Calculations for Examples 1, 2 and 3 in 'Auction design in the presence of collusion' by Gregory Pavlov

Example 1.

The optimal cutoff in the Myerson mechanism is defined as $\theta^* = (1 - F(\theta^*))/f(\theta^*)$ if such θ^* exists, and $\underline{\theta}$ otherwise. Hence, the optimal cutoff as a function of the lower bound of the support is

$$\theta^*(\underline{\theta}) = \begin{cases} \frac{1}{2}\underline{\theta} + \frac{1}{2} & \text{if} \quad 0 \le \underline{\theta} \le 1 \\ \underline{\theta} & \text{if} \quad \underline{\theta} > 1 \end{cases}.$$

Thus

$$\theta^*(\underline{\theta}) + \frac{F(\theta^*(\underline{\theta}))}{f(\theta^*(\underline{\theta}))} = 2\theta^*(\underline{\theta}) - \underline{\theta} = \begin{cases} 1 & \text{if} \quad 0 \le \underline{\theta} \le 1 \\ \underline{\theta} & \text{if} \quad \underline{\theta} > 1 \end{cases}.$$

The Myerson revenue as a function of $\underline{\theta}$ is

$$\Pi^*(\underline{\theta}) = \theta^*(\underline{\theta})\left(1 - F^2(\theta^*(\underline{\theta}))\right) + \int_{\theta^*(\underline{\theta})}^{\underline{\theta}+1} (1 - F(\theta))^2 \, d\theta$$

$$= \theta^*(\underline{\theta})\left(1 - (\theta^*(\underline{\theta}) - \underline{\theta})^2\right) + \frac{1}{3}(1 + \underline{\theta} - \theta^*(\underline{\theta}))^3.$$

Substituting the values for the optimal cutoffs it is easy to verify that

$$R^*(\underline{\theta}) = \frac{\Pi^*(\underline{\theta})}{(1 - F^2(\theta^*(\underline{\theta})))} = \begin{cases} \frac{(\underline{\theta}+1)(5-\underline{\theta})}{3(3-\underline{\theta})} & \text{if} \quad 0 \le \underline{\theta} \le 1 \\ \underline{\theta} + \frac{1}{3} & \text{if} \quad \underline{\theta} > 1 \end{cases}.$$

Note that $\theta^*(\underline{\theta}) + (F(\theta^*(\underline{\theta}))/f(\theta^*(\underline{\theta})))$ is greater than $R^*(\underline{\theta})$ if and only if $\underline{\theta} \le \frac{1}{2}\left(7 - \sqrt{33}\right) \approx 0.628$.

Example 2.

Denote $\underline{\theta} = 10$. From the analysis of Example 1 we know that the Myerson revenue cannot be achieved. By Theorem 4 the optimal price is

$$R^{**} = \theta^{**} + \frac{F(\theta^{**})}{f(\theta^{**})}, \text{ where } \theta^{**} = \arg\max_{\theta \in [\underline{\theta}, \theta']} \left(\theta + \frac{F(\theta)}{f(\theta)}\right)(1 - F^n(\theta)),$$

and where θ' is such that

$$\int_{\theta'}^{\overline{\theta}} \left(\widetilde{\theta} - \frac{1 - F\left(\widetilde{\theta}\right)}{f\left(\widetilde{\theta}\right)} - \left(\theta' + \frac{F(\theta')}{f(\theta')}\right)\right) dF^n\left(\widetilde{\theta}\right) = 0.$$

Substituting the formulas for F and f in the last equation gives:

$$2\int\limits_{\theta'}^{\overline{\theta}}\left(2\widetilde{\theta}-2\theta'-1\right)\left(\widetilde{\theta}-\underline{\theta}\right)d\widetilde{\theta}=0.$$

Solving for θ' yields $\theta'=\underline{\theta}+\frac{1}{4}\left(\sqrt{33}-5\right)\approx 10.186$.

Next note that

$$\left(\theta+\frac{F(\theta)}{f(\theta)}\right)\left(1-F^2(\theta)\right)=\left(2\theta-\underline{\theta}\right)\left(1-\left(\theta-\underline{\theta}\right)^2\right).$$

This function achieves maximum at $\theta=\frac{1}{6}\sqrt{\underline{\theta}^2+12}+\frac{5}{6}\underline{\theta}\approx 10.097$. Since this number is smaller than θ', we conclude that we found θ^{**}, and thus the optimal price is

$$R=\theta^{**}+\frac{F(\theta^{**})}{f(\theta^{**})}=\frac{1}{3}\sqrt{\underline{\theta}^2+12}+\frac{2}{3}\underline{\theta}\approx 10.194.$$

The principal's revenue is $R\left(1-F^2(\theta^{**})\right)\approx 10.098$.

Now consider an alternative cartel interim efficient allocation with respect to the asymmetric menu $c(p_1,p_2)=(R+\varepsilon)p_1+Rp_2$, where R is the same as above and $\varepsilon=0.05$, relative to the weight functions $W_1=\overline{W}$ and $W_2=\widetilde{W}$ where

$$\widetilde{W}(\theta_2)=\begin{cases}0 & \text{if} \quad \theta_2\in\left[\underline{\theta},\underline{\theta}+\frac{3}{4}\right)\\ 1 & \text{if} \quad \theta_2\in\left[\underline{\theta}+\frac{3}{4},\underline{\theta}+1\right]\end{cases}.$$

The virtual utility of the first agent is

$$V_1(\theta_1)=\theta_1+\frac{F(\theta_1)-W_1(\theta_1)}{f(\theta_1)}=\theta_1+\frac{F(\theta_1)}{f(\theta_1)}=2\theta_1-\underline{\theta}\text{ for every }\theta_1.$$

The virtual utility of the second agent is

$$V_2(\theta_2)=\theta_2+\frac{F(\theta_2)-W_2(\theta_2)}{f(\theta_2)}=\begin{cases}2\theta_2-\underline{\theta} & \text{if} \quad \theta_2\in\left[\underline{\theta},\underline{\theta}+\frac{3}{4}\right)\\ 2\theta_2-\underline{\theta}-1 & \text{if} \quad \theta_2\in\left[\underline{\theta}+\frac{3}{4},\underline{\theta}+1\right]\end{cases}.$$

The solution has to satisfy the monotonicity constraints IC^*. A standard approach is to drop the monotonicity constraints and to replace the virtual utilities V_1, V_2 by the *ironed* virtual utilities $\overline{V}_1$, $\overline{V}_2$ in the problem.[1] The solution will satisfy the monotonicity constraints and will also be optimal in the original problem.

[1] See for example Guesnerie and Laffont (1984), Myerson (1981).

The ironed virtual utility of the first agent $\overline{V}_1$ is the same as V_1, since the original virtual utility is strictly increasing. The ironed virtual utility of the second agent $\overline{V}_2$ is different, since V_2 has a downward jump at $\underline{\theta}+\frac{3}{4}$.

$$\overline{V}_2(\theta_2)=\begin{cases} 2\theta_2-\underline{\theta} & \text{if} \quad \theta_2\in\left[\underline{\theta},\underline{\theta}+\frac{1}{2}\right) \\ \underline{\theta}+1 & \text{if} \quad \theta_2\in\left[\underline{\theta}+\frac{1}{2},\underline{\theta}+1\right] \end{cases}.$$

Hence, the cartel interim efficient allocation with respect to the new menu (Σ, c) relative to the weights $\overline{W}, \widetilde{W}$ solves the following problem.

$$\begin{aligned}
&\max_p E\left[\left(\overline{V}_1(\theta_1)-(R+\varepsilon)\right)p_1(\theta)+\left(\overline{V}_2(\theta_2)-R\right)p_2(\theta)\right] \\
&\text{subject to} \\
&F^*: \ p(\theta)\in\Sigma \text{ for every } (\theta_1,\theta_2)\in\Theta; \\
&BB^*: \ E\left[\left(2\theta_1-(\underline{\theta}+1)-(R+\varepsilon)\right)p_1(\theta)+\left(2\theta_2-(\underline{\theta}+1)-R\right)p_2(\theta)\right]\geq 0.
\end{aligned}$$

The optimal allocation satisfies

$$(p_1(\theta_1,\theta_2),p_2(\theta_1,\theta_2))=\begin{cases}(0,0) & \text{if} \quad (\theta_1,\theta_2)\in I_{00} \\ (1,0) & \text{if} \quad (\theta_1,\theta_2)\in I_{10} \\ (0,1) & \text{if} \quad (\theta_1,\theta_2)\in I_{01}\end{cases},$$

where I_{00}, I_{10} and I_{01} are the following subsets of the type space $\Theta=\left[\underline{\theta},\overline{\theta}\right]^2$.

$$\begin{aligned}
I_{00}&=\left\{\overline{V}_1(\theta_1)-R-\varepsilon<0,\overline{V}_2(\theta_2)-R<0\right\}=\left\{\theta_1<\frac{1}{2}(R+\underline{\theta}+\varepsilon),\,\theta_2<\frac{1}{2}(R+\underline{\theta})\right\}, \\
I_{10}&=\left\{\overline{V}_1(\theta_1)-R-\varepsilon>\max\left\{\overline{V}_2(\theta_2)-R,0\right\}\right\}=\left\{\theta_1>\max\left\{\theta_2+\frac{1}{2}\varepsilon,\frac{1}{2}(R+\underline{\theta}+\varepsilon)\right\}\right\}, \\
I_{01}&=\left\{\overline{V}_2(\theta_2)-R>\max\left\{\overline{V}_1(\theta_1)-R-\varepsilon,0\right\}\right\}=\left\{\theta_2>\max\left\{\theta_1-\frac{1}{2}\varepsilon,\frac{1}{2}(R+\underline{\theta})\right\}\right\}.
\end{aligned}$$

Denote $r=R-\underline{\theta}$. The principal's expected revenue is

$$(\underline{\theta}+r+\varepsilon)\Pr\{(\theta_1,\theta_2)\in I_{10}\}+(\underline{\theta}+r)\Pr\{(\theta_1,\theta_2)\in I_{01}\},$$

where

$$\Pr\{(\theta_1,\theta_2)\in I_{10}\} = \left(\frac{1}{2}-\frac{1}{2}\varepsilon\right)+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}r\right)\left(\frac{1}{2}-\frac{1}{2}r\right) = \frac{5}{8}-\frac{1}{2}\varepsilon-\frac{1}{8}r^2,$$
$$\Pr\{(\theta_1,\theta_2)\in I_{01}\} = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\varepsilon\right)+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}r+\varepsilon\right)\left(\frac{1}{2}-\frac{1}{2}r\right) = \frac{3}{8}-\frac{1}{8}r^2+\frac{1}{2}\varepsilon-\frac{1}{4}r\varepsilon.$$

Substituting the values $(\varepsilon,r,\underline{\theta}) = \left(0.05,\frac{1}{3}\left(\sqrt{112}-10\right),10\right)$ we get the revenue of approximately 10.103.

The interim expected allocations are

$$E_{\theta_2}\left[p_1\left(\theta_1,\theta_2\right)\right] = \begin{cases} 0 & \text{if} \quad \theta_1\in\left(\underline{\theta},\underline{\theta}+\frac{1}{2}r+\frac{1}{2}\varepsilon\right) \\ \theta_1-\underline{\theta}-\frac{1}{2}\varepsilon & \text{if} \quad \theta_1\in\left(\underline{\theta}+\frac{1}{2}r+\frac{\varepsilon}{2},\underline{\theta}+\frac{1}{2}+\frac{1}{2}\varepsilon\right) \\ 1 & \text{if} \quad \theta_1\in\left(\underline{\theta}+\frac{1}{2}+\frac{1}{2}\varepsilon,\underline{\theta}+1\right) \end{cases},$$
$$E_{\theta_1}\left[p_2\left(\theta_1,\theta_2\right)\right] = \begin{cases} 0 & \text{if} \quad \theta_2\in\left(\underline{\theta},\underline{\theta}+\frac{1}{2}r\right) \\ \theta_2-\underline{\theta}+\frac{1}{2}\varepsilon & \text{if} \quad \theta_2\in\left(\underline{\theta}+\frac{1}{2}r,\underline{\theta}+\frac{1}{2}\right) \\ \frac{1}{2}+\frac{1}{2}\varepsilon & \text{if} \quad \theta_2\in\left(\underline{\theta}+\frac{1}{2},\underline{\theta}+1\right) \end{cases}.$$

The monotonicity constraints IC^* are satisfied since the interim expected allocations are nondecreasing.

Finally, we verify that the constraint BB^* holds.

$$\int_{\underline{\theta}}^{\underline{\theta}+1}\left(2\left(\theta_1-\underline{\theta}\right)-\left(1+r+\varepsilon\right)\right)E_{\theta_2}\left[p_1\left(\theta_1,\theta_2\right)\right]d\theta_1+\int_{\underline{\theta}}^{\underline{\theta}+1}\left(2\left(\theta_2-\underline{\theta}\right)-\left(1+r\right)\right)E_{\theta_1}\left[p_2\left(\theta_1,\theta_2\right)\right]d\theta_2$$
$$= \int_{\underline{\theta}+\frac{1}{2}r+\frac{1}{2}\varepsilon}^{\underline{\theta}+\frac{1}{2}+\frac{1}{2}\varepsilon}\left(2\left(\theta_1-\underline{\theta}\right)-\left(1+r+\varepsilon\right)\right)\left(\theta_1-\underline{\theta}-\frac{1}{2}\varepsilon\right)d\theta_1+\int_{\underline{\theta}+\frac{1}{2}+\frac{\varepsilon}{2}}^{\underline{\theta}+1}\left(2\left(\theta_1-\underline{\theta}\right)-\left(1+r+\varepsilon\right)\right)1d\theta_1$$
$$+\int_{\underline{\theta}+\frac{1}{2}r}^{\underline{\theta}+\frac{1}{2}}\left(2\left(\theta_2-\underline{\theta}\right)-\left(1+r\right)\right)\left(\theta_2-\underline{\theta}+\frac{1}{2}\varepsilon\right)d\theta_2+\int_{\underline{\theta}+\frac{1}{2}}^{\underline{\theta}+1}\left(2\left(\theta_2-\underline{\theta}\right)-\left(1+r\right)\right)\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)d\theta_2$$
$$= \frac{1}{4}r\varepsilon-\frac{1}{2}\varepsilon-r+\frac{1}{4}r^2+\frac{1}{12}r^3+\frac{1}{4}\varepsilon^2+\frac{1}{8}r^2\varepsilon+\frac{7}{24}.$$

Substituting the values $(\varepsilon,r) = \left(0.05,\frac{1}{3}\left(\sqrt{112}-10\right)\right)$ we get approximately 0.086, which is nonnegative as required.

Example 3.

Consider an alternative cartel interim efficient allocation relative to the same weight functions as in Example 2, with respect to a cost function described by the lower envelope (vex) over the following: $c(0,0)=0$, $c(1,0)=R+\varepsilon$, $c(0,1)=c\left(\frac{1}{2},\frac{1}{2}\right)=R$.

After replacing the virtual utilities V_1, V_2 with the ironed virtual utilities $\overline{V}_1$, $\overline{V}_2$ (as in Example 2), we can say that the cartel interim efficient allocation with respect to the new menu (Σ, c) relative to the weights $\overline{W}, \widetilde{W}$ solves the following problem.

$$
\begin{aligned}
&\max_p E\left[\overline{V}_1(\theta_1)\,p_1(\theta)+\overline{V}_2(\theta_2)\,p_2(\theta)-c(p(\theta))\right]\\
&\text{subject to}\\
&F^*:\ p(\theta)\in\Sigma \text{ for every } (\theta_1,\theta_2)\in\Theta;\\
&BB^*:\ E\left[(2\theta_1-(\underline{\theta}+1))\,p_1(\theta)+(2\theta_2-(\underline{\theta}+1))\,p_2(\theta)-c(p(\theta))\right]\geq 0.
\end{aligned}
$$

The optimal allocation satisfies

$$
(p_1(\theta_1,\theta_2),p_2(\theta_1,\theta_2))=\begin{cases}(0,0) & \text{if} \quad (\theta_1,\theta_2)\in I_{00}\\ (1,0) & \text{if} \quad (\theta_1,\theta_2)\in I_{10}\\ (0,1) & \text{if} \quad (\theta_1,\theta_2)\in I_{01}\\ \left(\frac{1}{2},\frac{1}{2}\right) & \text{if} \quad (\theta_1,\theta_2)\in I_{\frac{1}{2}\frac{1}{2}}\end{cases},
$$

where I_{00}, I_{10} and I_{01} are the following subsets of the type space $\Theta=\left[\underline{\theta},\overline{\theta}\right]^2$.

$$
\begin{aligned}
I_{00}&=\left\{\overline{V}_1(\theta_1)-R-\varepsilon<0,\,\overline{V}_2(\theta_2)-R<0 \text{ and } \frac{1}{2}\overline{V}_1(\theta_1)+\frac{1}{2}\overline{V}_2(\theta_2)-R<0\right\}\\
&=\left\{\theta_1<\frac{1}{2}(\underline{\theta}+R+\varepsilon),\,\theta_2<\frac{1}{2}(\underline{\theta}+R) \text{ and } \theta_1+\theta_2<\underline{\theta}+R\right\},\\
I_{10}&=\left\{\overline{V}_1(\theta_1)-R-\varepsilon>\max\left\{\frac{1}{2}\overline{V}_1(\theta_1)+\frac{1}{2}\overline{V}_2(\theta_2)-R,0\right\}\right\}\\
&=\left\{\begin{array}{ll}\theta_1>\frac{1}{2}(\underline{\theta}+R+\varepsilon) \text{ and } \theta_1>\theta_2+\varepsilon & \text{if} \quad \theta_2<\underline{\theta}+\frac{1}{2}\\ \theta_1>\underline{\theta}+\frac{1}{2}+\varepsilon & \text{if} \quad \theta_2>\underline{\theta}+\frac{1}{2}\end{array}\right\},\\
I_{01}&=\left\{\overline{V}_2(\theta_2)-R>\max\left\{\frac{1}{2}\overline{V}_1(\theta_1)+\frac{1}{2}\overline{V}_2(\theta_2)-R,0\right\}\right\}\\
&=\left\{\theta_2>\frac{1}{2}(\underline{\theta}+R),\,\theta_2>\theta_1 \text{ and } \theta_1<\underline{\theta}+\frac{1}{2}\right\},\\
I_{\frac{1}{2}\frac{1}{2}}&=\left\{\begin{array}{c}\frac{1}{2}\overline{V}_1(\theta_1)+\frac{1}{2}\overline{V}_2(\theta_2)-R>\max\left\{\overline{V}_1(\theta_1)-R-\varepsilon,\overline{V}_2(\theta_2)-R\right\},\\ \frac{1}{2}\overline{V}_1(\theta_1)+\frac{1}{2}\overline{V}_2(\theta_2)-R>0\end{array}\right\}\\
&=\left\{\begin{array}{ll}\theta_1+\theta_2>\underline{\theta}+R,\,0<\theta_1-\theta_2<\varepsilon & \text{if} \quad \theta_2<\underline{\theta}+\frac{1}{2}\\ \underline{\theta}+\frac{1}{2}<\theta_1<\underline{\theta}+\frac{1}{2}+\varepsilon & \text{if} \quad \theta_2>\underline{\theta}+\frac{1}{2}\end{array}\right\}.
\end{aligned}
$$

Denote $r = R - \underline{\theta}$. The principal's expected revenue is

$$\left(\underline{\theta} + r\right) \Pr\left\{(\theta_1, \theta_2) \in I_{10} \cup I_{\frac{1}{2}\frac{1}{2}} \cup I_{01}\right\} + \varepsilon \Pr\left\{(\theta_1, \theta_2) \in I_{10}\right\},$$

where

$$\begin{aligned}
&\Pr\left\{(\theta_1, \theta_2) \in I_{10} \cup I_{\frac{1}{2}\frac{1}{2}} \cup I_{01}\right\} = 1 - \frac{1}{4}r^2 - \frac{1}{4}r\varepsilon + \frac{1}{8}\varepsilon^2, \\
&\Pr\left\{(\theta_1, \theta_2) \in I_{10}\right\} = \frac{1}{2} - \varepsilon + \frac{1}{4}\left(1 - r + \varepsilon\right)\left(\frac{1}{2}r - \frac{1}{2}\varepsilon + \frac{1}{2}\right) = \frac{5}{8} - \frac{1}{8}r^2 - \varepsilon + \frac{1}{4}r\varepsilon - \frac{1}{8}\varepsilon^2.
\end{aligned}$$

Substituting the values $(\varepsilon, r, \underline{\theta}) = \left(0.05, \frac{1}{3}\left(\sqrt{112} - 10\right), 10\right)$ we get the revenue of approximately 10.105.

The interim expected allocations are

$$E_{\theta_2}\left[p_1\left(\theta_1, \theta_2\right)\right] = \begin{cases} 0 & \text{if} & \theta_1 \in \left(\underline{\theta}, \underline{\theta} + \frac{1}{2}r\right) \\ \theta_1 - \underline{\theta} - \frac{1}{2}r & \text{if} & \theta_1 \in \left(\underline{\theta} + \frac{1}{2}r, \underline{\theta} + \frac{1}{2}r + \frac{1}{2}\varepsilon\right) \\ \theta_1 - \underline{\theta} - \frac{1}{2}\varepsilon & \text{if} & \theta_1 \in \left(\underline{\theta} + \frac{1}{2}r + \frac{1}{2}\varepsilon, \underline{\theta} + \frac{1}{2}\right) \\ \frac{1}{2} + \frac{1}{2}\left(\theta_1 - \underline{\theta}\right) - \frac{1}{2}\varepsilon & \text{if} & \theta_1 \in \left(\underline{\theta} + \frac{1}{2}, \underline{\theta} + \frac{1}{2} + \frac{1}{2}\varepsilon\right) \\ 1 & \text{if} & \theta_1 \in \left(\underline{\theta} + \frac{1}{2} + \frac{1}{2}\varepsilon, \underline{\theta} + 1\right) \end{cases},$$

$$E_{\theta_1}\left[p_2\left(\theta_1, \theta_2\right)\right] = \begin{cases} 0 & \text{if} & \theta_2 \in \left(\underline{\theta}, \underline{\theta} + \frac{1}{2}r - \frac{1}{2}\varepsilon\right) \\ \theta_2 - \underline{\theta} - \frac{1}{2}r + \frac{1}{2}\varepsilon & \text{if} & \theta_2 \in \left(\underline{\theta} + \frac{1}{2}r - \frac{1}{2}\varepsilon, \underline{\theta} + \frac{1}{2}r\right) \\ \theta_2 - \underline{\theta} + \frac{1}{2}\varepsilon & \text{if} & \theta_2 \in \left(\underline{\theta} + \frac{1}{2}r, \underline{\theta} + \frac{1}{2}\right) \\ \frac{1}{2} + \frac{1}{2}\varepsilon & \text{if} & \theta_2 \in \left(\underline{\theta} + \frac{1}{2}, \underline{\theta} + 1\right) \end{cases}.$$

The monotonicity constraints IC^* are satisfied since the interim expected allocations are nondecreasing.

Finally, we verify that the constraint BB^* holds, i.e.

$$E\left[\left(2\theta_1 - \left(\underline{\theta} + 1\right)\right)p_1\left(\theta\right) + \left(2\theta_2 - \left(\underline{\theta} + 1\right)\right)p_2\left(\theta\right) - c\left(p\left(\theta\right)\right)\right] \geq 0.$$

Integration over the set I_{00} yields 0.

Integration over the set I_{10} yields:

$$\begin{aligned}
&\int\limits_{\underline{\theta}+\frac{1}{2}r+\frac{1}{2}\varepsilon}^{\underline{\theta}+\frac{1}{2}+\varepsilon} \int\limits_{\underline{\theta}}^{\theta_1-\varepsilon} \left(2\left(\theta_1 - \underline{\theta}\right) - \left(r + \varepsilon + 1\right)\right) d\theta_2 d\theta_1 + \int\limits_{\underline{\theta}+\frac{1}{2}+\varepsilon}^{\underline{\theta}+1} \int\limits_{\underline{\theta}}^{\underline{\theta}+1} \left(2\left(\theta_1 - \underline{\theta}\right) - \left(r + \varepsilon + 1\right)\right) d\theta_2 d\theta_1 \\
= \;& \frac{1}{24}r^3 - \frac{1}{8}r^2\varepsilon + \frac{1}{8}r^2 + \frac{1}{8}r\varepsilon^2 + \frac{3}{4}r\varepsilon - \frac{5}{8}r - \frac{1}{24}\varepsilon^3 + \frac{1}{8}\varepsilon^2 - \frac{3}{8}\varepsilon + \frac{5}{24}.
\end{aligned}$$

Integration over the set I_{01} yields:

$$\int_{\underline{\theta}+\frac{1}{2}r}^{\underline{\theta}+\frac{1}{2}} \int_{\underline{\theta}}^{\theta_2} \left(2\left(\theta_2-\underline{\theta}\right)-(r+1)\right) d\theta_1 d\theta_2 + \int_{\underline{\theta}+\frac{1}{2}}^{\underline{\theta}+1} \int_{\underline{\theta}}^{\underline{\theta}+\frac{1}{2}} \left(2\left(\theta_1-\underline{\theta}\right)-(r+1)\right) d\theta_1 d\theta_2$$
$$= \frac{1}{24}r^3+\frac{1}{8}r^2-\frac{3}{8}r+\frac{1}{12}.$$

Integration over the set $I_{\frac{1}{2}\frac{1}{2}}$ yields:

$$\int_{\underline{\theta}+\frac{1}{2}r-\frac{1}{2}\varepsilon}^{\underline{\theta}+\frac{1}{2}r} \int_{2\underline{\theta}+r-\theta_2}^{\theta_2+\varepsilon} \left(\left(\theta_1-\underline{\theta}\right)+\left(\theta_2-\underline{\theta}\right)-(1+r)\right) d\theta_1 d\theta_2$$
$$+\int_{\underline{\theta}+\frac{1}{2}r}^{\underline{\theta}+\frac{1}{2}} \int_{\theta_2}^{\theta_2+\varepsilon} \left(\left(\theta_1-\underline{\theta}\right)+\left(\theta_2-\underline{\theta}\right)-(1+r)\right) d\theta_1 d\theta_2$$
$$+\int_{\underline{\theta}+\frac{1}{2}}^{\underline{\theta}+1} \int_{\underline{\theta}+\frac{1}{2}}^{\underline{\theta}+\frac{1}{2}+\varepsilon} \left(\left(\theta_1-\underline{\theta}\right)+\left(\theta_2-\underline{\theta}\right)-(1+r)\right) d\theta_1 d\theta_2$$
$$= \frac{1}{4}r^2\varepsilon-\frac{1}{4}r\varepsilon^2-\frac{1}{2}r\varepsilon+\frac{1}{12}\varepsilon^3+\frac{1}{4}\varepsilon^2-\frac{1}{8}\varepsilon.$$

Summing up all the integrals we get

$$\frac{1}{12}r^3+\frac{1}{8}r^2\varepsilon+\frac{1}{4}r^2-\frac{1}{8}r\varepsilon^2+\frac{1}{4}r\varepsilon-r+\frac{1}{24}\varepsilon^3+\frac{3}{8}\varepsilon^2-\frac{1}{2}\varepsilon+\frac{7}{24}.$$

Substituting the values $(\varepsilon, r) = \left(0.05, \frac{1}{3}\left(\sqrt{112}-10\right)\right)$ we get approximately 0.086, which is nonnegative as required.