Supplementary Material

Supplement to "Full surplus extraction and within-period ex post implementation in dynamic environments"

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Appendix B: Further discussion

B.1 *Genericity*

This subsection studies assumptions with which strong and weak detectability are generic in the one-shot problem. Given (X, Θ^i, χ, S), whether strong and weak detectability are satisfied is determined by $\pi \in (\Delta(S))^{|X|\times|\Theta^i|}$. Define

$$\mathcal{SD}(X, \Theta^i, \chi, S) \equiv \{\pi \in (\Delta(S))^{|X|\times|\Theta^i|} : \Theta^i \text{ is strongly detectable with } (X, \Theta^i, \chi, S, \pi)\},$$

$$\mathcal{WD}(X, \Theta^i, \chi, S) \equiv \{\pi \in (\Delta(S))^{|X|\times|\Theta^i|} : \Theta^i \text{ is weakly detectable with } (X, \Theta^i, \chi, S, \pi)\}.$$

Note that $\mathcal{SD}(X, \Theta^i, \chi, S) \subset \mathcal{WD}(X, \Theta^i, \chi, S)$ because strong detectability implies weak detectability. Let L be the Lebesgue measure on $(\Delta(S))^{|X|\times|\Theta^i|}$ where we normalize $L((\Delta(S))^{|X|\times|\Theta^i|}) \equiv 1$. Strong (weak) detectability is *generically satisfied with* (X, Θ^i, χ, S) if $L(\mathcal{SD}(X, \Theta^i, \chi, S)) = 1$ $(L(\mathcal{WD}(X, \Theta^i, \chi, S)) = 1)$.

Theorem 5. *(i) Strong detectability is generically satisfied with* (X, Θ^i, χ, S) *if and only if*

$$|\Theta^i| \le |S|.$$

(ii) Weak detectability is generically satisfied with (X, Θ^i, χ, S) *if and only if*

$$\min_{x\in X} |\{\theta^i \in \Theta^i : \chi(\theta^i) \neq x\}| + 1 \le |S|.$$

Note that the inequality

$$\min_{x\in X} |\{\theta^i \in \Theta^i : \chi(\theta^i) \neq x\}| + 1 \le |\Theta^i| \tag{30}$$

holds, which implies that weak detectability is generically satisfied whenever strong detectability is. Furthermore, whenever there exist two or more distinct types that lead to the same allocation (i.e., there exists $\theta^i \neq \hat{\theta}^i$ such that $\chi(\theta^i) = \chi(\hat{\theta}^i)$), (30) is satisfied with strict inequality. Accordingly, Theorem 5 indicates that weak detectability is generic under a weaker condition than strong detectability is.

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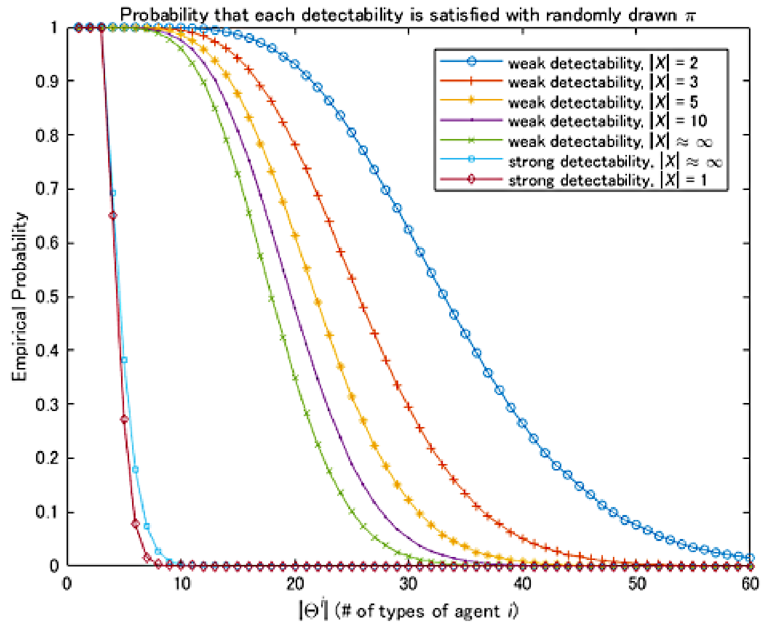

FIGURE 2. The empirical probability that strong and weak detectability are satisfied when (i) $|S| = 3$, and (ii) $\pi \in (\Delta(S))^{|X|\times|\Theta^i|}$ and $\chi \in X^{|\Theta^i|}$ are drawn uniformly at random.

B.2 *Numerical simulations*

The Lebesgue measure of $\mathcal{WD}(X, \Theta^i, \chi, S)$ is substantially larger than $\mathcal{SD}(X, \Theta^i, \chi, S)$. To illustrate this fact, we compute the Lebesgue measure of $\mathcal{WD}(X, \Theta^i, \chi, S)$ and $\mathcal{SD}(X, \Theta^i, \chi, S)$ numerically. We fix $|S| = 3$, change the variety of agent i's types ($|\Theta^i|$) and the variety of possible allocations ($|X|$), and draw (π, χ) uniformly at random. The result is shown as Figure 2. Each point in the figure represents the empirical probability that strong and weak detectability are satisfied when $\pi \in (\Delta(S))^{|X|\times|\Theta^i|}$ and $\chi \in X^{|\Theta^i|}$ are drawn uniformly at random. The approximation $|X| \approx \infty$ indicates that $|X|$ is so large that a different x is always assigned for a different θ^i, i.e., $\chi(\theta^i) \neq \chi(\hat{\theta}^i)$ for $\theta^i \neq \hat{\theta}^i$ with probability 1. We make the following observations:

(i) The larger the number of agent i's types ($|\Theta^i|$), the less likely that strong and weak detectability are satisfied. Note that when $|\Theta^i|$ is small and the corresponding dimensionality conditions of Theorem 5 are satisfied, strong and weak detectability are satisfied with probability 1, consistent with their genericity.

(ii) The larger the number of allocations ($|X|$), the less likely weak detectability is satisfied. In contrast, the larger is $|X|$, the more likely strong detectability is satisfied.

(iii) As $|\Theta^i|$ becomes larger, the chance of having strong detectability decreases much faster than for weak detectability. For example, in the case of $|X| \approx \infty$, (a) the difference in empirical probabilities that strong and weak detectability are satisfied is minimized and (b) neither detectability is generic if and only if $|\Theta^i| > 3$. Even in this case, when $|\Theta^i| = 10$, strong detectability is satisfied with a probability of only 0.26 percent, while the probability for weak detectability is 96.07 percent.

Of course, in applications, we rarely presume that (χ, π) is determined uniformly at random. However, this numerical simulation helps us to understand the extent to which weak detectability is more likely to be satisfied than strong detectability.

B.3 *Connection to the conventional notion of full surplus extraction*

One might think that the necessity discussed here is different from the conventional notion. Crémer and McLean (1988) fix a type space and a type distribution (which correspond to $(\Theta_t, \mu_t)_{t=0}^{T+1}$ in our setting), and provide a tight necessary and sufficient condition for full surplus extraction for arbitrary valuation functions. However, in this paper, we also fix an allocation rule and discuss the possibility of extraction (and implementation) for arbitrary sequences of flow valuation functions. These two notions for full surplus extraction seem different because we can ignore valuations that make the targeted allocation rule inefficient if our aim is full surplus extraction (or efficiency). However, these two necessity notions are equivalent: even when we fix an information structure and *one* agent's valuation function, still every allocation rule can be efficient.

LEMMA 3. *For all $(\Theta_t, \mu_t)_{t=0}^{T+1}$, $(v_t^i)_{t=0}^T$, and $(\chi_t)_{t=0}^{T+1}$, there exists $((v_t^j)_{j\neq i})_{t=0}^T$ such that $(\chi_t)_{t=0}^T$ is a unique efficient rule with respect to $(\Theta_t, \mu_t)_{t=0}^{T+1}$ and to $((v_t^j)_{j\in\mathcal{I}})_{t=0}^T$.*

Since we allow interdependent values, we can offset one agent's payoff by another agents'. For example, when $\mathcal{I} = \{i, -i\}$, in period T, to make χ_T uniquely efficient, we can set

$$v_T^{-i}(x_T, \theta_T) = \begin{cases} -v_T^i(x_T, \theta_T) + 1 & \text{if } x_T = \chi_T(\theta_T), \\ -v_T^i(x_T, \theta_T) & \text{otherwise.} \end{cases}$$

Similarly, we can choose a sequence of the other agents' valuation functions, $((v_t^j)_{j\neq i})_{t=0}^T$, which are not relevant to agent i's problem, to justify the efficiency of the targeted allocation rule for all t. Therefore, given $(\Theta_t, \mu_t)_{t=0}^{T+1}$, if there exist $(\chi_t)_{t=0}^{T+1}$ and $(v_t^i)_{t=0}^T$ such that we must leave information rent for i, then we cannot extract the full surplus with $((v_t^j)_{j\neq i})_{t=0}^T$ that makes $(\chi_t)_{t=0}^T$ an efficient allocation rule. Likewise, if there exist $(\chi_t)_{t=0}^{T+1}$ and $(v_t^i)_{t=0}^T$ such that we cannot satisfy wp-EPIC for i, then we cannot achieve efficiency with such $((v_t^j)_{j\neq i})_{t=0}^T$.

B.4 *Markovization*

"Markovity" of the environment largely simplifies our analysis because the true realization of the past types $\theta_{0:t-1}$ becomes independent of the period t problem (while the past reports matter because we use non-Markov payments). As Athey and Segal (2013) show, we can always "Markovize" the environment by rewriting the problem as one with a larger state space. However, strong and weak detectability are not independent of the state specification. The larger the state space becomes, the less likely these conditions are to be satisfied. Therefore, so as to apply our results to a non-Markov problem, we need to enlarge the state space until we obtain the Markovity of the environment (i.e., until the flow valuation function and the transition probability function become Markov), but no further.

B.5 *Non-Markov allocation rule*

In the manuscript, we assumed that the allocation rule $(\chi_t)_{t=0}^{T+1}$ is Markov. While Markovity of valuations and state transitions imply that there exists an efficient Markov allocation rule, if an inefficient allocation rule is targeted, this assumption is restrictive. This subsection discusses how our results can be generalized to the case of non-Markov allocation rules (χ_t depends not only on θ_t, but also on $\theta_{0:t-1}$).

For equilibrium payoff control by strong detectability, we need only a slight change. Since the targeted one-shot allocation rule changes according to the history of reports, we need to make the signal structure history-dependent; thus, we also need to modify the definition of backup sets so that they are also history-dependent. However, after these modifications, Theorem 1 provides a sufficient condition for satisfying the no-information-rent property.

Weak detectability should be modified more largely. For example, consider a two-stage problem in which $\mathcal{I} = \{i\}$, $\Theta_0^i = \{L_0, R_0\}$, $|\Theta_1^i| = 1$, $|X_0| = 1$, and $X_1 = \{l_1, r_1\}$. While no ex post signal is available, weak detectability is satisfied and any Markov allocation rule is implementable: (i) the mechanism does not have to distinguish $\theta_0^i = L_0$ or R_0 because $|X_0| = 1$ implies that they lead to the same allocation, and (ii) $|\Theta_1^i| = 1$ implies that there is no private information in period 1. However, in this environment, the signal structure does not guarantee implementability of a non-Markov allocation rule $\chi_1(L_0) = l_1$ and $\chi_1(R_0) = r_1$ because no signal is available.

To modify weak detectability to guarantee implementation of non-Markov allocation rules, we should distinguish type reports that may lead to different allocations not only in the current period, but also in the future. We incorporate a quotient set P generated by an equivalent relation $\sim$ on Θ^i into the definition of weak detectability: Θ^i is *weakly detectable* with $(X, \Theta^i, \chi, S, \pi, P)$ if, for all $\bar{\Theta}^i \subset \Theta^i$, there exists $\bar{\theta}^i \in \bar{\Theta}^i$ such that

$$\pi\big(\chi(\bar{\theta}^i), \bar{\theta}^i\big) \notin \mathrm{co}\big(\big\{\pi\big(\chi(\bar{\theta}^i), \hat{\theta}^i\big)\big\}_{\hat{\theta}^i \in \bar{\Theta}^i \setminus [\bar{\theta}^i]_P}\big), \tag{31}$$

where $[\bar{\theta}^i]_P$ denotes the equivalence class of $\bar{\theta}^i$ with respect to $\sim$. If the quotient set P is generated by the equivalence relation implied by $\theta^i \sim \hat{\theta}^i$ if and only if $\chi(\theta^i) = \chi(\hat{\theta}^i)$, then this modified version of weak detectability coincides with the previous one.

We also replace the signal structure in the original problem. We define $\Omega_t(\theta_{0:t-1}, \theta_t^{-i})$ to be the quotient set of Θ_t^i generated by the equivalence relation

$$\theta_t^i \sim \hat{\theta}_t^i \quad \text{if and only if}$$

$$\chi_\tau\big(\theta_{0:t-1}, \theta_t^i, \theta_t^{-i}, \theta_{t+1:\tau}\big) = \chi_\tau\big(\theta_{0:t-1}, \hat{\theta}_t^i, \theta_t^{-i}, \theta_{t+1:\tau}\big) \quad \text{for all } \tau = t, t+1, \dots, T+1.$$

We do not distinguish θ_t^i and $\hat{\theta}_t^i$ if and only if they lead to the same allocation not only in period t, but also in all future periods. Using $\Omega_t(\theta_{0:t-1}, \theta_t^{-i})$ as the quotient set for the modified version of weak detectability (and using modified backup sets discussed above), we obtain a new version of Theorem 2, which provides a sufficient condition for implementation of non-Markov allocation rules.

B.6 *Full surplus extraction in every period*

We have focused on period-0 full surplus extraction. Alternatively, we can consider mechanisms that extract full expected future surplus at the beginning of *each period*, i.e., a mechanism $(\chi_t, \psi_t)_{t=0}^{T+1}$ that satisfies

$$V_t^i(\theta_t) + \Psi_t^i(\theta_{0:t}) = 0 \tag{32}$$

for all $i \in \mathcal{I}$ and $\theta_{0:t} \in \Theta_{0:t}$.

However, in many cases, there are no such mechanisms because we can no longer exploit the intertemporal correlation. Recall that wp-EPIC of i at $\theta_{0:t}$ is satisfied if and only if

$$\begin{aligned} &V_t^i(\theta_t) + \Psi_t^i(\theta_{0:t}) \\ &\quad \geq v_t^i\big(\chi_t\big(\hat{\theta}_t^i, \theta_t^{-i}\big), \theta_t\big) + \psi_t^i(\theta_{0:t-1}, \theta_t) \\ &\qquad + \delta \cdot \mathbb{E}\big[V_{t+1}^i(\theta_{t+1}) + \Psi_{t+1}^i\big(\theta_{0:t-1}, \hat{\theta}_t^i, \theta_t^{-i}, \theta_{t+1}\big) | \chi_t\big(\hat{\theta}_t^i, \theta_t^{-i}\big), \theta_t\big] \end{aligned} \tag{33}$$

for all $\hat{\theta}_t^i \in \Theta_t^i$. Substituting (32) into (33), wp-EPIC of i at $\theta_{0:t}$ is reduced to

$$0 \geq v_t^i\big(\chi_t\big(\hat{\theta}_t^i, \theta_t^{-i}\big), \theta_t\big) + \psi_t^i\big(\theta_{0:t-1}, \hat{\theta}_t^i, \theta_t^{-i}\big)$$

for all $\hat{\theta}_t^i \in \Theta_t^i$. Hence, $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ leaves no information rent for i in all periods if and only if (i) for all t, the period-t static mechanism (χ_t, ψ_t^i) is ex post incentive compatible, and (ii) agent i's on-path ex post payoffs from (χ_t, ψ_t^i) are zero for every $\theta_t \in \Theta_t$. We cannot expect the existence of such static mechanisms in general.

B.7 *Dropping the full-support assumption*

We have assumed that all of $(\mu_t)_{t=0}^{T+1}$ have full support. Lemmas 1 and 2 indicate that under this assumption, the intraperiod correlation is "useless" for construction of a wp-EPIC mechanism because the intraperiod correlation does not appear in the necessary and sufficient condition.

In contrast, if $(\mu_t)_{t=0}^{T+1}$ does not satisfy the full-support assumption, the intraperiod correlation is sometimes helpful to sustain wp-EPIC. This is essentially because harsh punishments for the reports of zero-probability types are useful to achieve (static) ex post incentive compatibility. Let

$$\hat{\Theta}_t^i\big(x_{t-1}, \theta_t^{-i}\big) \equiv \big\{\theta_t^i \in \Theta_t^i : \exists \theta_{t-1} \in \Theta_{t-1} \text{ s.t. } \mu_t\big(\theta_t^i, \theta_t^{-i}; x_{t-1}, \theta_{t-1}\big) > 0\big\}.$$

Given (x_{t-1}, θ_t^{-i}), $\theta_t^i \notin \hat{\Theta}_t^i(x_{t-1}, \theta_t^{-i})$ does not happen with positive probability. Therefore, when such θ_t^i is reported, the central planner can punish agent i without changing on-path payoffs. In this case, it suffices to consider incentives for truthtelling against misreports within $\hat{\Theta}_t^i(x_{t-1}, \theta_t^{-i})$, rather than Θ_t^i, for satisfying wp-EPIC. Accordingly, an intraperiod correlation with shifting support (which is excluded from our main analysis by assuming full support) is helpful for both within-period ex post implementation and full surplus extraction, as the realization of θ_t^{-i} restricts the set of potentially profitable deviations.

B.8 *Combining our conditions with Liu (2018)*

We have proposed conditions different from Liu (2018). One may think that we can obtain a further weaker sufficient condition by combining our conditions with Liu's, and it may lead us to a necessary and sufficient condition. Examples 3 and 5 indicate that the former half is true: our theorems guarantee full surplus extraction in an environment in which Liu's theorem cannot (Example 3) and vice versa (Example 5). There also exists an environment in which full surplus extraction is possible, but neither our theorem nor Liu's condition is satisfied.

Example 6. This environment is a hybrid of Examples 3 and 5. Consider a three-stage problem where $|\Theta_0^{-i}| = |\Theta_1^{-i}| = |\Theta_2^i| = 1$, $\Theta_0^i = \{L_0, R_0\}$, $\Theta_1^i = \{A_1, B_1, C_1, D_1, M_1\}$, $\Theta_2^{-i} = \{E_2, F_2, G_2\}$, and $|X_t| = 1$ for $t = 0, 1, 2$. The state transitions μ_1 and μ_2 are summarized in Table 3. This environment is a modified version of Example 3: we added a new state M_1 to Θ_1^i, which has the role of B_1 in Example 5.

Same as Examples 3 and 5, since the allocation space is a singleton, implementability of the targeted allocation rule is trivial. We consider whether we can detect θ_0^i without leaving information rent.

In this example, the assumption of Theorem 1 is not met. Since no ex post signal is available in period 2, the backup set in period 1 is empty. Since

$$\frac{1}{2}\big[\mu_2(A_1) + \mu_2(D_1)\big] = \frac{1}{2}\big[\mu_2(B_1) + \mu_2(C_1)\big] = \mu_2(M_1) = (0.4, 0.3, 0.3), \tag{34}$$

Θ_1^i is not strongly detectable with $\Gamma_1^i(\varnothing)$. Hence, the assumption of Theorem 1 reduces to strong detectability with $\Gamma_0^i(\theta_0^{-i}, \varnothing)$. However, (i) $|\Theta_0^i| = 2$ implies that the agent's type space is not degenerate, but (ii) $|\Theta_1^{-i}| = 1$ implies that the signal space of $\Gamma_0^i(\theta_0^{-i}, \varnothing)$ is degenerate. Accordingly, Θ_0^i is not strongly detectable with $\Gamma_0^i(\varnothing)$.

Furthermore, Liu's way to use distant intertemporal correlations (described in Section 6.3) is not applicable either. We can derive that agent i's belief on θ_2^{-i} conditional on $\theta_0^i = L_0$ is

$$0.1 \cdot \big[\mu_2(A_1) + \mu_2(D_1)\big] + 0.3 \cdot \big[\mu_2(B_1) + \mu_2(C_1)\big] + 0.2 \cdot \mu_2(M_1) \tag{35}$$

and agent i's belief on θ_2^{-i} conditional on $\theta_0^i = R_0$ is

$$0.3 \cdot \big[\mu_2(A_1) + \mu_2(D_1)\big] + 0.1 \cdot \big[\mu_2(B_1) + \mu_2(C_1)\big] + 0.2 \cdot \mu_2(M_1). \tag{36}$$

	A_1	B_1	C_1	D_1	M_1
$\mu_1(\cdot; L_0)$	0.1	0.3	0.3	0.1	0.2
$\mu_1(\cdot; R_0)$	0.3	0.1	0.1	0.3	0.2

	E_2	F_2	G_2
$\mu_2(\cdot; A_1)$	0.6	0.2	0.2
$\mu_2(\cdot; B_1)$	0.4	0.5	0.1
$\mu_2(\cdot; C_1)$	0.4	0.1	0.5
$\mu_2(\cdot; D_1)$	0.2	0.4	0.4
$\mu_2(\cdot; M_1)$	0.4	0.3	0.3

TABLE 3. The state transitions $\mu_1 : \Theta_0^i \to \Theta_1^i$ and $\mu_2 : \Theta_1^i \to \Theta_2^{-i}$.

However, (34) indicates that (35) and (36) take the same value, $(0.4, 0.3, 0.3)$. Since L_0 and R_0 generate the same beliefs about the distribution of θ_2^{-i}, we cannot directly construct a Crémer–McLean lottery from the correlation between θ_0^i and θ_2^{-i}.

However, full surplus extraction is possible. For $\theta_1^i \in \{A_1, B_1, C_1, D_1\}$ $(= \Theta_1^i \setminus \{M_1\})$,

$$\mu_2^{-i}(\theta_1^i) \notin \text{co}\big(\{\mu_2^{-i}(\hat{\theta}_1^i)\}_{\hat{\theta}_1^i \in \Theta_1^i \setminus \{\theta_1^i\}}\big).$$

Accordingly, for each $\theta_1^i \in \{A_1, B_1, C_1, D_1\}$, there exists a lottery over $\theta_2^{-i} \in \Theta_2^{-i}$ such that (i) if agent i truthfully reports θ_1^i, his expected payoff is zero, and (ii) if agent i misreports some $\hat{\theta}_1^i \in \Theta_1^i \setminus \{\theta_1^i\}$, he receives a negative expected payoff. Using this lottery, we can prevent A_1, B_1, C_1, and D_1 from misreporting. Hence, we can provide an arbitrary continuation payoff vector $(U_1^i(A_1), U_1^i(B_1), U_1^i(C_1), U_1^i(D_1))$ to agent i, while preventing him from misreporting his type when his true type is $\theta_1^i \in \{A_1, B_1, C_1, D_1\}$. Furthermore, we can provide some constant subsidies (which are independent of θ_2^{-i}) when $\theta_1^i = M_1$ is reported, to prevent type M_1 agent from misreporting his type. Here, every $(U_1^i(A_1), U_1^i(B_1), U_1^i(C_1), U_1^i(D_1))$ is achievable with some $U_1^i(M)$.

Now we consider agent i's reporting problem in period 0. Since

$$\mu_1(M_1; L_0) = \mu_1(M_1; R_0) = 0.2,$$

continuation payoffs at (L_0, M_1) and (R_0, M_1) do not affect the incentive for reporting $\theta_0^i \in \Theta_0^i = \{L_0, R_0\}$. Using continuation payoffs at $\theta_1^i \in \{A_1, B_1, C_1, D_1\}$ as a Crémer–McLean lottery (specifically, providing a larger continuation payoff at (i) B_1 and C_1 when L_0 is reported, and (ii) A_1 and D_1 when R_0 is reported), we can detect the state of period 0 without leaving information rent. $\diamondsuit$

B.9 *The necessary and sufficient condition*

As discussed in Section 6.1, our main theorems do not provide the necessary and sufficient condition for full surplus extraction and the implementability of a targeted allocation rule because we have decreased the dimension of signal spaces to make the set of available continuation payoffs a linear space. In contrast, once we specify the set of achievable continuation payoffs without reducing the dimension, we can derive the necessary and sufficient conditions.

THEOREM 6. *Given $(v_t^i)_{t=0}^{T+1}$ and $(\chi_t)_{t=0}^{T+1}$, define $(\mathcal{U}_t^i)_{t=0}^{T+1}$ such that the following relationships hold:*

(i) *We have* $\mathcal{U}_t^i \subset \mathbb{R}^{|\Theta_t|}$ *for all* t.

(ii) *For* $t = T+1$,

$$\begin{aligned}\mathcal{U}_{T+1}^i \equiv \{U_{T+1}^i \in \mathbb{R}^{|\Theta_{T+1}|} : U_{T+1}^i(\theta_{T+1}^i, \theta_{T+1}^{-i}) \\ = U_{T+1}^i(\hat{\theta}_{T+1}^i, \theta_{T+1}^{-i}) \forall \theta_{T+1}^i, \hat{\theta}_{T+1}^i\}.\end{aligned} \tag{37}$$

(iii) *For* $t = 0, \ldots, T$, *if* $\mathcal{U}_{t+1}^i = \varnothing$, *then* $\mathcal{U}_{t+1}^i \equiv \varnothing$; *otherwise,* $U_t^i \in \mathcal{U}_t^i$ *if and only if there exists* $U_{t+1}^i : \Theta_t \to \mathcal{U}_{t+1}^i$ *such that*

$$U_t^i(\theta_t) = v_t^i(\chi_t(\theta_t), \theta_t) + \delta\mathbb{E}[U_{t+1}^i(\theta_t, \theta_{t+1}) | \chi_t(\theta_t), \theta_t] \tag{38}$$

$$\begin{aligned}&U_t^i(\theta_t) \geq v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t) + \delta\mathbb{E}[U_{t+1}^i(\hat{\theta}_t^i, \theta_t^{-i}, \theta_{t+1}) | \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t] \\ &\quad \textit{for all } \hat{\theta}_t^i \in \Theta_t^i.\end{aligned} \tag{39}$$

Then the following statements hold:

(a) *There exists* $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ *that satisfies wp-EPIC for* i *if and only if* $\mathcal{U}_0^i \neq \varnothing$.

(b) *There exists* $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ *that satisfies wp-EPIC and leaves no information rent for* i *if and only if* $(0, \ldots, 0) \in \mathcal{U}_0^i$.

Here, $\mathcal{U}_t^i$ is the set of ($|\Theta_t|$ length vectors of) continuation payoffs that the central planner can provide, preserving that wp-EPIC is satisfied (using some monetary transfers). However, it is difficult to obtain a necessary and sufficient condition on primitives because it is difficult to specify $(\mathcal{U}_t^i)_{t=0}^{T+1}$.

Appendix C: Infinite horizon

C.1 *Overview*

In the main body of this paper, we concentrated on a finite horizon. However, the techniques developed in our paper are also applicable to an infinite horizon. In this appendix, we provide a sufficient condition for full surplus extraction with an infinite horizon.

When we consider an infinite horizon, so as to make the mechanism individually rational (i.e., make each agent's on-path continuation payoff nonnegative after every history), it is important to prevent explosion of the payment. When the worst-case payment of agents goes to infinity as $t \to \infty$, (i) the one-shot deviation principle is not applicable because agents' flow payoffs may grow to infinity, potentially at a superexponential speed, and (ii) we cannot always use the "deposit scheme" to prevent the agents from leaving at $t \geq 1$ because we may need an infinite amount of "deposits" to keep the agents participating. In this section, we consider only uniformly bounded payment rules to guarantee that the one-shot deviation principle and the deposit scheme are applicable. To achieve this, we need a stronger condition to provide a uniform bound.

First, we assume the existence of a uniformly bounded payment rule $(g_t^i)_{t=0}^{\infty}$ that implements a targeted allocation rule and consider the revelation of agent i's initial type

	L	R
$\mu^i_{t+1}(\cdot; l)$	q_t	$1-q_t$
$\mu^i_{t+1}(\cdot; r)$	$1-q_t$	q_t

TABLE 4. The signal distribution $\mu^{-i}_{t+1} : \Theta^i_t \to \Delta(\Theta^{-i}_{t+1})$ discussed in Example 7.

θ^i_0. Even when the time horizon is infinite, we can initiate backward induction from a certain period, say $T < \infty$, to construct a sequence of backup sets $(B^{-i}_t)^{T+1}_{t=1}$ and obtain B^{-i}_1, just as in finite-horizon settings. If strong detectability with $\Gamma^i_0(\theta^{-i}_0, B^{-i}_1)$ is satisfied with obtained B^{-i}_1, we can detect θ^i_0 without leaving information rent. In this manner, we can obtain a counterpart to Theorem 1 for the infinite-horizon settings (Theorem 7).

To implement an efficient allocation rule in an infinite horizon, we need some additional assumptions. It is well known that when the correlation becomes weaker, we need to scale up the Crémer–McLean lottery to provide a sufficiently strong incentive for truthtelling. Therefore, when the correlation between agents' types converges to zero as $t \to \infty$, it is inevitable that the worst-case payment of the agents goes to infinity. In this case, we cannot induce truthtelling with a uniformly bounded payment rule. The following example illustrates this fact.

EXAMPLE 7. Consider the infinite-horizon environment $\delta \in (0,1)$, $\mathcal{I} = \{i, -i\}$, and, for all t, $\Theta^i_t = X_t = \{l, r\}$ and $\Theta^{-i}_{t+1} = \{L, R\}$. The variable χ_t is independent of θ^{-i}_t and it matches the allocation with agent i's type, i.e., $\chi_t(l) = l$ and $\chi_t(r) = r$. For all t, θ^i_t is drawn independently and identically distributed (i.i.d.) with equal probability, i.e., $\mu^i_t(l) = \mu^i_t(r) = 0.5$ after every (x_{t-1}, θ_{t-1}). In addition, the distribution of θ^{-i}_{t+1} depends on θ^i_t; thus, we can use it as a signal for θ^i_t. Specifically, we assume $\mu^{-i}_{t+1}(R; r) = \mu^{-i}_{t+1}(L; l) = q_t \in (1/2, 1)$ and $\mu^{-i}_{t+1}(L; r) = \mu^{-i}_{t+1}(R; l) = 1 - q_t$ (μ^{-i}_{t+1} is illustrated in Table 4). Note that $q_t \neq 1/2$ implies that, for all t, Θ^i_t is strongly detectable with $\Gamma^i_t(\theta^{-i}_t, \varnothing)$. Finally, we assume that $v^i_t(l, r) = v^i_t(r, l) = 1$ and $v^i_t(l, l) = v^i_t(r, r) = 0$ for all $t = \{0, \dots, T\}$, i.e., agent i dislikes the targeted allocation rule. With this environment, we consider agent i's incentive for truthtelling, assuming that agent $-i$ reports θ^{-i}_t truthfully.

Since (i) θ^i_{t+1} is independent of θ^i_t and (ii) θ_{t+2} is independent of θ^i_t, θ^{-i}_{t+1} is the only available signal for inducing truthtelling of θ^i_t. Hence, to consider wp-EPIC for i, without loss of generality, we can focus on payment rules that depend only on $(\theta^i_t, \theta^{-i}_{t+1})$, which can be written as $\phi^i_{t+1} : \Theta^i_t \times \Theta^{-i}_{t+1} \to \mathbb{R}$ (we set $\phi^i_0 \equiv 0$). Furthermore, taking advantage of the symmetry of l and r, we focus our attention on $(\phi^i_t)^\infty_{t=0}$ that satisfies

$$\phi^i_t(r, R) = \phi^i_t(l, L) = \bar{\phi}_t,$$

$$\phi^i_t(r, L) = \phi^i_t(l, R) = \underline{\phi}_t.$$

For a while, we assume that the one-shot deviation principle is applicable. Then $(\chi_t, \phi^i_t)^\infty_{t=0}$ is wp-EPIC for i if and only if

$$\delta\big(q_t \bar{\phi}_t + (1-q_t)\underline{\phi}_t\big) \geq 1 + \delta\big(q_t \underline{\phi}_t + (1-q_t)\bar{\phi}_t\big) \quad \text{for all } t$$

or, equivalently,

$$\bar{\phi}_t - \underline{\phi}_t \geq \frac{1}{\delta \cdot (2q_t - 1)} \quad \text{for all } t.$$

Now, assume $q_t = (1 + \delta^{2t})/2$ so that

$$\delta^t \cdot \frac{1}{\delta \cdot (2q_t - 1)} = \delta^{-(t+1)} \to \infty \quad \text{as } t \to \infty.$$

If the time horizon were finite, we could make the payment rule uniformly bounded, because $\bar{\phi}_t - \underline{\phi}_t$ could be bounded by δ^{-T}. As T increases, δ^{-T} also increases. However, as long as $T < \infty$, $\delta^{-T} < \infty$. Hence, the worst-case EPV, $M^i \equiv \min_{t,\theta_{0:t}}[V_t^i(\theta_t) + \Psi_t^i(\theta_{0:t})]$, were also bounded, and we could apply the one-shot deviation principle and the deposit scheme.

However, now, the time horizon is infinite; thus, $\delta^t \cdot (\bar{\phi}_t - \underline{\phi}_t)$ is unbounded. Then, even when flow valuation functions, v_t^i, are uniformly bounded, each agent's discounted flow payoff, $\delta^t \cdot (v_t^i(x_t, \theta_t) + y_t^i)$ is unbounded, because of the unboundedness of the payments, y_t^i. Now we have two problems: First, we cannot apply the one-shot deviation principle for the case of bounded payoffs; second, even with (a version of) the one-shot deviation principle, participation constraints in later periods cannot be satisfied by the deposit scheme. In the above mechanism, the present value of the worst-case future payment grows to infinity as $t \to \infty$. The deposit scheme is not applicable because agents can only make a finite payment in each period, while an infinite amount of deposit is necessary. $\diamondsuit$

In Example 7, $\delta^t \cdot (\bar{\phi}_t - \underline{\phi}_t)$ goes to infinity as $t \to \infty$ because the correlation between θ_t^i and θ_{t+1}^{-i} vanishes asymptotically; i.e., $q_t \to 1/2$ as $t \to \infty$. To avoid this, we first show that for each lower bound of correlation intensity, we can construct an upper bound of the incentive payment for the one-shot problem (Lemmas 5 and 6). In Example 7, the lower bound of correlation intensity corresponds to the value of $\epsilon > 0$ such that $q_t > 1/2 + \epsilon$, and we can obtain the upper bound on the incentive payments that depend on ϵ. Using this result for the one-shot problem, we show that if the correlation intensity is uniformly bounded, then we can provide a sufficiently strong incentive for truthtelling by a uniformly bound payment rule (Theorem 8).

The results for the infinite-horizon setting shown in this appendix are looser than our results for a finite horizon. For example, in Example 7, q_t may converge to $1/2$ much slower than the speed that δ^t goes to 0. In this case, while the worst-case payment may be unbounded, its present value is bounded; thus, both the one-shot deviation principle and the deposit scheme are applicable, and we may be able to implement the targeted allocation rule. However, the current statement is already very complex and we would not obtain additional qualitative results by dropping these assumptions. Hence, we focus on the case of the uniformly bounded correlation intensity and construct uniformly bounded payment rules.

C.2 *Environment*

We follow the notation of the manuscript; thus, we mention only the additional assumptions made for infinite-horizon settings. Since now we have an infinite horizon, agent i wants to maximize

$$\sum_{t=0}^{\infty} \delta^t \big[v_t^i(x_t, \theta_t) + y_t^i \big].$$

We additionally assume that $\delta < 1$ and v_t^i is uniformly bounded, i.e., there exists $\bar{v} \in \mathbb{R}$ such that $|v_t^i(x_t, \theta_t)| < \bar{v}$ for all i, t, x_t, and θ_t, so that $|\sum_{t=0}^{\infty} \delta^t v_t^i(x_t, \theta_t)| < \infty$ is always satisfied. Still the present value of the payment may be infinity, i.e., $|\sum_{t=0}^{\infty} \delta^t y_t^i| = \infty$, if we consider general payment rules $(\psi_t^i)_{t=0}^{\infty}$. However, we consider only uniformly bounded $(\psi_t^i)_{t=0}^{\infty}$, which ensures that the agents' discounted payoffs are finite. We also assume that $|X_t|$ and $|\Theta_t|$ are uniformly bounded.

Parallel to the manuscript, given a mechanism $(\chi_t, \psi_t)_{t=0}^{\infty}$, a sequence of past reports $\hat{\theta}_{0:t-1}$, and a (true) type profile of today θ_t, EPV terms are defined by

$$V_t^i\big(\theta_t; (\chi_k)_{k=0}^{\infty}\big) \equiv \mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t} v_s^i\big(\chi_s(\theta_s), \theta_s\big) \middle| (\chi_k)_{k=0}^{\infty}, \theta_t \right],$$

$$\Psi_t^i\big(\hat{\theta}_{0:t-1}, \theta_t; (\chi_k)_{k=0}^{\infty}\big) \equiv \mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t} \psi_s^i(\hat{\theta}_{0:t-1}, \theta_t, \theta_{t+1:s}) \middle| (\chi_k)_{k=0}^{\infty}, \theta_t \right].$$

Since we assume that v_t^i is uniformly bounded and focus on uniformly bounded ψ_t^is, V_t^i and Ψ_t^i are also uniformly bounded. Finiteness of V_t^i and Ψ_t^i also guarantee that the one-shot deviation principle used in Definition 1 (wp-EPIC) is also directly applicable to an infinite horizon. An *efficient allocation rule* $(\chi_t)_{t=0}^{\infty}$ maximizes $\mathbb{E}[\sum_{i \in \mathcal{I}} V_0^i(\theta_0; (\chi_t)_{t=0}^{\infty})]$.

C.3 *Deposit scheme*

First, we state the participation constraint we consider.

Definition 7 (wp-EPIR). A mechanism $(\chi_t, \psi_t^i)_{t=0}^{\infty}$ is *within-period ex post individually rational* (wp-EPIR) for i at $\theta_{0:t} \in \Theta_{0:t}$ if

$$V_t^i\big(\theta_t; (\chi_k)_{k=0}^{\infty}\big) + \Psi_t^i\big(\theta_{0:t}; (\chi_k)_{k=0}^{\infty}\big) \geq 0.$$

A mechanism $(\chi_t, \psi_t^i)_{t=0}^{\infty}$ is *wp-EPIR for i* if it is wp-EPIR for i for every t and $\theta_{0:t} \in \Theta_{0:t}$.

As we discussed in Section 3, when we have a finite horizon, whenever wp-EPIR is satisfied in period 0, we can keep wp-EPIR for every period without increasing information rent. To achieve this, the planner should collect a deposit in the initial period and return it in the last period. As long as $T < \infty$, this scheme does not need any additional assumptions.

We need to modify the scheme because the original one returns the deposit in the last period. However, given that $(\psi_t^i)_{t=0}^\infty$ is uniformly bounded, we can develop a similar scheme, which ensures that we can satisfy wp-EPIR without increasing information rents.

Lemma 4. *Suppose that $(\chi_t, \psi_t^i)_{t=0}^\infty$ is wp-EPIC for i and $(\psi_t^i)_{t=0}^\infty$ is uniformly bounded. Then there exists a uniformly bounded $(\bar{\psi}_t^i)_{t=0}^\infty$ such that $(\chi_t, \bar{\psi}_t^i)$ is wp-EPIC for i, is wp-EPIR for i, and*

$$\Psi_0^i(\theta_0) = \bar{\Psi}_0^i(\theta_0) \quad \textit{for all } \theta_0 \in \Theta_0.$$

In brief, since $(\psi_t^i)_{t=0}^\infty$ is uniformly bounded, $M^i \equiv \inf_{t,\theta_{0:t}}[V_t^i(\theta_{0:t}) + \Psi_t^i(\theta_{0:t})] > -\infty$. The central planner collects $-M^i$ as the deposit in the initial period and pays $-(1-\delta)\cdot M^i$ as interest in every period, as long as agent i stays in. Agent i does not leave the mechanism because EPV from the interest is $-M^i$, which is his worst-case EPV from the original mechanism, $(\chi_t, \psi_t^i)_{t=0}^\infty$. Note that unless we assume that $(\psi_t^i)_{t=0}^\infty$ is uniformly bounded, M^i may be $-\infty$, which means that it is impossible to make the agent deposit $-M^i$ in the initial period.

C.4 *Extraction*

With an infinite horizon, we cannot initiate the backward induction from the "last" period. However, we can always initiate backward induction at some time point (say $T < +\infty$) to construct a sequence of backup sets $(B_t^{-i})_{t=1}^{T+1}$. If we can obtain strong detectability with $\Gamma_0^i(\theta_0^{-i}, B_1^{-i})$ for every $\theta_0^{-i} \in \Theta_0^{-i}$, θ_0^i is detected without leaving information rent. Because the part of the payment rule that we use for extraction becomes zero after $T+1$, the uniform boundedness remains to be satisfied. Therefore, if, in addition, the existence of a uniformly bounded payment rule $(g_t^i)_{t=0}^\infty$ is ensured by some exogenous schemes, then full surplus extraction is possible. More formally, we have the following theorem.

Theorem 7. *Given $(\chi_t)_{t=0}^\infty$, suppose that there exist $T < +\infty$ and $(B_t^{-i})_{t=1}^{T+1}$ such that*

(i) for $t = T+1$, $B_{T+1}^{-i} = \varnothing$,

(ii) for $t = 1, 2, \ldots, T$, $\theta_t^{-i} \in B_t^{-i}$ if Θ_t^i is strongly detectable with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$.

Suppose also that there exists uniformly bounded $(g_t^i)_{t=0}^\infty$ that makes $(\chi_t, g_t^i)_{t=0}^\infty$ wp-EPIC for i and that Θ_t^i is strongly detectable with $\Gamma_0^i(\theta_0^{-i}, B_1^{-i})$ for all $\theta_0^{-i} \in \Theta_0^{-i}$. Then there exists uniformly bounded $(\psi_t^i)_{t=0}^\infty$ such that the associated mechanism $(\chi_t, \psi_t^i)_{t=0}^\infty$ is wp-EPIC and leaves no information rent for i.

The proof is parallel to Theorem 1 in the main paper, so it is omitted. Recall that Theorem 7 ensures the existence of the *uniformly bounded* payment rule $(\psi_t^i)_{t=0}^\infty$. Accordingly, we can apply Lemma 4 to make the mechanism satisfy wp-EPIR for i without increasing information rents, just as in a finite horizon.

The flow payment of the team mechanism of Athey and Segal (2013), $g_t^i(\theta_{0:t}) = \sum_{j \neq i} v_t^i(\chi_t(\theta_t), \theta_t^j)$ is uniformly bounded by $(I-1) \cdot \bar{v}$. Therefore, just as in a finite horizon, if we have private values, then the existence of "the uniformly bounded payment rule $(g_t^i)_{t=0}^{\infty}$ that makes $(\chi_t, g_t^i)_{t=0}^{\infty}$ wp-EPIC for i" is guaranteed.

C.5 *Strong detectability and correlation intensity*

Now we study the one-shot problem to show that the lower bound of the correlation intensity guarantees the boundedness of the incentive payment. Let ρ be the Euclidean distance. The type space Θ^i is d-strongly detectable if the Euclidean distance of the point and the convex hull in the definition of strong detectability is larger than some fixed $d > 0$.

Definition 8 (d-Strong detectability). Given $d > 0$, Θ^i is *d-strongly detectable with* $(X, \Theta^i, \chi, S, \pi)$ if, for all $\theta^i \in \Theta^i$,

$$\rho\big(\pi\big(\chi(\theta^i), \theta^i\big), \mathrm{co}\big(\big\{\pi\big(\chi(\theta^i), \hat{\theta}^i\big)\big\}_{\hat{\theta}^i \in \Theta^i \setminus \{\theta^i\}}\big)\big) > d. \tag{40}$$

Note that strong detectability with $(X, \Theta^i, \chi, S, \pi)$ is not satisfied if and only if the distance appearing in the left hand side of (40) is equal to zero. In other words, strong detectability (of Definition 4) corresponds to 0-strong detectability of Definition 8. The value of d specifies a lower bound of the strength of intertemporal correlations between agents' types. If we have stronger correlation, the scale of the Crémer–McLean lottery can be small. In the following lemma, we show that the payment rule of the one-shot mechanism has a uniform bound $MS(\bar{U}, \bar{u}, d)$, which depends only on the bound of expected payoffs $\bar{U}$, the bound of valuations $\bar{u}$, and the lower bound of correlation intensity d.

Lemma 5. *There exists $MS : \mathbb{R}^3_{++} \to \mathbb{R}_{++}$ that satisfies the following statements. Suppose that Θ^i is d-strongly detectable with $(X, \Theta^i, \chi, S, \pi)$. Then, for all $u^i : X \times \Theta^i \to \mathbb{R}$ and $U^i : \Theta^i \to \mathbb{R}$ such that $|u^i(x, \theta^i)| < \bar{u}$ for all $(x, \theta^i) \in X \times \Theta^i$ and $|U^i(\theta^i)| < \bar{U}$ for all $\theta^i \in \Theta^i$, there exists $p^i : \Theta^i \times S \to \mathbb{R}$ that satisfies*

$$U^i\big(\theta^i\big) = u^i\big(\chi\big(\theta^i\big), \theta^i\big) + \delta \cdot \mathbb{E}\big[p^i\big(\theta^i, s\big)|\chi\big(\theta^i\big), \theta^i\big] \tag{41}$$

for all $\theta^i \in \Theta^i$,

$$U^i\big(\theta^i\big) \geq u^i\big(\chi\big(\hat{\theta}^i\big), \theta^i\big) + \delta \cdot \mathbb{E}\big[p^i\big(\hat{\theta}^i, s\big)|\chi\big(\hat{\theta}^i\big), \theta^i\big] \tag{42}$$

for all $(\theta^i, \hat{\theta}^i) \in \Theta^i \times \Theta^i$, and

$$\big|p^i\big(\theta^i, s\big)\big| < MS(\bar{U}, \bar{u}, d)$$

for all $(\theta^i, s) \in \Theta^i \times S$.

C.6 *Weak detectability and correlation intensity*

Parallel to d-strong detectability, we define the d-weak detectability as follows.

Definition 9 (d-Weak detectability). Given $d > 0$, Θ^i is *d-weakly detectable with* $(X, \Theta^i, \chi, S, \pi)$,if for all $\bar{\Theta}^i \subset \Theta^i$, there exists $\bar{\theta}^i \in \bar{\Theta}^i$ such that

$$\rho\big(\pi\big(\chi(\bar{\theta}^i), \bar{\theta}^i\big), \mathrm{co}\big(\big\{\pi\big(\chi(\bar{\theta}^i), \hat{\theta}^i\big)\big\}_{\hat{\theta}^i \in \bar{\Theta}^i \text{ s.t. } \chi(\hat{\theta}^i) \neq \chi(\bar{\theta}^i)}\big)\big) > d.$$

Again, d represents the correlation intensity, and we can also develop an upper bound of payments $MW(\bar{u}, d)$ and an upper bound of expected payoffs $MU(\bar{u}, d)$ that depend only on the bound of valuations $\bar{u}$ and the lower bound of correlation intensity d.

Lemma 6. *There exist $MU : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ and $MW : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ that satisfy the following statements. Suppose that Θ^i is d-weakly detectable with $(X, \Theta^i, \chi, S, \pi)$. Then, for all $u^i : X \times \Theta^i \to \mathbb{R}$ such that $|u^i(x, \theta^i)| < \bar{u}$ for all $(x, \theta^i) \in X \times \Theta^i$, there exist $U^i : \Theta^i \to \mathbb{R}$ and $p^i : \Theta^i \times S \to \mathbb{R}$ that satisfy*

$$U^i(\theta^i) = u^i\big(\chi(\theta^i), \theta^i\big) + \delta \cdot \mathbb{E}\big[p^i(\theta^i, s) | \chi(\theta^i), \theta^i\big] \tag{41$'$}$$

for all $\theta^i \in \Theta^i$,

$$U^i(\theta^i) \geq u^i\big(\chi(\hat{\theta}^i), \theta^i\big) + \delta \cdot \mathbb{E}\big[p^i(\hat{\theta}^i, s) | \chi(\hat{\theta}^i), \theta^i\big] \tag{42$'$}$$

for all $(\theta^i, \hat{\theta}^i) \in \Theta^i \times \Theta^i$,

$$\big|U^i(\theta^i)\big| < MU(\bar{u}, d)$$

for all $\theta^i \in \Theta^i$, and

$$\big|p^i(\theta^i, s)\big| < MW(\bar{u}, d)$$

for all $(\theta^i, s) \in \Theta^i \times S$.

C.7 *Implementation*

We will develop a counterpart of Theorem 2 for an infinite horizon. To construct it, just as in the main paper, we first construct a dynamic mechanism for a finite horizon. However, different from the finite-horizon problems, we require that there exists an upper bound of the payment, which depends only on the intensity of the detectability (d) and length of the time horizon (T).

Definition 10 (d-Block). An interval $\{\underline{t}, \underline{t}+1, \ldots, \underline{t}+T\}$ is a *d-block along* $(\chi_t)_{t=\underline{t}}^{t+T}$ if there exists $(B_t^{-i})_{t=\underline{t}+1}^{t+T+1}$ such that the following statements hold:

(i) We have $B_t^{-i} \subset \Theta_t^{-i}$ for every $t \in \{\underline{t}+1, \ldots, \underline{t}+T+1\}$.

(ii) We have $B_{\underline{t}+T+1}^{-i} = \varnothing$.

(iii) For $t = \underline{t} + 1, \ldots, \underline{t} + T$, for all $\theta_t^{-i} \in B_t^{-i}$, Θ_t^i is d-strongly detectable with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$.

(iv) For $t = \underline{t}$, for all $\theta_{\underline{t}}^{-i} \in \Theta_{\underline{t}}^{-i}$, $\Theta_{\underline{t}}^i$ is d-strongly detectable with $\Gamma_{\underline{t}}^i(\theta_{\underline{t}}^{-i}, B_{\underline{t}+1}^{-i})$.

(v) For $t = \underline{t} + 1, \ldots, \underline{t} + T$ for all $\theta_t^{-i} \in \Theta_t^{-i} \setminus B_t^{-i}$, Θ_t^i is d-weakly detectable with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$.

Definition 10 involves that the assumptions of Theorems 1 and 2. However, in addition to Theorems 1 and 2, we explicitly state the lower bound of the correlation intensity, d. We regard each block as a problem with a finite horizon and construct a finite-horizon mechanism. Since assumptions of Theorems 1 and 2 are satisfied, we can satisfy wp-EPIC without leaving information rent; in other words, an agent's EPV in the initial period of each block can be set to zero. Hence, the beginning of a new block is equivalent to the termination of the world (as in a finite horizon) in the sense that agents' EPVs are set to zero regardless of the history. Accordingly, solving a finite-horizon problem for each block, we can implement a targeted allocation rule for an infinite-horizon problem.

We use Lemmas 5 and 6 instead of Lemmas 1 and 2; thus, we have a guarantee for the scale of payment rules: the incentive payment for $\{\underline{t}, \underline{t}+1, \ldots, \underline{t}+T\}$ is bounded by a constant, say, $MB(T, d)$, that depends only on the lower bound of correlation intensity d and the length of d-block T. The constant $MB(T, d)$ may go to infinity as the length of the d-block grows to infinity (i.e., $T \to \infty$). To guarantee that $(\psi_t^i)_{t=0}^\infty$ is uniformly bounded, we also need to assume the existence of the upper bound of the length of each d-block.

Theorem 8. *Consider an allocation rule $(\chi_t)_{t=0}^\infty$. Suppose that there exists $d > 0$, $L \in \mathbb{Z}_{++}$, and a partition $\mathcal{P}$ of the time horizon $\mathbb{Z}_+$ such that (i) each cell $P \in \mathcal{P}$ comprises a d-block and (ii) the length of each d-block is shorter than L, i.e., for all $P \in \mathcal{P}$, $|P| < L$. Then there exists a payment rule $(\psi_t^i)_{t=0}^\infty$ that makes $(\chi_t, \psi_t^i)_{t=0}^\infty$ wp-EPIC for i and leaves no information rent.*

Appendix D: Proofs

D.1 *Proof of Theorem 5*

Part (i) (genericity of strong detectability) Suppose that $|\Theta^i| \le |S|$. Then it follows from

$$\dim \mathrm{co}\big(\{\pi(\chi(\theta^i), \hat{\theta}^i)\}_{\hat{\theta}^i \in \Theta^i \setminus \{\theta^i\}}\big) \le \big|\Theta^i\big| - 2 < |S| - 1 = \dim \Delta(S)$$

that the measure of π that fails to satisfy (2) is zero, i.e., $L(\mathcal{SD}(X, \Theta^i, \chi, S)) = 1$.

To show that $L(\mathcal{SD}(X, \Theta^i, \chi, S)) < 1$ for $|\Theta^i| > |S|$, construct π as follows (it is depicted as Figure 3). Choose $\bar{\theta}^i \in \Theta^i$ arbitrarily and let $x^* \equiv \chi(\bar{\theta}^i)$. Let $o \equiv (1, 1, \ldots, 1)/|S|$ be the center of gravity of $\Delta(S)$. Define $\pi(x^*, \bar{\theta}^i) \equiv o$. Take $r > 0$ such that an r-open ball on $\Delta(S)$ of radius r centered at o, which is denoted by R, satisfies $R \subset \Delta(\Theta_{t+1}^{-i})$. Choose

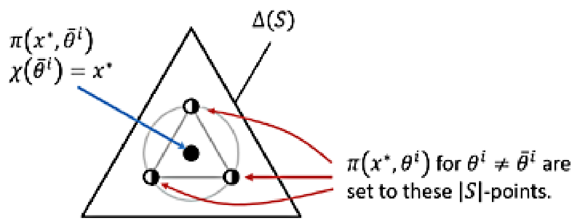

FIGURE 3. The signal distribution $\pi(x^*, \cdot)$ constructed in the proof of part (i) of Theorem 5. Choosing $\bar\theta^i$ for θ^i of (2), (2) is not satisfied for a ϵ-neighborhood of π for $\epsilon > 0$ small enough.

$|S|$ points on the boundary of R whose convex hull forms a $(|S|-1)$-dimensional simplex, and set $\pi(x^*, \theta^i)$ for $\theta^i \in \Theta^i \setminus \{\bar\theta^i\}$ to these points so that all of $|S|$ points are assigned at least one θ^i (such an assignment is feasible because $|\Theta^i \setminus \{\bar\theta^i\}| \geq |S|$). For $x \neq x^*$, $\pi(x, \cdot)$ is defined arbitrarily. Then, for $\epsilon > 0$ sufficiently small, (2) for $\bar\theta^i$ is not satisfied for an ϵ-neighborhood of π . Accordingly, $L(\mathcal{SD}(X, \Theta^i, \chi, S)) < 1$. □

Part (ii) (genericity of weak detectability) Suppose that $\min_{x\in X} |\{\theta^i \in \Theta^i : \chi(\theta^i) \neq x\}| + 1 \leq |S|$. Choose and fix $x^* \in \arg\min_{x\in X} |\{\theta^i \in \Theta^i : \chi(\theta^i) \neq x\}|$. Then it follows from

$$\arg\min_{x\in X}\left|\left\{\theta^i \in \Theta^i : \chi\left(\theta^i\right) \neq x\right\}\right| = \arg\max_{x\in X}\left|\left\{\theta^i \in \Theta^i : \chi\left(\theta^i\right) = x\right\}\right|$$

that $\hat\Theta^i \equiv \{\theta^i \in \Theta^i : \chi(\theta^i) = x^*\} \neq \varnothing$. Furthermore, $|\Theta^i \setminus \hat\Theta^i| = \min_{x\in X} |\{\theta^i \in \Theta^i : \chi(\theta^i) \neq x\}| \leq |S| - 1$.

If Θ^i is *not* weakly detectable with $(X, \Theta^i, \chi, S, \pi)$, there exists $\bar\Theta^i \subset \Theta^i$ such that, for all $\bar\theta^i \in \bar\Theta^i$, (31) is not satisfied. Let $\mathcal{C}(\bar\Theta^i)$ be the set of π such that, for all $\bar\theta^i \in \bar\Theta^i$, (31) is not satisfied. Clearly, $\bigcup_{\bar\Theta^i \subset \Theta^i} \mathcal{C}(\bar\Theta^i) = [\Delta(S)]^{|X|\times|\Theta^i|} \setminus \mathcal{WD}(X, \Theta^i, \chi, S)$ holds.

For $\bar\Theta^i$ such that $\bar\Theta^i \cap \hat\Theta^i \neq \varnothing$, for π to be in $\mathcal{C}(\bar\Theta^i)$, it is necessary that for all $\bar\theta^i \in \bar\Theta^i \cap \hat\Theta^i$,

$$\pi\left(x^*, \bar\theta^i\right) \in \mathrm{co}\left(\left\{\pi\left(x^*, \hat\theta^i\right)\right\}_{\hat\theta^i \in \bar\Theta^i \text{ s.t. } \chi(\hat\theta^i) \neq x^*}\right). \tag{43}$$

However, by assumption on x^*,

$$\begin{aligned}
\dim \mathrm{co}\left(\left\{\pi\left(x^*, \hat\theta^i\right)\right\}_{\hat\theta^i \in \bar\Theta^i \text{ s.t. } \chi(\hat\theta^i) \neq x^*}\right) &\leq \dim \mathrm{co}\left(\left\{\pi\left(x^*, \hat\theta^i\right)\right\}_{\hat\theta^i \in \Theta^i \text{ s.t. } \chi(\hat\theta^i) \neq x^*}\right) \\
&\leq \left|\Theta^i \setminus \hat\Theta^i\right| - 1 \\
&\leq \left(|S| - 1\right) - 1 \\
&< \dim \Delta(S),
\end{aligned}$$

indicating that the right hand side of (43) degenerates. Accordingly, $L(\mathcal{C}(\bar\Theta^i)) = 0$.

For $\bar\Theta^i$ such that $\bar\Theta^i \cap \hat\Theta^i = \varnothing$, it follows from $\bar\Theta^i \subset \Theta^i \setminus \hat\Theta^i$ that $|\bar\Theta^i| \leq |\Theta^i \setminus \hat\Theta^i| \leq |S| - 1$ holds. Take an arbitrary $\bar\theta^i \in \bar\Theta^i$. Then, for π to be in $\mathcal{C}(\bar\Theta^i)$, it is necessary that (31) is not

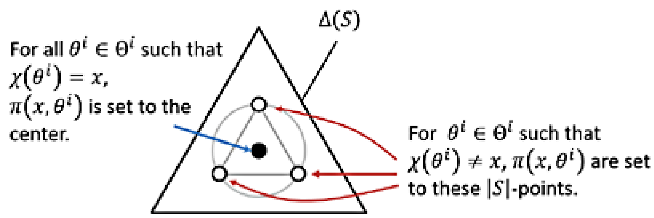

FIGURE 4. The signal distribution π constructed in the proof of part (ii) of Theorem 5. If we define $\pi(x,\cdot)$ in this manner for all $x \in X$, for $\bar{\Theta}^i = \Theta^i$, for all $\bar{\theta}^i \in \Theta^i$, (5) is not satisfied for an ϵ-neighborhood of π for $\epsilon > 0$ small enough.

satisfied with $\bar{\theta}^i$. However,

$$\begin{aligned}\dim \mathrm{co}\big(\{\pi(\chi(\bar{\theta}^i), \hat{\theta}^i)\}_{\hat{\theta}^i \in \bar{\Theta}^i \text{ s.t. } \chi(\hat{\theta}^i) \neq \chi(\bar{\theta}^i)}\big) &\leq |\bar{\Theta}^i| - 2 \\ &\leq (|S|-1) - 2 \\ &< \dim \Delta(S),\end{aligned}$$

indicating that the right hand side of (31) degenerates. Accordingly, $L(\mathcal{C}(\bar{\Theta}^i)) = 0$.

Finally,

$$0 \leq L\big((\Delta(S))^{|X|\times|\Theta^i|} \setminus \mathcal{WD}(X, \Theta^i, \chi, S)\big) = L\Bigg(\bigcup_{\bar{\Theta}^i \subset \Theta^i} \mathcal{C}(\bar{\Theta}^i)\Bigg) \leq \sum_{\bar{\Theta}^i \subset \Theta^i} L(\mathcal{C}(\bar{\Theta}^i)) = 0$$

implies that $L(\mathcal{WD}(X, \Theta^i, \chi, S)) = 1$.

To show that $L(\mathcal{WD}(X, \Theta^i, \chi, S)) < 1$ for $\min_{x\in X} |\{\theta^i \in \Theta^i : \chi(\theta^i) \neq x\}| + 1 > |S|$, construct π as follows (it is depicted as Figure 4). Again, let $o \equiv (1, 1, \ldots, 1)/|S|$ be the center of gravity of $\Delta(S)$, and take $r > 0$ such that an r-open ball on $\Delta(S)$ of radius r centered at o, which is denoted by R, satisfies $R \subset \Delta(S)$. For each $x \in X$, for $\theta^i \in \Theta^i$ such that $\chi(\theta^i) = x$, define $\pi(x, \theta^i) = o$. Choose $|S|$ points on the boundary of R whose convex hull forms an $(|S|-1)$-dimensional simplex, and set $\pi(x, \theta^i)$ for $\theta^i \in \Theta^i$ such that $\chi(\theta^i) \neq x$ to these points so that all of the $|S|$ points are assigned at least one θ^i (such an assignment is feasible because $|\{\theta^i \in \Theta^i : \chi(\theta^i) \neq x\}| \geq |S|$ for all x, by assumption). Then, for $\epsilon > 0$ sufficiently small, taking $\bar{\Theta}^i = \Theta^i$, (31) is not satisfied for all $\bar{\theta}^i \in \bar{\Theta}^i$ for an ϵ-neighborhood of π. Accordingly, $L(\mathcal{WD}(X, \Theta^i, \chi, S)) < 1$. □

D.2 *Proof of Lemma 3*

Choose some $k \in \mathcal{I} \setminus \{i\}$ arbitrarily. Define $v_t^j(x_t, \theta_t) \equiv 0$ for all $j \in \mathcal{I} \setminus \{i, k\}$, t, x_t, θ_t. Denoting the continuation value of the social welfare from period $t+1$ by

$$W_{t+1}(\theta_{t+1}) \equiv \mathbb{E}\left[\sum_{s=t+1}^{T} \delta^{s-(t+1)} \big\{v_s^i(\chi_s(\theta_s), \theta_s) + v_s^k(\chi_s(\theta_s), \theta_s)\big\} \,\Big|\, \theta_{t+1}; (\chi_s)_{s=t+1}^T\right],$$

$(\chi_t)_{t=0}^T$ is a unique efficient allocation rule if and only if

$$\{\chi_T(\theta_T)\} = \arg\max_{x_T \in X_T} \{v_T^i(x_T, \theta_T) + v_T^k(x_T, \theta_T)\} \tag{44}$$

holds for all $\theta_T \in \Theta_T$ and

$$\{\chi_t(\theta_t)\} = \arg\max_{x_t \in X_t} \{v_t^i(x_t, \theta_t) + v_t^k(x_t, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1})|x_t, \theta_t]\} \tag{45}$$

holds for all $t \in \{0, 1, \dots, T-1\}$ and $\theta_t \in \Theta_t$.

We construct $(v_t^k)_{t=0}^T$ backward. For period T, define $v_T^k : X_T \times \Theta_T \to \mathbb{R}$ by

$$v_T^k(x_T, \theta_T) \equiv \begin{cases} -v_T^i(x_T, \theta_T) + 1 & \text{if } x_T = \chi_T(\theta_T), \\ -v_T^i(x_T, \theta_T) & \text{otherwise.} \end{cases}$$

Then $v_T^i(x_T, \theta_T) + v_T^k(x_T, \theta_T) = 1$ if $x_T = \chi_T(\theta_T)$ and $v_T^i(x_T, \theta_T) + v_T^k(x_T, \theta_T) = 0$ otherwise. Therefore, (44) is satisfied for all θ_T.

After constructing $(v_s^k)_{s=t+1}^T$ to satisfy (44) for T and (45) for $t+1, \dots, T-1$, we can specify the value of W_{t+1}. For period t, define $v_t^k : X_t \times \Theta_t \to \mathbb{R}$ by

$$v_t^k(x_t, \theta_t) \equiv \begin{cases} -v_t^i(x_t, \theta_t) - \delta \mathbb{E}[W_{t+1}(\theta_{t+1})|x_t, \theta_t] + 1 & \text{if } x_t = \chi_t(\theta_t), \\ -v_t^i(x_t, \theta_t) - \delta \mathbb{E}[W_{t+1}(\theta_{t+1})|x_t, \theta_t] & \text{otherwise.} \end{cases}$$

Then

$$v_t^i(x_t, \theta_t) + v_t^k(x_t, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1})|x_t, \theta_t] = \begin{cases} 1 & \text{if } x_t = \chi_t(\theta_t), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, (45) is satisfied for all Θ_t.

Iterating this procedure, finally we obtain $((v_t^j)_{j \neq i})_{t=0}^T$ such that (44) holds for $\theta_T \in \Theta_T$, and (45) holds for all $t \in \{0, 1, \dots, T-1\}$ and $\theta_t \in \Theta_t$. With such $((v_t^j)_{j \neq i})_{t=0}^T$, $(\chi_t)_{t=0}^T$ is a unique efficient allocation rule. □

D.3 *Proof of Theorem 6*

If part Suppose that $\mathcal{U}_0^i \neq \varnothing$. Then $\mathcal{U}_t^i \neq \varnothing$ for all $t \in \{0, 1, \dots, T+1\}$. Take some $U_0^i \in \mathcal{U}_0^i$ arbitrarily. Then there exists $U_1^i : \Theta_0 \to \mathcal{U}_1^i$ such that (38) and (39) hold with $t = 0$. Similarly, for each $\theta_0 \in \Theta_0$, there exists $U_2^i(\cdot; \theta_0) : \Theta_1 \to \mathcal{U}_2^i$ such that (38) and (39) hold with $t = 1$. Iterating this process, finally we obtain $U_{T+1}^i(\cdot; \theta_{0:T}) \in \mathcal{U}_{T+1}^i$ such that $U_{T+1}^i(\theta_{T+1}^i, \theta_{T+1}^{-i}) = U_{T+1}^i(\hat{\theta}_{T+1}^i, \theta_{T+1}^{-i})$ for all $\theta_{T+1}^i, \hat{\theta}_{T+1}^i \in \Theta_{T+1}^i$. Define $\psi_t^i = 0$ for $t = 0, 1, \dots, T$ and $\psi_{T+1}^i(\theta_{0:T+1}) = U_{T+1}^i(\theta_{T+1}; \theta_{0:T})$.

Because agent i's period-$T+1$ payment does not depend on agent i's report,$(\chi_t, \psi_t^i)_{t=0}^{T+1}$ satisfies wp-EPIC in period $T+1$. Furthermore, (38) and (39) in t ensures wp-EPIC in t because, by construction of ψ_{T+1}^i,

$$U_t^i(\theta_t; \theta_{0:t-1}) = V_t^i(\theta_t) + \Psi_t^i(\theta_{0:t-1}, \theta_t)$$

holds for all $(\theta_{0:t-1}, \theta_t)$.

To prove part (ii), we should take $U_0^i \equiv (0, \dots, 0)$ at the beginning and then, finally, we obtain $V_0^i(\theta_0) + \Psi_0^i(\theta_0) = U_0^i(\theta_0) = 0$ for all θ_0. □

Only if part Suppose that there exists $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ that is wp-EPIC for i. Then consider $(\chi_t, \bar{\psi}_t^i)_{t=0}^{T+1}$ such that $\bar{\psi}_t^i = 0$ for $t = 0, 1, \ldots, T$ and

$$\bar{\psi}_{T+1}^i(\theta_{0:T+1}) = \sum_{t=0}^{T+1} \delta^{t-(T+1)} \psi_t^i(\theta_{0:t}).$$

By definition of wp-EPIC, $(\chi_t, \bar{\psi}_t^i)_{t=0}^{T+1}$ is also wp-EPIC for i.

In period $T+1$, $|X_{T+1}| = 1$ and $v_{T+1}^i(x_{T+1}, \theta_{T+1}) = 0$ for all θ_{T+1} by assumption. Hence, wp-EPIC for i at $(\theta_{0:T}, \theta_{T+1})$ and $(\theta_{0:T}, \hat{\theta}_{T+1}^i, \theta_{T+1}^{-i})$ implies

$$\bar{\Psi}_{T+1}^i(\theta_{0:T}, \theta_{T+1}) = \bar{\Psi}_{T+1}^i\big(\theta_{0:T}, \hat{\theta}_{T+1}^i, \theta_{T+1}^{-i}\big)$$

for all $\theta_{T+1}^i, \hat{\theta}_{T+1}^i \in \Theta_{T+1}^i$. Together with the fact that $V_{T+1}^i = 0$, we obtain $V_{T+1}^i(\cdot) + \bar{\Psi}_{T+1}^i(\theta_{0:T}, \cdot) \in \mathcal{U}_{T+1}^i$, where $\mathcal{U}_{T+1}^i$ is as defined in (37).

Suppose that $V_{t+1}^i(\cdot) + \bar{\Psi}_{t+1}^i(\theta_{0:t}, \cdot) \in \mathcal{U}_s^i$ for all $\theta_{0:t} \in \Theta_{0:t}$. Then, by definition of wp-EPIC, $V_t^i(\cdot) + \bar{\Psi}_t^i(\theta_{0:t-1}, \cdot) \in \mathcal{U}_t^i$ holds for all $\theta_{0:t-1} \in \Theta_{0:t-1}$, where $\mathcal{U}_t^i$ is as defined in (38) and (39).

Iterating this process, we can verify that $\mathcal{U}_t^i \neq \varnothing$ for all $t \in \{0, 1, \ldots, T+1\}$. In particular, if $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ leaves no information rent,

$$V_0^i(\theta_0) + \Psi_0^i(\theta_0) = V_0^i(\theta_0) + \bar{\Psi}_0^i(\theta_0) = 0$$

for all θ_0 and $V_0^i(\cdot) + \bar{\Psi}_0^i(\cdot) \in \mathcal{U}_0^i$ implies that $(0, \ldots, 0) \in \mathcal{U}_0^i$. □

D.4 *Proof of Lemma 4*

Suppose that $(\chi_t, \psi_t^i)_{t=0}^{\infty}$ is wp-EPIC and leaves no information rent for i, and $(\psi_t^i)_{t=0}^{\infty}$ is uniformly bounded. Define

$$\begin{aligned}
M^i &\equiv \inf_{t, \theta_{0:t} \in \Theta_{0:t}} \big[V_t^i(\theta_t) + \Psi_t^i(\theta_{0:t})\big], \\
\bar{\psi}_0^i(\theta_0) &\equiv \psi_0^i(\theta_0) + M^i - (1-\delta)M^i, \\
\bar{\psi}_t^i(\theta_{0:t}) &\equiv \psi_t^i(\theta_{0:t}) - (1-\delta)M^i.
\end{aligned}$$

Note that $M^i < \infty$ because the uniform boundedness of $(\psi_t^i)_{t=0}^{\infty}$ implies $|\Psi_t^i(\theta_{0:t})| < \infty$ for all t and $\theta_{0:t} \in \Theta_{0:t}$. Hence, clearly, $(\bar{\psi}_t^i)_{t=0}^{\infty}$ is uniformly bounded. Then, for $t = 0$,

$$\begin{aligned}
\bar{\Psi}_0^i(\theta_0) &= \Psi_0^i(\theta_0) + M^i - (1-\delta)\sum_{t=0}^{\infty} \delta^t M^i \\
&= \Psi_0^i(\theta_0) + M^i - M^i \\
&= \Psi_0^i(\theta_0).
\end{aligned}$$

Furthermore, for $t \geq 1$,

$$\bar{\Psi}_t^i(\theta_{0:t}) = \Psi_t^i(\theta_{0:t}) - (1-\delta)\sum_{s=0}^{\infty}\delta^s M^i$$
$$= \Psi_t^i(\theta_{0:t}) - M^i.$$

Hence,

$$\bar{\Psi}_0^i(\theta_0) = \Psi_0^i(\theta_0)$$

for all $\theta_0 \in \Theta_0$. Moreover,

$$V_t^i(\theta_t) + \bar{\Psi}_t^i(\theta_{0:t}) = V_t^i(\theta_t) + \Psi_t^i(\theta_{0:t}) - M^i \geq 0$$

for all $t \geq 1$ and $\theta_{0:t} \in \Theta_{0:t}$ by construction of M^i. Finally, since $(\chi_t, \bar{\psi}_t^i)_{t=0}^{\infty}$ is identical to $(\chi_t, \psi_t^i)_{t=0}^{\infty}$ up to constants, wp-EPIC of $(\chi_t, \psi_t^i)_{t=0}^{\infty}$ guarantees wp-EPIC of $(\chi_t, \bar{\psi}_t^i)_{t=0}^{\infty}$. □

D.5 *Proof of Lemma 5*

The construction of U^i and p^i is the same as that shown in the sufficiency part of Lemma 1. We will develop the bound of them.

Fix an arbitrary $\theta^i \in \Theta^i$. Since $\mathrm{co}(\{\pi(\chi(\theta^i), \hat{\theta}^i)\}_{\hat{\theta}^i \in \Theta^i \setminus \{\theta^i\}})$ is compact, there exists $m(\theta^i) \in \Delta(S)$ that satisfies

$$m(\theta^i) \in \arg\min_{\xi \in \Delta(S)} \rho\big(\xi, \pi\big(\chi(\theta^i), \theta^i\big)\big)$$
$$\text{s.t. } \xi \in \mathrm{co}\big(\big\{\pi\big(\chi(\theta^i), \hat{\theta}^i\big)\big\}_{\hat{\theta}^i \in \Theta^i \setminus \{\theta^i\}}\big).$$

Define $b : \Theta^i \to \Delta(S)$ by

$$b(\theta^i) \equiv \frac{\pi\big(\chi(\theta^i), \theta^i\big) - m(\theta^i)}{\rho\big(\pi\big(\chi(\theta^i), \theta^i\big), m(\theta^i)\big)}.$$

Then, clearly, $\|b(\theta^i)\| \equiv \sqrt{b(\theta^i) \cdot b(\theta^i)} = 1$. Furthermore,

$$\begin{aligned} & b(\theta^i) \cdot \big[\pi\big(\chi(\theta^i), \theta^i\big) - m(\theta^i)\big] \\ & = \rho\big(\pi\big(\chi(\theta^i), \theta^i\big), m(\theta^i)\big) \\ & = \rho\big(m(\theta^i), \mathrm{co}\big(\big\{\pi\big(\chi(\theta^i), \hat{\theta}^i\big)\big\}_{\hat{\theta}^i \in \Theta^i \setminus \{\theta^i\}}\big)\big) \\ & > d \end{aligned} \tag{46}$$

by d-strong detectability. Furthermore, since $m(\theta^i)$ is closest to $\pi(\chi(\theta^i), \theta^i)$, $\pi(\chi(\theta^i), \theta^i) - m(\theta^i)$ and $\pi(\chi(\theta^i), \hat{\theta}^i) - m(\theta^i)$ cannot make an acute angle. Therefore, for all $\hat{\theta}^i \in \Theta^i$,

$$b(\theta^i) \cdot \big[\pi\big(\chi(\theta^i), \hat{\theta}^i\big) - m(\theta^i)\big] \leq 0. \tag{47}$$

Combining (46) and (47), we have

$$b(\theta^i)\cdot\big[\pi\big(\chi(\theta^i),\theta^i\big)-\pi\big(\chi(\theta^i),\hat\theta^i\big)\big]>d.$$

Define $\lambda(\theta^i,\cdot):S\to\mathbb{R}$ by

$$\lambda\big(\theta^i,s\big)\equiv b\big(\theta^i,s\big)-b\big(\theta^i\big)\cdot\pi\big(\chi\big(\theta^i\big),\theta^i\big).$$

Then, clearly, $\mathbb{E}[\lambda(\theta^i,s)|\chi(\theta^i),\theta^i)]=0$ and $\mathbb{E}[\lambda(\theta^i,s)|\chi(\theta^i),\hat\theta^i]<-d$ holds for all $\hat\theta^i\in\Theta^i\setminus\{\theta^i\}$. In addition, since $\|b(\theta^i)\|=1$, $|\lambda(\theta^i,s)|\le 2$ for any s.

Following the sufficiency part of Lemma 1, let

$$p^i\big(\theta^i,s\big)\equiv\delta^{-1}\big[U^i\big(\theta^i\big)-u^i\big(\chi\big(\theta^i\big),\theta^i\big)\big]+\alpha\cdot\lambda\big(\theta^i,s\big).$$

Then (41) is always satisfied.

Finally, we determine the level of α. If agent i makes a truthful report, his expected payoff is $U^i(\theta^i,\theta^{-i})$. If he reports $\hat\theta^i$, it would be

$$\begin{aligned}&U^i\big(\hat\theta^i\big)+u^i\big(\chi\big(\hat\theta^i\big),\theta^i\big)-u^i\big(\chi\big(\hat\theta^i\big),\hat\theta^i\big)\\&\quad+\delta\cdot\alpha\cdot\mathbb{E}\big[\lambda\big(\hat\theta^i,s\big)|\chi\big(\hat\theta^i\big),\theta^i\big].\end{aligned}$$

Hence, setting

$$\alpha\equiv\frac{2\bar U+2\bar u}{\delta d},$$

(42) is clearly maintained.

Now we have

$$\begin{aligned}\big|p^i\big(\theta^i,s\big)\big|&\le\delta^{-1}(\bar U+\bar v)+2\cdot\frac{2\bar U+2\bar u}{\delta d},\\&\equiv MS(\bar U,\bar u,d)\end{aligned}$$

for all $(\theta^i,s)\in\Theta^i\times S$, as desired. □

D.6 *Proof of Lemma 6*

By an argument similar to the proof of Lemmas 2 and 5, we can construct a ordered partition $\{H(k)\}_{k=1}^K$ of Θ^i and corresponding payment rules $\{\lambda(k,\cdot):S\to\mathbb{R}\}_{k=1}^K$ that satisfy, for each $k=1,\ldots,K$, for all $\theta^i\in H(k)$,

$$\mathbb{E}\big[\lambda(k,s)|\chi\big(\theta^i\big),\hat\theta^i\big]\ge 0\quad\text{for }\hat\theta^i\in H(k),$$

$$\mathbb{E}\big[\lambda(k,s)|\chi\big(\theta^i\big),\hat\theta^i\big]<-d\quad\text{for }\hat\theta^i\in\bigcup_{l=k+1}^{K}H(l)$$

and $|\lambda(k,s)|\le 2$ for all k and $s\in S$.

We specify U^i and p^i in the same manner as the proof of Lemma 2. For $k=K$, we set

$$U^i(\theta^i)=u^i(\chi(\theta^i),\theta^i) \quad \text{for all } \theta^i\in H(K),$$
$$p^i(\theta^i,x)=0 \quad \text{for all } (\theta^i,s)\in H(K)\times S.$$

Defining $\check{p}(K)\equiv 1$, $|p^i(\theta^i,s)|<\check{p}(K)$ holds for all $(\theta^i,s)\in H(K)\times S$. We also define $U^i(\theta^i)$ for $\theta^i\in H(K)$ by (41). Then it follows from

$$U^i(\theta^i)=u^i(\chi(\theta^i),\theta^i)$$

that $|U^i(\theta^i)|<\bar{u}\equiv\check{U}(K)$ holds for any $\theta^i_t\in H(K)$.

Suppose that $p^i:\bigcup_{l=k+1}^K H(l)\times S\to\mathbb{R}$ and $U^i:\bigcup_{l=k+1}^K H(l)\to\mathbb{R}$ are constructed to satisfy (41) for $\theta^i\in\bigcup_{l=k+1}^K H(l)$ and (42) for $(\theta^i_t,\hat{\theta}^i_t)\in\bigcup_{l=k+1}^K H(l)\times\bigcup_{l=k+1}^K H(l)$, and

$$|p^i(\theta^i,s)|<\check{p}(k+1) \quad \text{for all } (\theta^i,s)\in\left(\bigcup_{l=k+1}^K H(l)\right)\times S,$$

$$|U^i(\theta^i)|<\check{U}^i(k+1) \quad \text{for all } \theta^i\in\bigcup_{l=k+1}^K H(l).$$

For $\theta^i\in H(k)$, let

$$\begin{aligned}
&p^i(\theta^i,s)\\
&\quad\equiv \max_{\check{\theta}^i\in H(k),\hat{\theta}^i\in\bigcup_{l=k+1}^K H(l)}\left\{\delta^{-i}\left[u^i(\chi(\hat{\theta}^i),\check{\theta}^i)-u^i(\chi(\check{\theta}^i),\check{\theta}^i)\right]+\mathbb{E}\left[p^i(\hat{\theta}^i,\tilde{s})|\chi(\hat{\theta}^i),\check{\theta}^i\right]\right\}\\
&\qquad+\alpha\cdot\lambda(k,s).
\end{aligned}$$

We should choose $\alpha\in\mathbb{R}_{++}$ sufficiently large to prevent upward misreports, i.e., any $\theta^i\in\bigcup_{l=k+1}^K H(l)$ has no incentive to misreport $\hat{\theta}^i\in H(k)$. If agent i sincerely reports $\theta^i\in\bigcup_{l=k+1}^K H(l)$, his expected payoff is $U^i(\theta^i)>-\check{U}^i(k+1)$. If he reports $\tilde{\theta}^i\in H(k)$, his expected payoff would be

$$\begin{aligned}
&u^i(\chi(\tilde{\theta}^i),\theta^i)\\
&\quad+\delta\cdot\max_{\check{\theta}^i\in H(k),\hat{\theta}^i\in\bigcup_{l=k+1}^K H(l)}\left\{\delta^{-1}\left[u^i(\chi(\hat{\theta}^i),\check{\theta}^i)-u^i(\chi(\check{\theta}^i),\check{\theta}^i)\right]+\mathbb{E}\left[p(\hat{\theta}^i,s)|\chi(\hat{\theta}^i),\check{\theta}^i\right]\right\}\\
&\quad+\delta\cdot\alpha\cdot\mathbb{E}\left[\lambda^i(k,s)|\chi(\tilde{\theta}^i),\theta^i\right].
\end{aligned}$$

Therefore, if we set

$$\alpha\equiv\frac{\check{U}(k+1)+3\bar{u}+\delta\check{p}(k+1)}{\delta d},$$

such misreports are prevented and then (42) is satisfied for $(\theta^i,\hat{\theta}^i)\in(\bigcup_{l=k}^K H(l))\times(\bigcup_{l=k}^K H(l))$. Furthermore,

$$|U^i(\theta^i)|<3\bar{u}+\delta\check{p}(k+1)\equiv\check{U}(k) \quad \text{for all } \theta^i\in\bigcup_{l=k}^K H(l),$$

$$\begin{aligned}\left|p^i(\theta^i, s)\right| &< \check{p}(k+1) + 2\delta^{-1}\bar{u} + 2 \cdot \frac{\check{U}(k+1) + 3\bar{u} + \delta \check{p}(k+1)}{\delta d}\\ &\equiv \check{p}(k) \quad \text{for all } (\theta^i, s) \in \left(\bigcup_{l=k}^{K} H(l)\right) \times S.\end{aligned}$$

The sequence of $(\check{U}(k), \check{p}(k))_{k=K}^{1}$ is determined by these first-order recurrence equations. Furthermore, it is clear that $\check{U}$ and $\check{p}$, are decreasing sequences. Let $M\Theta^i$ be the largest number of the states, i.e., the uniform upper bound of $(|\Theta_t^i|)_{t=0}^{\infty}$. Then it is also an upper bound of K, i.e., $K \leq M\Theta^i$ always holds. Defining

$$MW(\bar{U}, d) \equiv \check{U}(K - M\Theta),$$
$$MU(\bar{U}, d) \equiv \check{p}(K - M\Theta),$$

we have

$$\left|p^i(\theta^i, s)\right| < \check{p}(1) \leq MW(\bar{U}, d) \quad \text{for all } (\theta^i, s) \in \Theta^i \times S,$$
$$\left|U^i(\theta^i)\right| < \check{U}(1) \leq MU(\bar{U}, d) \quad \text{for all } \theta^i \in \Theta^i$$

as desired. □

D.7 *Proof of Theorem 8*

First, we show that for each d-block, we can construct a "finite-horizon mechanism" that implements the targeted allocation rule without leaving information rent and whose payment rule is bounded by $MB(T, d)$.

Lemma 7. *Suppose that $\{\underline{t}, \underline{t}+1, \ldots, \underline{t}+T\}$ is a d-block along $(\chi_t)_{t=\underline{t}}^{t+T}$ and there exists a mechanism $(\chi_t, g_t^i)_{t=\underline{t}+T+1}^{\infty}$, such that (i) $(\chi_t, g_t^i)_{t=\underline{t}+T+1}^{\infty}$ is wp-EPIC for i for $t = \underline{t}+T+1, \ldots$, (ii) $(g_t^i)_{t=\underline{t}+T+1}^{\infty}$ is independent of $\theta_{0:\underline{t}+T}$, and (iii) $(\chi_t, g_t^i)_{t=\underline{t}+T+1}^{\infty}$ satisfies*

$$V_{\underline{t}+T+1}^i(\theta_{\underline{t}+T+1}) + G_{\underline{t}+T+1}^i(\theta_{\underline{t}+T+1}) = 0 \quad \textit{for all } \theta_{\underline{t}+T+1} \in \Theta_{\underline{t}+T+1}.$$

Then there exists $(\phi_t^i)_{t=\underline{t}}^{t+T+1}$ such that the following statements hold:

(a) We have that $(\phi_t^i)_{t=\underline{t}}^{t+T+1}$ is independent of $\theta_{0:\underline{t}-1}$.

(b) There exist an upper bound of $(\phi_t^i)_{t=\underline{t}}^{t+T+1}$ that only depends on T and d, i.e.,

$$\left|\phi_t^i(\theta_{\underline{t}:t})\right| < MB(T, d) \quad \textit{for all } t \in \{\underline{t}, \ldots, \underline{t}+T+1\} \textit{ and } \theta_{\underline{t}:t} \in \Theta_{\underline{t}:t}.$$

(c) The combined mechanism $(\chi_t, \psi_t^i)_{t=\underline{t}}^{\infty}$ such that $\psi_t^i \equiv \phi_t^i$ for $t \in \{\underline{t}, \underline{t}+1, \ldots, \underline{t}+T\}$, $\psi_{\underline{t}+T+1}^i \equiv \phi_{\underline{t}+T+1}^i + g_{\underline{t}+T+1}^i$, and $\psi_t^i \equiv g_t^i$ for $t \in \{\underline{t}+T+1, \ldots\}$ is wp-EPIC for i for $t = \underline{t}, \underline{t}+1, \ldots$.

(d) The combined mechanism $(\chi_t, \psi_t^i)_{t=\underline{t}}^{\infty}$ *satisfies*

$$V_{\underline{t}}^i(\theta_{\underline{t}}) + \Psi_{\underline{t}}^i(\theta_{\underline{t}}) = 0 \quad \text{for all } \theta_{\underline{t}} \in \Theta_{\underline{t}}.$$

Proof. The construction of the mechanism is the same as that of Theorems 1 and 2. We will show the existence of the upper bound of the payments. For notational convenience, we prove only the case of $\underline{t} = 0$. (For the case of $\underline{t} \neq 0$, we can replace period t with period $\underline{t} + t$.)

Period T First consider the period-T problem. Suppose that to resolve the period-$T-1$ problem, we need to achieve some EPV that is no larger than $\bar{U}_0$ when $\theta_T^{-i} \in B_T^{-i}$. In this case, as in the proof of Theorem 1, we take

$$u_T^i\big(x_T, \theta_T^i; \theta_{0:T-1}, \theta_T^{-i}\big) = v_T^i(x_T, \theta_T)$$

and

$$\big|v_T^i(x_T, \theta_T)\big| < \bar{v}.$$

Therefore, by Lemma 5, the constructed incentive payment satisfies

$$\big|\phi_{T+1}^i\big(\theta_{0:T}, \theta_{T+1}^{-i}\big)\big| < MS(\bar{U}, \bar{v}; d) \equiv MS_0(\bar{U}; d).$$

When $\theta_T^{-i} \in \Theta_T^{-i} \setminus B_T^{-i}$ realizes, as in the proof of Theorem 2, we apply Lemma 6 (instead of Lemma 2). Again

$$u_T^i\big(x_T, \theta_T^i; \theta_{0:T-1}, \theta_T^{-i}\big) = v_T^i(x_T, \theta_T).$$

Therefore, the constructed incentive payment satisfies

$$\big|\phi_{T+1}^i\big(\theta_{0:T}, \theta_{T+1}^{-i}\big)\big| < MW(\bar{v}; d) \equiv MW_0(d).$$

Furthermore, agent i's EPV from $\theta_T^{-i} \in \Theta_T^{-i} \setminus B_T^{-i}$ satisfies

$$\begin{aligned}
&\big|v_T^i\big(\chi_T(\theta_T), \theta_T\big) + \delta\mathbb{E}\big[\phi_{T+1}^i\big(\theta_{0:T}, \theta_{T+1}^{-i}\big)|\chi_T(\theta_T), \theta_T\big]\big| \\
&= \big|V_T^i(\theta_T) + \delta\mathbb{E}\big[\Phi_{T+1}^i(\theta_{0:T+1})|\chi_T(\theta_T), \theta_T\big]\big| \\
&< MU(\bar{v}; d) \equiv MU_0(d).
\end{aligned}$$

Period $T-k$ (where $k \geq 1$) Make the following suppositions:

(i) If we specify the EPV that is no larger than $\bar{U}_{k-1}$ in period $T-(k-1)$ when $\theta_{T-(k-1)}^{-i} \in B_{T-(k-1)}^{-i}$ realizes, then

$$\begin{aligned}
&\big|\phi_t^i\big(\theta_{0:t-1}, \theta_t^{-i}\big)\big| < MS_{k-1}(\bar{U}_{k-1}; d) \\
&\quad \text{for } t \geq T-(k-1) \text{ whenever } \theta_{T-(k-1)}^{-i} \in B_{T-(k-1)}^{-i}.
\end{aligned}$$

(ii) We have

$$\begin{aligned}&\left|\phi_t^i\left(\theta_{0:t-1}, \theta_t^{-i}\right)\right| < MW_{k-1}(d)\\&\quad\text{for } t \geq T-(k-1) \text{ whenever } \theta_{T-(k-1)}^{-i} \in \Theta_{T-(k-1)}^{-i} \setminus B_{T-(k-1)}^{-i}.\end{aligned}$$

(iii) We have

$$\begin{aligned}&\left|V_{T-(k-1)}^i(\theta_{0:T-(k-1)}) + \delta\mathbb{E}\left[\Phi_{T-(k-2)}^i(\theta_{0:T-(k-2)})|\chi_{T-(k-1)}(\theta_{T-(k-1)}), \theta_{T-(k-1)}\right]\right|\\&\quad < MU_{k-1}(d) \quad \text{whenever } \theta_{T-(k-1)}^{-i} \in \Theta_{T-(k-1)}^{-i} \setminus B_{T-(k-1)}^{-i}.\end{aligned}$$

Suppose also that when $\theta_{T-k}^{-i} \in B_{T-k}^{-i}$ realizes, we specify some EPV that is not larger than $\bar{U}_k$. Then, for each $\theta_{T-k}^{-i} \in B_{T-k}^{-i}$, we apply Lemma 8 to obtain ϕ_{T-k+1} by specifying

$$\begin{aligned}&u_{T-k}^i\left(x_{T-k}, \theta_{T-k}^i; \theta_{0:T-k-1}, \theta_{T-k}^{-i}\right)\\&\quad = v_{T-k}^i(x_{T-k}, \theta_{T-k})\\&\qquad + \delta\mathbb{E}\left[\mathbf{1}_{\{\theta_{T-(k-1)}^{-i} \notin B_{T-(k-1)}^{-i}\}}\left(V_{T-(k-1)}^i(\theta_{0:T-(k-1)})\right.\right.\\&\qquad + \delta\mathbb{E}\left[\Phi_{T-(k-2)}^i(\theta_{0:T-(k-2)})|\chi_{T-(k-1)}(\theta_{T-(k-1)}), \theta_{T-(k-1)}\right]\left.\right)|\\&\qquad \chi_{T-k}(\theta_{T-k}), \theta_{T-k}\left.\right].\end{aligned} \tag{48}$$

Hence,

$$\left|u_{T-k}^i\left(x_{T-k}, \theta_{T-k}^i; \theta_{0:T-k-1}, \theta_{T-k}^{-i}\right)\right| < \bar{v} + \delta MU_{k-1}(d).$$

Accordingly, the obtained one-shot payment rule p_{T-k+1}^i satisfies

$$\left|p_{T-k+1}^i\left(\theta_{T-k}^i, \theta_{T-k+1}; \theta_{0:T-k-1}, \theta_{T-k}^{-i}\right)\right| < MS\left(\bar{U}_k, \bar{v} + \delta MU_{k-1}(d); d\right)$$

and, therefore,

$$\begin{aligned}&\left|\phi_t^i\left(\theta_{0:t-1}, \theta_t^{-i}\right)\right|\\&\quad < \max\left\{MS\left(\bar{U}_k, \bar{v} + \delta MU_{k-1}(d); d\right),\right.\\&\qquad \left. MS_{k-1}\left(MS\left(\bar{U}_k, \bar{v} + \delta MU_{k-1}(d); d\right); d\right), MW_{k-1}(d)\right\}\\&\quad \equiv MS_k(\bar{U}_k; d) \quad \text{for } t \geq T-k, \text{whenever } \theta_{T-k}^{-i} \in B_{T-k}^{-i}.\end{aligned}$$

When $\theta_{T-k}^{-i} \in \Theta_{T-k}^{-i} \setminus B_{T-k}^{-i}$ realizes, we also specify u_{T-k}^i by (48) to apply Lemma 6. Hence, the obtained p_{T-k+1}^i and U_{T-k}^i satisfy

$$\begin{aligned}\left|p_{T-k+1}^i\left(\theta_{T-k}^i, \theta_{T-k+1}; \theta_{0:T-k-1}, \theta_{T-k}^{-i}\right)\right| &< MW\left(\bar{v} + \delta MU_{k-1}(d); d\right),\\ \left|U_{T-k}^i\left(\theta_{T-k}^i; \theta_{0:T-k-1}, \theta_{T-k}^{-i}\right)\right| &< MU\left(\bar{v} + \delta MU_{k-1}(d); d\right).\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\left|\phi_t^i\left(\theta_{0:t-1}, \theta_t^{-i}\right)\right| \\
&\quad < \max\left\{MW\left(\bar{v} + \delta MU_{k-1}(d); d\right), \right. \\
&\qquad \left. MS_{k-1}\left(MW\left(\bar{v} + \delta MU_{k-1}(d); d\right); d\right), MW_{k-1}(d)\right\} \\
&\quad \equiv MW_k(d) \quad \text{for } t \geq T-k, \text{whenever } \theta_{T-k}^{-i} \in \Theta_{T-k}^{-i} \setminus B_{T-k}^{-i}
\end{aligned}$$

and

$$\begin{aligned}
&\left|V_{T-k}^i(\theta_{0:T-k}) + \delta \mathbb{E}\left[\Phi_{T-(k-1)}^i(\theta_{0:T-k}) | \chi_{T-k}(\theta_{T-k}), \theta_{T-k}\right]\right| \\
&\quad < MU\left(\bar{v} + \delta MU_{k-1}(d); d\right). \\
&\quad \equiv MU_k(d) \quad \text{whenever } \theta_{T-k}^{-i} \in \Theta_{T-k}^{-i} \setminus B_{T-k}^{-i}.
\end{aligned}$$

Period 0 The argument is similar to the analysis for period $T-k$, but we specify the EPV to be zero (i.e., $\bar{U}_0 = 0$) for all $\theta_0^{-i} \in \Theta_0^{-i}$. Hence,

$$\begin{aligned}
&\left|\phi_t^i\left(\theta_{0:t-1}, \theta_t^{-i}\right)\right| \\
&\quad < \max\left\{MS\left(0, \bar{v} + \delta MU_{T-1}(d); d\right), \right. \\
&\qquad \left. MS_{T-1}\left(MS\left(0, \bar{v} + \delta MU_{T-1}(d); d\right); d\right), MW_{T-1}(d)\right\} \\
&\quad \equiv MB(T, d) \quad \text{for all } t \text{ and } \left(\theta_{0:t-1}, \theta_t^{-i}\right) \in \Theta_{0:t-1} \times \Theta_t^{-i},
\end{aligned}$$

as desired. □

Proof of Theorem 8 For each d-block, we apply Lemma 7 to construct a finite-horizon mechanism that leaves no information rent. Then the EPV starting from $\bar{t}$, which is the initial period of the next d-block, is fixed to zero for all $\theta_{0:\bar{t}} \in \Theta_{0:\bar{t}}$, i.e.,

$$V_{\bar{t}}^i(\theta_{0:\bar{t}}) + \delta \mathbb{E}\left[\Phi_{\bar{t}+1}^i(\theta_{0:\bar{t}+1}) | \chi_{\bar{t}}(\theta_{\bar{t}}), \theta_{\bar{t}}\right] = 0 \quad \text{for all } \theta_{0:\bar{t}} \in \Theta_{0:\bar{t}}.$$

Accordingly, wp-EPIC of the finite-horizon mechanisms guarantees wp-EPIC of the combined infinite-horizon mechanism. Furthermore, by Theorem 7, the constructed infinite-horizon payment rule is uniformly bounded by $MB(L, d)$, as desired. □

References

Athey, Susan and Ilya Segal (2013), "An efficient dynamic mechanism." *Econometrica*, 81, 2463–2485. [3, 13]

Crémer, Jacques and Richard P. McLean (1988), "Full extraction of the surplus in Bayesian and dominant strategy auctions." *Econometrica*, 56, 1247–1257. [3]

Liu, Heng (2018), "Efficient dynamic mechanisms in environments with interdependent valuations: The role of contingent transfers." *Theoretical Economics*, 13, 795–829. [6]

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