Supplement to "Stochastic games with hidden states"

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S.1. Proof of Lemma B6

Pick an arbitrary belief μ. If

$$\frac{(1-\delta^{2^{|\Omega|}})2\overline{g}}{\delta^{2^{|\Omega|}}\overline{\pi}^{4^{|\Omega|}}} \geq \overline{g},$$

then the result obviously holds because we have $|\lambda\cdot v^{\omega}(\delta, s^{\omega}) - \lambda\cdot v^{\mu}(\delta, \tilde{s}^{\mu})| \leq \overline{g}$. So in what follows, we assume that

$$\frac{(1-\delta^{2^{|\Omega|}})2\overline{g}}{\delta^{2^{|\Omega|}}\overline{\pi}^{4^{|\Omega|}}} < \overline{g}.$$

Suppose that the initial prior is μ and players play the strategy profile $\tilde{s}^{\mu}$. Let $\Pr(h^t|\mu, \tilde{s}^{\mu})$ be the probability of h^t given the initial prior μ and the strategy profile $\tilde{s}^{\mu}$, and let $\mu^{t+1}(h^t|\mu, \tilde{s}^{\mu})$ denote the posterior belief in period $t+1$ given this history h^t. Let H^{*t} be the set of histories h^t such that $t+1$ is the first period at which the support of the posterior belief μ^{t+1} is in the set Ω^*. Intuitively, H^{*t} is the set of histories h^t such that players will switch their play to $s^{\mu^{t+1}}$ from period $t+1$ on, according to $\tilde{s}^{\mu}$.

Note that the payoff $v^{\mu}(\delta, \tilde{s}^{\mu})$ by the strategy profile $\tilde{s}^{\mu}$ can be represented as the sum of the two terms: The expected payoffs before the switch to s^{μ^t} occurs and the payoffs after the switch. That is, we have

$$\lambda\cdot v^{\mu}(\delta, \tilde{s}^{\mu}) = \sum_{t=1}^{\infty}\left(1-\sum_{\tilde{t}=0}^{t-1}\sum_{h^{\tilde{t}}\in H^{*\tilde{t}}}\Pr(h^{\tilde{t}}|\mu, \tilde{s}^{\mu})\right)(1-\delta)\delta^{t-1}E\left[\lambda\cdot g^{\omega^t}(a^t)|\mu, \tilde{s}^{\mu}\right]$$
$$+\sum_{t=0}^{\infty}\sum_{h^t\in H^{*t}}\Pr(h^t|\mu, \tilde{s}^{\mu})\delta^t\lambda\cdot v^{\mu^{t+1}(h^t|\mu,\tilde{s}^{\mu})}\left(\delta, s^{\mu^{t+1}(h^t|\mu,\tilde{s}^{\mu})}\right),$$

where the expectation operator is taken conditional on that the switch has not happened yet. Note that the term $1-\sum_{\tilde{t}=0}^{t-1}\sum_{h^{\tilde{t}}\in H^{*\tilde{t}}}\Pr(h^{\tilde{t}}|\mu, \tilde{s}^{\mu})$ is the probability that players still randomize all actions in period t because the switch has not happened by then. To simplify the notation, let ρ^t denote this probability. From Lemma B5, we know

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that

$$\lambda \cdot v^{\mu^{t+1}(h^t|\mu,\tilde{s}^\mu)}\big(\delta, s^{\mu^{t+1}(h^t|\mu,\tilde{s}^\mu)}\big) \geq v^*$$

for each $h^t \in H^{*t}$, where

$$v^* = \lambda \cdot v^\omega\big(\delta, s^\omega\big) - \frac{\big(1-\delta^{2^{|\Omega|}}\big)2\overline{g}}{\delta^{2^{|\Omega|}}\overline{\pi}^{4^{|\Omega|}}}.$$

Applying this and $\lambda \cdot g^{\omega^t}(a^t) \geq -2\overline{g}$ to the above equation, we obtain

$$\lambda \cdot v^\mu\big(\delta, \tilde{s}^\mu\big) \geq \sum_{t=1}^{\infty} \rho^t (1-\delta)\delta^{t-1}(-2\overline{g}) + \sum_{t=0}^{\infty} \sum_{h^t \in H^{*t}} \Pr\big(h^t|\mu, \tilde{s}^\mu\big)\delta^t v^*.$$

Using $\sum_{t=0}^{\infty}\sum_{h^t\in H^{*t}} \Pr(h^t|\mu,\tilde{s}^\mu)\delta^t = \sum_{t=1}^{\infty}(1-\delta)\delta^{t-1}\sum_{\tilde{t}=0}^{t-1}\sum_{h^{\tilde{t}}\in H^{*\tilde{t}}}\Pr(h^{\tilde{t}}|\mu,\tilde{s}^\mu) = \sum_{t=1}^{\infty}(1-\delta)\delta^{t-1}(1-\rho^t)$, we obtain

$$\lambda \cdot v^\mu\big(\delta, \tilde{s}^\mu\big) \geq (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\big\{\rho^t(-2\overline{g}) + \big(1-\rho^t\big)v^*\big\}. \tag{S1}$$

According to Lemma B4, the probability that the support reaches Ω^* within $4^{|\Omega|}$ periods is at least π^*. This implies that the probability that players still randomize all actions in period $4^{|\Omega|}+1$ is at most $1-\pi^*$. Similarly, for each natural number n, the probability that players still randomize all actions in period $n4^{|\Omega|}+1$ is at most $(1-\pi^*)^n$, that is, $\rho^{n4^{|\Omega|}+1} \leq (1-\pi^*)^n$. Then since ρ^t is weakly decreasing in t, we obtain

$$\rho^{n4^{|\Omega|}+k} \leq \big(1-\pi^*\big)^n$$

for each $n = 0, 1, \ldots$ and $k \in \{1, \ldots, 4^{|\Omega|}\}$. This inequality, together with $-2\overline{g} \leq v^*$, implies that

$$\rho^{n4^{|\Omega|}+k}(-2\overline{g}) + \big(1-\rho^{n4^{|\Omega|}+k}\big)v^* \geq \big(1-\pi^*\big)^n(-2\overline{g}) + \big\{1-\big(1-\pi^*\big)^n\big\}v^*$$

for each $n = 0, 1, \ldots$ and $k \in \{1, \ldots, 4^{|\Omega|}\}$. Plugging this inequality into (S1), we obtain

$$\lambda \cdot v^\mu\big(\delta, \tilde{s}^\mu\big) \geq (1-\delta)\sum_{n=1}^{\infty}\sum_{k=1}^{4^{|\Omega|}}\delta^{(n-1)4^{|\Omega|}+k-1}\begin{bmatrix} -\big(1-\pi^*\big)^{n-1}2\overline{g} \\ +\big\{1-\big(1-\pi^*\big)^{n-1}\big\}v^* \end{bmatrix}.$$

Since

$$\sum_{k=1}^{4^{|\Omega|}}\delta^{(n-1)4^{|\Omega|}+k-1} = \frac{\delta^{(n-1)4^{|\Omega|}}\big(1-\delta^{4^{|\Omega|}}\big)}{1-\delta},$$

we have

$$\lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \geq (1-\delta^{4|\Omega|}) \sum_{n=1}^{\infty} \delta^{(n-1)4|\Omega|} \begin{bmatrix} -(1-\pi^*)^{n-1} 2\overline{g} \\ +\{1-(1-\pi^*)^{n-1}\} v^* \end{bmatrix}$$

$$= -(1-\delta^{4|\Omega|}) \sum_{n=1}^{\infty} \{(1-\pi^*)\delta^{4|\Omega|}\}^{n-1} 2\overline{g}$$

$$+ (1-\delta^{4|\Omega|}) \sum_{n=1}^{\infty} [(\delta^{4|\Omega|})^{n-1} - \{(1-\pi^*)\delta^{4|\Omega|}\}^{n-1}] v^*.$$

Plugging in $\sum_{n=1}^{\infty}\{(1-\pi^*)\delta^{4|\Omega|}\}^{n-1} = 1/\{1-(1-\pi^*)\delta^{4|\Omega|}\}$ and $\sum_{n=1}^{\infty}(\delta^{4|\Omega|})^{n-1} = 1/(1-\delta^{4|\Omega|})$ gives

$$\lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \geq -\frac{(1-\delta^{4|\Omega|})2\overline{g}}{1-(1-\pi^*)\delta^{4|\Omega|}} + \frac{\delta^{4|\Omega|}\pi^*}{1-(1-\pi^*)\delta^{4|\Omega|}} v^*.$$

Subtracting both sides from $\lambda \cdot v^{\omega}(\delta, s^{\omega})$, we have

$$\begin{aligned} &\lambda \cdot v^{\omega}(\delta, s^{\omega}) - \lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \\ &\leq \frac{(1-\delta^{4|\Omega|})2\overline{g}}{1-(1-\pi^*)\delta^{4|\Omega|}} + \frac{\delta^{4|\Omega|}\pi^*(1-\delta^{2|\Omega|})2\overline{g}}{\{1-(1-\pi^*)\delta^{4|\Omega|}\}\delta^{2|\Omega|}\overline{\pi}^{4|\Omega|}} - \frac{(1-\delta^{4|\Omega|})\lambda \cdot v^{\omega}(\delta, s^{\omega})}{1-(1-\pi^*)\delta^{4|\Omega|}}. \end{aligned}$$

Since $\lambda \cdot v^{\omega}(\delta, s^{\omega}) \geq -\overline{g}$, then

$$\begin{aligned} &\lambda \cdot v^{\omega}(\delta, s^{\omega}) - \lambda \cdot v^{\mu}(\delta, \tilde{s}^{\mu}) \\ &\leq \frac{(1-\delta^{4|\Omega|})2\overline{g}}{1-(1-\pi^*)\delta^{4|\Omega|}} + \frac{\delta^{4|\Omega|}\pi^*(1-\delta^{2|\Omega|})2\overline{g}}{\{1-(1-\pi^*)\delta^{4|\Omega|}\}\delta^{2|\Omega|}\overline{\pi}^{4|\Omega|}} + \frac{(1-\delta^{4|\Omega|})\overline{g}}{1-(1-\pi^*)\delta^{4|\Omega|}} \\ &\leq \frac{(1-\delta^{4|\Omega|})3\overline{g}}{1-(1-\pi^*)} + \frac{\pi^*(1-\delta^{2|\Omega|})2\overline{g}}{\{1-(1-\pi^*)\}\delta^{2|\Omega|}\overline{\pi}^{4|\Omega|}} \\ &= \frac{(1-\delta^{4|\Omega|})3\overline{g}}{\pi^*} + \frac{(1-\delta^{2|\Omega|})2\overline{g}}{\delta^{2|\Omega|}\overline{\pi}^{4|\Omega|}}. \end{aligned}$$

Hence, the result follows.

S.1.1 *Proof of Lemma B11*

Pick a belief μ whose support is robustly accessible. Suppose that the initial prior is μ^{**}, the opponents play $\tilde{s}^{\mu}_{-i}$, and player i plays a best reply. Let ρ^t denote the probability that players $-i$ still randomize actions in period t. Then as in the proof of Lemma B6, we have

$$v_i^{\mu^{**}}(\tilde{s}^{\mu}_{-i}) \leq \sum_{t=1}^{\infty} \delta^{t-1}\{\rho^t \overline{g} + (1-\rho^t)K_i^{\mu}\},$$

because the stage-game payoff before the switch to s^{μ}_{-i} is bounded from above by $\overline{g}$ and the continuation payoff after the switch is bounded from above by $K^{\mu}_{i} = \max_{\tilde{\mu}\in\Delta^{\mu}} v^{\tilde{\mu}}_{i}(s^{\mu}_{-i})$.

As in the proof of Lemma B6, we have

$$\rho^{n4^{|\Omega|}+k} \leq (1-\pi^*)^n$$

for each $n = 0, 1, \ldots$ and $k \in \{1, \ldots, 4^{|\Omega|}\}$. This inequality, together with $\overline{g} \geq K^{\mu}_{i}$, implies that

$$\rho^{n4^{|\Omega|}+k}\overline{g} + (1-\rho^{n4^{|\Omega|}+k})v^{*}_{i} \leq (1-\pi^*)^n\overline{g} + \{1-(1-\pi^*)^n\}K^{\mu}_{i}$$

for each $n = 0, 1, \ldots$ and $k \in \{1, \ldots, 4^{|\Omega|}\}$. Plugging this inequality into the first inequality, we obtain

$$v^{\mu^{**}}_{i}(\tilde{s}^{\mu}_{-i}) \leq (1-\delta)\sum_{n=1}^{\infty}\sum_{k=1}^{4^{|\Omega|}}\delta^{(n-1)4^{|\Omega|}+k-1}\begin{bmatrix}(1-\pi^*)^{n-1}\overline{g} \\ +\{1-(1-\pi^*)^{n-1}\}K^{\mu}_{i}\end{bmatrix}.$$

Then as in the proof of Lemma B6, the standard algebra shows

$$v^{\mu^{**}}_{i}(\tilde{s}^{\mu}_{-i}) \leq \frac{(1-\delta^{4^{|\Omega|}})\overline{g}}{1-(1-\pi^*)\delta^{4^{|\Omega|}}} + \frac{\delta^{4^{|\Omega|}}\pi^* K^{\mu}_{i}}{1-(1-\pi^*)\delta^{4^{|\Omega|}}}.$$

Since

$$\frac{\delta^{4^{|\Omega|}}\pi^*}{1-(1-\pi^*)\delta^{4^{|\Omega|}}} = 1 - \frac{1-\delta^{4^{|\Omega|}}}{1-(1-\pi^*)\delta^{4^{|\Omega|}}},$$

we have

$$v^{\mu^{**}}_{i}(\tilde{s}^{\mu}_{-i}) \leq K^{\mu}_{i} + \frac{(1-\delta^{4^{|\Omega|}})(\overline{g}-K^{\mu}_{i})}{1-(1-\pi^*)\delta^{4^{|\Omega|}}}.$$

Since $1-(1-\pi^*)\delta^{4^{|\Omega|}} > 1-(1-\pi^*) = \pi^*$ and $K^{\mu}_{i} \geq -\overline{g}$, the result follows.

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