Supplementary Material

Supplement to "Learning dynamics with social comparisons and limited memory"

(*Theoretical Economics*, Vol. 14, No. 1, January 2019, 135–172)

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B.1 *Proofs for Section 4*

Proof of Lemma 1. Since the game is finite it has a mixed strategy Nash equilibrium, and for any $\nu > 0$ and any such Nash equilibrium $\hat{\alpha} \in \Delta(A)$, there is an open neighborhood $\mathcal{U}$ of $\hat{\alpha}$ in which every element is a $\nu/2$ equilibrium. For N sufficiently large there is a grid point $\alpha \in \Delta^N(A)$ in $\mathcal{U}$ and, consequently, for large enough N/M, if the learners are content with this grid point, it is ν-robust. We may choose N/M large enough that the behavior of the committed agents does not move the grid point outside of $\mathcal{U}$. $\square$

Proof of Lemma 2. The hypothesis $\nu < g$ implies that ν-best responses are strict best responses,[1] and for each pure opponent's action a^{-j} for which some a^j is the (unique) strict best response, there is a $\gamma \geq 0$ such that a^j is also a best response to any mixed strategy $\alpha^{-j} \in \Delta^N(A^{-j})$ such that $\alpha^{-j}(a^{-j}) \geq 1 - \gamma$. Because A^{-j} is finite, there is a $\overline{\gamma}$ such that for all $\gamma \in (0, \overline{\gamma})$ the previous conclusion holds for all such best responses a^j, which proves the statement. $\square$

Proof of Lemma 3. If z is 0-robust, all learners are content and are playing a best response to the unique $\alpha^{-j}(z) \in \mathcal{A}^j$. By Assumption 1, content learners in each population j must be playing the same best response $\hat{a}^j$ and so z is pure. At z then, $\alpha^j(\hat{a}^j) > 1 - M/N$ for each j, so $\hat{a}^j$ is a strict best response to $\hat{a}^{-j}$ and so $\hat{a}$ is a pure strategy Nash equilibrium. Conversely, suppose that $\hat{a}$ is a pure equilibrium, and that all learners in each population j are playing $\hat{a}^j$ and are content. Since $\hat{a}$ is strict, by Lemma 2, there is an N/M sufficiently large such that for each j, the action $\hat{a}^j$ is a strict best response to

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[1]Note that this is true even for $\nu = 0$.

 https://doi.org/10.3982/TE2626

any $\alpha^{-j}(\hat{a}^{-j}) > 1 - N/M$ and for such N/M, there is a 0-robust state for the learners to play $\hat{a}$. $\square$

B.2 *Auxiliary result for Section 4*

The following result was noted in Section 4.

Lemma B.1. *When $\epsilon > 0$, the Markov process P_ϵ generated by the low-information model is irreducible and aperiodic.*

Proof. Pick any state $\hat{z}$ where $D^j(\hat{z}) = N - \#\Xi^j$ for each population j. Start with any state z_t and take any agent state $x_t \in X(z_t)$. There is probability $\epsilon^{\#\mathcal{T}^j}$ that all learners tremble and $\#\mathcal{T}^j = N - \#\Xi^j$, so $D^j(z_{t+1}) = N - \#\Xi^j$ for $j = 1, 2$. Take $\alpha^j_{t+1} \in \mathcal{A}^j(\hat{z})$ and choose $\hat{x}_{t+1} \in X(\hat{z})$ with an action assignment $\hat{\sigma}^j$ consistent with α^j_{t+1}. Starting at $\hat{x}_{t+1}$ there is probability $(1/\#A^1)^{N-\#\Xi^1}(1/\#A^2)^{N-\#\Xi^2}$ that all agents play $\hat{\sigma}$. There is probability $(1-p)^{2N-\#\Xi^1-\#\Xi^2}$ that all learners are inactive, so they all stay discontent; hence, they enter $\hat{z}$. Next, starting at $\hat{x} \in X(\hat{z})$, there is positive probability that no learner trembles and is active, so that learners all remain discontent. Starting at any state, the fact that there is a positive probability of reaching a single state $\hat{z}$ where the system may rest for any length of time with positive probability implies that the system is irreducible and aperiodic. $\square$

B.3 *Proof of Lemma 7*

For each pure Nash equilibrium $\hat{a} = (\hat{a}^j, \hat{a}^{-j})$ of the game G, define $\underline{\rho}^j_{\hat{a}}(\nu)$ for player j to be the maximum probability $\alpha^{-j}(\hat{a}^{-j})$ such that $\hat{a}^j$ is not the *only* ν-best response to $\hat{a}^{-j}$. Analogously, let $\rho^j_{\hat{a}}(\nu)$ for player j be the supremum probability $\alpha^{-j}(\hat{a}^{-j})$ such that $\hat{a}^j$ is not a ν-best response to $\hat{a}^{-j}$. From Assumption 1, $\underline{\rho}^j_{\hat{a}}(0) = \rho^j_{\hat{a}}(0)$ for $j = 1, 2$, and by Assumption 2, $\rho^j_{\hat{a}}(0) > 0$ for $j = 1, 2$. By definition of equilibrium, $\underline{\rho}^j_{\hat{a}}(0) < 1$ for $j = 1, 2$. Then, since $\underline{\rho}^j_{\hat{a}}(0) = \rho^j_{\hat{a}}(0)$, it follows that for $j = 1, 2$ we have $(1 - \rho^j_{\hat{a}}(0)) < (1 - \underline{\rho}^1_{\hat{a}}(0)) + (1 - \underline{\rho}^2_{\hat{a}}(0))$ and $\underline{\rho}^j_{\hat{a}}(0), \rho^j_{\hat{a}}(0) < 1$. Notice that $\underline{\rho}^j_{\hat{a}}(\nu)$ is continuous at $\nu = 0$ by Assumption 3, and that $\rho^j_{\hat{a}}(\nu)$ is continuous at $\nu = 0$ by Assumptions 2 and 3. Hence for sufficiently small $\nu > 0$ for each j, we still have $(1 - \rho^j_{\hat{a}}(\nu)) < (1 - \underline{\rho}^1_{\hat{a}}(\nu)) + (1 - \underline{\rho}^2_{\hat{a}}(\nu))$ and $\underline{\rho}^j_{\hat{a}}(\nu), \rho^j_{\hat{a}}(\nu) < 1$. Since there are finitely many pure equilibria, we may choose $\overline{\nu}$ so that these conditions are satisfied at all such equilibria for all $\nu \leq \overline{\nu}$.

Take any $\nu \leq \overline{\nu}$. Since $(1 - \rho^j_{\hat{a}}(\nu)) < (1 - \underline{\rho}^1_{\hat{a}}(\nu)) + (1 - \underline{\rho}^2_{\hat{a}}(\nu))$ and $\underline{\rho}^j_{\hat{a}}(\nu), \rho^j_{\hat{a}}(\nu) < 1$, it must be that for sufficiently large $N - M$, we have $(N - M)(1 - \rho^j_{\hat{a}}(\nu)) + 3 < (N - M)[(1 - \underline{\rho}^1_{\hat{a}}(\nu)) + (1 - \underline{\rho}^2_{\hat{a}}(\nu))]$. Denote by $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) the smallest (resp. the largest) integer greater than or equal to x (resp. not larger than x) so that $\overline{r}^j_z = \lceil (N - M)(1 - \rho^j_{\hat{a}}(\nu)) \rceil$ and $\underline{r}^j_z = \lfloor (N - M)(1 - \underline{\rho}^j_{\hat{a}}(\nu)) \rfloor$. Since there are finitely many equilibria, there is, therefore, a constant Γ such that for $N - M \geq \Gamma$, we have $\overline{r}^j_z \leq \underline{r}^1_z + \underline{r}^2_z$. Since $M \geq 1$, there is a γ such that for $N/M \geq \gamma$, we have $N - M \geq \Gamma$. Since $\rho^j_{\hat{a}}(\nu) > 0$, a similar argument establishes that $\underline{r}^j_z \geq 1$ for $j = 1, 2$.

B.4 *Absorbing states with stochastic best response with inertia dynamic*

We next provide a proof that in acyclic games with a unique best response to each pure action of the opponent, the support of the limit invariant distribution for the stochastic best-response dynamic with inertia contains only singleton absorbing sets, i.e., pure Nash equilibria.

Lemma B.2. *Every state that does not correspond to a pure strategy Nash equilibrium is transient under best response with inertia dynamic.*

Proof. Fix a time t and suppose that the state does not correspond to a pure strategy equilibrium. There is positive probability that this period all agents of one player, say j, do not adjust their play while all agents of the other player $-j$ play the best response to the date-t state, and that at date $t+1$ all agents of j play the best response to the date $t+1$ state while all agents of player $-j$ hold their actions fixed. Thus, there is positive probability that play in each population corresponds to a pure strategy from period $t+2$ on. Because the game is finite and acyclic, the best response path from this state converges to a pure strategy Nash equilibrium in a number of steps no greater than $J \equiv \#A^1 \times \#A^2$. There is positive probability that the populations will take turns adjusting, all of the $-j$ agents adjusting in periods $t, t+2, t+4, \ldots$ and all of the j agents adjusting at $t+1, t+3, t+5, \ldots$, so this equilibrium has probability bounded away from 0 of being reached in $2+J$ steps, showing the initial time t state is transient. $\square$

B.5 *Analysis of Example 1 (Continued)*

We first show that the low information dynamic with $T=1$ can also predict a different equilibrium than best response with inertia even when the best response with inertia dynamic has a singleton stochastically stable set. Suppose that a player obtains $\kappa > 0$ instead of 0 when choosing (B,B) against (C,C). To escape from (B,B), now about $N/(11-\kappa)$ of one population needs to mutate so this is the radius of (B,B). Our dynamic selects (A,A) as it continues to have the largest radius among pure strategy equilibria. The set S equal to the union of (B,B), (C,C), and (D,D) still contains all stochastically stable states. Let S' be the union of (A,A) and (B,B). The radius of S' is about $N/(11-\kappa)$ of one population, since escaping from S' requires this amount of agents to tremble to move to (C,C) or (D,D), and the co-radius is about $N/11$. Because the radius of S' is larger than its co-radius, the stochastically stable states are in S'. Combining this with the fact that they also lie in S shows that the unique stable state is (B,B), although its radius is smaller than the radius of (A,A).

We next show that when $T > 16$ and $\nu > 0$ in the high-information dynamic, the stochastically stable set consists exactly of the three equilibria (B,B), (C,C), and (D,D).

The block game G_1^{BCD} has seven Nash equilibria: the pure equilibria (B,B), (C,C), and (D,D), the binary mixed equilibria $((\frac{10}{11}C, \frac{1}{11}D),(\frac{10}{11}C, \frac{1}{11}D))$, $((\frac{10}{11}B, \frac{1}{11}D),(\frac{10}{11}B, \frac{1}{11}D))$, and $((\frac{10}{11}B, \frac{1}{11}C),(\frac{10}{11}B, \frac{1}{11}C))$, and a mixed equilibrium in which players randomize uniformly across B, C, and D.[2] Since all ν-robust states of this dynamic do not belong to

[2] Notice that when analyzing ν-robust states there is a subset of ν-robust states in a neighborhood of each mixed equilibrium.

the same circuit, we have to analyze circuits of circuits, but first we must establish what the structure of the circuits is.[3]

First, the three pure ν-robust states corresponding to the equilibria (B, B), (C, C), and (D, D) form a circuit, since we can move from one of these equilibria to the next with resistance equal to the common radius of these equilibria, which is about $N/11$.

The mixed ν-robust states corresponding to a binary mixed equilibrium have a simple structure. Consider a binary mixed equilibrium. As weight shifts from one of the two actions for one of the players to the other until we reach an extremal point at which a further shift causes the other player no longer to be playing a ν-best response for both of his actions. The structure of these equilibria is that of a square: for each player, there is a sequence of consecutive grid points between the two actions for which the opponent's two actions are a ν-best response. The complete collection of mixed ν-robust states corresponding to the binary mixed equilibrium is then the Cartesian product of these two sets. Each of these collections forms a circuit, but these collections are also in a common circuit with the pure equilibria that we call the *pure/binary* circuit.[4]

The structure of the mixed ν-robust states corresponding to the mixed equilibrium over B, C, and D is more complicated, since shifts are no longer one-dimensional for each player. However, the least resistance from a ν-robust state in the pure/binary circuit to some ν-robust state corresponding to the completely mixed equilibrium is about $N/2$.[5] Since this is greater than $N/11$, none of the ν-robust states corresponding to the completely mixed equilibrium are in the pure/binary circuit. Moreover, transitions from these mixed ν-robust states to the pure/binary circuit all have resistance 1.

Finally, (A, A) lies also in a separate circuit. This is because the least resistance from a ν-robust state in the pure/binary circuit to (A, A) is about $N/2$.[6] Being greater than $N/11$ implies that (A, A) does not belong to the pure/binary circuit. We can move from (A, A) to any ν-robust state in the pure/binary circuit with resistance $N/3$.

We next need to compute the *modified resistance* of going from one circuit to the next circuit, which is the least resistance from one circuit to the next circuit minus the least resistance path out of the circuit. We can then define *circuits of circuits*, which are collections of circuits such that for any pair of circuits in the collection, we have a route from one to the other such that at each step the modified resistance of moving from one circuit to the next is the least resistance of moving from the one circuit to any other.

Although the structure of ν-robust states corresponding to the mixed equilibrium over B, C, and D involves several circuits, note that transitions from the pure/binary circuit to any circuit containing such ν-robust states have a modified resistance of

[3]Recall that a circuit is a set of ν-robust states such that for any pair of states z, z' there exists a least resistance chain from z to z'.

[4]Because they can be reached from the corresponding pure equilibria with resistance equal to about $N/11$, while within each collection corresponding to a binary mixed equilibrium, there is always a ν-robust state from which we can move to either of the two pure equilibria in the support of the mixed equilibrium with resistance 1.

[5]As half of one population may play the remaining action to make it a ν-best response and be in the memory set.

[6]Since if $1/2$ of one population is playing in the block BCD, one of those strategies must earn at least $19/6$, while playing (A, A) yields no more than $5/2$.

$N/2 - N/11$.[7] Moving on the other direction requires a modified resistance of no more than 1. Hence, from Theorem 10 of Levine and Modica (2016) we know that the stochastically stable set belongs to the pure/binary circuit, and within that circuit we look for the largest radii: the three pure equilibria.

References

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Co-editor George J. Mailath handled this manuscript.

Manuscript received 1 September, 2016; final version accepted 25 April, 2018; available online 4 May, 2018.

[7] Transitions from a pure ν-robust state to any ν-robust state corresponding to the mixed equilibrium over B, C, and D have resistance of about $N/2$ while the radius of such a pure ν-robust state is about $N/11$. Moving on the other direction requires a modified resistance of no more than 1.