Supplement to "Decentralized bargaining in matching markets: Efficient stationary equilibria and the core": Appendixes

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These appendixes discuss MPE multiplicity, the nongeneric cases of core match multiplicity and of boundary core payoffs, and the relationship to Okada (2011), and provide omitted proofs.

Appendix A: MPE multiplicity

This short section presents an economy in which there is a unique surplus maximizing match and multiple MPEs exist for all δ close to 1. Consider the four-player economy in Figure 8 with $p_a = p_b = 4/10$ and $p_c = p_d = 1/10$.

The economy clearly has a unique surplus maxmizing match since

$$\sigma_a + \sigma_d = \sigma_b + \sigma_c = 36 > 35.$$

Thus, an efficient MPE always exits for all δ close to 1. Consequently, a strongly efficient LMPE exists. However, for all δ close to 1, an inefficient MPE also exists with the proposal probabilities

$$\pi_{ad} = \pi_{bc} = \pi_{cd} = \pi_{dc} = 1.$$

By setting $V_a = V_b$ and $V_d = V_c$, value equations (1) for the inefficient equilibrium reduce to

$$V_a = \frac{4}{10}(35 - \delta V_c) + \frac{2}{10}\delta V_a(ab) + \frac{4}{10}\delta V_a(ad),$$
$$V_d = \frac{1}{10}(36 - \delta V_d) + \frac{1}{2}\delta V_d + \frac{4}{10}\delta V_d(ad).$$

Solving for subgame values establishes that

$$V_a = \frac{2(350 - 69\delta - 25\delta^2)}{5(5-\delta)(2-\delta)} \quad \text{and} \quad V_d = \frac{36 - 4\delta}{(5-\delta)(2-\delta)}.$$

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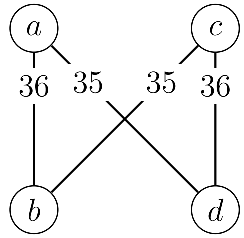

FIGURE 8. A four-player complete network with surplus heterogeneity.

Taking limits then implies that $\lim_{\delta\to 1} V_a = 128/5 = 25.6$ and $\lim_{\delta\to 1} V_d = 8$. Limit values then satisfy all the equilibrium incentive constraints, as

$$2V_a > 36, \qquad 2V_d < 36,$$
$$V_a + V_d < 35, \qquad 36 - V_d > 35 - V_a.$$

Since incentive constraints are strict and value functions are continuous, players strictly prefer to comply with the strategy for all δ close to 1. Thus, the proposed strategy is an MPE for all δ close to 1 and, consequently, an LMPE. This shows that multiple equilibria may exist even when the core match is unique. Intuitively, multiplicity may arise because directed search and partner selection bring about coordination problems, as players' bargaining powers are jointly determined by the entire profile of agreement probabilities.

APPENDIX B: MULTIVALUED CORE

The complications that arise when the core is multivalued (that is, when multiple matches are efficient) are closely related to those that occur when Rubinstein payoffs are on the boundary of the core, as any core payoff must be on the boundary of the core in such instances. When multiple matches are efficient, each efficient match is associated with a possibly different vector of Rubinstein payoffs. For any efficient match η, let $\sigma^\eta \in \mathbb{R}^{|N|}$ denote the vector of Rubinstein payoffs associated with the efficient match η. Consider an alternative efficient match $\gamma \neq \eta$. Shapley and Shubik (1971) establish that if the pair (η, σ^η) is a core outcome, then so is the pair (γ, σ^η), in that, for all players i, $\sigma_i^\eta + \sigma_{\eta(i)}^\eta = s_{i\eta(i)}$ and $\sigma_i^\eta + \sigma_{\gamma(i)}^\eta = s_{i\gamma(i)}$. As $\eta(i) \neq \gamma(i)$ for some player i, the core outcome (η, σ^η) must be on the boundary of the core, as players i and $\gamma(i)$ have a weakly profitable pairwise deviation.

Scenarios in which Rubinstein payoffs lie on the boundary of the core lead to complications. Our equilibrium construction can lead to payoffs that are outside the core for all $\delta < 1$, such that some player has a profitable deviation to offer inefficiently, even when limit payoffs belong to the core. We illustrate this in the following example.

Consider the four-player economy in which all matches are possible and generate a surplus of 1 as depicted in Figure 9. Suppose first that all players move with equal probability. If so, the match (ab, cd) is efficient, and for this match each player's Rubinstein

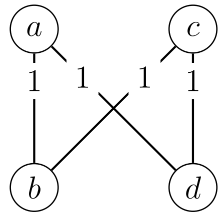

FIGURE 9. A four-player complete network with unit surplus.

payoff is 1/2. These payoffs belong to the boundary of the core. If we attempt the efficient MPE construction we use when players' payoffs are in the interior of the core, with certainty player a would offer to b, b would offer to a, c would offer to d, and d would offer to c. In this example, as is the case with core-interior Rubinstein payoffs, these offer strategies constitute an efficient MPE. For instance, given these strategies, continuation values for players b and d coincide whenever player a is selected as the proposer. Thus, player a is indifferent between offering to b or d, and offering to b is a best response for a as equilibrium play dictates.

Suppose now that players propose respectively with probabilities

$$p_a = p_b = 3/8 \quad \text{and} \quad p_c = p_d = 1/8.$$

The match (ab, cd) remains efficient, and for this match each player's Rubinstein payoff is 1/2. Thus, as before, Rubinstein payoffs are on the boundary of the core. However, if we now attempt the efficient MPE construction we use for interior Rubinstein payoffs, we no longer find an equilibrium. By complying with these strategies, all player still receive limit payoffs of 1/2, but for all $\delta < 1$, player a has a profitable deviation by offering to d. As d waits longer than b to be matched in expectation, d's continuation value is lower than b's for $\delta < 1$. Hence, a prefers to deviate and to offer to d. In the other efficient match (ad, cb), Rubinstein payoffs are 3/4 for a and b and 1/4 for c and d. These Rubinstein payoffs do not belong to the core, as c and d have a profitable pairwise deviation. If a were to offer to d, d were to offer to a, b were to offer to c, and c were to offer to b with certainty, player c would have a profitable deviation by offering to d.

This example is intended to illustrate the subtleties that may arise when Rubinstein payoffs are on the boundary of the core. Although there are no strictly profitable deviations in the limit as $\delta \to 1$, there may be strictly profitable deviations for all $\delta < 1$. Whether this happens depends on whether the sum of payoffs for each pair of efficiently matched players converges from above or below to the surplus they generate, which in turn depends on the fine details of the game. Nevertheless, there is one canonical case in which there always exists an efficient MPE when Rubinstein payoffs are on the boundary of the core. An assignment economy is said to be *simple* if $s_{ij} \in \{0, 1\}$ for all $i, j \in N$.

PROPOSITION 7. *Consider a simple assignment economy in which all players are selected to propose with equal probability. Then there exists a strongly efficient MPE if the Rubinstein payoffs associated with an efficient match belong to the core.*

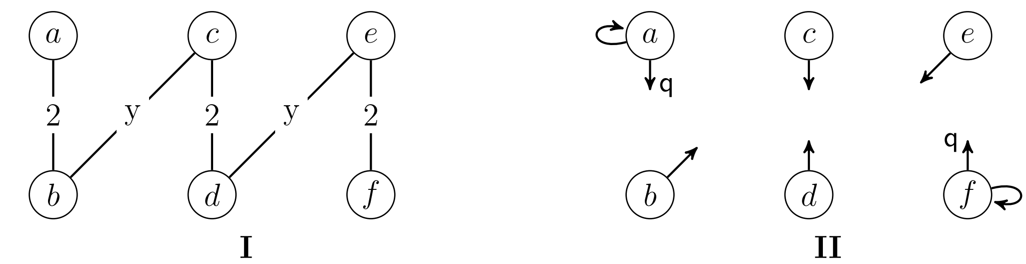

FIGURE 10. Panel I is the assignment economy; panel II is the agreement probabilities.

Symmetry in these settings suffices for the existence of strongly efficient MPE. In fact, since all players on one side of the market receive the same payoff and since delay destroys surplus, the sum of payoffs for each pair of efficiently matched players must converge from below to the surplus they generate. In general though, core match multiplicity may lead to discontinuities in equilibrium payoffs that would further complicate efficiency conclusions (as is the case in Example 1 when $y = 200$).

To conclude the discussion, we show an example in which core match multiplicity leads to MPE multiplicity and to additional delay frictions. Consider the six-player assignment economy depicted in panel I of Figure 10, in which agents a and f propose with probability 1/4, whereas all other players propose with probability 1/8. In such an example, the efficient matches are pinned down by the value of parameter y. We consider values of $y \in [2,3]$.

For all values of $y \in [2,3]$, Rubinstein payoffs do not belong to the core in any core match. We consider whether there can be an equilibrium in which a and f delay making an offer. Suppose that agents c and d agree with each other when proposing. If so, by delaying agent a may end up bargaining bilaterally with agent b provided that players c or d are selected before either b or e. As in this scenario a ends up in a strong position relative to b, player a could, in principle, prefer delay. To explore this possibility, we assume that agents a and f delay with probability $1-q$, and we look for conditions on q and y under which there is an equilibrium with the agreement probabilities shown in panel II of Figure 10. Finding agents' MPE values in the relevant subgames and taking the limit yields

$$\lim_{\delta\to 1} V_a(N) = \lim_{\delta\to 1} V_f(N) = \frac{16+q(17-3y)}{24+12q},$$

$$\lim_{\delta\to 1} V_b(N) = \lim_{\delta\to 1} V_e(N) = \frac{7+3y}{12},$$

$$\lim_{\delta\to 1} V_c(N) = \lim_{\delta\to 1} V_d(N) = 1.$$

All values are strictly positive for any $y \in [2,3]$ and any $q \in [0,1]$. Moreover, $\partial V_a/\partial q > 0$ for $y < 3$, but $\partial V_a/\partial q = 0$ for $y = 3$. Thus, players a and f do not delay and set $q = 1$ for $y < 3$. Yet there might be equilibrium delay for $y = 3$. In fact, an equilibrium exists in which $q = 0$ when $y = 3$. This discontinuity arises because multiple matches are efficient when $y = 3$. Although agent a delays, there is an efficient match in which he is unmatched.

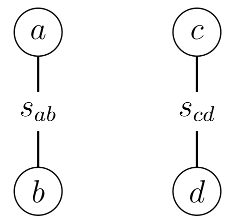

FIGURE 11. A four-player economy.

With heterogeneities, instances of multiple efficient matches are nongeneric. When the core match is unique, delay occurs only because of fundamental strategic reasons, as we documented in Examples 3 and 4.

APPENDIX C: RELATIONSHIP TO OKADA

There are some similarities between Okada (2011) and our paper. Both papers relate the existence of an efficient MPE in a non-cooperative bargaining game to whether different statistics belong to the core of an associated cooperative game. Nevertheless, the models are significantly different in a crucial dimension. Okada models coalitional bargaining, while we allow only pairs of players to bargain. The models are geared toward different applications (legislative bargaining for Okada; decentralized markets in our case) and, in this section, we argue that applying Okada's model to decentralized markets may lead to strange predictions.

Consider the four-player example shown in Figure 11. In terms of Okada's notation, this is a coalitional game with $N = \{a, b, c, d\}$, where

(i) $v(N) = s_{ab} + s_{cd}$

(ii) $v(a, b) = v(a, b, c) = v(a, b, d) = s_{ab}$

(iii) $v(c, d) = v(a, c, d) = v(b, c, d) = s_{cd}$

(iv) $v(S) = 0$ for any other coalition $S \subset N$.

For an efficient equilibrium as defined by Okada, each agent must make an acceptable proposal to the grand coalition with probability 1 if selected to propose. Unlike in our model, this option is available to agents, and by offering to the grand coalition, all players can reach agreement immediately, thereby eliminating any losses from agents' limited patience. By Okada's Proposition 3.1, in an efficient stationary equilibrium, the expected payoffs are given by the solution to the system of value equations

$$V_i = p_i\left[v(N) - \delta \sum_{j\in N\setminus i} V_j\right] + \delta V_i \sum_{j\in N\setminus i} p_j \quad \text{for all } i \in N.$$

In the limit as $\delta \to 1$, this yields expected payoffs

$$\lim_{\delta\to 1} V_i = \left(\frac{p_i}{p_a + p_b + p_c + p_d}\right) v(N) \quad \text{for all } i \in N.$$

Moreover, Okada shows that an efficient equilibrium exists only if these payoffs belong to the core of the associated cooperative game. In the limit, two necessary conditions for an efficient equilibrium are

$$V_i + V_{\eta(i)} \geq v\big(i, \eta(i)\big) \geq s_{i\eta(i)} \quad \text{for all } i \in N.$$

Substituting the efficient payoff characterization and rearranging, the conditions simplify to

$$(p_a + p_b)s_{cd} \geq (p_c + p_d)s_{ab} \quad \text{and} \quad (p_c + p_d)s_{ab} \geq (p_a + p_b)s_{cd}.$$

But if so, an efficient MPE exists only if $(p_a + p_b)s_{cd} = (p_c + p_d)s_{ab}$. This condition is a knife-edge case. Indeed, even if the condition was satisfied, any perturbation to the surpluses by some small independent noise terms (drawn from continuous distributions) would lead to the condition being violated with probability 1. The knife-edge nature of the condition is not an artifact of the example, but a general feature of Okada's setting in the context of assignment economies, which implies that efficient outcomes are very unlikely to occur with multilateral negotiations. Intuitively, having to agree with all players imposes further constraints on agreeable outcomes and restricts the scope for efficient negotiations. In contrast, in our setting, a strongly (and thus weakly) efficient MPE would exist for any values of $(p_a, p_b, p_c, p_d, s_{ab}, s_{cd})$, as Rubinstein payoffs would belong to the core for any such parameter values. Intuitively, with bilateral negotiations, non-core partners cannot affect bargaining outcomes and constrain efficiency when they generate no surplus with their alternative partners.

The example highlights the differences in the approach and the conclusions relative to Okada (2011). His model is most suitable for situations in which coalitions can jointly bargain. In contrast, ours is intended to capture decentralized markets in which buyer–seller pairs bargain in solitude. When this is the case, decentralized negotiations may actually lead to more efficient and arguably more plausible outcomes.

Appendix D: Omitted proofs

Proof of Remark 1. First, we establish part (a). By assumption, there is a unique preferred match at any active player set. Thus, for all $i \in A$ and all $A \subseteq N$, if $\max_{j\in A} s_{ij} > 0$, then $\arg\max_{j\in A} s_{ij}$ is a singleton. Moreover, i's continuation value when selected as the proposer satisfies

$$\lim_{\delta\to 0} v_i(A) = \lim_{\delta\to 0} \max\Big\{\delta V_i(A), \max_{j\in A\setminus i}\big\{s_{ij} - \delta V_j(A)\big\}\Big\} = \max_{j\in A\setminus i} s_{ij},$$

as $V_j(A) < \max_{k\in A} s_{jk} < \infty$ for all players $j \in A$ and active player sets $A \subseteq N$. Hence, in all MPEs for all δ close to 0, $\pi_{ij}(A) = 1$ if and only if $j = \arg\max_{j\in A} s_{ij}$. If we have that $\max_{k\in A} s_{ik} > 0$ at some active player set $A \subseteq N$, then there is a unique player $j = \arg\max_{k\in A} s_{ik}$ and for all δ close to zero,

$$\max_{k\in A}(s_{ij} - s_{ik}) > \delta \sum_{k\in A} s_{k\eta(k)}.$$

Thus, independently of the constraints imposed on subsequent matching, the expected social surplus is maximized by matching agent i to agent j, if agent i is selected as the proposer. Maximizing utilitarian welfare for δ all close to 0 simply amounts to setting $\pi_{ij}(A) = 1$ if and only if $j = \arg\max_{j\in A} s_{ij}$. So all MPEs maximize utilitarian welfare for all δ close to zero.[1]

Next, we establish part (b). Payoffs in any subgame $A \in C(N)$ of an efficient MPE are pinned down by Proposition 2 for any $\delta \in (0, 1)$. We show that if $s_{i\eta(i)} > s_{ij}$ for all $i \neq j$, complying with efficient strategies is an equilibrium when δ is sufficiently low. Recall that any player $j \in A$ accepts any offer that is worth at least $\delta V_j(A)$. Suppose, toward a contradiction, that some player $i \in A$ at some subgame $A \in C(N)$ has a profitable deviation that entails agreeing with $j \neq \eta(i)$ when all other agents play efficient strategies. For such an offer to be profitable for player i, it must be that

$$s_{ij} - \delta V_j(A) \geq s_{i\eta(i)} - \delta V_{\eta(i)}(A). \tag{12}$$

By taking limits on both sides of the inequality as δ converges to 0, we obtain

$$s_{ij} \geq s_{i\eta(i)}.$$

But this cannot be, as players strictly prefer their core match by assumption and so $s_{ij} < s_{i\eta(i)}$. Thus, any player $i \in A$ at any subgame $A \in C(N)$ does not have a profitable deviation when the discount factor is sufficiently low, which implies the existence of an efficient MPE for any δ close to 0.

Finally, we establish part (c). By contradiction, assume that an efficient MPE exists for all δ close to 0, but that $s_{ij} > s_{i\eta(i)}$ for some $j \neq \eta(i)$. If so, player i has a strictly profitable deviation from an efficient equilibrium if condition (12) holds strictly. But since $\delta V_i(A) \to 0$ and $\delta V_j(A) \to 0$, condition (12) must be strict for δ sufficiently low and, thus, player i must have a profitable deviation. □

Proof of Remark 2. To establish part (a), let u be a vector of core payoffs associated to the core match η. Consider two players $i, j \in N$ such that $\eta(i) = j$ and set

$$\frac{p_i}{p_j} = \frac{u_i}{s_{ij} - u_i}.$$

This condition ensures that i and j receive their core payoffs, u_i and u_j, if everyone plays the strategies characterized in the proof of Proposition 2. This removes at most $N/2$ degrees of freedom from the vector p. Thus, it is straightforward to find a probability vector p that satisfies the above condition for all $i \in N$.

Part (b) is a trivial consequence of the Rubinstein payoffs not being affected by proportional changes in probabilities. Part (c) is also straightforward. Let $U(S)$ denote the set of core payoffs when the surplus matrix is S. Observe that if the surplus changes from S to S', it must be that $s_{i\eta(i)} = s'_{i\eta(i)}$ for any $i \in N$. This is because the core match cannot

[1] If the preferred match is not unique, then a planner maximizing welfare may have preferences over preferred partners that differ from those of the proposer. Thus, for all δ close 0, there may be no welfare maximizing MPE.

change when S changes to S' and because $s_{i\eta(i)} \neq s'_{i\eta(i)}$ implies that any core payoff in S would not belong to S' (since $u_i + u_{\eta(i)} = s_{i\eta(i)}$ for any $u \in U(S)$). Thus, Rubinstein payoffs in the two markets must coincide, implying that

$$\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) = \left(\sigma'_1, \ldots, \sigma'_n\right) = \boldsymbol{\sigma}'.$$

The conclusion then follows immediately from these observations, since $\boldsymbol{\sigma} \in U(S) \subseteq U(S')$.

To prove part (d), it is useful to introduce the notions of an offer graph and a cyclical offer graph. For any subgame with active player set $A \subseteq N$ and any MPE, the *offer graph* (A, G) consists of a directed graph with vertices in A and with edges satisfying

$$ij \in G \quad \Leftrightarrow \quad i \in A \quad \text{and} \quad j \in \left\{k \mid \pi_{ik}(A) > 0\right\} \cup \eta(i).$$

We say that an offer graph is cyclical whenever there exists a subset of active players choosing to make offers so as to exchange their respective core partners with one another. Formally, an offer graph is *cyclical* if there exists a map $\varphi : N \to N$ and a set of players $F \subseteq P_k \cap A$ for $k \in \{1, 2\}$ such that

- $\varphi(i) = j \Rightarrow ij \in G$
- $\varphi(i) \neq \eta(i)$ for some $i \in F$
- $\{\varphi(i) | i \in F\} = \{\eta(i) | i \in F\}$.

Next, we establish that MPE offer graphs are never cyclical. If offers were cyclical, a subset of players who prefer offering to one another's core matches instead of their own core match would exist. These players would have to achieve a higher aggregate surplus by matching with non-core partners, thereby violating the efficiency properties of the core. Formally, suppose the offer graph is cyclical. By revealed preferences, for any player $i \in F$ and $\varphi(i)$ such that $\pi_{i\varphi(i)}(A) > 0$, subgame perfection requires that

$$s_{i\varphi(i)} - \delta V_{\varphi(i)} \geq s_{i\eta(i)} - \delta V_{\eta(i)}.$$

Furthermore, because of cyclicality, by summing over all players in F, we would have that

$$\sum_{i \in F} (s_{i\varphi(i)} - \delta V_{\varphi(i)}) \geq \sum_{i \in F} (s_{i\eta(i)} - \delta V_{\eta(i)}) \quad \Leftrightarrow \quad \sum_{i \in F} s_{i\varphi(i)} \geq \sum_{i \in F} s_{i\eta(i)}.$$

However, this leads to a contradiction, as the core match was assumed to be unique.

Finally, we establish that the core match always obtains with positive probability in an MPE without delay. The uniqueness of the core match and the nonnegativity of surpluses imply that all players on one side of the market are matched at the unique core allocation.[2] Fix an MPE without delay. No delay implies that every player with a positive value agrees with probability 1 when selected to propose in every possible subgame. Without loss of generality, suppose that $P_1 \cap A \geq P_2 \cap A$. If, for any A, there

[2]This is the only result in which the assumption on the nonnegativity of the surplus is substantive.

exists $i \in P_1 \cap A$ such that $\pi_{i\eta(i)}(A) > 0$, the conclusion obviously holds. Thus, assume that this is not the case. Then, for some A, $\pi_{i\eta(i)}(A) = 0$ for all $i \in P_1 \cap A$. Next, we show that this leads to a contradiction, as the offer graph would necessarily be cyclical. Pick any match φ satisfying $\varphi(i) = j$ for $\pi_{ij}(A) > 0$ and $\varphi(i) \neq \eta(i)$ for any $i \in P_1 \cap A$. Such a match exists because players in $P_1 \cap A$ do not delay and because $\pi_{i\eta(i)}(A) = 0$. Observe that since the core match is unique, $P_2 = \{\eta(i) | i \in P_1 \cap A\} \cap P_2$. Furthermore, by construction, it must be that $P_2 \supseteq \{\varphi(i) | i \in P_1 \cap A\} \cap P_2$. Since $\eta(i) \neq \eta(k)$ for any $i, k \in P_1 \cap A$, there must exist a set $F \subseteq P_2$ such that

$$\{\varphi(i) | i \in F\} = \{\eta(i) | i \in F\},$$

as otherwise a player $i \in P_1 \cap A$ would exist such that $\varphi(i) = \eta(i)$. This in turn implies the desired contradiction to the first part of the proposition, as the offer graph would necessarily be cyclical. □

Proof of Remark 3. For convenience, when $A = N$, value functions and proposal probabilities omit the dependence on the active player set A. First observe that players on one of the two core matches never delay in any weakly efficient LMPE for all δ close to 1. Delay on both core matches would require

$$\delta V_a + \delta V_b \geq s_{ab} \quad \text{and} \quad \delta V_c + \delta V_d \geq s_{cd}, \tag{13}$$

which violates feasibility since $\sum_{i \in N} V_i > s_{ab} + s_{cd}$. Thus, in any weakly efficient LMPE, there exists a core match in which no player delays. Call such a match $i\eta(i)$ so that $\pi_{ii} + \pi_{\eta(i)\eta(i)} = 0$. Next observe that players agree on at most one of the two non-core matches with positive probability in any weakly efficient LMPE for all δ close to 1. Agreement on both non-core matches would require

$$\delta V_a + \delta V_d = s_{ad} \quad \text{and} \quad \delta V_b + \delta V_c = s_{bc}.$$

But this would violate the weak efficiency of the limiting equilibrium, as

$$\lim_{\delta \to 1} \sum_{k \in N} V_k = s_{ad} + s_{bc} < s_{ab} + s_{cd}.$$

Thus, in any weakly efficient LMPE, there exists a non-core match with disagreement. As this link must involve either i or $\eta(i)$, it is without loss of generality to call such a match ij, so that $\pi_{ij} = 0$. This establishes that $\pi_{i\eta(i)} = 1$ and that $\pi_{jj} = 1$. Furthermore, there must be agreement in match $\eta(i)\eta(j)$. If, instead, we had that $\pi_{\eta(i)\eta(j)} + \pi_{\eta(j)\eta(i)} = 0$, the value equation for a player $k \in \{j, \eta(j)\}$ would simplify to

$$V_k = (1-2p)\delta V_k + 2p\delta V_k(\{j, \eta(j)\}) = (1-2p)\delta V_k + 2p\delta\sigma_k.$$

Thus, $\delta V_j + \delta V_{\eta(j)} < s_{i\eta(j)}$ and the equilibrium would be strongly efficient and not sequential. Thus, $\pi_{\eta(i)\eta(j)} + \pi_{\eta(j)\eta(i)} > 0$. Finally, observe that $\pi_{\eta(i)\eta(j)} = 0$. Otherwise,

$$s_{i\eta(i)} - \delta V_i = s_{\eta(j)\eta(i)} - \delta V_{\eta(j)} = \delta V_{\eta(i)},$$

where the first equality would hold by player $\eta(i)$'s indifference, while the latter would hold by player $\eta(j)$'s indifference. This implies that $\delta V_i + \delta V_{\eta(i)} \geq s_{i\eta(i)}$. But as j and $\eta(j)$ delay, the condition (13) would be satisfied and the values would be infeasible. Thus, we must have that $\pi_{\eta(i)i} = 1$ for all δ close to 1. This completely pins down the acceptance probabilities up to relabeling and, consequently, for $\pi_{\eta(j)\eta(i)} = q$, the value equations reduce to

$$\begin{aligned} s_{\eta(i)\eta(j)} &= \delta V_{\eta(i)} + \delta V_{\eta(j)}, \\ V_{\eta(i)} &= (1-p)\delta V_{\eta(i)} + p(s_{i\eta(i)} - \delta V_i), \\ V_{\eta(j)} &= (1-2p)\delta V_{\eta(j)} + 2p\delta V_{\eta(j)}\big(\{j, \eta(j)\}\big), \\ V_i &= (1-p-pq)\delta V_i + pq\delta V_i\big(\{i,j\}\big) + p(s_{i\eta(i)} - \delta V_{\eta(i)}), \\ V_j &= (1-2p-pq)\delta V_j + pq\delta V_j\big(\{i,j\}\big) + 2p\delta V_j\big(\{j,\eta(j)\}\big), \end{aligned} \tag{14}$$

where, obviously, for any $k, l \in N$, we have that

$$V_k(kl) = \frac{p}{1-\delta+2p\delta} s_{kl}.$$

First observe that $\eta(j)$'s value equation trivially implies that $V_{\eta(j)} \leq V_{\eta(j)}(\{j, \eta(j)\})$ for all $\delta \leq 1$. As j delays when $A = N$, $s_{j\eta(j)} - \delta V_j \leq \delta V_{\eta(j)}$. Thus, $s_{j\eta(j)} \leq \delta V_j + \delta V_{\eta(j)}(\{j, \eta(j)\})$. Toward a contradiction, suppose that $\delta V_j < \delta V_j(\{j, \eta(j)\})$. Then $s_{j\eta(j)} < \delta V_j(\{j, \eta(j)\}) + \delta V_{\eta(j)}(\{j, \eta(j)\})$ and $\eta(j)$ would have a profitable deviation delaying instead of offering to j in the subgame where only j and $\eta(j)$ are active. We therefore conclude that $\delta V_j \geq \delta V_j(\{j, \eta(j)\})$. From j's value function, this implies that $V_j(\{i,j\}) \geq V_j(\{j, \eta(j)\})$. Moreover, with equal proposal probabilities, this is equivalent to $s_{ij} \geq s_{j\eta(j)}$. By adding this inequality to the inequality that defines the core match, $s_{i\eta(i)} + s_{j\eta(j)} > s_{\eta(i)\eta(j)} + s_{ij}$, we further obtain that $s_{i\eta(i)} > s_{\eta(i)\eta(j)}$.

In any sequential LMPE, $\lim_{\delta\to 1} q = 0$. Taking limits of the value equations (14) as $\delta \to 1$ immediately delivers that

$$\lim_{\delta\to 1} V_{\eta(i)} = s_{\eta(i)\eta(j)} - \sigma_{\eta(j)}, \qquad \lim_{\delta\to 1} V_{\eta(j)} = \sigma_{\eta(j)},$$
$$\lim_{\delta\to 1} V_i = s_{i\eta(i)} - s_{\eta(i)\eta(j)} + \sigma_{\eta(j)}, \qquad \lim_{\delta\to 1} V_j = \sigma_j.$$

Observe that player $\eta(i)$ always possesses a deviation that sets $q = 0$ (namely, rejecting any offer from $\eta(j)$ when $A = N$). If so, i's and $\eta(i)$'s value functions reduce to

$$\begin{aligned} \hat{V}_i &= (1-p)\delta\hat{V}_i + p(s_{i\eta(i)} - \delta\hat{V}_{\eta(i)}), \\ \hat{V}_{\eta(i)} &= (1-p)\delta\hat{V}_{\eta(i)} + p(s_{i\eta(i)} - \delta\hat{V}_i) \end{aligned}$$

and $\eta(i)$ secures a payoff $\hat{V}_{\eta(i)} = ps_{i\eta(i)}/(1-\delta+2p\delta) \to \sigma_{\eta(i)}$. For $q > 0$ to be an equilibrium for all δ close to 1, such a deviation cannot be profitable. Thus, $V_{\eta(i)} \geq \hat{V}_{\eta(i)}$ for all δ close to 1 and

$$\lim_{\delta\to 1} V_{\eta(i)} = s_{\eta(i)\eta(j)} - \sigma_{\eta(j)} = s_{\eta(i)\eta(j)} - (s_{j\eta(j)}/2) \geq s_{i\eta(i)}/2 = \sigma_{\eta(i)} = \lim_{\delta\to 1} \hat{V}_{\eta(i)}.$$

This implies that $2s_{\eta(i)\eta(j)} \geq s_{j\eta(j)} + s_{i\eta(i)}$, which by efficiency and uniqueness of the core immediately implies that $s_{\eta(i)\eta(j)} > s_{ij}$. Thus, we conclude that

$$s_{i\eta(i)} > s_{\eta(i)\eta(j)} > s_{ij} \geq s_{j\eta(j)},$$

and, invoking our labeling convention, we deduce that $\eta(i) = a$, $i = b$, $j = c$, and $\eta(j) = d$.

To establish the final part of the result, we first find necessary conditions for the existence of a sequential LMPE, and then show that these conditions are also sufficient. Recall that the previous part of the proof establishes that a sequential LMPE exists only if

$$s_{ab} > s_{ad} > s_{bc} \geq s_{cd}. \tag{15}$$

For the proposed strategy profile to be an equilibrium, c and d must weakly prefer to delay instead of offering to each other, and so $\delta V_c + \delta V_d \geq s_{cd}$ for all δ sufficiently close to 1. Moreover, $\lim_{\delta\to 1} \delta(V_c + V_d) = s_{cd}$. Thus, the strategy is consistent with equilibrium behavior only if $\delta(V_c + V_d)$ converges to s_{cd} from above. By solving value equations (14), it is possible to show that

$$\lim_{\delta\to 1} \frac{\delta(V_c + V_d) - s_{cd}}{1-\delta} = \frac{s_{cd}(s_{bc} - s_{cd}) + 2s_{ad}(s_{bc} + s_{cd}) - s_{ab}(s_{bc} + 3s_{cd})}{2p\big[2(s_{ab} - s_{ad}) - (s_{bc} - s_{cd})\big]}. \tag{16}$$

If $s_{bc} = s_{cd}$, then the right hand side of (16) reduces to $-s_{cd}/p < 0$, which is not consistent with equilibrium behavior. Thus, $s_{bc} > s_{cd}$. Next observe that the denominator in (16) must be positive since $s_{ab} - s_{ad} > 0$ by (15) and since $s_{ab} - s_{ad} > s_{bc} - s_{cd}$ by definition of the core. Thus, as the denominator is always positive, (16) is satisfied if and only if the numerator is also positive. This requires that

$$\frac{s_{bc} - s_{cd}}{s_{ab} - s_{ad}} \geq 2\frac{s_{bc} + s_{cd}}{s_{ab} + s_{cd}}. \tag{17}$$

The first part of the proof also establishes that a strategy is consistent with weak efficiency only if $2s_{ad} \geq s_{ab} + s_{cd}$. However, if $s_{ad} = (s_{ab} + s_{cd})/2$, by substituting s_{ad} in (17) one obtains

$$\frac{s_{bc} - s_{cd}}{s_{ab} - s_{cd}} \geq 4\frac{s_{bc} + s_{cd}}{s_{ab} + s_{cd}},$$

which, with some rearrangements, in turn implies that

$$0 \geq 3(s_{ab} - s_{cd})(s_{bc} + s_{cd}) + 2s_{cd}(s_{ab} - s_{bc}),$$

which cannot hold by (15). Hence, $2s_{ad} > s_{ab} + s_{cd}$. Combining the above inequalities establishes that

$$s_{ab} > s_{ad} > (s_{ab} + s_{cd})/2 > s_{bc} > s_{cd}.$$

This establishes why the above condition is necessary for the existence of a sequential LMPE.

To show that this condition is also sufficient, we verify that no player can have a profitable deviation given the agreement probabilities pinned down in the first part of the proof. First, observe that c and b prefer delaying to offering to each other, as

$$\lim_{\delta\to 1} \delta(V_b + V_c) = s_{ab} - s_{ad} + \sigma_d + \sigma_c = s_{ab} + s_{cd} - s_{ad} > s_{bc},$$

where the last inequality holds by the uniqueness of the efficient match. By construction, d is indifferent between offering to a and delaying. Players c and d weakly prefer delaying to offering to each other as argued earlier in the proof. Players a and b weakly prefer offering to each other than delaying, as

$$\lim_{\delta\to 1} \frac{\delta(V_a + V_b) - s_{ab}}{1-\delta} = \frac{s_{cd} - 2s_{ad}}{2p} < 0,$$

which implies that $\delta V_a + \delta V_b \leq s_{ab}$ for all δ close to 1. Thus, for sufficiently high δ, a and b prefer offering to each other over delaying. As we have already established that b prefers delaying to offering to c, b's optimal offer strategy is to offer to a with probability 1 for all δ close to 1. Player a prefers offering to b than offering to d as

$$\lim_{\delta\to 1} \frac{s_{ab} - \delta V_b - s_{ad} - \delta V_d}{1-\delta} = \frac{2s_{ad} - s_{cd}}{2p} > 0.$$

Thus, it is optimal for a to offer to d with probability 1. Finally, mixing probabilities are consistent with a weakly efficient LMPE, as the probability that d and a agree converges to zero from above by

$$\lim_{\delta\to 1} \frac{q}{1-\delta} = \frac{2(2s_{ad} - s_{ab} - s_{cd})}{p(2s_{ab} - 2s_{ad} - s_{bc} + s_{cd})} > 0,$$

where the inequality holds because the numerator is positive by $2s_{ad} > s_{ab} + s_{cd}$, while the denominator is positive by $s_{ab} - s_{ad} > 0$ and $s_{ab} - s_{ad} > s_{bc} - s_{cd}$. All players thus best respond for δ close to 1, and so the condition we needed to show for the existence of a sequential LMPE is indeed sufficient. □

Proof of Proposition 7. Any simple assignment economy S can be represented by an unweighted bipartite network $L \subseteq P_1 \times P_2$ in which links capture the opportunity to generate a unit surplus. For any component of the network $\hat{L} \subseteq L$, let $\hat{L}_k \subseteq P_k$ denote the projection of $\hat{L}$ on P_k. The components of any such network must be of two types: (i) balanced components with the same number of players on both sides, $|\hat{L}_1| = |\hat{L}_2|$; (ii) unbalanced components with more players on one side $k \in \{1,2\}$, $|\hat{L}_k| > |\hat{L}_r|$. We begin by invoking a result implied by conclusions from Corominas-Bosch (2004).

Remark (Corominas-Bosch 2004). Any unweighted bipartite network $L \subseteq P_1 \times P_2$ possesses a subnetwork $L' \subseteq L$ such that the following statements hold:

(a) Any efficient match in L belongs to L'.

(b) In unbalanced components of L', the unique core payoff of all players on the long side is 0 and that of all players on the short side is 1.

(c) In balanced components of L', all players on one side receiving payoff $\beta \in [0,1]$ and all the remaining players receiving payoff $1-\beta$ is a core outcome.

By the assumptions imposed on the economy, observe that Rubinstein payoffs are $1/2$ in any efficient match. Thus, by the Corominas-Bosh remark, these payoffs are in the core if and only if all players who have a neighbor in L belong to balanced components in the resulting subnetwork L'. If so, pick any efficient match η. Suppose that any player $i \in N$ agrees with $\eta(i)$ with probability 1 when proposing, in any equilibrium path, active player set $A \in C^{\eta}(N)$. As in the proof of Proposition 2, these strategies imply that, in any equilibrium-path subgames $A \in C^{\eta}(N)$, the continuation value of any player $i \in A$ satisfies

$$V_i(A) = \frac{p}{1-\delta+2\delta p},$$

where p denotes the proposal probability of the representative player. If so, player i has no strictly profitable deviation when proposing, since all other players have the same continuation value as $\eta(i)$ and since

$$2V_i(A) = \frac{2p}{1-\delta(1-2p)} < 1 \quad \text{for all } \delta < 1.$$

Thus, the constructed strategies are an MPE. □

Proof of Remark 4. By Proposition 3 in the main document, a sufficient condition for the existence of an efficient MPE is that there exist no worker i and firm j who have a weakly profitable pairwise deviation when receiving their Rubinstein payoffs. As the efficient match is assortative, the core match of worker i is firm i. Thus, there is an assortative MPE if, for all $i \neq j$,

$$\frac{q_i}{p_i+q_i} S(i,i) + \frac{p_j}{p_j+q_j} S(j,j) > S(i,j),$$

where $S(k,k) = 0$ for all $k > \min\{w,f\}$.

If $w = f$, no agent is unmatched in the efficient match. Along with the condition that $p_i = q_i = p$, the above expression simplifies to

$$S(i,i) + S(j,j) > 2S(i,j) = S(i,j) + S(j,i), \tag{18}$$

where the equality follows from the condition that $S(i,j) = S(j,i)$. The existence of an efficient MPE then follows, as condition (18) holds by the increasing differences assumption (A3).[3]

To prove the second part of the remark, we show that there exist vertically differentiated markets for which there is no weakly efficiently LMPE whenever we relax one of three conditions in the statement of the result: (i) $w = f$; (ii) $p_i = q_i = p$ for all i; (iii) $S(i,j) = S(j,i)$ for all $i, j \leq \min\{w,f\}$. We do so by relying on the earlier results as well as the following two lemmas (which are proven below).

[3]This also follows by applying results from Eeckhout (2006).

LEMMA 1. *There is no weakly efficient LMPE in any market $S \in \bar{\mathcal{S}}$ satisfying (ii) and (iii) if $w = 2$, $f = 3$, and*

$$\max\{S(1,1)/2, S(1,3)\} + S(2,2) - S(2,3) < S(1,2), \qquad S(1,2)/2 < S(2,3).$$

LEMMA 2. *There is no weakly efficient LMPE if $w = f = 2$, $S(1,1) = 9$, $S(1,2) = S(2,1) = 6$, $S(2,2) = 4$, $p_1 = q_2 = 1/16$, and $p_2 = q_1 = 7/16$.*

Unbalanced Market. Lemma 1 identifies conditions on market $S \in \bar{\mathcal{S}}$ for the nonexistence of weakly efficient LMPE in markets satisfying (ii) and (iii), but violating (i). What remains to be shown is that these conditions are not vacuous and can be satisfied for some $S \in \bar{\mathcal{S}}$. Consider the economy

$$S(1,1) = 25; \qquad S(2,1) = S(1,2) = 20; \qquad S(2,2) = 16; \qquad S(1,3) = 12.$$

The economy trivially fulfills (A1), (A2), and (A3), implying that $S \in \bar{\mathcal{S}}$. Moreover, we have that $S(1,2)/2 = 10 < S(2,3) = 12$ and

$$\max\{S(1,1)/2, S(1,3)\} + S(2,2) - S(2,3) = 19 < S(1,2) = 20.$$

Thus the economy S satisfies the conditions of Lemma 1 and no weakly efficient LMPE exists.

Heterogeneous Probabilities. Lemma 2 provides an example in which conditions (i) and (iii) are satisfied, while condition (ii) is violated, and where there does not exist a weakly efficient LMPE.

Asymmetric Surpluses. Finally, consider the case in which $S(i,j) \neq S(j,i)$. Setting $w = f = 2$, we appeal directly to Remark 3 for a characterization of instances where there is no weakly efficient LMPE. To do so, it suffices to observe that the surpluses can satisfy (A1)–(A3) (and, thus, belong to $\bar{\mathcal{S}}$), while violating conditions for weak efficient LMPE existence identified in this result.

For convenience, in the proof of the next two lemmas, whenever $A = N$, we omit the dependence on the active player set A from value functions and proposal probabilities. □

PROOF OF LEMMA 1. By Proposition 3, an efficient MPE exists only if, for no worker–firm pair such that $i \neq j$, we have that

$$\frac{q_i}{p_i + q_i} S(i,i) + \frac{p_j}{p_j + q_j} S(j,j) < S(i,j),$$

where $S(k,k) = 0$ for $k > \min\{f, w\}$. But (ii) implies that this condition must be violated, as

$$S(2,2)/2 < S(1,2)/2 < S(2,3).$$

The first inequality holds by (A1) and the second holds by assumption. By Proposition 4, there is no strongly efficient LMPE if shifted Rubinstein payoffs are not in the core. In this case, the profiles of shifted Rubinstein payoffs are

$$\bar{\sigma}_1^w = \max\{S(1,1)/2, S(1,3)\} \quad \text{and} \quad \bar{\sigma}_2^w = S(2,3),$$
$$\bar{\sigma}_1^f = S(1,1) - \bar{\sigma}_1^w, \qquad \bar{\sigma}_2^f = S(2,2) - \bar{\sigma}_2^w \quad \text{and} \quad \bar{\sigma}_3^f = 0.$$

Thus, no strongly efficient LMPE exists, as $S(1,2) > \bar{\sigma}_1^w + \bar{\sigma}_2^f$.

By Proposition 5, any weakly efficient LMPE that is not strongly efficient must be a sequential LMPE. Next, we focus on ruling out the existence of a sequential LMPE. Recall that, by the proof of Proposition 2, we have that, at any active player set $A \in C(N)$, a player $i \in A \backslash E$ agrees with positive probability in any weakly efficient LMPE for all sufficiently high δ. So if only one worker is active at A, any weakly efficient LMPE is strongly efficient. So for the LMPE to be sequential, one core match must delay in the limit when $A = N$.

Let $A = N$. Suppose that worker 1 and firm 1 delay with probability 1 in the limit ($\beta_1^w = \beta_1^f = 0$). If so, with probability 1, worker 1 and firm 1 end up in the subgame $B_1 \subset N$ in which worker 2 and firm 2 exit. In this subgame, there is a unique MPE with limit payoffs

$$\bar{V}_1^w(B_1) = \max\{S(1,3), S(1,1)/2\} \quad \text{and} \quad \bar{V}_1^f(B_1) = S(1,1) - \bar{V}_1^w(B_1).$$

As this subgame is reached with probability 1, $\bar{V}_1^w = \bar{V}_1^w(B_1)$ and $\bar{V}_1^f = \bar{V}_1^f(B_1)$. In a weakly efficient LMPE, the probability that worker 2 and firm 3 agree must converge to zero. For worker 2 not to benefit by offering to firm 3 requires $\bar{V}_2^w \geq S(2,3)$. By Proposition 2, we also know that a weakly efficient LMPE would further require that $\bar{V}_2^w + \bar{V}_2^f = S(2,2)$. But for these conditions to hold at once, we would have that

$$S(2,2) - S(2,3) \geq \bar{V}_2^f.$$

Finally, in a weakly efficient LMPE, worker 1 must prefer delaying than offering to firm 2 for δ sufficiently high, which requires

$$\bar{V}_1^w + \bar{V}_2^f \geq S(1,2).$$

Combining these observations, we find that

$$\max\{S(1,3), S(1,1)/2\} + S(2,2) - S(2,3) \geq \bar{V}_1^w + \bar{V}_2^f \geq S(1,2),$$

which contradicts the assumption in the statement of our result. Thus, there is no sequential LMPE in which worker 1 and firm 1 delay.

Next, suppose instead that worker 2 and firm 2 delay with probability 1 in the limit, $\beta_2^w = \beta_2^f = 0$. If so, with probability 1, worker 2 and firm 2 end up in the subgame $B_2 \subset N$ in which worker 1 and firm 1 exit. In this subgame, there is a unique MPE with limit payoffs

$$\bar{V}_2^w(B_2) = S(2,3) \quad \text{and} \quad \bar{V}_2^f(B_2) = S(2,2) - \bar{V}_2^w(B_2).$$

As firm 3 delays with positive probability, $V_3^f(B_2) = 0$ for all δ sufficiently high. As this subgame is reached with probability 1, $\bar{V}_2^w = \bar{V}_2^w(B_2)$, $\bar{V}_2^f = \bar{V}_2^f(B_2)$, and $\bar{V}_3^f = \bar{V}_3^f(B_2)$. Suppose that there is a weakly efficient LMPE in which firm 3 and worker 1 agree with positive probability for all sufficiently high $\delta < 1$. If so, $S(1,3) \geq \delta(V_1^w + V_3^f)$. But in the limit, this implies that $\bar{V}_1^w = S(1,3)$. If so, however, worker 1 would benefit by offering to firm 2 with strictly positive probability in the limit by (A3), as

$$S(1,2) - \bar{V}_2^f = S(1,2) - S(2,2) + S(2,3) > S(1,3) = \bar{V}_1^w.$$

This contradicts the premise that this is a weakly efficient LMPE. Thus, firm 3 and worker 1 must reach agreement with probability 0 when all players are active for all sufficiently high δ.

Consider now the subgame in which worker 1 and firm 2 are not active. If so, for all δ sufficiently high, we have that worker 2's continuation value is $S(2,3)/\delta$. The latter follows because we have that $S(2,3) > S(2,1)/2$ by assumption, and because in the unique MPE of this subgame, firm 3 must mix between delaying and agreeing with worker 2 for all sufficiently high δ (as in Example 2 in the main document). Similarly, in the subgame in which worker 1 and firm 1 are not active, worker 2's continuation value is $S(2,3)/\delta$. Next, consider the value equation of worker 2 when all players are active, and recall that $\pi_{13}^w = \pi_{31}^f = 0$ for all sufficiently high δ. As we have characterized the value equation of worker 2 in every other subgame, the value equation at N simplifies to

$$V_2^w = p\big(\pi_{11}^f + \pi_{21}^f + \pi_{11}^w + \pi_{12}^w\big)S(2,3) + \big(1 - p\big(\pi_{11}^f + \pi_{21}^f + \pi_{11}^w + \pi_{12}^w\big)\big)\delta V_2^w.$$

From this, we conclude that for all sufficiently high $\delta < 1$, $\bar{V}_2^w < S(2,3)$. However, for worker 2 to delay with positive probability in the limit, it must be that for all sufficiently high $\delta < 1$,

$$\delta\big(\bar{V}_3^f + \bar{V}_2^w\big) \geq \delta\bar{V}_2^w \geq S(2,3),$$

which is a contradiction. So there can be no sequential LMPE. □

Proof of Lemma 2. As $w = f = 2$, shifted Rubinstein payoffs are the same as their Rubinstein payoffs. Moreover, these payoffs do not belong to the core, as, by the previous inequalities, $pS(1,1) + pS(2,2) < S(1,2)$. So there are no efficient MPEs by Propositions 3 and no strongly efficient LMPEs by Proposition 4. If so, any weakly efficient LMPE must be sequential by Proposition 5. Thus, we establish that there is no sequential LMPE. For notational ease, denote unconditional link-agreement probabilities as

$$v_{ij} = p_i\pi_{ij}^w + p_j\pi_{ji}^f \quad \text{for any } (i,j) \in W \times F.$$

First observe that in any weakly efficient LMPE that is not a (weakly) efficient MPE, we must have $\max\{v_{21}, v_{12}\} > 0$ for all $\delta < 1$ sufficiently high. Moreover, $\min\{v_{21}, v_{12}\} = 0$ for all $\delta < 1$ sufficiently high. If $v_{21} > 0$, then one of the following two conditions would hold:

$$\text{(a1) } S(1,2) - \delta V_2^w \geq S(1,1) - \delta V_1^w, \qquad \text{(a2) } S(1,2) - \delta V_1^f \geq S(2,2) - V_2^w.$$

If $v_{12} > 0$, then one of the following two conditions would hold:

$$\text{(b1) } S(1,2) - \delta V_1^w \geq S(2,2) - \delta V_2^w, \qquad \text{(b2) } S(1,2) - \delta V_2^f \geq S(1,1) - \delta V_1^f.$$

If (a1) and (b1) hold at once, then summing inequalities would yield a contradiction, as

$$2S(1,2) \geq S(1,1) + S(2,2).$$

But as the same argument applies when (a1) and (b2) hold, or when (a2) and (b1) hold, or when (a2) and (b2) hold, it must be that $\min\{v_{21}, v_{12}\} = 0$.

Define the following two-player active player sets, where the first entry denotes the active buyer and the second entry denotes the active seller:

$$B_1 = \{2,2\}, \qquad B_2 = \{2,1\}, \qquad B_3 = \{1,1\}, \qquad B_4 = \{1,2\}.$$

In the unique MPE of each of these subgames, we have that

$$\begin{aligned}
V_2^w(B_1) &= \frac{S(2,2)}{8(2-\delta)} \to \frac{1}{2} = \bar{V}_2^w(B_1) \quad \& \quad V_2^f(B_1) = \frac{7S(2,2)}{8(2-\delta)} \to \frac{7}{2} = \bar{V}_2^f(B_1),\\
V_2^w(B_2) &= \frac{S(1,2)}{2(8-7\delta)} \to 3 = \bar{V}_2^w(B_2) \quad \& \quad V_1^f(B_2) = \frac{S(1,2)}{2(8-7\delta)} \to 3 = \bar{V}_1^f(B_2),\\
V_1^w(B_3) &= \frac{7S(1,1)}{8(2-\delta)} \to \frac{63}{8} = \bar{V}_1^w(B_3) \quad \& \quad V_1^f(B_3) = \frac{S(1,1)}{8(2-\delta)} \to \frac{9}{8} = \bar{V}_1^f(B_3),\\
V_1^w(B_4) &= \frac{7S(2,1)}{2(8-\delta)} \to 3 = \bar{V}_1^w(B_4) \quad \& \quad V_2^f(B_4) = \frac{7S(2,1)}{2(8-\delta)} \to 3 = \bar{V}_2^f(B_4).
\end{aligned}$$

In any sequential LMPE, one of the two core pairs agrees in the limit, while the other does not by Propositions 2 and 5. Thus, four possible cases must be considered for δ sufficiently high:

Case A. $\pi_{21}^f + \pi_{22}^f < 1$, $\pi_{21}^w + \pi_{22}^w < 1$, and $\pi_{21}^f + \pi_{12}^w = 0$.

Case B. $\pi_{11}^f + \pi_{12}^f < 1$, $\pi_{11}^w + \pi_{12}^w < 1$, and $\pi_{21}^f + \pi_{12}^w = 0$.

Case C. $\pi_{21}^f + \pi_{22}^f < 1$, $\pi_{21}^w + \pi_{22}^w < 1$, and $\pi_{12}^f + \pi_{21}^w = 0$.

Case D. $\pi_{11}^f + \pi_{12}^f < 1$, $\pi_{11}^w + \pi_{12}^w < 1$, and $\pi_{12}^f + \pi_{21}^w = 0$.

Case A. For sufficiently high $\delta < 1$, the value equations of worker 2 and firm 2 amount to

$$\begin{aligned}
V_2^w &= (1 - v_{11})\delta V_2^w + v_{11}\delta V_2^w(B_1),\\
V_2^f &= (1 - v_{11} - v_{21})\delta V_2^f + v_{11}\delta V_2^w(B_1) + v_{21}\delta V_2^w(B_4).
\end{aligned}$$

Rearranging the first of these equations implies that $V_2^w < \delta V_2^w(B_1)$, while the second equation implies that $V_2^f < \delta V_2^f(B_1)$, where the latter holds for δ high enough, as

$\bar{V}_2^f(B_1) > \bar{V}_2^f(B_4)$. We, therefore, have that for all sufficiently high $\delta < 1$,

$$V_2^w + V_2^f < \delta\big(V_2^w(B_1) + V_2^f(B_1)\big) < S(2,2).$$

But as worker 2 and firm 2 delay in the limit, we must have $\delta(V_2^w + V_2^f) \geq S(2,2)$ for all sufficiently high $\delta < 1$, which is a contradiction. Hence, there is no such weakly efficient LMPE.

Case B. The argument here is identical to that for Case A. Writing out the value equations for firm 1 and worker 1, and rearranging them shows that $V_1^f < \delta V_1^f(B_3)$ and $V_1^w < \delta V_1^w(B_3)$. Combining these equations yields

$$V_1^f + V_1^w < \delta\big(V_1^f(B_3) + V_1^w(B_3)\big) < S(1,1).$$

But as firm 1 and worker 1 delay in the limit, $\delta(V_1^f + V_1^w) \geq S(1,1)$, which is a contradiction. So there is no such weakly efficient LMPE.

Case C. Writing out the value equations, we get

$$V_1^w = q_1\big(S(1,1) - \delta V_1^f\big) + \upsilon_{22}\delta V_1^w(B_3) + (1 - q_1 - \upsilon_{22})\delta V_1^w, \tag{19}$$

$$V_1^f = p_1\big(S(1,1) - \delta V_1^w\big) + \upsilon_{22}\delta V_1^f(B_3) + \upsilon_{12}\delta V_1^f(B_2) + (1 - p_1 - \upsilon_{22} - \upsilon_{12})\delta V_1^f, \tag{20}$$

$$V_2^w = \upsilon_{12}\delta V_2^w(B_2) + \upsilon_{11}\delta V_2^w(B_1) + (1 - \upsilon_{11} - \upsilon_{12})\delta V_2^w, \tag{21}$$

$$V_2^f = \upsilon_{11}\delta V_2^f(B_1) + (1 - \upsilon_{11})\delta V_2^f. \tag{22}$$

In any sequential LMPE, it must be that either $\upsilon_{22} = 0$ for all $\delta < 1$ sufficiently large or $\lim_{\delta\to 1} \upsilon_{22} = 0$. Allowing for these possibilities, there are three subcases to be considered. For all $\delta < 1$ sufficiently high, we could have (a) $\pi_{11}^w < 1$ and $\pi_{21}^f > 0$, (b) $\pi_{11}^w < 1$ and $\pi_{21}^f = 0$, or (c) $\pi_{11}^w = 1$ and $\pi_{21}^f > 0$. Furthermore, if $\pi_{11}^w < 1$, then

$$S(1,2) - \delta V_2^f = S(1,1) - \delta V_1^f, \tag{23}$$

while if $\pi_{21}^f > 0$, then

$$S(1,2) - \delta V_1^w = \delta V_2^f. \tag{24}$$

In subcase (a), both worker 1 and firm 2 play a mixed strategy. If so, the conjectured equilibrium is pinned down by a system of value equations that includes (19)–(24). Substituting (21) and (22) into (23) and (24) eliminates V_2^f and V_2^w. Rearranging these new equations creates expressions for V_1^w and V_1^f in terms of the mixing probability π_{11}^w only. Substituting these expressions into (19) and (20) to eliminate V_1^w and V_1^f then gives a system of two equations that depend only on the mixing probabilities (π_{11}^w, π_{21}^f, π_{22}^f, and π_{22}^w). Using these equations to eliminate π_{21}^f, we get an expression for π_{11}^w in terms of the parameters δ, π_{22}^f, and π_{22}^w. Taking limits and using that, in a sequential LMPE, we must have $\bar{\pi}_{22}^f + \bar{\pi}_{22}^w = 0$, we get that $\bar{\pi}_{11}^w = -\frac{1}{7}$. As mixing probabilities cannot be negative, this implies that we cannot have both worker 1 and firm 2 offering to each other with positive probability.

In subcase (b), worker 1 plays a mixed strategy but firm 2 does not. If so, the conjectured equilibrium is pinned down by a system of value equations that includes (19)–(23). Equation (22) identifies V_2^f in terms of parameters only and $\bar{V}_2^f = 7/2$. Manipulating (21), (19), and (20), we get an expression for V_1^f in terms of the mixing probabilities π_{11}^w, π_{22}^f, and π_{22}^w only. Moreover, using that $\bar{\pi}_{22}^f + \bar{\pi}_{22}^w = 0$ and $\bar{\pi}_{11}^w = 1$, we get that $\bar{V}_1^f = 3$. Combining these values with (23) yields a contradiction. So we cannot have that worker 1 offers to firm 2, but firm 2 does not offer to worker 1.

In subcase (c), firm 2 plays a mixed strategy, but worker 1 does not. If so, the conjectured equilibrium is pinned down by a system of value equations that includes (19)–(22) and (24). Combining (19) and (20) to eliminate V_1^f gives an expression for V_1^w in terms of mixing probabilities only. Substituting (22) into (24) to eliminate V_2^f gives a second expression for V_1^w just in terms of mixing probabilities. Using these two expression to eliminate V_1^w gives

$$\pi_{21}^f = -\frac{(7\delta-8)\big(\delta(83\delta-318)+192\big)\big(\delta\big(7\pi_{22}^f+\pi_{22}^w-16\big)+16\big)\big(\delta\big(7\pi_{22}^f+\pi_{22}^w-8\big)+16\big)}{7\delta\big(\delta\big(\delta\big(\delta\Phi+16\big(1701\pi_{22}^f+243\pi_{22}^w-5056\big)\big)-768\big(14\pi_{22}^f+2\pi_{22}^w-99\big)\big)-24{,}576\big)}, \tag{25}$$

for $\Phi = 7\delta(581\pi_{22}^f + 83\pi_{22}^w - 708) - 20{,}230\pi_{22}^f - 2890\pi_{22}^w + 34{,}592$.

If so, there are two further possibilities to consider: either $v_{22} > 0$ or $v_{22} = 0$. If $v_{22} > 0$, we have that $\delta(V_2^w + V_2^f) = S(2,2)$, as worker 2 and firm 2 must delay with positive probability for all sufficiently high $\delta < 1$ in any sequential LMPE. Substituting (22) and (21) into this expression to eliminate V_2^f and V_2^w, we get a second expression for π_{21}^f in terms of parameters only. Eliminating π_{21}^f by combining this equation with (25) yields an expression for π_{22}^w that is linear in π_{22}^f:

$$\pi_{22}^w = \Psi(\delta) - 7\pi_{22}^f.$$

Clearly, $\pi_{22}^w \le \Psi(\delta)$ as $\pi_{22}^f \ge 0$. Minor manipulations then establish that $\lim_{\delta\to 1}\Psi(\delta) = 0$ and

$$\lim_{\delta\to 1} \partial\Psi(\delta)/\partial\delta > 0.$$

This implies that $\Psi(\delta) < 0$ for all sufficiently high $\delta < 1$ and, thus, that $\pi_{22}^w < 0$ for all sufficiently high $\delta < 1$. But this is a contradiction and so $v_{22} = 0$.

Last, setting $v_{22} = 0$, (25) simplifies to

$$\pi_{21}^f = \frac{32(\delta-2)(\delta-1)(7\delta-8)\big(\delta(83\delta-318)+192\big)}{7\delta\big(\delta\big(\delta\big(\delta(1239\delta-8648)+20{,}224\big)-19{,}008\big)+6144\big)}.$$

This implies that $\bar{\pi}_{21}^{f} = 0$ and that

$$\lim_{\delta \to 1} \partial \pi_{21}^{f} / \partial \delta > 0.$$

Hence, for all sufficiently high values of $\delta < 1$, we would have $\pi_{21}^{f} < 0$. But again, this is a contradiction, as π_{21}^{f} is a probability, and so there is no sequential LMPE consistent with the proposal probabilities.

Case D. The argument is identical to that for Case C, but with worker 1 swapping roles with firm 2 and worker 2 swapping roles with firm 1. □

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