Supplement to "Information design and sequential screening with ex post participation constraint"

(*Theoretical Economics*, Vol. 15, No. 1, January 2020, 319–359)

Tibor Heumann
Department of Applied Economics, HEC Montreal

In this supplement, we offer some additional examples and discussions that complement the results in the main text. We first give an example of a learning process whereby the agent learns his type in two periods and the principal can generate higher profits than under the optimal static mechanism. This two-period learning process allows us to revisit the intuitions in our paper and to compare our results with those of Krähmer and Strausz (2015). We then show that the mechanism constructed in Section 5 is not unique by describing a mechanism in which the transfers and learning process differ from those in that section. Third, we revisit the ex post participation constraint studied in our paper and contrast it with a weaker version of the ex post participation constraint. Finally, we revisit the interpretation of the agent's utility in our model by studying a model in which the agent's valuation is subject to additional shocks, but the agent has no information about these additional shocks.

Appendix A: Two-stage information disclosure

Consider the model of an indivisible good whose cost is zero:

$$u(q,\theta)=q\cdot\theta \quad \text{and} \quad c(q,\theta)=0.$$

We assume that the prior distribution is uniform in $[0,1]$. In the optimal static mechanism, the principal makes a "take it or leave it" offer at a price of $1/2$; this yields expected profits of $1/4$. We consider the following information disclosure: $T=\{0,1\}$ and the signals are given by

$$s_0=\begin{cases}1 & \text{if } \theta\geq 1/2\\ 0 & \text{if } \theta<1/2\end{cases} \qquad \text{(A.1)}$$
$$s_1=\theta.$$

That is, in the first period, the agent learns a threshold on his type, and in the second period, he learns his exact type.

Consider the following mechanism. At the moment when the agent learns whether his type is above or below $1/2$, the principal offers two contracts. Under the first contract, the agent buys the object at price $1/2$. Under the second contract, the agent—after

Tibor Heumann: tibor.heumann@hec.ca

Available at http://econtheory.org. https://doi.org/10.3982/TE2818

learning his type—is allowed with probability ε to buy the object at price $1/4$. In other words, under this second contract, with probability ε, the principal offers the good at price $1/4$, and with probability $1-\varepsilon$, the principal does not sell the object (at all).

The outcome of the mechanism will be as follows. If the agent's valuation is above $1/2$, then he buys the object at price $1/2$. If the agent's valuation is between $1/4$ and $1/2$, then he buys the object at price $1/4$ with probability ε. If the agent's valuation is below $1/4$, then he does not buy the object.

Because types higher than $1/2$ are forced to choose before they learn their valuation, they choose to buy the object because then their expected rent is $1/4$ (which, for ε small enough, is higher than with the lottery). Ex post, however, there are some types close to $1/2$ who wish they had received the lottery outcome. The principal's expected revenues are $(1/2)\cdot 1/2+(1/4\cdot\varepsilon)\cdot 1/4$, which clearly exceed $1/4$ (or the expected profits under the optimal static mechanism). Formally, we have the following lemma.

Lemma A1 (Two-stage information disclosure). *Under the information disclosure described by (A.1), the principal can obtain expected profits of* $(1+\varepsilon^2)/4$ *for* $\varepsilon>0$.

Lemma A1 shows that the principal can do better than under the static mechanism even if the learning process reveals the agent his type in only two periods. The intuition is similar to that for Theorem 1. The agent is given a lower bound on his type and must report this lower bound *before* he learns his final valuation. The agent's expected rents from truthfully reporting that his type is above the threshold are greater than those from misreporting that his type is below the threshold.

This example demonstrates that the intuitions behind the results of our model extend to environments where the learning process must be simpler, as when it reveals the agent's type after two periods rather than continuously. Furthermore, the difference between our results and those of Krähmer and Strausz (2015) does not hinge on the continuous-time learning process of our model. Although that process allows the principal to maximize her own profits when the participation constraint must be satisfied ex post, simpler learning processes also allow the principal to generate more profits than under the optimal static mechanism.

Appendix B: Nonuniqueness of transfers and information disclosure

Next we show that the information disclosure and transfers in an optimal mechanism are not uniquely defined. We maintain the same payoff structure as in Section 5.8.2, namely,

$$u(q,\theta)=q\cdot\theta;\qquad c(q,\theta)=q^2/2;\qquad \Theta=[0,1];\qquad f(\theta)=1.$$

The quantities in the optimal mechanism must remain the same as the $q^*(\theta)$ already calculated in Section 5.8.2, because, by Proposition 3, the quantities are uniquely defined across all optimal mechanisms. However, we show that the principal can implement these quantities and generate the same profits using the transfers

$$\tilde{x}'(\theta) \triangleq \begin{cases} u\big(q^*(\theta), \theta\big) & \text{if } \theta \leq \tilde{\theta}' \\ u\big(q^*(\theta), \theta\big) - \int_{\tilde{\theta}'}^{\theta} \left(\frac{\partial u\big(q^*(s), s\big)}{\partial \theta} + \varepsilon \right) ds & \text{if } \theta > \tilde{\theta}'. \end{cases} \tag{B.1}$$

Here ε is small (in the numerical example, $\varepsilon = 0.2$). The transfers are constructed as in (17a) but for $\theta > \tilde{\theta}'$, they grow more rapidly than the rate dictated by the envelope theorem. The threshold $\tilde{\theta}'$ is found much as before; it is the unique solution to

$$\mathbb{E}_{\theta}^{f}\left[\left\{\int_{\tilde{\theta}'}^{\theta} \left(\frac{\partial u\big(q^*(s), s\big)}{\partial \theta} ds + \varepsilon \right)\right\}^{+}\right] = \max_{\theta' \in \Theta} \mathbb{E}_{\theta}^{f}\big[\big\{u\big(q^*(\theta'), \theta\big) - u\big(q^*(\theta'), \theta'\big)\big\}^{+}\big].$$

The left-hand side of this equality represents the agent's expected rents under truth-telling, while the right-hand side is the lower bound (as characterized in Lemma 2) on the agent's rents.

The learning process is

$$s_t(\theta) = \begin{cases} 1 & \text{if } \big[t < \tilde{\theta}' \text{ and } \theta = t\big] \text{ or } \big[t \geq \tilde{\theta}' \text{ and } \theta = \big(1 + \tilde{\theta}' - t\big)\big] \\ 0 & \text{otherwise.} \end{cases} \tag{B.2}$$

In other words, the learning process for types $t \leq \tilde{\theta}'$ is the same as in upward disclosure: at each moment in time, the agent learns a lower bound on his type. After time $\tilde{\theta}'$ (i.e., when $t \geq \tilde{\theta}'$), the learning process becomes *downward* disclosure, which is the natural counterpart of upward disclosure. Under downward disclosure, higher types learn their type earlier. For example, type $\theta = 1$ learns his type at time $t = \tilde{\theta}'$ and type $\theta = \tilde{\theta}'$ learns his type at time $t = 1$.

The IC constraints when the information disclosure is (B.2) can be written analogously to those in Lemma 3. Toward that end, we define the function τ as

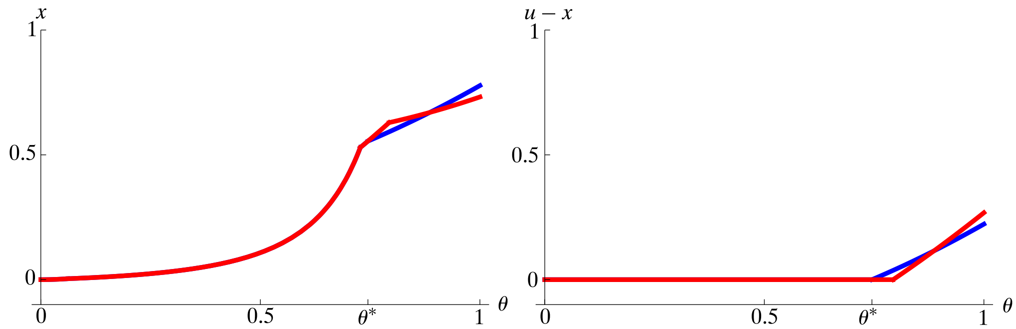
$$\tau(\theta) \triangleq \begin{cases} \theta & \text{if } \theta < \tilde{\theta}' \\ 1 - \theta + \tilde{\theta}' & \text{if } \theta \geq \tilde{\theta}'. \end{cases}$$

Thus, $\tau(\theta)$ gives the time at which type θ learn his type. For the learning process described by (B.2), an allocation policy $\{q(\theta), x(\theta)\}_{\theta \in \Theta}$ is incentive compatible if and only if each of the following statements holds:

$$\forall \theta'', \theta' \in \Theta \quad \text{with } \tau\big(\theta'\big) \geq \tau\big(\theta''\big), \quad u\big(q\big(\theta'\big), \theta''\big) - x\big(\theta'\big) \leq u\big(q\big(\theta''\big), \theta''\big) - x\big(\theta''\big) \tag{B.3}$$

$$\begin{aligned} \forall \theta' \in \Theta, \quad & \mathbb{E}_{\theta}^{f}\big[u\big(q\big(\theta'\big), \theta\big) - x\big(\theta'\big) | \tau(\theta) \geq \tau\big(\theta'\big)\big] \\ & \leq \mathbb{E}_{\theta}^{f}\big[u\big(q(\theta), \theta\big) - x(\theta) | \tau(\theta) \geq \tau\big(\theta'\big)\big]. \end{aligned} \tag{B.4}$$

In other words, type θ'' can pretend to be any type θ' that learns his type later (i.e., $\tau(\theta') > \tau(\theta'')$). But if a type θ'' mimics a type θ' that learns earlier (i.e., $\tau(\theta') < \tau(\theta'')$), then the agent must mimic type θ' *without* knowing the exact realization of his type (since he knows only that his type satisfies $\tau(\theta) > \tau(\theta')$).



(a) Transfers (b) Agent's Rents

FIGURE B1. Quadratic cost model. Plotted in blue are the transfers and agent's rents in the optimal mechanism under upward disclosure; in red, the transfers dictated by (B.1) with $\varepsilon = 0.2$.

It is easy to check numerically, using (B.3) and (B.4), that the mechanism is incentive compatible when $\varepsilon = 0.2$. In Figure B1, we plot the rents and transfers in a mechanism in which the learning is as in (B.2) and the transfers are as in (B.1). We compare this with the transfers and rents obtained under the optimal mechanism in Section 5.8.2. For low types, the transfers are the same and the agent earns zero rents; in fact, we have proved that this statement holds for all optimal mechanisms (see Proposition 3). For high types, in contrast, the rents and informational rents differ: the transfers (B.1) yield higher (resp., lower) informational rents for high types (resp., low types). Nonetheless, the agent's ex ante expected rents are the same under both transfers, which is to be expected given that the principal's profits must remain unchanged.

APPENDIX C: EX POST PARTICIPATION CONSTRAINT

We now compare the participation constraint studied in our paper with a weaker version of the ex post participation constraint. In this weaker version of the ex post participation constraint, the mechanism must offer the agent a strategy that leaves him with nonnegative rents ex post; however, the mechanism may leave the agent with negative rents if he uses a strategy different than the one "suggested" by the principal.

We now provide a formal expression of the principal's problem when he needs to satisfy this weaker form of the participation constraint. We denote by $\text{supp}(\mu) \subset \Theta \times S^T$ the support of the distribution $\mu \in \Delta(\Theta \times S^T)$.[1] The principal's profit-maximization problem when he needs to satisfy this weaker version of the ex post participation constraint is

$$\Pi \triangleq \max_{M \in \mathcal{M}} \mathbb{E}^{\mu}_{(\theta,s)}\big[x\big(\tilde{\sigma}(s)\big) - c\big(q\big(\tilde{\sigma}(s)\big), \theta\big)\big] \tag{C.1a}$$

$$\text{s.t. } \tilde{\sigma} \in \arg\max_{\sigma \in \Sigma} \mathbb{E}^{\mu}_{(\theta,s)}\big[u\big(q\big(\sigma(s)\big), \theta\big) - x\big(\sigma(s)\big)\big] \tag{C.1b}$$

$$u\big(q\big(\tilde{\sigma}(s)\big), \theta\big) - x\big(\tilde{\sigma}(s)\big) \geq 0 \quad \text{for all } (\theta, s) \in \text{supp}(\mu). \tag{C.1c}$$

[1] That is, $\text{supp}(\mu) \subset \Theta \times S^T$ is the smallest closed set such that it has measure 1 under μ.

Here the strategy $\tilde{\sigma}$ leaves the agent with nonnegative rents (i.e., (C.1c)) and it is optimal for the agent to use the strategy $\tilde{\sigma}$ (i.e., (C.1b)).[2] However, if the agent uses a different strategy, then the agent may be left with nonnegative rents. Here we would say that $\tilde{\sigma}$ (that solves (C.1b)) is the strategy that the principal "suggests" to the agent (or the truth-telling strategy in a direct mechanism).

The problem we solve in (1) imposes a stronger constraint on the principal's maximization problem than (C.1). While under some conditions on the learning process both versions of the ex post participation constraint coincide, in general, they are not equivalent.[3] We believe that the strong version of the participation constraint (i.e., the version we studied in the main text) is more adequate to model situations in which the agent can opt out of the mechanism ex post, for example, due to consumer protection laws.

We now show via an example that when the principal's problem is given by (C.1), she may achieve higher profits than when her problem is given by (1). Thus, both problems are not equivalent. Moreover, using this example, we also show that when the principal's problem is given by (C.1), upward disclosure may fail to be an optimal learning process.

Suppose the agent's type space is $\Theta = [0,1] \cup \{-M\}$, where $M \in \mathbb{R}$ is a large positive number. We assume that, with probability ε, the agent's type is $\theta = -M$ and with probability $(1-\varepsilon)$ his type is uniformly distributed in $[0,1]$. The set of feasible quantities is $Q = [0,1]$. The agent's utility is $u(\theta, q) = q \cdot \theta$. The learning process is as follows. Time is indexed by $T = [0,1] \cup \{2\}$ and the signal observed by the agent at time t is given by

$$s_t(\theta) = \begin{cases} 1 & \text{if } t = \theta \\ 1 & \text{if } t = 2 \text{ and } \theta = -M \\ 0 & \text{otherwise.} \end{cases} \tag{C.2}$$

In other words, the agent first learns whether his type is in [0,1] as in upward disclosure and then he learns whether his type is $\theta = -M$. We remark that this learning process is *not* equal to upward disclosure because when the agent's type is $\theta = -M$, the agent does not know his type until the end of the mechanism (i.e., $t = 2$).

[2]Note that strategy $\tilde{\sigma}$ leaves the agent with nonnegative rents for every history of signals and types that is in the support of μ. These events have probability 1 of occurring, so if the agent uses strategy $\tilde{\sigma}$, he is left with nonnegative rents with probability 1.

[3]For example, under the learning processes studied by Krähmer and Strausz (2015), both versions of the participation constraint are equivalent. The class of learning processes studied by them had the following characteristics: there were two periods, the agent learns his type with the second signal (i.e., $s_2 = \theta$), and the distribution over signals has full support. Without loss of generality, they focus on direct mechanisms. The full support assumption implies that every possible sequence of reports is consistent with being a truthful report. Therefore, the optimal mechanism that solves (C.1) must guarantee that the agent gets nonnegative rents after any sequence of reports. For this reason, in the optimal mechanism that solves (C.1), the agent would never opt out of the mechanism ex post in any case (that is, regardless of the messages he sends to the principal). Thus, the optimal mechanism that solves (C.1) could also be implemented when the principal's problem is (1) (where a priori the ex post participation constraint is stronger). For this reason, even if they studied the weaker version of the ex post participation constraint (as in (C.1)), the agent would still be guaranteed nonnegative rents ex post regardless of the messages he sends to the principal.

We assume that the message space is $Z = \{0, 1\}$. We construct a mechanism in which the allocation implemented depends only on the first time the agent reports $z = 1$. In other words, if the agent reports $z = 1$ at more than one t, then the allocation depends only on the first time in which $z = 1$ was reported. We denote by $\{(q(t), x(t))\}_{t\in T}$ the allocation implemented when the agent reports $z = 1$ for the first time at t. We consider the allocation rule

$$q(t) = \begin{cases} 1 & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \begin{cases} t & \text{if } t \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

If the agent reports $z = 1$ (for the first time) at $t \in [0, 1]$, then $q(t) = 1$ and $x(t) = t$. If the agent never reports $z = 1$ or he reports $z = 1$ at $t = 2$, then $q = x = 0$.

The agent's strategy consists of finding the optimal time to report $z = 1$ given the history of signals he has observed. If the agent reports $z = 1$ when $s_t = 1$, then we say the agent reported truthfully. If the agent reports truthfully, then the allocation implemented would be efficient and the principal would get the full surplus.

If the agent can opt out of the mechanism ex post regardless of the reported messages (as in (1)), then the agent's optimal strategy would report $\sigma(t) = 1$ at $t = 0$. This implies that the outcome of the mechanism would be $q = 1$ at a transfer $x = 0$. If the realization of his type is $\theta = -M$, then he would opt out of the mechanism ex post. We conclude that the mechanism is not incentive compatible when the agent can opt out of the mechanism ex post. We remark that, under this mechanism, if the agent can opt out of the mechanism ex post (as in (1)), then the mechanism would be efficient but the agent would get the full surplus.

If the agent can incur losses when he reports untruthfully (as in (C.1)), then this mechanism would be incentive compatible. To prove this, first observe that

$$\forall \theta \in \Theta, \quad q(\theta) \cdot \theta - x(\theta) = 0.$$

So if the agent reports truthfully, then he gets weakly positive rents ex post. Now observe that if the agent reports $z = 1$ at time $t < \theta$ (i.e., he reports $z = 1$ before he has observed the signal $s_t = 1$), then

$$\mathbb{E}\big[q(t) \cdot \theta - x(t) | \theta \notin [0, t]\big] \leq q(t)\big((1 - \varepsilon) + \varepsilon(-M)\big) - x(t) < 0,$$

where the last inequality is satisfied if M is large enough. In other words, if there is a positive probability that the agent's type can be $\theta = -M$ (and M is large enough), then he cannot profits from misreporting his type to pay a lower transfer. The agent cannot profitably misreport his type because the losses when his type is $\theta = -M$ are larger than the expected rents when his type is $\theta \in [0, 1]$. In this case, the mechanism would be incentive compatible and so the principal would be able to extract the full surplus.[4] We conclude that this mechanism solves (C.1).

We give some final remarks. First, our discussion shows that when the principal's problem is given by (C.1), she may achieve higher profits than when her problem is given by (1). Second, if the learning process was upward disclosure (instead of (C.2)), the principal would not be able to extract the full surplus when the participation constraint is as in (C.1c). Therefore, upward disclosure may not be the optimal information disclosure when the principal's problem is given by (C.1). Third, the existence of a type that derives negative utility from having the good (i.e., $\theta = -M$) is not critical for the arguments. Essentially the same argument goes through if the type space was $\Theta = [M, M+1] \cup \{0\}$, with M large enough. (We use the negative number because it is more suggestive of the arguments we make.)

Appendix D: Discussion of the agent's ex post valuation

In this section, we revisit the interpretation of the agent's utility. Suppose that the agent's valuation is determined by a two-dimensional type $(\theta, \phi) \in \mathbb{R}^2$, and the two types are independently distributed with densities f and g, respectively. We denote the agent's utility by $\tilde{u}(q, \theta, \phi)$ and let

$$u(q, \theta) \triangleq \mathbb{E}_{\phi}^{g}\big[\tilde{u}(q, \theta, \phi)\big].$$

In other words, $u(q, \theta)$ is the agent's expected utility conditional on θ. We continue to assume that the principal's cost depends only on θ (we assume this only to simplify the exposition). Furthermore, suppose that the principal must design a learning process that is independent of ϕ (i.e., the signals $\{s_t\}_{t \in T}$ are independent of ϕ). In other words, the shock ϕ affects the agent's utility, but he cannot learn about this shock from the signals he observes. The rest of the description of the model remains the same as in Section 2.

The analysis in our paper goes through completely unchanged in this augmented version of the model because the uncertainty about ϕ is irrelevant for the design of the mechanism. Formally, the addition of ϕ into the model does not change the agent's optimal strategy. The agent's expected utility conditional on any set of signals $\{s_{t'}\}_{t' \leq t}$ can be written as

$$\mathbb{E}_{(\theta,s),\phi}^{\mu,g}\big[\tilde{u}(q, \theta, \phi) | \{s_{t'}\}_{t' \leq t}\big] = \mathbb{E}_{(\theta,s)}^{\mu}\big[\mathbb{E}_{\phi}^{g}\big[\tilde{u}(q, \theta, \phi)\big] | \{s_{t'}\}_{t' \leq t}\big] = \mathbb{E}_{(\theta,s)}^{f}\big[u(q, \theta) | \{s_{t'}\}_{t' \leq t}\big].$$

The first equality is implied by the assumption that ϕ is independent of θ and $\{s_{t'}\}_{t' \leq t}$; the second equality is by the definition of $u(q, \theta)$. Thus, the agent's maximization problem is the same as simply replacing $\tilde{u}(q, \theta, \phi)$ with $u(q, \theta)$. In this augmented model, $u(q, \theta)$ is the agent's estimate utility conditional on all the information available to him and the principal. This utility may still be subject to shocks after he commits to buying the good (modeled by ϕ).

[4] We showed only that the agent does not profit from misreporting a type lower than his true type (i.e., reporting $z = 1$ when he has not yet observed $s_t = 1$). However, it is straightforward that the agent does not profit from misreporting a type higher than his true type.

The ex post participation constraint dictates that the principal must disclose all the information about the good to the agent before he decides to opt out of the mechanism (summarized by θ). In the real estate example discussed in the Introduction, the type θ is interpreted as the agent's estimate valuation of the house at the moment he buys the house. Because the buyer can inspect the house before committing to buying it, the principal cannot conceal information about the house at the time the buyer commits to buying the house. Of course, after buying the house, unexpected contingencies may arise that may change the buyer's valuation (e.g., due to an earthquake). These contingencies would be represented by ϕ. However, they are irrelevant from the perspective of the mechanism's design because the buyer cannot learn about these contingencies. In other words, after he buys the house his valuation may change, but the principal cannot conceal any information from the agent at the time the agents commits to paying the house.

References

Krähmer, Daniel and Roland Strausz (2015), "Optimal sales contracts with withdrawal rights." *Review of Economic Studies*, 82, 762–790. [1, 2, 5]

Co-editor Thomas Mariotti handled this manuscript.

Manuscript received 14 March, 2017; final version accepted 5 August, 2019; available online 8 August, 2019.