Supplementary Material

Supplement to "Macro-financial volatility under dispersed information"

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S1. Analysis in Section 6

S1.1 *Equilibrium system*

As in Section 5, we derive the equilibrium system in four steps.

Step 1. Derive the Wold representation for the signal system under Assumption 3. Given the AR(1) processes for a_t and u_t, the signal representation follows

$$X_{it} = H(L)\eta_{it} \equiv \begin{bmatrix} \dfrac{1}{1-\rho_a L} & 1 & 0 \\ \dfrac{\pi_1(L)}{\bigl(1-\pi_2(L)\bigr)(1-\rho_a L)} & 0 & \dfrac{1}{\bigl(1-\pi_2(L)\bigr)(1-\rho_u L)} \end{bmatrix} \begin{bmatrix} \epsilon_{at} \\ \epsilon_{it} \\ \epsilon_{ut} \end{bmatrix},$$

and so the spectral density for the signal is

$$\begin{aligned} S_x(z) &= H(z)\Sigma_{\boldsymbol{\eta}} H(z^{-1})^{\intercal} \\ &= \begin{bmatrix} \dfrac{\sigma_a^2}{(1-\rho_a z)(1-\rho_a z^{-1})} + \sigma_i^2 & \dfrac{\pi_1(z^{-1})}{1-\pi_2(z^{-1})} \dfrac{1}{(1-\rho_a z)(1-\rho_a z^{-1})} \sigma_a^2 \\ \dfrac{\pi_1(z)}{1-\pi_2(z)} \dfrac{1}{(1-\rho_a z)(1-\rho_a z^{-1})} \sigma_a^2 & \dfrac{\dfrac{\pi_1(z)\pi_1(z^{-1})}{(1-\rho_a z)(1-\rho_a z^{-1})}\sigma_a^2 + \dfrac{\sigma_u^2}{(1-\rho_u z)(1-\rho_u z^{-1})}}{\bigl(1-\pi_2(z)\bigr)\bigl(1-\pi_2(z^{-1})\bigr)} \end{bmatrix}. \end{aligned}$$

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Using the method presented in Appendix S3, we can first factorize the spectral density in a lower triangular form

$$\tilde{\Gamma}(z) = \begin{bmatrix} \sigma_w \dfrac{z-\lambda_w}{1-\rho_a z} & 0 \\ \dfrac{\sigma_a^2}{\sigma_w} \dfrac{\pi_1(z) z}{\big(1-\pi_2(z)\big)(1-\lambda_w z)(1-\rho_a z)} & \dfrac{1}{\sigma_w} \dfrac{\tilde{\pi}_1(z)}{1-\pi_2(z)} \dfrac{1-\rho_a z}{1-\lambda_w z} \end{bmatrix},$$

where the constants $\lambda_w \in (0,1)$ and σ_w are determined by the univariate spectral factorization of the first signal a_{it} in the frequency domain,

$$\sigma_w^2 \frac{(1-\lambda_w z)\big(1-\lambda_w z^{-1}\big)}{(1-\rho_a z)\big(1-\rho_a z^{-1}\big)} = \frac{\sigma_a^2}{(1-\rho_a z)\big(1-\rho_a z^{-1}\big)} + \sigma_i^2.$$

It follows that

$$\sigma_w^2 (1-\lambda_w z)\big(1-\lambda_w z^{-1}\big) = \sigma_a^2 + \sigma_i^2 (1-\rho_a z)\big(1-\rho_a z^{-1}\big).$$

Matching coefficients on the two sides of the equality yields

$$\lambda_w = \frac{1}{2\rho_a}\Big[\big(1+\tau+\rho_a^2\big) - \sqrt{\tau^2 + 2\tau + 2\tau\rho_a^2 + 1 - 2\rho_a^2 + \rho_a^4}\Big],$$

and $\sigma_w^2 = \frac{\rho_a \sigma_i^2}{\lambda_w}$. Here $\tau \equiv \sigma_a^2/\sigma_i^2 \in (0,\infty)$ denotes the relative volatility of the aggregate shock to the idiosyncratic shock. It is easy to verify that $0 < \lambda_w < \rho_a < 1$ and $\lim_{\sigma_i \to \infty} \lambda_w = \rho_a$.

Define the function $\tilde{\pi}_1(z)$ by

$$\tilde{\pi}_1(z)\tilde{\pi}_1\big(z^{-1}\big) = \frac{\pi_1(z)\pi_1\big(z^{-1}\big)\sigma_a^2\sigma_i^2}{(1-\rho_a z)\big(1-\rho_a z^{-1}\big)} + \frac{(1-\lambda_w z)\big(1-\lambda_w z^{-1}\big)\sigma_u^2\sigma_w^2}{(1-\rho_a z)\big(1-\rho_a z^{-1}\big)(1-\rho_u z)\big(1-\rho_u z^{-1}\big)}. \tag{S1.1}$$

A stationary equilibrium requires that the endogenous function $\pi_1 \in \mathbf{H}^2(\mathbb{D})$. It is then clear that the right-hand side of (S1.1) is a well defined spectral density supported by a stationary process. Then by the Paley–Wiener theorem (e.g., Lindquist and Picci 2015, Theorem 4.4.1), there exists a Wold spectral factor $\tilde{\pi}_1(z) \in \mathbf{H}^2(\mathbb{D})$ that satisfies the factorization (S1.1). Using a similar argument, we can show that the function $\frac{\tilde{\pi}_1(z)}{1-\pi_2(z)} \in \mathbf{H}^2(\mathbb{D})$. Hence, the matrix $\tilde{\Gamma}(z)$ is a valid spectral factor in $\mathbf{H}^2(\mathbb{D})$ that satisfies $S_x(z) = \tilde{\Gamma}(z)\tilde{\Gamma}^{\intercal}(z^{-1})$. The determinant of $\tilde{\Gamma}(z)$ is given by

$$\det \tilde{\Gamma}(z) = \frac{\tilde{\pi}_1(z)}{1-\pi_2(z)} \frac{z-\lambda_w}{1-\lambda_w z}.$$

As in Section 5, we restrict our attention to the equilibrium such that $\frac{\tilde{\pi}_1(z)}{1-\pi_2(z)}$ has no roots in the open unit disk. To derive the Wold fundamental representation, we need to remove the root at $z = \lambda_w \in (0,1)$. Using the Blaschke matrix $B(z)$ in Step 2 of Appendix S3, we set

$$\Gamma(z) = \tilde{\Gamma}(z) V^{-1} B(z),$$

where

$$V = \begin{bmatrix} \sqrt{\dfrac{h^2}{1+h^2}} & \sqrt{\dfrac{1}{1+h^2}} \\ \sqrt{\dfrac{1}{1+h^2}} & -\sqrt{\dfrac{h^2}{1+h^2}} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{bmatrix}, \qquad B(z) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1-\lambda_w z}{z-\lambda_w} \end{bmatrix}.$$

Here the constant

$$h \equiv \frac{\pi_1(\lambda_w)\lambda_w \sigma_a^2}{\tilde{\pi}_1(\lambda_w)(1-\rho_a\lambda_w)^2}$$

is endogenous and will be determined in equilibrium. The unitary matrix V is symmetric and satisfies $V = V^\intercal = V^{-1}$, and $\det V = -1$. We then obtain the Wold fundamental matrix

$$\Gamma(z) = \begin{bmatrix} \sigma_w \dfrac{z-\lambda_w}{1-\rho_a z} V_{11} & \sigma_w \dfrac{1-\lambda_w z}{1-\rho_a z} V_{12} \\ \Gamma_\pi^{(1)}(z) & \Gamma_\pi^{(2)}(z) \end{bmatrix},$$

where we define

$$\Gamma_\pi^{(1)}(z) \equiv \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z) z}{\big(1-\pi_2(z)\big)(1-\lambda_w z)(1-\rho_a z)} V_{11} + \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(z)}{1-\pi_2(z)} \frac{1-\rho_a z}{1-\lambda_w z} V_{12}$$

$$\Gamma_\pi^{(2)}(z) \equiv \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z) z}{\big(1-\pi_2(z)\big)(z-\lambda_w)(1-\rho_a z)} V_{12} + \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(z)}{1-\pi_2(z)} \frac{1-\rho_a z}{z-\lambda_w} V_{22}.$$

We compute that

$$\Gamma^{-1}(z) = \begin{bmatrix} G_1(z) \dfrac{\sigma_a^2}{\sigma_w} \dfrac{\pi_1(z)}{\tilde{\pi}_1(z)} + G_2(z) \dfrac{1}{\sigma_w} & -\dfrac{1-\pi_2(z)}{\tilde{\pi}_1(z)} \sigma_w G_3(z) \\ -\left[G_4(z) \dfrac{\sigma_a^2}{\sigma_w} \dfrac{\pi_1(z)}{\tilde{\pi}_1(z)} + G_5(z) \dfrac{1}{\sigma_w} \right] & \dfrac{1-\pi_2(z)}{\tilde{\pi}_1(z)} \sigma_w G_6(z) \end{bmatrix},$$

where we define

$$G_1(z) = -V_{12} \frac{z}{(z-\lambda_w)(1-\rho_a z)}, \qquad G_2(z) = -V_{22} \frac{1-\rho_a z}{z-\lambda_w}$$

$$G_3(z) = -V_{12} \frac{1-\lambda_w z}{1-\rho_a z}, \qquad G_4(z) = -V_{11} \frac{z}{(1-\lambda_w z)(1-\rho_a z)}$$

$$G_5(z) = -V_{12} \frac{1-\rho_a z}{1-\lambda_w z}, \qquad G_6(z) = -V_{11} \frac{z-\lambda_w}{1-\rho_a z}.$$

Note that all $G_1(z), \dots, G_6(z)$ are independent of the endogenous price signal except for the constant in V. We also define the functions that will be repeatedly used later:

$$\Gamma_I^{(1)}(z) = G_1(z)\frac{\sigma_a^2}{\sigma_w}\frac{\pi_1(z)}{\tilde{\pi}_1(z)} + G_2(z)\frac{1}{\sigma_w}, \qquad \Gamma_I^{(3)}(z) \equiv \sigma_w G_3(z)\frac{\pi_1(z)}{\tilde{\pi}_1(z)}$$
$$\Gamma_I^{(2)}(z) = G_4(z)\frac{\sigma_a^2}{\sigma_w}\frac{\pi_1(z)}{\tilde{\pi}_1(z)} + G_5(z)\frac{1}{\sigma_w}, \qquad \Gamma_I^{(4)}(z) \equiv \sigma_w G_6(z)\frac{\pi_1(z)}{\tilde{\pi}_1(z)}.$$

By the Paley–Wiener theorem and the fact that $\tilde{\pi}_1(z)$ is analytic in the open unit disk and Wold fundamental, these functions are analytic in the open unit disk.[1]

Step 2. Solve for the equilibrium quantities. We conjecture that $y_{it} = M_y(L)\eta_{it}$, where $M_y(z) = [M_y^a(z), M_y^i(z), M_y^u(z)]$ and $M_y^a(z)$, $M_y^i(z)$, and $M_y^u(z)$ are all in $\mathbf{H}^2(\mathbb{D})$. Aggregation leads to aggregate output $y_t = M_y(z)I_y\eta_{it}$, where I_y is defined earlier. Using the Wiener–Hopf prediction formula, we derive that

$$\mathbb{E}_{it}[y_t] = \left[\psi_y^{(1)}(L), \psi_y^{(2)}(L)\right]_+ \Gamma^{-1}(L)H(L)\eta_{it}$$

in terms of innovations, where the z-transform of the operator $\psi_y = [\psi_y^{(1)}, \psi_y^{(2)}]$ is given by

$$\psi_y(z) = z^{-1}S_{yx}(z)\left(\Gamma^{-1}\left(z^{-1}\right)\right)^{\intercal}. \tag{S1.2}$$

The annihilation is given by $[\psi_y^{(1)}(z)]_+ = \psi_y^{(1)}(z) - P_y^{(1)}(z)$ and $[\psi_y^{(2)}(z)]_+ = \psi_y^{(2)}(z) - P_y^{(2)}(z)$, where $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ denote the negative powers of z in the Laurent series expansions of $\psi_y^{(1)}(z)$ and $\psi_y^{(2)}(z)$, respectively. There are no explicit formulas for $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ in general.

Using (S1.2), $y_t = M_y(z)I_y\eta_{it}$, and the cross-spectrum

$$\begin{aligned} S_{yx} &= M_y(z)I_y\Sigma_\eta H^{\intercal}\left(z^{-1}\right) \\ &= \left[M_y^a, 0, M_y^u\right]\begin{bmatrix} \sigma_a^2 & & \\ & \sigma_u^2 & \\ & & \sigma_u^2 \end{bmatrix}\begin{bmatrix} \frac{1}{1-\rho_a z^{-1}} & \frac{\pi_1\left(z^{-1}\right)}{\left(1-\pi_2\left(z^{-1}\right)\right)\left(1-\rho_a z^{-1}\right)} \\ 1 & 0 \\ 0 & \frac{1}{\left(1-\pi_2\left(z^{-1}\right)\right)\left(1-\rho_u z^{-1}\right)} \end{bmatrix}, \end{aligned}$$

we can derive

$$\begin{aligned} \psi_y^{(1)}(z) &= M_y^a(z)\sigma_a^2 A_n^{(1)}(z) - M_y^u(z)\sigma_u^2 A_n^{(2)}(z) \\ \psi_y^{(2)}(z) &= -M_y^a(z)\sigma_a^2 A_n^{(3)}(z) + M_y^u(z)\sigma_u^2 A_n^{(4)}(z), \end{aligned}$$

[1] Sayed and Kailath (2001) summarized the property of the Wold fundamental matrix implied by the Paley–Wiener theorem.

where we define

$$A_n^{(1)}(z) = \frac{1}{1-\rho_a z^{-1}}\big[\Gamma_I^{(1)}(z^{-1}) - \Gamma_I^{(3)}(z^{-1})\big], \qquad A_n^{(2)}(z) = \frac{1}{1-\rho_u z^{-1}}\frac{1}{\pi_1(z^{-1})}\Gamma_I^{(3)}(z^{-1})$$

$$A_n^{(3)}(z) = \frac{1}{1-\rho_a z^{-1}}\big[\Gamma_I^{(2)}(z^{-1}) - \Gamma_I^{(4)}(z^{-1})\big], \qquad A_n^{(4)}(z) = \frac{1}{1-\rho_u z^{-1}}\frac{1}{\pi_1(z^{-1})}\Gamma_I^{(4)}(z^{-1}).$$

Substituting the preceding expression for $\mathbb{E}_{it}[y_t]$ into (19) and matching coefficients for η_{it}, we obtain

$$M_y^a(z) = \frac{1}{\xi}\frac{1}{1-\rho_a z} + \frac{1}{1-\rho_a z}\big[G_y^{(1)}(z) - A_y^{(1)}(z) + G_y^{(2)}(z) - A_y^{(2)}(z)\big]\theta \tag{S1.3}$$

$$M_y^i(z) = \frac{1}{\xi} + \big[G_y^{(1)}(z) - A_y^{(1)}(z)\big]\theta \tag{S1.4}$$

$$M_y^u(z) = \frac{1}{1-\rho_u z}\frac{\theta}{\pi_1(z)}\big[G_y^{(2)}(z) - A_y^{(2)}(z)\big], \tag{S1.5}$$

where we define

$$G_y^{(1)}(z) = \psi_y^{(1)}(z)\Gamma_I^{(1)}(z) - \psi_y^{(2)}\Gamma_I^{(2)}(z), \qquad A_y^{(1)}(z) = P_y^{(1)}(z)\Gamma_I^{(1)}(z) - P_y^{(2)}\Gamma_I^{(2)}(z)$$

$$G_y^{(2)}(z) = \psi_y^{(2)}(z)\Gamma_I^{(4)}(z) - \psi_y^{(1)}(z)\Gamma_I^{(3)}(z), \qquad A_y^{(2)}(z) = P_y^{(2)}(z)\Gamma_I^{(4)}(z) - P_y^{(1)}\Gamma_I^{(3)}(z).$$

Here $\Gamma_I^{(1)}(z), \dots, \Gamma_I^{(4)}(z)$ are defined earlier.

Using equations (S1.3) and (S1.5) and the definition of $G_y^{(1)}(z)$ and $G_y^{(2)}(z)$, we can derive that

$$\begin{bmatrix} Q_1(z) & Q_2(z) \\ Q_3(z) & Q_4(z) \end{bmatrix}\begin{bmatrix} M_y^a(z) \\ M_y^u(z) \end{bmatrix} = \begin{bmatrix} \frac{1}{\xi} - A_y^{(1)}(z)\theta - A_y^{(2)}(z)\theta \\ -A_y^{(2)}(z)\theta \end{bmatrix}, \tag{S1.6}$$

where we define

$$Q_1(z) = (1-\rho_a z) - \theta\sigma_a^2 H_a(z), \qquad Q_2(z) = \theta\sigma_u^2 H_u(z)$$

$$Q_3(z) = \theta\sigma_a^2 H_d(z), \qquad Q_4(z) = (1-\rho_u z)\pi_1(z) - \theta\sigma_u^2 H_c(z)$$

and

$$H_a(z) = A_n^{(1)}(z)\big(\Gamma_I^{(1)}(z) - \Gamma_I^{(3)}(z)\big) + A_n^{(3)}(z)\big(\Gamma_I^{(2)}(z) - \Gamma_I^{(4)}(z)\big)$$

$$H_u(z) = A_n^{(2)}(z)\big(\Gamma_I^{(1)}(z) - \Gamma_I^{(3)}(z)\big) + A_n^{(4)}(z)\big(\Gamma_I^{(2)}(z) - \Gamma_I^{(4)}(z)\big)$$

$$H_c(z) = A_n^{(4)}(z)\Gamma_I^{(4)}(z) + A_n^{(2)}\Gamma_I^{(3)}(z)$$

$$H_d(z) = A_n^{(3)}(z)\Gamma_I^{(4)}(z) + A_n^{(1)}\Gamma_I^{(3)}(z).$$

Once $\pi_1(z)$ and $\pi_2(z)$ are known, we can use the system (S1.6) to determine $M_y^a(z)$ and $M_y^u(z)$. Equation (S1.4) then determines $M_y^i(z)$.

As in the proof of Theorem 2, we deduce that $d_t = M_d(L)\eta_{it}$, $n_{it} = M_n(L)\eta_{it}$, and $b_{it} = M_b(L)\eta_{it}$, where

$$\begin{aligned}
M_d(z) &= \left[\frac{1}{\alpha_6}\left(1-\frac{\alpha_7}{\alpha}\right)M_y^a(z) + \frac{\alpha_7}{\alpha\alpha_6}\frac{1}{1-\rho_a z}, 0, \frac{1}{\alpha_6}\left(1-\frac{\alpha_7}{\alpha}\right)M_y^u(z)\right] \\
M_n(z) &= \frac{1}{\alpha}\left[M_y^a(z) - \frac{1}{1-\rho_a z}, M_y^i(z) - 1, M_y^u(z)\right] \\
M_b(z) &= \alpha_4 M_d(z) + \alpha_5 M_n(z).
\end{aligned}$$

Each component of these vectors is in $\mathbf{H}^2(\mathbb{D})$.

Step 3. We proceed to the financial side of the model. We need to compute several conditional expectations for χ_{it} in (31). First, we use the Wiener–Hopf formula to derive

$$\alpha_3\mathbb{E}_{it}\big[s_{it+2}^h\big] = \alpha_3\big[\psi_s(L)\big]_+\Gamma^{-1}(L)X_{it},$$

where the z-transform of the operator ψ_s is given by $\psi_s(z) = z^{-1}S_{sx}(z)(\Gamma^{-1}(z^{-1}))^\intercal$ and

$$\alpha_3\big[\psi_s^{(1)}(z)\big]_+ = \alpha_3\psi_s^{(1)}(z) - P_s^{(1)}(z), \qquad \alpha_3\big[\psi_s^{(2)}(z)\big]_+ = \alpha_3\psi_s^{(2)}(z) - P_s^{(2)}(z).$$

Here $P_s^{(1)}(z)$ and $P_s^{(2)}(z)$ denote the negative powers of z in the Laurent series expansions of $\alpha_3\psi_s^{(1)}(z)$ and $\alpha_3\psi_s^{(2)}(z)$, respectively. It follows that

$$\alpha_3\big[\psi_s^{(1)}(z), \psi_s^{(2)}(z)\big]_+\Gamma^{-1}(z) = \left[G_s^{(1)}(z) - A_s^{(1)}(z), \frac{1-\pi_2(z)}{\pi_1(z)}\big(G_s^{(2)}(z) - A_s^{(2)}(z)\big)\right],$$

where

$$\begin{aligned}
G_s^{(1)}(z) &= \sigma_i^2 z^{-1}\alpha_3 M_s^i(z)\big[\Gamma_I^{(1)}(z)\Gamma_I^{(1)}\big(z^{-1}\big) + \Gamma_I^{(2)}(z)\Gamma_I^{(2)}\big(z^{-1}\big)\big] \\
G_s^{(2)}(z) &= \sigma_i^2 z^{-1}\alpha_3 M_s^i(z)\big[-\Gamma_I^{(3)}(z)\Gamma_I^{(1)}\big(z^{-1}\big) - \Gamma_I^{(4)}(z)\Gamma_I^{(2)}\big(z^{-1}\big)\big]
\end{aligned}$$

and

$$A_s^{(1)}(z) = P_s^{(1)}(z)\Gamma_I^{(1)}(z) - P_s^{(2)}(z)\Gamma_I^{(2)}(z), \qquad A_s^{(2)}(z) = P_s^{(2)}(z)\Gamma_I^{(4)}(z) - P_s^{(1)}(z)\Gamma_I^{(3)}(z).$$

It is easy to verify that Lemma 3 continues to hold, which implies

$$\begin{aligned}
G_s^{(1)}(z) &= \sigma_i^2\frac{1-\lambda_s}{z(1-\lambda_s z)}\pi_1(z)\big[\Gamma_I^{(1)}(z)\Gamma_I^{(1)}\big(z^{-1}\big) + \Gamma_I^{(2)}(z)\Gamma_I^{(2)}\big(z^{-1}\big)\big] \\
G_s^{(2)}(z) &= \sigma_i^2\frac{1-\lambda_s}{z(1-\lambda_s z)}\pi_1(z)\big[-\Gamma_I^{(3)}(z)\Gamma_I^{(1)}\big(z^{-1}\big) - \Gamma_I^{(4)}(z)\Gamma_I^{(2)}\big(z^{-1}\big)\big].
\end{aligned}$$

Second, the Wiener–Hopf formula gives

$$\mathbb{E}_{it}[q_{t+1}] = \big[\psi_q(L)\big]_+\Gamma^{-1}(L)X_{it},$$

where the z-transform of the operator ψ_q is given by

$$\psi_q(z) = \frac{1}{z}[0,1]S_x(z)\big(\Gamma^{-1}\big(z^{-1}\big)\big)^\intercal = \frac{1}{z}[0,1]\Gamma(z) = z^{-1}\big[\Gamma_\pi^{(1)}(z), \Gamma_\pi^{(2)}(z)\big],$$

where $\Gamma_{\pi}^{(1)}(z)$ and $\Gamma_{\pi}^{(2)}(z)$ are defined earlier. Since $z=0$ is the only inside pole of $\psi_q(z)$, it follows from the lemma in Appendix A of Hansen and Sargent (1980) that

$$\left[\psi_q(L)\right]_+\Gamma^{-1}(z) = z^{-1}[0,1] - P_q(z)\Gamma^{-1}(z),$$

where

$$P_q(z) = z^{-1}\left[\frac{1}{\sigma_w}\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}V_{12}, \frac{1}{\sigma_w}\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}\left(-\frac{1}{\lambda_w}\right)V_{22}\right].$$

Thus,

$$\begin{aligned}\mathbb{E}_{it}[q_{t+1}] = &-z^{-1}\frac{1}{\sigma_w}\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}\left[V_{12}\Gamma_I^{(1)}(z) + \frac{1}{\lambda_w}V_{22}\Gamma_I^{(2)}(z)\right]a_{it}\\ &+ z^{-1}\frac{1}{\sigma_w}\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}\left[V_{12}\Gamma_I^{(3)}(z) + \frac{1}{\lambda_w}V_{22}\Gamma_I^{(4)}(z)\right]\frac{1-\pi_2(z)}{\pi_1(z)}q_t.\end{aligned}$$

Third, the Wiener–Hopf formula gives

$$\mathbb{E}_{it}[d_{t+1}] = \left[\psi_d(L)\right]_+\Gamma^{-1}(L)X_{it},$$

where the z-transform of the operator ψ_d is given by

$$\psi_d(z) = \left[\psi_d^{(1)}(z), \psi_d^{(2)}(z)\right] = z^{-1}S_{dx}(z)\left(\Gamma^{-1}\left(z^{-1}\right)\right)^{\intercal},$$

and $[\psi_d^{(1)}(z)]_+ = \psi_d^{(1)}(z) - P_d^{(1)}(z)$ and $[\psi_d^{(2)}(z)]_+ = \psi_s^{(2)}(z) - P_d^{(2)}(z)$. Here $P_d^{(1)}(z)$ and $P_d^{(2)}(z)$ denote the negative powers of z in the Laurent series expansions of $\psi_d^{(1)}(z)$ and $\psi_d^{(2)}(z)$, respectively. As in Step 2, we can compute that

$$\begin{aligned}\psi_d^{(1)}(z) &= z^{-1}\left[M_d^a(z)A_n^{(1)}(z)\sigma_a^2 - M_d^u(z)A_n^{(2)}(z)\sigma_u^2\right]\\ \psi_d^{(2)}(z) &= z^{-1}\left[-M_d^a(z)A_n^{(3)}(z)\sigma_a^2 + M_d^u(z)A_n^{(4)}(z)\sigma_u^2\right].\end{aligned}$$

It follows that

$$\mathbb{E}_{it}[d_{t+1}] = \left[G_d^{(1)}(L) - A_d^{(1)}(L), \frac{1-\pi_2(z)}{\pi_1(z)}\left(G_d^{(2)}(L) - A_d^{(2)}(L)\right)\right]X_{it},$$

where

$$G_d^{(1)}(z) = \psi_d^{(1)}(z)\Gamma_I^{(1)}(z) - \psi_d^{(2)}\Gamma_I^{(2)}(z), \qquad G_d^{(2)}(z) = \psi_d^{(2)}(z)\Gamma_I^{(4)}(z) - \psi_d^{(1)}(z)\Gamma_I^{(3)}(z)$$

and

$$A_d^{(1)}(z) = P_d^{(1)}(z)\Gamma_I^{(1)}(z) - P_d^{(2)}(z)\Gamma_I^{(2)}(z), \qquad A_d^{(2)}(z) = P_d^{(2)}(z)\Gamma_I^{(4)}(z) - P_d^{(1)}(z)\Gamma_I^{(3)}(z).$$

Finally, the Wiener–Hopf formula gives

$$\mathbb{E}_{it}[\Delta b_{it+1}] = \left[\psi_b(L)\right]_+\Gamma^{-1}(L)X_{it},$$

where the z-transform of the operator ψ_b is given by

$$\psi_b(z) = \left[\psi_b^{(1)}(z), \psi_b^{(2)}(z)\right] = z^{-1}(z-1)S_{bx}(z)\left(\Gamma^{-1}\left(z^{-1}\right)\right)^{\intercal},$$

and $[\psi_b^{(1)}(z)]_+ = \psi_b^{(1)}(z) - P_b^{(1)}(z)$ and $[\psi_b^{(2)}(z)]_+ = \psi_b^{(2)}(z) - P_b^{(2)}(z)$. Here $P_b^{(1)}(z)$ and $P_b^{(2)}(z)$ denote the negative powers of z in the Laurent series expansions of $\psi_b^{(1)}(z)$ and $\psi_b^{(2)}(z)$, respectively. It follows that

$$\left[\psi_b^{(1)}(z), \psi_b^{(2)}(z)\right]_+ \Gamma^{-1}(z) = \left[G_b^{(1)}(z) - A_b^{(1)}(z), \frac{1-\pi_2(z)}{\pi_1(z)}\left(G_b^{(2)}(z) - A_b^{(2)}(z)\right)\right],$$

where

$$G_b^{(1)}(z) = \psi_b^{(1)}(z)\Gamma_I^{(1)}(z) - \psi_b^{(2)}\Gamma_I^{(2)}(z), \qquad G_b^{(2)}(z) = \psi_b^{(2)}(z)\Gamma_I^{(4)}(z) - \psi_b^{(1)}(z)\Gamma_I^{(3)}(z)$$

and

$$A_b^{(1)}(z) = P_b^{(1)}(z)\Gamma_I^{(1)}(z) - P_b^{(2)}(z)\Gamma_I^{(2)}(z), \qquad A_b^{(2)}(z) = P_b^{(2)}(z)\Gamma_I^{(4)}(z) - P_b^{(1)}(z)\Gamma_I^{(3)}(z).$$

As in Step 2, we can also derive that

$$\begin{aligned}
\psi_b^{(1)}(z) &= z^{-1}(z-1)\left[M_b^a(z)A_n^{(1)}(z)\sigma_a^2 - M_b^u(z)A_n^{(2)}(z)\sigma_u^2 + \Gamma_I^{(1)}\left(z^{-1}\right)M_b^i(z)\sigma_i^2\right] \\
\psi_b^{(2)}(z) &= z^{-1}(z-1)\left[-M_b^a(z)A_n^{(3)}(z)\sigma_a^2 + M_b^u(z)A_n^{(4)}(z)\sigma_u^2 - \Gamma_I^{(2)}\left(z^{-1}\right)M_b^i(z)\sigma_i^2\right].
\end{aligned}$$

Step 4. Derive the equilibrium system for $\pi_1(z)$ and $\pi_2(z)$. By Step 3, we obtain an expression for χ_{it}. Matching coefficients of $X_{it} = [a_{it}, q_t]^{\intercal}$ with those in (32), we obtain the equilibrium conditions for $\pi_1(z)$ and $\pi_2(z)$,

$$\begin{aligned}
\pi_1(z) = {} & \frac{(1-\lambda_s)}{z(1-\lambda_s z)}\left[\Gamma_I^{(1)}(z)\Gamma_I^{(1)}\left(z^{-1}\right) + \Gamma_I^{(2)}(z)\Gamma_I^{(2)}\left(z^{-1}\right)\right]\sigma_i^2\pi_1(z) \\
& - A_s^{(1)}(z) + \frac{R^{(1)}(z)}{z(1-\lambda_s z)}
\end{aligned} \tag{S1.7}$$

and

$$\begin{aligned}
\pi_2(z) = {} & \frac{1-\pi_2(z)}{z(1-\lambda_s z)\pi_1(z)}\left\{(\lambda_s - 1)\left[\Gamma_I^{(1)}\left(z^{-1}\right)\Gamma_I^{(3)}(z) + \Gamma_I^{(2)}\left(z^{-1}\right)\Gamma_I^{(4)}(z)\right]\sigma_i^2\pi_1(z)\right. \\
& \left. - z(1-\lambda_s z)A_s^{(2)}(z) + R^{(2)}(z)\right\} + z^{-1}\beta,
\end{aligned} \tag{S1.8}$$

where $R^{(1)}(z)$ and $R^{(2)}(z)$ are defined as

$$\begin{aligned}
R^{(1)}(z) = {} & \left\{-\beta\frac{1}{\sigma_w}\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}z^{-1}\left(V_{12}\Gamma_I^{(1)}(z) + \frac{1}{\lambda_w}V_{22}\Gamma_I^{(2)}(z)\right)\right. \\
& \left. + (1-\beta)\left[G_d^{(1)}(z) - A_d^{(1)}(z)\right] + \left[G_b^{(1)}(z) - A_b^{(1)}(z)\right]\right\} z(1-\lambda_s z)
\end{aligned}$$

and

$$R^{(2)}(z) = \left\{ \beta \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(0)}{1-\pi_2(0)} z^{-1} \left(V_{12}\Gamma_I^{(3)}(z) + \frac{1}{\lambda_w} V_{22}\Gamma_I^{(4)}(z) \right) + (1-\beta)\left[G_d^{(1)}(z) - A_d^{(1)}(z) \right] + \left[G_b^{(2)}(z) - A_b^{(2)}(z) \right] \right\} z(1-\lambda_s z).$$

Define an operator $\mathcal{T}$ that maps the vector of functions $[\pi_1(z), \pi_2(z)]$ to the vector of functions that are equal to the expressions on the right-hand sides of (S1.7) and (S1.8). Since the signal system contains endogenous prices, many variables in these expressions depend on $[\pi_1(z), \pi_2(z)]$ in a complicated way. Thus, the operator $\mathcal{T}$ is nonlinear in general. The equilibrium functions $\pi_1(z)$ and $\pi_2(z)$ correspond to the fixed point of $\mathcal{T}$ in $\mathbf{H}^2(\mathbb{D})$. Moreover, we use (S1.8) to derive that

$$\begin{aligned} &\frac{\pi_1(z)}{1-\pi_2(z)} \\ &\quad = \frac{1}{(1-\lambda_s z)(z-\beta)} \big\{ -z(1-\lambda_s z) A_s^{(2)}(z) + R^{(2)}(z) \\ &\qquad + \left[z(1-\lambda_s z) - (1-\lambda_s)\left(\Gamma_I^{(1)}\left(z^{-1}\right)\Gamma_I^{(3)}(z) + \Gamma_I^{(2)}\left(z^{-1}\right)\Gamma_I^{(4)}(z)\right)\sigma_i^2 \right] \pi_1(z) \big\}. \end{aligned} \tag{S1.9}$$

We also have to ensure that $\frac{\pi_1(z)}{1-\pi_2(z)} \in \mathbf{H}^2(\mathbb{D})$ in equilibrium. Note that our triangular spectral factorization method also sheds light on the rationale behind the nonlinearity and the nonrational representation of the equilibrium. Specifically, the nonlinearity arises from the first step of the spectral factorization in which a new function $\tilde{\pi}_1(z)$ is created and the integrity of the original function $\frac{\pi_1(L)}{(1-\pi_2(L))}$ cannot be preserved. By comparison, Assumption 2 in Section 5 leads to a spectral factorization with no additional endogenous function. It also preserves the integrity of the original functions $\frac{\pi_1(L)}{(1-\pi_2(L))}$ as a whole. A similar argument also applies to Kasa et al. (2014), as their signal system is square so that factorization does not need the first step, avoiding the complication.

S1.2 *Numerical methods*

The equilibrium is characterized by the fixed point of the operator $\mathcal{T}$. Due to the endogeneity of the price signal, this operator is nonlinear and, thus, the model does not admit a solution in the form of rational functions. We now approximate the true model solution, which is in the form of MA(∞), by finite-order ARMA(p, q) processes in the time domain or by rational functions in the frequency domain. Rational functions also allow us to evaluate the annihilation operator tractably using the lemma in Appendix A of Hansen and Sargent (1980). The numerical method involves the following steps.

Step 1. We begin by an initial guess for $\pi_1(z)$ in the form of an irreducible rational function,

$$\pi_1(z) = \sigma_\pi \frac{\prod_{i=1}^{q}(1+\theta_i z)}{\prod_{j=1}^{p}(1-\rho_j z)}, \tag{S1.10}$$

where p and q are the orders of the ARMA representation, and σ_π, θ_i, and $|\rho_j| < 1$ are constants. Given the initial guess, we solve for the canonical factorization (S1.1) to obtain

$$\tilde{\pi}_1(z) = \sigma_{\tilde{\pi}} \frac{\prod_{i=1}^{m+1}(1+\hat{\theta}_i z)}{(1-\rho_a z)(1-\rho_u z)\prod_{j=1}^{p}(1-\rho_j z)}, \tag{S1.11}$$

where $m = \max(p, q)$, and $\sigma_{\tilde{\pi}}$ and $\hat{\theta}_i$ are determined by the factorization

$$\begin{aligned}
&\sigma_{\tilde{\pi}}^2 \prod_{i=1}^{m+1}(1+\hat{\theta}_i z)\big(1+\hat{\theta}_i z^{-1}\big) \\
&\quad = \sigma_a^2\sigma_i^2\sigma_\pi^2 \prod_{i=1}^{q}(1+\theta_i z)\big(1+\theta_i z^{-1}\big)(1-\rho_u z)\big(1-\rho_u z^{-1}\big) \\
&\qquad + \sigma_u^2\sigma_w^2(1-\lambda_w z)\big(1-\lambda_w z^{-1}\big)\prod_{j=1}^{p}(1-\rho_j z)\big(1-\rho_j z^{-1}\big).
\end{aligned} \tag{S1.12}$$

In particular, set $|\hat{\theta}_i| < 1\ \forall i = 1, 2, \ldots, m+1$.

In addition, we take an initial guess for the constant $\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}$.

Step 2. Solve for the decision rules for quantities on the real side of the economy. We use (S1.6) to derive $M_y^a(z)$ and $M_y^u(z)$. We need to compute $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ by using the lemma in Hansen and Sargent (1980). Given the guess for $\pi_1(z)$ in (S1.10) and (S1.11), and the expressions for $\psi_y^{(1)}(z)$ and $\psi_y^{(2)}(z)$ derived in Step 2 of Section S1.1, we deduce that $-\hat{\theta}_1, \ldots,$ and $-\hat{\theta}_{m+1}$ are the poles of $\psi_y^{(1)}(z)$ and $\psi_y^{(2)}(z)$ that are inside the unit disk. Thus, we have

$$P_y^{(1)}(z) = \sum_{k=1}^{m+1} \frac{\psi_{k,y}}{z+\hat{\theta}_k}, \qquad P_y^{(2)}(z) = \sum_{k=1}^{m+1} f_k \frac{\psi_{k,y}}{z+\hat{\theta}_k}$$

$$f_k \equiv \frac{V_{11}}{V_{12}} \frac{1+\lambda_w\hat{\theta}_k}{\hat{\theta}_k+\lambda_w}, \quad k = 1, 2, \ldots, m+1,$$

where each $\psi_{k,y}$ is a constant defined as

$$\psi_{k,y} = \lim_{z\to -\hat{\theta}_k} (z+\hat{\theta}_k)\big[M_y^a(z)\sigma_a^2 A_n^{(1)}(z) - M_y^u(z)\sigma_u^2 A_n^{(2)}(z)\big],$$

provided that all poles $\{-\hat{\theta}_k\}_{k=1}^{m+1}$ inside the unit disk are distinct. No constant $\psi_{k,y}$ can be solved numerically using the preceding formula because $M_y^a(z)$ and $M_y^u(z)$ are unknown functions to be determined. We use the method below to determine all $\psi_{k,y}$.

Plugging the guess for $\pi_1(z)$ and the expressions above for $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ (taking all unknown constant $\psi_{k,y}$ as given) into (S1.6), we obtain the linear system

$$\begin{bmatrix} Q_1(z) & Q_2(z) \\ \tilde{Q}_3(z) & \tilde{Q}_4(z) \end{bmatrix} \begin{bmatrix} M_y^a(z) \\ M_y^u(z) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\xi} - A_y^{(1)}(z)\theta - A_y^{(2)}(z)\theta \\ -\displaystyle\prod_{j=1}^{p}(1-\rho_j z)A_y^{(2)}(z)\theta \end{bmatrix} \equiv \begin{bmatrix} C_y^{(1)}(z) \\ C_y^{(2)}(z) \end{bmatrix},$$

where

$$\tilde{Q}_3(z) = \theta\sigma_a^2 \prod_{j=1}^{p}(1-\rho_j z)H_d(z)$$

$$\tilde{Q}_4(z) = \prod_{i=1}^{q}(1+\theta_i z)(1-\rho_u z)\sigma_\pi - \theta\sigma_u^2 \prod_{j=1}^{p}(1-\rho_j z)H_c(z).$$

Solving this linear system yields

$$\begin{bmatrix} M_y^a(z) \\ M_y^u(z) \end{bmatrix} = \frac{1}{D_1^2(z)\big[\tilde{Q}_4(z)Q_1(z) - Q_2(z)\tilde{Q}_3(z)\big]} \times \begin{bmatrix} D_1(z)\tilde{Q}_4(z)C_y^{(1)}(z) - D_1(z)Q_2(z)C_y^{(2)}(z) \\ -D_1(z)\tilde{Q}_3(z)C_y^{(1)}(z) + D_1(z)Q_1(z)C_y^{(2)}(z) \end{bmatrix},$$

where we define

$$D_1(z) = \prod_{i=1}^{m+1}(1+\hat{\theta}_i z)(z+\hat{\theta}_i).$$

We can verify that the above solutions for $M_y^a(z)$ and $M_y^u(z)$ are irreducible rational functions. That is, the numerator and denominator are pure polynomial functions.

The denominator function $D_y(z) \equiv D_1^2(z)[\tilde{Q}_4(z)Q_1(z) - Q_2(z)\tilde{Q}_3(z)]$ determines the existence and uniqueness of a stationary equilibrium. The necessary condition for the existence requires that $D_y(z)$ has precisely $m+1$ roots inside the open unit disk. We verify this condition in every iteration in our numerical computations. Let $\{z_j\}_{j=1}^{m+1}$ denote all the inside roots of $D_y(z)$. To pin down the vector of constants $\psi_y = [\psi_{1,y}, \ldots, \psi_{m+1,y}]^\intercal$, we use the system of $m+1$ equations

$$D_1(z_j)\tilde{Q}_4(z_j)C_y^{(1)}(z_j) - D_1(z_j)Q_2(z_j)C_y^{(2)}(z_j) = 0, \quad j = 1, 2, \ldots, m+1,$$

which gives a linear system for ψ_y,

$$A^c \psi_y = C^c,$$

where A^c is an $(m+1)\times(m+1)$ matrix of constants and C^c is an $(m+1)$-dimensional vector of constants. We derive this system by substituting $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ (which depend on ψ_y) into $A_y^{(i)}(z)$ and $C_y^{(i)}(z)$, $i=1,2$. For simplicity, we omit the detailed algebra here. The idea is that the solution for ψ_y must remove the poles of $D_y(z)$ inside the open unit disk so that the solutions for $M_y^a(z)$ and $M_y^u(z)$ are analytic inside the open unit disk. If the matrix A^c is invertible, the solution is unique. We verify this condition in every iteration of our numerical computations. Given the solutions for $M_y^a(z)$ and $M_y^u(z)$, we solve for $M_y^i(z)$ using (S1.4). We can also solve for $M_b(z)$, $M_n(z)$, and $M_d(z)$ using the formulas derived in Step 2 of Section S1.1.

Step 3. We compute all annihilated functions of negative powers of z on the financial side of the model using the Hansen–Sargent lemma. Let $\{z_k\}_{k=1}^{m+2} = \{0, -\hat{\theta}_1, \ldots, -\hat{\theta}_{m+1}\}$ denote the set of poles inside the unit disk. Provided that all poles are distinct, we have

$$P_s^{(1)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,s}^{(1)}}{z-z_k}, \qquad P_s^{(2)}(z) = -\sum_{k=1}^{m+2} \frac{\psi_{k,s}^{(2)}}{z-z_k}$$

$$P_d^{(1)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,d}^{(1)}}{z-z_k}, \qquad P_d^{(2)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,d}^{(2)}}{z-z_k}$$

$$P_b^{(1)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,b}^{(1)}}{z-z_k}, \qquad P_b^{(2)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,b}^{(2)}}{z-z_k},$$

where the constants are given by

$$\psi_{k,s}^{(1)} = \lim_{z\to z_k} (z-z_k)\big[z^{-1}\alpha_3 M_s^i(z)\Gamma_I^{(1)}\big(z^{-1}\big)\big]\sigma_i^2$$

$$\psi_{k,s}^{(2)} = \lim_{z\to z_k} (z-z_k)\big[z^{-1}\alpha_3 M_s^i(z)\Gamma_I^{(2)}\big(z^{-1}\big)\big]\sigma_i^2,$$

and

$$\psi_{k,d}^{(1)} = \lim_{z\to z_k} (z-z_k)z^{-1}\big[M_d^a(z)A_n^{(1)}(z)\sigma_a^2 - M_d^u(z)A_n^{(2)}(z)\sigma_u^2\big]$$

$$\psi_{k,d}^{(2)} = \lim_{z\to z_k} (z-z_k)z^{-1}\big[M_d^u(z)A_n^{(4)}(z)\sigma_u^2 - M_d^a(z)A_n^{(3)}(z)\sigma_a^2\big]$$

$$\psi_{k,b}^{(1)} = \lim_{z\to z_k} (z-z_k)(z-1)z^{-1}\big[M_b^a(z)A_n^{(1)}(z)\sigma_a^2 - M_b^u(z)A_n^{(2)}(z)\sigma_u^2 + M_b^i(z)\Gamma_I^{(1)}\big(z^{-1}\big)\sigma_i^2\big]$$

$$\psi_{k,b}^{(2)} = \lim_{z\to z_k} (z-z_k)(z-1)z^{-1}\big[M_b^u(z)A_n^{(4)}(z)\sigma_u^2 - M_b^a(z)A_n^{(3)}(z)\sigma_a^2 - M_b^i(z)\Gamma_I^{(2)}\big(z^{-1}\big)\sigma_i^2\big].$$

Given the guess of $\pi_1(z)$ in (S1.10) and the solutions for $M_y(z)$, $M_d(z)$, $M_n(z)$, and $M_b(z)$ in the previous step, we can compute the constants $\psi_{k,d}^{(1)}$, $\psi_{k,d}^{(2)}$, $\psi_{k,b}^{(1)}$, and $\psi_{k,b}^{(2)}$ for $k=1,2,\ldots,m+2$. The other constants $\psi_{k,s}^{(1)}$ and $\psi_{k,s}^{(2)}$ are solved in the next step. We cannot use the formulas above to determine $\psi_{k,s}^{(1)}$ and $\psi_{k,s}^{(2)}$ because $M_s^i(z)$ is an unknown function to be determined in equilibrium. We can verify that

$$\psi_{k,s}^{(2)} = h_k \psi_{k,s}^{(1)}$$

$$h_k = \begin{cases} \dfrac{V_{12}}{V_{22}} \dfrac{1-\lambda_w z_k}{z_k - \lambda_w}, & \text{if } z_k = 0 \\ \dfrac{V_{11}}{V_{12}} \dfrac{1-\lambda_w z_k}{z_k - \lambda_w}, & \text{else.} \end{cases}$$

Thus, we only need to solve for $\psi_{k,s}^{(1)}$, $k = 1, \ldots, m+2$.

Step 4. Solve for the update of $\pi_1(z)$ and $\pi_2(z)$ using (S1.7) and (S1.8). Given the guess for $\pi_1(z)$ in (S1.10), we can verify that $R^{(1)}(z)$ is an analytic rational function. Let $R_D^{(1)}(z)$ denote the denominator polynomial function of $R^{(1)}(z)$ in its irreducible form. Since $R^{(1)}(z)$ is analytic, $R_D^{(1)}(z) \neq 0$ inside the open unit disk. We can write

$$R_D^{(1)}(z) = R_D^{(1)}(0) \prod_{i=1}^{g} (1 + z_i z),$$

where g denotes the degree of $R_D^{(1)}(z)$ and $-z_i^{-1}, \ldots, -z_g^{-1}$ are the g roots of $R_D^{(1)}(z)$ that are outside the open unit disk. Using the definition of the unitary matrix V, we can show that the denominator of the rational function $z(1-\lambda_s z) - (1-\lambda_s)\sigma_i^2[\Gamma_I^{(1)}(z)\Gamma_I^{(1)}(z^{-1}) + \Gamma_I^{(2)}(z)\Gamma_I^{(2)}(z^{-1})]$ in the irreducible form is given by

$$D_1(z) = \prod_{k=1}^{m+1} (1 + \hat{\theta}_k z)(z + \hat{\theta}_k).$$

Notice that some factors in $D_1(z)$ and $R_D^{(1)}(z)$ may be identical. We define $D_2(z)$ as their least common multiple.

We now rewrite (S1.7) as

$$\pi_1(z) = \frac{D_2(z)\big[R^{(1)}(z) - z(1-\lambda_s z) A_s^{(1)}(z)\big]}{D_2(z)\big[z(1-\lambda_s z) - (1-\lambda_s)\sigma_i^2\big[\Gamma_I^{(1)}(z)\Gamma_I^{(1)}\big(z^{-1}\big) + \Gamma_I^{(2)}(z)\Gamma_I^{(2)}\big(z^{-1}\big)\big]\big]}, \quad \text{(S1.13)}$$

where both the numerator and the denominator are pure polynomial functions. Let $\pi_1^D(z)$ denote the denominator function. The existence and uniqueness of a stationary equilibrium solution for $\pi_1(z)$ is determined by the roots of $\pi_1^D(z)$. More specifically, to determine the $m+2$-dimensional vector of unknown constants $\psi_s = [\psi_{1,s}^{(1)}, \ldots, \psi_{m+2,s}^{(1)}]^\intercal$, we need $\pi_1^D(z)$ to have precisely $m+2$ distinct roots inside the open unit disk. We verify this condition in every iteration of the numerical computation. Without risk of confusion, let $\{\hat{z}_k\}_{k=1}^{m+2}$ denote the set of distinct roots of $\pi_1^D(z)$ that are inside the open unit disk.

We then pin down ψ_s by removing the poles $\{\hat{z}_k\}_{k=1}^{m+2}$ and evaluating the numerator polynomial

$$D_2(\hat{z}_k)\big[R^{(1)}(\hat{z}_k) - \hat{z}_k(1-\lambda_s \hat{z}_k) A_s^{(1)}(\hat{z}_k)\big] = 0 \quad \forall k = 1, 2, \ldots, m+2,$$

which leads to the linear system

$$A^{\pi}\psi_s = C^{\pi},$$

where we have used the definition of $A_s^{(1)}(z)$ and the expression of $P_s^{(1)}(z)$ derived in Step 2. We deduce that A^{π} is an $(m+2)\times(m+2)$ matrix with elements given by

$$A^{\pi}(k,i) = \frac{\Gamma_I^{(1)}(\hat{z}_k)D_2(\hat{z}_k)}{\hat{z}_k - z_i} + \frac{\Gamma_I^{(2)}(\hat{z}_k)D_2(\hat{z}_k)}{\hat{z}_k - z_i}h_i$$

for $k = 1, 2, \ldots, m+2$ and $i = 1, 2, \ldots, m+2$, and $z_i \in \{0, -\hat{\theta}_1, \ldots, -\hat{\theta}_{m+1}\}$. The kth element of $(m+2)\times 1$ vector C^{π} is given by

$$C^{\pi}(k) = R^{(1)}(\hat{z}_k)D_2(\hat{z}_k) \quad \forall k = 1, 2, \ldots, m+2.$$

If A^{π} is full rank, the solution is indeed unique. Again, we verify this condition in every iteration.

Once we determine ψ_s, we update the guess for $\pi_1(z)$ using the solution in (S1.13). Given this solution for $\pi_1(z)$, we use (S1.9) to solve for $\frac{\pi_1(z)}{1-\pi_2(z)}$. Observe that the numerator on the right-hand side of (S1.9) is analytic inside the open unit disk, but we still need to remove the pole at $z = \beta$. We set the constant $\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}$ to remove this pole. That is,

$$\phi(\beta)\pi_1(\beta) - \beta(1-\lambda_s\beta)A_s^{(2)}(\beta) + R^{(2)}(\beta) = 0,$$

where

$$\phi(z) = z(1-\lambda_s z) - (1-\lambda_s)\big[\Gamma_I^{(1)}\big(z^{-1}\big)\Gamma_I^{(3)}(z) + \Gamma_I^{(2)}\big(z^{-1}\big)\Gamma_I^{(4)}(z)\big]\sigma_i^2.$$

This leads to the solution for the constant:

$$\begin{aligned}\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)} = {} & \frac{\sigma_w}{\beta(1-\lambda_s\beta)\left[V_{12}\Gamma_I^{(3)}(\beta) + \dfrac{1}{\lambda_w}V_{22}\Gamma_I^{(4)}(\beta)\right]} \\ & \times \big\{\beta(1-\lambda_s\beta)\big(A_s^{(2)}(\beta) - (1-\beta)\big[G_d^{(1)}(\beta) - A_d^{(1)}(\beta)\big] \\ & - \big[G_b^{(2)}(\beta) - A_b^{(2)}(\beta)\big]\big) - \phi(\beta)\pi_1(\beta)\big\}. \end{aligned} \tag{S1.14}$$

We use this solution to update the initial guess for $\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}$. Finally, we iterate until convergence.

In summary, we employ the following iterative algorithm (Algorithm 1) to solve the model.

S1.3 *Macro-financial disconnection*

In the extension of Section 6.2, information is segregated between groups as agents in one group receive no signal about the other group's shocks. Let $\mathcal{I}_{it}^{p}$ and $\mathcal{I}_t^{n}$ denote the

Algorithm 1 Numerical Approximation of Equilibrium.

Step 0. Begin with a guess for p, q, σ_π, $\pi_c \equiv \frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}$, $\{\theta_i\}_{i=1}^q$, $\{\rho_j\}_{j=1}^p$ with $|\rho_j| < 1\ \forall j$.

Step 1. Set $m = \max\{p, q\}$ and compute $\sigma_{\hat{\pi}}$ and $\{\hat{\theta}_i\}_{i=1}^m$ using (S1.12).

Step 2. Solve for the functions $M_y(z)$, $M_d(z)$, $M_b(z)$, and $M_n(z)$.

Step 3. Let $\pi_1^A(z)$ and π_c^+ be the expressions on the right-hand sides of (S1.13) and (S1.14), respectively.

Step 4. Update the initial guess using

$$\pi_1^+(z) = \sigma_\pi^+ \frac{\prod_{i=1}^{q}(1+\theta_i^+ z)}{\prod_{j=1}^{p}(1-\rho_j^+ z)},$$

where σ_π^+, θ_i^+, and ρ_j^+ are the solution to the problem

$$\min_{\sigma_\pi, \theta_i, \rho_j} \sum_{n=1}^{N} \left|\pi_1^+(n) - \pi_1^A(n)\right|^2,$$

where $\pi_1^+(n)$ and $\pi_1^A(n)$ are the coefficients of the moving average expansion of $\pi_1^+(z)$ and $\pi_1^A(z)$, with $N = 70$.

Step 5. Iterate Steps 0–4 until $\max\{|\rho_j^+ - \rho_j|, |\theta_i^+ - \theta_i|, |\sigma_\pi^+ - \sigma_\pi|\} < 10^{-3}\ \forall i, j$.

Step 6. Compute $\epsilon = \max\{||\pi_1^+(z) - \pi_1^A(z)||_{\mathbf{H}^2}, |\pi_c^+ - \pi_c|\}$; if $\epsilon < 10^{-5}$, stop; otherwise, set $p := p+1$, $q := q+1$, and repeat Steps 0–5.

information set for agents in participating island i and any nonparticipating island, respectively. Then conditional expectations of the other group's shocks are equal to their unconditional mean, i.e.,

$$\begin{aligned} &\mathbb{E}_j\left[F(L)\epsilon_{at}^p|\mathcal{I}_t^n\right] = 0; \qquad \mathbb{E}_j\left[F(L)\epsilon_{it}|\mathcal{I}_t^n\right] = 0 \quad \forall i \in I_p, j \in I_n \\ &\mathbb{E}_i\left[F(L)\epsilon_{at}^n|\mathcal{I}_{it}^p\right] = 0; \qquad \mathbb{E}_i\left[F(L)\epsilon_{jt}|\mathcal{I}_{it}^p\right] = 0 \quad \forall i \in I_p, j \in I_n \end{aligned} \tag{S1.15}$$

for any square-summable lag polynomial $F(L)$. Then we can use (26) to characterize the equity market equilibrium[2]

$$q_t = \int_{i\in I_p} \mathbb{E}_i\left[\alpha_3 s_{it+2}^h + \Delta b_{it+1}|\mathcal{I}_{it}^p\right] di + \int_{i\in I_p} \mathbb{E}_i\left[\beta q_{t+1} + (1-\beta) d_{t+1}|\mathcal{I}_{it}^p\right] di + u_t. \tag{S1.16}$$

[2]The log-linearized coefficients determined by the steady state will be different from the basic model. In particular, the production side remains the same, while a redistriburion of consumption occurs between participating and nonparticipating islands.

Given property (S1.15) and our information structure, (S1.16) implies that we can focus on the equilibrium in which the equity prices are driven by

$$q_t = M_q^p(L)\epsilon_{at}^p + M_q^u(L)\epsilon_{ut},$$

which resembles (33). Intuitively, the stock price does not respond to fluctuations of nonparticipants' TFP shocks. Moreover, the information structure and the dynamic interactions between shareholding choices s_{it}^h and q_t ((35) and (36)) remains the same as in the basic model. Therefore, the unit root result in the equity price volatility is still valid, although the quantitative outcome depends on the participation measure κ and the modified real equilibrium.

Next, we characterize the log-linearized equilibrium in the real economy,

$$y_{it} = \frac{1}{\xi}\big(a_t^p + \epsilon_{it}\big) + \theta\mathbb{E}_i\big[\kappa y_t^p + (1-\kappa)y_t^n|\mathcal{I}_{it}^p\big] \quad \forall i \in I_p$$

$$y_{jt} = \frac{1}{\xi}\big(a_t^n + \epsilon_{jt}\big) + \theta\mathbb{E}_j\big[\kappa y_t^p + (1-\kappa)y_t^n|\mathcal{I}_t^n\big] \quad \forall j \in I_n,$$

where $y_t = \kappa y_t^p + (1-\kappa)y_t^n$, and $y_t^p = \frac{1}{\kappa}\int_{i\in I_p} y_{it}\, di$ and $y_t^n = \frac{1}{1-\kappa}\int_{j\in I_n} y_{jt}\, dj$ are log-linearized group aggregates. We conjecture that the "segregated" equilibrium decision rules follow

$$y_{it} = M_y^p(L)\epsilon_{at}^p + M_y^{i,p}(L)\epsilon_{it} + M_y^u(L)\epsilon_{ut} \quad \forall i \in I_p$$

$$y_{jt} = M_y^n(L)\epsilon_{at}^n + M_y^{j,n}(L)\epsilon_{jt} \quad \forall j \in I_n.$$

We then use (S1.15) and the fact that $\mathbb{E}_j[\epsilon_{ut}|\mathcal{I}_t^n] = 0$ to get

$$y_{it} = \frac{1}{\xi}\big(a_t^p + \epsilon_{it}\big) + \theta\kappa\mathbb{E}_i\big[y_t^p|\mathcal{I}_{it}^p\big] \quad \forall i \in I_p \tag{S1.17}$$

$$y_{jt} = \frac{1}{\xi}\big(a_t^n + \epsilon_{jt}\big) + \theta(1-\kappa)y_t^n \quad \forall j \in I_n. \tag{S1.18}$$

Note that (S1.17) resembles (19), which leads to the decision rule for total output on participating islands,

$$y_t^p = \frac{1}{\kappa}\int_{i\in I_p} y_{it}\, di = M_y^p(L)\epsilon_{at}^p + M_y^u(L)\epsilon_{ut}.$$

Meanwhile, aggregating (S1.18) produces a simple solution for y_t^n:

$$y_t^n = \frac{1}{1-\kappa}\int_{j\in I_n} M_y^n(L)\epsilon_{at}^n\, dj = \frac{1}{\xi\big[1-\theta(1-\kappa)\big](1-\rho_a L)}\epsilon_{at}^n.$$

It is then more transparent to write the equilibrium aggregate output as

$$y_t = \kappa\big[M_y^p(L)\epsilon_{at}^p + M_y^u(L)\epsilon_{ut}\big] + (1-\kappa)\frac{1}{\xi\big[1-\theta(1-\kappa)\big](1-\rho_a L)}\epsilon_{at}^n.$$

Since $M_y^p(L)$ and $M_y^u(L)$ are determined by (S1.17), which is almost equivalent to the equilibrium condition in the basic model except for the appearance of the κ parameter, the solution for $M_y^p(L)$ and $M_y^u(L)$ is invariant up to the changes in κ and the steady-state coefficients.

S2. Frequency domain methods

In this section, we introduce some mathematical background for the frequency domain methods. We study casual covariance stationary real-valued equilibrium processes that have an MA(∞) representation. For example, the aggregate output process in the model of Section 3 can be written as

$$y_t = \sum_{j=0}^{\infty} M_j \varepsilon_{a,t-j}, \tag{S2.1}$$

where $\{M_j\}_{j=0}^{\infty}$ is square summable, i.e., $\sum_{j=0}^{\infty} |M_j|^2 < \infty$. Solving for the infinite sequence of $\{M_j\}_{j=0}^{\infty}$ is a daunting task. The idea of the frequency domain method is to transform this problem into an equivalent problem of solving for an analytical function in the Hardy space. To define this space, we recall that $\mathbb{C}$ denotes the complex plan, $\mathbb{T}$ denotes the unit circle, and $\mathbb{D}$ denotes the open unit disk.

Definition S1. The Hardy space $\mathbf{H}^2(\mathbb{D})$ is the class of analytical functions g in the unit disk $\mathbb{D}$ satisfying

$$\left\{ \frac{1}{2\pi} \sup_{0 \le r < 1} \int_{-\pi}^{\pi} \left| g\left(re^{i\omega}\right) \right|^2 d\omega \right\}^{1/2} < \infty.$$

It can be verified that the expression on the preceding inequality defines a norm on $\mathbf{H}^2(\mathbb{D})$, denoted as $\|g\|_{\mathbf{H}^2}$. The Hardy space can also be viewed as a certain closed vector subspace of the complex L^2 space for the unit circle $\mathbb{T}$. This connection is provided by the fact that the radial limit

$$\tilde{g}\left(e^{i\omega}\right) = \lim_{r \uparrow 1} g\left(re^{i\omega}\right)$$

exists for almost all $\omega \in [-\pi, \pi]$. The function $\tilde{g}$ belongs to the space $L^2(\mathbb{T})$ of functions $f : \mathbb{T} \to \mathbb{C}$ with the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\omega}\right) \overline{f_2\left(e^{i\omega}\right)}\, d\omega, \quad f_1, f_2 \in L^2(\mathbb{T}).$$

Then we have

$$\|g\|_{\mathbf{H}^2} = \|\tilde{g}\|_{L^2} = \lim_{r \uparrow 1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| g\left(re^{i\omega}\right) \right|^2 d\omega \right\}^{1/2} < \infty.$$

Denote by $\mathbf{H}^2(\mathbb{T})$ the vector subspace of $L^2(\mathbb{T})$, consisting of all limit functions $\tilde{g}$, when g varies in $\mathbf{H}^2(\mathbb{D})$.

THEOREM S1 (Katznelson (1976)). *We have $f \in \mathbf{H}^2(\mathbb{T})$ if and only if $f \in L^2(\mathbb{T})$ and $\hat{f}_n = 0$ for all $n < 0$, where $\hat{f}_n$ is the Fourier coefficient of a function that is f integrable on the unit circle:*

$$\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(e^{i\omega}\big)e^{-i\omega n}\, d\omega, \quad n = 0, \pm 1, \pm 2, \ldots .$$

Suppose that $\tilde{g} \in \mathbf{H}^2(\mathbb{T})$ and $\tilde{g}$ has Fourier coefficients $\{a_n\}$ with $a_n = 0$ for all $n < 0$. We define

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1.$$

The following theorem ensures that $g \in \mathbf{H}^2(\mathbb{D})$. Thus, we have a bijection between $\mathbf{H}^2(\mathbb{D})$ and $\mathbf{H}^2(\mathbb{T})$.

THEOREM S2. *If $f(z)$ is an analytic function in $\mathbb{D}$ and its Laurent expansion is*

$$f(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then $f \in \mathbf{H}^2(\mathbb{D})$ if and only if $\{b_n\}_{n=0}^{\infty}$ is square summable, i.e., $\sum_{n=0}^{\infty} |b_n|^2 < \infty$. When this condition is satisfied,

$$\sum_{n=0}^{\infty} |b_n|^2 = \|f\|_{\mathbf{H}^2}.$$

We call the map from the sequence $\{b_n\}_{n=0}^{\infty}$ to $f(z)$ a z-transform. Theorem S2 also allows us to give an equivalent definition of the Hardy space $\mathbf{H}^2(\mathbb{D})$ as the class of analytical functions $f : \mathbb{D} \to \mathbb{C}$, which are the z-transforms of some square-summable sequences. Thus, solving for $\{M_j\}_{j=0}^{\infty}$ in (S2.1) is equivalent to solving for a function $M(z)$ in the hardy space $\mathbf{H}^2(\mathbb{D})$. In particular, we can write $y_t = M(L)\epsilon_{at}$, where $M(z) \in \mathbf{H}^2(\mathbb{D})$ is the object we solve for. We can use Theorem S2 to compute the variance of y_t easily because

$$\mathrm{Var}(y_t) = \sigma_a^2 \sum_{j=0}^{\infty} M_j^2 = \sigma_a^2 \big\|M(z)\big\|_{\mathbf{H}^2}.$$

Finally, a rational function $f(z) \in \mathbf{H}^2(\mathbb{D})$ if and only if $f(z)$ is analytic in the closed unit disk. In particular, poles are not allowed on the unit circle.

S3. COMPUTING EXPECTATIONS IN THE FREQUENCY DOMAIN

We present our approach in a general framework. Suppose that the signal is an ℓ-dimensional variable X_t, defined in terms of infinite-order moving average processes.[3]

[3]We can extend the definition to contain information about future innovations (e.g., Bacchetta and van Wincoop 2008).

Let $\mathbb{C}$ denote the complex plane, let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and let $\mathbb{D}$ denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

Definition S2 (Signal representation). The ℓ-dimensional real-valued signal process $\{X_t\}$ is linearly regular and admits representation

$$\underset{\ell\times 1}{X_t} = \underset{\ell\times k}{H(L)}\,\underset{k\times 1}{\eta_t}\,, \quad \ell \le k,$$

where L denotes the lag operator, $\{\eta_t\}$ represents structural Gaussian innovations with mean zero and covariance matrix Σ_η, and $H(z)$ is an $\ell \times k$ matrix analytic function defined on the open unit disk $\mathbb{D}$ in the matrix-valued Hardy space $\mathbf{H}^2(\mathbb{D})$.[4]

We call $H(\cdot)$ the signal matrix or the transfer function as in the mathematics literature. To simplify the signal extraction problem, it is useful to assume a maximal rank condition for the signal process so that no redundant information is contained in X_t.

Assumption 4. *The ℓ-dimensional signal process X_t has maximal rank, i.e., the rank of its associated spectral density $f_x(\omega)$ equals its dimension,*

$$\operatorname{rank}\big(f_x(\omega)\big) = \ell$$

for almost all $\omega \in [-\pi, \pi]$.

An methodological contribution of our paper is that we study a non-square signal representation in that $\ell < k$. The existing literature focuses on the case of square signal representations with $\ell = k$ (e.g., Kasa et al. 2014 and Rondina and Walker 2020). To use the Wiener–Hopf prediction formula, we need the Wold fundamental representation for the signal process. For the case of non-square signal representation, finding the Wold representation is nontrivial. We use spectral factorization techniques to solve this problem.

S3.1 *A two-step spectral factorization procedure*

Our goal is to find a Wold representation for $\{X_t\}$. We are looking for an outer analytic matrix function $\Gamma(\cdot)$ in the Hardy space $\mathbf{H}^2(\mathbb{D})$ such that[5]

$$\underset{\ell\times 1}{X_t} = \underset{\ell\times \ell}{\Gamma(L)}\,\underset{\ell\times 1}{e_t}\,, \qquad f_x(\omega) = \Gamma\big(e^{-i\omega}\big)\Gamma^*\big(e^{-i\omega}\big), \quad \omega \in [-\pi, \pi],$$

[4] See the Appendix S2 for the definition of the Hardy space. This definition can be easily extended to matrix cases; see Lindquist and Picci (2015, Appendix B.2).

[5] The function $\Gamma(z)$ is also called a *canonical* or *fundamental* spectral factor. We refer readers to Lindquist and Picci (2015, Chapter 4) for characterizations of outer functions. One prominent feature of outer functions is that they cannot have zeros inside the unit disk. Note that Lindquist and Picci 2015 use the engineering definition of $z = e^{i\omega}$ so that the analytic region is reversed compared with this paper, but all analytic results remain valid.

where the asterisk denotes the conjugate transpose, $\{e_t\}$ is some mutually uncorrelated Wold (fundamental) innovation process with mean zero and an identity covariance matrix, f_x is the spectral density, and $\Gamma(\cdot)$ is an outer analytic function.[6]

For the square signal case with $\ell = k$, we can directly apply the Beurling–Blaschke factorization method to derive the Wold representation as in Kasa et al. (2014) and Rondina and Walker (2020). However, this method does not apply to the non-square case with $\ell < k$. We propose a two-step spectral factorization procedure. In Step 1 we apply the convolution theorem to find the spectral density $f_x(\omega)$ of the signal process $\{X_t\}$. Then we use the Rozanov (1967) theorem to find a lower triangular decomposition of $f_x(\omega)$. In Step 2 we apply the Beurling–Blaschke factorization method to the lower triangular matrix. Due to the length constraints, we omit the algebraic derivations in this section. These details are contained in the Appendix S4.

Before describing the two-step procedure, we start with the following well known result in time series, the proof of which is omitted for brevity.

LEMMA S1. *Suppose that X_t is the vector of signals defined in Definition S2 and that Assumption 4 holds. Moreover, the transfer function $H(z)$ is a non-square matrix function with dimension $k > \ell$. Then the spectral density $f_x(\omega)$ is an $\ell \times \ell$ matrix function defined on $[-\pi, \pi]$ and*

$$f_x(\omega) = H(e^{-i\omega})\Sigma_\eta H^*(e^{-i\omega}) = H(z)\Sigma_\eta H(z^{-1})^\intercal, \quad z = e^{-i\omega},$$

where the superscript $\intercal$ denotes the transpose of a matrix. Furthermore, $f_x(\omega)$ is a Hermitian normal matrix that is nonnegative definite for almost all $\omega \in [-\pi, \pi]$. If we extend the definition of z to the entire complex plane $\mathbb{C}$, then the autocovariance generating function is given by $S_x(z) = H(z)\Sigma_\eta H(z^{-1})^\intercal$, but without the Hermitian nonnegativeness property for general $z \in \mathbb{C}$.

Lemma S1 allows us to transform the non-square signal transfer matrix function into the square spectral density matrix $f_x(\omega)$. Based on this lemma, the first step of the spectral factorization procedure is to decompose $f_x(\omega)$ into triangular matrix functions using Rozanov's (1967) analytical method.

Step 1. Given an $\ell \times \ell$ spectral density matrix $f_x(\omega)$ with full rank almost everywhere, construct an $\ell \times \ell$ lower triangular matrix function $\widetilde{\Gamma}(e^{-i\omega})$ such that

$$f_x(\omega) = \widetilde{\Gamma}(e^{-i\omega})\widetilde{\Gamma}^*(e^{-i\omega}),$$

where

$$\widetilde{\Gamma}(z) = \begin{bmatrix} \widetilde{\Gamma}_{11}(z) & 0 & \dots & 0 \\ \widetilde{\Gamma}_{21}(z) & \widetilde{\Gamma}_{22}(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\Gamma}_{\ell 1}(z) & \widetilde{\Gamma}_{\ell 2}(z) & \dots & \widetilde{\Gamma}_{\ell\ell}(z) \end{bmatrix}.$$

[6] Note that the Wold fundamental innovations can have nondiagonal, nonnormalized covariance matrices. Using the unitary eigendecomposition of the covariance matrix, we can obtain the orthonormal Wold representations with an identity covariance matrix.

If $f_x(\omega)$ is rational, then all elements of the matrix function are rational and analytic in the closed unit disk $\mathbb{T} \cup \mathbb{D}$, and, hence, in the $\mathbf{H}^2(\mathbb{D})$ space. Moreover, $\widetilde{\Gamma}(e^{-i\omega})$ has full rank in $\mathbb{D}$ except for at most a finite number of points.

If the determinant of the analytic matrix $\widetilde{\Gamma}(z)$ vanishes at finitely many points inside the unit disk, it is not a Wold spectral factor. Without loss of generality, let $\{z_1, z_2, \ldots, z_n\}$ be the finite set of distinct points such that $\det(\widetilde{\Gamma}(z_j)) = 0$, $|z_j| < 1$, $j \in \{1, 2, \ldots, n\}$. Let $\overline{z}_j$ denote the conjugate of z_j. We assume that all zeros are of order 1 (this property is generic).

The second step of our spectral factorization method employs a multivariate version of the Beurling–Blaschke factorization theorem to remove any zeros inside the unit disk.

Step 2. The Wold spectral factor $\Gamma(z)$ can be obtained by the factorization for Hardy space functions as

$$\Gamma(z) = \widetilde{\Gamma}(z) \prod_{j=1}^{n} V_j^{-1} B_j(z),$$

where the $\ell \times \ell$ Blaschke matrices $B_j(z)$ are (inverse) inner matrix functions of the form

$$B_j(z) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \dfrac{1 - \bar{z}_j z}{z - z_j} \end{bmatrix},$$

and the constant unitary matrix V_j is given by the singular value decomposition of $\widetilde{\Gamma}(z)$ evaluated at the zeros

$$\widetilde{\Gamma}(z_j) = U_j D V_j,$$

where D is a diagonal matrix containing the singular values.

The constant unitary matrices V_j remove the unwelcome poles brought in by the Blaschke factors. There are different ways to compute these matrices, and we use the eigendecomposition method. In particular, the orthonormal column vectors of V_j can be directly picked from normalized linear independent eigenvectors of the Hermitian matrix $G_j(z_j) = \widetilde{\Gamma}^*(z_j)\widetilde{\Gamma}(z_j)$, which are automatically pairwise-orthogonal for distinct eigenvalues. For more complicated systems, the eigenvectors can be found easily using symbolic toolboxes in Matlab or Mathematica.

S3.2 *Wiener–Hopf prediction formula*

Using the Wold representation for the signal process, we can compute the conditional expectations given the history of signals. Since agents in our model need to perform optimal linear filtering to estimate unobserved shocks, we use the Wiener–Hopf prediction formula, a generalization of the Wiener–Kolmogorov forecasting formula.

Consider any random vector Θ_t satisfying $\Theta_t = G(L)\eta_t$, where $G(z)$ is a matrix analytic function in some matrix-valued Hardy space. We wish to compute the conditional expectation $\mathbb{E}[L^m\Theta_t|\{X_{t-n}\}_{n=0}^{\infty}]$ given the history of signals $\{X_{t-n}\}_{n=0}^{\infty}$, where m is any integer. The Wiener–Hopf prediction formula gives

$$\mathbb{E}\big[L^m\Theta_t|\{X_{t-n}\}_{n=0}^{\infty}\big] = \Xi(L)X_t,$$

where the analytic matrix function $\Xi(z)$ is given by

$$\Xi(z) = \big[z^m S_{\Theta x}(z)\big(\Gamma^{-1}\big(z^{-1}\big)\big)^{\intercal}\big]_{+}\Gamma^{-1}(z).$$

Here $\Gamma(z)$ is the Wold spectral factor derived in the previous subsection and $S_{\Theta x}(z) = G(z)\Sigma_{\eta}H(1/z)^{\intercal}$ is the covariance generating function. The annihilation operator $[\cdot]_+$ is linear and is used to remove the principal part of the Laurent series expansion of the analytic functions around a common region of convergence.[7] This formula reduces to the Wiener–Kolmogorov formula when $\Theta_t = X_t$ so that $\Xi(z) = [z^m\Gamma(z)]_+\Gamma^{-1}(z)$. If the forecast objects follow geometrically discounted processes, the formula reduces to the Hansen–Sargent optimal prediction formula.

S4. Algebraic derivation on spectral factorization in Appendix S3

Derivations in step 1 Since $f_x(\omega)$ is rational, it has a constant, maximal rank of ℓ except at a finite number of points on the unit circle $\mathbb{T}$. To develop the triangular factorization of the spectral density, we need the following lemma from Rozanov (1967) on rational functions.

Lemma S2. *Every nonnegative (real) rational function $f(\omega)$ of $e^{-i\omega}$ can be represented in the form*

$$f(\omega) = \frac{\left|P\big(e^{-i\omega}\big)\right|^2}{\left|Q\big(e^{-i\omega}\big)\right|^2} = \frac{P\big(e^{-i\omega}\big)\overline{P\big(e^{-i\omega}\big)}}{Q\big(e^{-i\omega}\big)\overline{Q\big(e^{-i\omega}\big)}} = \frac{P(z)\overline{P(z)}}{Q(z)\overline{Q(z)}}$$

for $z \in \mathbb{T}$. The polynomial functions $P(z)$ and $Q(z)$ have no zeros in the open unit disk. If f satisfies

$$f(\omega) = f(-\omega),$$

then the coefficients of $P(z)$ and $Q(z)$ can be chosen all real.

See Rozanov (1967, Lemma 10.1) for the proof.

If we extend $f(z)$ to be a complex function in the entire complex plane, the preceding lemma implies that it can be factorized in a "symmetric" way such that if λ_i is a root for $f(z)$, so is the conjugate inverse $1/\overline{\lambda_i}$.

[7] See Kailath et al. (2000) for a textbook proof of the Wiener–Hopf prediction formula. Hansen and Sargent (1980) provide a practical method of computing the annihilation operator using elementary complex analysis.

Now consider the $\ell \times \ell$ spectral density matrix $f_x(\omega)$. By definition it is Hermitian, normal, and nonnegative definite for almost all ω. For simplicity, we drop the x subscript and write the f matrix as

$$f(\omega) = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1\ell} \\ f_{21} & f_{22} & \dots & f_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ f_{\ell 1} & f_{\ell 2} & \dots & f_{\ell\ell} \end{bmatrix}.$$

Using Sylvester's criterion for the nonnegative definite matrix, define the family of leading principal minors as $M_j(\omega)$, $j = 1, 2, \dots, \ell$. By definition, $M_j(\omega) \geq 0$ a.e. and $M_1(\omega) = f_{11} \geq 0$ a.e.

Next we implement elementary row operations on the matrix. Adding to the rth row ($r = 2, 3, \dots, \ell$) the first row, multiplied by $-\frac{f_{r1}}{f_{11}}$, yields

$$f(\omega) = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1\ell} \\ 0 & f_{22} - f_{12}\dfrac{f_{21}}{f_{11}} & \dots & f_{2\ell} - f_{1\ell}\dfrac{f_{21}}{f_{11}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & f_{\ell 2} - f_{12}\dfrac{f_{\ell 1}}{f_{11}} & \dots & f_{\ell\ell} - f_{1\ell}\dfrac{f_{\ell 1}}{f_{11}} \end{bmatrix}.$$

Similarly, adding to the jth column ($j = 2, 3, \dots, \ell$) from the first column multiplied by $-\frac{f_{1j}}{f_{11}}$, we have

$$f^{(2)}(\omega) = \begin{bmatrix} f_{11} & 0 & \dots & 0 \\ 0 & f_{22} - f_{12}\dfrac{f_{21}}{f_{11}} & \dots & f_{2\ell} - f_{1\ell}\dfrac{f_{21}}{f_{11}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & f_{\ell 2} - f_{12}\dfrac{f_{\ell 1}}{f_{11}} & \dots & f_{\ell\ell} - f_{1\ell}\dfrac{f_{\ell 1}}{f_{11}} \end{bmatrix} = \begin{bmatrix} f_{11} & \mathbf{0} \\ \mathbf{0} & g^{(2)} \end{bmatrix},$$

where the elements of matrix $g^{(2)} = [g_{rj}^{(2)}]$ have the form $g_{rj}^{(2)} = f_{rj} - \frac{f_{r1} f_{1j}}{f_{11}}$.

Notice that the diagonal element $g_{22}^{(2)}$ satisfies $g_{22}^{(2)}(\omega) = \frac{M_2(\omega)}{M_1(\omega)}$ a.e. If we denote $g^{(1)} = f^{(1)} = f$, then $f^{(2)}$ is obtained by using the row–column transformations on $f^{(1)}$. Now consider the matrix

$$g^{(2)} = \begin{bmatrix} f_{22} - f_{12}\dfrac{f_{21}}{f_{11}} & \dots & f_{2\ell} - f_{1\ell}\dfrac{f_{21}}{f_{11}} \\ \vdots & \ddots & \vdots \\ f_{\ell 2} - f_{12}\dfrac{f_{\ell 1}}{f_{11}} & \dots & f_{\ell\ell} - f_{1\ell}\dfrac{f_{\ell 1}}{f_{11}} \end{bmatrix}.$$

We apply the same transformation for $g^{(2)}$ to eliminate its first row and column except the leading coefficient, yielding

$$g^{(2)} = \begin{bmatrix} f_{22} - f_{12}\dfrac{f_{21}}{f_{11}} & \mathbf{0} \\ \mathbf{0} & g^{(3)} \end{bmatrix}.$$

It is easy to verify that $g^{(3)}_{33}(\omega) = \frac{M_3(\omega)}{M_2(\omega)}$. We then arrive at a new $\ell \times \ell$ matrix:

$$f^{(3)}(\omega) = \begin{bmatrix} f_{11} & 0 & \mathbf{0} \\ 0 & f_{22} - f_{12}\dfrac{f_{21}}{f_{11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & g^{(3)} \end{bmatrix}.$$

Continue this process until we reach a diagonal matrix $f^{(\ell)}(\omega)$, admitting the form

$$f^{(\ell)}(\omega) = \begin{bmatrix} h_{11} & & & \\ & h_{22} & & \\ & & \ddots & \\ & & & h_{\ell\ell} \end{bmatrix}.$$

It is easy to see that the diagonal elements are

$$h_{11}(\omega) = M_1(\omega), \qquad h_{rr}(\omega) = \frac{M_r(\omega)}{M_{r-1}(\omega)}, \quad r = 2, 3, \ldots, \ell.$$

It follows that $f(\omega)$ admits the following logical disk unit-like (LDU-like) decomposition.

The spectral density $f_x(\omega)$ can be decomposed as $f_x = gf^{(\ell)}g^*$, where the matrix function $g(\omega)$ is lower triangular with diagonal elements equal to 1:

$$g(\omega) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ g_{21} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{\ell 1} & g_{\ell 2} & \ldots & 1 \end{bmatrix}.$$

The off-diagonal nonzero elements are defined as $g_{rj} = \frac{g^{(j)}_{rj}}{h_{jj}}$, $r > j$, where $g^{(l)}_{rl}$ is determined by the recursion

$$g^{(1)}_{rj} = f_{rj}, \qquad g^{(i)}_{rj} = g^{(i-1)}_{rj} - \frac{g^{(i-1)}_{r,i-1} - g^{(i-1)}_{i-1,j}}{g^{(i-1)}_{i-1,i-1}}, \quad i = 2, 3, \ldots, j.$$

Since the elements of $f_x(\omega)$ are rational functions, the matrix transformation implies that elements of g and $f^{(\ell)}$ are rational as well. Next we define $g_{rj}(\omega) = \frac{P_{rj}(z)}{Q_{rj}(z)}$, where $z = e^{-i\omega}$. We extend the definition of z to the entire complex plane and fix a column

$j \in \{1, 2, \ldots, \ell\}$. Let $\alpha_p^{(j)}$, $p = 1, 2, \ldots$, denote the roots of the set of polynomials $\{Q_{rj}(z) : r = 1, \ldots, \ell\}$ that are located inside the unit circle, counting multiplicities. Define

$$c_j(z) = \prod_p \big(z - \alpha_p^{(j)}\big), \qquad D_j(z) = \frac{h_{jj}(z)}{\left|c_j(z)\right|^2}.$$

Note that $D_j(z)$ is nonnegative by construction. We can use Lemma S2 to decompose $D_j(z)$ as

$$D_j(z) = \left|\frac{\Phi_j(z)}{\Psi_j(z)}\right|^2 = \frac{\Phi_j(z)\Phi_j\left(\frac{1}{z}\right)}{\Psi_j(z)\Psi_j\left(\frac{1}{z}\right)}$$

on the unit circle, where we can choose $\Phi_j(z)$ and $\Psi_j(z)$ such that they have no zeros inside the unit disk (when extending the definition of z to the entire complex plane). The second equality follows from the real-coefficients assumption. If the polynomials have complex-valued coefficients, we need to conjugate the coefficients accordingly.

Now set

$$\widetilde{\Gamma}_{rj}(z) = g_{rj}(z)c_j(z)\frac{\Phi_j(z)}{\Psi_j(z)}, \quad r = 1, \ldots, \ell,$$

where $z = e^{-i\omega}$. Continuing this construction for all columns of g, we obtain the desired matrix $\widetilde{\Gamma}(z)$ such that $f_x(\omega) = \widetilde{\Gamma}(e^{-i\omega})\widetilde{\Gamma}^*(e^{-i\omega})$, where all elements of the matrix function

$$\widetilde{\Gamma}(z) = \begin{bmatrix} \widetilde{\Gamma}_{11}(z) & 0 & \ldots & 0 \\ \widetilde{\Gamma}_{21}(z) & \widetilde{\Gamma}_{22}(z) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\Gamma}_{\ell 1}(z) & \widetilde{\Gamma}_{\ell 2}(z) & \ldots & \widetilde{\Gamma}_{\ell\ell}(z) \end{bmatrix}$$

are analytic in the closed unit disk and, hence, in the $\mathbf{H}^2(\mathbb{D})$ space.

Derivations of step 2 In Step 1, we obtain

$$f_x(\omega) = \widetilde{\Gamma}\big(e^{-i\omega}\big)\widetilde{\Gamma}^*\big(e^{-i\omega}\big).$$

The Beurling–Blaschke factorization theorem states that every $\widetilde{\Gamma}(z) \in \mathbf{H}^2(\mathbb{D})$ can be written in the form

$$\widetilde{\Gamma}(z) = \Gamma(z)Q(z), \tag{S4.1}$$

where $Q(z)$ is an $\ell \times \ell$ matrix inner function. The proof of this theorem can be found in Rudin (1986, Theorem 17.17); the matrix generalization of this theorem can be found in Lindquist and Picci (2015, Theorem 4.6.5–4.6.8). The factorization is unique up to constant unitary matrices.[8] Since $\widetilde{\Gamma}(z)$ is rational, the outer function $\Gamma(z)$ is also rational

[8]The conditional uniqueness corresponds only to orthonormal Wold innovations. In fact, given a Wold representation $X_t = \Gamma(L)v_t$, the transformation $X_t = \Gamma(L)\Sigma\Sigma^{-1}v_t$ is also Wold fundamental provided that the constant matrix Σ is invertible. In this case, the Wiener–Hopf formula is modified to contain Σ.

as well. A rational outer function is completely characterized by the location of its zeros. That is, a rational function $\Gamma(z)$ is an outer function if and only if $\det(\Gamma(z)) \neq 0 \; \forall |z| < 1$. Hence, the inner function $Q(z)$ can be reduced to the Blaschke matrices satisfying

$$Q(z) = \prod_{j=1}^{n} \widetilde{B}_j(z) V_j, \tag{S4.2}$$

where $\widetilde{B}_j$ satisfies

$$\widetilde{B}_j(z) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dfrac{z - z_j}{1 - \bar{z}_j z} \end{bmatrix} = B_j^{-1}(z),$$

and z_j are zeros of $\det(Q(z))$ or $\det(\widetilde{\Gamma}(z))$ satisfying $|z_j| < 1$. Here V_j are constant unitary matrices. In other words, the singular part of the rational inner function is absent (see Rudin 1986, Theorem 17.9 and Lindquist and Picci 2015, Theorem 4.6.11). Compared with the general definition of the Blaschke factors, we implicitly assume there are no zeros at $z = 0$ and omit the norm terms $\frac{\bar{z}_j}{|z_j|}$ since finite Blaschke products have no convergence issues. Combining (S4.1) and (S4.2), we have

$$\Gamma(z) = \widetilde{\Gamma}(z) \prod_{j=1}^{n} V_j^{-1} \big[\widetilde{B}_j(z)\big]^{-1} = \widetilde{\Gamma}(z) \prod_{j=1}^{n} V_j^{-1} B_j(z).$$

Note that the Blacheke inner function satisfies $Q(z)Q^*(z) = I \; \forall |z| = 1$ on the unit circle. The spectral density is preserved under the factorization

$$\Gamma(z)\Gamma^*(z) = \widetilde{\Gamma}(z) \prod_{j=1}^{n} V_j^{-1} B_j(z) \prod_{j=1}^{n} B_j^*(z) \big(V_j^{-1}\big)^* \widetilde{\Gamma}^*(z) = f_x(\omega),$$

where $z = e^{-i\omega}$. Moreover, all zeros inside the unit disk are removed because

$$\begin{aligned} \det\big(\Gamma(z)\big) &= \det\big(\widetilde{\Gamma}(z)\big) \prod_{j=1}^{n} \det\big(V_j^{-1}\big) \prod_{j=1}^{n} \frac{1 - \bar{z}_j z}{z - z_j} \\ &= \Upsilon(z) \prod_{j=1}^{n} (z - z_j) \prod_{j=1}^{n} \det\big(V_j^{-1}\big) \prod_{j=1}^{n} \frac{1 - \bar{z}_j z}{z - z_j} \\ &= \Upsilon(z) \prod_{j=1}^{n} \det\big(V_j^{-1}\big) \prod_{j=1}^{n} (1 - \bar{z}_j z) \\ &\neq 0 \quad \forall |z| < 1, \end{aligned}$$

where $\Upsilon(z) = \frac{\det(\widetilde{\Gamma}(z))}{\prod_{j=1}^{n}(z-z_j)}$ has no zeros inside the unit disk by construction. Unfortunately, the right multiplication of the Blaschke matrices also brought poles ($z = z_j$) for the element in the $\widetilde{\Gamma}(z)$ matrix that has no inside zeros. To maintain the analyticity inside the unit disk so that $\Gamma(z) \in \mathbf{H}^2_{\ell\times\ell}(\mathbb{D})$, we need to get rid of these by-product poles. We remove these poles inside the unit disk by setting appropriate constant unitary matrices V_j.

In practice, V_j can be obtained by the singular value decomposition in a sequential procedure. For $j = 1$, we have

$$\Gamma_1(z) = \widetilde{\Gamma}(z)V_1^{-1}B_1(z).$$

Without the constant unitary matrix V_1, the matrix transformation is

$$\widetilde{\Gamma}(z)B_1(z) = \begin{bmatrix} \widetilde{\Gamma}_{11}(z) & 0 & \dots & 0 \\ \widetilde{\Gamma}_{21}(z) & \widetilde{\Gamma}_{22}(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\Gamma}_{\ell 1}(z) & \widetilde{\Gamma}_{\ell 2}(z) & \dots & \widetilde{\Gamma}_{\ell\ell}(z) \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dfrac{1-\bar{z}_1 z}{z-z_1} \end{bmatrix}.$$

It is clear the potential poles can only appear in the last column if we assume that $\widetilde{\Gamma}_{\ell\ell}(z)$ has no zeros at $z = z_1$. To remove this pole, we follows Rozanov (1967) by employing the singular value decomposition (SVD) for $\widetilde{\Gamma}(z)$ at $z = z_1$:

$$\widetilde{\Gamma}(z_1) = U_1 D_1 V_1 = U_1 \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} V_1.$$

By definition, the unitary matrices U_1 and V_1 are given by the (unitary) eigendecomposition

$$G_(z_1) = \widetilde{\Gamma}(z_1)\widetilde{\Gamma}^*(z_1) = U_1\bar{D}_1U_1^*, \qquad \hat{G}_(z_1) = \widetilde{\Gamma}^*(z_1)\widetilde{\Gamma}(z_1) = V_1\hat{D}_1V_1^*.$$

Such decomposition always exists as $G_(z_1)$ and $\hat{G}_(z_1)$ are Hermitian and nonnegative definite by construction. The diagonal matrices $\bar{D}_1$ and $\hat{D}_1$ contain eigenvalues of $G_(z_1)$ and $\hat{G}_(z_1)$, which are not necessarily distinct. The diagonal matrix D_1 in the SVD contains the singular values of $\widetilde{\Gamma}(z)$. The nonzero singular values $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$ are the square root of the nonzero eigenvalues of $G_(z_1)$ and $\hat{G}_(z_1)$, which are not necessarily distinct. Since we know that $\det(\widetilde{\Gamma}(z_1)) = 0$,

$$\det(G_(z_1)) = \det\big(\widetilde{\Gamma}(z_1)\big)\det\big(\widetilde{\Gamma}(z_1)^*\big) = 0.$$

Therefore, there exists at least one singular value in D_1 that is zero, i.e., $p < d$.[9] Now evaluate $\Gamma_1(z)$ at $z = z_1$:

$$\begin{aligned}\Gamma_1(z_1) &= \widetilde{\Gamma}(z_1)V_1^{-1}B_1(z_1) = U_1 D_1 V_1 V_1^{-1} B_1(z_1)\\ &= U_1 \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dfrac{1-\bar{z}_1 z_1}{z_1 - z_1} \end{bmatrix}.\end{aligned}$$

Since the last column of D_1 is identically zero, the pole at $\frac{1-\bar{z}_1 z_1}{z_1 - z_1}$ vanishes at $z = z_1$. In other words, $\Gamma_1^{(i,j)}(z_1) < \infty$ are all well defined without poles. Alternatively, condition (S4.3) ensures that zeros at $z = z_1$ are removed as well.

Now consider the second step $j = 2$:

$$\Gamma_2(z) = \Gamma_1(z)V_2^{-1}B_2(z).$$

Without the constant unitary matrix V_2,

$$\Gamma_1(z)B_2(z) = \Gamma_1(z) \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dfrac{1-\bar{z}_2 z}{z - z_2} \end{bmatrix}$$

would have poles in the last column. Note that $\Gamma_1(z)$ is no longer lower triangular after the first step transformation. To remove these poles at $z = z_2$, we employ the SVD again,

$$\begin{aligned}\Gamma_2(z_2) &= \Gamma_1(z_2)V_2^{-1}B_1(z_2) = U_2 D_2 V_2 V_2^{-1} B_2(z_2)\\ &= U_2 \begin{bmatrix} \widetilde{\lambda}_1 & 0 & \dots & 0 \\ 0 & \widetilde{\lambda}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dfrac{1-\bar{z}_2 z_2}{z_2 - z_2} \end{bmatrix},\end{aligned}$$

where $\{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_{\widetilde{p}}\}$ are the nonzero singular values. Again, there exists at least one zero in the diagonal of the D_2 matrix ($\widetilde{p} < d$), since $\det(\Gamma_1(z_2)) = 0$. Arranging the zeros in the last positions of the diagonal, it follows immediately that $\Gamma_2^{(i,j)}(z_1) < \infty$ are all well defined without poles, since the last column of D_2 is identically zero and the poles introduced by $\frac{1-\bar{z}_2 z_2}{z_2 - z_2}$ vanish.

[9]The rank loss generally depends on the multiplicity of zeros in $\det(\widetilde{\Gamma}(z_1))$.

Continuing this sequential procedure for all z_j, it follows that $\Gamma(z)$ is analytic (componentwise) at $z = \{z_1, z_2, \dots, z_n\}$ inside the unit disk. By (S4.3), we conclude that $\Gamma(z)$ is indeed a Wold (outer) spectral factor. The underlying construction can be trivially extended to the case with higher-order zeros; see Rozanov (1967, p. 47). In particular, the location of the Blaschke factor $\frac{1-\bar{z}_j z}{z-z_j}$ (along the diagonal) is inconsequential as long as we put the zero in the corresponding diagonal position of D_j.

A working example of a 2×3 *signal system* To illustrate the use of our method, we consider an alternative specification of the 2×3 signal system. Let the signal representation be

$$X_{it} = H(L)\eta_{it} \equiv \begin{bmatrix} \dfrac{1}{1-\rho_a L} & 1 & 0 \\ F(L) & 0 & F(L) \end{bmatrix} \begin{bmatrix} \epsilon_{at} \\ \epsilon_{it} \\ \epsilon_{ut} \end{bmatrix},$$

where $F(z)$ is some an outer function in $\mathbf{H}^2(\mathbb{D})$.

Step 1. The spectral density $f_x(\omega)$ is given by

$$f_x(\omega) \equiv \begin{bmatrix} \dfrac{1}{(1-\rho_a z)\big(1-\rho_a z^{-1}\big)}\sigma_a^2 + \sigma_i^2 & \dfrac{F\big(z^{-1}\big)}{(1-\rho_a z)}\sigma_a^2 \\ \dfrac{F(z)}{\big(1-\rho_a z^{-1}\big)}\sigma_a^2 & F(z)F\big(z^{-1}\big)\big[\sigma_a^2 + \sigma_u^2\big] \end{bmatrix},$$

where $z = e^{-i\omega}$. The leading principal minors are given by

$$M_1(\omega) = f_{11}(\omega) = \frac{(1-\lambda_w z)\big(1-\lambda_w z^{-1}\big)}{(1-\rho_a z)\big(1-\rho_a z^{-1}\big)}\sigma_w^2$$

$$M_2(\omega) = \det\big(f_x(\omega)\big) = \frac{F(z)F\big(z^{-1}\big)}{(1-\rho_a z)\big(1-\rho_a z^{-1}\big)}\big[\sigma_g^2(1-\lambda_w)\big(1-\lambda_w z^{-1}\big) - \sigma_a^4\big],$$

where we define $\sigma_p^2 = \sigma_a^2 + \sigma_u^2$ and $\sigma_g^2 = \sigma_w^2\sigma_p^2$, $\lambda_w \in (0,1)$. Using Lemma S2,

$$\sigma_g^2(1-\lambda_w)\big(1-\lambda_w z^{-1}\big) - \sigma_a^4 = \sigma_h^2(1-\lambda_h z)\big(1-\lambda_h z^{-1}\big).$$

The new parameters σ_h and λ_h satisfy $\lambda_h = \frac{\lambda_w \sigma_g^2}{\sigma_h^2}$ and $\sigma_h^2(1+\lambda_h^2) = \sigma_g^2(1+\lambda_w^2) - \sigma_a^4$. In particular, we can pick a real $\lambda_h \in (0,1)$. Then the spectral density admits the decomposition

$$f_x(\omega) = \begin{bmatrix} 1 & 0 \\ g_{21}(\omega) & 1 \end{bmatrix} \begin{bmatrix} h_{11}(\omega) & 0 \\ 0 & h_{22}(\omega) \end{bmatrix} \begin{bmatrix} 1 & g_{21}^*(\omega) \\ 0 & 1 \end{bmatrix}.$$

The diagonal elements h_{11} and h_{22} are given by

$$h_{11}(\omega) = M_1(\omega); \qquad h_{22}(\omega) = \frac{M_2(\omega)}{M_1(\omega)}.$$

In addition, we use the recursion formula to get $g_{21}(\omega) = \frac{g_{21}^{(1)}}{h_1 1} = \frac{f_{21}}{h_{11}}$. Therefore,

$$g_{21}(\omega) = \frac{\sigma_a^2}{\sigma_w^2} \frac{F(z)(1-\rho_a z)}{(1-\lambda_w z)\left(1-\lambda_w z^{-1}\right)}.$$

Now fix the first column $j = 1$. We know the only inside pole is at $z = \lambda_w$ in g_{21}. This implies

$$C_1(z) = (z - \lambda_w), \qquad D_1(z) = \frac{h_{11}(z)}{\left|C_1(z)\right|^2} = \left|\frac{\Phi_1(z)}{\Psi_1(z)}\right|^2.$$

Hence $\frac{\Phi_1(z)}{\Psi_1(z)} = \frac{\sigma_w}{1-\rho_a z}$. This in turn implies

$$\tilde{\Gamma}_{11}(z) = g_{11} C_1(z) \frac{\Phi_1(z)}{\Psi_1(z)} = \sigma_w \frac{z-\lambda_w}{1-\rho_a z}, \qquad \tilde{\Gamma}_{21}(z) = g_{21} C_1(z) \frac{\Phi_1(z)}{\Psi_1(z)} = \frac{\sigma_a^2}{\sigma_w} \frac{F(z) z}{(1-\lambda_w z)}.$$

We repeat this procedure for the second column. Notice that the second column of g is constants; therefore, $C_2(z) = 1$ and $\frac{\Phi_2(z)}{\Psi_2(z)} = \frac{\sigma_h}{\sigma_w} \frac{F(z)(1-\lambda_h z)}{(1-\lambda_w z)}$. In the end, we obtain the lower triangular matrix

$$\tilde{\Gamma}(z) = \begin{bmatrix} \sigma_w \dfrac{z-\lambda_w}{1-\rho_a z} & 0 \\ \dfrac{\sigma_a^2}{\sigma_w} \dfrac{F(z) z}{(1-\lambda_w z)} & \dfrac{\sigma_h}{\sigma_w} \dfrac{F(z)(1-\lambda_h z)}{(1-\lambda_w z)} \end{bmatrix}.$$

Clearly, $\tilde{\Gamma}(z) \in \mathbf{H}^2_{2\times 2}(\mathbb{D})$.

Step 2. We remove the inside zeros at $z = \lambda_w$ to achieve the Wold fundamental representation. Using the Blaschke factorization, we have $\Gamma(z) = \tilde{\Gamma}(z) V_1^{-1} B(z)$, where

$$B(z) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1-\lambda_w z}{z-\lambda_w} \end{bmatrix}$$

and V_1 satisfies the unitary eigendecomposition of $\hat{G}(\lambda_w) = \widetilde{\Gamma}^*(\lambda_w)\widetilde{\Gamma}(\lambda_w) = V_1 \hat{D}_1 V_1^*$. It is easy to check that eigenvalues of the Hermitian matrix $\hat{G}(\lambda_w)$ are distinct. Therefore, we can pick two eigenvectors from the two eigenvalues, which are necessarily orthogonal by the spectral theorem. Normalizing these two eigenvectors yields the unitary matrix as desired,

$$V_1 = \begin{bmatrix} \sqrt{\dfrac{h^2}{1+h^2}} & \sqrt{\dfrac{1}{1+h^2}} \\ \sqrt{\dfrac{1}{1+h^2}} & -\sqrt{\dfrac{h^2}{1+h^2}} \end{bmatrix},$$

where $h = \frac{\sigma_a^2}{\sigma_h} \frac{\lambda_w}{(1-\lambda_h \lambda_w)}$. The resulting matrix $\Gamma(z)$ is the Wold fundamental matrix

$$\Gamma(z) = \begin{bmatrix} \sigma_w \dfrac{z-\lambda_w}{1-\rho_a z} V_1^{(11)} & \sigma_w \dfrac{1-\lambda_w z}{1-\rho_a z} V_1^{(12)} \\ F(z) \dfrac{\sigma_h V_1^{(12)}}{\sigma_w} & F(z) \dfrac{V_1^{(12)} \sigma_a^2}{\sigma_w (1-\lambda_h \lambda_w)} \end{bmatrix}.$$

Finally, we can transform $\Gamma(z)$ into an upper triangular form by right multiplication of another unitary matrix V_2,

$$V_2 = \begin{bmatrix} \sqrt{\dfrac{1}{1+x^2}} & \sqrt{\dfrac{x^2}{1+x^2}} \\ -\sqrt{\dfrac{x^2}{1+x^2}} & \sqrt{\dfrac{1}{1+x^2}} \end{bmatrix},$$

where $x = \frac{\sigma_h (1-\lambda_h \lambda_w)}{\sigma_a^2}$. After some algebraic simplifications, we obtain

$$\Gamma(z) = \begin{bmatrix} \dfrac{\sigma_h}{\sigma_p} \dfrac{1-\lambda_h z}{1-\lambda_w z} & \dfrac{\sigma_a^2}{\sigma_p} \dfrac{1}{1-\rho_a z} \\ 0 & F(z) \sigma_p \end{bmatrix}.$$

Since we assume $F(z)$ is outer, i.e., has no roots in the open unit disk, $\Gamma(z)$ is the Wold representation.

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