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Efficient and strategy-proof mechanism under general constraints

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This study investigates efficient and strategy-proof mechanisms for allocating indivisible goods under constraints. First, we examine a setting without endowments. In this setting, we introduce a class of constraints—ordered accessibility—for which the serial dictatorship (SD) mechanism is Pareto-efficient (PE), individually rational (IR), and group strategy-proof (GSP). Then we prove that accessibility is a necessary condition for the existence of PE, IR, and GSP mechanisms. Moreover, we show an example where the SD mechanism with a dynamically constructed order satisfies PE, IR, and GSP if one school has an arbitrary accessible constraint and each of the other schools has a capacity constraint. Second, we examine a setting with endowments. We find that the generalized matroid is a necessary and sufficient condition on the constraint structure for the existence of a mechanism that is PE, IR, and strategy-proof. We also demonstrate that a top trading cycles mechanism satisfies PE, IR, and GSP under any generalized matroid constraint. Finally, we observe that any two out of the three properties—PE, IR, and GSP—can be achieved under general constraints.

Keywords. Matching with constraints, efficient matching, generalized matroid, strategy-proofness.

JEL classification. C78, D47, D71.

1. Introduction

Our focus is on the problem of allocating indivisible goods among agents in the presence of constraints. For example, when assigning schools to students, each school should satisfy not only the usual capacity constraints, but also meet diversity requirements, including type-specific quotas (Abdulkadiroğlu and Sönmez (2003)) and proportionality constraints (Nguyen and Vohra (2019)). Additionally, schools may have

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minimal quotas to determine the minimum number of students required for their operations. In the case of refugee resettlement (Delacrétaz, Kominers, and Teytelboym (2023)), the central authority must consider factors such as heterogeneous family sizes and other requirements—such as job training and language classes—resulting in multidimensional knapsack constraints. In student–project assignment problems (Abraham, Irving, and Manlove (2007)) in which an instructor can offer multiple projects, certain subsets of projects may share common quotas, as both projects and instructors have capacity constraints.

Our goal is to characterize those constraints that admit the existence of allocation mechanisms that are Pareto-efficient (PE), individual rational (IR), and strategy-proof (SP) for the agents. PE is a natural efficiency requirement and IR ensures that agents have incentives to participate in the mechanism. SP is often considered desirable because it eliminates the need for participating agents to engage in sophisticated reasoning; truthful reporting of preferences becomes a dominant strategy. We also examine group strategy-proofness (GSP), which is a stronger requirement than SP, as GSP mechanisms are robust to manipulation by groups of agents.[1]

We consider two settings. In the first, agents are not endowed with any goods, as in the case of school choice. In the second case, some agents are endowed with a good, such as in the case with teacher reassignment (Combe, Tercieux, and Terrier (2022), Combe, Dur, Tercieux, Terrier, and Ünver (2022)). Refugee resettlement would fit into either setting (Delacrétaz, Kominers, and Teytelboym (2023)).

Before summarizing our results, we will establish a context: agents will be referred to as students, and objects are seats within schools. Constraints on how students must be assigned to schools, beyond the obvious requirement that no school exceeds its capacity, will be referred to as feasibility constraints.

For the no-endowment setting, there are a variety of PE, IR, and GSP mechanisms for allocating students to schools that satisfy various feasibility constraints. For example, Delacrétaz, Kominers, and Teytelboym (2023) proposed a modified version of a top trading cycle (TTC) mechanism for multidimensional knapsack constraints. Kamada and Kojima (2023) introduced *general upper bound (hereditary or downward-closed)* constraints. This class also yields the existence of PE, IR, and GSP mechanisms. However, there is no PE, IR, and GSP mechanism for arbitrary constraints. For example, a desired mechanism may not exist under proportionality constraints (see Example 3).

Our result delineates the boundary between what is possible and what is not. We show that the SD mechanism with a dynamically constructed order satisfies PE, IR, and GSP if one school has an *accessible* constraint and each of the other schools has a capacity constraint (Theorem 2). Furthermore, we prove that accessibility is a necessary condition (in a maximal domain sense) to guarantee the existence of a mechanism that satisfies PE, IR, and GSP (Theorem 3). Moreover, a PE, IR, and GSP mechanism exists when the feasibility constraints satisfy a property called σ-*accessibility* for some permutation σ of the students (Theorem 1).

[1]Instances of coordinated reporting to manipulate school choice mechanisms have been documented by Pathak and Sönmez (2008).

An example of a σ-accessible constraint is when a school requires that the number of minority students matched to it must be at least half the number of majority students matched to it. This constraint is σ-accessible for a permutation σ in which the minority students are ahead of the majority students. The σ-accessible constraints also arise in school choice in China (Huang (2021)). In China, each district contains multiple schools, and although students can apply to schools in other districts, the government imposes limits on the proportion of cross-district students in schools. Note that these constraints are accessible but not downward-closed. In contrast, every downward-closed constraint is σ-accessible for any σ, and every σ-accessible constraint is accessible.

Now let us turn to the setting with endowments. Here, IR requires that each student be assigned to a school that is at least as good as her endowment. In general, there is no PE, IR, and SP mechanism under arbitrary constraints. Delacrétaz, Kominers, and Teytelboym (2023) provide an example with multidimensional knapsack constraints (see Example 4). This raises the question of which constraint structure is essential for the existence of PE, IR, and SP mechanisms. We show that the feasibility constraints being generalized matroid (g-matroid) is both a "necessary and sufficient" condition to guarantee existence.

To establish sufficiency, we modify the TTC with M-convex set constraints (TTC-M) mechanism introduced by Suzuki, Tamura, and Yokoo (2018) (Theorem 4). Our modification of TTC-M not only handles the constraints covered by Suzuki, Tamura, and Yokoo (2018). but also accommodates a wider range of more complex constraints, as detailed in Section 1.1. To establish the necessity of the g-matroid condition, we provide an example of a market in which a single school has a constraint that is not a g-matroid, and for which no PE, IR, and SP mechanism exists (Theorem 5).

1.1 *Related work*

Our study is closely related to the papers by Suzuki, Tamura, and Yokoo (2018), Suzuki, Tamura, Yahiro, Yokoo, and Zhang (2023). These studies explored settings with endowments and a generalized TTC, where the distributional constraint is represented by an M-convex set on the vector of the number of students assigned to each school. Suzuki, Tamura, and Yokoo (2018), Suzuki et al. (2023) proposed the TTC-M mechanism and proved that it is PE, IR, and GSP. We make two major contributions to the literature.

First, we identify that a g-matroid is a necessary condition of the constraint structure for the existence of mechanisms that satisfy the three desirable properties. This finding partially addresses the open question posed by Suzuki et al. (2023). In addition, g-matroid is an important concept in the literature on indivisible goods allocation problems with monetary transfers. Kelso and Crawford (1982) introduced the gross substitutes condition and showed that a competitive equilibrium exists under this condition. The key fact is that a demand correspondence derived from the gross substitutes condition forms a g-matroid for every price vector (Gul and Stacchetti (1999), Fujishige and Yang (2003), Nguyen and Vohra (2024)).

Second, our model generalizes theirs because constraints are imposed on the matched student–school pairs. An example of such constraints can be found in academic hiring, where each student (or applicant) has multiple labels based on their expertise, and each school (or university) provides an upper and lower quota on each label (Huang (2010), Fleiner and Kamiyama (2016), Yokoi (2017)). Another example is a model in which a student has multiple types, but is allocated as one of her types (Kurata, Hamada, Iwasaki, and Yokoo (2017)). This model includes important real-life applications, such as affirmative action in India (Sönmez and Yenmez (2022)) and Brazil (Aygün and Bó (2021)).

Another difference from the model proposed by Suzuki, Tamura, and Yokoo (2018), Suzuki et al. (2023) is that our model includes outside options and allows for unmatched agents. Therefore, our model is flexible enough to include house allocation with existing tenants (Abdulkadiroğlu and Sönmez (1999)) and kidney exchanges (Roth, Sönmez, and Ünver (2004)) as special cases. In addition, our TTC generalizes the "'you request my house—I get your turn" (YRMH-IGYT) mechanism (Abdulkadiroğlu and Sönmez (1999)) and the top trading cycles and chains (TTCC) mechanism with the SP and PE chain rule (Roth, Sönmez, and Ünver (2004)).

Hafalir, Kojima, and Yenmez (2023) studied the existence of a desired mechanism that weakly improves a distributional objective upon the initial matching. They showed that if the distributional objective satisfies a notion of discrete concavity, called *pseudo* $M^\natural$*-concavity*, their generalized TTC satisfies (constrained) PE, IR, and SP. It should be noted that the set of matchings that weakly improves the distributional objective upon the initial matching forms a g-matroid if the distributional objective satisfies pseudo $M^\natural$-concavity.

Kamiyama (2013) explored the case where the outside option is assumed to be worst for every student (every school is acceptable to any student). He showed that a mechanism, called the generalized serial dictatorship with project closures (GSDPC), satisfies PE and SP for general constraints. The GSDPC sequentially assigns each student to her best school to the extent that the remaining students can be feasibly assigned. It is not difficult to see that the GSDPC satisfies GSP. Furthermore, in the setting without endowments, any mechanism is IR; hence, the GSDPC satisfies PE, IR, and GSP.

Imamura and Kawase (2024) studied PE under a general constraint and, in particular, provided a method for checking whether a given matching is Pareto-efficient. They identified that a matroid is a necessary and sufficient condition for the constraint to characterize the set of PE matchings by serial dictatorship (SD). They also introduced the constrained serial dictatorship (CSD) to check PE under general constraints. The CSD is almost the same as the GSDPC; however, it also considers IR. Hence, the CSD can be viewed as a PE and IR mechanism, but it is not SP.

The field of matching under constraints has grown rapidly (Abdulkadiroğlu and Sönmez (2003), Biró, Fleiner, Irving, and Manlove (2010), Hafalir, Yenmez, and Yildirim (2013), Ehlers, Hafalir, Yenmez, and Yildirim (2014), Kamada and Kojima (2015, 2017), Kawase and Iwasaki (2020)) with a primary focus on stability or fairness. However, our study emphasizes the importance of PE. Several studies examined PE mechanisms under constraints (Root and Ahn (2020), Yokote (2022), Delacrétaz, Kominers, and Teytelboym (2023)). In particular, Delacrétaz, Kominers, and Teytelboym (2023) studied PE, IR, and SP mechanisms under multidimensional knapsack constraints. As previously highlighted, they established that desired mechanisms do not exist when endowments are present and do exist when they are not. These findings can be derived from our results.

2. Preliminaries

2.1 *Model*

A market is a tuple $(I, S, (\succ_i)_{i\in I}, (\mathcal{F}_s)_{s\in S}, \omega)$. $I = \{1, 2, \dots, n\}$ is a finite set of students, and S is a finite set of schools. Each student i has a strict preference $\succ_i$ over $S \cup \{\varnothing\}$, where $\varnothing$ means being unmatched (or an outside option). We write $x \succeq_i x'$ if either $x \succ_i x'$ or $x = x'$ holds. $\mathcal{F}_s$ is the family of subsets of students that school s can accept; $\omega\colon I \to S \cup \{\varnothing\}$ is an endowment function, where $\omega(i) = s$ denotes that the endowment of i is $s \in S \cup \{\varnothing\}$. In a setting without endowments, we assume that $\omega(i) = \varnothing$ for all $i \in I$.

A *matching* μ is a subset of $I \times S$ such that each student i appears at most in one pair of μ; that is, $|\mu \cap \{(i, s) : s \in S\}| \leq 1$ for all $i \in I$. For each $i \in I$, we write $\mu(i)$ to denote the school to which i is assigned at μ, that is, $\mu(i) = s$ if $(i, s) \in \mu$ and $\mu(i) = \varnothing$ if $(i, s) \notin \mu$ for all $s \in S$. Similarly, for each $s \in S$, we write $\mu(s)$ to denote the set of students assigned to s at μ, that is, $\mu(s) = \{i \in I : (i, s) \in \mu\}$. A matching is called *feasible* if $\mu(s) \in \mathcal{F}_s$ for all $s \in S$. For notational simplicity, we sometimes add unmatched pairs $(i, \varnothing)$ to a matching, but we ignore such pairs.

Let μ_0 denote the endowment matching (or initial matching), that is, $\mu_0(i) = \omega(i)$ for all $i \in I$. We assume that the endowment matching is feasible, that is, $\mu_0 \in \mathcal{F}$.

2.2 *Constraints*

The aggregated constraint is sometimes represented by $\mathcal{F} = \{X \subseteq I \times S : X(s) \in \mathcal{F}_s\ (\forall s \in S)\}$, where $X(s) = \{i \in I : (i, s) \in X\}$.[2] Using this notation, a matching μ is feasible if and only if $\mu \in \mathcal{F}$. In addition, we will also consider a distributional constraint $\mathcal{F} \subseteq I \times S$ that may not be expressible through individual constraints $(\mathcal{F}_s)_{s\in S}$.

Let E be a ground set. A family of subsets $\mathcal{F} \subseteq 2^E$ is a *matroid* if it satisfies the following three properties: (i) $\emptyset \in \mathcal{F}$; (ii) if $X \in \mathcal{F}$ and $X' \subseteq X$, then $X' \in \mathcal{F}$; (iii) if $X, Y \in \mathcal{F}$ and $|X| < |Y|$, then $y \in Y \setminus X$ exists such that $X \cup \{y\} \in \mathcal{F}$. If individual constraint $\mathcal{F}_s$ is a matroid for every $s \in S$, then the aggregated constraint $\mathcal{F}$ is also a matroid. Given a matroid $\mathcal{F}$, an element $B \in \mathcal{F}$ is called a *base* if B is an inclusion-wise maximal subset of E in $\mathcal{F}$. According to property (iii), all the bases of a given matroid have the same cardinality. The collection of all the bases is called the *matroid base family*. The matroid base family can be characterized as a nonempty family of subsets $\mathcal{B} \subseteq 2^E$ that satisfies the following property: for any $B, B' \in \mathcal{B}$ and $b \in B \setminus B'$, there exists $b' \in B' \setminus B$ such that $(B \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$.

Matroid constraints include many real-life examples of constraints. Abdulkadiroğlu and Sönmez (2003) formally studied type-specific quotas to address student diversity

[2] Note that $X \in \mathcal{F}$ may not be a matching because some students may appear multiple times.

requirements within schools. Kamada and Kojima (2015) studied the regional maximum quotas in the context of medical residency matching in Japan. These constraints are special cases of a matroid.

A nonempty family of subsets $\mathcal{F} \subseteq 2^E$ is a g-matroid if, for any $X, Y \in \mathcal{F}$ and $e \in X \setminus Y$, it holds that

(i) $X \setminus \{e\}$ and $Y \cup \{e\} \in \mathcal{F}$ or

(ii) there is $e' \in Y \setminus X$ such that $(X \setminus \{e\}) \cup \{e'\}$ and $(Y \cup \{e\}) \setminus \{e'\}$ are in $\mathcal{F}$.

Alternatively, a g-matroid can be characterized by another property (Murota and Shioura (1999), Tardos (1985)): for any $X, Y \in \mathcal{F}$ and $e \in X \setminus Y$, it holds that

(i) $X \setminus \{e\} \in \mathcal{F}$ or $(X \setminus \{e\}) \cup \{e'\} \in \mathcal{F}$ for some $e' \in Y \setminus X$ and

(ii) $Y \cup \{e\} \in \mathcal{F}$ or $(Y \cup \{e\}) \setminus \{e'\} \in \mathcal{F}$ for some $e' \in Y \setminus X$.

Moreover, a g-matroid can be represented by $\mathcal{F} = \{S \subseteq E : p(S) \leq |X \cap S| \leq q(S)\ (\forall X \subseteq E)\}$, with a *paramodular* pair (p, q) (Frank (2011)). Here, a pair (p, q) is called paramodular if

(i) p is supermodular (i.e., $p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y)$ for all $X, Y \subseteq E$)

(ii) q is submodular (i.e., $q(X) + q(Y) \geq q(X \cup Y) + q(X \cap Y)$ for all $X, Y \subseteq E$)

(iii) p, q satisfy cross-inequality (i.e., $p(X) - q(Y) \geq p(X \setminus Y) - q(Y \setminus X)$ for all $X, Y \subseteq E$).

A g-matroid is also called an $M^\natural$-convex family because the corresponding set of 0–1 vectors is an $M^\natural$-convex set as a subset of $\mathbb{Z}^E$ (Murota (2016)). The subsequent proposition gives useful subclasses of g-matroids. Refer to (Yokoi, 2017, Proposition 17) for its proof.

Proposition 1. *Let $\mathcal{L} \subseteq 2^E$ be a* laminar family[3] *and let $\ell_L, u_L \in \mathbb{Z}_{\geq 0}$ for each $L \in \mathcal{L}$. Then a family $\mathcal{F} = \{X \subseteq E : \ell_L \leq |X \cap L| \leq u_L\ (\forall L \in \mathcal{L})\}$ is a g-matroid if $\mathcal{F} \neq \emptyset$.*

It is not difficult to see that a g-matroid is a class that includes both a matroid and a matroid base family. Moreover, a nonempty family of subsets $\mathcal{F} \subseteq 2^E$ is a g-matroid if and only if there exists a matroid base family $\mathcal{B} \subseteq 2^{E'}$ with $E \subseteq E'$ such that $\mathcal{F} = \{B \cap E : B \in \mathcal{B}\}$ (Tardos (1985)). Additionally, for a g-matroid $\mathcal{F}$ and $\ell, u \in \mathbb{Z}_{\geq 0}$, its truncation $\mathcal{F}_\ell^u = \{X \in \mathcal{F} : \ell \leq |X| \leq u\}$ is also a g-matroid if $\mathcal{F}_\ell^u \neq \emptyset$ (Tardos (1985)). If individual constraint $\mathcal{F}_s$ is a g-matroid for every $s \in S$, then the aggregated distributional constraint is also a g-matroid.

A family of subsets $\mathcal{F} \subseteq 2^E$ belongs to the class of *general upper bound* (or independence system) if $X \subseteq Y \in \mathcal{F}$ implies $X \in \mathcal{F}$. A family of subsets $\mathcal{F} \subseteq 2^E$ is called *accessible* if for any $X \in \mathcal{F} \setminus \{\emptyset\}$, there exists $e \in X$ such that $X \setminus \{e\} \in \mathcal{F}$. By definition, any

[3]A family $\mathcal{L} \subseteq 2^E$ is called a laminar family if, for any $X, Y \in \mathcal{L}$, either $X \cap Y = \emptyset$, $X \subseteq Y$, or $X \supseteq Y$.

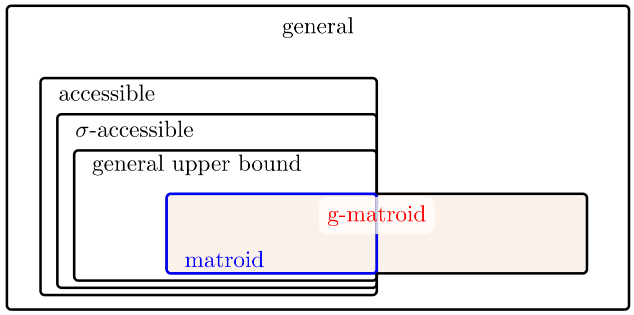

FIGURE 1. Classes of constraints we deal with in this study.

nonempty accessible set system must contain the empty set. For an order σ of E, a family of subsets $\mathcal{F} \subseteq 2^E$ is called *σ-accessible* if for any $X \in \mathcal{F} \setminus \{\emptyset\}$, we have $X \setminus \{e\} \in \mathcal{F}$ for $e \in \arg\max\{\sigma^{-1}(e) : e \in X\}$. By definition, every general upper bound is σ-accessible for any σ, and every σ-accessible set system (for some σ) is accessible. In addition, these classes are distinct, as $\{\emptyset, \{1\}, \{1, 2\}\}$ is σ-accessible for $\sigma = (1, 2)$, but not general upper bound, and $\{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ is accessible, but not σ-accessible for any σ.

Figure 1 illustrates the relationship among classes of constraints.

2.3 *Properties*

A matching μ is said to *Pareto dominate* μ' if $\mu(i) \succeq_i \mu'(i)$ for all $i \in I$ and $\mu(i) \succ_i \mu'(i)$ for some $i \in I$. A feasible matching μ is called *Pareto-efficient* (PE) if there is no feasible matching μ' that Pareto dominates μ. Additionally, a feasible matching μ is called *individually rational* (IR) if $\mu(i) \succeq_i \mu_0(i)$ for all $i \in I$.

A mechanism ψ is a map from a preference profile to a feasible matching. A mechanism is PE and IR if it always produces a feasible matching that fulfills the conditions of PE and IR, respectively.

A mechanism ψ is *strategy-proof* (SP) if for every preference profile $\succ_I$, there is no $i \in I$ and her preference $\succ'_i$ such that $\psi[\succ'_i, \succ_{-i}](i) \succ_i \psi[\succ_I](i)$, where $\succ_I = (\succ_j)_{j\in I}$ and $\succ_{-i} = (\succ_j)_{j\in I\setminus\{i\}}$. Intuitively, SP requires that no student can be assigned to a strictly preferred school by misreporting her preference. Similarly, the mechanism ψ is *group strategy-proof* (GSP) if, for every preference profile $\succ_I$, there is no $I' \in 2^I \setminus \{\emptyset\}$ and their preference profile $\succ_{I'}$ such that $\psi[\succ'_{I'}, \succ_{-I'}](i) \succeq_i \psi[\succ_I](i)$ for all $i \in I'$ and $\psi[\succ'_{I'}, \succ_{-I'}](i) \succ_i \psi[\succ_I](i)$ for some $i \in I'$, where $\succ'_{I'} = (\succ'_j)_{j\in I'}$ and $\succ_{-I'} = (\succ_j)_{j\in I\setminus I'}$. In other words, GSP requires that no group of students can make each member weakly better off and that at least one student in the group is strictly better off by jointly misreporting her preferences. Clearly, GSP is a stronger property than SP.

A mechanism is *nonbossy* if no student can influence the assignment of others without changing her own assignment by misreporting her preference. Formally, for every preference profile $\succ_I$, $i \in I$, and preference $\succ'_i$, $\psi[\succ_I](i) = \psi[\succ'_i, \succ_{-i}](i)$ implies $\psi[\succ_I] = \psi[\succ'_i, \succ_{-i}]$. Pápai (2000) showed that a mechanism is GSP under unit capacity constraint if and only if it is SP and nonbossy. It is easy to verify that this equivalence still holds under any constraints in our model.

2.4 *Applications*

In this section, we examine some applications on matching under constraints and show that our results can be used to check the existence of a desired mechanism in each case.[4]

Reassignment of teachers with distributional concerns Combe et al. (2022), Combe, Tercieux, and Terrier (2022) studied a teacher reassignment market and focused on improving distributional welfare over the initial matching μ_0. Each teacher $i \in I$ has a type $\tau(i)$ that represents her characteristics, such as experience. Each school s has a quota q_s and a type ranking $\rhd_s$ over the types $\Theta := \{\tau(i) : i \in I\} \cup \{\theta_\varnothing\}$. We assume that $\tau(i) \rhd_s \theta_\varnothing$ for all $i \in \mu_0(s)$ and $s \in S$. A matching μ is *status quo improving* if it is IR for each teacher, and *Lorenz dominates* the initial matching for each school s (i.e., $\tau(i) \rhd_s \theta_\varnothing$ for all $i \in \mu(s)$ and $|\{i \in \mu(s) : \tau(i) \trianglerighteq_s \theta\}| \geq |\{i \in \mu_0(s) : \tau(i) \trianglerighteq_s \theta\}|$ for all type $\theta \in \Theta$). A matching is *status quo improving teacher optimal* (SI teacher optimal) if it is status quo improving and not Pareto dominated for teachers by any other status quo improving matching. Combe et al. (2022) provided a variant of TTC, which is SI teacher optimal and SP.

Their existence result can be derived from our findings.[5] For each school s, define a constraint as a family of subsets of students that Lorenz dominate the students matched to s in the initial matching. Then SI teacher optimality is equivalent to the conjunction of IR and PE in a setting with endowments. The key fact is that the constraint for each school forms a g-matroid, enabling the application of Theorem 4. Moreover, our result can strengthen their result from SP to GSP.

Note that the constraint of Lorenz domination for each school s can be represented by a g-matroid of the form in Proposition 1 by setting $\mathcal{L} = \{L_\theta : \theta \in \Theta, \theta \trianglerighteq_s \theta_\varnothing\}$ and

- $L_{\theta_\varnothing} = \{i \in I : \theta_\varnothing \rhd_s \tau(i)\}$, $u_{\theta_\varnothing} = \ell_{\theta_\varnothing} = 0$, and
- $L_\theta = \{i \in I : \tau(i) \trianglerighteq_s \theta\}$, $u_\theta = q_s$, $\ell_\theta = |\{i \in \mu_0(s) : \tau(i) \trianglerighteq_s \theta\}|$ for each $\theta \in \Theta$ with $\theta \rhd_s \theta_\varnothing$.

It is possible to construct a more general g-matroid constraint by using different values for the upper and lower bounds. For example, setting $u_\theta = |\{i \in \mu_0(s) : \tau(i) \trianglerighteq_s \theta\}| + 1$ for the most experienced type θ would prevent allocating too many such teachers to one school.

As seen above, our necessary and sufficient condition enables us to appropriately extend a model while preserving the existence of the desired mechanism.

Proportionality ceiling constraint The proportionality ceiling constraint arises from school choice in a Chinese district. In this context, the government has imposed a

[4]For additional existing models not discussed in this paper, please refer to the working paper version for details: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4844451.

[5]The model studied by Combe, Tercieux, and Terrier (2022) is a special case where unmatched teachers and schools with vacant seats are not allowed in the initial matching, and different students cannot have the same type. Thus, our findings can also derive the existence result of Combe, Tercieux, and Terrier (2022).

proportionality ceiling that states the number of students from outside a district assigned to a school should not exceed a certain fraction of the total number of students assigned (Huang (2021)).

For example, let us consider a setting in which there are two students from within the district, denoted i_1, i_2, and two students from outside the district, denoted u_1, u_2. If the proportion to be guaranteed is half, the constraint for a school s with capacity 2 is

$$\hat{\mathcal{F}}_s = \{\emptyset, \{i_1\}, \{i_2\}, \{i_1, i_2\}, \{i_1, u_1\}, \{i_1, u_2\}, \{i_2, u_1\}, \{i_2, u_2\}\}.$$

The constraint $\hat{\mathcal{F}}_s$ is σ-accessible with respect to $\sigma = (i_1, i_2, u_1, u_2)$.

In general, the proportionality ceiling constraint is σ-accessible, where σ is an order in which students from within the district are listed before those from outside the district. Theorem 1 immediately implies the existence of PE, IR, and GSP mechanisms in a setting without endowments.

Moreover, similar constraints also appear in various other applications such as resource allocation during a pandemic (Dur, Morrill, and Phan (2021)) and dynamic matching (Bando and Kawasaki (2021)). These details are discussed in Section 3.3.

Proportionality constraint Maintaining a certain balance in the student body is a common practical requirement. Specific ratios or percentages often define this balance. For example, in 2003, the Cambridge, Massachusetts, public school district implemented a policy requiring that the percentage of students from families of low socioeconomic status be within a range of 15 percent of the district's overall proportion (Nguyen and Vohra (2019)).

Consider the same students provided in the example of proportionality ceiling constraint. Suppose instead of a proportionality ceiling constraint, a proportionality constraint is imposed that the number of the two types of students must be equal. Then the constraint for school s becomes

$$\hat{\mathcal{F}}_s = \{\emptyset, \{i_1, u_1\}, \{i_1, u_2\}, \{i_2, u_1\}, \{i_2, u_2\}\}.$$

This constraint is inaccessible; thus, a proportional constraint is inaccessible in general. Thus, Theorem 3 immediately implies the nonexistence of the desired mechanism under these constraints in a setting without endowments.

The negative findings on proportionality constraints can be associated with the nonexistence of stable matchings. These details are discussed in Section 3.3.

3. Setting without endowments

In this section, we consider a setting without endowments. We first prove that, for any order σ of students, the SD mechanism with σ satisfies PE, IR, and GSP if the constraints are σ-accessible. We then observe that PE, IR, and SP mechanisms may not exist even when the constraints are accessible. Furthermore, we demonstrate that accessibility is a necessary condition for the existence of PE, IR, and GSP mechanisms.

Algorithm 1: Serial dictatorship (SD) with σ.

input : a market $(I, S, (\succ_i)_{i \in I}, \mathcal{F})$ and $\sigma \in \Sigma$
output: a matching

Let $\mu^{(0)} \leftarrow \emptyset$;
for $k \leftarrow 1, 2, \ldots, |I|$ **do**
    $r \leftarrow \arg\max_{\succ_{\sigma(k)}} \{s \in S \cup \{\varnothing\} : \mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{F}\}$;
    if $r \in S$ **then** $\mu^{(k)} \leftarrow \mu^{(k-1)} \cup \{(\sigma(k), r)\}$;
    else $\mu^{(k)} \leftarrow \mu^{(k-1)}$;
return $\mu^{(|I|)}$;

3.1 *SD mechanism for accessible constraints*

Let Σ be the set of all permutations of the students. The SD mechanism considers students one by one in a predetermined order $\sigma \in \Sigma$. In each step of the mechanism, the current student is given the opportunity to select her most preferred school from the remaining available schools, subject to the imposed constraint. Once the student makes a choice, the student is fixed on her assignment to the school of her choice. The SD mechanism is formally described in Algorithm 1.

The SD mechanism is IR because each student can at least choose the option of being unmatched. Furthermore, the mechanism is GSP because of the sequential nature of the mechanisms. Indeed, as each student selects her preferred school in her turn, there is no room for a group of students to coordinate and manipulate the outcome strategically.

Unfortunately, the SD mechanism does not satisfy PE under general constraints, even when there is only one school.[6] To observe this, consider a market with $I = \{1, 2\}$, $S = \{s\}$, $\succ_1 = \succ_2 = (s\varnothing)$, and $\mathcal{F}_s = \{\emptyset, \{1, 2\}\}$. In this market, the SD mechanism outputs a matching in which no student matches to s, regardless of the order. However, the unique PE and IR matching is the matching in which both students are matched to school s. The essential reason why the SD mechanism fails to output the matching is that the constraint $\mathcal{F}_s$ is not accessible.

By contrast, the SD mechanism is PE if the individual constraints are σ-accessible for a common $\sigma \in \Sigma$. We prove this fact for a more general case in which the distributional constraint is σ-accessible. A distributional constraint $\mathcal{F}$ is σ-accessible if $\mu \setminus \{(i, \mu(i))\} \in \mathcal{F}$ for any feasible nonempty matching $\mu \in \mathcal{F} \setminus \{\emptyset\}$ and $i \in \arg\max\{\sigma^{-1}(i) : i \in I, \mu(i) \neq \varnothing\}$. Note that, for any σ-accessible individual constraints $(\mathcal{F}_s)_{s \in S}$, the aggregated constraint $\mathcal{F}$ is also σ-accessible. Indeed, we have the following theorem.

Theorem 1. *If the distributional constraint is σ-accessible for an order $\sigma \in \Sigma$, the SD mechanism (Algorithm 1) with σ satisfies PE, IR, and GSP.*

[6] In contrast, the CSD (Imamura and Kawase (2024)) is PE, IR, and GSP for any market consisting of only one school s. The mechanism is PE and IR in general. In addition, it is GSP since each student can only indicate whether she desires the school s, and misreporting affects the outcome only when it makes the agent worse off.

Proof. The SD mechanism satisfies IR and GSP, as we have stated above. Therefore, it is sufficient to prove that it also satisfies PE. Suppose, on the contrary, that the SD mechanism outputs a matching μ that is not PE. Then there exists a feasible matching μ' ($\neq \mu$) that Pareto dominates μ. Let k be the smallest index such that $\mu(\sigma(k)) \neq \mu'(\sigma(k))$. Then we have $\mu(\sigma(j)) = \mu'(\sigma(j))$ for $j = 1, 2, \dots, k-1$ and $\mu'(\sigma(k)) \succ_{\sigma(k)} \mu(\sigma(k))$. This leads to a contradiction because $\sigma(k)$ could have chosen $\mu'(\sigma(k))$ on her turn in the SD mechanism. □

We remark that σ-accessibility is not a necessary condition to guarantee the existence of a PE, IR, and GSP mechanism.

Example 1. Let $I = \{1, 2, 3\}$, $S = \{s_1, s_2\}$, $\mathcal{F}_{s_1} = \{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$, and $\mathcal{F}_{s_2} = \{X \subseteq I : |X| \leq 1\}$. Note that $\mathcal{F}_{s_1}$ is not σ-accessible for any σ.

However, this market admits a PE, IR, and GSP mechanism. Indeed, the SD mechanism, which employs the order $(1, 2, 3)$ if s^1 is the most preferable school for student 1 and order $(1, 3, 2)$ otherwise, satisfies PE, IR, and GSP. This is because student 1 is assigned to the most preferable school, and the sets $\{X \setminus \{1\} : 1 \in X \in \mathcal{F}_{s_1}\}$ and $\{X : 1 \notin X \in \mathcal{F}_{s_1}\}$ are $(2, 3)$- and $(3, 2)$-accessible, respectively.

More generally, if one school s^* has an arbitrary accessible constraint $\mathcal{F}_{s^*}$ and each of the other schools $s \in S \setminus \{s^*\}$ has a capacity constraint $\mathcal{F}_s = \{X \subseteq I : |X| \leq q_s\}$, then the SD mechanism with a dynamically constructed order (which is formally described in Algorithm 2) satisfies PE, IR, and GSP.

Theorem 2. *If one school s^* has an accessible constraint $\mathcal{F}_{s^*}$ and each of the other schools $s \in S \setminus \{s^*\}$ has a capacity constraint $\mathcal{F}_s = \{X \subseteq I : |X| \leq q_s\}$, then Algorithm 2 satisfies PE, IR, and GSP.*

Algorithm 2: Serial dictatorship (SD) with a dynamically constructed order.

input : a market $(I, S, (\succ_i)_{i \in I}, (\mathcal{F}_s)_{s \in S})$ where $\mathcal{F}_{s^*}$ is accessible and $\mathcal{F}_s = \{X \subseteq I : |X| \leq q_s\}$ $(\forall s \in S \setminus \{s^*\})$ and $\sigma \in \Sigma$
output: a matching

Let $\mu^{(0)} \leftarrow \emptyset$ and $P \leftarrow I$;
for $k \leftarrow 1, 2, \dots, |I|$ **do**
  if $\exists i \in P$ *such that* $\mu^{(k-1)}(s^*) \cup \{i\} \in \mathcal{F}_{s^*}$ **then**
    Let $i^{(k)}$ be the first such an i according to the order of σ;
  else Pick the first $i^{(k)} \in P$ according to the order of σ;
  Let $r \leftarrow \arg\max_{\succ_{i^{(k)}}} \{s \in S \cup \{\varnothing\} : \mu^{(k-1)} \cup \{(i, s)\} \in \mathcal{F}\}$;
  if $r \in S$ **then** $\mu^{(k)} \leftarrow \mu^{(k-1)} \cup \{(i^{(k)}, r)\}$;
  else $\mu^{(k)} \leftarrow \mu^{(k-1)}$;
  $P \leftarrow P \setminus \{i^{(k)}\}$;
return $\mu^{(|I|)}$;

Proof. The mechanism is IR since each student can at least choose the option of being unmatched. The mechanism is GSP because each student selects her preferred school in her turn.

To prove PE, suppose to the contrary that the mechanism outputs a matching μ that is not PE. Let μ' be a feasible matching that Pareto dominates μ. Let $k^* \in \{1, 2, \dots, |I|\}$ be the smallest integer such that $\mu'(i^{(k^*)}) \succ_{i^{(k^*)}} \mu(i^{(k^*)})$. Then, regardless of whether $\mu'(i^{(k^*)}) = s^*$ or not, $i^{(k^*)}$ can select $\mu'(i^{(k^*)}) = s^*$ in the k^*th round of Algorithm 2. This contradicts the assumption and, thus, the mechanism is PE. □

3.2 *Impossibility for inaccessible constraints*

If more than one school does not have a capacity constraint, a PE, IR, and SP mechanism may not exist even when the constraints are accessible.

Example 2. Let $I = \{1, 2\}$ and $S = \{s_1, s_2\}$. The constraint $\mathcal{F}_s$ of each school s is defined as

$$\mathcal{F}_{s_1} = \{\emptyset, \{2\}, \{1, 2\}\} \quad \text{and} \quad \mathcal{F}_{s_2} = \{\emptyset, \{1\}, \{1, 2\}\}.$$

Note that $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_2}$ are accessible. Suppose that the true preference $\succ_i$ of each student i is given as

$$\succ_1 = (s_1 s_2 \emptyset) \quad \text{and} \quad \succ_2 = (s_2 s_1 \emptyset).$$

It is not difficult to see that there exist only two PE and IR matchings for their true preferences: $\mu_1 := \{(1, s_1), (2, s_1)\}$ and $\mu_2 := \{(1, s_2), (2, s_2)\}$. If student 1 misreports her preference as $\succ_1' = (s_1 \emptyset s_2)$, whereas student 2 reports her true preference $\succ_2$, then μ_1 is the unique PE and IR matching. Conversely, if student 1 reports her true preference $\succ_1$, and student 2 misreports her preference as $\succ_2' = (s_2 \emptyset s_1)$, then μ_2 is the unique PE and IR matching. Therefore, in any PE and IR mechanism, either student 1 or 2 can benefit from misreporting their preferences. This means that no mechanism can simultaneously satisfy PE, IR, and SP for the market.

Consequently, accessibility is insufficient to guarantee the existence of a mechanism that satisfies PE, IR, and SP. Nevertheless, accessibility is a necessary condition in a maximal domain sense for the existence of a mechanism that satisfies PE, IR, and GSP.

We first observe that a simple market does not admit a desired mechanism. We then prove that any market does not admit a desired mechanism if there is one school with an inaccessible constraint and another school with a unit capacity constraint.

Example 3. Suppose there are two students 1, 2 and two schools s_1, s_2. The constraint $\mathcal{F}_s$ on each school s is defined as

$$\mathcal{F}_{s_1} = \{\emptyset, \{1, 2\}\} \quad \text{and} \quad \mathcal{F}_{s_2} = \{\emptyset, \{1\}, \{2\}\}.$$

The constraint $\mathcal{F}_{s_1}$ could appear as a proportional constraint; for example, the number of male and female students who match with a particular school must be equal. This market does not admit a mechanism that simultaneously satisfies PE, IR, and SP.

To obtain a contradiction, suppose that ψ is a mechanism that satisfies PE, IR, and SP. We define eight preference profiles:

- $P^{(1)} = (s_2\varnothing s_1, s_1\varnothing s_2)$
- $P^{(2)} = (s_2 s_1\varnothing, s_1\varnothing s_2)$
- $P^{(3)} = (s_2 s_1\varnothing, s_1 s_2\varnothing)$
- $P^{(4)} = (s_2\varnothing s_1, s_1 s_2\varnothing)$
- $Q^{(1)} = (s_1\varnothing s_2, s_2\varnothing s_1)$
- $Q^{(2)} = (s_1\varnothing s_2, s_2 s_1\varnothing)$
- $Q^{(3)} = (s_1 s_2\varnothing, s_2 s_1\varnothing)$
- $Q^{(4)} = (s_2\varnothing s_1, s_2 s_1\varnothing)$.

In profile $P^{(1)}$, student 1 prefers s_2, followed by $\varnothing$ and then s_1. Student 2 prefers s_1, followed by $\varnothing$ and then s_2. Based on PE and IR, we derive that $\psi[P^{(1)}] = \{(1, s_2)\}$. For profile $P^{(2)}$, the outcome $\psi[P^{(2)}]$ must be $\{(1, s_1), (2, s_1)\}$ or $\{(1, s_2)\}$ by PE and IR. However, $\psi[P^{(2)}] = \{(1, s_1), (2, s_1)\}$ is impossible by SP because it incentivizes student 1 to misreport so that the preference profile becomes $P^{(1)}$. Hence, we obtain $\psi[P^{(2)}] = \{(1, s_2)\}$. Similarly, we have $\psi[P^{(3)}] = \{(1, s_2)\}$ by PE, IR, and SP. This implies $\psi[P^{(4)}] = \{(1, s_2)\}$ by PE, IR, and SP.

Applying similar reasoning, we can determine that $\psi[Q^{(1)}] = \psi[Q^{(2)}] = \psi[Q^{(3)}] = \{(2, s_2)\}$. In addition, we can conclude that $\psi[Q^{(4)}] = \{(2, s_2)\}$ by PE, IR, and SP. However, this contradicts SP because it incentivizes student 2 to misreport at $P^{(4)}$.

Theorem 3. *Fix a set of students I with $|I| \geq 2$, a set of schools S with $|S| \geq 2$, and a school $s^* \in S$ with the constraint $\mathcal{F}_{s^*}$. Suppose that $\mathcal{F}_{s^*}$ is not accessible. Then there must exist a market $(I, S, (\mathcal{F}_s)_{s\in S})$ with $s^* \in S$ and $\mathcal{F}_s = \{X \subseteq I : |X| \leq 1\}$ for all $s \in S \setminus \{s^*\}$ such that no mechanism simultaneously satisfies PE, IR, and GSP.*

Proof. We first consider the case where $|S| = 2$. We consider a market in which $S = \{s^*, t\}$ and $\mathcal{F}_t = \{X \subseteq I : |X| \leq 1\}$. As $\mathcal{F}_{s^*}$ is not accessible, there exists a nonempty $X^* \in \mathcal{F}_{s^*}$ such that $X^* \setminus \{i\} \notin \mathcal{F}_{s^*}$ for all $i \in X^*$. Note that X^* must contain at least two students because $\emptyset \in \mathcal{F}_{s^*}$ by assumption. Suppose, to the contrary, that there exists a mechanism ψ that satisfies PE, IR, and GSP. Note that ψ is also nonbossy because it is GSP.

For each $i \in X^*$, we define $P^{(i)}$ as a preference profile such that $P^{(i)}_i = (t\varnothing s^*)$, $P^{(i)}_j = (s^*\varnothing t)$ for each $j \in X^* \setminus \{i\}$, and $P^{(i)}_j = (\varnothing s^* t)$ for each $j \in I \setminus X^*$. By PE and IR, student i must be matched with school t at $P^{(i)}$ (i.e., $(i, t) \in \psi[P^{(i)}]$). In addition, at least one student $j \in X^* \setminus \{i\}$ is unmatched at $P^{(i)}$ (i.e., $(j, s^*) \notin \psi[P^{(i)}]$) because $X^* \setminus \{i\} \notin \mathcal{F}_{s^*}$. We draw an arrow from each student $i \in X^*$ to an agent $j \in X^* \setminus \{i\}$ who is unmatched at $P^{(i)}$. Then there must be at least one cycle. Let $(i_1, i_2, \ldots, i_k)$ be such a cycle, where $(i_{\ell+1}, s^*) \notin \psi[P^{(i_\ell)}]$ for $\ell = 1, 2, \ldots, k$ (we use i_{k+1} to represent i_1 for simplicity). Note that $k \geq 2$ because there is no self-loop. For each $j \in \{1, 2, \ldots, k\}$, we define the preference profiles $\hat{P}^{(i_j)}$, $\hat{\hat{P}}^{(i_j)}$, and $Q^{(i_j)}$ as

Table 1. Preference profiles in the proof of Theorem 3.

	$\{i_1, \dots, i_k\} \setminus \{i_j, i_{j+1}\}$	i_j	i_{j+1}	$X^* \setminus \{i_1, \dots, i_k\}$	$I \setminus X^*$
$P^{(i_j)}$	$(s^*\varnothing t)$	$(t\varnothing s^*)$	$(s^*\varnothing t)$	$(s^*\varnothing t)$	$(\varnothing s^* t)$
$\hat{P}^{(i_j)}$	$(s^*\varnothing t)$	$(ts^*\varnothing)$	$(s^*\varnothing t)$	$(s^*\varnothing t)$	$(\varnothing s^* t)$
$\hat{\hat{P}}^{(i_j)}$	$(s^*\varnothing t)$	$(ts^*\varnothing)$	$(s^* t\varnothing)$	$(s^*\varnothing t)$	$(\varnothing s^* t)$
$Q^{(i_j)}$	$(s^*\varnothing t)$	$(ts^*\varnothing)$	$(ts^*\varnothing)$	$(s^*\varnothing t)$	$(\varnothing s^* t)$
R	$(ts^*\varnothing)$	$(ts^*\varnothing)$	$(ts^*\varnothing)$	$(s^*\varnothing t)$	$(\varnothing s^* t)$

- $\hat{P}^{(i_j)}_{i_j} = (ts^*\varnothing)$ and $\hat{P}^{(i_j)}_i = P^{(i_j)}_i$ for each $i \in I \setminus \{i_j\}$
- $\hat{\hat{P}}^{(i_j)}_{i_{j+1}} = (s^* t\varnothing)$ and $\hat{\hat{P}}^{(i_j)}_i = \hat{P}^{(i_j)}_i$ for each $i \in I \setminus \{i_{j+1}\}$
- $Q^{(i_j)}_{i_{j+1}} = (ts^*\varnothing)$ and $Q^{(i_j)}_i = \hat{\hat{P}}^{(i_j)}_i$ for each $i \in I \setminus \{i_{j+1}\}$.

The preference profiles are summarized in Table 1. By PE and SP, we have $\psi[\hat{P}^{(i_j)}](i_j) = t$ for $j = 1, 2, \dots, k$. Hence, by nonbossiness, $\psi[\hat{P}^{(i_j)}] = \psi[P^{(i_j)}]$ for $j = 1, 2, \dots, k$. Moreover, by a similar argument, we also have $\psi[Q^{(i_j)}] = \psi[\hat{\hat{P}}^{(i_j)}] = \psi[\hat{P}^{(i_j)}] = \psi[P^{(i_j)}]$.

Now, let us consider the preference profile R such that $R_i = (ts^*\varnothing)$ for each $i \in \{i_1, \dots, i_k\}$, $R_i = (s^*\varnothing t)$ for each $i \in X^* \setminus \{i_1, \dots, i_k\}$, and $R_i = (\varnothing s^* t)$ for each $i \in I \setminus X^*$. Recall that school t has a capacity of 1. By symmetry, we may assume, without loss of generality, that no student other than i_k is matched to t in $\psi[R]$ (i.e., $\psi[R](i_k) = t$ or $\psi[R] = \{(i, s^*) : i \in X^*\}$). For each $j \in \{1, 2, \dots, k\}$, let $R^{(j)}$ be the preference profile such that $R^{(j)}_i = (ts^*\varnothing)$ for each $i \in \{i_j, \dots, i_k\}$, $R^{(j)}_i = (s^*\varnothing t)$ for each $i \in X^* \setminus \{i_j, \dots, i_k\}$, and $R^{(j)}_i = (\varnothing s^* t)$ for each $i \in I \setminus X^*$. Note that $R^{(1)} = R$ and $R^{(k-1)} = Q^{(i_{k-1})}$. By PE, IR, and GSP, it is not difficult to see that $\psi[R] = \psi[R^{(1)}] = \psi[R^{(2)}] = \cdots = \psi[R^{(k-1)}] = \psi[Q^{(i_{k-1})}]$. This implies that $(i_{k-1}, t) \in \psi[Q^{(i_{k-1})}] = \psi[R]$. However, this contradicts the assumption that no student other than i_k is matched to t in $\psi[R]$. Hence, it can be concluded that no mechanism simultaneously satisfies PE, IR, and GSP.

The case where $|S| > 2$ can be proved in the same way by setting $\varnothing \succ_i s$ for all $i \in I$ and $s \in S \setminus \{s^*, t\}$. □

By combining Theorem 2 and Theorem 3, we can conclude that accessibility is a necessary and sufficient condition for the existence of a PE, IR, and GSP mechanism when there are at least two schools and at most one school does not have a capacity constraint.

3.3 *Relation to stability*

Finally, we discuss the relationship between the results obtained in this section and the existence of stable matchings. Note that stability is a stronger condition than IR, but not comparable to PE. In addition, it is clear that the mechanism that always outputs the endowment matching satisfies both IR and SP. In order to discuss stability, we introduce a model to allocate indivisible goods with priorities. In this model, each school s is endowed with a priority, which is represented by a choice function over sets of students. Let $\text{Ch}_s: 2^I \to 2^I$ be the choice function of $s \in S$, where $\text{Ch}_s(X) \subseteq X$ for all $X \subseteq I$. The choice function Ch_s induces the feasibility constraint $\mathcal{F}_s = \{X \subseteq I : \text{Ch}_s(X) = X\}$. The condition $\text{Ch}_s(X) = X$ is called individual rationality of school s. A matching μ is *stable* if it is individually rational for both sides and there exists no $(i, s) \in I \times S$ such that $s \succ_i \mu(i)$ and $i \in \text{Ch}_s(\mu(s) \cup \{i\})$.

We introduce conditions that impose restrictions on the priorities. A choice function Ch satisfies *path-independence* (Plott (1973)) if for any sets of students X and Y, we have $\text{Ch}(X \cup Y) = \text{Ch}(\text{Ch}(X) \cup \text{Ch}(Y))$. Furthermore, a choice function Ch satisfies *unidirectional substitutes and complements conditions* (Huang (2021), Dur, Morrill, and Phan (2021)) if there exists an ordered type $t: I \to \mathbb{R}$ such that for any $X \subseteq I$ and $i \in \text{Ch}(X)$, the following conditions hold: (a) $\{i' \in \text{Ch}(X) \setminus \{i\} : t(i') = t(i)\} \subseteq \{i' \in \text{Ch}(X \setminus \{i\}) : t(i') = t(i)\}$ and (b) $\{i' \in \text{Ch}(X) : t(i') < t(i)\} \setminus \{i\} = \{i' \in \text{Ch}(X \setminus \{i\}) : t(i') < t(i)\}$.

When every choice function satisfies path-independence, a stable matching exists (Roth (1984), Aygün and Sönmez (2013)). Intuitively, a path-independent choice function rules out complementarities, which are associated with the nonexistence of stable matchings. However, Huang (2021) demonstrated that a choice function can accommodate a specific type of complementarity. When every choice function satisfies unidirectional substitutes and complements conditions for a common t, a stable matching still exists. Note that a path-independent choice function C induces a general upper bound since $C(X) = X$ implies $C(Y) = Y$ for all $Y \subseteq X$.[7] Moreover, a choice function that satisfies unidirectional substitutes and complements induces a σ-accessible constraint, as discussed in a similar manner to the arguments presented in Section 2.4.[8]

An inaccessible constraint is associated with stronger complementarities. A choice function Ch with the following complementarities leads to an inaccessible constraint: there exists $X \subseteq I$ with $\text{Ch}(X) \neq \emptyset$ such that for any $i \in \text{Ch}(X)$, we have $\text{Ch}(\text{Ch}(X) \setminus \{i\}) \subsetneq \text{Ch}(X) \setminus \{i\}$. The set $\text{Ch}(X)$ with such an X is inaccessible in the feasibility constraint induced by Ch. This type of complementarity is encountered in choice functions under proportional constraints and lower bounds, and is also observed in matchings involving couples. The presence of this complementarity is known to lead to the nonexistence of a stable matching (Nguyen and Vohra (2019), Biró et al. (2010), Ehlers et al. (2014), Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2016), Fragiadakis and Troyan (2017)). Importantly, this complementarity not only implies the absence of stable matchings but also rules out the existence of mechanisms that satisfy the properties of PE, IR, and GSP, as required by our necessity of accessibility.

[7] If a path-independent choice function induces a matroid constraint, it satisfies the law of aggregate demand (Yokoi (2019)). Consequently, this class of choice functions guarantees the existence of stable and SP mechanisms (Hatfield and Milgrom (2005)).

[8] Bando and Kawasaki (2021) introduced a broader class of choice functions and studied dynamic matching. The constraints induced by the choice functions are also σ-accessible.

4. Setting with endowments

In this section, we establish that a g-matroid is a maximal domain for the existence of PE, IR, and SP mechanisms in a setting with endowments. To demonstrate this, we first prove that a TTC mechanism satisfies PE, IR, and GSP if the constraints are g-matroid. Subsequently, we construct a market that permits no PE, IR, and SP mechanisms for each constraint $\mathcal{F}_{s^*}$ that is not a g-matroid.

4.1 *Motivating example*

We begin with the following example, a simplified version of one found in Delacrétaz, Kominers, and Teytelboym (2023), that illustrates that no mechanism can simultaneously achieve PE, IR, and SP under general constraints. Specifically, Delacrétaz, Kominers, and Teytelboym (2023) demonstrated that no mechanism satisfies PE, IR, and SP under multidimensional knapsack constraints.[9]

Example 4. Suppose that there are three students, 1, 2, 3, and three schools, s_1, s_2, s_3. The preference $\succ_i$ of each student i is given as

$$\succ_1 = (s_3 s_1 s_2 \varnothing), \qquad \succ_2 = (s_3 s_1 s_2 \varnothing), \qquad \succ_3 = (s_2 s_3 \varnothing s_1).$$

For this preference, student 1 prefers school s_3 the most and least prefers the outside option $\varnothing$. The constraint $\mathcal{F}_s$ of each school s is given as

$$\mathcal{F}_{s_1} = \{\emptyset, \{1\}, \{2\}, \{3\}\}, \qquad \mathcal{F}_{s_2} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\}, \qquad \mathcal{F}_{s_3} = \{\emptyset, \{1\}, \{2\}, \{3\}\}.$$

Here, $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_3}$ are (unit) capacity constraints, whereas $\mathcal{F}_{s_2}$ is not. Indeed, $\{1, 2\}, \{3\} \in \mathcal{F}_{s_2}$, but $\{1, 3\}, \{2, 3\} \notin \mathcal{F}_{s_2}$. Constraints such as $\mathcal{F}_{s_2}$ appear as budget constraints (e.g., student 3 requires more scholarship money). The endowments of students 1 and 2 are s_2, and the endowment of student 3 is s_3.

It is not difficult to see that there exist only two PE and IR matchings:

$$\mu_1 = \{(1, s_3), (2, s_1), (3, s_2)\} \quad \text{and} \quad \mu_2 = \{(1, s_1), (2, s_3), (3, s_2)\}.$$

Here, if student 1 misreports her preference as $\succ'_1 = (s_3 s_2 \varnothing s_1)$ whereas the other students report their true preferences, then μ_1 is a unique PE and IR matching. Similarly, if student 2 misreports her preference as $\succ'_2 = (s_3 s_2 \varnothing s_1)$ whereas the other students report their true preferences, then μ_2 is a unique PE and IR matching. Hence, in any PE and IR mechanism, either student 1 or 2 can be better off by misreporting his/her preference, depending on whether the outcome for true reporting is μ_1 or μ_2.

The example raises the question of which constraint structure is crucial for the existence of PE, IR, and SP mechanisms. We identify that a *generalized matroid* (g-matroid) is a "necessary and sufficient" condition of constraints to guarantee existence.

[9] In the model with multidimensional knapsack constraints, there is a finite set of service D. Each family $i \in I$ has service needs $\nu^i = (\nu^i_d) \in \mathbb{Z}^{|D|}_{\geq 0}$. Each location $s \in S$ has a service capacity profile $\kappa^s = (\kappa^s_d) \in \mathbb{Z}^{|D|}_{\geq 0}$. The constraint of each school s is represented by $\mathcal{F}_s \equiv \{I' \subseteq I : \sum_{i \in I'} \nu^i_d \leq \kappa^s_d$ for all $d \in D\}$.

4.2 *Mechanism for g-matroid constraints*

We provide a TTC mechanism that satisfies PE, IR, and GSP when the constraints are g-matroid. We derive this mechanism by utilizing the TTC-M mechanism introduced by Suzuki, Tamura, and Yokoo (2018), Suzuki et al. (2023). The TTC-M mechanism maintains PE, IR, and GSP for any distributional constraint that can be represented by an M-convex set on the vector of the number of students assigned to each school. Let $\chi_e \in \{0, 1\}^E$ be the eth unit vector. A set of integer vectors $\mathcal{V} \subseteq \mathbb{Z}^E_{\geq 0}$ is an M-convex set if, for all $v, v' \in \mathcal{V}$ and all $e \in E$ with $v_e > v'_e$, there exists $f \in E$ with $v_f < v'_f$ such that $v - \chi_e + \chi_f \in \mathcal{V}$ and $v' + \chi_e - \chi_f \in \mathcal{V}$ (Murota (2003)).

Note that the TTC-M mechanism cannot be directly applied to our setting. The primary reason for this is that in our setting, the constraints are not imposed on the number of students assigned to each school, but rather on the matched student–school pairs. In addition, our setting allows students to be unmatched, whereas their model does not.

To utilize the TTC-M mechanism, we construct a virtual market $(I, \tilde{S}, (\tilde{\succ}_i)_{i\in I}, \tilde{\mathcal{F}}, \tilde{\omega})$ from the given market $(I, S, (\succ_i)_{i\in I}, \mathcal{F}, \omega)$. The set of schools in the virtual market is defined as the set of student–school pairs $\tilde{S} := \{(i, s) : i \in I,\ s \in S \cup \{\varnothing\}\}$. Each student $i \in I$ has a strict preference $\tilde{\succ}_i$ over $\tilde{S}$ such that for any $(i_1, s_1), (i_2, s_2) \in \tilde{S}$, we have

(i) $(i_1, s_1) \tilde{\succ}_i (i_2, s_2) \iff s_1 \succ_i s_2$ if $i_1 = i_2 = i$

(ii) $(i_1, s_1) \tilde{\succ}_i (i_2, s_2)$ if $i_1 = i$ and $i_2 \neq i$.

The distributional constraint $\tilde{\mathcal{F}} \subseteq \mathbb{Z}^{\tilde{S}}_{\geq 0}$ is defined as

$$\tilde{\mathcal{F}} := \left\{ \nu \in \{0, 1\}^{\tilde{S}} : \sum_{(i,s)\in\tilde{S}} \nu_{(i,s)} = |I| \text{ and } \left\{(i, s) \in I \times S : \nu_{(i,s)} = 1\right\} \in \mathcal{F} \right\}.$$

The endowment function satisfies $\tilde{\omega}(i) = (i, \omega(i))$ for each $i \in I$. We will demonstrate that $\tilde{\mathcal{F}}$ is an M-convex set if $\mathcal{F}$ is a g-matroid.

The TTC-M mechanism runs on the virtual market as follows. Let $\rhd$ be a common priority order over the students I. Without loss of generality, we may assume that $1 \rhd 2 \rhd \cdots \rhd n$. In each round, every (virtual) school $(i, s) \in \tilde{S}$ selects a student. If (i, s) belongs to the endowment matching, then it selects i. Otherwise, (i, s) selects the highest priority student among the students i' for which (i, s) can be added to the current matching by removing $(i', \omega(i'))$ without violating feasibility. This mechanism gives the selected student the right to obtain a seat. Each student selects the right to obtain her top applicable school seat. Subsequently, students with such rights can trade seats among themselves by constructing trading cycles. Implement the trade indicated by this cycle, and all the involved students are removed from the market. If any students remain, the procedure continues.

For clarity, we provide an example of how our TTC mechanism works.

Example 5. Let $I = \{1, 2, 3, 4, 5\}$ and $S = \{s_1, s_2\}$. Suppose that students 1 and 2 prefer $s_2, s_1, \varnothing$ in this order, and students 3, 4, and 5 prefer $s_1, s_2, \varnothing$ in this order. The constraints

Algorithm 3: Generalized TTC.

input : a market $(I, S, (\succ_i)_{i\in I}, \mathcal{F}, \omega)$
output: a matching $\tilde{\mu}$

1 Let $\mu^{(0)} \leftarrow \{(i, \omega(i)) : i \in I\}$, $\tilde{\mu}^{(0)} \leftarrow \emptyset$, and $I^{(0)} \leftarrow I$;
2 **for** $k \leftarrow 1, 2, \dots$ **do**
3   **if** $I^{(k-1)} = \emptyset$ **then return** $\tilde{\mu}^{(k-1)}$;
4   **foreach** $i \in I^{(k-1)}$ **do**
5   Let $S_i^{(k)} \leftarrow \{s \in S \cup \{\varnothing\} : (\mu^{(k-1)} \setminus \{(i', \omega(i'))\}) \cup \tilde{\mu}^{(k-1)} \cup \{(i, s)\} \in \mathcal{F}\ (\exists i' \in I^{(k-1)})\}$;
6   Let $p_i^{(k)}$ be the most preferred school in $S_i^{(k)}$ for i;
7   i points to $(i, p_i^{(k)})$;
8   **foreach** $(i, s) \in \{(i, p_i^{(k)}) : i \in I^{(k-1)}\}$ **do**
9   **if** $(i, s) \in \mu^{(k-1)}$ **then** (i, s) points to i;
10   **else**
11   Let $I_{(i,s)}^{(k)} \leftarrow \{i' \in I^{(k-1)} : (\mu^{(k-1)} \setminus \{(i', \omega(i'))\}) \cup \tilde{\mu}^{(k-1)} \cup \{(i, s)\} \in \mathcal{F}\}$;
12   (i, s) points to the most prioritized (smallest index) student in $I_{(i,s)}^{(k)}$;
13   Identify a cycle $(i_1, (i_1, p_{i_1}^{(k)}), i_2, (i_2, p_{i_2}^{(k)}), \dots, i_r, (i_r, p_{i_r}^{(k)}))$;
14   $\mu^{(k)} \leftarrow \mu^{(k-1)} \setminus \{(i_1, \omega(i_1)), \dots, (i_r, \omega(i_r))\}$;
15   $\tilde{\mu}^{(k)} \leftarrow \tilde{\mu}^{(k-1)} \cup \{(i_1, p_{i_1}^{(k)}), \dots, (i_r, p_{i_r}^{(k)})\}$;
16   $I^{(k)} \leftarrow I^{(k-1)} \setminus \{i_1, \dots, i_r\}$;

$\mathcal{F}$ is a g-matroid that is defined as the aggregation of

$$\mathcal{F}_{s_1} = \{I' \subseteq I : |I' \cap \{2, 3, 5\}| \le 1\} \quad \text{and} \quad \mathcal{F}_{s_2} = \{I' \subseteq I : 1 \le |I'| \le 2\}.$$

Let the endowments be $(\omega(1), \omega(2), \omega(3), \omega(4), \omega(5)) = (s_1, s_1, s_2, \varnothing, \varnothing)$, that is, the endowment matching is $\mu^{(0)} = \{(1, s_1), (2, s_1), (3, s_2)\}$.

In round 1 of Algorithm 3, student 1 points to $(1, s_2)$, $(1, s_2)$ points to 1, student 2 points to $(2, s_2)$, $(2, s_2)$ points to 1, and so on (see Figure 2a). Note that $\{(2, s_1), (3, s_2), (2, s_2)\}$ is in $\mathcal{F}$ although it is not a matching. The cycle identified at line 13 is $(1, (1, s_2))$. Hence, we obtain $\mu^{(1)} = \{(2, s_1), (3, s_2)\}$, $\tilde{\mu}^{(1)} = \{(1, s_2)\}$, and $I^{(1)} = \{2, 3, 4, 5\}$.

In round 2, the cycle identified at line 13 is $(2, (2, s_2), 3, (3, s_1))$ (see Figure 2b). Thus, we obtain $\mu^{(2)} = \emptyset$, $\tilde{\mu}^{(2)} = \{(1, s_2), (2, s_2), (3, s_1)\}$, and $I^{(2)} = \{4, 5\}$.

In round 3, there are two cycles $(4, (4, s_1))$ and $(5, (5, \varnothing))$ (see Figure 2c). Note that student 5 cannot point to s_1, as student 3 was matched to s_1 in round 2, and, therefore, $s_1 \notin S_5^{(3)}$. The trades indicated by these cycles are implemented in rounds 3 and 4. Consequently, we obtain the matching $\tilde{\mu}^{(4)} = \{(1, s_2), (2, s_2), (3, s_1), (4, s_1)\}$.

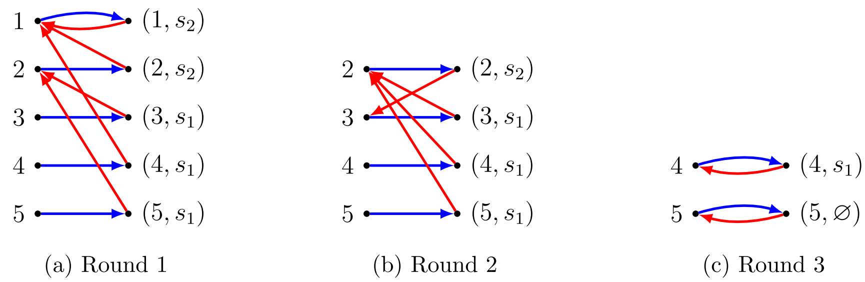

FIGURE 2. Cycles obtained by the TTC in Example 5. The blue and red arrows represent the relationship to which students and virtual schools are pointing, respectively. Virtual schools that have not been pointed to by any student are omitted.

Note that a trading cycle can be interpreted as an alternating cycle in the exchange graph of a g-matroid intersection. This correspondence can be established by constructing an instance of the g-matroid intersection problem where the common ground set is the set of student–school pairs $\tilde{S}$. One g-matroid is the distributional constraint $\tilde{\mathcal{F}}$, and the other is a partition matroid $\mathcal{M}$ that ensures that each student appears at most once. In other words, $X \in \mathcal{M}$ if $|X \cap \{(i, s) \in \tilde{S} : s \in S \cup \{\varnothing\}\}| \leq 1$ for all $i \in I$. For a feasible matching μ, the exchange graph is a directed bipartite graph with bipartition μ and $\tilde{S} \setminus \mu$. A pair $(y, x) \in \mu \times (\tilde{S} \setminus \mu)$ is an arc if $(\mu \setminus \{y\}) \cup \{x\} \in \tilde{\mathcal{F}}$ and $(x, y) \in (\tilde{S} \setminus \mu) \times \mu$ is an arc if $(\mu \setminus \{y\}) \cup \{x\} \in \mathcal{M}$. To preserve the feasibility of matching after trading, it is sufficient to select a cycle in the exchange graph that does not contain shortcuts (Murota (1996)). A standard method for selecting such a cycle is to select a shortest cycle. However, such a selection rule does not satisfy strategy-proofness (Imamura and Kawase (2024)). The TTC-M mechanism instead selects cycles without shortcuts by utilizing the priority order.

Formally, our TTC mechanism is described in Algorithm 3. At the beginning of round k, the set of remaining students is $I^{(k-1)}$, and each student $i \in I^{(k-1)}$ is matched with $\mu^{(k-1)}(i) = (i, \omega(i))$. Each student $i \in I \setminus I^{(k-1)}$ exits the market matched with $\tilde{\mu}^{(k-1)}(i)$. The set of schools to which student $i \in I^{(k-1)}$ has a chance of being matched with is represented as $S_i^{(k)}$. Then each student $i \in I^{(k-1)}$ points to $(i, p_i^{(k)})$, where $p_i^{(k)}$ is the most preferred school in $S_i^{(k)}$. Each virtual school (i, s) points to the most prioritized student i' who (i, s) can add by removing $(i', \omega(i'))$.

We prove the following theorem.

THEOREM 4. *The generalized TTC mechanism (Algorithm 3) satisfies PE, IR, and GSP if the distributional constraints form a g-matroid. Additionally, Algorithm 3 can be implemented to run in time $O(|I|^2 \cdot |S|)$ if we assume that the feasibility of a matching can be checked in a constant time.*

PROOF. Recall that the TTC-M mechanism satisfies PE, IR, and GSP when the distributional constraint is represented by an M-convex set on the vector of the number of students assigned to each school (Suzuki et al. (2023)). Therefore, to demonstrate that

Algorithm 3 satisfies PE, IR, and GSP, it is sufficient to prove that $\tilde{\mathcal{F}}$ is an M-convex set if $\mathcal{F}$ is a g-matroid. Suppose that $\mathcal{F}$ is a g-matroid. Then $\mathcal{F}' = \{\nu \subseteq \tilde{S} : \nu \cap (I \times S) \in \mathcal{F}\}$ is also a g-matroid by definition. Further, $\tilde{\mathcal{F}}$ can be obtained from $\mathcal{F}'$ by truncating it with cardinality $|I|$ (i.e., $\tilde{\mathcal{F}} = \{\nu \in \mathcal{F}' : |\nu| = |I|\}$), and such a truncation induces a matroid base family (Tardos (1985)). As the class of matroid base families is a subclass of M-convex sets (Murota (2016)), $\mathcal{F}$ is an M-convex set.

Next we discuss the computational complexity of Algorithm 3. As at least one student is fixed in each iteration, the number of iterations is at most $O(|I|)$. The running time of each iteration is $O(|I| \cdot |S|)$. Therefore, the total running time is at most $O(|I|^2 \cdot |S|)$. □

4.3 *Impossibility for non-g-matroid constraints*

Next we demonstrate that the g-matroid structure is necessary for the existence of a mechanism that satisfies PE, IR, and SP.

Theorem 5. *Fix a set of students I, a set of schools S with $|S| \geq 3$, and a school s^* with the constraint $\mathcal{F}_{s^*}$. Suppose that $\mathcal{F}_{s^*}$ is not a g-matroid. Then there must exist a market $(I, S, (\mathcal{F}_s)_{s \in S}, \omega)$ with $s^* \in S$ and $\mathcal{F}_s = \{X \subseteq I : |X| \leq 1\}$ for all $s \in S \setminus \{s^*\}$ such that no mechanism simultaneously satisfies PE, IR, and SP.*

Proof. As $\mathcal{F}_{s^*}$ is not a g-matroid, there exist subsets X and Y in $\mathcal{F}_{s^*}$ and a student e in $X \setminus Y$, such that we have the alternatives

(i) $X \setminus \{e\} \notin \mathcal{F}_{s^*}$ and $(X \setminus \{e\}) \cup \{e'\} \notin \mathcal{F}_{s^*}$ for any $e' \in Y \setminus X$

(ii) $Y \cup \{e\} \notin \mathcal{F}_{s^*}$ and $(Y \cup \{e\}) \setminus \{e'\} \notin \mathcal{F}_{s^*}$ for any $e' \in Y \setminus X$.

Here, we provide the proof for the case in which (i) holds. We defer the proof for the case when (ii) holds to Appendix A, as it can be demonstrated in a similar manner.

Suppose that there exist $X, Y \in \mathcal{F}_{s^*}$ and $e \in X \setminus Y$ such that $X \setminus \{e\} \notin \mathcal{F}_{s^*}$ and $(X \setminus \{e\}) \cup \{f\} \notin \mathcal{F}_{s^*}$ for any $f \in Y \setminus X$. Let $Z \in \mathcal{F}_{s^*}$ be a set of students such that $(X \cap Y) \subseteq Z \subseteq (X \cup Y) \setminus \{e\}$. Such a set Z must exist because Y satisfies the condition. Among all sets Z that satisfy this condition, we select a set that maximizes $|X \cap Z|$.

We consider two cases separately: (a) $|X \setminus Z| = 1$ and (b) $|X \setminus Z| \geq 2$.

Case (a): $|X \setminus Z| = 1$ In this case, we have $X \cap Z = X \setminus \{e\}$. In addition, we have $|Z \setminus X| \geq 2$ because $(X \setminus \{e\}) \cup J = Z \in \mathcal{F}_{s^*}$ by setting $J = Z \setminus X$. We select two students $x, y \in Z \setminus X$ arbitrarily (see Figure 3). We consider a market in which the set of schools is $S = \{s^*, t, u\}$ and $\mathcal{F}_t = \mathcal{F}_u = \{I' \subseteq I : |I'| \leq 1\}$. Additionally, let the endowments be $\omega(e) = t$, $\omega(i) = s^*$ for each $i \in Z$, and $\omega(i) = \varnothing$ for each $i \notin Z \cup \{e\}$. The endowment matching μ_0 for this market is feasible because $\mu_0(s^*) = Z$, $|\mu_0(t)| = 1$, and $|\mu_0(u)| = 0 \leq 1$.

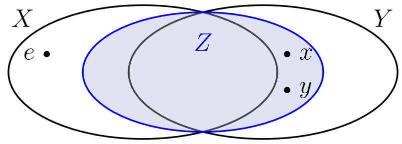

Figure 3. Case (a).

Suppose that the students' preferences are given as

- $\succ_e = (s^* t \cdots)$
- $\succ_x = (tus^* \cdots)$
- $\succ_y = (tus^* \cdots)$
- $\succ_i = (s^* \cdots)$ for each $i \in X \setminus \{e\}$
- $\succ_i = (\varnothing s^* \cdots)$ for each $i \in Z \setminus (X \cup \{x, y\})$
- $\succ_i = (\varnothing \cdots)$ for each $i \notin X \cup Z$.

Let μ_x be the matching such that x matches to u and every other student matches to her most favorite school (or her outside option). Similarly, let μ_y be the matching such that y matches to u and every other student matches to her most favorite school. Then μ_x and μ_y are feasible since $\mu_x(s^*) = \mu_y(s^*) = X$. Furthermore, we can observe that only μ_x and μ_y are PE and IR. By symmetry, we can assume, without loss of generality, that a mechanism outputs μ_x. Suppose that x misreports her preference as $t \succ'_x s^* \succ'_x \cdots$. With this misreporting, the unique PE and IR matching is μ_y. Hence, any PE and IR mechanism cannot satisfy SP.

Case (b): $|X \setminus Z| \geq 2$ Let e' be an arbitrary student in $X \setminus (Z \cup \{e\})$ (see Figure 4). We consider a market in which the set of schools is $S = \{s^*, t, u\}$ and $\mathcal{F}_t = \mathcal{F}_u = \{I' \subseteq I : |I'| \leq 1\}$. In addition, let the endowments be $\omega(e) = t$, $\omega(e') = u$, $\omega(i) = s^*$ for each $i \in Z$, and $\omega(i) = \varnothing$ for each $i \in I \setminus (Z \cup \{e, e'\})$. The endowment matching μ_0 for this market is feasible because $\mu_0(s^*) = Z$ and $|\mu_0(t)| = |\mu_0(u)| = 1$.

Suppose that students' preferences are defined as

- $\succ_e = (us^* t \cdots)$
- $\succ_{e'} = (s^* tu \cdots)$
- $\succ_i = (s^* \cdots)$ for each $i \in X \cap Z$
- $\succ_i = (\varnothing s^* \cdots)$ for each $i \in Z \setminus X$

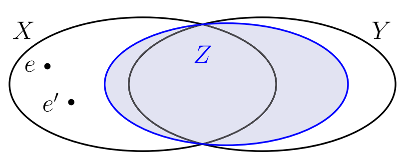

Figure 4. Case (b).

- $\succ_i = (s^* \varnothing \cdots)$ for each $i \in X \setminus (Z \cup \{e, e'\})$
- $\succ_i = (\varnothing \cdots)$ for each $i \notin X \cup Z$.

Let μ be the matching produced by a PE, IR, and SP mechanism. By IR, we have $X \cap Z \subseteq \mu(s^*) \subseteq X \cup Z$. If $e \notin \mu(s^*)$, then we must have $\mu(s^*) \subseteq Z$ by the maximality of $|X \cap Z|$. Hence, $\mu(e) \neq s^*$ implies $\mu(e') \neq s^*$. Let us consider three subcases depending on $\mu(e)$.

Case (b1): $\mu(e) = t$. In this case, $\mu(e') \neq s^*$ and $\mu(e') = u$. This means that μ is not PE because e and e' can be better off by swapping their allocated schools, which is a contradiction.

Case (b2): $\mu(e) = s^*$. Suppose that e misreports s^* as being unacceptable (i.e., submitting $\succ'_e = (ut \cdots)$). Then e must be matched with u in any PE and IR matching, which contradicts SP.

Case (b3): $\mu(e) = u$. In this case, $\mu(e') \neq s^*$ and $\mu(e') = t$. Suppose that e' misreports that t as being unacceptable (i.e., submitting $\succ'_e = (s^* u \cdots)$). Then e' must be matched with s^* because there exists a unique PE and IR matching $\{(i, s^*) : i \in X\}$, which contradicts SP. □

5. Discussion and conclusion

5.1 *Relationship between the two settings*

We discuss the relationship between the settings, which can be summarized as shown in Table 2. Recall that the endowments are assumed to be feasible in both settings. In the setting with endowments, any feasible matching in $\mathcal{F}$ can be set as the initial matching μ_0. In contrast, in the setting without endowments, the initial matching μ_0 is restricted to the empty matching, but it implies that the empty matching must be feasible in this setting. Thus, the necessary or sufficient conditions of one setting cannot be simply applied to the other setting.

To make this difference clearer, let us assume that the empty matching is feasible in the setting with endowments as well. Then the necessary and sufficient condition for the existence of a desired mechanism in this setting becomes a matroid. Since any matroid constraint is σ-accessible for every σ, this is a sufficient condition for the existence of a desired mechanism in the setting without endowments.

Table 2. Relationship between settings for the existence of a desired mechanism.

Setting	Assumption	Initial Endowment	Condition
Without endowments	$\emptyset \in \mathcal{F}$	$\mu_0 = \emptyset$	(σ-)accessible
With endowments	$\mathcal{F} \neq \emptyset$	$\mu_0 \in \mathcal{F}$	g-matroid
Including both	$\emptyset \in \mathcal{F}$	$\mu_0 \in \mathcal{F}$	matroid

5.2 *Two out of PE, IR, and GSP*

In both settings, with and without endowments, any two of the three properties PE, IR, and GSP can be achieved under general constraints. It is evident that the mechanism that always outputs the endowment matching satisfies both IR and GSP. To satisfy PE and GSP, we can utilize a generalized SD mechanism that sequentially assigns each student to her best school in a predetermined order, ensuring that the remaining students can be feasibly assigned. To observe that the outcome μ of the mechanism is PE, suppose, to the contrary, that there exists a feasible matching μ' that is a Pareto improvement of μ. Let i^* be the first student assigned to a school other than $\mu(i^*)$ in the mechanism. Then $\mu'(i^*) \succ_{i^*} \mu(i^*)$; however, this contradicts the behavior of the generalized SD mechanism. Additionally, the mechanism is GSP because if a student does not select her preferred school in her turn, she will not receive another chance to do so. This mechanism is equivalent to the GSDPC proposed by Kamiyama (2013). PE and IR can be achieved by using the CSD mechanism (Imamura and Kawase (2024)). The CSD mechanism sequentially assigns each student to her best school in a predetermined order, while ensuring that the remaining students can be assigned to produce a feasible IR matching. Clearly, this mechanism satisfies IR. The property of PE follows from the fact that a matching is PE if it is PE under the IR constraint. Note that the CSD mechanism is not SP because each student is assigned to a school depending on the preferences of the later students.

5.3 *Conclusion*

This study investigated the existence of efficient and strategy-proof mechanisms in indivisible goods allocation problems under general constraints.

In the setting without endowments, we demonstrated that the SD mechanism satisfies PE, IR, and GSP if the constraints are σ-accessible for a common σ. We also proved that accessibility is a necessary condition to ensure the existence of PE, IR, and GSP mechanisms. Identifying the most general class of constraints under which PE, IR, and SP mechanisms exist remains open. In a setting with endowments, we revealed that the g-matroid is a maximal domain under which we can guarantee the existence of a PE, IR, and SP mechanism. The same statement holds true even if we replace SP with GSP.

In a setting without endowments, we formulate an integer linear program (ILP) to determine the existence of PE, IR, and SP mechanisms for a given market. In the case where $I = \{1, 2, 3\}$, $S = \{s_1, s_2\}$, $\mathcal{F}_{s_1} = \{X \subseteq I : |X| \neq 2\}$, and $\mathcal{F}_{s_2} = \{X \subseteq I : |X| \leq 1\}$, the Gurobi solver with the ILP revealed that no such mechanism exists. The irreducible inconsistent subsystem obtained for the market contains relationships among 43 preferences, making it challenging to discern its underlying structure. Whether accessibility is necessary for the existence of PE, IR, and SP mechanisms remains for future research.

In a setting with endowments, Delacrétaz, Kominers, and Teytelboym (2023) presented stronger nonexistence results under multidimensional knapsack constraints. For example, the desired mechanism does not exist even when PE and IR are replaced by the property that a mechanism Pareto improves upon every Pareto-inefficient endowment. We call this property *Pareto-improving* (PI). Formally, a mechanism φ is PI if, for any preference profile $\succ_I$ at which the endowment matching μ_0 is Pareto-inefficient,

$\varphi[\succ_I](i) \succeq_i \mu_0(i)$ for all $i \in I$ and $\varphi[\succ_I](i) \succ_i \mu_0(i)$ for some $i \in I$. PI is a weaker requirement than the conjunction of PE and IR. Delacrétaz, Kominers, and Teytelboym (2023) showed by example that no PI and SP mechanism exists under multidimensional knapsack constraints. In contrast, a PI and SP mechanism exists in Example 4. Thus, we are left with the question, "Which class of constraints is necessary and sufficient for the existence of PI and SP mechanisms?"

Finally, let us discuss the case in which the endowment matching μ_0 is infeasible. In this case, no IR matchings exist, especially when every student prefers her own endowment the most. Therefore, we have no option but to abandon IR. Moreover, abandoning IR is a natural choice when allocating chores in a setting without endowments. Nevertheless, even without IR, we can still attain PE and GSP by employing the GSDPC mechanism under any constraints, as long as at least one feasible matching exists.

Appendix A: Omitted part of the proof of Theorem 5

Here, we provide the proof of Theorem 5 for the case when (ii) holds.

Suppose that there exist $X, Y \in \mathcal{F}_{s^*}$ and $e \in X \setminus Y$ such that $Y \cup \{e\} \notin \mathcal{F}_{s^*}$ and $(Y \cup \{e\}) \setminus \{f\} \notin \mathcal{F}_{s^*}$ for any $f \in Y \setminus X$. Let $Z \in \mathcal{F}_{s^*}$ be a set of students such that $(X \cap Y) \cup \{e\} \subseteq Z \subseteq X \cup Y$. Such a set Z must exist because X satisfies the condition. Among all sets Z that satisfy this condition, we select a set that minimizes $|X \cap Z|$.

We consider two cases separately: (c) $|X \cap Z| = |X \cap Y| + 1$ and (d) $|X \cap Z| \geq |X \cap Y| + 2$.

Case (c): $|X \cap Z| = |X \cap Y| + 1$ In this case, we have $X \cap Z = (X \cap Y) \cup \{e\}$. In addition, we have $|Y \setminus Z| \geq 2$ because $(Y \cup \{e\}) \setminus J = Z \in \mathcal{F}_{s^*}$ by setting $J = Y \setminus Z$. We select two students $x, y \in Y \setminus Z$ arbitrarily (see Figure 5). We consider a market in which the set of schools is $S = \{s^*, t, u\}$ and $\mathcal{F}_t = \mathcal{F}_u = \{I' \subseteq I : |I'| \leq 1\}$. Additionally, let the endowments be $\omega(e) = t$, $\omega(i) = s^*$ for each $i \in Y$ and $\omega(i) = \varnothing$ for each $i \notin Y \cup \{e\}$. The endowment matching μ_0 for this market is feasible because $\mu_0(s^*) = Y$, $|\mu_0(t)| = 1$, and $|\mu_0(u)| = 0 \leq 1$.

Suppose that the students' preferences are given as

- $\succ_e = (s^* t \cdots)$
- $\succ_x = (t u s^* \cdots)$
- $\succ_y = (t u s^* \cdots)$
- $\succ_i = (s^* \cdots)$ for each $i \in Z \setminus \{e\}$
- $\succ_i = (\varnothing s^* \cdots)$ for each $i \in Y \setminus (Z \cup \{x, y\})$
- $\succ_i = (\varnothing \cdots)$ for each $i \notin Z \cup Y$.

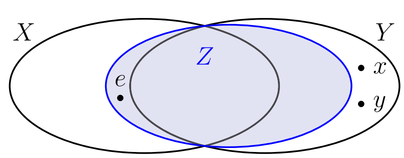

Figure 5. Case (c).

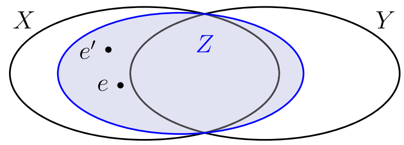

Figure 6. Case (d).

Let μ_x be the matching such that x matches to u and every other student matches to her most favorite school (or her outside option). Similarly, let μ_y be the matching such that y matches to u and every other student matches to her most favorite school. Then μ_x and μ_y are feasible because $\mu_x(s^*) = \mu_y(s^*) = Z$. Furthermore, we can observe that only μ_x and μ_y are PE and IR. By symmetry, we can assume, without loss of generality, that a mechanism outputs μ_x. Suppose that x misreports her preference as $t \succ'_x s^* \succ'_x \cdots$. With this misreporting, the unique PE and IR matching is μ_y. Hence, any PE and IR mechanism cannot satisfy SP.

Case (d): $|X \cap Z| \geq |X \cap Y| + 2$ Let e' be an arbitrary student in $Z \setminus (Y \cup \{e\})$ (see Figure 6). We consider a market in which the set of schools is $S = \{s^*, t, u\}$ and $\mathcal{F}_t = \mathcal{F}_u = \{I' \subseteq I : |I'| \leq 1\}$. In addition, let the endowments be $\omega(e) = u$, $\omega(e') = t$, $\omega(i) = s^*$ for each $i \in Y$, and $\omega(i) = \varnothing$ for each $i \in I \setminus (Y \cup \{e, e'\})$. The endowment matching μ_0 for this market is feasible because $\mu_0(s^*) = Y$ and $|\mu_0(t)| = |\mu_0(u)| = 1$.

Suppose that students' preferences are defined as

- $\succ_e = (s^* t u \cdots)$
- $\succ_{e'} = (u s^* t \cdots)$
- $\succ_i = (s^* \cdots)$ for each $i \in Z \cap Y$
- $\succ_i = (\varnothing s^* \cdots)$ for each $i \in Y \setminus Z$
- $\succ_i = (s^* \varnothing \cdots)$ for each $i \in Z \setminus (Y \cup \{e, e'\})$
- $\succ_i = (\varnothing \cdots)$ for each $i \notin Z \cup Y$.

Let μ be the matching produced by a PE, IR, and SP mechanism. By IR, we have $Z \cap Y \subseteq \mu(s^*) \subseteq Z \cup Y$. If $e \in \mu(s^*)$, then we must have $X \cap Z \subseteq \mu(s^*)$ by the minimality of $|X \cap Z|$. Hence, $\mu(e') \neq s^*$ implies $\mu(e) \neq s^*$. Let us consider three subcases depending on $\mu(e')$:

Case (d1): $\mu(e') = t$. In this case, $\mu(e) \neq s^*$ and $\mu(e) = u$. This means that μ is not PE because e and e' can be better off by swapping their allocated schools, which is a contradiction.

Case (d2): $\mu(e') = s^*$. Suppose that e' misreports s^* as being unacceptable (i.e., submitting $\succ'_{e'} = (ut \cdots)$). Then e' must be matched with u in any PE and IR matching, which contradicts SP.

Case (d3): $\mu(e') = u$. In this case, $\mu(e) \neq s^*$ and $\mu(e) = t$. Suppose that e misreports t as being unacceptable (i.e., submitting $\succ'_e = (s^* u \cdots)$). Then e must be matched with s^* because there exists a unique PE and IR matching $\{(i, s^*) : i \in Z\}$, which contradicts SP.

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