Supplement to Noisy Talk

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In this supplement, we provide a formal statement of the claim that noise enables communication when it would otherwise not have been possible, even once we move beyond the uniform-quadratic case (see section 4.1.2). Return to the general framework introduced in section 2.1, and assume that Crawford and Sobel's monotonicity condition (M) is satisfied.¹ The following constraint on preferences is also needed:

UNBOUNDED BIAS: For all θ , $\lim_{b\to\infty} a^S(\theta, b) = \infty$.

Recall that $a^{S}(\theta, b)$ denotes the (unique) ideal action of a sender of type θ and bias b. Unbounded bias guarantees that, in the CS model, communication ceases to be possible if the sender's bias is large enough. We can state out result.

Proposition 1 There exist b^*, b^{**} with $0 < b^* < b^{**}$ such that for all $b \in [b^*, b^{**})$, for all $\epsilon \in (0, 1)$, there is an equilibrium of the noise model that is better for the receiver than all equilibria of the CS model.

Proof. First, recall that we use $a_{CS}(\underline{\theta}, \overline{\theta})$ to denote the best response of the receiver if he knows only that θ lies between $\underline{\theta}$ and $\overline{\theta}$, i.e. for $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$,

$$a_{CS}\left(\underline{\theta},\overline{\theta}\right) = \begin{cases} \arg\max\int_{\underline{\theta}}^{\overline{\theta}} U^{R}\left(a,\theta\right) f\left(\theta\right) d\theta & \text{if } \underline{\theta} < \overline{\theta} \\ a^{R}\left(\theta\right) & \text{if } \underline{\theta} = \overline{\theta} \end{cases}$$

Next, define b^* to be the unique value of b that solves

$$U^{S}(a_{CS}(0,0),0,b) = U^{S}(a_{CS}(0,1),0,b).$$
(1)

 $^{^1 \}mathrm{See}$ page 1444 of Crawford and Sobel [1], or the discussion preceding the proof of Proposition 5 in the appendix.

To show that such a b^* exists, observe that

$$U^{S}(a_{CS}(0,0),0,0) > U^{S}(a_{CS}(0,1),0,0)$$

(since $a_{CS}(0,0)$ maximizes $U^{S}(\cdot,0,0)$). Next, define $\check{b} > 0$ as the solution to $a^{S}(0,b) = a_{CS}(0,1)$ (the existence of \check{b} follows from unbounded bias); then we have

$$U^{S}\left(a_{CS}\left(0,0\right),0,\check{b}\right) < U^{S}\left(a_{CS}\left(0,1\right),0,\check{b}\right)$$

It follows from the Intermediate Value Theorem that there is some b (with 0 < b < b) that satisfies (1). Uniqueness is implied by $U_{13}^S(a, \theta, b) > 0$. Note that $U^S(a_{CS}(0, 0), 0, b) \leq U^S(a_{CS}(0, 1), 0, b)$ for all $b \geq b^*$.

Proposition 1 follows easily from the following two lemmas. The first establishes conditions under which there is no communicative CS equilibrium, and the second shows when it is possible to construct a noise equilibrium with more steps than the most informative CS equilibrium.

Lemma 1 For all $b \ge b^*$, there is no communicative CS equilibrium.

Proof. Suppose there is a communicative CS equilibrium for some $b' \ge b^*$. Then there is a communicative two-step equilibrium with a critical type $\theta_1 \in (0, 1)$. Clearly θ_1 solves

$$U^{S}(a_{CS}(0,\theta_{1}),\theta_{1},b') = U^{S}(a_{CS}(\theta_{1},1),\theta_{1},b'), \qquad (2)$$

and so by (M) if $\tilde{\theta}_1 \in (0, \theta_1)$ and $\tilde{\theta}_2 > \tilde{\theta}_1$ solve

$$U^{S}\left(a_{CS}\left(0,\tilde{\theta}_{1}\right),\tilde{\theta}_{1},b'\right) = U^{S}\left(a_{CS}\left(\tilde{\theta}_{1},\tilde{\theta}_{2}\right),\tilde{\theta}_{1},b'\right),\tag{3}$$

we have $\tilde{\theta}_2 < 1$. Further, $0 < \tilde{\theta}_1 < \tilde{\theta}_2$ along with (1) imply that $a^S(\tilde{\theta}_1, b') < a_{CS}(\tilde{\theta}_1, \tilde{\theta}_2)$. Now consider $\tilde{\theta}_1 \to 0$. Since preferences never coincide for any state of the world, $\tilde{\theta}_2$ remains bounded away from zero. Therefore, by continuity, we get $U^S(a_{CS}(0,0), 0, b') = U^S(a_{CS}(0, \theta'), 0, b')$, for some $\theta' \in (0, 1)$, so that $a_{CS}(0, 0) < a^S(0, b') < a_{CS}(0, \theta')$. But $U^S(a_{CS}(0, 0), 0, b') \leq U^S(a_{CS}(0, 1), 0, b')$, establishing a contradiction.

Lemma 2 Suppose that $a^{S}(0,b) < a_{CS}(0,1)$. Then for all $\epsilon \in (0,1)$ there is an equilibrium partition of the noise model with two steps.

Proof. Consider a two-step partition $\{[0, \theta), [\theta, 1]\}$, and let types belonging to the first step randomize according to G over [0, 1), and types belonging to the second step send message m = 1. Then, conditional on receiving message $m \neq 1$, the posterior probability that the

message has been sent by error equals

$$\mu(\theta, \epsilon) = \frac{g(m) \epsilon}{g(m) \epsilon + g(m) (1 - \epsilon) F(\theta)}$$
$$= \frac{\epsilon}{\epsilon + (1 - \epsilon) F(\theta)}.$$

Note that the receiver's optimal response to receiving message 1 is $a_{CS}(\theta, 1)$. Let $\alpha(\theta)$ denote the receiver's best response to receiving a message in [0, 1) given the use of messages postulated above, i.e.

$$\begin{split} \alpha\left(\theta\right) &= \arg\max_{a} \left(\frac{\left(1-\epsilon\right)F(\theta)}{\epsilon+\left(1-\epsilon\right)F\left(\theta\right)} \int_{0}^{\theta} U^{R}\left(a,t\right)\frac{f\left(t\right)}{F\left(\theta\right)}dt + \frac{\epsilon}{\epsilon+\left(1-\epsilon\right)F(\theta)} \int_{0}^{1} U^{R}\left(a,t\right)f\left(t\right)dt\right). \end{split}$$

Clearly,

 $\alpha\left(\theta\right) < a_{CS}\left(0,1\right) \text{ for all } \theta \in \left(0,1\right),$

since $\epsilon \in (0, 1)$. Furthermore, since $\lim_{\theta \to 0} \mu(\theta) = 1$, we have

$$\lim_{\theta \to 0} \alpha\left(\theta\right) = a_{CS}\left(0,1\right).$$

It follows from the continuity of a^S , α and a_{CS} that there is some $\tilde{\theta}$ such that for $\theta \in (0, \tilde{\theta})$

$$a^{S}(\theta, b) < \alpha(\theta) < a_{CS}(\theta, 1)$$

and so

$$U^{S}\left(\alpha\left(\theta\right),\theta,b\right) - U^{S}\left(a_{CS}\left(\theta,1\right),\theta,b\right) > 0.$$

But

 $\alpha(1) < a_{CS}(1,1) < a^{S}(1,b),$

giving us

 $U^{S}(\alpha(1), 1, b) - U^{S}(a_{CS}(1, 1), 1, b) < 0.$

By the Intermediate Value Theorem, then, for some $\theta^* \in (0, 1)$ we have

$$U^{S}(\alpha(\theta^{*}),\theta^{*},b) - U^{S}(a_{CS}(\theta^{*},1),\theta^{*},b) = 0$$

At θ^* then, the sender is indifferent between sending any message in [0, 1) and sending 1, and so the strategies described above specify an equilibrium, with partition $\{[0, \theta^*), [\theta^*, 1]\}$

To complete the proof of Proposition 1, suppose that $b \ge b^*$, so the maximal CS equilibrium partition is the trivial one, with one step. It is easy to see that $a^S(0, b^*) < a_{CS}(0, 1)$,

and by continuity there is some $b^{**} > b^*$ such that $a^S(0, b^{**}) = a_{CS}(0, 1)$. Thus for all $b \in [b^*, b^{**})$, Lemma 2 implies that there exists an equilibrium of the noise model with two steps. Since there is no communicative CS equilibrium in this range, these equilibria are better for the receiver than all CS equilibria.

References

 Crawford, V.P. and J. Sobel (1982), "Strategic Information Transmission." *Econometrica* 50, 1431–1451.