Supplementary Material

Supplement to "Competing mechanisms in markets for lemons"

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S.A. Restriction to incentive-compatible allocations

We show in what follows that in the analysis of competitive equilibria, we can restrict our attention, without loss of generality, to incentive-compatible allocations on and off the path, as claimed in the main paper following (4). To see this, consider a tuple $(m, \underline{\lambda}, \overline{\lambda}) \notin \mathcal{M}^{\mathrm{IC}}$ and consider the following reporting game. A high-type seller and a low-type seller, respectively, choose to report being of high type with probability $\overline{\alpha}$ and $\underline{\alpha}$ so as to maximize

$$\begin{aligned}\underline{\hat{u}}\big(m|\underline{\lambda}^c, \overline{\lambda}^c\big) &= \underline{\alpha}\big(\overline{t}_m\big(\underline{\lambda}^c, \overline{\lambda}^c\big) - \overline{x}_m\big(\underline{\lambda}^c, \overline{\lambda}^c\big)\underline{c}\big) + (1-\underline{\alpha})\big(\underline{t}_m\big(\underline{\lambda}^c, \overline{\lambda}^c\big) - \underline{x}_m\big(\underline{\lambda}^c, \overline{\lambda}^c\big)\underline{c}\big)\\ \hat{\overline{u}}\big(m|\underline{\lambda}^c, \overline{\lambda}^c\big) &= \overline{\alpha}\big(\overline{t}_m\big(\underline{\lambda}^c, \overline{\lambda}^c\big) - \overline{x}_m\big(\underline{\lambda}^c, \overline{\lambda}^c\big)\overline{c}\big) + (1-\overline{\alpha})\big(\underline{t}_m\big(\underline{\lambda}^c, \overline{\lambda}^c\big) - \underline{x}_m\big(\underline{\lambda}^c, \overline{\lambda}^c\big)\overline{c}\big)\end{aligned}$$

for some $\underline{\lambda}^c$ and $\overline{\lambda}^c$ denoting the sellers' conjecture over the expected low and high messages, respectively. An equilibrium of this game is a pair of reporting strategies, $\underline{\alpha}(m, \underline{\lambda}^c, \overline{\lambda}^c) = \arg\max_{\underline{\alpha}} \underline{\hat{u}}(m|\underline{\lambda}^c, \overline{\lambda}^c)$ and $\overline{\alpha}(m, \underline{\lambda}^c, \overline{\lambda}^c) = \arg\max_{\overline{\alpha}} \hat{\overline{u}}(m|\underline{\lambda}^c, \overline{\lambda}^c)$ for $\underline{\lambda}^c$ and $\overline{\lambda}^c$ solving the consistency condition between the conjectured number of low and high messages and the equilibrium fraction of such messages. That is,

$$\begin{aligned}\overline{\lambda}^c &= \overline{\lambda}\overline{\alpha}\big(m, \underline{\lambda}^c, \overline{\lambda}^c\big) + \underline{\lambda}\underline{\alpha}\big(m, \underline{\lambda}^c, \overline{\lambda}^c\big)\\ \underline{\lambda}^c &= \overline{\lambda}(1-\overline{\alpha})\big(m, \underline{\lambda}^c, \overline{\lambda}^c\big) + \underline{\lambda}(1-\underline{\alpha})\big(m, \underline{\lambda}^c, \overline{\lambda}^c\big).\end{aligned}$$

Let $\overline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})$ and $\underline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})$ denote the solutions to this system of equations and consider an alternative mechanism $\hat{m}$ such that

$$\begin{aligned}\overline{x}_{\hat{m}}(\underline{\lambda}, \overline{\lambda}) = {}& \overline{\alpha}\big(m, \underline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda}), \overline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})\big)\overline{x}_m\big(\underline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda}), \overline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})\big)\\ &+ \big(1 - \overline{\alpha}\big(m, \underline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda}), \overline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})\big)\big)\underline{x}_m\big(\underline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda}), \overline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})\big)\\ \overline{t}_{\hat{m}}(\underline{\lambda}, \overline{\lambda}) = {}& \overline{\alpha}\big(m, \underline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda}), \overline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})\big)\overline{t}_m\big(\underline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda}), \overline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})\big)\\ &+ \big(1 - \overline{\alpha}\big(m, \underline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda}), \overline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})\big)\big)\underline{t}_m\big(\underline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda}), \overline{\lambda}^c(m, \underline{\lambda}, \overline{\lambda})\big)\end{aligned}$$

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$$\begin{aligned}
\underline{x}_{\hat m}(\underline{\lambda},\overline{\lambda}) &= \underline{\alpha}\big(m,\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})\big)\overline{x}_m\big(\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})\big)\\
&\quad + \big(1-\underline{\alpha}\big(m,\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})\big)\big)\underline{x}_m\big(\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})\big),\\
\underline{t}_{\hat m}(\underline{\lambda},\overline{\lambda}) &= \underline{\alpha}\big(m,\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})\big)\overline{t}_m\big(\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})\big)\\
&\quad + \big(1-\underline{\alpha}\big(m,\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})\big)\big)\underline{t}_m\big(\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})\big).
\end{aligned}$$

Evidently, mechanism $\hat m$ achieves the same allocation as mechanism m and, consequently, yields the same payoff for buyers and for sellers. Moreover, $\hat m$ is incentive compatible and feasible, that is, $(\hat m,\underline{\lambda},\overline{\lambda}) \in \mathcal{M}^{\mathrm{IC}}$. To see the first property, note that the reporting strategy of each seller under the original mechanism m is optimal, which directly implies that truthful reporting under mechanism $\hat m$ has to be optimal. To see that the second property also holds, note that the feasibility of m implies that the trading probabilities of m are also feasible from an ex ante point of view, i.e., that conditions (9)–(11) are satisfied. Given that expected trading probabilities of mechanism $\hat m$ are equivalent to those of m, mechanism $\hat m$ is also feasible from an ex ante point of view. By Proposition 3.2, feasibility can then be satisfied ex post. Taken together, this demonstrates that any non-incentive-compatible mechanism can be replicated by an incentive-compatible mechanism.[1]

S.B. Proof of Lemma 3.4 (further details)

We consider the case $\overline{U} < (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$ and $\underline{v}-\underline{c} \le \overline{v}-\overline{c}$. First, we want to show that on the domain $\overline{\lambda} \in (0,L]$, the buyer's expected payoff is strictly concave. The buyer's expected payoff on this domain is given by

$$\tilde\pi(\overline{\lambda}) = \overline{\lambda}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c}) + \left(1-\frac{\overline{\lambda}}{1-e^{-\overline{\lambda}}}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}\right)(\underline{v}-\underline{c}) - \overline{\lambda}\overline{U} - \ln\left(\frac{1-e^{-\overline{\lambda}}}{\overline{\lambda}}\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)\underline{U}.$$

Since $\underline{x} > \overline{x} =$ for $\underline{\lambda},\overline{\lambda} > 0$ (by Lemma 3.3), the fact that the low-type incentive constraint (16) holds as equality immediately implies that the high-type incentive constraint (17) is slack. At a solution of $P^{\mathrm{aux}'}$ with $\overline{\lambda} > 0$, $\overline{\lambda}$ is thus the unconstrained maximizer of $\tilde\pi$ on $(0,L]$. We show next that the function $\tilde\pi$ is strictly concave in $\overline{\lambda}$. Note that

$$\frac{\partial^2\tilde\pi}{\partial\overline{\lambda}^2} = e^{\overline{\lambda}}\frac{2\big(e^{\overline{\lambda}}-1\big)-\overline{\lambda}\big(e^{\overline{\lambda}}+1\big)}{\big(e^{\overline{\lambda}}-1\big)^3}(\underline{v}-\underline{c})\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} + \frac{\overline{\lambda}^2e^{\overline{\lambda}}-\big(e^{\overline{\lambda}}-1\big)^2}{\overline{\lambda}^2\big(e^{\overline{\lambda}}-1\big)^2}\underline{U}.$$

[1] As an example, suppose m is such that in the equilibrium of the reporting game, we have $\overline{\alpha}(m,\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})) = \underline{\alpha}(m,\underline{\lambda}^c(m,\underline{\lambda},\overline{\lambda}),\overline{\lambda}^c(m,\underline{\lambda},\overline{\lambda})) = 1$, that is, all sellers weakly prefer to report being of high type. Here, $\overline{x}_{\hat m}(\underline{\lambda},\overline{\lambda}) = \underline{x}_{\hat m}(\underline{\lambda},\overline{\lambda}) = \overline{x}_m(0,\underline{\lambda}+\overline{\lambda})$ and $\overline{t}_{\hat m}(\underline{\lambda},\overline{\lambda}) = \underline{t}_{\hat m}(\underline{\lambda},\overline{\lambda}) = \overline{t}_m(0,\underline{\lambda}+\overline{\lambda})$. Mechanism $\hat m$ can be interpreted as a posted price, potentially coupled with a trading probability, making sellers' payoff independent of their report and, therefore, satisfying incentive compatibility trivially.

The numerator of the first term of the derivative is equal to zero at $\overline{\lambda} = 0$ and strictly decreasing for all $\overline{\lambda} > 0$,

$$\frac{\partial\big(2(e^{\overline{\lambda}} - 1) - \overline{\lambda}(e^{\overline{\lambda}} + 1)\big)}{\partial\overline{\lambda}} = -\overline{\lambda}e^{\overline{\lambda}} \underbrace{\left(1 - \frac{1}{\overline{\lambda}}(1 - e^{-\overline{\lambda}})\right)}_{>0} < 0,$$

so that this first term is strictly negative. As we show next, the numerator of the second term is also strictly negative: $\overline{\lambda}^2 e^{\overline{\lambda}} < (e^{\overline{\lambda}} - 1)^2$. To see this, notice first that the inequality can be rewritten as $1 - e^{\overline{\lambda}} + \overline{\lambda}e^{\overline{\lambda}/2} < 0$. The term $1 - e^{\overline{\lambda}} + \overline{\lambda}e^{\overline{\lambda}/2}$ is equal to zero at $\overline{\lambda} = 0$ and is strictly decreasing in $\overline{\lambda}$ for all $\overline{\lambda} > 0$,

$$\frac{\partial\big(1 - e^{\overline{\lambda}} + \overline{\lambda}e^{\frac{\overline{\lambda}}{2}}\big)}{\partial\overline{\lambda}} = -e^{\overline{\lambda}}\left[1 - e^{-\frac{\overline{\lambda}}{2}} - \frac{\overline{\lambda}}{2}e^{-\frac{\overline{\lambda}}{2}}\right] < 0,$$

where the term in the square bracket is the probability of at least two arrivals when the queue length is $\overline{\lambda}/2$ and, therefore, is strictly positive. This establishes $\partial^2\tilde{\pi}/\partial\overline{\lambda}^2 < 0$; that is, $\tilde{\pi}(\overline{\lambda})$ is strictly concave on the domain $(0, L]$.

Given the strict concavity of $\tilde{\pi}(\overline{\lambda})$ on $(0, L]$, $P^{\text{aux}'}$ can at most have two solutions: one at $\overline{\lambda} > 0$ and possibly one at $\overline{\lambda} = 0$. In what follows, we show that there cannot be a solution with $\overline{\lambda} = 0$, which establishes the claim that the solution of $P^{\text{aux}'}$ is unique. To this end, we need to characterize the properties of possible solutions with $\overline{\lambda} = 0$. Recall that the above expression of $\tilde{\pi}(\overline{\lambda})$ was only valid for $\overline{\lambda} > 0$; hence, when $\overline{\lambda} = 0$ we need to consider $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$. Ignoring incentive constraints for a moment, the value of $\underline{\lambda}$ that maximizes $\hat{\pi}(\underline{\lambda}, 0) = (1 - e^{-\underline{\lambda}})(\underline{v} - \underline{c}) - \underline{\lambda}\underline{U}$ is $\ln((\underline{v} - \underline{c})/\underline{U})$. Given $\overline{\lambda} = 0$, the high-type incentive constraint (16) can always be satisfied by picking some value of $\overline{x}$ weakly smaller than $(\underline{U} - \overline{U})/(\overline{c} - \underline{c})$. Alternatively, the high-type incentive constraint (17) is satisfied at $\underline{\lambda} = \ln((\underline{v} - \underline{c})/\underline{U})$ if and only if

$$\frac{1}{\ln\left(\frac{\underline{v} - \underline{c}}{\underline{U}}\right)}\left(1 - \frac{\underline{U}}{\underline{v} - \underline{c}}\right) \geq \frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}. \tag{S.0}$$

- Suppose first that this inequality is not satisfied, i.e., the value of $\underline{\lambda}$ that maximizes $\hat{\pi}(\underline{\lambda}, 0)$ is too large. The optimal value of $\underline{\lambda}$ is then given by L, defined earlier by the implicit condition $(1 - e^{-L})/L = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$. The buyer's payoff associated to this value of $\underline{\lambda}$ is

$$\hat{\pi}(L, 0) = L\left(\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}(\underline{v} - \underline{c}) - \underline{U}\right).$$

  Since $\underline{v} - \underline{c} \leq \overline{v} - \overline{c}$ and $\underline{U} > \overline{U}$, $\hat{\pi}(L, 0)$ is strictly smaller than the corresponding expression when all low types are swapped with high types, given by

$$\hat{\pi}(0, L) = L\left(\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}(\overline{v} - \overline{c}) - \overline{U}\right).$$

Since the pair $(0, L)$ is an admissible solution for $(\underline{\lambda}, \overline{\lambda})$, we cannot have $\overline{\lambda} = 0$ at a solution of $P^{\text{aux}'}$; hence, a contradiction.

- Suppose next that (S.0) is satisfied so that $\ln((\underline{v} - \underline{c})/\underline{U})$ is an admissible value for $\underline{\lambda}$. Toward a contradiction, suppose that we have $\underline{\lambda} = \ln((\underline{v} - \underline{c})/\underline{U})$ and $\overline{\lambda} = 0$ at a solution of $P^{\text{aux}'}$. We need to distinguish two cases:
  - Suppose first that $\underline{U}/(\underline{v} - \underline{c}) > (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$. Consider the tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ with $\underline{\lambda}' = \underline{\lambda}$ and $\underline{x}' = (1 - e^{-\underline{\lambda}})/\underline{\lambda}$, so that the payoff with low-type sellers remains unchanged, and let $\overline{x}' = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$. Given these restrictions, the tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ satisfies the incentive compatibility constraints and, if $\overline{\lambda}' > 0$, yields a strictly positive payoff with high-type sellers:

$$\overline{\lambda}'\big[\overline{x}'(\overline{v} - \overline{c}) - \overline{U}\big] = \overline{\lambda}\left[\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}(\overline{v} - \overline{c}) - \overline{U}\right].$$

  A strictly positive value of $\overline{\lambda}'$ is feasible (that is, consistent with (19), using the values specified above for $\underline{\lambda}', \underline{x}', \overline{x}'$) if

$$\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} \leq \frac{\underline{U}}{\underline{v} - \underline{c}} \frac{1}{\overline{\lambda}'}\big(1 - e^{-\overline{\lambda}'}\big)$$

  for some $\overline{\lambda}' > 0$. In the case under consideration, $(\underline{U}/(\underline{v} - \underline{c}) > (\underline{U} - \overline{U})/(\overline{c} - \underline{c}))$, this is indeed the case, implying that there is no solution of $P^{\text{aux}'}$ with $\overline{\lambda} = 0$.

  - Consider next the case $\underline{U}/(\underline{v} - \underline{c}) \leq (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$. Let $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ be specified again with $\underline{\lambda}' = \underline{\lambda}$, $\underline{x}' = (1 - e^{-\underline{\lambda}})/\underline{\lambda}$, while the value of $\overline{x}'$ is now set so that (19) is satisfied with equality:

$$\overline{x}' = e^{-\underline{\lambda}} \frac{1}{\overline{\lambda}'}\big(1 - e^{-\overline{\lambda}'}\big) = \frac{\underline{U}}{\underline{v} - \underline{c}} \frac{1}{\overline{\lambda}'}\big(1 - e^{-\overline{\lambda}'}\big) < \frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}.$$

  The above inequality implies that the low-type incentive constraint (16) is satisfied for all $\overline{\lambda}' > 0$.[2] The difference in payoff between the two mechanisms is given by

$$\hat{\pi}\left(\ln\left(\frac{\underline{v} - \underline{c}}{\underline{U}}\right), 0\right) - \hat{\pi}\left(\ln\left(\frac{\underline{v} - \underline{c}}{\underline{U}}\right), \overline{\lambda}'\right)$$
$$= e^{-\underline{\lambda}}\big(1 - e^{-\overline{\lambda}'}\big)(\overline{v} - \overline{c}) - \overline{\lambda}'\overline{U} = \overline{\lambda}'\left[\frac{1}{\overline{\lambda}'}\big(1 - e^{-\overline{\lambda}'}\big)\frac{\overline{v} - \overline{c}}{\underline{v} - \underline{c}}\underline{U} - \overline{U}\right].$$

  Since $\underline{U} > \overline{U}$ and $\overline{v} - \overline{c} \geq \underline{v} - \underline{c}$ in the case under consideration (b), the above expression is strictly positive for $\overline{\lambda}'$ small enough.[3]

We have thus shown that there can be no solution of $P^{\text{aux}'}$ with $\overline{\lambda} = 0$ and, hence, that the solution of $P^{\text{aux}'}$ is unique.

[2]The other incentive compatibility constraint (17) is also satisfied because the low-type sellers' trading probability remains unchanged.

[3]Recall that $\lim_{x \to 0}(1 - e^{-x})/x = 1$.

S.C. Proof of Proposition 4.1 (further details)

In what follows, we show that pooling sellers in a single submarket leads to strictly more meetings than when sellers are divided into separate submarkets. To see this, suppose there are $N \geq 2$ submarkets and let b_n (s_n) denote the measure of buyers posting in (sellers selecting) market $n = 1, 2, \ldots, N$. Consider two markets n and n', and let $b_{n,n'} = b_n + b_{n'}$, $s_{n,n'} = s_n + s'_n$ and $\gamma_{n,n'} = b_n / b_{n,n'}$, $\sigma_{n,n'} = s_n / s_{n,n'}$. The total number of meetings in these two markets is then given by

$$T_{n,n'} = b_{n,n'}\big[\gamma_{n,n'}\big(1 - e^{-\frac{\sigma_{n,n'} s_{n,n'}}{\gamma_{n,n'} b_{n,n'}}}\big) + (1 - \gamma_{n,n'})\big(1 - e^{-\frac{(1-\sigma_{n,n'}) s_{n,n'}}{(1-\gamma_{n,n'}) b_{n,n'}}}\big)\big].$$

The derivative with respect to $\gamma_{n,n'}$ is

$$\frac{\partial T_{n,n'}}{\partial \gamma_{n,n'}} = -b_{n,n'}\left[\left(1 + \frac{\sigma_{n,n'} s_{n,n'}}{\gamma_{n,n'} b_{n,n'}}\right) e^{-\frac{\sigma_{n,n'} s_{n,n'}}{\gamma_{n,n'} b_{n,n'}}} - \left(1 + \frac{(1-\sigma_{n,n'}) s_{n,n'}}{(1-\gamma_{n,n'}) b_{n,n'}}\right) e^{-\frac{(1-\sigma_{n,n'}) s_{n,n'}}{(1-\gamma_{n,n'}) b_{n,n'}}}\right].$$

The first term inside the square bracket shows the probability of at most one arrival when the arrival rate is $(\sigma_{n,n'} s_{n,n'})/(\gamma_{n,n'} b_{n,n'})$, while the second term inside the square bracket shows the corresponding probability when the arrival rate is $((1 - \sigma_{n,n'}) s_{n,n'})/((1 - \gamma_{n,n'}) b_{n,n'})$. When $\gamma_{n,n'} < \sigma_{n,n'}$, we have $(\sigma_{n,n'} s_{n,n'})/(\gamma_{n,n'} b_{n,n'}) > ((1 - \sigma_{n,n'}) s_{n,n'})/((1 - \gamma_{n,n'}) b_{n,n'})$. Since the probability of at most one arrival $e^{-x} + xe^{-x}$ is strictly decreasing in the arrival rate x, it follows that the first term inside the square bracket is strictly smaller than the second term. This implies that $\partial T_{n,n'} / \partial \gamma_{n,n'} > 0$. For $\gamma_{n,n'} > \sigma_{n,n'}$, the opposite implication holds. The total number of meetings is thus maximized when $\gamma_{n,n'} = \sigma_{n,n'}$.

This implies that the total number of meetings is maximized when the queue length is equal in each of the two submarkets, which is equivalent to merging the two submarkets n and n'. By iterating the argument, it follows that when there are N submarkets, the total number of meetings is maximized by merging all of them. This implies that the competitive search equilibrium maximizes the total number of meetings and, consequently, under the stated conditions, maximizes total surplus.

S.D. Proof of Proposition 4.2

For the private value case $\underline{v} = \overline{v}$, Eeckhout and Kircher (2010, Lemma 2 and Proposition 3) show that the claims in the statement of Proposition 4.2 are satisfied. In what follows, we thus assume $\underline{v} < \overline{v}$.

Under urn–ball matching, the probability for a seller to be chosen at random from those sellers arriving at the buyer when the overall queue length is $\underline{\lambda} + \overline{\lambda}$ is given by $(1 - e^{-\underline{\lambda} - \overline{\lambda}})/(\underline{\lambda} + \overline{\lambda})$. Hence, trading probabilities of bilateral menus must satisfy the constraints

$$\underline{x}, \overline{x} \leq \frac{1}{\underline{\lambda} + \overline{\lambda}}\big(1 - e^{-\underline{\lambda} - \overline{\lambda}}\big). \tag{S.1}$$

Moreover, trading probabilities and transfers must not condition on the reports of sellers other than the one selected by the buyer. When deriving the competitive search equilibrium we ignore this restriction and verify that, with the constraints (S.1) in place, the set of solutions of the buyers' auxiliary optimization problem are such that trading probabilities and transfers are indeed independent of the reports of other sellers arriving at the same buyer. Given that the statements in Appendix A.2 in the main paper continue to hold, at any equilibrium mechanism, the associated trading probabilities, transfers, and queue lengths have to be a solution to $P^{\text{aux}'}$ subject to the additional constraints (S.1). It is useful to notice that these constraints imply the two previous feasibility constraints (10) and (11). We call the resulting problem $P^{\text{aux}''}$.

We can first show that under the stated restriction on the class of feasible mechanisms, at any solution of $P^{\text{aux}''}$, $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ with $\underline{\lambda} > 0$, we must have $\underline{x}(\underline{v} - \underline{c}) - \underline{U} \geq \overline{x}(\overline{v} - \overline{c}) - \overline{U}$. Otherwise there exists a profitable deviation by replacing all low types with high types, that is, by setting $\underline{\lambda}' = 0$ and $\overline{\lambda}' = \underline{\lambda} + \overline{\lambda}$, while keeping the respective trading probabilities unchanged. By a perfectly symmetric argument, we must have $\underline{x}(\underline{v} - \underline{c}) - \underline{U} \leq \overline{x}(\overline{v} - \overline{c}) - \overline{U}$ at any solution of $P^{\text{aux}''}$ with $\overline{\lambda} > 0$. Taken together, this implies that an optimal menu attracting both types of seller must satisfy $\underline{x}(\underline{v} - \underline{c}) - \underline{U} = \overline{x}(\overline{v} - \overline{c}) - \overline{U}$.

The property on relative payoffs with each type of seller further implies that market utilities must always be such that $\overline{U} < (\overline{v} - \overline{c})/(\underline{v} - \underline{c})\underline{U}$. Recall that if this inequality is not satisfied, buyers can make at most zero profits with high-type sellers and, as a consequence, weakly prefer not to attract them. Alternatively, at a solution of $P^{\text{aux}''}$, the buyer's payoff with each low-type sellers must be strictly positive, i.e., $\underline{x}(\underline{v} - \underline{c}) - \underline{U} > 0$. If that is not the case, it would be strictly optimal for a buyer to increase $\underline{x}$, combined with a decrease of $\underline{\lambda} + \overline{\lambda}$ so as to satisfy (S.1), and thereby make a strictly positive profit. We thus have $\underline{x}(\underline{v} - \underline{c}) - \underline{U} > 0 \geq \overline{x}(\overline{v} - \overline{c}) - \overline{U}$, which, by the previous argument, implies $\overline{\lambda} = 0$. However, if $\overline{\lambda} = 0$ at all solutions of $P^{\text{aux}''}$, we must have $\overline{U} = 0$ and, therefore, $\overline{U} < (\overline{v} - \overline{c})/(\underline{v} - \underline{c})\underline{U}$.

We can then show that at a solution of $P^{\text{aux}''}$, the trading probabilities $\underline{x}$ and $\overline{x}$ satisfy (S.1) as an equality whenever, respectively, $\underline{\lambda} > 0$ and $\overline{\lambda} > 0$. Toward a contradiction, suppose first that $\overline{\lambda} > 0$ and $\overline{x} < (1 - e^{-\underline{\lambda} - \overline{\lambda}})/(\underline{\lambda} + \overline{\lambda})$. In this case, the low-type incentive constraint (7) must be binding, as otherwise the buyer could increase his payoff by increasing $\overline{x}$. By $\underline{x} > \overline{x}$ (no pooling), this implies that the high-type incentive constraint (8) is slack, and, by $\overline{U} < (\overline{v} - \overline{c})/(\underline{v} - \underline{c})\underline{U}$, that the profit with each high-type seller, $\overline{x}(\overline{v} - \overline{c}) - \overline{U}$, is strictly positive. Consider then an alternative tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ with $\underline{x}' = (1 - e^{-\underline{\lambda} - \overline{\lambda} - \varepsilon})/(\underline{\lambda} + \overline{\lambda} + \varepsilon)$, $\overline{x}' = \overline{x}$, $\underline{\lambda}' = 0$, and $\overline{\lambda}' = \underline{\lambda} + \overline{\lambda} + \varepsilon$. As can be verified, the tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ satisfies all constraints as long as ε is small enough and, given that $\underline{x}(\underline{v} - \underline{c}) - \underline{U} \leq \overline{x}(\overline{v} - \overline{c}) - \overline{U}$, strictly increases the buyer's payoff by (at least) $\varepsilon[\overline{x}(\overline{v} - \overline{c}) - \overline{U}] > 0$. We thus have a contradiction. For the case $\underline{x} < (1 - e^{-\underline{\lambda} - \overline{\lambda}})/(\underline{\lambda} + \overline{\lambda})$, an even simpler argument holds. Here a small increase in $\underline{x}$ does not violate any constraints and leads to a strict increase of the buyer's payoff.

We therefore have $\underline{x} = (1 - e^{-\underline{\lambda} - \overline{\lambda}})/(\underline{\lambda} + \overline{\lambda})$ if $\underline{\lambda} > 0$ and $\overline{x} = (1 - e^{-\underline{\lambda} - \overline{\lambda}})/(\underline{\lambda} + \overline{\lambda})$ if $\overline{\lambda} > 0$. Since for $\underline{\lambda}, \overline{\lambda} > 0$ this implies $\underline{x} = \overline{x}$ and since pooling cannot be optimal by the argument above, at a solution of $P^{\text{aux}''}$, the buyer attracts at most one type. Consider first the possibility of attracting low-type sellers. With $\underline{x} = (1 - e^{-\underline{\lambda}})/\underline{\lambda}$, this problem amounts to maximizing $(1-e^{-\underline{\lambda}})(\underline{v}-\underline{c})-\underline{\lambda U}$ over $\underline{\lambda}$ subject to the high-type incentive constraint

$$\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right)\geq\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}. \tag{S.2}$$

Similarly, an optimal mechanism that attracts high-type sellers must be such that the associated queue length $\overline{\lambda}$ maximizes $(1-e^{-\overline{\lambda}})(\overline{v}-\overline{c})-\overline{\lambda U}$ subject to the low-type incentive constraint

$$\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right)\leq\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}. \tag{S.3}$$

For mechanisms that attract low-type sellers, notice that (S.2) cannot be binding at a solution of $P^{\text{aux}''}$, since the buyer would strictly prefer to attract the same queue length of high types instead of low types, thereby making a strictly larger profit. The optimal queue length of low types therefore solves the unconstrained problem of maximizing $(1-e^{-\underline{\lambda}})(\underline{v}-\underline{c})-\underline{\lambda U}$ and is, hence, characterized by the first-order condition

$$e^{-\underline{\lambda}}(\underline{v}-\underline{c})=\underline{U}\quad\Leftrightarrow\quad\underline{\lambda}=\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right). \tag{S.4}$$

For a contract that attracts high-type sellers, we need to distinguish two cases, namely whether the low-type incentive constraint (S.3) is binding or is not binding. Suppose first that it is slack. Then the buyer's problem of finding the best mechanism that attracts only high-type sellers is analogous that for low-type sellers and the optimal value of $\overline{\lambda}$ is pinned down by

$$e^{-\overline{\lambda}}(\overline{v}-\overline{c})=\overline{U}\quad\Leftrightarrow\quad\overline{\lambda}=\ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right). \tag{S.5}$$

This solution satisfies (S.3) if and only if

$$\frac{1}{\ln\left(\dfrac{\overline{v}-\overline{c}}{\overline{U}}\right)}\left(1-\frac{\overline{U}}{\overline{v}-\overline{c}}\right)\leq\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}. \tag{S.6}$$

If (S.6) is violated, the optimal $\overline{\lambda}$ is pinned down by the low-type incentive constraint (S.3).

Equilibrium

(i) To construct an equilibrium, consider first the possibility that only low-type sellers trade in equilibrium. Here $\overline{U}=0$, while $\underline{U}=e^{-\underline{\lambda}^p}(\underline{v}-\underline{c})$. A buyer's equilibrium payoff is then given by

$$\pi\left(\underline{\lambda}^p,0\right)=\left(1-e^{-\underline{\lambda}^p}-\underline{\lambda}^p e^{-\underline{\lambda}^p}\right)(\underline{v}-\underline{c}).$$

Given $\overline{U}=0$, the best contract that attracts only high-type sellers is clearly such that $p=\overline{c}$ and $\overline{\lambda}=+\infty$, as this minimizes the price and maximizes the trading probability for a buyer. The payoff from this deviating contract is given by

$$\pi(0,+\infty) = \lim_{\overline{\lambda}\to+\infty}\left(1-e^{-\overline{\lambda}}\right)(\overline{v}-\overline{c}) = \overline{v}-\overline{c}.$$

An equilibrium in which only low quality is traded thus exists if and only if[4]

$$\left(1-e^{-\underline{\lambda}^p}-\underline{\lambda}^p e^{-\underline{\lambda}^p}\right)(\underline{v}-\underline{c}) \geq \overline{v}-\overline{c}.$$

(ii)(a) Consider now the possibility of an equilibrium in which both types of sellers trade and suppose first that market utilities are such that condition (S.6) is satisfied. In equilibrium, buyers need to be indifferent between posting the best contract that attracts only high types and the best contract that attracts only low types. Let γ denote the fraction of buyers who attract high types and let $1-\gamma$ denote the fraction who trade with low types. The seller–buyer ratios in the high and low quality market are then given by $\underline{\lambda} = \underline{\lambda}^p/(1-\gamma)$ and $\overline{\lambda} = \overline{\lambda}^p/\gamma$. These values of $\underline{\lambda}$ and $\overline{\lambda}$ are optimal for a buyer if they, respectively, satisfy conditions (S.4) and (S.5). This is the case if $\overline{U} = e^{-\overline{\lambda}^p/\gamma}(\overline{v}-\overline{c})$ and $\underline{U} = e^{-\underline{\lambda}^p/(1-\gamma)}(\underline{v}-\underline{c})$. A buyer's payoff associated to the contract with low-type sellers and high-type sellers is then, respectively, given by

$$\left(1-e^{-\frac{\underline{\lambda}^p}{1-\gamma}}-\frac{\underline{\lambda}^p}{1-\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right)(\underline{v}-\underline{c}) \quad \text{and} \quad \left(1-e^{-\frac{\overline{\lambda}^p}{\gamma}}-\frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\overline{\lambda}^p}{\gamma}}\right)(\overline{v}-\overline{c}).$$

Indifference requires that γ is such that these payoffs are the same, i.e.,

$$\left(1-e^{-\frac{\overline{\lambda}^p}{\gamma}}-\frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\overline{\lambda}^p}{\gamma}}\right)(\overline{v}-\overline{c}) = \left(1-e^{-\frac{\underline{\lambda}^p}{1-\gamma}}-\frac{\underline{\lambda}^p}{1-\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right)(\underline{v}-\underline{c}). \tag{S.7}$$

Note that $1-e^{-x}-xe^{-x}$, the probability of at least two arrivals given arrival rate x, strictly increases in x. This implies that the left-hand side of (S.7) strictly decreases in γ, while the right-hand side strictly increases in γ. A solution to this equation exists if, for $\gamma\to 0$, the left-hand side exceeds the right-hand side, i.e., $\overline{v}-\overline{c} > (1-e^{-\underline{\lambda}^p}-\underline{\lambda}^p e^{-\underline{\lambda}^p})(\underline{v}-\underline{c})$, and, for $\gamma\to 1$, the right-hand side exceeds the left-hand side, i.e., $(1-e^{-\overline{\lambda}^p}-\overline{\lambda}^p e^{-\overline{\lambda}^p})(\overline{v}-\overline{c}) < \underline{v}-\underline{c}$. If the solution exists, it is unique. In such cases, the solution, denoted by γ^*, is incentive compatible for low-type sellers if (S.6) is satisfied, which, substituting the values of $\underline{U}$ and $\overline{U}$, can be rewritten as[5]

$$\frac{1}{\frac{\overline{\lambda}^p}{\gamma^*}}\left(1-e^{-\frac{\overline{\lambda}^p}{\gamma^*}}\right)(\overline{c}-\underline{c}) \leq e^{-\frac{\underline{\lambda}^p}{1-\gamma^*}}(\underline{v}-\underline{c}) - e^{-\frac{\overline{\lambda}^p}{\gamma^*}}(\overline{v}-\overline{c}). \tag{S.8}$$

[4]Note that under this condition, the price offered to low-type sellers is strictly smaller than $\overline{c}$. In particular, $p = \underline{c}+(\underline{\lambda}^p e^{-\underline{\lambda}^p})/(1-e^{-\underline{\lambda}^p})(\underline{v}-\underline{c}) < \overline{c}$ if $(1-e^{-\underline{\lambda}^p})(\overline{c}-\underline{c}) > \underline{\lambda}^p e^{-\underline{\lambda}^p}(\underline{v}-\underline{c}) \Leftrightarrow (1-e^{-\underline{\lambda}^p}-\underline{\lambda}^p e^{-\underline{\lambda}^p})(\underline{v}-\underline{c}) > (1-e^{-\underline{\lambda}^p})(\underline{v}-\overline{c})$, which is satisfied by $(1-e^{-\underline{\lambda}^p})(\underline{v}-\overline{c}) < \overline{v}-\overline{c}$.

[5]Note that a necessary condition for (S.7) and (S.8) both to be satisfied is $\overline{v}-\overline{c} < \underline{v}-\underline{c}$. Suppose this is not the case. Then (S.7) implies $\overline{\lambda}^p/\gamma^* < \underline{\lambda}^p/(1-\gamma^*)$, which in turn implies $e^{-\underline{\lambda}^p/(1-\gamma^*)}(\underline{v}-\underline{c}) < e^{-\overline{\lambda}^p/\gamma^*}(\overline{v}-\overline{c})$, making the right-hand side of (S.8) negative.

(ii)(b) Consider next the possibility of an equilibrium in which the incentive compatibility of low-type sellers is satisfied with equality. The optimal values of $\overline{\lambda}$ and $\underline{\lambda}$ are then, respectively, given by $\overline{\lambda}^p/\gamma$ and $\underline{\lambda}^p/(1-\gamma)$ if $\underline{U} = e^{-\underline{\lambda}^p/(1-\gamma)}(\underline{v}-\underline{c})$ and $\overline{U}$ is such that $(1-e^{-\overline{\lambda}^p/\gamma})/(\overline{\lambda}^p/\gamma) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, that is,

$$\overline{U} = e^{-\frac{\underline{\lambda}^p}{1-\gamma}}(\underline{v}-\underline{c}) - \frac{1}{\frac{\overline{\lambda}^p}{\gamma}}\left(1-e^{-\frac{\overline{\lambda}^p}{\gamma}}\right)(\overline{c}-\underline{c}). \tag{S.9}$$

With this, a buyer's payoff when attracting high types is given by

$$\left(1-e^{-\frac{\overline{\lambda}^p}{\gamma}}\right)(\overline{v}-\underline{c}) - \frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}(\underline{v}-\underline{c}).$$

The indifference condition of the buyer is thereby

$$\left(1-e^{-\frac{\overline{\lambda}^p}{\gamma}}\right)(\overline{v}-\underline{c}) - \frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}(\underline{v}-\underline{c}) = \left(1-e^{-\frac{\underline{\lambda}^p}{1-\gamma}} - \frac{\underline{\lambda}^p}{1-\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right)(\underline{v}-\underline{c}). \tag{S.10}$$

Let

$$A \equiv \left(1-e^{-\frac{\overline{\lambda}^p}{\gamma}}\right)(\overline{v}-\underline{c}) - \frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}(\underline{v}-\underline{c}) - \left(1-e^{-\frac{\underline{\lambda}^p}{1-\gamma}} - \frac{\underline{\lambda}^p}{1-\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right)(\underline{v}-\underline{c})$$

be defined as the difference between the left- and right-hand sides of the above equation. As can be verified, A is strictly increasing in γ on $[0, \overline{\lambda^p}/(\overline{\lambda^p}+\underline{\lambda}^p)]$ with $\lim_{\gamma\to 0} A = -\infty$ and $\lim_{\gamma\to\overline{\lambda^p}/(\overline{\lambda^p}+\underline{\lambda}^p)} = (1-e^{-\overline{\lambda}^p-\underline{\lambda}^p})(\overline{v}-\underline{v}) > 0$, which implies that (S.10) has a unique solution on $(0, \overline{\lambda^p}/(\overline{\lambda^p}+\underline{\lambda}^p))$. Let this solution be denoted by $\hat{\gamma}$.

Suppose now that $\overline{v}-\overline{c} > (1-e^{-\underline{\lambda}^p} - \underline{\lambda}^p e^{-\underline{\lambda}^p})(\underline{v}-\underline{c})$ and that the solution of (S.7), γ^*, does not satisfy (S.8), so that neither of the previous two types of equilibria exists. An equilibrium characterized by (S.10) exists if $\underline{U}$ and $\overline{U}$ are such that the low-type incentive constraint is indeed binding. Since, by construction, the low-type incentive constraint is satisfied with equality at $\overline{\lambda}^p/\hat{\gamma}$, this is the case if the solution to the unrestricted problem, $\overline{\lambda}^* = \ln((\overline{v}-\overline{c})/\overline{U})$, is strictly smaller than $\frac{\overline{\lambda}^p}{\hat{\gamma}}$; that is,

$$\overline{U} > e^{-\frac{-\overline{\lambda}^p}{\hat{\gamma}}}(\overline{v}-\overline{c}).$$

By substituting (S.9) for $\overline{U}$ and using condition (S.10), after some rearranging, the above inequality can be written as

$$\left(1-e^{-\frac{\overline{\lambda}^p}{\hat{\gamma}}} - \frac{\overline{\lambda}^p}{\hat{\gamma}}e^{-\frac{\overline{\lambda}^p}{\hat{\gamma}}}\right)(\overline{v}-\overline{c}) > \left(1-e^{-\frac{\underline{\lambda}^p}{1-\hat{\gamma}}} - \frac{\underline{\lambda}^p}{1-\hat{\gamma}}e^{-\frac{\underline{\lambda}^p}{1-\hat{\gamma}}}\right)(\underline{v}-\underline{c}). \tag{S.11}$$

Notice that this inequality is satisfied with equality at $\hat{\gamma} = \gamma^*$ and that at γ^*, we have

$$A = \left(1 - e^{-\frac{\overline{\lambda}^p}{\gamma^*}}\right)(\overline{v} - \underline{c}) - \frac{\overline{\lambda}^p}{\gamma^*} e^{-\frac{\underline{\lambda}^p}{1-\gamma^*}}(\underline{v} - \underline{c}) - \left(1 - e^{-\frac{\overline{\lambda}^p}{\gamma^*}} - \frac{\overline{\lambda}^p}{\gamma} e^{-\frac{\overline{\lambda}^p}{\gamma^*}}\right)(\overline{v} - \overline{c})$$

$$= \frac{\overline{\lambda}^p}{\gamma^*}\left[\frac{1}{\frac{\overline{\lambda}^p}{\gamma^*}}\left(1 - e^{-\frac{\overline{\lambda}^p}{\gamma^*}}\right)(\overline{c} - \underline{c}) - \left(e^{-\frac{\underline{\lambda}^p}{1-\gamma^*}}(\underline{v} - \underline{c}) - e^{-\frac{\overline{\lambda}^p}{\gamma^*}}(\overline{v} - \overline{c})\right)\right].$$

The above term is strictly positive by violation of condition (S.8), which implies that $\hat{\gamma} < \gamma^*$. Since, as noted earlier, the left-hand side of (S.11) strictly decreases in $\hat{\gamma}$, while the right-hand side strictly increases in $\hat{\gamma}$, this implies that (S.11) is indeed satisfied under the stated conditions and, hence, that the low-type incentive constraint is binding. The equilibrium therefore exists.

These three cases cover the whole parameter space, demonstrating that an equilibrium with the properties specified in Proposition 4.3 exists. □

References

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