Supplementary Material

Supplement to "High frequency repeated games with costly monitoring": Appendix

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In this Appendix, we provide the proofs of the main theorem of the paper.

A.1. Proof of Lemma 4

By applying an affine transformation on the payoffs of the players, we can assume without loss of generality (w.l.o.g.) that $u(\beta) = (0, R)$ and $u(\gamma) = (R, 0)$, and then J_η is the line segment that connects $(\eta, R - 2\eta)$ and $(R - 2\eta, \eta)$. We prove that all the points on J_η are equilibrium payoffs.[1]

We construct an equilibrium in public strategies in which the expected discounted payoff after every public history is in J_η. The mixed-action pair α^n that the players play along the equilibrium path is either β or γ. Whenever the repeated game payoff x^n satisfies $x_1^n < \frac{R-\eta}{2}$, i.e., whenever x^n is in the upper half of the line segment J_η, the players play $\alpha^n = \beta$; otherwise they play $\alpha^n = \gamma$. Since the sum of payoffs of both players in both β and γ is R, while the sum of payoffs of both players in each point on J_η is $R - \eta$, the players must spend at every stage an expected amount of $(1 - r^\Delta)\eta$ on monitoring. The expected amount spent on monitoring by player i at stage n is $p_i^n c_i$. Consequently, define

$$p_i := \frac{(1 - r^\Delta)\eta}{c_i} \tag{7}$$

and instruct player 1 (resp. player 2) to monitor player 2 (resp. player 1) with probability p_1 (resp. p_2) in every stage in which the players play the mixed-action pair β (resp. γ).

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[1] The attentive reader will note that due to the affine transformation on payoffs, we in fact prove Lemma 4 only in the case that $u_1(\gamma) - u_1(\beta) = u_2(\beta) - u_2(\gamma)$. This assumption simplifies the calculations and highlights the main ideas of the proof.

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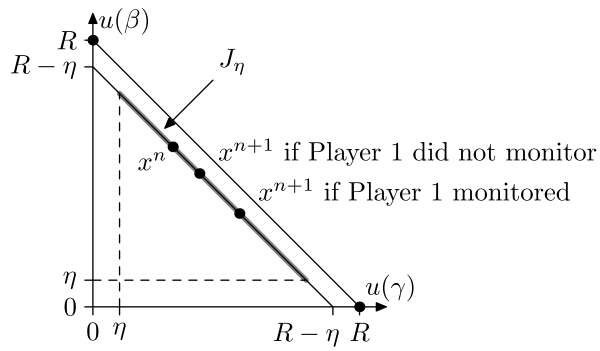

FIGURE 9. The construction in the proof of Lemma 4.

Condition A4 implies that $p_i < 1$. By Condition A3, we have $p_i > \frac{2(1-r^\Delta)}{r^\Delta \eta}$ for $i \in \{1,2\}$. Due to the discussion in Section 5.2 (see (3)), a deviation of player 2 (resp. player 1) to an action outside the support of β_2 (resp. γ_1) is not profitable, provided it triggers a punishment at the minmax level.

We now turn to the formal definition of the proposed equilibrium. For every stage $n = 1, 2, \ldots$, if $x_1^n < \frac{R-\eta}{2}$, then the following statements hold:

- We have $\alpha^n = \beta$: the players play the mixed-action pair β.
- We have $p_1^n = p_1$ and $p_2^n = 0$. That is, only player 1 monitors and he does it with probability p_1 given in (7).
- If player 1 monitors player 2, then x^{n+1} is given by (see Figure 9)

$$x_1^{n+1} := \frac{x_1^n + c_1}{r^\Delta}, \qquad x_2^{n+1} := R - \eta - \frac{x_1^n + c_1}{r^\Delta}.$$

- If player 1 does not monitor player 2, then x^{n+1} is given by

$$x_1^{n+1} := \frac{x_1^n}{r^\Delta}, \qquad x_2^{n+1} := R - \eta - \frac{x_1^n}{r^\Delta}.$$

If $x_1^n \geq \frac{R-\eta}{2}$, the play is defined analogously: the players play the mixed-action pair γ; player 1 does not monitor player 2; player 2 monitors player 1 with probability p_2 given in (7); if player 2 monitors player 1 at stage n, then

$$x_1^{n+1} := R - \eta - \frac{x_2^n + c_2}{r^\Delta}, \qquad x_2^{n+1} := \frac{x_2^n + c_2}{r^\Delta},$$

while if player 2 does not monitor player 1 at stage n, then

$$x_1^{n+1} := R - \eta - \frac{x_2^n}{r^\Delta}, \qquad x_2^{n+1} := \frac{x_2^n}{r^\Delta}.$$

Since $D_i^n = 0$ for every $n \in \mathbf{N}$, it is sufficient to verify that Conditions C2–C6 are satisfied. Since $x_i \geq \eta > 0 \geq v_i$ for every $x \in J_\eta$ and every $i = 1, 2$, Condition C2 holds. The definition of p_i and Condition A1 imply that Condition C3 holds. The verification that

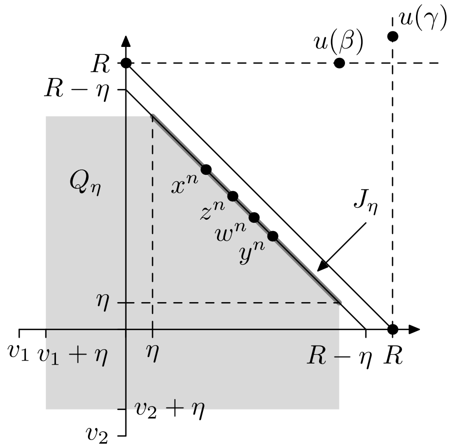

FIGURE 10. The construction in the proof of Lemma 4.

Conditions C4–C6 hold follows by simple algebraic manipulations. We provide here the verification of Condition C4. Assume then that $x_1^n \le \frac{R-\eta}{2}$, so that $\alpha^n = \beta$ and $p_1^n = p_1$. Since player 1 plays a best response at β, we have $u_1(a_1, \alpha_2^n) = 0$ for every $a_1 \in \operatorname{supp}(\alpha_1^n)$. Since $D_1^n = 0$, Condition C4 translates to $x_i^n = r^\Delta x_i^{n+1} - c_i \cdot \mathbf{1}_{I_i^n}$, which holds by the definition of x_1^{n+1}. Regarding player 2, since he is indifferent at β, we have $u_2(\alpha_1^n, a_2) = R$ for every $a_2 \in \operatorname{supp}(\alpha_2^n)$. Since $D_2^n = 0$, Condition C4 translates to

$$x_2^n = (1-r^\Delta)R + r^\Delta\left(p_1\left(R - \eta - \frac{x_1^n + c_1}{r^\Delta}\right) + (1-p_1)\left(R - \eta - \frac{x_1^n}{r^\Delta}\right)\right).$$

Since $x_2^n = R - \eta - x_1^n$, after cancelling the term $R - x_1^n$ from both of its sides, this equation reduces to $p_1 c_1 = (1-r^\Delta)\eta$, which holds by the definition of p_1.

A.2. PROOF OF LEMMA 5

By applying an affine transformation on the payoffs of the players, we can assume that $R_1^{(1)} = R_2^{(2)}$ and $u_1(\beta) = u_2(\gamma) = 0$. We prove the following result, which implies Lemma 5.

LEMMA 9. *Let $\beta = (\beta_1, \beta_2)$ and $\gamma = (\gamma_1, \gamma_2)$ be two mixed-action pairs and let $R > 0$ such that the following conditions hold:*

(i) Player 1 plays a best response at β, $u_1(\beta) \ge 0$, and $u_2(\beta_1, a_2) \ge R$ for every action $a_2 \in \operatorname{supp}(\beta_2)$.

(ii) Player 2 plays a best response at γ, $u_2(\gamma) \ge 0$, and $u_1(a_1, \gamma_2) \ge R$ for every action $a_1 \in \operatorname{supp}(\gamma_1)$.

Then the pentagon Q_η whose extreme points are (see Figure 10) $(v_1 + \eta, v_2 + \eta)$, $(v_1 + \eta, R - 2\eta)$, $(R - 2\eta, v_2 + \eta)$, $(\eta, R - 2\eta)$, and $(R - 2\eta, \eta)$ is a subset of $NE(r, c_1, c_2, \Delta)$, provided that the parameters r, c_1, c_2, Δ, and η satisfy Conditions A1–A4.

Proof. Let $\xi \in Q_\eta$. We construct an equilibrium with payoff ξ. The construction is similar to the construction in the proof of Lemma 4 and uses burning-money processes. Recall that $p_i = \frac{(1-r^\Delta)\eta}{c_i}$ for $i \in \{1,2\}$.

Fix a Nash equilibrium α^* in the base game. The play in the first stages depends on three parameters: a payoff vector $x \in Q_\eta$ close to J_η and two nonnegative integers k_1 and k_2. We first describe the play in the first $k := \max\{k_1, k_2\}$ stages, and then explain how to choose the parameters x, k_1, and k_2.

The players play as follows:

- They play the mixed action α^* for k stages.
- In the first k_1 stages, player 1 monitors player 2, and in the first k_2 stages, player 2 monitors player 1. If, for example, $k_1 < k_2$, then in the first k_1 stages, both players monitor each other, and in the following $k_2 - k_1$ stages, player 2 monitors player 1 while player 1 does not monitor player 2.
- From stage $k+1$ onward, the players implement an equilibrium with payoff x.

The payoff to each player i is then $(1 - r^{k\Delta})u_i(\alpha^*) + r^{k\Delta}x_i - (1 - r^{k_i\Delta})c$: in the first k stages, the players play α^*; in the first k_i stages, player i monitors player j, and the continuation payoff at stage k is x. We choose the parameters x, k_1, and k_2 to satisfy the following conditions.

Condition D1. We have $\xi_i = (1 - r^{k\Delta})u_i(\alpha^*) + r^{k\Delta}x_i - (1 - r^{k_i\Delta})c$.

Condition D2. We have $x_i \geq \eta$ for $i \in \{1,2\}$.

Condition D3. We have $R - \eta - 2c \leq x_1 + x_2 \leq R - \eta$.

Conditions D2 and D3 ensure that x is close to J_η: there is $y \in J_\eta$ that dominates x and satisfies $y_i - x_i \leq 3c$. Fix then such $y \in J_\eta$ and set

$$x^1 := y, \qquad D^1 := y - x.$$

For every stage $n > k$, if $x_1^n \leq \frac{R-\eta}{2}$, the following statements hold:

- We have $\alpha^n = \beta$: the players play the mixed-action pair β.
- If $D_i^n \geq c_i$, then $p_i^n = 1$: a player with a high debt monitors the other player (and burns money).
- If $D_1^n < c_1$, then $p_1^n = p_1$; if $D_2^n < c_2$, then $p_2^n = 0$: Only player 1 monitors with positive probability. Recall that player 1 plays a best response at β, so that he cannot gain by deviating from β_1; hence he does not have to be monitored.
- If player 1 monitors player 2 and finds out that player 2 played an action $a_2 \notin \mathrm{supp}(\beta_2)$, then from stage $n+1$ onward he switches to a punishment strategy that reduces player 2's payoff to $v_2 + \eta$.

In case $x_1^n > \frac{R-\eta}{2}$, the play is analogous: the players play the mixed-action pair γ, a player with a debt of at least c_i monitors the other with probability 1; if player 1's (resp.

	If $x_1^n \leq \frac{R-\eta}{2}$ and ...	x^{n+1}	$r^\Delta D_1^{n+1}$	$r^\Delta D_2^{n+1}$
1	$D_1^n \geq c_1, D_2^n \geq c_2$	w^n	$D_1^n - c_1 + (1-r^\Delta)u_1(\beta) + (1-r^\Delta)\eta$	$D_2^n - c_2 + (1-r^\Delta)(u_2(\beta_1, a_2) - R)$
2	$D_1^n \geq c_1, D_2^n < c_2$	w^n	$D_1^n - c_1 + (1-r^\Delta)u_1(\beta) + (1-r^\Delta)\eta$	$D_2^n + (1-r^\Delta)(u_2(\beta_1, a_2) - R)$
3	$D_1^n < c_1, I_1^n, D_2^n \geq c_2$	y^n	$D_1^n + (1-r^\Delta)u_1(\beta)$	$D_2^n - c_2 + \frac{c_1}{\eta}(u_2(\beta_1, a_2) - R)$
4	$D_1^n < c_1, I_1^n, D_2^n < c_2$	y^n	$D_1^n + (1-r^\Delta)u_1(\beta)$	$D_2^n + \frac{c_1}{\eta}(u_2(\beta_1, a_2) - R)$
5	$D_1^n < c_1, \neg I_1^n, D_2^n \geq c_2$	z^n	$D_1^n + (1-r^\Delta)u_1(\beta)$	$D_2^n - c_2 + (1-r^\Delta)(u_2(\beta) - R)$
6	$D_1^n < c_1, \neg I_1^n, D_2^n < c_2$	z^n	$D_1^n + (1-r^\Delta)u_1(\beta)$	$D_2^n + (1-r^\Delta)(u_2(\beta) - R)$

FIGURE 11. The continuation payoff and the debt.

player 2's) debt is lower than c_1 (resp. c_2), then he does not monitor player 2 (resp. monitors player 2 with probability p_2); and if player 2 monitors player 1 and finds out that player 1 played an action outside the support of γ_1, then he switches to a minmax strategy against player 1.

It is left to define the processes $(x^n)_{n\in\mathbf{N}}$ and $(D^n)_{n\in\mathbf{N}}$ so that (a) the discounted payoff is x and (b) no player has an incentive to deviate. We define these two processes recursively. Suppose that $x^n \in J_\eta$ and $D^n \in \mathbb{R}_+^2$ have already been defined, and assume that $x_i^n - D_i^n \geq v_i + \eta$ for $i = 1, 2$. If $x_1^n \leq \frac{R-\eta}{2}$, define $y^n, z^n, w^n \in J_\eta$ as (see Figure 10)

$$w_1^n := \frac{x_1^n}{r^\Delta} + \eta(1-r^\Delta) = \frac{R - x_2^n}{r^\Delta} - \eta, \qquad w_2^n := \frac{x_2^n - (1-r^\Delta)R}{r^\Delta},$$

$$y_1^n := \frac{x_1^n + c}{r^\Delta}, \qquad y_2^n := R - \eta - \frac{x_1^n + c}{r^\Delta},$$

$$z_1^n := \frac{x_1^n}{r^\Delta}, \qquad z_2^n := R - \eta - \frac{x_1^n}{r^\Delta}.$$

The payoff vectors y^n and z^n are the continuation payoffs when player 1's debt is lower than c_1. These quantities are similar to x^{n+1} in the proof of Lemma 4, where y^n (resp. z^n) was the continuation payoff when player 1 monitored (resp. did not monitor) player 2. The vector w^n is the continuation payoff when player 1's debt is higher than c_1. Note that these three vectors are on J_η.

When $x_1^n > \frac{R-\eta}{2}$, the roles of the two players are exchanged: they play the mixed-action pair γ, player 2 monitors player 1 with positive probability, and the continuation payoffs w^n, y^n, and z^n are defined analogously.

The continuation payoff x^{n+1} and the debt D^{n+1} are given by the table in Figure 11. For the sake of convenience we provide the quantity $r^\Delta D_i^{n+1}$ for $i = 1, 2$. When player 1 monitors player 2, the action a_2 that player 2 plays at stage n is common knowledge, and D^{n+1} can depend on it. Recall that the event that player i monitors player j at stage n is denoted by I_i^n.

We explain below the intuition behind the definition of the burning-money process for player 1.

- Whenever a player monitors the other to burn money (i.e., $D_i^n \geq c_i$, which implies $p_i^n = 1$), his debt decreases by c_i. For instance, in lines 1 and 2, the first part of $r^\Delta D_1^{n+1}$ is $D_1^n - c_i$.

- When $D_1^n \geq c_i$, as in lines 1 and 2, the last part of $r^\Delta D_1^{n+1}$ is $(1-r^\Delta)\eta$. The reason for adding this term is that the construction assumes that the stage payoff is $(0, R)$ or $(R, 0)$, so that the sum of stage payoffs of the two players is R. However, the sum of payoffs in all points in J_η is $R - \eta$. Since both the continuation payoff and the current payoff should be on J_η, we need to add $(1-r^\Delta)\eta$ to the debt of the players.

- Whenever $D_1^n \geq c_1$, the continuation payoff w^{n+1} is defined to satisfy Condition C6 for player 1. Indeed, using the definition of w^n and D^n,

$$
\begin{aligned}
&(1-r^\Delta)u_1(\beta) - c_1 + r^\Delta w_1 - r^\Delta D_1^{n+1} \\
&\quad = (1-r^\Delta)u_1(\beta) - c_1 + r^\Delta\left(\frac{x_1^n}{r^\Delta} + \eta(1-r^\Delta)\right) \\
&\qquad - \left(D_1^n - c_1 + (1-r^\Delta)u_1(\beta) + (1-r^\Delta)\eta\right) \\
&\quad = x_1^n - D_1^n.
\end{aligned}
$$

- Whenever player 1 decides randomly whether to monitor player 2 (lines 3–6 in the table in Figure 11), the continuation payoff x^{n+1} is given as in the proof of Lemma 4. In case player 1 monitors player 2, the continuation payoff is y^n; otherwise it is z^n. The continuation payoffs were chosen to ensure that Condition C4 holds. We verify this for line 3:

$$
\begin{aligned}
&(1-r^\Delta)u_1(\beta) + r^\Delta y_1 - r^\Delta D_1^{n+1} \\
&\quad = (1-r^\Delta)u_1(\beta) - c_1 + r^\Delta y_1 - \left[D_1^n + (1-r^\Delta)u_1(\beta)\right] \\
&\quad = (1-r^\Delta)u_1(\beta) - c_1 + r^\Delta\left[\frac{x_1^n + c_1}{r^\Delta}\right] - \left[D_1^n + (1-r^\Delta)u_1(\beta)\right] \\
&\quad = x_1^n - D_1^n.
\end{aligned}
$$

We now verify that this definition satisfies the conditions listed in Section 5.3. In the first k stages, no player can profit by deviating, and so we need to verify these conditions only from stage k onward. We first check that Condition C2 holds. One can verify that if $D_i^n \geq c_i$, then $x_i^{n+1} - D_i^{n+1} \geq x_i^n - D_i^n$, and the result follows by induction; otherwise, $D_i^{n+1} \leq \frac{c_i}{r^\Delta}$. When $x_1^n \leq \frac{R_\eta}{2}$, we have $y_1^n, z_1^n \geq x_1^n \geq \eta$, and therefore $x_1^{n+1} - D_1^{n+1} \geq 0$. Since $x_2^n \leq \frac{R}{2}$, it follows that $x_2^{n+1} - D_2^{n+1} \geq 0$. The definition of p_i together with Condition A3 implies that Condition C3 is satisfied. The verification that Conditions C4–C6 hold amounts to substituting the quantities defined above in the relevant equations, as illustrated above. $\square$

A.3. Proof of Lemma 6

We denote $u(a) = (A, B)$, and distinguish between four cases that are handled separately (see Figure 12).

Case 1. We have $u_1(a) \geq t_1^2$ and $u_2(a) \geq t_2^1$.

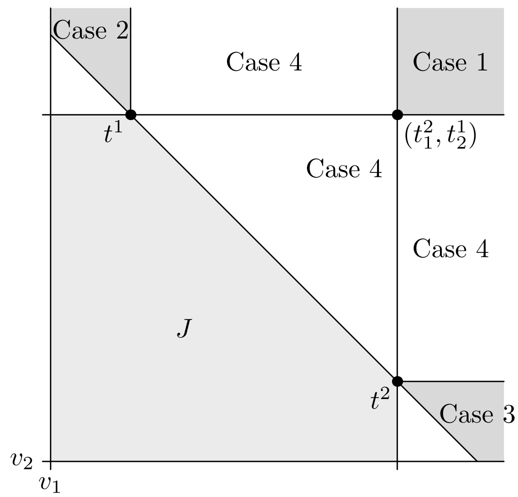

FIGURE 12. The four cases in the proof of Lemma 5.

Case 2. We have $u_1(a) < t_1^1$ and $u_2(a) > t_2^1$.

Case 3. We have $u_2(a) < t_2^2$ and $u_1(a) > t_1^2$, which is analogous to Case 2.

Case 4. Cases 1–3 do not hold.

The construction in this section does not employ burning-money processes. Rather, we use a recursive construction: we identify a set J_1 of payoff vectors, which can be arbitrarily close to J', and for every payoff vector $g \in J_1$, we define a one-shot auxiliary game in which (a) the payoffs are the stage payoff in the base game plus a continuation payoff, (b) the continuation payoff, which depends on the players' choices, are in $J \cup J_1$, and (c) there is an equilibrium whose payoff is g and in which each player monitors the other with probability p that satisfies (3).

A.4. CASE 1: $u_1(a) \geq t_1^2$ AND $u_2(a) \geq t_2^1$

Roughly, we prove that all the points in the triangle whose extreme points are t^1, t^2, and (t_1^2, t_2^1) are in $E(r, c, \Delta)$, provided that c_1, c_2, and Δ are sufficiently small. Fix $\eta < \min\{\frac{t_1^2 - u_1(\beta^*)}{7}, \frac{t_2^1 - u_2(\gamma^*)}{7}\}$. Denote by $-\alpha$ the slope of the line segment $[t^1, t^2]$. Set (see Figure 13)

$$w^1 := (t_1^1 + \eta, t_2^1 - 2\alpha\eta), \qquad w^2 := (t_1^2 - 2\eta, t_2^2 + \alpha\eta) \tag{8}$$

$$s^1 := (t_1^1 + 2\eta, t_2^1 - 3\alpha\eta), \qquad s^2 := (t_1^2 - 3\eta, t_2^2 + 2\alpha\eta) \tag{9}$$

$$\widehat{z} := (t_1^2 - 4\eta, t_2^1 - 4\alpha\eta).$$

Since J is an asymptotic set of Nash equilibrium payoffs, Lemma 5 implies that all the points in the pentagon J_0 whose extreme points are (v_1, v_2), (v_1, w_2^1), (w_1^2, v_2), w^1, and w^2

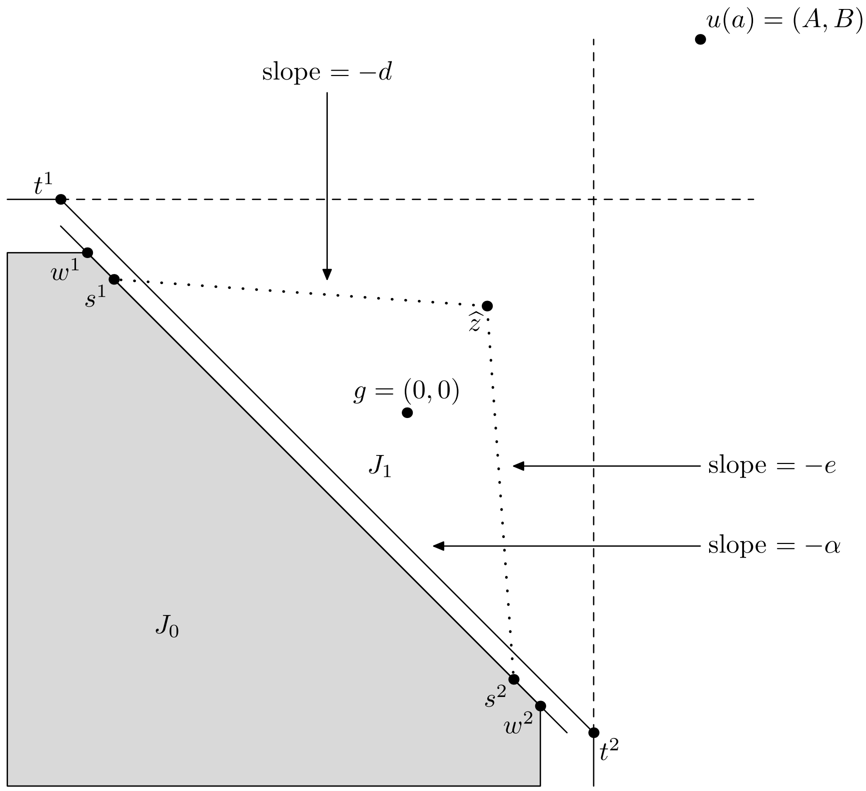

FIGURE 13. Case 1.

are in $E(r, c_1, c_2, \Delta)$, provided that c_1, c_2, and Δ are sufficiently small. Denote the slope of the line segment $[s^1, \widehat{z}]$ by $-d := -\frac{\alpha\eta}{t_1^2 - u_1(\beta^*) - 6\eta}$, and denote the slope of the line segment $[\widehat{z}, s^2]$ by $-e := -\frac{t_2^1 - u_2(\gamma^*) - 6\alpha\eta}{\eta}$ (see Figure 13).

By the choice of η, we can state the following condition.

Condition E0. We have $s_1^1 < \widehat{z}_1$ and $s_2^2 < \widehat{z}_2$.

Hence, $e > d > 0$.

Assume that c_1, c_2, and Δ are sufficiently small to satisfy Conditions A1–A4, as well as the following conditions for $i \in \{1, 2\}$.

Condition E1. We have $1 - r^\Delta < c_i < \frac{\eta}{6}$.

Condition E2. We have $\frac{1}{\eta} c_i < \frac{1}{4} Be$, $\frac{1}{4} Ad < \frac{\eta}{2(1 - r^\Delta)}$.

Condition E3. We have $\frac{2c_i}{dr^\Delta}$, $\frac{2ec_i}{r^\Delta} < \eta$.

Condition E4. We have $r^\Delta > \frac{1}{2}$ and $4(1 - r^\Delta) < \eta$.

We show that all points in the triangle J_1, whose extreme points are s^1, s^2, and $\widehat{z}$, are in $E(r, c_1, c_2, \Delta)$.

		Player 2: Don't Monitor	Player 2: Monitor
Player 1	Don't Monitor	$(1-r^\Delta)A - r^\Delta x$ $(1-r^\Delta)B - r^\Delta y$	$(1-r^\Delta)A - r^\Delta \zeta_1$ $(1-r^\Delta)B + r^\Delta d\zeta_1 - c_2$
	Monitor	$(1-r^\Delta)A + r^\Delta \frac{\zeta_2}{e} - c_1$ $(1-r^\Delta)B - r^\Delta \zeta_2$	$(1-r^\Delta)A - c_1$ $(1-r^\Delta)B - c_2$

FIGURE 14. The game $G(\zeta_1, \zeta_2, x, y)$.

		Player 2: Don't Monitor	Player 2: Monitor
Player 1	Don't Monitor	$-x, -y$	$-\zeta_1, d\zeta_1$
	Monitor	$\frac{\zeta_2}{e}, -\zeta_2$	$0, 0$

FIGURE 15. The continuation payoffs that underlie the game $G(\zeta_1, \zeta_2, x, y)$.

Fix a point g in the triangle J_1, and for the calculations below add a constant to the payoff so that $g = (0, 0)$. We now describe a 2×2 one-shot auxiliary game $G(\zeta_1, \zeta_2, x, y)$ whose payoffs depend on four positive real numbers ζ_1, ζ_2, x, and y, and an equilibrium in that game that yields the payoff $(0, 0)$.

- Each player has two actions, "monitor" and "don't monitor."
- The payoff function is given by the table in Figure 14, in which at each entry, player 1's payoff appears at the top and player 2's payoff appears at the bottom.

The payoff is calculated as if, in the original repeated game with costly observation, the players play the pure action pair a, each player chooses whether to monitor the other player, and the continuation payoffs, which depend on the identity of the players who chose to monitor, are given by the matrix in Figure 15.

Because the slopes of the line segments that define J_1 are $-d$ and $-e$, the vectors $(\frac{\zeta_2}{e}, -\zeta_2)$ and $(-\zeta_1, d\zeta_1)$ are in $J_0 \cup J_1$, provided that ζ_1 and ζ_2 are sufficiently small. Recall that $g = (0, 0)$ is in J_1. We have to ensure that $x, y \in [0, \eta]$ so that $(-x, -y)$ is in $J_0 \cup J_1$.

Set

$$p := \frac{2(1-r^\Delta)}{r^\Delta \eta}.$$

We find positive numbers ζ_1, ζ_2, x, and y such that the pair of strategies in which each player monitors the other with probability p is an equilibrium of $G(\zeta_1, \zeta_2, x, y)$ with payoff $(0, 0)$. By (3), this implies that in the repeated game, no player can profit by a deviation to an action that he is supposed to play with probability 0, provided such a deviation leads to a punishment at the maxmin level. By solving the indifference conditions of the players, we obtain that if

$$\zeta_2 = \frac{e(c_1 - (1-r^\Delta)A)}{r^\Delta(1-p)}, \tag{10}$$

$$\zeta_1 = \frac{c_2 - (1-r^\Delta)B}{dr^\Delta(1-p)}, \tag{11}$$

$$x = \frac{(1-r^\Delta)Ad(1-p) - p(c_2 - (1-r^\Delta)B)}{d(1-p)^2}, \tag{12}$$

$$y = \frac{(1-r^\Delta)B(1-p) - pe(c_1 - (1-r^\Delta)A)}{(1-p)^2}, \tag{13}$$

then having both players monitor each other with probability p is an equilibrium of $G(\zeta_1, \zeta_2, x, y)$.

Condition E1 implies that ζ_1 and ζ_2 are positive; Conditions E2 and E4 imply that x and y are positive; Conditions E3 and E4 imply that ζ_1 and ζ_2 are smaller than η; and Condition E2 implies that x and y are smaller than η. This concludes the proof for Case 1.

A.5. Case 2: $u_1(a) < t_1^2$ and $u_2(a) > t_2^1$

The proof in this case is similar to the proof in Case 1, with a different definition of $\widehat{z}$, and the calculations are slightly more cumbersome. Recall that $u(a) = (A, B)$, so that in this case $A < t_1^1$ and $B > t_2^1$ (see Figure 16). The slope of the line segment $[u(a), t^2]$ is $\frac{t_2^2 - B}{t_1^2 - A} < 0$. Fix $\eta > 0$ sufficiently small to satisfy $\eta < -\frac{t_2^2 - B}{t_1^2 - A}$, and define the points w^1, w^2, s^1, and s^2 as in Case 1 (see (8) and (9)). Set

$$e := -\frac{t_2^2 - B}{t_1^2 - A} - \eta > 0.$$

Consider the line with slope $-e$ that passes through s^2, and let $\widehat{z}$ be the point on this line that satisfies $\widehat{z}_2 = t_2^1 - 4\eta$. Let d be the slope of the line segment $[s^1, \widehat{z}]$. Then $0 < d < e < \infty$. Suppose that Conditions E2–E4 hold for the d and e defined here, as well as the following two conditions.

Condition E5. We have $ec_1 < \frac{1}{8}$, $e(1-r^\Delta)(-A) < \frac{1}{8}$, and $c_2 < 4d\eta$.

Condition E6. We have $d(-A)\eta^2 > 8e(c_2 + dc_1)$.

Denote by J_1 the triangle whose extreme points are s^1, s^2, and $\widehat{z}$. We prove that all the points in the triangle J_1 are in $E(r, c_1, c_2, \Delta)$, provided that c_1, c_2, and Δ are sufficiently small.

Fix a point $g \in J_1$ and for the calculation below add a constant to the payoffs so that $g = (0,0)$. Because $g = (0,0)$ is below the line segment $[u(a), t^2]$,

$$\frac{B}{-A} > e + \eta.$$

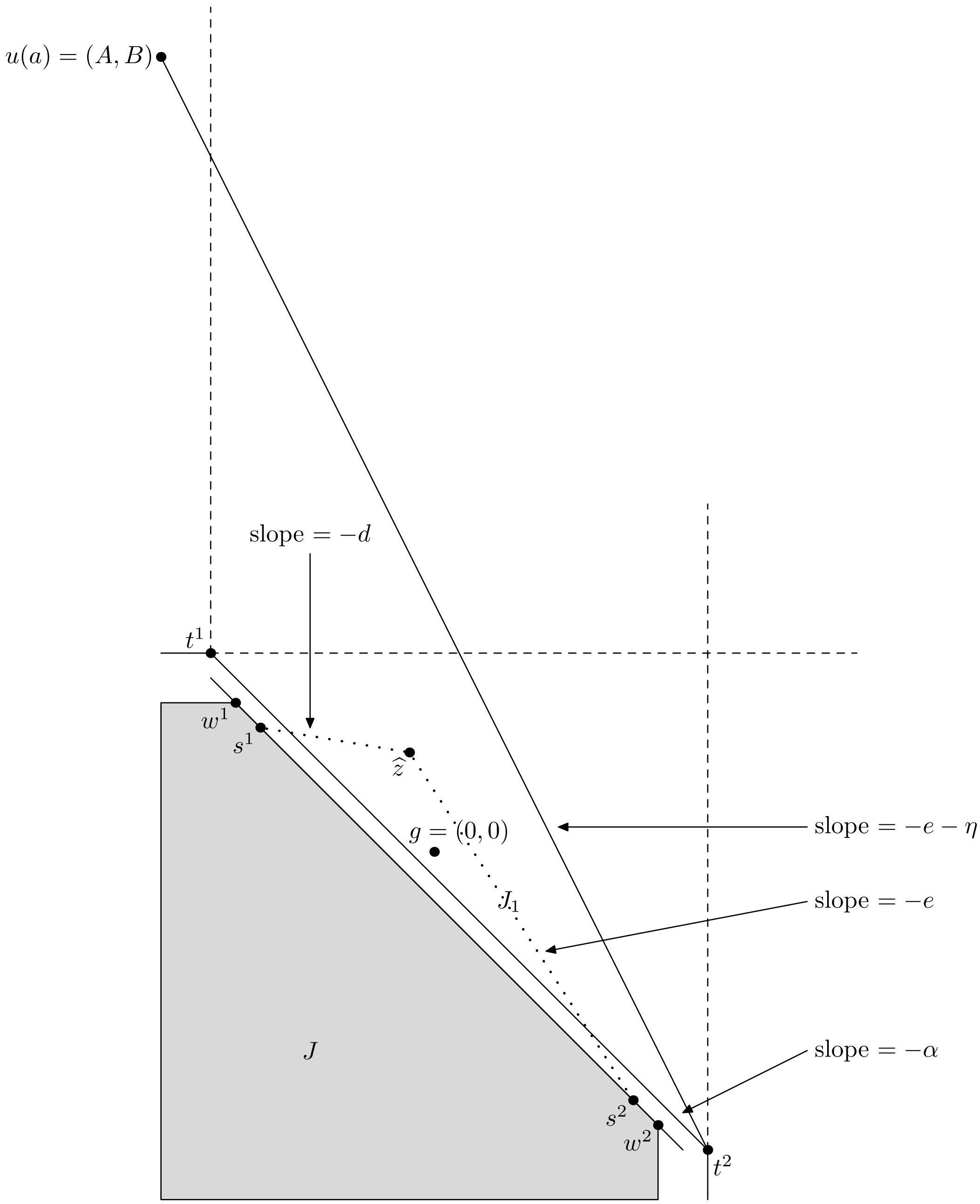

FIGURE 16. The setup in Case 2.

Consider the 2×2 one-shot auxiliary game $G(\zeta_1, \zeta_2, x, y)$ that is defined in Figure 15. Having each player monitor the other with probability $p = \frac{3(1-r^\Delta)}{r^\Delta \eta}$ is an equilibrium in the game $G(\zeta_1, \zeta_2, x, y)$ that yields payoff $(0, 0)$, where ζ_1, ζ_2, x, and y are given by (10)–(13). Because A is negative, x is negative as well.

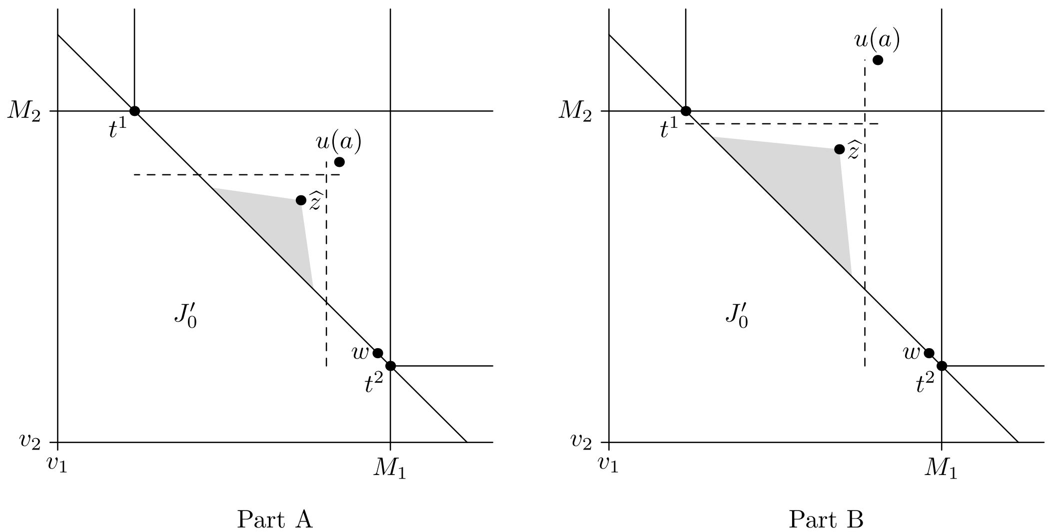

FIGURE 17. The proof in Case 4.

Condition E1 implies that ζ_1 is positive and together with Condition E3 it implies that it is less than η. Plainly ζ_2 is positive, and Conditions E1 and E5 imply that is it less than η. Condition E2 implies that y is positive and Conditions E1 and E4 implies that it is less than η. As mentioned above, x is negative, and Conditions E1 and E5 imply that it is larger than $-\eta$.

To complete the proof we need to show that $(-x, y)$ lies in $J_0 \cup J_1$. To this end we show that $\frac{-y}{-x} > -e$. By (12) and (13), this inequality reduces to

$$(1-r^\Delta)dB(1-p) - ped\big(c_1 - (1-r^\Delta)A\big) > pe\big(c_2 - (1-r^\Delta)B\big) - (1-r^\Delta)Ade(1-p).$$

Since $p = \frac{3(1-r^\Delta)}{r^\Delta \eta}$, we can divide all terms by $(1-r^\Delta)$, so that this inequality is equivalent to

$$d(1-p)(B+Ae) + pe(dA+B) > \frac{2e(c_2+dc_1)}{r^\Delta \eta},$$

which holds by Condition E6.

A.6. CASE 4

To solve Case 4, we use Case 1 and jointly controlled lotteries. Specifically, by Case 1, the grey area in Figure 17 is in $NE(r, c_1, c_2, \Delta)$, provided that c_1, c_2, and Δ are sufficiently small.

To complete Case 4, we show that the set $NE(r, c_1, c_2, \Delta)$ is almost convex. Indeed, suppose that at the first stage, the players jointly choose between, say, implementing as an equilibrium payoff a vector close to $\widehat{z}$ or a vector close to t^2. This is done as follows.

- Let $a_1, a_1' \in A_1$ and $a_2, a_2' \in A_2$ be two distinct actions of the two players. At the first stage, player 1 (resp. player 2) chooses either a_1 or a_1' (resp. a_2 or a_2') with equal probabilities, and both players monitor each other.
- If, at the first stage, one of the players fails to monitor the other or fails to play one of the designated actions, both players switch to punishment strategies.
- Otherwise, according to the realized action pair at the first stage, the players implement from the second stage onward one of the following payoff vectors as an equilibrium payoff, where $w = t^2 + (-\eta, \eta)$.

Action pair	Player 1's payoff	Player 2's payoff
(a_1, a_2)	$\widehat{z}_1 - (1-r^\Delta)u_1(a_1, a_2)$	$\widehat{z}_2 - (1-r^\Delta)u_2(a_1, a_2)$
(a_1, a_2')	$w_1 - (1-r^\Delta)u_1(a_1, a_2')$	$w_2 - (1-r^\Delta)u_2(a_1, a_2')$
(a_1', a_2)	$w_1 - (1-r^\Delta)u_1(a_1', a_2)$	$w_2 - (1-r^\Delta)u_2(a_1', a_2)$
(a_1', a_2')	$\widehat{z}_1 - (1-r^\Delta)u_1(a_1', a_2')$	$\widehat{z}_2 - (1-r^\Delta)u_2(a_1', a_2')$

A.7. Public perfect equilibria

In the construction of Nash equilibria, we used threats of punishment. In this section, we modify the proof of Lemma 5 so that the implementation of the vector $\xi := (v_1 + \eta, v_2 + \eta)$ does not involve noncredible threats. As is common in the literature, the implementation of a credible punishment is accomplished by having the players lower their payoffs for a fixed number of stages and return to the equilibrium play afterward.

The implementation of ξ in the proof of Lemma 5 includes a first phase that lasts k stages, in which the players follow an equilibrium α^* of the base game and partially monitor each other. We change only the implementation of this phase.

Suppose w.l.o.g. that $k_1 \le k_2$, so that $k = k_2$. Thus, player 1 monitors player 2 in the first k_1 stages, and player 2 monitors player 1 in the first k_2 stages.

- In the first k stages, player 1 plays a minmax mixed action β_1.
- In the first k_1 stages, player 2 plays a minmax mixed action β_2.
- In the following $k_2 - k_1$ stages, player 2 plays a best response γ_2 against β_1.

If no deviation occurs, the expected payoff to player 1 in the first k stages, given the public history, is

$$\delta_1 := \sum_{n=1}^{k_1}(1-r^\Delta)r^{(n-1)\Delta}u_1(a_1^n, a_2^n) + \sum_{n=k_1+1}^{k_2}(1-r^\Delta)r^{(n-1)\Delta}u_1(a_1^n, \gamma_2) - (1-r^{k_1\Delta})c,$$

and the expected payoff to player 2 in the first k stages, given the public history, is

$$\delta_2 := \sum_{n=1}^{k_1}(1-r^\Delta)r^{(n-1)\Delta}u_2(a_1^n, a_2^n) + \sum_{n=k_1+1}^{k_2}(1-r^\Delta)r^{(n-1)\Delta}u_2(a_1^n, \gamma_2) - (1-r^{k_2\Delta})c.$$

The continuation payoff x is a random variable that satisfies

$$\xi = \delta + r^{k_2 \Delta} x,$$

where $\delta = (\delta_1, \delta_2)$. Provided Δ is small, x is in Q_η and satisfies $d(x, J_\eta) \leq 2\eta$.

Whenever a deviation is observed, the players restart implementing ξ with the above construction. It is left to the reader to verify that the construction is indeed a public perfect equilibrium.

Reference

Fudenberg, Drew, David M. Kreps, and Eric S. Maskin (1990), "Repeated games with long-run and short-run players." *The Review of Economic Studies*, 57, 555–573. [1]

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