Supplement to "Observational learning in large anonymous games": Omitted proofs

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S.1. Proof of Lemma 6

Take a limit point $x = (x_0, x_1)$ with $v_0(x_0) > 0$ and $v_1(x_1) < 0$. In the limit, agents want their action to go against the state of the world. Now the simple strategy $\widetilde{\sigma}^T$ is

$$\widetilde{\sigma}^T(\widetilde{\xi}, s) = \begin{cases} 1 & \text{if } \widetilde{\xi} = 1 \text{ and } l(s) \leq \underline{k}^T \equiv \dfrac{v_0(E_{\sigma^T}[X_0])}{-v_1(E_{\sigma^T}[X_1])} \dfrac{\mathbf{P}_{\sigma^T}(\widetilde{\xi} = 1 \mid \theta = 0)}{\mathbf{P}_{\sigma^T}(\widetilde{\xi} = 1 \mid \theta = 1)} \\ 1 & \text{if } \widetilde{\xi} = 0 \text{ and } l(s) \leq \overline{k}^T \equiv \dfrac{v_0(E_{\sigma^T}[X_0])}{-v_1(E_{\sigma^T}[X_1])} \dfrac{\mathbf{P}_{\sigma^T}(\widetilde{\xi} = 0 \mid \theta = 0)}{\mathbf{P}_{\sigma^T}(\widetilde{\xi} = 0 \mid \theta = 1)} \\ 0 & \text{otherwise.} \end{cases}$$

Given this simple strategy, the approximate improvement is given by

$$\begin{aligned}
\Delta^T &= \frac{1}{2} \sum_{\theta \in \{0,1\}} \big[\mathbf{P}_{\widetilde{\sigma}^T}(a_i = 1 \mid \theta) - E_{\sigma^T}[X_\theta]\big] \cdot v_\theta\big(E_{\sigma^T}[X_\theta]\big) \\
&= \frac{1}{2} \sum_{\theta \in \{0,1\}} \big[\varepsilon + (1 - 2\varepsilon)\big[\pi_\theta^T G_\theta(\underline{k}^T) + (1 - \pi_\theta^T) G_\theta(\overline{k}^T)\big] - E_{\sigma^T}[X_\theta]\big] \cdot v_\theta\big(E_{\sigma^T}[X_\theta]\big) \\
&= \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta\big(E_{\sigma^T}[X_\theta]\big)\big[\varepsilon + (1 - 2\varepsilon)\big[\pi_\theta^T\big[G_\theta(\underline{k}^T) - 1\big] + (1 - \pi_\theta^T) G_\theta(\overline{k}^T)\big]\big] \\
&\quad + v_\theta\big(E_{\sigma^T}[X_\theta]\big)\big[(1 - 2\varepsilon)\pi_\theta - E_{\sigma^T}[X_\theta]\big] \\
&= \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta\big(E_{\sigma^T}[X_\theta]\big)\big[(1 - 2\pi_\theta^T)\varepsilon + (1 - 2\varepsilon)\big[\pi_\theta^T\big[G_\theta(\underline{k}^T) - 1\big] + (1 - \pi_\theta^T) G_\theta(\overline{k}^T)\big]\big] \\
&\quad + \frac{1}{2} \sum_{\theta \in \{0,1\}} v_\theta\big(E_{\sigma^T}[X_\theta]\big)\big[\pi_\theta - E_{\sigma^T}[X_\theta]\big].
\end{aligned}$$

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Thus,

$$\begin{aligned}\Delta^T = {} & \frac{1}{2}\big[(1-2\pi_0^T)\varepsilon + (1-2\varepsilon)\big[-\pi_0^T\big[1-G_0(\underline{k}^T)\big] + (1-\pi_0^T)G_0(\overline{k}^T)\big]\big]\cdot v_0(E_{\sigma^T}[X_0]) \\ & + \frac{1}{2}\big[(1-2\pi_1^T)\varepsilon + (1-2\varepsilon)\big[-\pi_1^T\big[1-G_1(\underline{k}^T)\big] + (1-\pi_1^T)G_1(\overline{k}^T)\big]\big]\cdot v_1(E_{\sigma^T}[X_1]) \\ & + \frac{1}{2}\sum_{\theta\in\{0,1\}} v_\theta(E_{\sigma^T}[X_\theta])\big[\pi_\theta - E_{\sigma^T}[X_\theta]\big] \\ = {} & \frac{1}{2}\bigg[(1-2\varepsilon)(1-\pi_0^T)\bigg[G_0(\overline{k}^T) - \frac{-v_1(E_{\sigma^T}[X_1])}{v_0(E_{\sigma^T}[X_0])}\frac{(1-\pi_1^T)}{(1-\pi_0^T)}G_1(\overline{k}^T)\bigg]\bigg]\cdot v_0(E_{\sigma^T}[X_0]) \\ & + \frac{1}{2}\bigg[(1-2\varepsilon)\pi_1^T\bigg[\frac{v_0(E_{\sigma^T}[X_0])}{-v_1(E_{\sigma^T}[X_1])}\frac{\pi_0^T}{\pi_1^T}\big[1-G_0(\underline{k}^T)\big] - \big[1-G_1(\underline{k}^T)\big]\bigg]\bigg]\cdot v_1(E_{\sigma^T}[X_1]) \\ & + \frac{1}{2}(1-2\pi_0^T)\varepsilon\cdot v_0(E_{\sigma^T}[X_0]) + \frac{1}{2}(1-2\pi_1^T)\varepsilon\cdot v_1(E_{\sigma^T}[X_1]) \\ & + \frac{1}{2}\sum_{\theta\in\{0,1\}} v_\theta(E_{\sigma^T}[X_\theta])\big[\pi_\theta - E_{\sigma^T}[X_\theta]\big] \\ = {} & \frac{1}{2}\big[(1-2\pi_0^T)\varepsilon + (1-2\varepsilon)(1-\pi_0^T)\big[G_0(\overline{k}^T) - (\overline{k}^T)^{-1}G_1(\overline{k}^T)\big]\big]\cdot v_0(E_{\sigma^T}[X_0]) \\ & + \frac{1}{2}\big[(2\pi_1^T-1)\varepsilon + (1-2\varepsilon)\pi_1^T\big[\big[1-G_1(\underline{k}^T)\big] - \underline{k}^T\big[1-G_0(\underline{k}^T)\big]\big]\big]\cdot\big(-v_1(E_{\sigma^T}[X_1])\big) \\ & + \frac{1}{2}\sum_{\theta\in\{0,1\}} v_\theta(E_{\sigma^T}[X_\theta])\big[\pi_\theta - E_{\sigma^T}[X_\theta]\big].\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{T\to\infty}\Delta^T = {} & \frac{1}{2}\big[(1-2x_0)\varepsilon + (1-2\varepsilon)(1-x_0)\big[G_0(\overline{k}) - (\overline{k})^{-1}G_1(\overline{k})\big]\big]\cdot v_0(x_0) \\ & + \frac{1}{2}\big[(2x_1-1)\varepsilon + (1-2\varepsilon)x_1\big[\big[1-G_1(\underline{k})\big] - \underline{k}\big[1-G_0(\underline{k})\big]\big]\big]\cdot\big(-v_1(x_1)\big).\end{aligned}$$

Again, Corollary 2 leads directly to

$$\begin{aligned}& \big[(1-2\varepsilon)(1-x_0)\big[G_0(\overline{k}) - (\overline{k})^{-1}G_1(\overline{k})\big] - \varepsilon(2x_0-1)\big]\cdot v_0(x_0) \\ & \quad + \big[(1-2\varepsilon)x_1\big[\big[1-G_1(\underline{k})\big] - \underline{k}\big[1-G_0(\underline{k})\big]\big] - \varepsilon(1-2x_1)\big]\cdot\big(-v_1(x_1)\big) \le 0.\end{aligned}$$

S.2. Proof of Lemma 7

Let $\widetilde{\mathrm{NE}}_\delta = \{x\in[0,1]^2 : d(x, \mathrm{NE}_{(\underline{l},\bar{l})}) \le \delta\}$ be the set of all points that are δ-close to elements of $\mathrm{NE}_{(\underline{l},\bar{l})}$ and let L^ε denote the set of limit points in a game with mistake probability $\varepsilon > 0$. I show first the following lemma, which is analogous to Lemma 11 in the main paper.

Lemma 11′ (Limit set approaches $\mathrm{NE}_{(\underline{l},\bar{l})}$). *For any $\delta > 0$, $\exists\tilde{\varepsilon} > 0$, $L^\varepsilon \subseteq \widetilde{\mathrm{NE}}_\delta$ $\forall\varepsilon < \tilde{\varepsilon}$.*

Proof. The proof is by contradiction. Assume that there exists (i) a sequence of mistake probabilities $\{\varepsilon^n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty}\varepsilon^n = 0$ and (ii) an associated sequence $\{x^n\}_{n=1}^{\infty}$ with $x^n \in L^{\varepsilon^n}$ for all n, but (iii) $x^n \notin \widetilde{\mathrm{NE}}_\delta$ for all n. Since $x^n \in [0,1]^2$ for all n, this sequence has a convergent subsequence $\{x^{n_m}\}_{m=1}^{\infty}$ with $\lim_{m\to\infty} x^{n_m} = \bar{x} = (\bar{x}_0, \bar{x}_1)$. If $v_0(\bar{x}_0) = v_1(\bar{x}_1) = 0$, then $\bar{x} \in \mathrm{NE}$, so for m large enough, $x^{n_m} \in \widetilde{\mathrm{NE}}_\delta$. Then it must be the case that $v_\theta(\bar{x}_\theta) \neq 0$ for some θ.

Assume that $v_1(\bar{x}_1) > 0$. Pick $\tilde{m}$ large enough so that $v_1(x_1^{n_m}) > 0$ for all $m > \tilde{m}$. For all m with $v_0(x_0^{n_m}) \geq 0$, Lemma 4 implies that $x^{n_m} = (1-\varepsilon^{n_m}, 1-\varepsilon^{n_m})$. So if $v_0(x_0^{n_m}) \geq 0$ infinitely often, then $\bar{x} = (1,1)$. As a result, $\bar{x} \in \mathrm{NE}$, so for m large enough, $x^{n_m} \in \widetilde{\mathrm{NE}}_\delta$.

Take next all m with $v_0(x_0^{n_m}) < 0$. By Lemma 5, (3) must hold:

$$\begin{aligned}
&\frac{-v_0(x_0^{n_m})}{2}\Big[\overbrace{(1-2\varepsilon^{n_m})}^{\to 1}\overbrace{x_0^{n_m}\big[G_0(\underline{k}^{n_m}) - (\underline{k}^{n_m})^{-1}G_1(\underline{k}^{n_m})\big]}^{\geq 0} - \overbrace{\varepsilon(1-2x_0)}^{\to 0}\Big] \\
&\quad + \frac{v_1(x_1^{n_m})}{2}\Big[\underbrace{(1-2\varepsilon^{n_m})}_{\to 1}\underbrace{(1-x_1^{n_m})\big[\big[1-G_1(\overline{k}^{n_m})\big] - \overline{k}^{n_m}\big[1-G_0(\overline{k}^{n_m})\big]\big]}_{\geq 0} \\
&\quad - \underbrace{\varepsilon^{n_m}(2x_1^{n_m}-1)}_{\to 0}\Big] \leq 0. \qquad \text{(S.1)}
\end{aligned}$$

Proposition 3 guarantees both that $[[1-G_1(\overline{k}^{n_m})] - \overline{k}^{n_m}[1-G_0(\overline{k}^{n_m})]] \geq 0$ and that $[G_0(\underline{k}^{n_m}) - (\underline{k}^{n_m})^{-1}G_1(\underline{k}^{n_m})] \geq 0$. Then, as (S.1) shows, when $\varepsilon^{n_m} \to 0$, only nonnegative terms may remain. Assume that $\overline{k} = -[v_0(\bar{x}_0)(1-\bar{x}_0)]/[v_1(\bar{x}_1)(1-\bar{x}_1)] < \bar{l}$. Then, for ε small enough, $\overline{k}^{n_m} < \bar{l}$. Proposition 3 implies that

$$\lim_{m\to\infty}\big[\big[1-G_1(\overline{k}^{n_m})\big] - \overline{k}^{n_m}\big[1-G_0(\overline{k}^{n_m})\big]\big] > 0.$$

To summarize, whenever $\overline{k} < \bar{l}$, (S.1) is not satisfied for small enough ε^{n_m}. It must be the case then that $\overline{k} \geq \bar{l}$. Similarly, if $\underline{k} > \underline{l}$, then

$$\lim_{m\to\infty}\big[G_0(\underline{k}^{n_m}) - (\underline{k}^{n_m})^{-1}G_1(\underline{k}^{n_m})\big] > 0$$

for small enough ε^{n_m}. It must be the case then that $\underline{k} \leq \underline{l}$.

Analogous arguments (using also Lemma 6) lead to the same result for the case with $v_1(\bar{x}_1) < 0$. As a result, $\bar{x} \in \mathrm{NE}_{(\underline{l},\bar{l})}$, so for m large enough, $x^{n_m} \in \widetilde{\mathrm{NE}}_\delta$.

The rest of the proof is identical to the proof of Proposition 2 in the paper. □

S.3. Example 4: Standard observational learning with mistakes

This corresponds to Example 4 in the paper. Utility is given by $u(1,X,1) = u(0,X,0) = 1$ and $u(1,X,0) = u(0,X,1) = 0$. Each agent observes his immediate predecessor: $M = 1$. The signal structure is described by $\nu_1[(0,s)] = s^2$ and $\nu_0[(0,s)] = 2s - s^2$ with $s \in (0,1)$.

Proof of Example 4. Let $\pi \equiv \Pr(\xi = 1 \mid \theta = 1)$. An agent who observes $\xi = 1$ chooses action 1 if and only if $\frac{\pi}{1-\pi}\frac{s}{1-s} \geq 1 \Leftrightarrow s \geq 1 - \pi$. Similarly, an agent who observes $\xi =$

0 chooses action 1 if and only if $\frac{1-\pi}{\pi}\frac{s}{1-s} \geq 1 \Leftrightarrow s \geq \pi$. As a result, the likelihood that somebody who observes a sample (that is, not agent 1) will choose the right action is given by

$$\begin{aligned}
\Pr\big(a_i = 1 \mid \theta = 1, Q(i) \neq 1\big) &= \frac{1}{T-1}\sum_{t=2}^{T}\Pr(a_t = 1 \mid \theta = 1) \\
&= \varepsilon + (1-2\varepsilon)\big[\pi \Pr(s \geq 1-\pi) + (1-\pi)\Pr(s \geq \pi)\big] \\
&= \varepsilon + (1-2\varepsilon)\big[\pi\big[1-(1-\pi)^2\big] + (1-\pi)\big[1-\pi^2\big]\big] \\
&= \varepsilon + (1-2\varepsilon)\big[\pi - \pi\big(1+\pi^2-2\pi\big) + 1 - \pi - \pi^2 + \pi^3\big] \\
&= \varepsilon + (1-2\varepsilon)\big[\pi - \pi - \pi^3 + 2\pi^2 + 1 - \pi - \pi^2 + \pi^3\big] \\
&= \varepsilon + (1-2\varepsilon)\big(1 - \pi + \pi^2\big).
\end{aligned}$$

Reordering yields

$$\Pr(a_1 = 1 \mid \theta = 1) + \sum_{t=2}^{T}\Pr(a_t = 1 \mid \theta = 1) = \sum_{t=1}^{T-1}\Pr(a_t = 1 \mid \theta = 1) + \Pr(a_T = 1 \mid \theta = 1).$$

Then

$$\begin{aligned}
\varepsilon + (1-2\varepsilon)\big(1-\pi+\pi^2\big) - \pi - \frac{\Pr(a_T = 1 \mid \theta = 1) - \Pr(a_1 = 1 \mid \theta = 1)}{T-1} &= 0 \\
\varepsilon + (1-2\varepsilon)\big(1-\pi+\pi^2\big) - \pi - \Delta &= 0 \\
(1-2\varepsilon)\pi^2 - 2(1-\varepsilon)\pi + 1 - \varepsilon - \Delta &= 0,
\end{aligned}$$

where I define $\Delta \equiv \frac{\Pr(a_T=1|\theta=1)-\Pr(a_1=1|\theta=1)}{T-1}$. Then

$$\begin{aligned}
\pi &= \frac{2(1-\varepsilon) \pm \sqrt{4(1-\varepsilon)^2 - 4(1-2\varepsilon)(1-\varepsilon-\Delta)}}{2(1-2\varepsilon)} \\
&= \frac{1-\varepsilon-\sqrt{(1-\varepsilon)^2-(1-2\varepsilon)(1-\varepsilon-\Delta)}}{1-2\varepsilon}.
\end{aligned}$$

Note that $\lim_{T\to\infty}\Delta = 0$. Then

$$\begin{aligned}
\pi &\to \frac{1-\varepsilon-\sqrt{(1-\varepsilon)^2-(1-2\varepsilon)(1-\varepsilon)}}{1-2\varepsilon} \\
&= \frac{1-\varepsilon}{1-2\varepsilon}\left(1-\sqrt{1-\frac{1-2\varepsilon}{1-\varepsilon}}\right) = \frac{1-\varepsilon}{1-2\varepsilon}\left(1-\sqrt{\frac{\varepsilon}{1-\varepsilon}}\right).
\end{aligned}$$

Also, as $T \to \infty$, $\pi - \Pr(a_i = 1 \mid \theta) \to 0$. Then $x_1 = \lim_{T\to\infty}\Pr(a_i = 1 \mid \theta) = \frac{1-\varepsilon}{1-2\varepsilon}(1 - \sqrt{\frac{\varepsilon}{1-\varepsilon}})$. □

S.4. Example 8: Multiple equilibria in a coordination game

Proof of Example 8. Consider a sequence of symmetric strategy profiles $\{\sigma^T(s,\xi)\}$, where $\sigma^T(s,\xi)=\sigma(s,\xi)$ does not change with T and is given by

$$\sigma(s,\xi)=\begin{cases}1 & \text{if } s=1\\ 0 & \text{if } s=0\\ \xi & \text{if } s=1/2.\end{cases}$$

Let $\pi\equiv\Pr(\xi=1\mid\theta=1)$. Under $\sigma(s,\xi)$, the likelihood that somebody who observes a sample (that is, not agent 1) chooses action 1 is given by

$$\begin{aligned}\Pr\bigl(a_i=1\mid\theta=1,Q(i)\neq 1\bigr)&=\frac{1}{T-1}\sum_{t=2}^{T}\Pr(a_t=1\mid\theta=1)\\ &=\varepsilon+(1-2\varepsilon)\bigl[\Pr(s=1)+\Pr(s=1/2)\pi\bigr]\\ &=\varepsilon+(1-2\varepsilon)\bigl[(1-\gamma)/100+99/100\pi\bigr].\end{aligned}$$

Reordering yields

$$\Pr(a_1=1\mid\theta=1)+\sum_{t=2}^{T}\Pr(a_t=1\mid\theta=1)=\sum_{t=1}^{T-1}\Pr(a_t=1\mid\theta=1)+\Pr(a_T=1\mid\theta=1).$$

Then

$$\begin{aligned}&\frac{\sum_{t=2}^{T}\Pr(a_t=1\mid\theta=1)}{T-1}-\frac{\sum_{t=1}^{T-1}\Pr(a_t=1\mid\theta=1)}{T-1}\\ &\quad=\frac{\Pr(a_T=1\mid\theta=1)-\Pr(a_1=1\mid\theta=1)}{T-1},\end{aligned}$$

so

$$\begin{aligned}\Pr\bigl(a_i=1\mid\theta=1,Q(i)\neq 1\bigr)-\pi&=\frac{\Pr(a_T=1\mid\theta=1)-\Pr(a_1=1\mid\theta=1)}{T-1}\\ \varepsilon+(1-2\varepsilon)\bigl[(1-\gamma)/100+99/100\pi\bigr]-\pi&=\Delta.\end{aligned}$$

Then

$$\begin{aligned}\varepsilon-2\varepsilon\bigl[(1-\gamma)/100+99/100\pi\bigr]+(1-\gamma)/100-1/100\pi&=\Delta\\ \varepsilon-2\varepsilon(1-\gamma)/100-\varepsilon 198/100\pi+(1-\gamma)/100-1/100\pi&=\Delta\\ +(1-\gamma)/100+\bigl[1-(1-\gamma)/50\bigr]\varepsilon-(1/100+198/100\varepsilon)\pi&=\Delta\\ +(1-\gamma)+\bigl[100-2(1-\gamma)\bigr]\varepsilon-(1+198\varepsilon)\pi&=100\Delta.\end{aligned}$$

Then

$$\pi = \frac{(1-\gamma) + \left[100 - 2(1-\gamma)\right]\varepsilon - 100\Delta}{1 + 198\varepsilon}.$$

Proposition 1 guarantees that as the number of agents grows large, the average action is close to its expectation. For low enough ε and large enough T, approximately $X_0|\sigma \xrightarrow{p} \gamma$ and $X_1|\sigma \xrightarrow{p} 1-\gamma$. Then

$$\frac{\Pr(\theta = 1 \mid \xi = 1)}{\Pr(\theta = 0 \mid \xi = 1)} \approx \frac{1-\gamma}{\gamma}.$$

So the sample is informative about the state of the world. To sum up, there is ε small and T large such that σ is indeed an equilibrium. □

S.5. Proving Lemma 12

I illustrate first the effect of different values of $\gamma > 1$ on sampling probabilities. Figure S.1 presents an agent in position 21. The black line shows the probability of observing a predecessor in position $\tau < 21$ when $\gamma = 8$. With probability higher than 0.998, the agent observes one of his three immediate predecessors. The distribution becomes flatter as γ decreases. The red line shows the distribution when $\gamma = 1.05$. In this case, the agent in position 21 observes his immediate predecessor twice as often as he observes the first agent in the sequence. As $\gamma \to 1$, sampling approaches uniform random sampling. Instead, as $\gamma \to \infty$, sampling approaches observing the immediate predecessor.

Next, I present the proof of Lemma 12.

Proof of Lemma 12. A strategy σ_i induces $\rho_\theta(\xi) = \mathbf{P}_{\sigma_i}(a_i \mid \theta, \xi)$. For the rest of this section, I fix the state of the world θ and drop its index. Then a strategy σ_i induces a vector $(\rho(\varnothing), \rho(0), \rho(1))$. Because of mistakes, $\varepsilon < \rho(\xi) < 1-\varepsilon$ for all $\xi \in \{0, 1, \varnothing\}$.

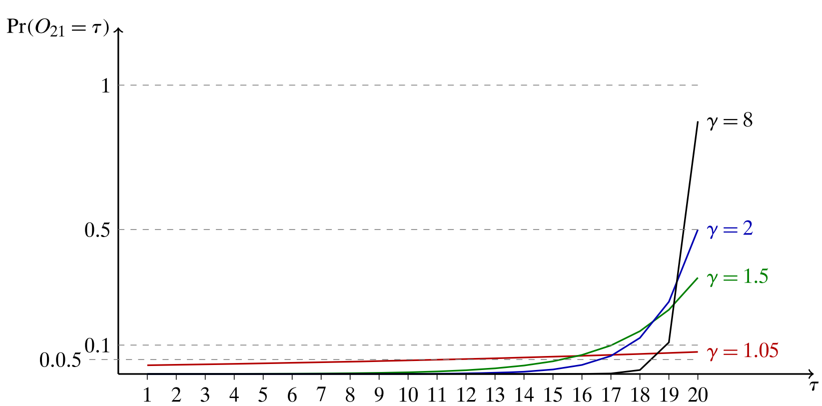

Figure S.1. Probabilities of different predecessors being observed: geometric sampling.

Assume first that $\gamma > 1$. The first agent in the sequence chooses action 1 with probability $\rho(\varnothing)$. For $t \geq 2$,

$$\begin{aligned}
\mathbf{P}_\sigma(a_t = 1) &= \Pr(\xi_t = 0)\Pr(a_t = 1 \mid \xi_t = 0) + \Pr(\xi_t = 1)\Pr(a_t = 1 \mid \xi_t = 1) \\
&= \Pr(\xi_t = 0)\rho(0) + \Pr(\xi_t = 1)\rho(1) \\
&= \big[1 - \Pr(\xi_t = 1)\big]\rho(0) + \Pr(\xi_t = 1)\rho(1) \\
&= \rho(0) + \big[\rho(1) - \rho(0)\big]\Pr(\xi_t = 1) \\
&= \rho(0) + \big[\rho(1) - \rho(0)\big]\sum_{\tau<t}\Pr(O_t = \tau)\mathbb{1}\{a_\tau = 1\} \\
&= \rho(0) + \big[\rho(1) - \rho(0)\big]\sum_{\tau=1}^{t-1}\frac{\gamma-1}{\gamma}\frac{\gamma^\tau}{\gamma^{t-1}-1}a_\tau.
\end{aligned}$$

Define the weighted sum of the past history by $p_t \equiv \sum_{\tau=1}^{t-1}\frac{\gamma-1}{\gamma}\frac{\gamma^\tau}{\gamma^{t-1}-1}a_\tau$ for $t \geq 2$. This concept plays a key role in the model:

$$\mathbf{P}_\sigma(a_t = 1) = \rho(0) + \big[\rho(1) - \rho(0)\big]p_t.$$

This weighted sum has a recursive nature:

$$\begin{aligned}
p_{t+1} &= \sum_{\tau=1}^{t}\frac{\gamma-1}{\gamma}\frac{\gamma^\tau}{\gamma^t-1}a_\tau = \frac{\gamma^{t-1}-1}{\gamma^t-1}\left[\sum_{\tau=1}^{t-1}\frac{\gamma-1}{\gamma}\frac{\gamma^\tau}{\gamma^{t-1}-1}a_\tau\right] + \frac{\gamma-1}{\gamma}\frac{\gamma^t}{\gamma^t-1}a_t \\
&= \frac{\gamma^{t-1}-1}{\gamma^t-1}p_t + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}a_t.
\end{aligned}$$

In expectation,

$$\begin{aligned}
E[p_{t+1} \mid I_t] &= \frac{\gamma^{t-1}-1}{\gamma^t-1}E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}E[a_t \mid I] \\
&= \frac{\gamma^{t-1}-1}{\gamma^t-1}E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}\big[\rho(0) + \big[\rho(1)-\rho(0)\big]E[p_t \mid I]\big] \\
&= \frac{\gamma^t-1+\gamma^{t-1}-\gamma^t}{\gamma^t-1}E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}\big[\rho(0) + \big[\rho(1)-\rho(0)\big]E[p_t \mid I]\big] \\
&= E[p_t \mid I] + \frac{\gamma^{t-1}-\gamma^t}{\gamma^t-1}E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}\big[\rho(0) + \big[\rho(1)-\rho(0)\big]E[p_t \mid I]\big] \\
&= E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}\big[\rho(0) - \big[1+\rho(0)-\rho(1)\big]E[p_t \mid I]\big] \\
&= E[p_t \mid I] + \frac{\gamma^t-\gamma^{t-1}}{\gamma^t-1}\big[1+\rho(0)-\rho(1)\big]\big[\rho^* - E[p_t \mid I]\big].
\end{aligned}$$

Let $\rho^* \equiv \frac{\rho(0)}{1+\rho(0)-\rho(1)}$.[1] Then

$$\begin{aligned}
E[p_{t+1} \mid I] - \rho^* &= E[p_t \mid I] - \rho^* - \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1}\big[1+\rho(0)-\rho(1)\big]\big[E[p_t \mid I] - \rho^*\big] \\
&= \left[1 - \frac{\gamma^t - \gamma^{t-1}}{\gamma^t - 1}\big[1+\rho(0)-\rho(1)\big]\right]\big[E[p_t \mid I] - \rho^*\big] \\
&= \Bigg[1 - \underbrace{\frac{\gamma-1}{\gamma}\frac{\gamma^t}{\gamma^t-1}}_{(*)}\underbrace{\big[1+\rho(0)-\rho(1)\big]}_{(**)}\Bigg]\big[E[p_t \mid I] - \rho^*\big]. \qquad \text{(S.2)}
\end{aligned}$$

I next provide bounds for the terms $(*)$ and $(**)$ in (S.2):

$$\begin{aligned}
2\varepsilon \le 1+\rho(0)-\rho(1) \le 2-2\varepsilon \\
\frac{\gamma-1}{\gamma} \le \frac{\gamma-1}{\gamma}\frac{\gamma^t}{\gamma^t-1} \le 1.
\end{aligned}$$

With these bounds, I can also bound the whole term in brackets in (S.2):

$$\begin{aligned}
\frac{\gamma-1}{\gamma}2\varepsilon \le \frac{\gamma-1}{\gamma}\frac{\gamma^t}{\gamma^t-1}\big[1+\rho(0)-\rho(1)\big] \le 2-2\varepsilon \\
\frac{\gamma-1}{\gamma}2\varepsilon - 1 \le \frac{\gamma-1}{\gamma}\frac{\gamma^t}{\gamma^t-1}\big[1+\rho(0)-\rho(1)\big] - 1 \le 1-2\varepsilon \\
\left|1 - \frac{\gamma-1}{\gamma}\frac{\gamma^t}{\gamma^t-1}\big[1+\rho(0)-\rho(1)\big]\right| \le 1 - \frac{\gamma-1}{\gamma}2\varepsilon.
\end{aligned}$$

This leads to a simple bound over time:

$$\begin{aligned}
\big|E[p_{t+n} \mid I_t] - \rho^*\big| &= \prod_{\tau=t}^{t+n-1}\left|1 - \frac{\gamma-1}{\gamma}\frac{\gamma^t}{\gamma^t-1}\big[1+\rho(0)-\rho(1)\big]\right|\big|E[p_t \mid I_t] - \rho^*\big| \\
&\le \left(1 - \frac{\gamma-1}{\gamma}2\varepsilon\right)^{n-1}.
\end{aligned}$$

In particular,

$$\begin{aligned}
\big|E[p_{t+n} \mid a_t = 1] - \rho^*\big| \le \left(1 - \frac{\gamma-1}{\gamma}2\varepsilon\right)^{n-1} \\
\big|E[p_{t+n}] - \rho^*\big| \le \left(1 - \frac{\gamma-1}{\gamma}2\varepsilon\right)^{t+n-1}.
\end{aligned}$$

[1] Note that $\rho(0) > \varepsilon$ and $\rho(1) < 1-\varepsilon$, so $1+\rho(0)-\rho(1) \ge 1+\varepsilon-(1-\varepsilon) = 2\varepsilon$ and so $1+\rho(0)-\rho(1) \ne 0$.

So finally,

$$\begin{aligned}
\left|E[p_{t+n} \mid I_t] - E[p_{t+n}]\right| &\le \left|E[p_{t+n} \mid a_t = 1] - \rho^*\right| + \left|E[p_{t+n}] - \rho^*\right| \\
&\le \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n-1} + \left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{t+n-1} \\
&\le 2\left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n-1},
\end{aligned}$$

and turning this into probabilities yields

$$\begin{aligned}
\left|\mathbf{P}_\sigma(a_{t+n} = 1 \mid a_t = 1) - \mathbf{P}_\sigma(a_{t+n} = 1)\right| &= \big|\rho(0) + \left[\rho(1) - \rho(0)\right] E[p_{t+n} \mid a_t = 1] \\
&\qquad - \left[\rho(0) + \left[\rho(1) - \rho(0)\right] E[p_{t+n}]\right]\big| \\
&= \left|\left[\rho(1) - \rho(0)\right]\left[E[p_{t+n} \mid a_t = 1] - E[p_{t+n}]\right]\right| \\
&\le 2\left|\left[E[p_{t+n} \mid a_t = 1] - E[p_{t+n}]\right]\right| \\
&\le 4\left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n-1} \\
&\le \frac{4}{1 - \frac{\gamma - 1}{\gamma} 2\varepsilon}\left(1 - \frac{\gamma - 1}{\gamma} 2\varepsilon\right)^{n}.
\end{aligned}$$

Next assume that $\gamma = 1$. Then

$$\mathbf{P}_\sigma(a_t = 1) = \rho(0) + \left[\rho(1) - \rho(0)\right] \frac{1}{t-1} \sum_{\tau=1}^{t-1} a_\tau.$$

Define now $p_t \equiv \frac{1}{t-1} \sum_{\tau=1}^{t-1} a_\tau$ for $t \ge 2$, which leads to

$$p_{t+1} = \frac{1}{t} \sum_{\tau=1}^{t} a_\tau = \frac{t-1}{t} \sum_{\tau=1}^{t-1} a_\tau + \frac{1}{t} a_t = \frac{t-1}{t} p_t + \frac{1}{t} a_t.$$

In expectation,

$$\begin{aligned}
E[p_{t+1} \mid I_t] &= \frac{t-1}{t} E[p_t \mid I] + \frac{1}{t} E[a_t \mid I] \\
&= \frac{t-1}{t} E[p_t \mid I] + \frac{1}{t} E\left[\rho(0) + \left[\rho(1) - \rho(0)\right] p_t \mid I\right] \\
&= \frac{1}{t}\left[t - 1 + \rho(1) - \rho(0)\right] E[p_t \mid I] + \frac{1}{t} \rho(0),
\end{aligned}$$

so in this case,

$$\begin{aligned} E[p_{t+1} \mid I_t] - \rho^* &= \frac{1}{t}\big[t - 1 + \rho(1) - \rho(0)\big]E[p_t \mid I] + \frac{1}{t}\rho(0) - \rho^* \\ &= \frac{1}{t}\big[\rho(0) - \big[1 + \rho(0) - \rho(1)\big]E[p_t \mid I]\big] + E[p_t \mid I] - \rho^* \\ &= \frac{1}{t}\big[1 + \rho(0) - \rho(1)\big]\big[\rho^* - E[p_t \mid I]\big] + E[p_t \mid I] - \rho^* \\ &= \bigg[1 - \frac{1}{t}\big[1 + \rho(0) - \rho(1)\big]\bigg]\big[E[p_t \mid I] - \rho^*\big]. \end{aligned}$$

Then

$$E[p_{t+n} \mid I_t] - \rho^* = \big[E[p_t \mid I] - \rho^*\big]\prod_{\tau=0}^{n}\bigg[1 - \frac{1}{t+\tau}\big[1 + \rho(0) - \rho(1)\big]\bigg].$$

I present without proof the following remark.

Remark 1. Let $0 < a_n < 1$ for all n. Then $\prod_{\tau=0}^{\infty} a_n > 0 \Leftrightarrow \sum_{\tau=0}^{\infty}(1 - a_n) < \infty$.

Then it suffices to show that

$$\sum_{\tau=0}^{n}\frac{1}{t+\tau}\big[1 + \rho(0) - \rho(1)\big] = \big[1 + \rho(0) - \rho(1)\big]\sum_{\tau=0}^{n}\frac{1}{t+\tau} = \infty$$

and follow the same steps as in the case with $\gamma > 1$. □

S.6. Proof of Lemma 13

I show Proposition 1 by proving that $X|\sigma^T - E[X|\sigma^T]$ converges to zero in L^2 norm. The variance $V(\sigma^\tau)$ as defined by (5) is bounded above by

$$V(\sigma^\tau) \le \frac{1}{T}\left(1 + 4(1 - 2\varepsilon^M)^{-1}\frac{(1 - 2\varepsilon^M)^{\frac{1}{M}}}{1 - (1 - 2\varepsilon^M)^{\frac{1}{M}}}\right).$$

Note that $\lim_{T\to\infty} 4(1 - 2\varepsilon^{M(T)})^{-1} = 4$ and $\lim_{T\to\infty}(1 - 2\varepsilon^{M(T)})^{\frac{1}{M(T)}} = 1$. Then the bound converges to zero whenever $\lim_{T\to\infty} T[1 - (1 - 2\varepsilon^{M(T)})^{\frac{1}{M(T)}}] = \infty$. I need to show that for any $K < \infty$, there exists a $\widetilde{T} < \infty$ such that $T[1 - (1 - 2\varepsilon^{M(T)})^{\frac{1}{M(T)}}] \ge K$ for all $T \ge \widetilde{T}$. This simplifies to

$$\left(1 - \frac{K}{T}\right)^{M(T)} \ge 1 - 2\varepsilon^{M(T)} \quad \forall T \ge \widetilde{T}.$$

Since $(1 - \frac{K}{T})^{M(T)} \ge 1 - \frac{KM}{T}$, it suffices to show that

$$1 - \frac{KM}{T} \ge 1 - 2\varepsilon^{M(T)} \quad \Leftrightarrow \quad \frac{\varepsilon^{M(T)}}{M} \ge \frac{K}{2}\frac{1}{T},$$

where $M(T)$ is $o(\log(T))$. Then, for any constant $c \geq 0$, there is T large enough such that $M(T) \leq c\log(T)$. Pick $c = (-2\log(\varepsilon))^{-1}$. Note next that the function ε^x/x is decreasing. Then, for T large, $\frac{\varepsilon^{M(T)}}{M(T)} \geq \frac{\varepsilon^{(-2\log(\varepsilon))^{-1}\log(T)}}{(-2\log(\varepsilon))^{-1}\log(T)}$. As a result, it suffices to show that for T large enough,

$$
\begin{aligned}
\frac{\varepsilon^{[(-2\log(\varepsilon))^{-1}\log(T)]}}{\left(-2\log(\varepsilon)\right)^{-1}\log(T)} &\geq \frac{K}{2}\frac{1}{T} \\
\varepsilon^{(-2\log(\varepsilon))^{-1}\log(T)} &\geq \frac{K}{2}\frac{1}{T}\left(-2\log(\varepsilon)\right)^{-1}\log(T) \\
T^{(-2\log(\varepsilon))^{-1}\log(\varepsilon)} &\geq \frac{1}{-4\log(\varepsilon)}K\frac{\log(T)}{T} \\
T^{-\frac{1}{2}} &\geq \frac{1}{-4\log(\varepsilon)}K\frac{\log(T)}{T} \\
\frac{T^{\frac{1}{2}}}{\log(T)} &\geq \frac{1}{-4\log(\varepsilon)}K.
\end{aligned}
$$

The left hand side goes to infinity and the right hand side is constant. Then there always exists a T such that this holds. This shows the first part of Proposition 1.

Next, I focus on the second part of Proposition 1. Equation (6) in the paper now becomes

$$
\Pr\left(\left|X|\sigma^T - X|\widetilde{\sigma}^T\right| \geq \frac{n}{T}\right) \leq \left[\left(1-2\varepsilon^{M(T)}\right)^{\frac{1}{M(T)}}\right]^n,
$$

which holds for all n.

Let $n = \lceil(-2\log(\varepsilon))^{-1}\log(T)T^{\frac{3}{4}}\rceil$. As $(1-2\varepsilon^M)^{\frac{1}{M}} \leq 1$, then

$$
\begin{aligned}
\Pr\left(\left|X|\sigma^T - X|\widetilde{\sigma}^T\right| \geq \frac{n}{T}\right) &\leq \left[\left(1-2\varepsilon^{M(T)}\right)^{\frac{1}{M(T)}}\right]^n \\
&\leq \left[\left(1-2\varepsilon^{M(T)}\right)^{\frac{1}{M(T)}}\right]^{(-2\log(\varepsilon))^{-1}\log(T)T^{\frac{3}{4}}} \\
&\leq \left(1-2\varepsilon^{(-2\log(\varepsilon))^{-1}\log(T)}\right)^{\frac{(-2\log(\varepsilon))^{-1}\log(T)T^{\frac{3}{4}}}{(-2\log(\varepsilon))^{-1}\log(T)}} \\
&= \left(1-2T^{-\frac{1}{2}}\right)^{T^{\frac{3}{4}}},
\end{aligned}
$$

where I have used the fact that $M(T)$ is $o(\log(T))$, so $M(T) \leq (-2\log(\varepsilon))^{-1}\log(T)$ for T large enough. Moreover, I also used the fact that $(1-2\varepsilon^M)^{\frac{1}{M}}$ is increasing in M.

I need to show that for all $b > 0$, there exists $\widetilde{T}$, such that $\Pr(|X|\sigma^T - X|\widetilde{\sigma}^T| \geq b) < b$ for all $T > \widetilde{T}$. Then it suffices to show that $\lim_{T\to\infty}\frac{n}{T} = 0$ and $\lim_{T\to\infty}(1 - 2T^{-\frac{1}{2}})^{T^{\frac{3}{4}}} = 0$.

So first, note that

$$\frac{n}{T} \le \frac{\left(-2\log(\varepsilon)\right)^{-1}\log(T)T^{\frac{3}{4}}+1}{T} = \frac{1}{\left(-2\log(\varepsilon)\right)}\frac{\log(T)}{T^{\frac{1}{4}}} + \frac{1}{T} \to 0,$$

so $\lim_{T\to\infty}\frac{n}{T}=0$.

Second, note that $\lim_{T\to\infty}(1-2T^{-\frac{1}{2}})^{T^{\frac{3}{4}}}=0 \Leftrightarrow \lim_{T\to\infty}T^{\frac{3}{4}}\log(1-2T^{-\frac{1}{2}})=-\infty$. So using l'Hôpital's rule,

$$\lim_{T\to\infty}\frac{\log\left(1-2T^{-\frac{1}{2}}\right)}{T^{-\frac{3}{4}}} = \lim_{T\to\infty}\frac{\frac{1}{1-2T^{-\frac{1}{2}}}(-2)\left(-\frac{1}{2}\right)T^{-\frac{3}{2}}}{-\frac{3}{4}T^{-\frac{7}{4}}} = \lim_{T\to\infty}-\frac{4}{3}\frac{T^{\frac{1}{4}}}{1-2T^{-\frac{1}{2}}} = -\infty.$$

This finishes the proof of the second part of Proposition 1.

Lemma 10 also needs some adjustment to allow for M to grow with T. Equation (8) from the paper becomes

$$\begin{aligned}\pi_\theta^T - E_{\sigma^T}[X_\theta] = \frac{1}{T}\Bigg[&\sum_{\tau=1}^{M(T)-1}\overbrace{\mathbf{P}_{\sigma^T}(a_\tau=1)}^{\le 1}\left(\sum_{t=\tau}^{\tau+M(T)-1}\overbrace{t^{-1}-1}^{\le 1}\right)\\ &-\sum_{\tau=T-M(T)+1}^{T}\underbrace{\mathbf{P}_{\sigma^T}(a_\tau=1)}_{\le 1}\underbrace{\left(1-\frac{T-\tau}{M(T)}\right)}_{\le 1}\Bigg]\\ \le{}& \frac{2M(T)}{T}.\end{aligned}$$

Since $M(T)$ is $o(\log(T))$, then, $\pi_\theta^T - E_{\sigma^T}\to 0$. This adapts Lemma 10 to the case with growing M. The rest of Proposition 2 does not change.

S.7. Many states of the world and many actions

S.7.1 *The model*

States and Actions There are N_θ equally likely states of the world $\theta\in\Theta=\{1,2,\ldots,N_\theta\}$. Agents must choose between N_a possible actions $a\in\mathcal{A}=\{1,2,\ldots,N_a\}$. Let $X^a\equiv\frac{1}{T}\sum_{j\in\mathcal{I}}\mathbb{1}\{a_j=a\}$ denote the proportion of agents who choose action a, with realizations $x^a\in[0,1]$. The vector $X=(X^1,X^2,\ldots,X^{N_a})$ denotes the proportion of agents who choose each action. Agent i obtains utility $u(a_i,X,\theta):\mathcal{A}\times[0,1]\times\Theta\to\mathbb{R}$, where $u(a_i,X,\theta)$ is a continuous function in X.

Private Signals Conditional on the true state of the world, signals are i.i.d. across individuals and distributed according to F_θ. I assume that F_θ and $F_{\tilde\theta}$ are mutually absolutely continuous for any two $\theta,\tilde\theta\in\Theta$. Then no perfectly revealing signals occur with positive probability, and the likelihood ratio (Radon–Nikodym derivative) $l_{\tilde\theta,\theta}(s)\equiv\frac{dF_{\tilde\theta}}{dF_\theta}(s)$ exists.

I also define a likelihood ratio that indicates how likely one state is, relative to all other states:

$$l_\theta(s) = \left(\sum_{\tilde\theta \neq \theta} l_{\tilde\theta,\theta}(s)\right)^{-1}.$$

Let $G_\theta(l) \equiv \Pr(l_\theta(S) \leq l \mid \theta)$. I modify the assumption of signals being of unbounded strength as follows.

Definition (Signal strength). Signal strength is *unbounded* if $0 < G_\theta(l) < 1$ for all likelihood ratios $l \in (0,\infty)$ and for all states $\theta \in \Theta$.

Sampling Strategies and Mistakes The sampling rule does not change. A strategy is now a function $\sigma_i : \mathcal{S} \times \Xi \to [\varepsilon, 1-(N_a-1)\varepsilon]^{N_a}$ that specifies a probability vector $\sigma_i(s,\xi)$ for choosing each action given the information available. For example, $\sigma_i^a(s,\xi)$ indicates the probability of choosing action $a \in \mathcal{A}$, after receiving signal s and sample ξ.

Definition of Social Learning I modify the definition of NE to allow for many states and actions. I say that x_θ corresponds to a Nash equilibrium of the stage game (and denote it by $x_\theta \in \mathrm{NE}^\theta$) whenever $u(a, x_\theta, \theta) > u(a^*, x_\theta, \theta)$ for some $a, a^* \in \mathcal{A} \Rightarrow x_\theta^{a^*} = 0$. Then $x \in \mathrm{NE}$ whenever $x_\theta \in \mathrm{NE}^\theta$ for all $\theta \in \Theta$.

S.7.2 *Results*

Existence and Convergence of Average Action The proofs of Lemma 1 and Proposition 1 extend directly to a context with many actions and many states. I need to adapt the notation. The random variable $X|\sigma$ is now a matrix. Each element $X_\theta^a|\sigma$ is a random variable that denotes the proportion of agents choosing action a in state θ. So the random variable $X|\sigma = (X_1|\sigma, X_2|\sigma, \ldots, X_{N_\theta}|\sigma)$ has realizations $x = (x_1, x_2, \ldots, x_{N_\theta})$, where each x_θ is itself a vector: $x_\theta = (x_\theta^1, x_\theta^2, \ldots, x_\theta^{N_a})$.

Utility Convergence In what follows, I provide modified expressions for the expected utility, the utility of the expected average action, and the approximate utility of a deviation. These expressions apply to contexts with many actions and many states.

Agents' expected utility under symmetric profile σ^T is simply

$$u(\sigma^T) \equiv E_{\sigma^T}\big[u(a_i, X, \theta)\big] = \frac{1}{N_\theta}\sum_{\theta\in\Theta} E_{\sigma^T}\Bigg[\sum_{a\in\mathcal{A}} X_\theta^a \cdot u(a, X_\theta, \theta)\Bigg].$$

Define the *utility of the expected average action* $\bar u^T$ by

$$\bar u^T \equiv \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}} E_{\sigma^T}\big[X_\theta^a\big] \cdot u\big(a, E_{\sigma^T}[X_\theta], \theta\big).$$

Define the *approximate utility of the deviation* $\widetilde{u}^T$ by

$$\widetilde{u}^T \equiv \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}} \mathbf{P}_{\widetilde\sigma^T}(a_i = a \mid \theta) \cdot u\big(a, E_{\sigma^T}[X_\theta], \theta\big).$$

The proofs of Lemmas 2 and 3, as well as Corollary 1, extend directly to a context with many actions and many states.

Corollary 2′ (The approximate improvement). *Let the* approximate improvement Δ^T *be given now by*

$$\Delta^T \equiv \widetilde{u}^T - \bar{u}^T = \frac{1}{N_\theta} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}} \big[\mathbf{P}_{\widetilde{\sigma}^T}(a_i = a \mid \theta) - E_{\sigma^T}\big[X_\theta^a\big]\big] \cdot u\big(a, E_{\sigma^T}[X_\theta], \theta\big).$$

The proof of Corollary 2′ extends directly to a context with many actions and many states.

S.7.3 *Alternative Strategy 1: Always follow a given action*

I present next a version of Lemma 4 that applies to many actions and many states. Let action $a^* \in \mathcal{A}$ be weakly dominant if

$$u\big(a^*, x_\theta, \theta\big) \geq u(a, x_\theta, \theta) \quad \text{for all } a \in \mathcal{A} \text{ and for all } \theta \in \Theta.$$

Let action $a^* \in \mathcal{A}$ be strictly dominant if

$$u\big(a^*, x_\theta, \theta\big) > u(a, x_\theta, \theta) \quad \text{for all } a \in \mathcal{A} \text{ and for all } \theta \in \Theta.$$

Lemma 4′ (Dominance). *If action $a^* \in \mathcal{A}$ is strictly dominant, then $x_\theta^{a^*} = 1 - (N_a - 1)\varepsilon$ for all $\theta \in \Theta$. Assume instead that action $a^* \in \mathcal{A}$ is weakly dominant. If there exists state $\theta \in \Theta$ with $u(a^*, x_\theta, \theta) > u(\tilde{a}, x_\theta, \theta)$, then $x_\theta^{\tilde{a}} = \varepsilon$.*

Proof. Consider the alternative strategy of always choosing action a^*. Because of mistakes, this means a^* is chosen with probability $1 - (N_a - 1)\varepsilon$. Then the improvement is

$$\begin{aligned}
\Delta^T &= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \bigg[\big[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}\big] u\big(a^*, x_\theta, \theta\big) + \sum_{a \neq a^*} \big(\varepsilon - x_\theta^a\big) \cdot u(a, x_\theta, \theta)\bigg] \\
&= \frac{1}{N_\theta} \sum_{\theta \in \Theta} \bigg[\big[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}\big] u\big(a^*, x_\theta, \theta\big) - \sum_{a \neq a^*} \big(x_\theta^a - \varepsilon\big) \cdot u(a, x_\theta, \theta)\bigg].
\end{aligned}$$

Note that $x_\theta^a - \varepsilon \geq 0$ for all a, θ. Then

$$\begin{aligned}
&\big[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}\big] u\big(a^*, x_\theta, \theta\big) - \sum_{a \neq a^*} \big(x_\theta^a - \varepsilon\big) \cdot u(a, x_\theta, \theta) \\
&\quad \geq \big[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}\big] u\big(a^*, x_\theta, \theta\big) - \sum_{a \neq a^*} \big(x_\theta^a - \varepsilon\big) \cdot u\big(a^*, x_\theta, \theta\big) \\
&\quad = \bigg[\big[1 - (N_a - 1)\varepsilon - x_\theta^{a^*}\big] - \sum_{a \neq a^*} \big(x_\theta^a - \varepsilon\big)\bigg] \cdot u\big(a^*, x_\theta, \theta\big)
\end{aligned}$$

$$= \underbrace{\left[1-(N_a-1)\varepsilon-\sum_{a\in\mathcal{A}}x_\theta^a+(N_a-1)\varepsilon\right]}_{=0}\cdot u\big(a^*,x_\theta,\theta\big)$$

$$=0.$$

Recall that $\Delta^T\le 0$ by Corollary 2. Moreover, $\Delta^T\ge 0$. Then $\Delta^T=0$. Also, as each term in Δ^T is weakly positive, then all terms in Δ^T must be zero:

$$\big[1-(N_a-1)\varepsilon-x_\theta^{a^*}\big]u\big(a^*,x_\theta,\theta\big)-\sum_{a\neq a^*}\big(x_\theta^a-\varepsilon\big)\cdot u(a,x_\theta,\theta)=0.$$

Assume next that for some action $\tilde{a}\in\mathcal{A}$ in some state $\theta\in\Theta$, $u(a^*,x_\theta,\theta)>u(\tilde{a},x_\theta,\theta)$. Then

$$\begin{aligned}
0&=\big[1-(N_a-1)\varepsilon-x_\theta^{a^*}\big]u\big(a^*,x_\theta,\theta\big)-\sum_{a\neq a^*}\big(x_\theta^a-\varepsilon\big)\cdot u(a,x_\theta,\theta)\\
&\ge\left[1-(N_a-1)\varepsilon-x_\theta^{a^*}-\sum_{a\neq a^*,a\neq\tilde{a}}\big(x_\theta^a-\varepsilon\big)\right]u\big(a^*,x_\theta,\theta\big)-\big(x_\theta^{\tilde{a}}-\varepsilon\big)u(\tilde{a},x_\theta,\theta)\\
&=\big[1-\varepsilon-\big(1-x_\theta^{\tilde{a}}\big)\big]u\big(a^*,x_\theta,\theta\big)-\big(x_\theta^{\tilde{a}}-\varepsilon\big)u(\tilde{a},x_\theta,\theta)\\
&=\big(x_\theta^{\tilde{a}}-\varepsilon\big)u\big(a^*,x_\theta,\theta\big)-\big(x_\theta^{\tilde{a}}-\varepsilon\big)u(\tilde{a},x_\theta,\theta)\\
&=\big(x_\theta^{\tilde{a}}-\varepsilon\big)\big[u\big(a^*,x_\theta,\theta\big)-u(\tilde{a},x_\theta,\theta)\big].
\end{aligned}$$

To sum up,

$$\big(x_\theta^{\tilde{a}}-\varepsilon\big)\overbrace{\big[u\big(a^*,x_\theta,\theta\big)-u(\tilde{a},x_\theta,\theta)\big]}^{>0}\le 0.$$

So $x_\theta^{\tilde{a}}=\varepsilon$. Similarly, if $u(a^*,x_\theta,\theta)>u(a,x_\theta,\theta)$ for all $a\in\mathcal{A}$ and for all $\theta\in\Theta$, then $x_\theta^{a^*}=1-(N_a-1)\varepsilon$. □

S.7.4 *Alternative Strategy 2: Improve upon a sampled agent*

Consider a possible limit point $x=(x_1,x_2,\ldots,x_{N_\theta})$. Assume that action $\tilde{a}$ is not optimal in state θ^*: $u(a^*,x_{\theta^*},\theta^*)>u(\tilde{a},x_{\theta^*},\theta^*)$, but it is still played in the limit: $x_{\theta^*}^{\tilde{a}}>\varepsilon$. As in the case with two states, let $\widetilde{\xi}$ denote the action of one individual selected at random from the sample. Consider an alternative simple strategy $\widetilde{\sigma}$, which makes the agent choose the action

$$a_i(\widetilde{\xi},s)$$

$$=\begin{cases} a^* & \text{if } \widetilde{\xi}=\tilde{a} \text{ and}\\ & l_{\theta^*}(s)\ge k^T\equiv\dfrac{-\bar{u}}{u\big(a^*,E_{\sigma^T}[X_{\theta^*}],\theta^*\big)-u\big(\tilde{a},E_{\sigma^T}[X_{\theta^*}],\theta^*\big)}\dfrac{1}{\mathbf{P}_{\sigma^T}\big(\widetilde{\xi}=\tilde{a}|\theta=\theta^*\big)}\\ \widetilde{\xi} & \text{otherwise.}\end{cases}$$

I provide next a version of Lemma 5 in the paper that applies to many actions and many states.

LEMMA 5′ (Improvement principle). *Take any limit point* $x \in L$ *with* $u(a^*, x_{\theta^*}, \theta^*) > u(\tilde{a}, x_{\theta^*}, \theta^*)$. *Then*

$$\widetilde{\Delta}(\varepsilon) + \frac{1-(N_a-1)\varepsilon}{N_\theta}\left[x_{\theta^*}^{\tilde{a}} \cdot \left[u\left(a^*, x_{\theta^*}, \theta^*\right) - u\left(\tilde{a}, x_{\theta^*}, \theta^*\right)\right]\right]$$
$$\times \left[\left[1 - G_{\theta^*}(\bar{k})\right] - \bar{k}\left[1 - \widetilde{G}_{\theta^*}(\bar{k})\right]\right] \leq 0 \qquad \text{(S.3)}$$

with

$$\bar{k} = -\bar{u}\left[\left(u\left(a^*, x_{\theta^*}, \theta^*\right) - u\left(\tilde{a}, x_{\theta^*}, \theta^*\right)\right)x_{\theta^*}^{\tilde{a}}\right]^{-1}$$
$$\widetilde{\Delta}(\varepsilon) = \frac{\varepsilon}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\left[1-(N_a-1)x_\theta^a\right]u(a, x_\theta, \theta)\right].$$

See Section S.7.5 for the proof.

The term $[[1 - G_{\theta^*}(\bar{k})] - \bar{k}[1 - \widetilde{G}_{\theta^*}(\bar{k})]] \geq 0$ in (S.3) decreases in $\bar{k}$ (as shown later in Proposition 3). Moreover, with signals of unbounded strength, this term is strictly positive. Then, whenever $x_\theta^{\tilde{a}} > 0$, there is potential for improvement. The existence of mistakes may present such an improvement. Note, however, that $\lim_{\varepsilon\to 0}\widetilde{\Delta}(\varepsilon) = 0$. Then when mistakes are unlikely, the potential for improvement dominates in (S.3).

S.7.5 *Proof of Lemma 5′*

Let $\rho_\theta^T(a|\tilde{a}) \equiv \mathbf{P}_{\sigma^T}(a_i = a|\theta, \widetilde{\xi} = \tilde{a})$. In general, the improvement is given by

$$\Delta^T = \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\Bigg[\varepsilon + \left[1-(N_a-1)\varepsilon\right]\sum_{a'\in\mathcal{A}}\rho_\theta\left(a|a'\right)\mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = a'|\theta\right)$$
$$- E_{\sigma^T}\left[X_\theta^a\right]\Bigg]u\left(a, E_{\sigma^T}[X_\theta], \theta\right)$$
$$= \left[\frac{\varepsilon}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}u\left(a, E_{\sigma^T}[X_\theta], \theta\right)\right]$$
$$+ \frac{1-(N_a-1)\varepsilon}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\sum_{a'\in\mathcal{A}}\rho_\theta\left(a|a'\right)\mathbf{P}_{\sigma^T}\left(\widetilde{\xi} = a'|\theta\right)u\left(a, E_{\sigma^T}[X_\theta], \theta\right)\right]$$
$$- \frac{1-(N_a-1)\varepsilon}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}E_{\sigma^T}\left[X_\theta^a\right]u\left(a, E_{\sigma^T}[X_\theta], \theta\right)\right]$$
$$- \frac{(N_a-1)\varepsilon}{N_\theta}\left[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}E_{\sigma^T}\left[X_\theta^a\right]u\left(a, E_{\sigma^T}[X_\theta], \theta\right)\right].$$

Let

$$\widetilde{\Delta}^T(\varepsilon) \equiv \frac{\varepsilon}{N_\theta}\Bigg[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}} u\big(a, E_{\sigma^T}[X_\theta], \theta\big) - (N_a - 1)\Bigg[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}} E_{\sigma^T}\big[X_\theta^a\big] u\big(a, E_{\sigma^T}[X_\theta], \theta\big)\Bigg]\Bigg]$$
$$= \frac{\varepsilon}{N_\theta}\Bigg[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}} \big[1 - (N_a - 1)E_{\sigma^T}\big[X_\theta^a\big]\big] u\big(a, E_{\sigma^T}[X_\theta], \theta\big)\Bigg]$$

and

$$J(\varepsilon) \equiv \frac{1 - (N_a - 1)\varepsilon}{N_\theta}.$$

Then

$$\Delta^T = \widetilde{\Delta}^T(\varepsilon) + J(\varepsilon)\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\Bigg[\sum_{a'\in\mathcal{A}} \rho_\theta\big(a|a'\big)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = a'|\theta\big) - E_{\sigma^T}\big[X_\theta^a\big]\Bigg]$$
$$\times u\big(a, E_{\sigma^T}[X_\theta], \theta\big). \tag{S.4}$$

However,

$$= \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\Bigg[\sum_{a'\in\mathcal{A}} \rho_\theta\big(a|a'\big)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = a'|\theta\big) - E_{\sigma^T}\big[X_\theta^a\big]\Bigg] u\big(a, E_{\sigma^T}[X_\theta], \theta\big)$$
$$= \frac{1}{N_\theta}\Bigg[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\sum_{a'\in\mathcal{A}} \rho_\theta\big(a|a'\big)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = a'|\theta\big) u\big(a, E_{\sigma^T}[X_\theta], \theta\big)\Bigg]$$
$$- \frac{1}{N_\theta}\Bigg[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}} E_{\sigma^T}\big[X_\theta^a\big] u\big(a, E_{\sigma^T}[X_\theta], \theta\big)\Bigg]$$
$$= \frac{1}{N_\theta}\Bigg[\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\sum_{a'\in\mathcal{A}} \rho_\theta\big(a|a'\big)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = a'|\theta\big) u\big(a, E_{\sigma^T}[X_\theta], \theta\big)\Bigg]$$
$$- \frac{1}{N_\theta}\Bigg[\sum_{\theta\in\Theta}\sum_{a'\in\mathcal{A}} E_{\sigma^T}\big[X_\theta^{a'}\big] u\big(a', E_{\sigma^T}[X_\theta], \theta\big)\Bigg]$$
$$= \frac{1}{N_\theta}\sum_{\theta\in\Theta}\sum_{a'\in\mathcal{A}}\Bigg[\sum_{a\in\mathcal{A}} \rho_\theta\big(a|a'\big)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = a'|\theta\big) u\big(a, E_{\sigma^T}[X_\theta], \theta\big)$$
$$- E_{\sigma^T}\big[X_\theta^{a'}\big] u\big(a', E_{\sigma^T}[X_\theta], \theta\big)\Bigg].$$

As a result, the improvement in (S.4) can be expressed as

$$\Delta^T = \widetilde{\Delta}^T(\varepsilon) + J(\varepsilon)\sum_{\theta\in\Theta}\sum_{a'\in\mathcal{A}}\Bigg[\sum_{a\in\mathcal{A}} \rho_\theta\big(a|a'\big)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = a'|\theta\big) u\big(a, E_{\sigma^T}[X_\theta], \theta\big)$$
$$- E_{\sigma^T}\big[X_\theta^{a'}\big] u\big(a', E_{\sigma^T}[X_\theta], \theta\big)\Bigg].$$

In particular, for the simple strategy $\widetilde{\sigma}$,

$$\begin{aligned}\Delta^T &= \widetilde{\Delta}^T(\varepsilon) + J(\varepsilon)\sum_{\theta\in\Theta}\big[\rho_\theta\big(a^*|\tilde{a}\big)\mathbf{P}_{\sigma^T}(\widetilde{\xi}=\tilde{a}|\theta)u\big(a^*, E_{\sigma^T}[X_\theta],\theta\big)\\
&\quad + \big[1-\rho_\theta\big(a^*|\tilde{a}\big)\big]\mathbf{P}_{\sigma^T}(\widetilde{\xi}=\tilde{a}|\theta)u\big(\tilde{a}, E_{\sigma^T}[X_\theta],\theta\big) - E_{\sigma^T}\big[X_\theta^{\tilde{a}}\big]u\big(\tilde{a}, E_{\sigma^T}[X_\theta],\theta\big)\big]\\
&= \widetilde{\Delta}^T(\varepsilon) + J(\varepsilon)\sum_{\theta\in\Theta}\big[\rho_\theta\big(a^*|\tilde{a}\big)\mathbf{P}_{\sigma^T}(\widetilde{\xi}=\tilde{a}|\theta)\big[u\big(a^*, E_{\sigma^T}[X_\theta],\theta\big) - u\big(\tilde{a}, E_{\sigma^T}[X_\theta],\theta\big)\big]\\
&\quad + \big[\mathbf{P}_{\sigma^T}(\widetilde{\xi}=\tilde{a}|\theta) - E_{\sigma^T}\big[X_\theta^{\tilde{a}}\big]\big]u\big(\tilde{a}, E_{\sigma^T}[X_\theta],\theta\big)\big].\end{aligned}$$

Let

$$\widetilde{\widetilde{\Delta}}^T \equiv J(\varepsilon)\sum_{\theta\in\Theta}\big[\mathbf{P}_{\sigma^T}(\widetilde{\xi}=\tilde{a}|\theta) - E_{\sigma^T}\big[X_\theta^{\tilde{a}}\big]\big]u\big(\tilde{a}, E_{\sigma^T}[X_\theta],\theta\big)].$$

Then

$$\begin{aligned}\Delta^T &= \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T\\
&\quad + J(\varepsilon)\sum_{\theta\in\Theta}\big[\rho_\theta\big(a^*|\tilde{a}\big)\mathbf{P}_{\sigma^T}(\widetilde{\xi}=\tilde{a}|\theta)\big[u\big(a^*, E_{\sigma^T}[X_\theta],\theta\big) - u\big(\tilde{a}, E_{\sigma^T}[X_\theta],\theta\big)\big]\big]\\
&= \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T\\
&\quad + J(\varepsilon)\Bigg[\sum_{\theta\in\Theta,\theta\neq\theta^*}\big[\rho_\theta\big(a^*|\tilde{a}\big)\mathbf{P}_{\sigma^T}(\widetilde{\xi}=\tilde{a}|\theta)\big[u\big(a^*, E_{\sigma^T}[X_\theta],\theta\big) - u\big(\tilde{a}, E_{\sigma^T}[X_\theta],\theta\big)\big]\big]\\
&\quad + \rho_{\theta^*}\big(a^*|\tilde{a}\big)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi}=\tilde{a}|\theta^*\big)\big[u\big(a^*, E_{\sigma^T}[X_{\theta^*}],\theta^*\big) - u\big(\tilde{a}, E_{\sigma^T}[X_{\theta^*}],\theta^*\big)\big]\Bigg].\end{aligned}$$

Now let

$$-\bar{u} \equiv \min_{a\in\mathcal{A},a'\in\mathcal{A},\theta\in\Theta,x_\theta\in[0,1]^{N_a}}\big[u(a,x_\theta,\theta) - u\big(a',x_\theta,\theta\big)\big].$$

This minimum exists since there is a finite number of states and actions, and the utility functions are continuous in X. Then

$$\big[u\big(a^*, E_{\sigma^T}[X_{\theta^*}],\theta^*\big) - u\big(\tilde{a}, E_{\sigma^T}[X_{\theta^*}],\theta^*\big)\big] \geq -\bar{u}.$$

Then

$$\begin{aligned}\Delta^T \geq \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T + J(\varepsilon)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi}=\tilde{a}|\theta^*\big)\big[u\big(a^*, E_{\sigma^T}[X_{\theta^*}],\theta^*\big) - u\big(\tilde{a}, E_{\sigma^T}[X_{\theta^*}],\theta^*\big)\big]\\
\times\left[-\frac{\bar{u}\sum_{\theta\in\Theta,\theta\neq\theta^*}\big[\rho_\theta\big(a^*|\tilde{a}\big)\mathbf{P}_{\sigma^T}(\widetilde{\xi}=\tilde{a}|\theta)\big]}{\mathbf{P}_{\sigma^T}\big(\widetilde{\xi}=\tilde{a}|\theta^*\big)\big[u\big(a^*, E_{\sigma^T}[X_{\theta^*}],\theta^*\big) - u\big(\tilde{a}, E_{\sigma^T}[X_{\theta^*}],\theta^*\big)\big]} + \rho_{\theta^*}\big(a^*|\tilde{a}\big)\right]\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T + J(\varepsilon)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = \tilde{a}|\theta^*\big)\big[u\big(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*\big) - u\big(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*\big)\big] \\
&\quad \times \Bigg[\rho_{\theta^*}\big(a^*|\tilde{a}\big) - k^T \sum_{\theta\in\Theta, \theta\neq\theta^*} \big[\rho_\theta\big(a^*|\tilde{a}\big)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = \tilde{a}|\theta\big)\big]\Bigg] \\
&\geq \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T + J(\varepsilon)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = \tilde{a}|\theta^*\big)\big[u\big(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*\big) - u\big(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*\big)\big] \\
&\quad \times \Bigg[\rho_{\theta^*}\big(a^*|\tilde{a}\big) - k^T \sum_{\theta\in\Theta, \theta\neq\theta^*} \rho_\theta\big(a^*|\tilde{a}\big)\Bigg] \\
&= \Delta^T_* \equiv \widetilde{\Delta}^T(\varepsilon) + \widetilde{\widetilde{\Delta}}^T + J(\varepsilon)\mathbf{P}_{\sigma^T}\big(\widetilde{\xi} = \tilde{a}|\theta^*\big)\big[u\big(a^*, E_{\sigma^T}[X_{\theta^*}], \theta^*\big) - u\big(\tilde{a}, E_{\sigma^T}[X_{\theta^*}], \theta^*\big)\big] \\
&\quad \times \big[\big[1 - G_{\theta^*}\big(k^T\big)\big] - k^T\big[1 - \widetilde{G}_{\theta^*}\big(k^T\big)\big]\big].
\end{aligned}
$$

Note that $\lim_{T\to\infty} \widetilde{\widetilde{\Delta}}^T = 0$. Let $\widetilde{\Delta}(\varepsilon) \equiv \lim_{T\to\infty} \widetilde{\Delta}^T(\varepsilon)$. Finally, note that, as in the proof in the paper, $\lim_{T\to\infty} k^T = \bar{k}$. Then

$$
\begin{aligned}
\lim_{T\to\infty} \Delta^T_* = \widetilde{\Delta}(\varepsilon) + \frac{1-(N_a-1)\varepsilon}{N_\theta}\big[x^{\tilde{a}}_{\theta^*}\big[u\big(a^*, x_{\theta^*}, \theta^*\big) - u\big(\tilde{a}, x_{\theta^*}, \theta^*\big)\big]\big] \\
\times \big[\big[1 - G_{\theta^*}(\bar{k})\big] - \bar{k}\big[1 - \widetilde{G}_{\theta^*}(\bar{k})\big]\big].
\end{aligned}
$$

S.7.6 *Strategic learning*

Lemmas 4′ and 5′ are the main building blocks to show how Proposition 2 also applies to a context with many states and many actions. I present this formally.

Proposition 2′ (Strategic learning). *Assume signals are of unbounded strength. Then there is strategic learning.*

The proof of Proposition 3′ requires modifying Proposition 3 and Lemma 11 in the paper. With these results in hand, the proof of Proposition 2′ is analogous to the proof of Proposition 2 in the main text. Lemma 11 extends directly to a context with many actions and many states. I present next a version of Proposition 3 in the paper that applies to many states of the world.

Proposition 3′. *For all $l \in (\underline{l}, \bar{l})$, $G_\theta(l)$ satisfies*

$$
l > \frac{G_\theta(l)}{\widetilde{G}_\theta(l)} \quad \text{and} \quad l < \frac{1-G_1(l)}{1-G_0(l)}. \tag{S.5}
$$

Moreover, if $k' \geq k$, then

$$
\big[1 - G_1(k)\big] - k\big[1 - G_0(k)\big] \geq \big[1 - G_1\big(k'\big)\big] - k'\big[1 - G_0\big(k'\big)\big]. \tag{S.6}
$$

Proof. The proof follows that from Proposition 11 in Monzón and Rapp (2014), but here the likelihood ratio G_θ indicates how likely state θ is relative to all other states. Note first that

$$\begin{aligned} l_\theta(s)^{-1} &= \sum_{\tilde\theta\neq\theta} l_{\tilde\theta,\theta}(s) = \sum_{\tilde\theta\neq\theta} \frac{dF_{\tilde\theta}}{dF_\theta}(s) \\ dF_\theta(s) l_\theta(s)^{-1} &= \sum_{\tilde\theta\neq\theta} dF_{\tilde\theta}(s) \\ dF_\theta(s) &= l_\theta(s)\sum_{\tilde\theta\neq\theta} dF_{\tilde\theta}(s). \end{aligned}$$

Recall that $\widetilde{G}_\theta(L) \equiv \sum_{\tilde\theta\neq\theta} \Pr(l_\theta(s)\leq L \mid \tilde\theta)$:

$$\begin{aligned} G_\theta(L) &= \int_{\{S\in\mathcal{S}:l_\theta(s)\leq L\}} dF_\theta = \int_{\{S\in\mathcal{S}:l_\theta(s)\leq L\}} l_\theta(s)\sum_{\tilde\theta\neq\theta} dF_{\tilde\theta}(s) \\ &< \int_{\{S\in\mathcal{S}:l_\theta(s)\leq L\}} L\sum_{\tilde\theta\neq\theta} dF_{\tilde\theta}(s) = L\sum_{\tilde\theta\neq\theta}\int_{\{S\in\mathcal{S}:l_\theta(s)\leq L\}} dF_{\tilde\theta}(s) \\ &= L\widetilde{G}_\theta(L). \end{aligned}$$

Similarly,

$$\begin{aligned} 1-G_\theta(L) &= \int_{\{S\in\mathcal{S}:l_\theta(s)> L\}} dF_\theta = \int_{\{S\in\mathcal{S}:l_\theta(s)> L\}} l_\theta(s)\sum_{\tilde\theta\neq\theta} dF_{\tilde\theta}(s) \\ &> \int_{\{S\in\mathcal{S}:l_\theta(s)> L\}} L\sum_{\tilde\theta\neq\theta} dF_{\tilde\theta}(s) = L\sum_{\tilde\theta\neq\theta}\int_{\{S\in\mathcal{S}:l_\theta(s)> L\}} dF_{\tilde\theta}(s) \\ &= L\big[1-\widetilde{G}_\theta(L)\big]. \end{aligned}$$

This shows that (S.5) holds. I move next to the second part. Take $k' > k$:

$$\begin{aligned} \big[1-G_\theta(k)\big]-\big[1-G_\theta(k')\big] = G_\theta(k')-G_\theta(k) &= \int_{S\in\mathcal{S}:k\leq l_\theta(S)\leq k'} dF_\theta \\ &= \int_{S\in\mathcal{S}:k\leq l_\theta(S)\leq k'} l_\theta(S)\sum_{\tilde\theta\neq\theta} dF_{\tilde\theta} \\ &\geq k\int_{S\in\mathcal{S}:k\leq l_\theta(S)\leq k'} \sum_{\tilde\theta\neq\theta} dF_{\tilde\theta} = k\big[\widetilde{G}_\theta(k')-\widetilde{G}_\theta(k)\big] \\ &= k\big[1-\widetilde{G}_\theta(k)\big]-k\big[1-\widetilde{G}_\theta(k')\big] \\ &\geq k\big[1-\widetilde{G}_\theta(k)\big]-k'\big[1-\widetilde{G}_\theta(k')\big]. \end{aligned}$$

Then

$$
\begin{gathered}
\left[1-G_\theta(k)\right]-\left[1-G_\theta\left(k'\right)\right] \geq k\left[1-\widetilde{G}_\theta(k)\right]-k'\left[1-\widetilde{G}_\theta\left(k'\right)\right] \\
\left[1-G_\theta(k)\right]-k\left[1-\widetilde{G}_\theta(k)\right] \geq\left[1-G_\theta\left(k'\right)\right]-k'\left[1-\widetilde{G}_\theta\left(k'\right)\right].
\end{gathered}
$$

This shows that (S.6) holds. □

References

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