[X] \geq Q_\tau[Y] \quad &\iff \quad X \succcurlyeq_\tau Y \quad \iff \quad X \succcurlyeq_{(\pi,u)} Y \\ &\iff \quad E_\pi\big[u(X)\big] \geq E_\pi\big[u(Y)\big]. \end{aligned} \tag{116}$$

This can be seen in a simple state space, with only two states, i.e., $\Omega = \{\omega_1, \omega_2\}$.

Now we prove that assertion (116) does not hold. A quantile preference $\succcurlyeq_\tau$ over random variables $X : \Omega \to \mathbb{R}$ is defined by a number $\tau \in (0, 1)$ and probability over Ω defined by $p = \Pr[\omega = \omega_1]$. Consider random variables X and Y such that $X(\omega_1) \leq X(\omega_2)$ and $Y(\omega_1) > Y(\omega_2)$. Then

$$Q_\tau[X] = \begin{cases} X(\omega_1) & \text{if } \tau \leq p, \\ X(\omega_2) & \text{if } \tau > p \end{cases} \quad \text{and} \quad Q_\tau[Y] = \begin{cases} Y(\omega_2) & \text{if } \tau \leq 1-p, \\ Y(\omega_1) & \text{if } \tau > 1-p. \end{cases}$$

For concreteness, assume $\tau = \frac{1}{3}$ and $p = \frac{1}{2}$, so that $\tau \leq p$ and $\tau \leq 1 - p$.

For a contradiction, suppose that we have found subjective beliefs π over Ω, defined by $\pi = \Pr[\omega = \omega_1]$, and strictly increasing utility function $u : \mathbb{R} \to \mathbb{R}$ such that (116) holds. Let X' be a constant random variable, i.e., $X'(\omega_1) = X'(\omega_2) = x'$. Thus, $Q_\tau[X'] = x' = u^{-1}(E_\pi[u(X')])$. By (116), we must have, for any random variable X such that $X'(\omega_1) \leq X'(\omega_2)$,

$$x' \geq Q_\tau[X] = X(\omega_1) \quad \iff \quad u(x') \geq \pi u\big[X(\omega_1)\big] + (1-\pi)u\big[X(\omega_2)\big].$$

We claim that this implies that $\pi = 1$. Indeed, for a contradiction, assume that $\pi < 1$. Pick $X(\omega_1) = 0 < x' = 1$ and $u[X(\omega_2)] > \frac{u(x')-\pi u(X(\omega_1)}{1-\pi} = \frac{u(1)-\pi u(0)}{1-\pi}$, which implies $X(\omega_2) > x' > X(\omega_1)$ and $u(x') < \pi u[X(\omega_1)] + (1-\pi)u[X(\omega_2)]$, contradicting (116). This shows that $\pi = 1$.

Since $\pi = 1$, for any random variable X, $E_\pi[u(X)] = u(X(\omega_1))$. Consider the random variables X and Y defined by $1 = X(\omega_1) \leq X(\omega_2) = 2$ and $Y(\omega_1) = 3 > Y(\omega_2) = 0$. Then

$$u(1) = E_\pi\big[u(X)\big] < E_\pi\big[u(Y)\big] = u(3) \quad \text{but } 1 = Q_\tau[X] > Q_\tau[Y] = 0,$$

which again contradicts (116) and shows that this equivalence is not possible.

[46] This subsection has been developed to address a question posed by an anonymous reviewer.

This proof works for an arbitrary state space and shows that the problem already arises if we restrict ourselves to binary lotteries, i.e., random variables that take only two variables. In this case, it is enough to substitute the two states by a fixed partition of $\Omega = \Omega_1 \cup \Omega_2$.

B.2 *Restricting the set of random variables*

Despite of the main negative general point of the equivalence between QP and EU for binary lotteries discussed above, when one restricts the utility function *and* the class of random variables, it may be possible to obtain equivalence of QP and EU for given utility functions and beliefs π in this restricted setting.

Both QP and EU are characterized the one parameter capturing risk attitude. Thus, we concentrate on studying the connection through this parameter.

We begin our study of the relationship between QP and the EU preferences by specifying the class of utility functions. We consider the Constant Relative Risk Aversion (CRRA):

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \text{for } \gamma > 0, \gamma \neq 1. \tag{117}$$

Moreover, let us restrict ourselves to log-normal variables X such that $\ln(X) \sim N(\mu, \sigma)$ to describe lotteries.

First, we calculate the certainty equivalent for a γ-CRRA EU maximizer is

$$\frac{v^{1-\gamma}}{1-\gamma} = \mathrm{E}\big[u(X)\big] = \frac{\exp\left[(1-\gamma)\mu + \frac{1}{2}\sigma^2(1-\gamma)^2\right]}{(1-\gamma)}$$
$$\Rightarrow \quad \ln(v) = \mu + \frac{1}{2}\sigma^2(1-\gamma) \quad \Rightarrow \quad v = \exp\left[\mu + \frac{1}{2}\sigma^2(1-\gamma)\right], \tag{118}$$

which depends on γ, μ, and σ^2.

Second, for QP, the certainty equivalent is given by

$$\mathrm{Q}_\tau[X] = \exp(\mu + \sigma q_\tau), \tag{119}$$

where $q_\tau = F_N^{-1}(\tau)$ is the τ-quantile of a standard normal variable (F_N is the c.d.f. of a standard normal variable), and depends on τ, μ, and σ.

It is interesting to determine when the certainty equivalent for the two preferences coincide. From (118) and (119), this happens if and only if

$$\mu + \frac{1}{2}\sigma^2(1-\gamma) = \mu + \sigma q_\tau \Leftrightarrow \frac{1}{2}\sigma(1-\gamma) = q_\tau \Leftrightarrow \tau = F_N\left(\frac{\sigma}{2}(1-\gamma)\right). \tag{120}$$

If we fix the standard deviation σ of the random variable that we are considering, then (120) defines a map between γ, the risk aversion parameter in the EU model, and τ, the risk aversion parameter in the QP model. This map will be further discussed below.

However, note that we can achieve equivalence between the two preference structures even if they do not yield the same certainty equivalent, which was imposed to obtain (120). Indeed, this equivalence can be established by considering a subset of random variables where the orders implied by both preferences coincide. We will investigate this equivalence within some restricted sets. We start by analyzing the case with two risky choices, then extend our analysis to include risk-free alternatives.

B.2.1 *Two risky choices* Let X and Y be two random variables such that $\ln(X) \sim N(\mu, \sigma_X)$ and $\ln(Y) \sim N(\mu, \sigma_Y)$, i.e., they are two log-normal variables that differ only in their variance, but not their average. Therefore, if $\gamma \neq 1$, from equations (118) and (119), we have the following choices:

$$X \succsim_{EU} Y \iff \mu + \frac{1}{2}\sigma_X^2(1-\gamma) \geq \mu + \frac{1}{2}\sigma_Y^2(1-\gamma) \iff \begin{cases} \sigma_X \geq \sigma_Y & \text{if } \gamma < 1, \\ \sigma_X \leq \sigma_Y & \text{if } \gamma > 1 \end{cases}$$

$$X \succsim_{\tau} Y \iff \exp(\mu + \sigma_X q_\tau) \geq \exp(\mu + \sigma_Y q_\tau) \iff \begin{cases} \sigma_X \geq \sigma_Y & \text{if } \tau > \frac{1}{2}, \\ \sigma_X \leq \sigma_Y & \text{if } \tau < \frac{1}{2}. \end{cases}$$

This shows that, if we compare *only* log-normal random variables with the same average, τ-quantile preferences for $\tau < \frac{1}{2}$ are *equivalent* to CRRA EU with $\gamma > 1$: in both models, the DM prefers the random variable with *lower* variance. Of course, this domain of choices is very restrictive and does not allow even a distinction between EU preferences with different parameters.

Notice that $\tau < \frac{1}{2} \iff q_\tau < 0$. Since we understand $\tau < \frac{1}{2}$ as risk aversion, the choice under the QP makes sense: the DM prefers the random variable with lower variance.[47]

It is illustrative to verify what happens when $\gamma = 1$. In this case, $\mathrm{E}[u(X)] = \mathrm{E}[\ln(X)] = \mu$. Thus, this DM is indifferent between log-normal variables with the same mean but different variance, even though she is risk averse. Notice that the same indifference occurs if $\gamma = 0$. For the quantile model, indifference occurs only if $\tau = \frac{1}{2}$.[48]

It is natural to consider a slightly larger domain, which includes also risk-free variables. We consider this next.

B.2.2 *Including risk-free alternatives* Since all EU and QP lead to the same certain equivalent for the risk-free variables, namely the value that they assume with probability one, the only interesting case that remains to analyze is the comparison between a log-normal variable X such that $\ln(X) \sim N(\mu, \sigma_X)$ and a risk-free variable Y such that $\Pr[Y = y] = 1$.

Consider a γ-CRRA EU preference, for $\gamma > 1$, and a τ-quantile preference, for $\tau < \frac{1}{2}$. Then

$$X \succsim_{EU} Y \iff \exp\left[\mu + \frac{1}{2}\sigma_X^2(1-\gamma)\right] \geq y,$$

[47]It should be noted that a CRRA with $\gamma \in (0,1)$ is also considered risk averse, since $u''(x) = -\gamma x^{\gamma-1} < 0$. However, when confronted with two log-normal variables with same mean, a EU DM with $\gamma < 1$ prefers would prefer the random variable with *larger* risk aversion. For quantile preferences, this happens only if $\tau > \frac{1}{2}$, which characterizes risk loving. Notice also that if $\tau = \frac{1}{2}$ the DM is indifferent between X and Y.

[48]Remember that we are excluding the cases in which $\tau \in \{0, 1\}$. If we were to consider these cases, then $\tau = 0$ would lead to $\mathrm{Q}_\tau[X] = 0$ for all log-normal variables X.

$$X \succcurlyeq_\tau Y \quad \Longleftrightarrow \quad \exp(\mu + \sigma_X q_\tau) \geq y.$$

The two preferences may differ if

$$\mu + \frac{1}{2}\sigma_X^2(1-\gamma) \gtreqless \mu + \sigma_X q_\tau \quad \Longleftrightarrow \quad \frac{1}{2}\sigma_X(1-\gamma) \gtreqless q_\tau.$$

For a given $\gamma > 1$ and $\tau < \frac{1}{2}$, there exists σ such that $\frac{1}{2}\sigma(1-\gamma) = q_\tau$. If $\sigma_X < \sigma$,

$$\begin{aligned} &\frac{1}{2}\sigma_X(1-\gamma) > \frac{1}{2}\sigma(1-\gamma) = q_\tau \\ \Rightarrow \quad &\exp\left[\mu + \frac{1}{2}\sigma_X^2(1-\gamma)\right] > \exp(\mu + \sigma_X q_\tau). \end{aligned} \tag{121}$$

In this case, there are risk-free variables Y for which the preferences would be

$$X \succ_{EU} Y \succ_\tau X.$$

This means that the EU DM prefers the risky variable X over the risk-free alternative Y, while the quantile DM prefers the risk-free Y. If $\sigma_X > \sigma$, we can have the reverse, i.e.,

$$X \prec_{EU} Y \prec_\tau X.$$

Notice, however, that this discrepancy would happen only for certain values of risk-free lotteries. If we exclude some risk-free lotteries, the two preferences may agree. This suggests the following procedure.

Suppose that we fix $\gamma > 1$ and a set of random variables $\ln(X) \sim N(\mu, \sigma_X)$ such that $\sigma_X \in [0, \bar{\sigma}]$. What should be our choice of τ and what intervals of risk-free comparisons would make the two preferences agree?

Choose τ such that $q_\tau = \frac{\bar{\sigma}}{2}(1-\gamma) < 0$. Since $\sigma_X \leq \bar{\sigma}$, we want to exclude risk-free random variables with values y satisfying

$$\exp(\mu) \geq \exp\left[\mu + \frac{1}{2}\sigma_X^2(1-\gamma)\right] > y > \exp(\mu + \sigma_X q_\tau) \geq \exp(\mu + \bar{\sigma} q_\tau),$$

i.e., we want to exclude y such that $\ln(y) \in (\mu + \bar{\sigma} q_\tau, \mu)$. In other words, fixing $\sigma = \bar{\sigma}$ and $\gamma > 1$, let $\tau = F_N(\frac{\sigma}{2}(1-\gamma)) \Leftrightarrow q_\tau = \frac{\sigma}{2}(1-\gamma)$. Then the γ-CRRA EU preference agrees with the τ-quantile preferences for all log-normal variables X such that $\ln(X) \sim N(\mu, \sigma_X)$, with $\sigma_X \in [0, \sigma]$ and risk-free variables taking values $y \leq \mu + \frac{\sigma^2}{2}(1-\gamma)$.

It is useful to illustrate the map $\gamma \mapsto \tau$ defined by (120), that is,

$$\tau = m(\sigma, \gamma) = F_N\left(\frac{\sigma}{2}(1-\gamma)\right),$$

for $\sigma = \frac{1}{2}, 1, 2$. If $\sigma = \frac{1}{2}$ and $\gamma = 2.65$, we have $\tau = 0.34$, as illustrated in Figure 3. If $\sigma = 2$, then $\gamma = 1.41$ gives $\tau = 0.34$. If $\sigma = 1$, then $\gamma = 1.83$ gives $\tau = 0.34$. Notice that this means that the τ-quantile preference for $\tau = 0.34$ is *the same* as the γ-CRRA for $\gamma = 1.41$, 1.83 and 2.65 for *different* sets of random variables. Table 1 illustrates these different set of

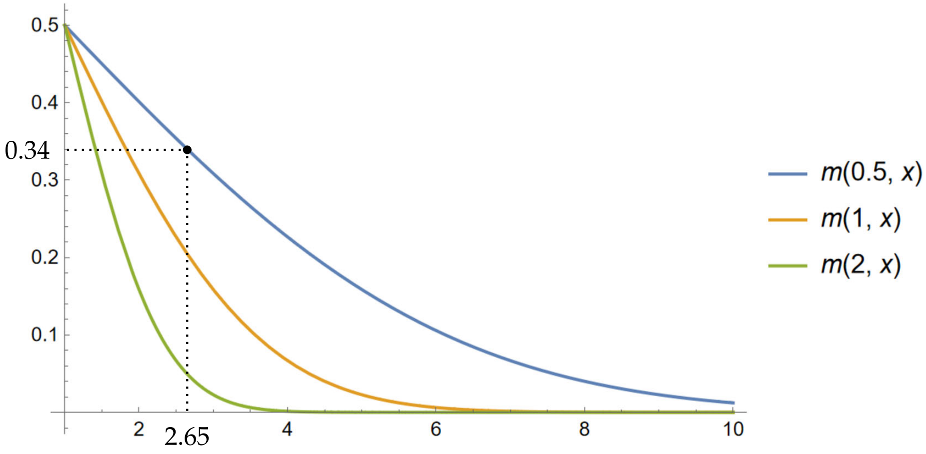

FIGURE 3. Map between $\gamma \mapsto \tau = m(\sigma, \gamma)$, for $\sigma = \frac{1}{2}, 1, 2$.

random variables. The table reveals the trade-off: if we want to include higher values of variance in our set of allowable log-normal variables, we have to give-up some risk-free lotteries, *and* chose a higher γ.[49]

We can also illustrate the corresponding preferences for a set of random variables. Consider the log-normal variables X^i such that $X^i \sim N(2, \sigma_i)$, for $i = 1, 2, 3, 4$, where $\sigma_4 = 0.3$, $\sigma_3 = 0.7$, $\sigma_2 = 1.5$, $\sigma_1 = 3$. The certainty equivalents of those lotteries for different preferences are shown in Table 2.

Consider now the risk-free lotteries Y^j, for $j = 1, 2, 3$, where $\Pr[Y^j = y_j] = 1$, for $y_1 = 2$, $y_2 = 3.5$, $y_3 = 5.5$, $y_4 = 6.7$, so that $y_1 < 3.24 < y_2 < 4.89 < y_3 < 6.01 < y_4$; compare with values for risk-free lotteries in Table 1. Let us denote by $\succcurlyeq_i$ the γ_i-CRRA EU, for $i = 1, 2, 3$, where $\gamma_1 = 1.41$, $\gamma = 1.82$, and $\gamma_3 = 2.65$. Let $\succcurlyeq_\tau$ denote the quantile preference for $\tau = 0.34$. Of course, for all preferences $k \in \{1, 2, 3, \tau\}$, $X^4 \succ_k X^3 \succ_k X^2 \succ_k X^1$ and $Y^4 \succ_k Y^3 \succ_k Y^2 \succ_k Y^1$. The preferences differ, however, in how X^i and Y^j are compared. The ranking are as follows:

$$X^4 \succ_1 X^3 \succ_1 Y^4 \succ_1 Y^3 \succ_1 X^2 \succ_1 Y^2 \succ_1 Y^1 \succ_1 X^1;$$
$$X^4 \succ_2 Y^4 \succ_2 X^3 \succ_2 Y^3 \succ_2 Y^2 \succ_2 X^2 \succ_2 Y^1 \succ_2 X^1;$$
$$X^4 \succ_3 Y^4 \succ_3 Y^3 \succ_3 X^3 \succ_3 Y^2 \succ_3 Y^1 \succ_3 X^2 \succ_3 X^1;$$

TABLE 1. Range of random variables X and Y such that $\ln(X) \sim N(\mu, \sigma_X)$, for $\mu = 2$, and $\Pr[Y = y] = 1$ associated to $\tau = 0.34$.

σ	γ	$-\sigma q_\tau$	Interval of σ_X	Interval of y
0.5	2.65	−0.21	[0, 0.5]	$(-\infty, 6.01] \cup [7.39, +\infty)$
1	1.82	−0.41	[0, 1]	$(-\infty, 4.89] \cup [7.39, +\infty)$
2	1.41	−0.82	[0, 2]	$(-\infty, 3.24] \cup [7.39, +\infty)$

[49]The value 7.39 that appears 3 times in Table 1 is just an approximation of $\exp(\mu) = \exp(2)$.

Table 2. CE for γ-CRRA EU and τ-QP for $\gamma = 1.41$, 1.82, 2.65, and $\tau = 0.34$.

σ_i / Certaint Equivalents:	$\gamma_1 = 1.41$	$\gamma_2 = 1.82$	$\gamma_3 = 2.65$	0.34-QP
$\sigma_4 = 0.3$	7.25	7.12	6.86	6.53
$\sigma_3 = 0.7$	6.68	6.04	4.93	5.54
$\sigma_2 = 1.5$	4.66	2.94	1.15	3.98
$\sigma_1 = 3.0$	1.17	0.18	0.00	2.14

$$Y^4 \succ_\tau X^4 \succ_\tau X^3 \succ_\tau Y^3 \succ_\tau X^2 \succ_\tau Y^2 \succ_\tau X^1 \succ_\tau Y^1.$$

We will see now how these rankings confirm the previous predictions. Observe that if $\sigma = 0.5$ and $\gamma_1 = 1.41$, the variables that belong to the set of permissible values is $\{X^4, Y^3, Y^2, Y^1\}$. Indeed, $X^4 \succ_1 Y^3 \succ_1 Y^2 \succ_1 Y^1$ and $X^4 \succ_\tau Y^3 \succ_\tau Y^2 \succ_\tau Y^1$. On the other hand, $\{X^3, X^2, X^1, Y^4\}$ are not permissible since these variables lead to inconsistencies with $\succcurlyeq_\tau$:

$$\begin{aligned} &\text{for } X^3: && X^3 \succ_1 Y^4 \text{ but } Y^4 \succ_\tau X^3;\\ &\text{for } X^2: && X^2 \succ_1 Y^2 \text{ but } Y^2 \succ_\tau X^2;\\ &\text{for } X^1: && Y^1 \succ_1 X^1 \text{ but } X^1 \succ_\tau Y^1;\\ &\text{for } Y^4: && X^4 \succ_1 Y^4 \text{ but } Y^4 \succ_\tau X^4. \end{aligned}$$

Similarly, if if $\sigma = 1$ and $\gamma_2 = 1.82$, the variables that belong to the set of permissible values is $\{X^4, X^3, Y^2, Y^1\}$. Indeed, $X^4 \succ_2 X^3 \succ_2 Y^2 \succ_2 Y^1$ and $X^4 \succ_\tau X^3 \succ_\tau Y^2 \succ_\tau Y^1$. On the other hand, $\{X^2, X^1, Y^4, Y^3\}$ are not permissible, since these variables lead to inconsistencies with $\succcurlyeq_\tau$:

$$\begin{aligned} &\text{for } X^2: && X^2 \succ_2 Y^2 \text{ but } Y^2 \succ_\tau X^2;\\ &\text{for } X^1: && Y^1 \succ_2 X^1 \text{ but } X^1 \succ_\tau Y^1;\\ &\text{for } Y^4: && X^4 \succ_2 Y^4 \text{ but } Y^4 \succ_\tau X^4;\\ &\text{for } Y^3: && X^3 \succ_2 Y^3 \text{ but } Y^3 \succ_\tau X^3. \end{aligned}$$

Finally, if if $\sigma = 2$ and $\gamma_3 = 2.65$, the variables that belong to the set of permissible values is $\{X^4, X^3, X^2, Y^1\}$. Indeed, $X^4 \succ_3 X^3 \succ_3 X^2 \succ_3 Y^1$ and $X^4 \succ_\tau X^3 \succ_\tau X^2 \succ_\tau Y^1$. On the other hand, $\{X^1, Y^4, Y^3, Y^2\}$ are not permissible, since these variables lead to inconsistencies with $\succcurlyeq_\tau$:

$$\begin{aligned} &\text{for } X^1: && Y^1 \succ_3 X^1 \text{ but } X^1 \succ_\tau Y^1;\\ &\text{for } Y^4: && X^4 \succ_3 Y^4 \text{ but } Y^4 \succ_\tau X^4;\\ &\text{for } Y^3: && Y^3 \succ_3 X^3 \text{ but } X^3 \succ_\tau Y^3;\\ &\text{for } Y^2: && Y^2 \succ_3 X^2 \text{ but } X^2 \succ_\tau Y^2. \end{aligned}$$

The discussion in this Appendix has important practical implications for empirically identifying and separating the QP from the EU. For example, when designing an experiment to identify and estimate the risk attitude and compare these models, one needs enough variation in the lotteries to be able to separate them. de Castro et al. (2022c) consider binary lotteries with fixed payoffs, but consider substantial variation in the corresponding probabilities to identify the parameters of these two models.

B.3 *Comparison with Epstein–Zin*

Quantile preferences (QP) are representatives of monotone preferences studied by Bommier, Kochov, and Le Grand (2017). Thus, in particular, QP has also the recursive representation established by their Lemma 1, namely

$$U(c,m)=W\big(c,I\big(m\circ U^{-1}\big)\big),$$

where W is a time aggregator and I is a certainty equivalent.[50] Although the usual *specification* of Epstein and Zin preferences is not monotonic, in general, as Bommier, Kochov, and Le Grand (2017) show, the above equation may suggest that QP are also a subclass of the Epstein and Zin (1989) general preferences. It turns out that this is not the case.

Epstein and Zin (1989, p. 944) called their certainty equivalent as *mean value* functional as map from the set of measures to $\mathbb{R}_+$ "which is consistent with first- and second-degree stochastic dominance and satisfies" $\mu(\delta_x)=x$, $\forall x\in\mathbb{R}_+$.

While the quantile certainty equivalent is consistent with first-degree stochastic dominance, it does not satisfy second-degree stochastic dominance. To see this, consider the following example.

Example B.1. Let Y be a risk-free lottery that pays 100 for sure. Let X be a mean-preserving spread of Y, such as the following: $X=99+p=100-(1-p)$ with probability $p\in(0,1)$ and $X=100+p$ with probability $1-p$. It is clear that $\mathrm{E}[X]=100$ and $\mathrm{E}[u(X)]\le u(\mathrm{E}[X])=u(100)=u(Y)$ for any concave u. However,

$$\mathrm{Q}_\tau[X]=\begin{cases}99+p & \text{if } \tau\le p,\\ 100+p & \text{if } \tau>p.\end{cases}$$

Thus, $\mathrm{Q}_\tau[X]>\mathrm{Q}_\tau[Y]=100$ if $\tau>p$. Finally, note that while Y stochastically dominates X in the second degree for any $p\in(0,1)$, a τ-quantile maximizer may prefer X if $\tau>p$. ◊

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