Simple Contracts with Adverse Selection and Moral Hazard: Online Appendix

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Contents

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1 Binding Participation Constraint

In the paper, we assumed that only the agent's limited liability, but not the participation constraint was binding. This appendix complements the paper by considering cases in which the participation constraint binds.

For simplicity, we assume that there are two efforts, two outputs, and two types ($\Theta = \{A, B\}$). Suppose without loss of generality that

$$\frac{\Delta c^A}{\Delta p^A} \geq \frac{\Delta c^B}{\Delta p^B}.$$

A feasible mechanism is a four-dimensional vector (s^A, b^A, s^B, b^B) satisfying the constraints (IC$_1^\theta$)-(IC$_3^\theta$), (LL$^\theta$) and (IR$^\theta$). The following proposition shows that with only two type, optimal mechanism may entail full separation.

Proposition 1. *In a model with two types, two levels of efforts and two outputs, there are economies where full separation is the optimal mechanism.*

Proof. Let $(s^i, b^i)_{i\in\{A,B\}}$ be any feasible mechanism such that $b^A \neq b^B$. We will show whether we can or cannot dominate this mechanism by a pooling one. Let us divide the analysis in several cases.

Case $b^B > b^A$**.** Notice that constraint (IC$_2^\theta$) is equivalent to $p_1^A(b^A - b^B) \geq -(s^A - s^B) \geq p_1^B(b^A - b^B)$. Then, $p_1^B \geq p_1^A$ and $s^A > s^B$. Increase b^A or decrease s^A in order to keep the expected benefit of type A (i.e., $s^A + p_1^A b^A$) constant and $b^B \geq b^A$. We can do this until we get the new contract $(\widetilde{s}^A, \widetilde{b}^A)$ satisfying

$$s^B + p_1^B b^B = \widetilde{s}^A + p_1^B \widetilde{b}^A \text{ or } \widetilde{s}^A = s^B \text{ and } \widetilde{b}^A = b^B.$$

In both case, the mechanism formed by the only contract $(\widetilde{s}^A, \widetilde{b}^A)$ satisfies all the constraints (here it is important that $\widetilde{b}^A \geq b^A$) and generates the same expected cost to the principal.

Case $b^A > b^B$**.** Analogous to the previous case, $p_1^A \geq p_1^B$ and $s^B > s^A$. If $s^A + p_1^B b^A \geq c_1^B$, then keeping only contract (s^A, b^A) satisfies all constraints and reduces the principal's expected cost. Suppose that $c_1^B > s^A + p_1^B b^A$. Now decrease b^A and increase s^A keeping the expected benefit of type A (i.e., $s^A + p_1^A b^A$) constant. We can do this until the new contract $(\widetilde{s}^A, \widetilde{b}^A)$ satisfies:

Subcase $c_1^B = \widetilde{s}^A + p_1^B \widetilde{b}^A$**.** The mechanism formed by the only contract $(\widetilde{s}^A, \widetilde{b}^A)$ satisfies all constraints if $\widetilde{b}^A \geq \frac{\Delta c^A}{\Delta p^A}$ and generates weaker expected cost to the principal.

Subcase $\widetilde{b}^A = \frac{\Delta c^A}{\Delta p^A}$**.** We have that

$$s^A + p_1^A b^A = \widetilde{s}^A + p_1^A \frac{\Delta c^A}{\Delta p^A} \geq s^B + p_1^A b^B \geq s^B + p_1^B b^B \geq c_1^B > \widetilde{s}^A + p_1^B \frac{\Delta c^A}{\Delta p^A}.$$

Now increase b^B and decrease s^B keeping the expected benefit of type B (i.e., $s^B + p_1^B b^B$) constant. We can do this until the new contract $(\widetilde{s}^B, \widetilde{b}^B)$ satisfies:

Subcase $\widetilde{s}^B = \widetilde{s}^A$**.** We have that $\widetilde{b}^B = \widetilde{b}^A$ and the mechanism formed by the only contract $(\widetilde{s}^A, \widetilde{b}^A)$ satisfies all constraints and generates weaker expected cost to the principal.

Subcase $\widetilde{s}^A + p_1^A \frac{\Delta c^A}{\Delta p^A} = \widetilde{s}^B + p_1^A \widetilde{b}^B$**.** We have that

$$s^A + p_1^A b^A = \widetilde{s}^A + p_1^A \frac{\Delta c^A}{\Delta p^A} = \widetilde{s}^B + p_1^A \widetilde{b}^B \geq \widetilde{s}^B + p_1^B \widetilde{b}^B = s^B + p_1^B b^B \geq c_1^B > \widetilde{s}^A + p_1^B \frac{\Delta c^A}{\Delta p^A}.$$

Notice that the mechanism formed only by contract $(\widetilde{s}^B, \widetilde{b}^B)$ satisfies all constraints if and only if constraint (IC$_1^A$) is satisfied, i.e., $\widetilde{b}^B \geq \frac{\Delta c^A}{\Delta p^A}$.

Suppose that $\widetilde{b}^B < \frac{\Delta c^A}{\Delta p^A}$. The constraint that can be violated is (IC$_3^\theta$), which can be written as

$$\begin{array}{l} s^A + p_1^A b^A \geq s^B + p_1^A b^B - \Delta p^A b^B + \Delta c^A \\ s^B + p_1^B b^B \geq s^A + p_1^B b^A - \Delta p^B b^A + \Delta c^B \end{array}.$$

Since $\frac{\Delta c^A}{\Delta p^A} > \widetilde{b}^B \geq b^B$, (IC$_3^A$) is stronger than (IC$_2^A$) and, since $b^A \geq \frac{\Delta c^A}{\Delta p^A} \geq \frac{\Delta c^B}{\Delta p^B}$, (IC$_3^B$) is slack in the presence of (IC$_2^B$). Repeating the same steps above but replacing (IC$_2^A$) by (IC$_3^A$), we have the constraints:

$$s^A + p_1^A b^A = \widetilde{s}^A + p_1^A \frac{\Delta c^A}{\Delta p^A} \geq s^B + p_1^A b^B - \Delta p^A b^B + \Delta c^A \geq s^B + p_1^B b^B \geq c_1^B > \widetilde{s}^A + p_1^B \frac{\Delta c^A}{\Delta p^A}.$$

Therefore, in this case, the principal's problem is to find $(\widetilde{s}^A, s^B, b^B)$ to minimize the expected cost subject to

$$\begin{array}{l} \frac{\Delta c^B}{\Delta p^B} \leq b^B \leq \frac{\Delta c^A}{\Delta p^A} \\ \widetilde{s}^A + p_1^A \frac{\Delta c^A}{\Delta p^A} \geq c_1^A \\ \widetilde{s}^A + p_1^A \frac{\Delta c^A}{\Delta p^A} \geq s^B + p_0^A b^B + \Delta c^A \\ s^B + p_1^B b^B \geq c_1^B \\ s^B + p_1^B b^B \geq \widetilde{s}^A + p_1^B \frac{\Delta c^A}{\Delta p^A}. \end{array}$$

By reducing $\widetilde{s}^A$ and s^B the principal will benefit. We can reduce $\widetilde{s}^A$ until $\widetilde{s}^A + p_1^A \frac{\Delta c^A}{\Delta p^A} = c_1^A$ or $\widetilde{s}^A = 0$. We can reduce s^B until (i) $s^B = 0$ or (ii) $s^B + p_1^B b^B = c_1^B$ or (iii) $s^B + p_1^B b^B = \widetilde{s}^A + p_1^B \frac{\Delta c^A}{\Delta p^A}$.

If (i) holds, then by the first and last constraints of the minimization problem above we must have $b^B = \frac{\Delta c^A}{\Delta p_A}$ and $\widetilde{s}^A = 0$.

If (ii) holds, then increase b^B and decrease s^B to keep (ii) true. Thus, $b^B = \frac{\Delta c^A}{\Delta p^A}$ or $s^B = 0$ (which again implies $b^B = \frac{\Delta c^A}{\Delta p^A}$) or (iii) holds. If $p_0^A \leq p_1^B$, then (iii) cannot hold.

Suppose that $p_0^A > p_1^B$. As we see above, there are two possibilities: $\widetilde{s}^A + p_1^A \frac{\Delta c^A}{\Delta p^A} = c_1^A$ or $\widetilde{s}^A = 0$. Notice that if $\widetilde{s}^A + p_1^A \frac{\Delta c^A}{\Delta p^A} = c_1^A$, then $s^B + p_1^B b^B = c_1^B$ if and only if the third constraint of the minimization problem above is binding. In this case, the solution of the problem is the solution of the system of equations:

$$\begin{array}{l} s^B + p_0^A b^B = c_0^A \\ s^B + p_1^B b^B = c_1^B \end{array}$$

which is

$$b^B = \frac{c_0^A - c_1^B}{p_0^A - p_1^B} \text{ and } s^B = c_0^A - p_0^A b^B = c_1^B - p_1^B b^B.$$

From this analysis we get the following possible examples where the optimal mechanism entails full separation.

Counter-example 1. We need parameters such that

$$\begin{array}{l} p_0^A > p_1^B \\ \frac{\Delta c^A}{\Delta p^A} > \frac{c_0^A - c_1^B}{p_0^A - p_1^B} \geq \frac{\Delta c^B}{\Delta p^B} \\ \frac{c_1^B}{p_1^B} \geq \frac{\Delta c^A}{\Delta p^A} = \frac{c_1^A}{p_1^A}. \end{array}$$

In this case the optimal contract should be

$$\left(0, \frac{\Delta c^A}{\Delta p^A}\right) \text{ and } \left(c_1^B - p_1^B \frac{c_0^A - c_1^B}{p_0^A - p_1^B}, \frac{c_0^A - c_1^B}{p_0^A - p_1^B}\right).$$

Counter-example 2. We need parameters such that

$$\begin{array}{l} p_0^A > p_1^B \\ \frac{\Delta c^A}{\Delta p^A} > \frac{c_0^A - c_1^B}{p_0^A - p_1^B} \geq \frac{\Delta c^B}{\Delta p^B} \\ \min\left\{\frac{c_1^A}{p_1^A}, \frac{c_1^B}{p_1^B}\right\} \geq \frac{\Delta c^A}{\Delta p^A} \\ c_1^B \geq c_1^A - (p_1^A - p_1^B)\frac{\Delta c^A}{\Delta p^A}, \end{array}$$

which is equivalent to

$$\frac{\Delta c^A}{\Delta p^A} \geq \frac{c_1^A - c_1^B}{p_1^A - p_1^B}.$$

In this case the optimal contract should be

$$\left(c_1^A - p_1^A \frac{\Delta c^A}{\Delta p^A}, \frac{\Delta c^A}{\Delta p^A}\right) \text{ and } \left(c_1^B - p_1^B \frac{c_0^A - c_1^B}{p_0^A - p_1^B}, \frac{c_0^A - c_1^B}{p_0^A - p_1^B}\right).$$

Counter-example 3. We need parameters such that

$$\begin{array}{l} p_0^A > p_1^B \\ \frac{\Delta c^A}{\Delta p^A} > b^B \geq \frac{\Delta c^B}{\Delta p^B} \\ \frac{c_1^B}{p_1^B} \geq \frac{\Delta c^A}{\Delta p^A} \geq \frac{c_1^A}{p_1^A}, \end{array}$$

where

$$b^B = \frac{p_1^A \frac{\Delta c^A}{\Delta p^A} - \Delta c^A - c_1^B}{p_0^A - p_1^B} = \frac{p_0^A \frac{\Delta c^A}{\Delta p^A} - c_1^B}{p_0^A - p_1^B}.$$

In this case the optimal contract should be

$$\left(0, \frac{\Delta c^A}{\Delta p^A}\right) \text{ and } \left(c_1^B - p_1^B b^B, b^B\right).$$

□

2 Random Mechanisms

Following Kadan et al. (2017), a *contract* is a function that specifies a distribution of transfer to the agent conditional on each possible output, effort and type. According to Kadan et al. (2017), a random mechanism is any $(\omega, \mathcal{E})$ such that $\omega : X \times E \times \Theta \to \Delta(\mathbb{R})$and $\mathcal{E} : \Theta \to \Delta(E)$ are transition probability,[1] where $\Delta(\mathbb{R})$ and $\Delta(E)$ represent the space of lotteries on $\mathbb{R}$ and E. Given a mechanism $(\omega, \mathcal{E})$ a type-θ agent gets expected payoff

$$U(\theta) := \int_E \left[\sum_{i=1}^N \int_{\mathbb{R}} r d\omega(r|x_i, e, \theta) p_e^\theta(x_i) - c_e^\theta\right] d\mathcal{E}(e|\theta). \tag{1}$$

The feasible mechanism has to satisfy the following IC, IR, and LL constraints:

$$U(\theta) \geq \int_E \sup_{e \in E} \left[\sum_{i=1}^N \int_{\mathbb{R}} r d\omega(r|x_i, \hat{e}, \hat{\theta}) p_e^\theta(x_i) - c_e^\theta\right] d\mathcal{E}(\hat{e}|\hat{\theta}), \quad \forall \theta, \hat{\theta}, \tag{IC}$$

[1]A transition probability is a mapping, γ say, from A into $\Delta(B)$ such that, for every measurable $M \subset B$, $\gamma(M|a)$ is a measurable function of $a \in A$

$$U(\theta) \geq 0, \ \ \forall \theta, \tag{IR}$$

$$\text{supp } \{\omega(\cdot|x_i, e, \theta)\} \subset \mathbb{R}_+, \ \ \forall \theta, e, i, \tag{LL}$$

where supp $\{\lambda\}$ means the support of the measure λ. An *optimal* mechanism maximizes the principal's expected profit

$$\int_\Theta \int_E \sum_{i=1}^N p_e^\theta(x_i) \left[x_i - \int_\mathbb{R} r d\omega(r|x_i, e, \theta)\right] d\mathcal{E}(e|\theta) d\mu(\theta) \tag{2}$$

among mechanisms that satisfy IC, IR and LL.

For each random mechanism $(\omega, \mathcal{E})$, let us define the following expected payments (with some abuse of notation):

$$w_i^{\theta,e} = \int r d\omega(r|x_i, e, \theta) \text{ and } w_i^\theta = \int \int r d\omega(r|x_i, \hat{e}, \theta) d\mathcal{E}(\hat{e}|\theta) = \int w_i^{\theta,\hat{e}} d\mathcal{E}(\hat{e}|\theta)$$

for each θ, e, i. We will use this notation throughout in what follows.

Remark 1. A *deterministic contract* is a function that specifies a transfer to the agent conditional on each possible output. A deterministic mechanism specifies a contract and an effort recommendation for each type. That is, a mechanism is a pair of measurable functions $w : \Theta \times X \to \mathbb{R}$ and $e : \Theta \to E$, so that a type-θ agent is recommended effort $e(\theta)$ and gets paid $w^\theta(x)$ in case of output x.

The next theorem extends Theorem 1 for random mechanisms.

Theorem 1. *Suppose MS holds and E is finite.*[2] *There exists an essentially unique optimal stochastic mechanism that offers a single deterministic contract to all types.*

Proof of Theorem 1

Let us start with a basic and important lemma that shows that, in every IC random mechanism, the agent is indifferent among all efforts in the support of the effort recommendation distribution.

Lemma 1. *Suppose that $(\omega, \mathcal{E})$ is a (random) mechanism satisfying IC. Then, for every $\theta \in \Theta$ and e in the support of $\mathcal{E}(\cdot|\theta)$,*

$$U(\theta) = \sum_i p_{e,i}^\theta w_i^\theta - c_e^\theta,$$

where $U(\theta)$ is defined by (1).

Proof. From (IC), taking $\hat{\theta} = \theta$, it is straightforward to see that $U(\theta) \geq \sum_i p_{e,i}^\theta w_i^\theta - c_e^\theta$, for all $e \in E$. Suppose, by absurd, that there exists e in the support of $\mathcal{E}(\cdot|\theta)$ for which this last inequality is strict. Then, we could reallocate the weight from e to some $\hat{e}$ in the support of $\mathcal{E}(\cdot|\theta)$ that satisfies

$$\sum_i \int r d\omega(r|x_i, \hat{e}, \theta) p_{\hat{e},i}^\theta - c_{\hat{e}}^\theta > \sum_i p_{e,i}^\theta w_i^\theta - c_e^\theta.$$

This would generate a new effort distribution with a higher payoff to the agent, i.e., a profit deviation for the agent from the original effort recommendation of the principal. This is a contradiction. □

The following result will be important in order to establish existence of an optimal random mechanism, by allowing us to restrict the set of possible contracts to a compact set.

[2] We can extend the proof for the general case of compact metric space with burden of extra technical assumptions on the space of feasible mechanisms.

Lemma 2. *Let $(\omega, \mathcal{E})$ be a mechanism satisfying IC and LL. Suppose that $w_i^\theta > \frac{\Delta x}{\underline{p}^2}$ for some i and positive measure set of θ. Then $(\omega, \mathcal{E})$ is not optimal.*

Proof. The proof has two steps. First, it shows that if an incentive-compatible mechanism offers a high enough expected payment in one state, then every other contract must also have a high enough expected payment in some state (otherwise, everyone would prefer the former contract). Second, it shows that any contract that makes a high enough expected payment in some state is dominated by than the null contract.
Step 1. Suppose the mechanism offers an expected contract $\tilde{w} = (\tilde{w}_1, ..., \tilde{w}_N)$ with $\tilde{w}_i > \frac{\Delta x}{\underline{p}^2}$ for some output i. Let θ be a type that picks a contract with expected contract w^θ. By Lemma 1, this type's incentive-compatibility constraint gives:

$$\sum_{i=1}^{N} p_{e,i}^\theta w_i^\theta - c_e^\theta \geq \sum_{i=1}^{N} p_{e,i}^\theta \tilde{w}_i - c_e^\theta$$

for all e in the support of $\mathcal{E}(\cdot|\theta)$. Since $\max\left\{w_1^\theta, ..., w_N^\theta\right\} \geq \sum_{i=1}^N w_i^\theta p_{e,i}^\theta$ and $\tilde{w}_i \geq 0$ for all i, this inequality implies the following:

$$\max_{i \in \{1,...,N\}} \left\{w_i^\theta\right\} \geq \underline{p}\tilde{w}_i > \frac{\Delta x}{\underline{p}},$$

where the last inequality uses $\tilde{w}_i > \frac{\Delta x}{\underline{p}^2}$.
Step 2. We now show that this mechanism gives the principal a lower payoff than offering the contract that always pays zero to all types. Since the probability is bounded below by $\underline{p}$ and payments are non-negative, the principal's payoff from offering $w_i^{\theta,e}$ to type θ is

$$\int_E \left[\sum_{i=1}^{N} p_{e,i}^\theta \left(x_i - w_i^{\theta,e}\right)\right] d\mathcal{E}(e|\theta) \leq x_N - \underline{p}w_i^\theta, \quad \forall i.$$

Let $e_0 \in \arg\max c_e^\theta$. The principal's payoff from offering type θ a zero payment in all states is $\sum_{i=1}^N p_{e^0,i}^\theta x_i \geq x_1$. Combining these two inequalities, we obtain the following necessary condition for w^θ to give a weakly higher payoff to the principal than the null contract:

$$w_i^\theta \leq \frac{\Delta x}{\underline{p}}, \quad \forall i.$$

Thus, if

$$w_i^\theta > \frac{\Delta x}{\underline{p}}$$

for some i and positive measure set of θ, then the principal is strictly better offering the null contract. □

Let $(\omega, \mathcal{E})$ be a feasible mechanism. Let $\mathcal{M} := \left\{w^\theta : \theta \in \Theta\right\}$ denote the set of all expected contracts in this mechanism. By Lemma 2, there is no loss of generality in assuming that $\mathcal{M}$ is bounded. Its closure, $\bar{\mathcal{M}}$, is compact. There are three cases to consider:

Case 1) $\sum_{i=1}^N h_i(x_i - w_i^\theta) \geq 0$**, for all** $\theta \in \Theta$**.** Let $w^+ \in \arg\max_{w \in \bar{\mathcal{M}}} \sum_{i=1}^N h_i w_i$, which exists because $\bar{\mathcal{M}}$ is compact (from the previous lemma and E is compact) and the objective function is a continuous linear functional. Let $e^+(\theta)$ be an effort that maximizes the agent's payoff under contract w^+ (see the online appendix for existence). Then, using Lemma 1, the agent's payoff with the effort e chosen in the support of $\mathcal{E}$ cannot exceed the agent's payoff with effort $e^+(\theta)$, which by MS, can be written as

$$\left[I\left(e,\theta\right)-I\left(e^+(\theta),\theta\right)\right]\sum_{i=1}^{N}h_i w_i^+ \geq c^{\theta}_{e^+(\theta)}-c^{\theta}_{e}, \tag{3}$$

with strict inequality in case e is a suboptimal effort choice for type θ under contract w^+. Similarly, because e is in the support of the agent's effort choice with contract w^θ,

$$\left[I\left(e,\theta\right)-I\left(e^+(\theta),\theta\right)\right]\sum_{i=1}^{N}h_i w_i^\theta \leq c^{\theta}_{e^+(\theta)}-c^{\theta}_{e}. \tag{4}$$

Combining (3) and (4), we obtain

$$\begin{aligned}\left[I\left(e,\theta\right)-I\left(e^+(\theta),\theta\right)\right]\textstyle\sum_{i=1}^{N}h_i w_i^\theta &\leq c^{\theta}_{e^+(\theta)}-c^{\theta}_{e}\\ &\leq \left[I\left(e,\theta\right)-I\left(e^+(\theta),\theta\right)\right]\textstyle\sum_{i=1}^{N}h_i w_i^+,\end{aligned} \tag{5}$$

where the last inequality is strict for e in the support of $\mathcal{E}(\cdot|\theta)$ which is a suboptimal effort choice for type θ under contract w^+. By the definition of w^+,

$$\sum_{i=1}^{N}h_i w_i^+ \geq \sum_{i=1}^{N}h_i w_i^\theta$$

for all θ. Therefore, if $\sum_{i=1}^{N}h_i w_i^+ > \sum_{i=1}^{N}h_i w_i^{\theta,e}$, it follows from MS and (5) that $I\left(e,\theta\right)\geq I\left(e^+(\theta),\theta\right)$. If $\sum_{i=1}^{N}h_i w_i^+ = \sum_{i=1}^{N}h_i w_i^\theta$, then e cannot be a suboptimal effort choice in its support for type θ under contract w^+ since in this case the second inequality being strict would lead to contradiction. Therefore, we can take $c^{\theta}_{e^+(\theta)}\geq c^{\theta}_{e}$, which, again, gives $I\left(e,\theta\right)\geq I\left(e^+(\theta),\theta\right)$.

We now establish that replacing contract $\omega(\cdot|x_i,e,\theta)$ by w^+ increases the principal's payoff from type θ. We first show that, holding effort fixed, the principal is better off with the substitution of contracts. Since w^+ is the limit of sequence in $\mathcal{M}$, the agent's utility is continuous, and the original mechanism is incentive compatible, it follows that

$$\int_E\int_{\mathbb{R}} v_e^\theta(r)d\omega(r|\cdot,e,\theta)d\mathcal{E}(e|\theta)\geq \int_E v_e^\theta(w^+)d\mathcal{E}(e|\theta). \tag{6}$$

Substitute the expression for the agent's payoff, multiply both sides by -1, and add $\int_E\sum_{i=1}^{N}p^\theta_{e,i}x_i d\mathcal{E}(e|\theta)$ to both sides to write:

$$\int_E\sum_{i=1}^{N}p^\theta_{e,i}(x_i-w_i^+)d\mathcal{E}(e|\theta)\geq\int_E\sum_{i=1}^{N}p^\theta_{e,i}(x_i-\int_{\mathbb{R}}rd\omega(r|x_i,e,\theta))d\mathcal{E}(e|\theta), \tag{7}$$

which states that, holding effort $\mathcal{E}(\cdot|\theta)$ fixed, the principal gets a higher profit with contract w^+ than with $\omega(\cdot|x_i,e,\theta)$.

To show that the change in effort also benefits the principal, notice that since $\int_E I\left(e,\theta\right)d\mathcal{E}(e|\theta)\geq I\left(e^+(\theta),\theta\right)$ and $w^+\in\bar{\mathcal{M}}$ (so that $\sum_{i=1}^{N}h_i(x_i-w_i^+)\geq 0$), the following inequality holds:

$$\left[\int_E I\left(e,\theta\right)d\mathcal{E}(e|\theta)-I\left(e^+(\theta),\theta\right)\right]\sum_{i=1}^{N}h_i\left(x_i-w_i^+\right)\geq 0.$$

Use MS to rewrite this inequality as

$$\sum_{i=1}^{N} p^{\theta}_{e^{+}(\theta),i}\left(x_i - w_i^{+}\right) \geq \int_E \sum_{i=1}^{N} p^{\theta}_{e,i}\left(x_i - w_i^{+}\right) d\mathcal{E}(e|\theta), \tag{8}$$

which states that the principal gains from the change in effort.

Combining (7) and (8) establishes that the principal's profit from θ with the new contract exceeds her profit with the original contract:

$$u^{\theta}_{e^{+}(\theta)}(w^{+}) \geq \int_E \sum_{i=1}^{N} p^{\theta}_{e,i}(x_i - \int_{\mathbb{R}} r d\omega(r|x_i, e, \theta)) d\mathcal{E}(e|\theta).$$

By construction, the mechanism (w^{+}, e^{+}) is incentive compatible and satisfies LL. Moreover, from the previous argument, it raises the principal's payoff pointwise (i.e. it raises the principal's payoff conditional on each type).

Case 2) $\sum_{i=1}^{N}(x_i - w_i^{\theta})h_i \leq 0$, for all $\theta \in \Theta$. The proof of case 2 is similar to the one of case 1, except that, instead of substituting all contracts by the one that maximizes $\sum_{i=1}^{N} h_i w_i$, we substitute them by the one that minimizes this expression. Formally, let $w^{-} \in \arg\min_{w \in \bar{\mathcal{M}}} \sum_{i=1}^{N} h_i w_i$ and let $e^{-}(\theta)$ be an effort that maximizes the agent's payoff under contract w^{-}.

As in case 1, incentive compatibility gives

$$\begin{aligned} \left[I(e,\theta)\, d\mathcal{E}(e|\theta) - I\left(e^{-}(\theta), \theta\right)\right] \textstyle\sum_{i=1}^{N} h_i w_i^{\theta} d\mathcal{E}(e|\theta) &\leq c^{\theta}_{e^{-}(\theta)} - c^{\theta}_{e} \\ &\leq \left[I(e,\theta) - I\left(e^{-}(\theta), \theta\right)\right] \textstyle\sum_{i=1}^{N} h_i w_i^{-}. \end{aligned} \tag{9}$$

Since $w^{-} \in \bar{\mathcal{M}}$, it satisfies

$$\sum_{i=1}^{N} h_i w_i^{-} \leq \sum_{i=1}^{N} h_i w_i^{\theta,e},$$

so that, by inequality (9), it follows from same argument as in case 1 that $I\left(e^{-}(\theta), \theta\right) \geq \int_E I(e,\theta)\, d\mathcal{E}(e|\theta)$. As in case 1, incentive compatibility implies that, holding effort in the support of $\mathcal{E}(\cdot|\theta)$ fixed, the principal's profit is higher with contract w^{-} than with $\omega(\cdot|x_i, e, \theta)$:

$$\int_E \sum_{i=1}^{N} p^{\theta}_{e,i}(x_i - w_i^{-}) d\mathcal{E}(e|\theta) \geq \int_E \sum_{i=1}^{N} p^{\theta}_{e,i}(x_i - \int_{\mathbb{R}} r d\omega(r|x_i, e, \theta)) d\mathcal{E}(e|\theta). \tag{10}$$

Next, we show that the change in effort also benefits the principal. Because $I\left(e^{-}(\theta), \theta\right) \geq \int_E I(e,\theta)\, d\mathcal{E}(e|\theta)$, and because $w^{-} \in \bar{\mathcal{M}}$, the following inequality holds:

$$\left[\int_E I(e,\theta)\, d\mathcal{E}(e|\theta) - I\left(e^{-}(\theta), \theta\right)\right] \sum_{i=1}^{N} h_i (x_i - w_i^{-}) \geq 0.$$

Use MS to rewrite this inequality as

$$\sum_{i=1}^{N} p^{\theta}_{e^{-}(\theta),i}(x_i - w_i^{-}) \geq \int_E \sum_{i=1}^{N} p^{\theta}_{e,i}(x_i - w_i^{-}) d\mathcal{E}(e|\theta), \tag{11}$$

which shows that the principal gains from the change in effort. Combining (10) and (11) establishes that the principal's profit from θ with the new contract exceeds her profit with the original contract:

$$u^{\theta}_{e^-(\theta)}(w^-) \geq \int_E \sum_{i=1}^N p^{\theta}_{e,i}(x_i - \int_{\mathbb{R}} r d\omega(r|x_i, e, \theta)) d\mathcal{E}(e|\theta).$$

Therefore, the mechanism (w^-, e^-) is incentive compatible, satisfies LL, and increases the principal's payoff pointwise relative to the original mechanism.

Case 3) There exist $\theta_+, \theta_- \in \Theta$ for which $\sum_{i=1}^N h_i(x_i - w_i^{\theta_+}) \geq 0 \geq \sum_{i=1}^N h_i(x_i - w_i^{\theta_-})$. First, we establish that, because of the risk neutrality and limited liability, introducing "scaled down versions" of the contracts from the original mechanism preserves incentive compatibility, meaning that no type would benefit from deviating to such a contract. More precisely, let

$$\mathcal{N} = \{\alpha w^{\theta};\ \theta \in \Theta \text{ and } \alpha \in [0, 1]\}$$

denote the menu of contracts obtained by introducing scaled down versions of all contracts in $\mathcal{M}$. Then, for all $\theta, \hat{\theta} \in \Theta$ and $\alpha \in [0, 1]$,

$$\begin{aligned} \int_E v_e^{\theta}(w^{\theta,e}) d\mathcal{E}(e|\theta) &\geq \int_E \sup_{e \in E} \left[\sum_{i=1}^N \int_{\mathbb{R}} r d\omega(r|x_i, \hat{e}, \hat{\theta}) p_e^{\theta}(x_i) - c_e^{\theta}\right] d\mathcal{E}(\hat{e}|\hat{\theta}) \\ &\geq \int_E \sup_{e \in E} \left[\sum_{i=1}^N \int_{\mathbb{R}} \alpha r d\omega(r|x_i, \hat{e}, \hat{\theta}) p_e^{\theta}(x_i) - c_e^{\theta}\right] d\mathcal{E}(\hat{e}|\hat{\theta}), \end{aligned}$$

where the first inequality follows from incentive compatibility of the original mechanism and the second inequality follows from $d\omega(r|x_i, \hat{e}, \hat{\theta}) \geq \alpha d\omega(r|x_i, \hat{e}, \hat{\theta})$ (by LL and the fact that $\alpha \leq 1$). Therefore, there is no loss of generality in assuming that the principal offers the menu of contracts $\mathcal{N}$ rather than $\mathcal{M}$.

Let $w^0 \in \mathcal{N}$ be a contract that satisfies

$$\sum_{i=1}^N h_i(x_i - w_i^0) = 0. \tag{12}$$

We claim that w^0 exists. Indeed, suppose first that $\sum_{i=1}^N h_i x_i \geq 0$. Then, because $\sum_{i=1}^N h_i x_i \leq \sum_{i=1}^N h_i w_i^{\theta_-}$, there exists $\alpha^0 \in [0, 1]$ such that

$$\sum_{i=1}^N h_i x_i = \alpha^0 \sum_{i=1}^N h_i w_i^{\theta_-}.$$

Similarly, suppose that $\sum_{i=1}^N h_i x_i \leq 0$. Then, because $\sum_{i=1}^N h_i x_i \geq \sum_{i=1}^N h_i w_i^{\theta_+, e_+}$, there exists $\alpha^0 \in [0, 1]$ such that

$$\sum_{i=1}^N h_i x_i = \alpha^0 \sum_{i=1}^N h_i w_i^{\theta_+, e_+}.$$

As in cases 1 and 2, incentive compatibility implies that, holding effort fixed, the principal's profit is higher with contract w^0 than with $\omega(\cdot|x_i, e, \theta)$:

$$\int_E \sum_{i=1}^N p^{\theta}_{e,i}(x_i - w_i^0) d\mathcal{E}(e|\theta) \geq \int_E \sum_{i=1}^N p^{\theta}_{e,i}(x_i - \int_{\mathbb{R}} r d\omega(r|x_i, e, \theta)) d\mathcal{E}(e|\theta). \tag{13}$$

Let $e^0(\theta)$ be an effort that maximizes the type θ's payoff under contract w^0. We claim that changing efforts from $\mathcal{E}(\cdot|\theta)$ to $e^0(\theta)$ does not affect the principal's profit. To see this, multiply both sides of equation (12) by

$\int_E I(e,\theta)\, d\mathcal{E}(e|\theta) - I\left(e^0(\theta),\theta\right)$ to write:

$$\left[\int_E I(e,\theta)\, d\mathcal{E}(e|\theta) - I\left(e^0(\theta),\theta\right)\right]\sum_{i=1}^N h_i\left(x_i - w_i^0\right) = 0.$$

Using MS, we can rewrite this equality as

$$\sum_{i=1}^N p^\theta_{e^0(\theta),i}(x_i - w_i^0) = \int_E \sum_{i=1}^N p^\theta_{e,i}(x_i - w_i^0) d\mathcal{E}(e|\theta), \tag{14}$$

which shows that the principal gets the same payoff with both effort profiles.

Combining (13) and (14) establishes that the mechanism (w^0, e^0) is incentive compatible, satisfies LL, and raises the principal's payoff point-wise relative to the original mechanism. The essential uniqueness claim is analogous to the proof of Theorem 1.

Now extend the robustness of the result on the optimality of a single contract to the assumptions of multiplicative separability in the space of random mechanisms.

Proposition 2. *Let E and Θ be finite. For generic economies satisfying MS, there is a neighborhood around it for which the optimal mechanism is unique and offers a single contract to all types. Moreover, this contract pays zero in all but one state.*

Proof of Proposition 2

We first introduce some notation. Let $\mathcal{P}$ denote the space of distributions satisfying $p^\theta_e(x_i) > \underline{p}$ for some $\underline{p} > 0$ (defined in Assumption 2), let $\#\Theta$ denote the number of elements in Θ and, for notation simplicity, let $\Delta c^\theta_{e,\hat{e}} := c^\theta_e - c^\theta_{\hat{e}}$. For the (finite-dimensional) distribution $\left(p^\theta_{e,i}\right)$ and cost function (c^θ_e), take any of the equivalent Euclidean norms.

Let $\Psi : \mathcal{P} \times \Delta\left(E^{\#\Theta}\right) \mapsto \mathbb{R}^{\#\Theta\times N\times\#E}$ denote the feasibility correspondence:

$$\Psi(p,\mathcal{E}) := \left\{\begin{array}{l} \tilde{w} \in \mathbb{R}_+^{\#\Theta\times N\times\#E};\ \forall \hat{e}\in E, \forall\theta,\hat{\theta}\in\Theta \\ \sum_{i=1}^N \left[\int p^\theta_{e,i}\tilde{w}_i^{\theta,e} d\mathcal{E}(e|\theta) - \int p^\theta_{\hat{e},i}\tilde{w}_i^{\hat{\theta},e} d\mathcal{E}(e|\hat{\theta})\right] \geq \int c^\theta_e d\mathcal{E}(e|\theta) - c^\theta_{\hat{e}} \end{array}\right\},$$

that is, the set of incentive compatible mechanisms under p, where $\Delta\left(E^{\#\Theta}\right)$ is the set of family of distributions $\{d\mathcal{E}(\cdot|\theta);\theta\in\Theta\}$. When the effort profile is deterministic, we will denote it simply as $e(\theta)$. Let $\Gamma : \mathcal{P}\times\Delta\left(E^{\#\Theta}\right) \mapsto \times\mathbb{R}^{\#\Theta\times N\times\#E}$ denote the policy correspondence of the principal's program:

$$\Gamma(p,\mathcal{E}) = \underset{\tilde{w}\in\Psi(p,\mathcal{E})}{\arg\max} \quad \sum_\theta \mu^\theta \sum_{i=1}^N \int p^\theta_{e,i}(x_i - \tilde{w}_i^{\theta,e}) d\mathcal{E}(e|\theta) \ ,$$

and $V : \mathcal{P}\times\Delta\left(E^{\#\Theta}\right) \to \mathbb{R}$ denote its optimal value:

$$V(p,\mathcal{E}) = \max_{\tilde{w}\in\Psi(p,\mathcal{E})} \sum_\theta \mu^\theta \sum_{i=1}^N \int p^\theta_{e,i}(x_i - \tilde{w}_i^{\theta,e}) d\mathcal{E}(e|\theta).$$

Throughout the proof, we take any of the equivalent Euclidean norms for contracts $w \in \mathbb{R}^{\#\Theta\times N\times\#E}$.

Lemma 3. *For each $\mathcal{E}\in\Delta\left(E^{\#\Theta}\right)$, $V(\cdot,\mathcal{E})$ is a continuous function and $\Gamma(\cdot,\mathcal{E})$ is a upper semi-continuous correspondence at any p for which the interior of $\Psi(p,\mathcal{E})$ is non-empty.*

Proof. As shown in the existence part of the proof of Theorem 1, we can assume, without loss of generality, that all feasible contracts belong to $[0, L]^{\#\Theta\times N\times\#E}$ for some $L > 0$. Notice that $\Psi(\cdot, \mathcal{E})$ is a compact-valued correspondence. The proof proceeds by verifying the conditions for the Maximum Theorem (Berge, 1963). For this, we show that $\Psi(\cdot, \mathcal{E})$ is a continuous correspondence.

(a) Ψ is upper semi-continuous ("u.s.c."): Let (p^n) be a sequence of distributions in $\mathcal{P}$ converging to p. For any $\tilde{w}_n \in \Psi(p_n, \mathcal{E})$, the finiteness of E and Θ, the compactness of $[0, L]^N$ (and passing to a convergent subsequence if necessary), we can suppose that $\tilde{w}_n$ converges to $\tilde{w} \in [0, L]^{\#\Theta\times N\times\#E}$. By the continuity of the objective function and the constraints of the maximization problem that defines Γ, we have that $\tilde{w} \in \Psi(p, \mathcal{E})$. Therefore, Ψ is u.s.c.

(b) Ψ is lower semi-continuous ("l.s.c."): Let (p^n) be a sequence of distributions in $\mathcal{P}$ that converges to p. Let $\tilde{w}$ be an interior point of $\Psi(p, \mathcal{E})$, i.e.,

$$\sum_{i=1}^{N}\left[\int p^{\theta}_{e,i}\tilde{w}^{\theta,e}_i d\mathcal{E}(e|\theta) - \int p^{\theta}_{\hat{e},i}\tilde{w}^{\hat{\theta},e}_i d\mathcal{E}(e|\hat{\theta})\right] > \int c^{\theta}_e d\mathcal{E}(e|\theta) - c^{\theta}_{\hat{e}},$$

for all $(\hat{\theta}, \hat{e}) \notin \{(\theta, e); \theta \in \Theta \text{ and } e \text{ in the support of } \mathcal{E}(\cdot|\theta)\}$.

Then, for n sufficiently large we have that the previous inequality is also true for p^n instead of p. This implies that the constant sequence $\tilde{w} \in \Psi(p_n, \mathcal{E})$ converges to $\tilde{w} \in \Psi(p, \mathcal{E})$, which shows that Ψ is l.s.c. Let w be a frontier point of $\Psi(p, \mathcal{E})$. Since $\Psi(p, \mathcal{E})$ is a convex set with a non-empty interior, we can find a sequence (w_k) in the interior of $\Psi(p, \mathcal{E})$ converging to w. Now, for every n we can then find k_n such that w_{k_n} belongs to the interior of $\Psi(p_n, \mathcal{E})$. Since (w_{k_n}) is a subsequence of (w_k), it also converges to w, establishing that Ψ is l.s.c.

Because the objective function of the maximization program in $V(p, \mathcal{E})$ is continuous and $\Psi(\cdot, \mathcal{E})$ is a continuous correspondence, it follows from the Maximum Theorem that $V(\cdot, \mathcal{E})$ is a continuous function and $\Gamma(\cdot, \mathcal{E})$ is u.s.c. □

In what follows we use the convention that $V(p, \mathcal{E}) = -\infty$ when $\Psi(p, \mathcal{E}) = \emptyset$.

Corollary 1. *Let $\mathcal{E}_i \in \Delta\left(E^{\#\Theta}\right)$, $i = 1, 2$. If $V(p, \mathcal{E}_1) > V(p, \mathcal{E}_2)$ and $\Psi(p, \mathcal{E}_1)$ has non-empty interior for some distribution $p \in \mathcal{P}$, then there exists a neighborhood $\mathcal{N}$ of p such that $V(\tilde{p}, \mathcal{E}_1) > V(\tilde{p}, \mathcal{E}_2)$, for all $\tilde{p} \in \mathcal{N}$.*

Proof. To obtain a contradiction, let (p_n) be a sequence converging to p such that

$$V(p_n, \mathcal{E}_2) \geq V(p_n, \mathcal{E}_1)$$

for all $n \in \mathbb{N}$. Let $w_n \in \Psi(p_n, \mathcal{E}_2)$ be a sequence that attains value $V(p_n, \mathcal{E}_2)$. Passing to a convergent subsequence if necessary, let $w = \lim_n w_n$. Since Ψ compact-valued correspondence, $w \in \Psi(p, \mathcal{E}_2)$. Hence, $V(p, \mathcal{E}_2)$ is at least as high as the value attained at w. By Lemma 3 (and passing convergent subsequence if necessary), $\lim_{n\to\infty} V(p_n, \mathcal{E}_1) = V(p, \mathcal{E}_1)$, and, therefore,

$$V(p, \mathcal{E}_2) \geq V(p, \mathcal{E}_1),$$

which contradicts the hypothesis that $V(p, \mathcal{E}_1) > V(p, \mathcal{E}_2)$. □

Lemma 4. *Let $\mathcal{P}_{MS}$ be the set of distributions in $\mathcal{P}$ that satisfy MS. Then, the subset of $\mathcal{P}_{MS}$ for which the optimal contract is non-null and pays a positive amount in one state only is generic (i.e., open and dense in $\mathcal{P}_{MS}$).*

Proof. The proof is as in Lemma 3 of the paper. □

Lemma 5. *For a generic set of distributions in $\mathcal{P}_{MS}$ and a generic set of cost functions, there is a neighborhood in $\mathcal{P}$ for which the optimal mechanism is implemented by only one contract that pays a positive amount in only one state.*

Proof. The proof is as in Lemma 4 of the paper. □

3 Lower Bound on Profits

In this section we obtain a general lower bound on the principal's loss by restricting herself to offering a single contract (i.e., offering pooling mechanisms only).

Suppose there are finitely many effort levels and types. Let $p = \{p^{\theta}_{e,i}; e \in E, \theta \in \Theta, x_i \in X\}$ denote the vector of probabilities of the economy, and let $(e(\theta))_{\theta\in\Theta}$ denote a profile of efforts recommended to each type.

Define the maximum profit when the principal is restricted to offering pooling mechanisms as

$$\begin{array}{rll} \pi^{pol}(p) = & \max\limits_{w,e(\theta)} & \sum_{\theta} \mu^{\theta} \sum_i p^{\theta}_{e(\theta),i}(x_i - w_i) \\ \text{s. t.} & & \sum_i \Delta p^{\theta}_{e(\theta),\tilde{e},i} w_i \geq \Delta c^{\theta}_{e(\theta),\tilde{e}}, \quad \forall \theta, \tilde{e} \\ & & w_i \geq 0, \quad \forall i, \end{array} \tag{15}$$

where

$$\Delta p^{\theta}_{e,\tilde{e}} = p^{\theta}_{e} - p^{\theta}_{\tilde{e}} \text{ and } \Delta c^{\theta}_{e,\tilde{e}} = c^{\theta}_{e} - c^{\theta}_{\tilde{e}}.$$

For each contract $w \in \mathbb{R}^N_+$, let $\Theta_e(w)$ be the set of types that are recommended effort e, i.e.,[3]

$$\Theta_e(w) \subset \{\theta \in \Theta; \Delta p^{\theta}_{e,\tilde{e}} \cdot w \geq \Delta c^{\theta}_{e,\tilde{e}}, \forall \tilde{e}\},$$

$\partial\Theta_{e,\tilde{e}}(w)$ be the set of types that are indifferent between $e \neq \tilde{e}$, i.e.,

$$\partial\Theta_{e,\tilde{e}}(w) = \{\theta \in \Theta_e(w); \Delta p^{\theta}_{e,\tilde{e}} \cdot w = \Delta c^{\theta}_{e,\tilde{e}}\},$$

where, for convenience, we are using dot to represent the inner product. Let us also denote

$$\partial\Theta^{+}_{e,\tilde{e}}(w) = \{\theta \in \partial\Theta_e(w); \Delta c^{\theta}_{e,\tilde{e}} > 0\} \text{ and } \partial\Theta^{-}_{e,\tilde{e}}(w) = \partial\Theta_{e,\tilde{e}}(w) \backslash \partial\Theta^{+}_{e,\tilde{e}}(w).$$

We can write the Lagrangian

$$L(w, p, \lambda) = \sum_{e\in E} \sum_{\theta\in\Theta_e(w)} \mu^{\theta} p^{\theta}_{e} \cdot (x - w) + \sum_{e\in E} \sum_{\theta\in\Theta_e(w)} \sum_{\tilde{e}\in E} \lambda^{\theta}_{e,\tilde{e}} \Delta p^{\theta}_{e,\tilde{e}} \cdot w + \sum_{\theta} \xi^{\theta} \cdot w,$$

where $\lambda^{\theta}_{e,\tilde{e}}, \xi^{\theta} \geq 0$ are the Lagrangian multipliers.

Suppose that the distribution p satisfies MS and, without loss of generality, the optimal pooling contract w^* pays out all in state N, i.e., $w^* = (0, ..., w^*_N)$. We have to consider two cases:

(i) $w^*_N > 0$. Notice that the IC constraint is equivalent to

$$\frac{\Delta c^{\theta_+}_{e,\tilde{e}}}{\Delta p^{\theta_+}_{e,\tilde{e},N}} = w^*_N = \frac{\Delta c^{\theta_-}_{f,\tilde{f}}}{\Delta p^{\theta_-}_{f,\tilde{f},N}}$$

[3] Notice that $\{\Theta_e(w); e \in E\}$ forms a partition of Θ.

for all $\theta^+ \in \partial\Theta^+_{e,\tilde{e}}(w^*)$ and $\theta^- \in \partial\Theta^-_{f,\tilde{f}}(w^*)$ such that $\Delta p^{\theta_-}_{f,\tilde{f},N} < 0$ (notice that if $\theta^- \in \partial\Theta^-_{f,\tilde{f}}(w^*)$ and $\Delta p^{\theta_-}_{f,\tilde{f},N} = 0$, then $\Delta c^{\theta_-}_{f,\tilde{f}} = 0$ and the IC constraint trivially holds). Hence, without loss of generality we can assume that $\partial\Theta_{e,\tilde{e}}(w^*) = \partial\Theta^+_{e,\tilde{e}}(w^*)$. In particular, the regularity (Slater) condition for the existence of Lagrangian multipliers is trivially satisfied.

Denoting $\Theta^*_e = \Theta_e(w^*)$ and $\partial\Theta^*_{e,\tilde{e}} = \partial\Theta^+_{e,\tilde{e}}(w^*)$, the first-order condition and the complementary slackness condition imply

$$-\sum_{e\in E}\sum_{\theta\in\Theta^*_e}\mu^\theta p^\theta_{e,N} + \sum_{e,\tilde{e}\in E}\sum_{\theta\in\partial\Theta^*_{e,\tilde{e}}}\lambda^\theta_{e,\tilde{e}}\Delta p^\theta_{e,\tilde{e},N} = 0,$$

which implies that

$$k\sum_{e,\tilde{e}\in E}\sum_{\theta\in\partial\Theta^*_{e,\tilde{e}}}\lambda^\theta_{e,\tilde{e}} \le \sum_{e,\tilde{e}\in E}\sum_{\theta\in\partial\Theta^*_{e,\tilde{e}}}\lambda^\theta_{e,\tilde{e}}\Delta p^\theta_{e,\tilde{e},N} = \sum_{e\in E}\sum_{\theta\in\Theta^*_e}\mu^\theta p^\theta_{e,N},$$

where

$$k = \min\left\{\Delta p^\theta_{e,\tilde{e},N} = \frac{\Delta c^\theta_{e,\tilde{e}}}{w^*_N}; \theta\in\partial\Theta^*_{e,\tilde{e}}, e,\tilde{e}\in E\right\} \ge \frac{1}{w^*_N}\min\{\Delta c^\theta_{e,\tilde{e}} > 0; \theta\in\Theta, e\neq\tilde{e}\}.$$

If $\min\{\Delta c^\theta_{e,\tilde{e}} > 0; \theta\in\Theta, e\neq\tilde{e}\} = \overline{\Delta c} > 0$, then

$$w^*_N\sum_{e,\tilde{e}\in E}\sum_{\theta\in\partial\Theta^*_{e,\tilde{e}}}\lambda^\theta_{e,\tilde{e}} \le \frac{w^*_N}{k} = \frac{(w^*_N)^2}{\overline{\Delta c}}.$$

(ii) $w^*_N = 0$. Notice that the IC constraint is equivalent to $\Delta c^\theta_{e,\tilde{e}} = 0$, for all $\theta\in\partial\Theta_{e,\tilde{e}}(w^*)$. In this case we can ignore the IC constraint and consider only the positiveness constraint. Hence, the first-order and the complementary slackness conditions imply

$$-\sum_{e\in E}\sum_{\theta\in\Theta^*_e}\mu^\theta p^\theta_{e,N} + \xi_N = 0.$$

Notice that the Lagrangian is a linear function of p for each w. Hence, taking the maximum of these functions indexed by w, we get a convex functional of p. By the envelope theorem, the Gateaux derivative of $\pi^{pol}(p)$ is given by

$$\delta_h\pi^{pol}(p) = \sum_{e\in E}\sum_{\theta\in\Theta^*_e}\mu^\theta h^\theta_e\cdot(x-w^*) + \sum_{e,\tilde{e}\in E}\sum_{\theta\in\partial\Theta^*_{e,\tilde{e}}}\lambda^\theta_{e,\tilde{e}}\Delta h^\theta_{e,\tilde{e},N}w^*_N,$$

when $w^*\neq 0$ and

$$\delta_h\pi^{pol}(p) = \sum_{e\in E}\sum_{\theta\in\Theta^*_e}\mu^\theta h^\theta_e\cdot x,$$

when $w^* = 0$, where h is the incremental probability vector, i.e. $h = \hat{p} - p$, where $\hat{p}$ is alternative probability vector. Therefore,

$$\left|\delta_h\pi^{pol}(p)\right| \le \left(||x-w^*|| + 2\frac{(w^*_N)^2}{\overline{\Delta c}}\right).||h||,$$

where $||\cdot||$ is the maximum norm and the constant between parentheses on the right hand side is the slope of a convex function $\pi^{pol}(p)$, which is the lower bound of the principal's payoff.

4 Optimal Screening under Distributions Close to MS

This appendix presents a family of distributions for which the optimal mechanism offers a single contract to all types if and only if the distribution satisfies MS. This example shows that the genericity requirement in Proposition **??** is needed since, in this (non-generic) case, offering a menu of contracts is optimal for distributions arbitrarily close to MS.

There are two effort levels ($e \in \{0,1\}$), two evenly-distributed types ($\theta \in \{A,B\}$), and three outputs ($x_i = i$, $i = 1,2,3$). For both types, outputs are uniformly distributed under low effort:

$$p_0^A = p_0^B = p_0 = (1/3, 1/3, 1/3).$$

With high effort, type A delivers the intermediate output, whereas type B obtains a distribution supported on the extreme points:

$$p_1^A = (0,1,0)$$
$$p_1^B = (1-\lambda, 0, \lambda),$$

where $\lambda \in [1/2, 1]$.[4] Note that MS is satisfied if and only if $\lambda = 1/2$.

Let $\mathcal{E} = (e_A, e_B) \in \{0,1\}^2$ be an arbitrary vector of recommended efforts. We will now compute the principal's expected profit $\pi_{\mathcal{E}}$ for each effort vector $\mathcal{E}$.

- $\mathcal{E} = (0,0)$. The relevant constraints are the low effort recommendation for both types:

  $$p_0 \cdot w^0 \geq c_0 \text{ and } \left(p_0 - p_1^\theta\right) \cdot w^0 \geq c_0 - c_1^\theta$$

  for all $\theta \in \{A,B\}$, where $\cdot$ represents the canonical inner product. If $c_0 \leq c_1^\theta$, then $w^0 = (c_0, c_0, c_0)$ is the optimal (pooling) contract, and the expected profit is $\pi^*_{(0,0)} = p_0 \cdot \left[(1,2,3) - w^0\right] = 2 - c_0$.

- $\mathcal{E} = (1,0)$. The relevant constraints are the high effort recommendation for type A:

  $$p_1^A \cdot w^A \geq c_1^A \text{ and } \left(p_1^A - p_0\right) \cdot w^A \geq c_1^A - c_0$$

  and the low effort recommendation for type B:

  $$p_0 \cdot w^A \geq c_0 \text{ and } \left(p_0 - p_1^B\right) \cdot w^A \geq c_0 - c_1^B.$$

  If $c_0 \leq \frac{c_1^A}{3}$, then the optimal (pooling) contract is $w^A = (0, \frac{3\left(c_1^A - c_0\right)}{2}, 0)$, and the profit equals:

  $$\pi^*_{(1,0)} = \frac{1}{2}\left(p_1^A + p_0\right) \cdot \left[(1,2,3) - w^A\right] = 2 - \frac{5}{4}\left(c_1^A - c_0\right).$$

- $\mathcal{E} = (0,1)$. The relevant constraints are the high effort recommendation for type B:

  $$p_1^B \cdot w^B \geq c_1^B \text{ and } \left(p_1^B - p_0\right) \cdot w^B \geq c_1^B - c_0$$

[4] These probabilities are not strictly positive and, therefore, do not satisfy the assumption we used to ensure existence of an optimal mechanism. Nevertheless, as we show below, an optimal mechanism exists here. Moreover, it is straightforward to show by continuity that small perturbations lead to strictly positive distributions under which the results below go through. We present this case here to simplify calculations.

and the low effort recommendation for type A:

$$p_0 \cdot w^B \geq c_0 \text{ and } \left(p_0 - p_1^A\right) \cdot w^B \geq c_0 - c_1^A.$$

If $c_0 \leq \frac{c_1^B}{3\lambda}$, then the optimal (pooling) contract is $w^B = (0, 0, \frac{3\left(c_1^B - c_0\right)}{3\lambda - 1})$, and the expected profit is:

$$\pi^*_{(0,1)} = \frac{1}{2}\left(p_0 + p_1^B\right) \cdot \left[(1,2,3) - w^B\right] = \frac{3}{2} + \lambda - \frac{3\lambda\left(c_1^B - c_0\right)}{2(3\lambda - 1)}.$$

If $c_0 > \frac{c_1^B}{3\lambda}$, the optimal pooling contract is $w^B = (0, 0, 3c_0)$, and the expected profit equals:

$$\pi^{pool}_{(0,1)} = \frac{1}{2}\left(p_0 + p_1^B\right) \cdot \left[(1,2,3) - w^B\right] = \frac{3}{2} + \lambda - \frac{3\lambda c_0}{2}.$$

We now show that for every $\lambda > 1/2$, the principal can do better by offering different contracts to each type. Taking the contracts:

$$w_s^A = (0, 3c_0, 0) \text{ and } w_s^B = (0, 0, \frac{3\left(c_1^B - c_0\right)}{3\lambda - 1}),$$

the principal's expected profit is

$$\begin{aligned}\pi^*_{(0,1)} &= \tfrac{1}{2}\left[\left(p_0 + p_1^B\right) \cdot (1,2,3) - p_0 \cdot w_s^A - p_1^B \cdot w_s^B\right] \\ &= \tfrac{3}{2} + \lambda - \tfrac{c_0}{2} - \tfrac{3\lambda\left(c_1^B - c_0\right)}{2(3\lambda - 1)} > \pi^{pool}_{(0,1)}.\end{aligned}$$

Notice that IC constraints are satisfied since

$$p_0 \cdot (w_s^A - w_s^B) \geq 0 \geq p_B^1 \cdot (w_s^A - w_s^B)$$

and type A chooses low effort if and only if

$$\left(p_0 - p_1^A\right) \cdot w_s^A \geq c_0 - c_1^A$$

or

$$c_0 \leq \frac{c_1^A}{3}.$$

- $\mathcal{E} = (1, 1)$. The relevant constraints are the high effort recommendation for both types:

$$p_1^\theta \cdot w^{AB} \geq c_1^\theta \text{ and } \left(p_1^\theta - p_0\right) \cdot w^{AB} \geq c_1^\theta - c_0,$$

for $\theta \in \{A, B\}$. If $c_0 \leq \frac{c_1^A}{3}$ and $c_0 > \frac{c_1^B}{3\lambda}$, then the optimal (pooling) contract is $w^{AB} = (0, \frac{3\left(c_1^A - c_0\right)}{2}, 3c_0)$, and the principal's expected profits are:

$$\pi^*_{(1,1)} = \frac{1}{2}\left(p_1^A + p_1^B\right) \cdot \left[(1,2,3) - w^{AB}\right] = \frac{3}{2} + \lambda - \frac{3\left(c_1^A - c_0\right)}{4} - \frac{3\lambda c_0}{2}.$$

Proposition 3. *Let $c_1^B = 0$. For every $\lambda > 1/2$, there exists (c_0, c_1^A) satisfying $0 < c_0 \leq \frac{c_1^A}{3}$ such that the principal's optimal mechanism offers a different contract to each type.*

Proof. From the computations above, we have

$$\begin{aligned}\pi^*_{(0,0)} &= 2 - c_0\\ \pi^*_{(1,0)} &= 2 - \tfrac{5}{4}\left(c_1^A - c_0\right)\\ \pi^*_{(0,1)} &= \tfrac{3}{2} + \lambda - \tfrac{c_0}{2} + \tfrac{3\lambda c_0}{2(3\lambda-1)}\\ \pi^*_{(1,1)} &= \tfrac{3}{2} + \lambda - \tfrac{3\left(c_1^A - c_0\right)}{4} - \tfrac{3\lambda c_0}{2}.\end{aligned}$$

Then, for each $\lambda > 1/2$, if we take c_1^A large enough, $\pi^*_{(0,1)}$ is largest value among the profits above. But, as seen previously, the optimal mechanism offers a different contract to each type in this case. □

Note that if $\lambda = 1/2$, then $\pi^*_{(0,1)} = \pi^*_{(0,0)}$ and pooling is optimal (as must be the case since the distribution satisfies MS in this case).

5 Regulation and Procurement

This section formally describes the regulation and procurement example described in the main text and provides the versions of Theorem 1 and 2 for that example.

Recall that a contract is a function that specifies a transfer to the firm conditional on each possible cost C. A mechanism is a pair of measurable functions $w : \{C_1, ..., C_N\} \times \Theta \to \mathbb{R}$ and $e : \Theta \to \mathbb{R}$ specifying, for each reported type, a recommended effort and a transfer for each cost realization. Given a mechanism (w, e), a type-θ manager gets payoff

$$U(\theta) := \sum_{i=1}^{N} p^{\theta}_{e(\theta)}\left(C_i\right) w^{\theta}\left(C_i\right) - c^{\theta}_{e(\theta)}. \tag{16}$$

As usual, the mechanism must satisfy the IC and IR constraints:

$$U\left(\theta\right) \geq \sum_{i=1}^{N} p^{\theta}_{\hat{e}}\left(C_i\right) w^{\hat{\theta}}\left(C_i\right) - c^{\theta}_{\hat{e}}, \quad \forall \theta, \hat{\theta}, \hat{e}, \tag{IC}$$

$$U\left(\theta\right) \geq 0, \quad \forall \theta. \tag{IR}$$

The manager is protected by limited liability, so that payments are non-negative:

$$w^{\theta}(C) \geq 0, \quad \forall C. \tag{LL}$$

We impose the technical conditions from Assumption 2 (with C instead of x).

Since, by the accounting convention described in the text, the regulator fully reimburses the firm's cost realization, the regulator's expected payment to type θ equals

$$\sum_{i=1}^{N} p^{\theta}_{e(\theta)}\left(C_i\right)\left[C_i + w^{\theta}\left(C_i\right)\right].$$

Because the government uses distortionary taxation to raise public funds, the regulator faces a shadow cost of public funds $\lambda > 0$. The net surplus of consumers/taxpayers is

$$S - (1+\lambda)\sum_{i=1}^{N} p^{\theta}_{e(\theta)}\left(C_i\right)\left[C_i + w^{\theta}\left(C_i\right)\right]. \tag{17}$$

A utilitarian regulator maximizes the sum of the expected utility of the firm's manager (16) and the consumers' net surplus (17):

$$S-(1+\lambda)\sum_{i=1}^{N}p_{e(\theta)}^{\theta}\left(C_{i}\right)C_{i}-c_{e(\theta)}^{\theta}-\lambda\sum_{i=1}^{N}p_{e(\theta)}^{\theta}\left(C_{i}\right)w^{\theta}\left(C_{i}\right). \tag{18}$$

As the last term of this expression shows, because taxation is distortionary ($\lambda > 0$), leaving rents the regulated firm is costly. Moreover, because each dollar reimbursed to the firm has an additional cost of λ, cutting the firm's cost increases social surplus by $1+\lambda$. The first-best effort minimizes $(1+\lambda)\sum_{i=1}^{N}p_{e}^{\theta}\left(C_{i}\right)C_{i}+c_{e}^{\theta}$.

If types were observable and the firm did not have limited liability, the first best would be implemented by making the firm the residual claimant of the social gain from cutting costs and extracting the entire surplus:

$$w^{\theta}(C_{j})=(1+\lambda)\left[\sum_{i=1}^{N}p_{e^{FB}(\theta)}^{\theta}(C_{i})C_{i}-C_{j}\right],$$

where $e^{FB}(\theta)$ is the first-best effort. This contract is no longer feasible when types are unobservable since the manager would always pretend to have a high effort cost. It also violates violates LL since w is non-degenerate and has mean zero.

The main difference between this model and the principal-agent model considered previously is that, while the principal only cares about her own payoff, a utilitarian regulator also cares about the manager's payoffs. Because the regulator internalizes the manager's effort cost, their preferences are not perfectly misaligned. However, to avoid distortionary taxation, the regulator would still like to leave as little rents as possible to the firm's manager.

It may also be desirable to include a free disposal constraint (FD), requiring the firm's compensation for cutting costs not to exceed the amount cut:

$$w^{\theta}\left(C_{j}\right)-w^{\theta}\left(C_{i}\right)\leq C_{i}-C_{j} \tag{FD}$$

for all j and i with $i > j$. FD requires the firm's compensation for cutting costs not to exceed the amount cut. It must be satisfied, for example, if the firm's manager can secretly inflate firm costs.

Unlike output in the principal-agent model, a higher cost decreases the principal's payoff in this model. We therefore say that the cost distribution satisfies MLRP if, for any e_L, e_H, and θ with $I(e_H,\theta) > I(e_L,\theta)$, $\frac{p_{e_H}^{\theta}(C)}{p_{e_L}^{\theta}(C)}$ is *decreasing* in C.

In order to rewrite the procurement model using similar terminology as in the paper, perform the change of variables:

$$x_i := S-(1+\lambda)C_i.$$

We will write contracts in terms of the taxpayer's net surplus x, instead of the firm's production cost C by letting $W^{\theta}(x) := w^{\theta}\left(\frac{S-x}{1+\lambda}\right)$. The distribution of the taxpayer's net surplus x is determined by $q_{e}^{\theta}(x_i) := p_{e}^{\theta}\left(\frac{S-x_i}{1+\lambda}\right)$. Notice that if p_e^{θ} satisfies MS, so does q_e^{θ}. With some abuse of notation, we will write p_e^{θ} for the probability distribution function associated with the distribution function q_e^{θ}.

The regulator's payoff is

$$\sum_{i=1}^{N}p_{e(\theta),i}^{\theta}\left(x_{i}-\lambda W_{i}^{\theta}\right)-c_{e(\theta)}^{\theta}, \tag{19}$$

whereas the type-θ manager's payoff is

$$\sum_{i=1}^{N}p_{e(\theta),i}^{\theta}W_{i}^{\theta}-c_{e(\theta)}^{\theta},$$

where we write W_i^{θ} for $W^{\theta}(x_i)$, for all i and θ.

Proposition 4. *Suppose MS holds.*

a) *There exists an essentially unique optimal mechanism that offers a single contract to all types.*

b) *If the cost distribution also satisfies MLRP, there exists an essentially unique optimal FD-mechanism that offers all types the contract* $w(C) = \max\left\{\bar{C} - C; \quad 0\right\}$*, for some* $\bar{C}$*.*

Proof. **a)** Let (w, e) be a feasible mechanism. It is straightforward to adapt Lemma 1 to show that there is no loss of generality in assuming that contracts are uniformly bounded. As in the proof of Theorem 1, let $\mathcal{M} := \left\{W^\theta : \theta \in \Theta\right\}$ denote the set of all contracts in this mechanism, and let $\bar{\mathcal{M}}$ denote its closure, which is compact.

Let

$$W^- \in \arg\min_{W \in \bar{\mathcal{M}}} \sum_{i=1}^{N} h_i W_i \tag{20}$$

and, for each type, let

$$e^-(\theta) \in \arg\max_e \sum_{i=1}^{N} p_{e,i}^\theta W_i^- - c_e^\theta. \tag{21}$$

Existence of W^- and $e^-(\theta)$ follow from the arguments in Theorem 1.

Use IC to write

$$\begin{aligned} \left[I\left(e(\theta),\theta\right) - I\left(e^-\left(\theta\right),\theta\right)\right] \textstyle\sum_{i=1}^N h_i W_i^\theta &\leq c_{e^-(\theta)}^\theta - c_{e(\theta)}^\theta \\ &\leq \left[I\left(e(\theta),\theta\right) - I\left(e^-\left(\theta\right),\theta\right)\right] \textstyle\sum_{i=1}^N h_i W_i^- \end{aligned} . \tag{22}$$

Since W^- solves (20), it satisfies

$$\sum_{i=1}^{N} h_i W_i^- \leq \sum_{i=1}^{N} h_i W_i^\theta,$$

so that, by (22), $I\left(e^-\left(\theta\right),\theta\right) \geq I\left(e(\theta),\theta\right)$.[5] That is, offering W^- yields a distribution of net surplus x that first-order stochastically dominates any other contract in $\bar{\mathcal{M}}$.

We first show that, holding effort $e(\theta)$ fixed, the regulator's payoff is higher with contract W^- than with W^θ. Since W^- is the limit of sequence in $\mathcal{M}$, the agent's utility is continuous, and the original mechanism is incentive compatible, it follows that

$$\sum_{i=1}^{N} p_{e(\theta),i}^\theta W_i^\theta - c_{e(\theta)}^\theta \geq \sum_{i=1}^{N} p_{e(\theta),i}^\theta W_i^- - c_{e(\theta)}^\theta.$$

With some algebraic manipulations, we can rewrite this inequality as

$$\sum_{i=1}^{N} \left(x_i - \lambda W_i^-\right) p_{e(\theta),i}^\theta - c_{e(\theta)}^\theta \geq \sum_{i=1}^{N} \left(x_i - \lambda W_i^\theta\right) p_{e(\theta),i}^\theta - c_{e(\theta)}^\theta, \tag{23}$$

which shows that, holding effort constant, the regulator obtains a higher payoff with W^- than with W^θ for $e(\theta)$ fixed.

Next, we show that changing effort from $e(\theta)$ to $e^-(\theta)$ also increases the regulator's payoff. Let $\Delta W_i^- = W_i^- - W_{i-1}^-$, so that

$$\sum_{i=1}^{N} h_i \left(\frac{x_i}{1+\lambda} - W_i^-\right) = -\sum_{i=1}^{N} H(x_i) \left(\frac{\Delta x_i}{1+\lambda} - \Delta W_i^-\right) \leq 0, \tag{24}$$

where $H(x_i) = \sum_{j \leq i} h(x_j)$ and the equality uses summation by parts, $x_0 = 0 = W_0^-$, and the inequality uses

[5] The argument here is equivalent to one used in the proof of Theorem 1.

$H(x) \geq 0$ for all x and FD. Use (24) to obtain:

$$\begin{array}{rcl} \left[I\left(e\left(\theta\right),\theta\right)-I\left(e^{-}\left(\theta\right),\theta\right)\right]\sum_{i=1}^{N}h_{i}x_{i} & \geq & (1+\lambda)\left[I\left(e\left(\theta\right),\theta\right)-I\left(e^{-}\left(\theta\right),\theta\right)\right]\sum_{i=1}^{N}h_{i}W_{i}^{-} \\ & \geq & c_{e^{-}(\theta)}^{\theta}-c_{e(\theta)}^{\theta}+\lambda\left[I\left(e\left(\theta\right),\theta\right)-I\left(e^{-}\left(\theta\right),\theta\right)\right]\sum_{i=1}^{N}h_{i}W_{i}^{-} \end{array}, \tag{25}$$

where the first inequality follows from $I\left(e^{-}\left(\theta\right),\theta\right) \geq I\left(e(\theta),\theta\right)$ and some algebraic manipulations, whereas the second inequality follows from the fact that $e^{-}(\theta)$ maximizes type θ's effort under contract W^{-} (program 21). Rearranging (25), we obtain:

$$-\left[I\left(e^{-}\left(\theta\right),\theta\right)-I\left(e\left(\theta\right),\theta\right)\right]\sum_{i=1}^{N}h_{i}\left(x_{i}-\lambda W_{i}^{-}\right) \geq c_{e^{-}(\theta)}^{\theta}-c_{e(\theta)}^{\theta}.$$

Using MS, this inequality can be written as

$$\sum_{i=1}^{N}p_{e^{-}(\theta),i}^{\theta}\left(x_{i}-\lambda W_{i}^{-}\right)-c_{e^{-}(\theta)}^{\theta} \geq \sum_{i=1}^{N}p_{e(\theta),i}^{\theta}\left(x_{i}-\lambda W_{i}^{-}\right)-c_{e(\theta)}^{\theta}, \tag{26}$$

which establishes that the change in effort from $e(\theta)$ to $e^{-}(\theta)$ increases the regulator's payoff. Combining inequalities (23) and (26) concludes the proof.

b) If (W, e) is an optimal FD mechanism, it must solve the following program

$$\min_{W}\sum_{i=1}^{N}W_{i}\int_{\Theta}p_{e(\theta),i}^{\theta}d\mu(\theta)$$

subject to

$$\sum_{i=1}^{N}h_{i}W_{i}=K, \tag{IC'}$$

$$W_{i} \geq 0, \tag{LL}$$

$$\frac{x_{i}-x_{i-1}}{1+\lambda} \geq W_{i}-W_{i-1}. \tag{M}$$

Using the same arguments as in proof of Theorem 2 of the paper, it follows that there exists $\bar{x}$ such that

$$w(x)=\left\{\begin{array}{c} 0 \text{ if } x \leq \bar{x} \\ \frac{x-\bar{x}}{1+\lambda} \text{ if } x>\bar{x} \end{array}\right. .$$

Rewriting in terms of costs, we obtain $w(C)=\max\left\{\bar{C}-C;\ \ 0\right\}$, where $\bar{C} \equiv \frac{S-\bar{x}}{1+\lambda}$. □

References

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