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Multinary group identification

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Group identification refers to the problem of classifying individuals into groups (e.g., racial or ethnic classification). We consider a multinary group identification model where memberships to three or more groups are simultaneously determined based on individual opinions on who belong to what groups. Our main axiom requires that membership to each group, say the group of J's, should depend only on the opinions on who is a J and who is not (that is, independently of the opinions on who is a K or an L). This shares the spirit of Arrow's independence of irrelevant alternatives and, therefore, is termed *independence of irrelevant opinions*. Our investigation of multinary group identification and the independence axiom reports a somewhat different message from the celebrated impossibility result by Arrow (1951). We show that the independence axiom, together with symmetry and non-degeneracy, implies the liberal rule (each person self-determines her own membership). This characterization provides a theoretical foundation for the self-identification method commonly used for racial or ethnic classifications.

Keywords. Group identification, independence of irrelevant opinions, symmetry, liberalism, one-vote rules.

JEL classification. C0, D70, D71, D72.

1. Introduction

Ethnic and racial classification is an important issue in countries with diverse demographic characteristics such as China, India, Russia, the United Kingdom, and the United States. It serves as a basis for evaluating public policies in terms of equal opportunity and anti-discrimination. A key element complicating the classification is the fact that a large number of ethnic groups are identified simultaneously. For instance,

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the 2011 U.K. Census recognizes 18 ethnicity categories[1] and the 2010 China Census recognizes 56.[2]

Group identification (Kasher and Rubinstein 1997) formalizes the problem of classifying individuals. However, the literature largely focuses on the binary case (Samet and Schmeidler 2003, Sung and Dimitrov 2005, Dimitrov et al. 2007, Houy 2007, Miller 2008, Çengelci and Sanver 2010, Ju 2010, 2013). In the binary model, there is a collective, say J, whose membership is to be determined; that is, individuals need to be divided into the group of J's and the group of non-J's. Each person has an opinion on who are J's and who are non-J's. The question is how to aggregate individual opinions and identify the two groups.

The binary model can be applied to multinary problems once they are transformed appropriately. However, such transformations invoke an important principle underpinning the binary model. To illustrate, suppose that there are three groups, J, K, and L. Person i may view person j not to be in J but to be in K or L. The binary model treats the two cases in the same way, leading to the same decision on group J. Thus, implicit in the binary model is the principle that the identification of the group under question should not be tainted by irrelevant opinions on the other groups, which is reminiscent of Arrow's independence of irrelevant alternatives (Arrow 1951).[3] We propose this principle as an axiom, termed *independence of irrelevant opinions*, for social decision rules over multinary group identification problems. Despite the wide use of multinary classifications in the aforementioned countries and the great attention directed to the independence axiom in social choice theory, as far as we know, there has been no earlier investigation of multinary group identification, not to speak of the independence axiom therein. Our main objective is to scrutinize independence of irrelevant opinions in multinary group identification.

In our model, there are three or more groups and each person needs to be identified as a member of one of the groups. Taking as input individual opinions on who belong to what groups, a (social decision) rule determines memberships to the groups. Our main axiom for rules—independence of irrelevant opinions—requires that membership to each group be decided based solely on the opinions on who belongs to that group and who does not (that is, independently of the opinions on who belongs to the other groups). It is a variant of Arrow's independence axiom for preference aggregation rules and is vacuous in the binary group identification, as is Arrow's independence when there are only two alternatives.

We show that independence of irrelevant opinions, together with the basic condition of *non-degeneracy* (there should be no person who is always put in one fixed group, regardless of opinions), implies a simple method of identifying each person using only one vote (Theorem 1). We call these rules the one-vote rules, noting the connection

[1]Office for National Statistics (http://www.ons.gov.uk/ons/guide-method/measuring-equality/equality/ethnic-nat-identity-religion/ethnic-group/index.html). Retrieved July 25, 2014.

[2]National Bureau of Statistics of the People's Republic of China (http://www.stats.gov.cn/tjsj/pcsj/rkpc/6rp/indexch.htm). Retrieved July 25, 2014.

[3]See also Hansson (1969) and Fishburn (1970) for their discussion on the role of the independence axiom in Arrow's impossibility theorem.

with the one-vote rules in the binary model (Miller 2008). For example, a dictatorial rule determines each person's membership following the dictator's opinion; each person is a J when, and only when, the dictator believes so. Another example is the liberal rule, according to which everyone self-determines her membership. There are many other one-vote rules. However, when *symmetry* (the names of persons should not matter; Samet and Schmeidler 2003) is added, the liberal rule is the unique rule satisfying the three axioms (Theorem 2). Therefore, our investigation of multinary group identification and the independence axiom offers a somewhat different message from the well-known impossibility result in preference aggregation theory by Arrow (1951) and Blau (1957).

The liberal rule, or the self-identification method, is the most common way to identify one's ethnicity and race. For example, the 2011 U.K. Census uses this method and the Office for National Statistics of the U.K. government explains the reason as follows:

> Membership of an ethnic group is something that is subjectively meaningful to the person concerned, and this is the principal basis for ethnic categorization in the United Kingdom. So, in ethnic group questions, we are unable to base ethnic identification upon objective, quantifiable information as we would, say, for age or gender. And this means that we should rather ask people which group they see themselves as belonging to.[4]

That ethnic classification can be subjective and hence controversial is precisely the reason why each person concerned should report an opinion not just about herself but about all the other persons concerned. Nevertheless, the Office does not provide a more fundamental basis for the self-identification method or principles underlying it. Our characterization of the liberal rule by independence of irrelevant opinions, symmetry, and non-degeneracy reveals what those principles can be and serves as a formal justification.

Our results rest chiefly on independence of irrelevant opinions and the assumption that there are three or more groups. In the binary model, independence of irrelevant opinions has no bite and there are numerous rules other than the liberal rule satisfying both symmetry and non-degeneracy. The consent rules due to Samet and Schmeidler (2003) are examples. The liberal rule is a special case in the latter family, with minimum consent quotas. Depending on the choice of consent quotas, a consent rule can also be "democratic" in that everyone's vote counts equally; e.g., the majority rule with consent quota $\frac{n+1}{2}$, where n is the number of persons.

One important reason why independence of irrelevant opinions turns out to be so strong is that social decisions in our model identify each person as a member of exactly one group; thus, decisions partition the set of persons into groups (everyone belongs to one and only one group). This single-membership requirement can be relaxed by allowing the case of no membership or the case of multiple memberships. In either case, our results no longer hold and the family of rules satisfying independence of irrelevant

[4]Office for National Statistics, *Ethnic Group Statistics: A Guide for the Collection and Classification of Ethnicity Data* (2003, p. 9).

opinions and non-degeneracy becomes quite diverse, including various extensions of the consent rules. Section 5.3 discusses this issue in detail.[5]

In the literature on preference aggregation, Arrow's independence axiom, together with a few fairly mild axioms, implies a rather unequal distribution of decision power: only a single person or a group of persons is decisive (Arrow 1951, Blau 1957, 1972, Guha 1972, Mas-Colell and Sonnenschein 1972). In the literature on aggregation of equivalence relations, Fishburn and Rubinstein (1986a, 1986b) and Dimitrov et al. (2012) consider a variant of Arrow's independence axiom (Fishburn and Rubinstein call it binary independence) and establish similar results. In contrast to these, our independence axiom admits more diverse power distributions, including both an equal distribution of power as in the liberal rule and the most unequal distribution of power as in the dictatorial rules.

Of particular relevance to our investigation is Miller (2008). He studies binary identification problems in a model where the group whose membership is to be decided can vary. He characterizes the family of one-vote rules (similarly defined in the binary setup) but his results are based on an axiom of "separability," requiring that decisions across groups be consistent with respect to the conjunction and disjunction of groups (J and K, J or K).[6] A proper comparison of our paper and Miller (2008) requires an extended model that subsumes both. In our companion paper, Cho and Ju (2015), we introduce an extended setup where social decision rules need to identify not only two or more groups, but all derived groups that are obtained by conjunction or disjunction of the basic groups. Using this extended setup, we find that an independence axiom,[7] much stronger than our independence of irrelevant opinions, is implicitly assumed in Miller (2008) and, together with his separability axiom, plays a critical role. Although the family of rules our set of axioms characterizes is similar to Miller's (2008), neither his strong independence nor separability is used in our results. Moreover, independence of irrelevant opinions, non-degeneracy, and a certain unanimity axiom characterize a larger family of rules than the one-vote rules characterized by Miller (2008).[8]

2. The model

There are n persons, each of whom needs to be identified as a member of one of m groups. Let $N \equiv \{1, \dots, n\}$ be the set of persons and let $G \equiv \{1, \dots, m\}$ be the set of groups. We assume, unless noted otherwise, that $n \geq 2$ and $m \geq 3$. Each person $i \in N$ has an *opinion* on who she believes are the members of each of these groups. The opinion is represented by a list $P_i \equiv (P_{ij})_{j \in N} \in G^N$, where for all $j \in N$

[5]Cho and Ju (2015) study the extended model without the single-membership property and the extended consent rules. The role of the single-membership property is discussed more explicitly there. See also footnote 11.

[6]Meet separability requires the equivalence of (i) the conjunction of the two decisions for group "J" and group "K", and (ii) the decision for group "J and K". Join separability requires the equivalence of (i) the disjunction of the two decisions for group "J" and group "K", and (ii) the decision for group "J or K".

[7]It requires, for instance, that the decision on group "J and K" be independent of the opinions on group "J" or the opinions on group "K", which are quite relevant.

[8]See Theorem 1 in Cho and Ju (2015).

and all $k \in G$, $P_{ij} = k$ when person i views person j as a member of group k.[9] Individual opinions $P_1, \ldots, P_n$ constitute an (identification) *problem* $P \equiv (P_{ij})_{i,j \in N}$, an $n \times n$ matrix. Let $\mathcal{P} \equiv G^{N \times N}$ be the set of all problems. A *domain* $\mathcal{D} \subseteq \mathcal{P}$ is a nonempty subset of $\mathcal{P}$. We call $\mathcal{P}$ the *universal domain*. When $m = 2$, our model reduces to the standard, binary group identification model (Kasher and Rubinstein 1997, Samet and Schmeidler 2003). Let $G_{\mathcal{D}} \equiv \{P_{ij} : P \in \mathcal{D} \text{ and } i, j \in N\}$.

A *decision* is a profile $x \equiv (x_i)_{i \in N} \in G^N$, where, for all $i \in N$ and all $k \in G$, $x_i = k$ means that person i belongs to group k. Given a domain $\mathcal{D}$, a social decision rule, briefly a *rule* $f\colon \mathcal{D} \to G^N$ associates with each problem in $\mathcal{D}$ a decision. For example, a *plurality rule* puts each person i in the group for which she wins the most votes, namely, group k such that for all $k' \in G$, $|\{j \in N : P_{ji} = k\}| \geq |\{j \in N : P_{ji} = k'\}|$ and when the inequality holds with equality, the tie-breaking condition $k \leq k'$ is satisfied.[10] Different tie-breaking methods lead to different plurality rules.

In the binary model, the consent rules (Samet and Schmeidler 2003) allow each person i to determine her own membership if her opinion about herself wins a sufficient consent from others (that is, the number of persons agreeing with i, $P_{ji} = P_{ii}$, is no less than a given quota). When the consent quota is 1, everyone self-determines her own membership; i.e., for all $P \in \mathcal{D}$ and all $i \in N$, i belongs to group P_{ii}. This rule is called the *liberal rule*. When a consent rule is not liberal (i.e., with a quota of 2 or above), a person may not win a sufficient consent from others for the group she claims to be a member of. With insufficient consent, she fails to self-determine her membership, which in the binary model, means that she belongs to the other group. In our multinary model, this case of insufficient consent is indeterminate since there are two or more other groups.[11] Hence, none of the consent rules except for the liberal rule is well defined in our model.

Nevertheless, we can define similar rules by introducing a mapping $\delta\colon G \to G$, associating with all $k \in G$ the *default decision against membership to group* k, denoted by $\delta_k \in G$. Group δ_k serves as the default membership for any person who considers herself a member of group k but fails to win a sufficient consent from others. For all $k \in G$,

[9] Each person $i \in N$ views each person $j \in N$ as belonging to some group in G. Thus, opinions are required to be complete. This can be too demanding when, e.g., a person should provide an opinion on a large number of persons. To accommodate incomplete opinions, we can define P_i as an element of $(G \cup \{\varnothing\})^N$, where for all $j \in N$, $P_{ij} = \varnothing$ means that person i is not sure of person j's membership. The definitions of "decision" and "rule" are similarly modified to include $\varnothing$. In this new setup, we can define independence of irrelevant opinions as requiring independent decision making for the groups in G. While the interpretation is different, the case of incomplete opinions is very similar to the case that allows for no membership. Section 5.3.1 discusses the latter. As shown there, independence of irrelevant opinions is now substantially weaker and is satisfied by not just the one-vote rules, but also a subfamily of the consent rules with default decisions and some peculiar rules. Footnote 17 elaborates on this issue.

[10] Any linear ordering of the groups can be used as a tie-breaking method.

[11] To allow for this indeterminacy, we can define decision x as an element of $(G \cup \{\varnothing\})^N$, where for all $i \in N$, $x_i = \varnothing$ means that person i's membership is not determined. Then a rule is a mapping $f : \mathcal{D} \to (G \cup \{\varnothing\})^N$. In this setup, we may adapt the consent rules as follows: each person i belongs to group P_{ii} if she wins a sufficient consent from others; her membership is undetermined (i.e., she belongs to group $\varnothing$) otherwise. It is evident that the consent rules so defined satisfy independence of irrelevant opinions (to be introduced below) as well as non-degeneracy. Thus, our main results do not hold in this setup. Also, the model with indeterminacy is mathematically equivalent to the one in Section 5.3.1, where a person may not belong to any of the groups in G.

let $q_k \in \mathbb{N}$ be the *consent quota for group* k. The *consent rule with quotas* $q \equiv (q_k)_{k\in G}$ *and default decisions* $\delta \equiv (\delta_k)_{k\in G}$, denoted by $f^{q,\delta}$, is defined as follows: for all $P \in \mathcal{D}$ and all $i \in N$ with $P_{ii} = k$,

(i) if $|\{j \in N : P_{ji} = k\}| \geq q_k$, then $f_i^{q,\delta}(P) = k$; and

(ii) otherwise, $f_i^{q,\delta}(P) = \delta_k$.[12]

Under the consent rule $f^{q,\delta}$, each person $i \in N$ belongs to either the group of her own choice (P_{ii}) or the opposite ($\delta_{P_{ii}}$). Clearly, when $q_1 = \cdots = q_m = 1$, $f^{q,\delta}$ coincides with the liberal rule, whatever δ is.

When there is a status quo group $\kappa \in G$ to which all persons initially belong, one can define a consent rule that determines regrouping of all members in the status quo group by setting, for all $k \in G$, $\delta_k = \kappa$. Then each person belongs to either the group of her choice (P_{ii}) or the status quo group (κ). Thus when she considers herself to be in the status quo group, her opinion is decisive for her own membership. She needs others' consent only when she considers herself not belonging to the status quo group. We discuss an extension of this idea and other related issues in Section 5.2.

Axioms

Should a person belong to J because many others believe that she belongs to K rather than to L? Should membership to a group depend on opinions on the other groups? The answer, obviously, will differ from context to context.

If there is an exogenous, universally accepted relationship among groups, the answer may be affirmative. For instance, suppose that we seek to identify a group whose membership has two tiers—regular and honor. Thus, three groups to be determined are non-members, regular members, and honor members. It seems inappropriate to require that i's regular membership be independent of who views her as an honor member or as a non-member. With more and more people approving her as an honor member, person i receives greater support for regular membership (as well as for honor membership). The three groups are ordered in terms of the level of club membership. When such an acknowledged relationship is present among groups, it is necessary to take account of opinions on honor membership as well as those on regular membership when identifying regular membership.

By contrast, in the context of ethnic classification, there is no order over groups that defines their relationship. Moreover, each ethnicity is treated as an independent entity, the identity of which should not be compromised by the other ethnicities. Hence it is natural to require that membership to an ethnicity should be decided independently of opinions on the other ethnicities. This is the context in which our independence axiom is meaningful. Consider two problems, P and P', such that all persons agree on

[12]Our definition permits the possibility that for some $k \in G$, $\delta_k = k$. For such k, whenever $P_{ii} = k$, $f^{q,\delta}$ puts person i in group k. Also, note that in the binary model of Samet and Schmeidler (2003), our definition coincides with their definition of consent rules once $q_k + q_{\delta_k} \leq n + 2$ is added. This inequality is needed for their monotonicity axiom to hold.

membership to group k; that is, each person i considers each person j to be a member of group k at P if and only if she does so at P'. Their opinions may differ in memberships to the other groups, but if this difference is viewed as irrelevant when identifying group k, it is reasonable to require that $f(P)$ and $f(P')$ agree on group-k membership.

Independence of Irrelevant Opinions. Let $P, P' \in \mathcal{D}$ and $k \in G_{\mathcal{D}}$. Suppose that for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$. Then for all $i \in N$, $f_i(P) = k$ if and only if $f_i(P') = k$.

It is evident that the liberal rule satisfies independence of irrelevant opinions. Other rules, as simple as the liberal rule, also satisfy this axiom. They are characterized in Section 4. The consent rules with default decisions do not necessarily satisfy the independence axiom; membership to the default group against membership to group k relies on the opinions on group k. Nevertheless, on some restricted domains, they do. Here is an example.

Example 1. Suppose that people used to be just "Earthians," but now they seek to divide into several tribes. Their identities are determined based on individual opinions. Let G be the set of all tribes and let ν be the null group, which denotes the group of persons who belong to none of the tribes. *Assume that everyone believes that each person belongs to one of these tribes.* Then their opinions give rise to the restricted domain $\mathcal{D}_\nu \equiv \{P \in \mathcal{P} : \text{ for all } i, j \in N, P_{ij} \in G\backslash\{\nu\}\}$. On this domain, all consent rules with the constant default decision of ν satisfy independence of irrelevant opinions. This is because all problems in $\mathcal{D}_\nu$ share the common opinion on the null group ν: no one ever believes anyone, including herself, to be in the null group. Thus, with respect to this null group, independence of irrelevant opinions has no bite; with respect to the other non-null groups $k \in G\backslash\{\nu\}$, it is evident from the definition that membership to group k depends only on the opinions on group k (who they believe belong to k or not). $\Diamond$

We also consider the following fairly standard axioms in the group identification literature. Given a permutation $\pi: N \to N$ and a problem $P \in \mathcal{P}$, let $P_\pi \equiv (P_{\pi(i),\pi(j)})_{i,j\in N}$ be the problem obtained from P by changing names of persons through π. Name changes shift no fundamental content. Thus, it is reasonable to require that the decision be unaffected by such changes (Samet and Schmeidler 2003). Let $f_\pi(P) \equiv (f_{\pi(i)}(P))_{i\in N}$.

Symmetry. For all $P \in \mathcal{D}$ and all permutations $\pi: N \to N$ such that $P_\pi \in \mathcal{D}$, $f(P_\pi) = f_\pi(P)$.

Our next axiom concerns decisions for "unanimous" opinion profiles: if all persons consider all persons belonging to one group, say group k, then all persons should be classified into group k. For each $k \in G$, let $k_{n\times n}$ and $k_{1\times n}$ be the problem and the decision consisting of only k's.

Unanimity. For all $k \in G$ such that $k_{n\times n} \in \mathcal{D}$, $f(k_{n\times n}) = k_{1\times n}$.

A rule may be "degenerate" for a person in that there is one fixed group into which the rule always classifies her, regardless of opinions. We require that such degeneracy occur for no person. Clearly, this is weaker than unanimity.

Non-degeneracy. For all $i \in N$, there are $P, P' \in \mathcal{D}$ such that $f_i(P) \neq f_i(P')$.

3. Independence of irrelevant opinions and decomposability

A problem contains binary information on membership to all groups. Thus, we may "decompose" the problem into multiple binary problems, obtain binary decisions for the latter, and combine them into a single decision. The decision so obtained may or may not be the same as the decision a rule assigns to the initial problem. Below we establish that independence of irrelevant opinions is "almost" equivalent to requiring that this be the case.

More precisely, let $\mathcal{B} \equiv \{0,1\}^{N \times N}$. Given $P \in \mathcal{P}$, for all $k \in G$, let $B^{P,k} \in \mathcal{B}$ be the binary problem concerning group k derived from P; i.e., for all $i, j \in N$, (i) if $P_{ij} = k$, then $B^{P,k}_{ij} = 1$, and (ii) if $P_{ij} \neq k$, then $B^{P,k}_{ij} = 0$. A (binary) *approval function* $\varphi \colon \mathcal{B} \to \{0,1\}^N$ associates with each binary problem $B \in \mathcal{B}$ a binary decision, namely, a profile of 0's and 1's, where for all $i \in N$, $\varphi_i(B) = 0$ means the disapproval of i's membership and $\varphi_i(B) = 1$ means the approval of i's membership. For all binary problems $B \in \mathcal{B}$, let $\bar{B} \equiv 1_{n\times n} - B$ be the *dual problem* of B. Likewise, for all binary decisions $x \in \{0,1\}^N$, let $\bar{x} \equiv 1_{1\times n} - x$ be the *dual decision* of x.

Using these definitions, each problem $P \in \mathcal{P}$ can be decomposed into m binary problems, $B^{P,1}, \dots, B^{P,m}$. The next axiom requires that the decision for problem P be identical to the combination of m binary decisions for the m binary problems assigned by an approval function.

Decomposability. There is an approval function φ such that for all $P \in \mathcal{D}$, all $i \in N$, and all $k \in G$, $f_i(P) = k$ if and only if $\varphi_i(B^{P,k}) = 1$.

In this case, we say that f *is represented by* φ.

We show that an approval function representing a decomposable rule satisfies the following properties. The approval function φ is *m-unit-additive* if for all m binary problems $B^1, \dots, B^m \in \mathcal{B}$,

$$\sum_{k\in G} B^k = 1_{n\times n} \quad \text{implies} \quad \sum_{k\in G} \varphi(B^k) = 1_{1\times n}.$$

It is *unanimous* if $\varphi(0_{n\times n}) = 0_{1\times n}$ and $\varphi(1_{n\times n}) = 1_{1\times n}$. The *dual of* φ, denoted φ^d, is the approval function such that for all $B \in \mathcal{B}$, $\varphi^d(B) = \overline{\varphi(\overline{B})}$. We say that φ is *self-dual* if $\varphi = \varphi^d$. Finally, φ is *monotonic* if for all $B, B' \in \mathcal{B}$ such that $B \leq B'$, $\varphi(B) \leq \varphi(B')$.

Proposition 1. *Consider the universal domain (i.e., $\mathcal{D} = \mathcal{P}$). An approval function represents a decomposable rule if and only if it is m-unit-additive. Also, if an approval function is m-unit-additive, then it is unanimous, self-dual, and monotonic.*

Proof. First, we prove the "if and only if" statement. Note that for all $P \in \mathcal{P}$, $\sum_{k\in G} B^{P,k} = 1_{n\times n}$ and that for all $B^1, \dots, B^m \in \mathcal{B}$ with $\sum_{k\in G} B^k = 1_{n\times n}$, there is $P \in \mathcal{P}$ such that $B^1 = B^{P,1}, \dots, B^m = B^{P,m}$. This observation is enough to prove the "if" part. Next, suppose that an approval function φ represents a decomposable rule f. Let $B^1, \dots, B^m \in \mathcal{B}$ be such that $\sum_{k\in G} B^k = 1_{n\times n}$. There is $P \in \mathcal{P}$ such that for all $k \in G$, $B^k = B^{P,k}$. Let $i \in N$ and $k^* \equiv f_i(P)$. Since φ represents f, $f_i(P) = k^*$ implies $\varphi_i(B^{k^*}) = \varphi_i(B^{P,k^*}) = 1$; and for all $k \in G\backslash\{k^*\}$, $f_i(P) \neq k$ implies $\varphi_i(B^k) = \varphi_i(B^{P,k}) = 0$. Thus, $\sum_{k\in G} \varphi_i(B^k) = 1$ and φ is m-unit-additive.

Assume, henceforth, that φ is m-unit-additive. To prove that φ is unanimous, let $i \in N$. Let $B^1 \equiv 1_{n\times n}$, and for all $k \in G\backslash\{1\}$, let $B^k \equiv 0_{n\times n}$. Let $s \equiv \varphi_i(1_{n\times n})$ and $t \equiv \varphi_i(0_{n\times n})$. Since $\sum_{k\in G} B^k = 1_{n\times n}$, $1 = \sum_{k\in G} \varphi_i(B^k) = s + (m-1)t$. Since $s, t \in \{0, 1\}$ and $m \geq 3$, it follows that $s = 1$ and $t = 0$.

To prove that φ is self-dual, let $B \in \mathcal{B}$. Let $B^1 \equiv B$, $B^2 \equiv \overline{B}$, and, for all $k \in G\backslash\{1, 2\}$, let $B^k \equiv 0_{n\times n}$. Since $\sum_{k\in G} B^k = 1_{n\times n}$, then by m-unit-additivity and unanimity, $1_{1\times n} = \sum_{k\in G} \varphi(B^k) = \varphi(B) + \varphi(\overline{B})$. This gives $\varphi(B) = \overline{\varphi(\overline{B})}$.

Finally, to prove that φ is monotonic, let $B, B' \in \mathcal{B}$ be such that $B \leq B'$. Let $i \in N$. If $\varphi_i(B) = 0$, then trivially, $\varphi_i(B) \leq \varphi_i(B')$. Thus, assume that $\varphi_i(B) = 1$. Let $B^1 = B$ and $B^2 = \overline{B'}$. Let $B^3, \dots, B^m \in \mathcal{B}$ be such that $\sum_{k\in G} B^k = 1_{n\times n}$ (such $B^3, \dots, B^m$ exist because $\overline{B} \geq \overline{B'}$ and $B^1 + B^2 = B + \overline{B'} \leq 1_{n\times n}$). Since $\sum_{k\in G} \varphi_i(B^k) = 1$ and $\varphi_i(B^1) = 1$, $0 = \varphi_i(B^2) = \varphi_i(\overline{B'})$. Since φ is self-dual, $\varphi_i(B') = \overline{\varphi_i(\overline{B'})} = 1$. □

A rule f is *independent of irrelevant opinions* if and only if it can be represented by m approval functions $(\varphi^k)_{k\in G}$. To see this, for all $k \in G$, define the approval function $\varphi^k : \mathcal{B} \to \{0, 1\}^N$ as follows: for all $B \in \mathcal{B}$ and all $i \in N$, (i) $\varphi_i^k(B) = 1$ if for some $P \in \mathcal{P}$ such that $B^{P,k} = B$, $f_i(P) = k$, and (ii) $\varphi_i^k(B) = 0$ if for some $P \in \mathcal{P}$ such that $B^{P,k} = B$, $f_i(P) \neq k$. Then $(\varphi^k)_{k\in G}$ are well defined if and only if f is independent of irrelevant opinions.[13] Further, f is represented by $(\varphi^k)_{k\in G}$; i.e., for all $P \in \mathcal{P}$, all $i \in N$, and all $k \in G$, $f_i(P) = k$ if and only if $\varphi_i^k(B^{P,k}) = 1$. In addition to the existence of approval functions $(\varphi_k)_{k\in G}$ representing f, decomposability requires that they be identical ($\varphi^1 = \cdots = \varphi^m$). Therefore, decomposability implies independence of irrelevant opinions. The converse does not hold. As we show below, the essential difference between the two axioms is non-degeneracy. To prove this, we use the following lemma.

Lemma 1. *On the universal domain (i.e., $\mathcal{D} = \mathcal{P}$), independence of irrelevant opinions and non-degeneracy together imply unanimity.*

Proof. Let f be a rule satisfying independence of irrelevant opinions and non-degeneracy. Then there are approval functions $(\varphi^k)_{k\in G}$ representing f. Now we proceed in three steps.

Step 1: For all $i \in N$, all $P \in \mathcal{P}$, and all $\ell \in G\backslash\{f_i(P)\}$, $\varphi_i^\ell(0_{n\times n}) = 0$. Let $i \in N$ and $P \in P$. Let $k \equiv f_i(P)$. Let $\ell, h \in G\backslash\{k\}$ be distinct. Let $P' \in \mathcal{P}$ be such that for all $j, j' \in N$,

[13] For all $k \in G$, φ^k is well defined if and only if the following holds: whenever for some $P \in \mathcal{P}$, $B^{P,k} = B$ and $f_i(P) = k$, there is no $P' \in \mathcal{P}$ such that $B^{P',k} = B$ and $f_i(P') \neq k$. This is precisely what independence of irrelevant opinions requires.

(i) $P'_{jj'} = k$ if and only if $P_{jj'} = k$, and (ii) $P'_{jj'} = h$ if and only if $P_{jj'} \neq k$. By independence of irrelevant opinions, $f_i(P') = f_i(P) = k$, so that $f_i(P') \neq \ell$. Then $\varphi_i^\ell(0_{n\times n}) = \varphi_i^\ell(B^{P',\ell}) = 0$.

Step 2: For all $i \in N$ and all $k \in G$, $\varphi_i^k(0_{n\times n}) = 0$. Let $i \in N$. By non-degeneracy, there are $P, P' \in \mathcal{P}$ such that $f_i(P) \neq f_i(P')$. Let $k \equiv f_i(P)$ and $\ell \equiv f_i(P')$. By Step 1, for all $h \in G\backslash\{k\}$, $\varphi_i^h(0_{n\times n}) = 0$. Similarly, for all $h \in G\backslash\{\ell\}$, $\varphi_i^h(0_{n\times n}) = 0$.

Step 3: The rule f is unanimous. Suppose, by contradiction, that for some $i \in N$ and some $k \in G$, $f_i(k_{n\times n}) \neq k$. Let $\ell \equiv f_i(k_{n\times n})$. Then $\varphi_i^\ell(0_{n\times n}) = \varphi_i^\ell(B^{k_{n\times n},\ell}) = 1$, contradicting Step 2. □

Now we prove the logical relation between independence of irrelevant opinions and decomposability.

Proposition 2. *On the universal domain (i.e., $\mathcal{D} = \mathcal{P}$), the combination of independence of irrelevant opinions and non-degeneracy is equivalent to decomposability.*

Proof. We already noted that decomposability implies independence of irrelevant opinions. When a rule is decomposable, by Proposition 1, the approval function representing it is unanimous. Therefore, the rule is also unanimous and, hence, non-degenerate.

To prove the converse, let f be a rule satisfying independence of irrelevant opinions and non-degeneracy. Then f is represented by a profile of m approval functions $(\varphi^k)_{k\in G}$. By Lemma 1, f is unanimous. Now we proceed in two steps.

Step 1: For all $i \in N$ and all $P \in \mathcal{P}$, $f_i(P)$ is one of the entries of P. Suppose, by contradiction, that for some $i \in N$ and $P \in \mathcal{P}$, $f_i(P)$ is not one of the entries of P; i.e., for some k such that $B^{P,k} = 0_{n\times n}$, $f_i(P) = k$. Let $\ell \in G$ be one of the entries of P and consider $\ell_{n\times n} \in \mathcal{P}$. Then $B^{P,k} = 0_{n\times n} = B^{\ell_{n\times n},k}$. Thus applying independence of irrelevant opinions to P and $\ell_{n\times n}$, $f_i(P) = k$ implies $f_i(\ell_{n\times n}) = k$, which contradicts unanimity.

Step 2: It follows that $\varphi^1 = \varphi^2 = \cdots = \varphi^m$. Suppose, by contradiction, that there are $k, \ell \in G$ such that $\varphi^k \neq \varphi^\ell$. Then there are $B \in \mathcal{B}$ and $i \in N$ such that $\varphi_i^k(B) \neq \varphi_i^\ell(B)$. Without loss of generality, assume that $\varphi_i^k(B) = 0$ and $\varphi_i^\ell(B) = 1$. Let $h \in G\backslash\{k, \ell\}$. Let $P \in \mathcal{P}$ be such that for all $j, j' \in N$, (i) $P_{jj'} = h$ if and only if $B_{jj'} = 0$; and (ii) $P_{jj'} = k$ if and only if $B_{jj'} = 1$. Similarly, let $P' \in \mathcal{P}$ be such that for all $j, j' \in N$, (i) $P'_{jj'} = h$ if and only if $B_{jj'} = 0$; and (ii) $P'_{jj'} = \ell$ if and only if $B_{jj'} = 1$. By construction, $B^{P,k} = B^{P',\ell} = B$. Since $\varphi_i^k(B) = 0$ and $\varphi_i^\ell(B) = 1$, it follows that $f_i(P) \neq k$ and $f_i(P') = \ell$. By Step 1, $f_i(P) \neq k$ implies $f_i(P) = h$. Note that $B^{P,h} = B^{P',h}$. Hence, applying independence of irrelevant opinions to P and P', $f_i(P) = h$ implies $f_i(P') = h$, which contradicts $f_i(P') = \ell$. □

Note that by Proposition 2 and Lemma 1, decomposability also implies unanimity.

4. Main results

In this section, we present our main characterization results. We first characterize the rules satisfying independence of irrelevant opinions and non-degeneracy. These rules are represented by the "one-vote" approval functions that Miller (2008) introduces in the binary identification model. An approval function φ is a *one-vote approval function* if for all $i \in N$, there are $j, h \in N$ such that for all $B \in \mathcal{B}$, $\varphi_i(B) = B_{jh}$. A rule f is a *one-vote rule* if for all $i \in N$, there are $j, h \in N$ such that for all $P \in \mathcal{P}$, $f_i(P) = P_{jh}$. The one-vote rules are decomposable, represented by one-vote approval functions; moreover, they are the only decomposable rules.

Theorem 1. *Let f be a rule on the universal domain (i.e., $\mathcal{D} = \mathcal{P}$). Then the following statements are equivalent.*

(i) The rule f is independent of irrelevant opinions and is non-degenerate.

(ii) The rule f is decomposable.

(iii) The rule f is a one-vote rule.

Proof. We use the following notation in the proof. Consider a binary problem $B \in \mathcal{B}$. Let $|B| \equiv \sum_{i,j\in N} B_{ij}$ be the number of 1's in B and call it the *size* of B. Binary problem B is a *unit binary problem* if $|B| = 1$. For all $i, j \in N$, let $U^{ij} \in \mathcal{B}$ be the unit binary problem such that $U^{ij}_{ij} = 1$.

By Proposition 2, we only have to show the equivalence of (ii) and (iii). We only prove the non-trivial implication, "(ii) implies (iii)". Consider a decomposable rule represented by an approval function φ. It suffices to show that φ is a one-vote approval function. Let $i \in N$. We proceed in two steps.

Step 1: There are $j, h \in N$ such that $\varphi_i(U^{jh}) = 1$. By Proposition 1, φ is m-unit-additive, unanimous, self-dual, and monotonic. Suppose, by contradiction, that for all $B \in \mathcal{B}$,

$$|B| = 1 \quad \text{implies} \quad \varphi_i(B) = 0. \tag{1}$$

We prove by induction on the size of binary problems that for all $B \in \mathcal{B}$, $\varphi_i(B) = 0$.

Let $\ell \in \mathbb{N}$ be such that $\ell < n^2$ and assume that for all $B \in \mathcal{B}$,

$$|B| \le \ell \quad \text{implies} \quad \varphi_i(B) = 0. \tag{2}$$

Let $B \in \mathcal{B}$ be such that $|B| = \ell + 1$. Then $|\bar{B}| = n^2 - \ell - 1$ and there are B^1 and B^2 such that $|B^1| = 1$, $|B^2| = \ell$, and $B^1 + B^2 + \bar{B} = 1_{n\times n}$. By m-unit-additivity and unanimity, $\varphi_i(B^1) + \varphi_i(B^2) + \varphi_i(\bar{B}) = 1$. Since by the induction hypothesis (2), $\varphi_i(B^1) = \varphi_i(B^2) = 0$, we obtain $\varphi_i(\bar{B}) = 1$. By self-duality, $\varphi_i(B) = 0$. Hence, for all $B \in \mathcal{B}$,

$$|B| \le \ell + 1 \quad \text{implies} \quad \varphi_i(B) = 0.$$

Therefore, (1) and the induction argument prove that for all $B \in \mathcal{B}$, $\varphi_i(B) = 0$. In particular, $\varphi_i(1_{n\times n}) = 0$, which contradicts unanimity of φ.

Step 2: For all $B \in \mathcal{B}$, $\varphi_i(B) = 1$ if and only if $B_{jh} = 1$. Let $j, h \in N$ be such that $\varphi_i(U^{jh}) = 1$. Let $B \in \mathcal{B}$. If $B_{jh} = 1$, then since $B \ge U^{jh}$, monotonicity implies that $\varphi_i(B) \ge \varphi_i(U^{jh}) = 1$. If $B_{jh} = 0$, then since $B \le \overline{U^{jh}}$, monotonicity and self-duality imply that $\varphi_i(B) \le \varphi_i(\overline{U^{jh}}) = 0$. □

In Theorem 1, non-degeneracy plays only a minor role of preventing membership of anyone from being pre-determined. Without this axiom, independence of irrelevant opinions alone characterizes the family of rules that are one-vote rules when restricted to those persons without pre-determined membership (see Section 5.1). In contrast, if we weaken independence of irrelevant opinions or relax the single-membership requirement, then quite a diverse family of rules, including variants of the consent rules by Samet and Schmeidler (2001), become available (see Sections 5.2 and 5.3).

When $n \geq 3$, among the one-vote rules, there is only one symmetric rule: the liberal rule.

Theorem 2. *Assume that there are at least three persons ($n \geq 3$). Let f be a rule on the universal domain (i.e., $\mathcal{D} = \mathcal{P}$). Then the following statements are equivalent.*

(i) The rule f is independent of irrelevant opinions, non-degenerate, and symmetric.

(ii) The rule f is decomposable and symmetric.

(iii) The rule f is the liberal rule.

Proof. We prove that (ii) implies (iii). Let f be a rule satisfying decomposability and symmetry. By Theorem 1, it is a one-vote rule. Then there is a function $h\colon N \to N \times N$ such that for all $P \in \mathcal{P}$ and all $i \in N$, $f_i(P) = P_{h(i)}$. Now symmetry implies that h satisfies the following condition: for all permutations $\pi : N \to N$ and all $i \in N$,

$$h\big(\pi(i)\big) = \big(\pi\big(h_1(i)\big), \pi\big(h_2(i)\big)\big). \tag{3}$$

It is enough to show that for all $i \in N$, $h(i) = (i, i)$. Suppose, by contradiction, that there is $i \in N$ such that $h(i) \neq (i, i)$. Let $(j, k) \equiv h(i)$. Without loss of generality, assume that $k \neq i$. Since $n \geq 3$, there is $\ell \in N\backslash\{i, k\}$. Let $\pi\colon N \to N$ be the transposition of k and ℓ. Then $h(\pi(i)) = h(i) = (j, k)$ but $(\pi(h_1(i)), \pi(h_2(i))) = (\pi(j), \pi(k)) = (j, \ell)$, contradicting (3). □

Remark 1. When $n = 2$, there are other, non-liberal one-vote rules satisfying the axioms in Theorem 2. In fact, parts (i) and (ii) of Theorem 2 are equivalent to the following statement: (iii′) (a) the rule f is the liberal rule, or (b) f is such that for all $P \in \mathcal{P}$, $f(P) = (P_{21}, P_{12})$, or (c) for all $P \in \mathcal{P}$, $f(P) = (P_{12}, P_{21})$, or (d) for all $P \in \mathcal{P}$, $f(P) = (P_{22}, P_{11})$. Therefore, when there are only two persons, four rules satisfy the axioms in parts (i) or (ii).

Proof of Remark 1. Let $n = 2$. As in the proof of Theorem 2, we can obtain a function $h : N \to N \times N$. By symmetry, h satisfies (3). Since $n = 2$, $h(1)$ determines $h(2)$ as well: letting $\pi : N \to N$ be the transposition of 1 and 2, it follows that $h(2) = h(\pi(1)) = (\pi(h_1(1)), \pi(h_2(1)))$. For instance, if $h(1) = (1, 2)$, then $h(2) = (2, 1)$. Since $h(1) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, we can define h in four different ways, thus obtaining the four rules in Remark 1. □

Remark 2. On the restricted domain in Example 1, there are numerous consent rules, far from being liberal, which satisfy all axioms in parts (i) and (ii). The logical independence of the three axioms in part (i) (also the two axioms in part (ii)) are established by the following examples: one-vote rules (satisfying all but symmetry), "uniformly degenerate"[14] rules (satisfying all but non-degeneracy), and plurality rules (satisfying all but independence of irrelevant opinions).

5. Concluding remarks

In the binary group identification model, independence of irrelevant opinions is vacuous; decomposability is also mild since it coincides with self-duality. However, with three or more groups, the two axioms become very demanding as shown by Theorems 1 and 2. The contrasting consequences of these axioms in the binary and multinary setups are similar to those of Arrovian preference aggregation with two alternatives and with three or more alternatives.[15]

Now we conclude by discussing several issues.

5.1 *Dropping non-degeneracy*

Consider circumstances where some are "legacy members" of a certain group and their membership, which is pre-determined, cannot be altered. The membership of the other persons is to be decided based on the opinions of all persons including the legacy members. Non-degeneracy is not totally desirable in these circumstances. In this section, we show that even without non-degeneracy, characterizations similar to our main results obtain.

In the absence of non-degeneracy, there may be a person with pre-determined membership; she always belongs to a certain group regardless of opinions. Given a rule f, person $i \in N$ is *bound* if her membership is pre-determined, that is, for all $P, P' \in \mathcal{D}$, $f_i(P) = f_i(P')$; otherwise, she is *unbound*. Likewise, one can also define bound or unbound persons for an approval function. It turns out that without non-degeneracy, our characterizations still hold for unbound persons.

In Sections 2 and 3, we introduced (i) unanimity and decomposability of rules; and (ii) m-unit additivity, unanimity, and self-duality of approval functions. We can define weaker versions of those properties by restricting the scope of application to unbound persons. For instance, a rule f is *decomposable for unbound persons* if there is an approval function φ such that for all $P \in \mathcal{D}$, all unbound persons $i \in N$, and all $k \in G$, $f_i(P) = k$ if and only if $\varphi_i(B^{P,k}) = 1$; an approval function φ is *m-unit-additive for unbound persons* if for all unbound persons $i \in N$ and all m binary problems $B^1, \dots, B^m \in \mathcal{B}$, $\sum_{k\in G} B^k = 1_{n\times n}$ implies $\sum_{k\in G} \varphi_i(B^k) = 1$.

[14] A rule is uniformly degenerate if there is one fixed group to which everyone always belongs. Uniformly degenerate rules are also discussed in Section 5.1.

[15] When there are three or more alternatives, independence of irrelevant alternatives, transitivity, and unanimity (or Pareto principle) imply dictatorship (Arrow's impossibility theorem; Arrow 1951). When there are two alternatives, independence of irrelevant alternatives and transitivity are vacuous, and there are numerous non-dictatorial aggregation rules that perform well in terms of, e.g., "monotonicity" and "anonymity."

With only minor changes to the proofs, we can drop non-degeneracy and state all results in Section 3 in terms of these weaker properties. In particular, it follows that independence of irrelevant opinions is equivalent to decomposability for unbound persons. Then using the proof of Theorem 1, we can show that independence of irrelevant opinions (or decomposability for unbound persons) alone characterizes the family of rules that are one-vote rules as far as unbound persons are concerned. Formally, a rule f is a *one-vote rule for unbound persons* if for all unbound persons $i \in N$, there are $j, h \in N$ such that for all $P \in \mathcal{P}$, $f_i(P) = P_{jh}$.

PROPOSITION 3. *Let f be a rule on the universal domain (i.e., $\mathcal{D} = \mathcal{P}$). Then the following statements are equivalent.*

(i) The rule f is independent of irrelevant opinions.

(ii) The rule f is decomposable for unbound persons.

(iii) The rule f is a one-vote rule for unbound persons.

Proposition 3 indicates that in Proposition 2, non-degeneracy only serves to exclude those rules admitting a bound person; combining the axiom with independence of irrelevant opinions yields no further implication.

Symmetry requires that all persons be treated in the same way. Therefore, if the axiom is added, we cannot have both bound and unbound persons: either everyone is unbound or everyone is bound. In the former case, we only have the liberal rule; in the latter, there is a group to which all agents always belong. Say that a rule f is *uniformly degenerate* if there is $k \in G$ such that for all $P \in \mathcal{P}$, $f(P) = (k, \dots, k)$. Thus, the liberal rule and the uniformly degenerate rules are the only rules satisfying independence of irrelevant opinions (or decomposability for unbound persons) and symmetry.

PROPOSITION 4. *Assume that there are at least three persons ($n \geq 3$). Let f be a rule on the universal domain (i.e., $\mathcal{D} = \mathcal{P}$). Then the following statements are equivalent.*

(i) The rule f is independent of irrelevant opinions and symmetric.

(ii) The rule f is decomposable for unbound persons and symmetric.

(iii) The rule f is either liberal or uniformly degenerate.

PROOF. We prove that (ii) implies (iii). Let f be a rule satisfying decomposability for unbound persons and symmetry. By Proposition 3, it is a one-vote rule for unbound persons. We distinguish two cases.

Case 1: Everyone is unbound. In effect, non-degeneracy is imposed. We can proceed as in the proof of Theorem 2.

Case 2: At least one person is bound. Assume that $i \in N$ is bound. Then there is $k \in G$ such that for all $P \in \mathcal{P}$, $f_i(P) = k$. We show that for all $j \in N \setminus \{i\}$ and all $P \in \mathcal{P}$, $f_j(P) = k$. Let $j \in N \setminus \{i\}$. Let $\pi : N \to N$ be a permutation such that $\pi(i) = j$ and $\pi(j) = i$. By symmetry, $f_j(P) = f_{\pi(i)}(P) = f_i(P_\pi) = k$. □

When there are only two persons ($n = 2$), parts (i) and (ii) in Proposition 4 are equivalent to the following:

(iii$'$) A rule is either one of the four rules in Remark 1 or is uniformly degenerate.

5.2 *Weakening independence of irrelevant opinions*

5.2.1 *Weak independence* Independence of irrelevant opinions identifies quite a large number of problems as far as group-k membership is concerned. Let $k \in G$ and $P \in \mathcal{D}$. As we change P to another problem while keeping fixed the binary information on group k, non-k entries can change to any group in $G\backslash\{k\}$ in many different ways. The axiom requires treating all these changes of non-k opinions equally as far as group-k membership is concerned. We may ask, however, how the independence axiom is affected if we only allow more systematic changes in P, thus identifying a smaller number of problems that are "closer" to each other in terms of the information they contain. For instance, we may require two problems to preserve equality of non-k entries (i.e., equal entries in one problem remain equal in the other problem and unequal entries remain unequal). By restricting independence to such cases, we obtain the following axiom.

Weak Independence. Let $P, P' \in \mathcal{D}$ and $k \in G$. Suppose that (i) for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$; and (ii) for all $i, j, u, v \in N$, $P_{ij} = P_{uv}$ if and only if $P'_{ij} = P'_{uv}$. Then for all $i \in N$, $f_i(P) = k$ if and only if $f_i(P') = k$.[16]

Independence of irrelevant opinions applies to all problems $P, P' \in \mathcal{D}$ that satisfy condition (i) above and, therefore, is stronger than *weak independence*. Note that the two conditions (i) and (ii) mean that P' is obtained from P by applying a permutation $\gamma: G \to G$ with a fixed point k ($\gamma(k) = k$) to each entry of P, that is, $P' = \gamma(P) \equiv (\gamma(P_{ij}))_{i,j\in N}$. Call γ a *k-fixed groupwise permutation* and $\gamma(P)$ the *transformation of P by the k-fixed groupwise permutation γ*. Therefore, a rule satisfies weak independence if for all $k \in G$, group-k membership is invariant with respect to any transformation by a k-fixed groupwise permutation. There turn out to be a number of rules exhibiting this invariance. We provide some examples below. All these rules satisfy non-degeneracy and are quite distinct from the one-vote rules.

First is a subfamily of the consent rules with default decisions (defined in Section 2). Let $\ell_0 \in G$ and $q_0 \in \{1, \dots, n\}$. Let $q \equiv (q_k)_{k\in G}$ and $\delta \equiv (\delta_k)_{k\in G}$ be such that $q_1 = \cdots = q_m = q_0$ and $\delta_1 = \cdots = \delta_m = \ell_0$. Then under the consent rule $f^{q,\delta}$ with quotas q and default decisions δ, group-k membership is not affected by any k-fixed groupwise permutation of the entries of P; so $f^{q,\delta}$ satisfies weak independence.

Second is a family of rules obtained by mixing features of the liberal and plurality rules. Let $\ell_0 \in G$. Define a rule f^{ℓ_0} as follows: for all $P \in \mathcal{D}$ and all $i \in N$, (i) if for all $\ell \in G\backslash\{P_{ii}\}$, $|\{(u, v) \in N^2 : P_{uv} = P_{ii}\}| \geq |\{(u, v) \in N^2 : P_{uv} = \ell\}|$, then $f_i^{\ell_0}(P) = P_{ii}$, and (ii) otherwise, $f_i^{\ell_0}(P) = \ell_0$. Then it can be shown that for all $k \in G$, group-k membership is not affected by any k-fixed groupwise permutation of the entries of P.

[16] This weaker axiom was suggested by an anonymous referee.

5.2.2 *Initial membership and regrouping* We can extend our model to allow for "initial membership," and modify independence of irrelevant opinions accordingly. Consider a mapping $\sigma: N \to G$, associating with each person $i \in N$ her *initial membership* $\sigma_i \in G$. For all $k \in G$, let $q_k \in \mathbb{N}$ be the *consent quota for group* k. The *regrouping consent rule with initial membership* $\sigma \equiv (\sigma_i)_{i \in N}$ *and quotas* $q \equiv (q_k)_{k \in G}$, denoted by $f^{\sigma,q}$, is defined as follows: for all $P \in \mathcal{P}$ and all $i \in N$ with $P_{ii} = k$,

(i) if $|\{j \in N : P_{ji} = k\}| \geq q_k$, then $f_i^{\sigma,q}(P) = k$; and

(ii) otherwise, $f_i^{\sigma,q}(P) = \sigma_i$.

Thus each person $i \in N$ only belongs to the group of her self-opinion (P_{ii}) or her initial group (σ_i). She can always decide to stay in the initial group; she needs others' consent only when she claims a change. Although these rules are similar to the consent rules with default decisions, the two families are different. For instance, in the binary model, the regrouping consent rules do not coincide with the consent rules of Samet and Schmeidler (2003).

Under initial membership, independence of irrelevant opinions can be weakened by requiring the same independence only for changing the membership from the initial one. Whether a person remains in her initial group may depend on opinions on the other groups.

Regrouping Independence. Let $P, P' \in \mathcal{P}$ and $k \in G$. Suppose that for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$. Then for all $i \in N$ with $\sigma_i \neq k$, $f_i(P) = k$ if and only if $f_i(P') = k$.

Regrouping independence weakens independence of irrelevant opinions only slightly. But interestingly, it is satisfied by the regrouping consent rules, which are quite different from the one-vote rules and provide a more diversified menu. The regrouping consent rules satisfy unanimity (hence, non-degeneracy) and if for all $i, j \in N$, $\sigma_i = \sigma_j$, they also satisfy symmetry. When $q_1 = \cdots = q_m = 1$, $f^{\sigma,q}$ coincides with the liberal rule.

5.3 *Relaxing the single-membership requirement*

In our model, opinions and decisions partition the set of persons into groups. This "single-membership" requirement plays an important role in our analysis and makes independence of irrelevant opinions have strong implications. For instance, in our proofs, we frequently use the argument that a person belongs to a group if and only if she does not belong to any other groups. The single-membership requirement precludes two possibilities: (i) a person may not belong to any group and (ii) a person may belong to two or more groups. If one or both of the two cases are permitted, independence of irrelevant opinions becomes much weaker and a number of rules other than the one-vote rules satisfy it. To highlight this fact, we minimally depart from the model by weakening the single-membership requirement for decisions while maintaining it for opinions.

5.3.1 *Allowing for no membership* First, we allow for the possibility that a person does not belong to any group. Then a decision puts each person in at most one group. Formally, a *decision* is a profile $x \equiv (x_i)_{i\in N} \in (G \cup \{\varnothing\})^N$, where for all $i \in N$, $x_i = \varnothing$ means that person i does not belong to any group in G. Given a domain $\mathcal{D}$, a *rule* is a mapping $f : \mathcal{D} \to (G \cup \{\varnothing\})^N$. Independence of irrelevant opinions can be defined as follows.

Independence of Irrelevant Opinions. Let $P, P' \in \mathcal{D}$ and $k \in G$. Suppose that for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$. Then for all $i \in N$, $f_i(P) = k$ if and only if $f_i(P') = k$.

In addition to the one-vote rules, a subfamily of the consent rules with default decisions satisfy independence of irrelevant opinions. For all $k \in G$, let $\delta_k \equiv \varnothing$ be the default decision against group k. Let $\delta \equiv (\delta_k)_{k\in G}$. Then for any quotas $q \equiv (q_k)_{k\in G}$, the consent rule $f^{\delta,q}$ with default decisions δ and quotas q satisfies the axiom.

The one-vote rules and the above subfamily of the consent rules are represented by quite well-behaved approval functions; the approval functions satisfy unanimity, self-duality, and monotonicity. In general, however, independence of irrelevant opinions alone does not guarantee those properties in the present setup due to the possibility of a no-membership decision; hence implications in Proposition 1 do not hold. In fact, a rule satisfying independence of irrelevant opinions can be represented by a rather peculiar approval function. To illustrate this point, consider rule $\hat{f}$ defined as follows: for all $P \in \mathcal{D}$, all $i \in N$, and all $k \in G$, (i) $f_i(P) = k$ if $P_{ii} = k$ and all $j \in N\backslash\{i\}$, $P_{ij} \neq k$, and (ii) $f_i(P) = \varnothing$ otherwise. Evidently, $\hat{f}$ satisfies independence of irrelevant opinions. Now define an approval function $\hat{\varphi}$ as follows: for all $B \in \mathcal{B}$ and all $i \in N$, (i) $\hat{\varphi}_i(B) = 1$ if $B_{ii} = 1$ and all $j \in N\backslash\{i\}$, $B_{ij} = 0$, and (ii) $\hat{\varphi}_i(B) = 0$ otherwise. While $\hat{\varphi}$ represents $\hat{f}$, it violates unanimity, self-duality, and monotonicity, which, by Proposition 1, should be satisfied in the single-membership model.[17]

5.3.2 *Allowing for multiple memberships* Next, we consider the case where a person can belong to multiple groups. Here a *decision* is a profile $x \equiv (x_i)_{i\in N} \in (2^G\backslash\{\varnothing\})^N$, where, for all $i \in N$ and all $k \in G$, $k \in x_i$ means that person i belongs to group k. Thus, a decision puts each person in at least one group. Given a domain $\mathcal{D}$, a *rule* is a mapping $f : \mathcal{D} \to (2^G\backslash\{\varnothing\})^N$. In this context, independence of irrelevant opinions is defined as follows.

Independence of Irrelevant Opinions. Let $P, P' \in \mathcal{D}$ and $k \in G$. Suppose that for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$. Then for all $i \in N$, $k \in f_i(P)$ if and only if $k \in f_i(P')$.

To give an example of rules satisfying independence of irrelevant opinions, we can adapt the consent rules Samet and Schmeidler (2003) to the present setup. For all $k \in G$,

[17] When incomplete opinions are allowed as in footnote 9, a problem is an element of $(G \cup \{\varnothing\})^{N\times N}$ and a decision is an element of $(G \cup \{\varnothing\})^{N\times N}$. The consent rules with default decisions and the rule $\hat{f}$ in this section are only defined on $G^{N\times N}$, but they can easily be extended to $(G \cup \{\varnothing\})^{N\times N}$. The extensions thus obtained satisfy independence of irrelevant opinions because the axiom applies only to the groups in G, excluding $\varnothing$. Therefore, independence of irrelevant opinions becomes weaker if incomplete opinions are allowed.

let $s_k, t_k \in \mathbb{N}$. The *consent rule with* $(s, t) \equiv (s_k, t_k)_{k \in G}$, denoted by $f^{s,t}$, is defined as follows: for all $P \in \mathcal{D}$, all $i \in N$, and all $k \in G$,

(i) when $P_{ii} = k$, $k \in f_i(P)$ if and only if $|\{j \in N : P_{ji} = k\}| \geq s_k$; and

(ii) when $P_{ii} \neq k$, $k \notin f_i(P)$ if and only if $|\{j \in N : P_{ji} \neq k\}| \geq t_k$.

Clearly, all these consent rules satisfy independence of irrelevant opinions. However, under no conditions on (s, t), $f^{s,t}$ may put an agent in none of the groups in G, violating our definition of decisions in this section (the case of no membership is excluded). To avoid such cases, we may restrict (s, t). For instance, let $s_1 = \cdots = s_m = 1$ (so that a person is at least a member of the group she classifies herself into); or for all distinct $k, \ell \in G$, let s_k and t_ℓ be such that $\frac{n-s_k}{m-1} \geq n + 1 - t_\ell$ (so that whenever a person fails to win enough approval for the group she classifies herself into, she belongs to some other group).

A remark similar to that in Section 5.3.1 applies. When multiple memberships are allowed, implications in Proposition 1, which were obtained in the single-membership setup, do not hold. Therefore, in contrast to the consent rules, a rule may satisfy independence of irrelevant opinions and yet be represented by an anomalous approval function. For example, define rule $\tilde{f}$ as follows: for all $P \in \mathcal{D}$, all $i \in N$, and all $k \in G$, (i) $k \notin \tilde{f}_i(P)$ if for all $j \in N$, $P_{ji} = k$, and (ii) $k \in \tilde{f}_i(P)$ otherwise. Rule $\tilde{f}$ is represented by the approval function $\tilde{\varphi}$ defined as for all $B \in \mathcal{B}$ and all $i \in N$, $\tilde{\varphi}_i(B) = 0$ if for all $j \in N$, $B_{ji} = 1$; and $\tilde{\varphi}_i(B) = 1$ otherwise. Note that $\tilde{\varphi}$ violates unanimity, self-duality, and monotonicity, which are the three properties of approval functions implied by independence of irrelevant opinions in the single-membership setup (Proposition 1).

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