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When is a monotone function cyclically monotone?

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We provide sufficient conditions for a monotone function with a finite set of outcomes to be cyclically monotone. Using these conditions, we show that any monotone function defined on the domain of gross substitutes is cyclically monotone. The result also extends to the domain of generalized gross substitutes and complements.

Keywords. Monotone, cyclically monotone, nonconvex domain, gross substitutes, gross substitutes and complements, mechanism design, algebraic topology, homology, nerve theorem.

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1. Introduction

One of the major goals of mechanism design is to study the properties of optimal mechanisms that maximize a given objective such as revenue or welfare maximization. The difficulty in deriving such mechanisms results from the designer lacking information about the agents' preferences. Hence, a well designed mechanism should take into account the agents' ability to hide their privately held information, often called *incentive compatibility constraints.*

Previous work provides important insights into these constraints. Myerson (1981) showed that in standard private value settings with one-dimensional types, any *nondecreasing* allocation rule can be implemented; that is, there exists a payment rule that when combined with the allocation rule produces a direct mechanism where truthtelling is in the best interests of the agents. In multidimensional settings, Rochet (1987) showed that an allocation rule is implementable if and only if it is *cyclically monotone*.

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To define a cyclically monotone allocation rule $f:\mathbb{R}^N \to \mathbb{R}^N$, $N \geq 1$, consider a weighted graph with points in the domain of f being vertices and directed edges from any point t to any other point t'. With each directed edge, we associate weight $t \cdot (f(t) - f(t'))$. Allocation f is then cyclically monotone if along any cycle the sum of edge weights is nonnegative. If the weight of any cycle with two edges is nonnegative, the allocation rule satisfies a weaker condition called *monotone*.

Though Rochet's characterization and its modifications have been successfully used in auction theory, computer science, and matching theory, it is often tedious to verify.[1] An important contribution by Saks and Yu (2005) (also Bikhchandani et al. 2006) establishes an equivalence between cyclically montone and monotone conditions for convex domains with a finite set of outcomes. Their result greatly simplifies checking whether an allocation rule is implementable.[2]

Does the equivalence result extend to nonconvex domains? Ashlagi et al. (2010) showed that Saks and Yu's result cannot be extended beyond domains with convex closure if one requires equivalence between two conditions for every finite-valued randomized allocation. However, randomized allocations might not always be plausible. Further, Ashlagi et al.'s (2010) result does not preclude equivalence between monotone and cyclically monotone conditions on nonconvex domains for a given set of possible outcomes.

In this paper, we provide sufficient conditions on a domain, on a set of possible outcomes, and on a function that guarantee that if the function is monotone, then it is also cyclically monotone. Our two main conditions require the domain to be simply connected and the function to satisfy the local-to-global condition. The former condition ensures that the domain does not contain "holes" of a certain type. The latter condition ensures that if a function is a solution to a local optimization problem, it also delivers the global optimal.[3]

We then apply these conditions to study deterministic demand functions on the domain of gross substitutes. The domain of gross substitutes is an important nonconvex domain of agent preferences that has been extensively exploited in the mechanism design, matching, equilibrium, and algorithmic literatures (e.g., Ausubel and Milgrom 2002, Roth 1991, Gul and Stacchetti 1999, Paes Leme 2017). We establish that any monotone demand function defined on the domain of gross substitutes is also cyclically monotone. We further extend the equivalence between monotone and cyclically monotone conditions to the domain of generalized gross substitutes and complements, the domain that allows for multiple objects of the same type and some complementarities across objects (Sun and Yang 2006, Shioura and Yang 2015).

On the methodological side, we introduce some novel techniques to economics. The proof of our main result uses a version of the nerve theorem—a classical result in algebraic topology (see Björner 1995). To explain the result, let us consider a set that is covered by a finite system of closed subsets. Nerve is then a special weighted hypergraph

[1] See Lavi and Swamy (2009), Mishra and Roy (2013), and Carbajal and Mu'alem (2020). The cyclic monotonicity condition also has applications in revealed preference theory, producer theory, and spatial allocation (see Chambers and Echenique 2018, Kushnir and Lokutsievskiy 2019).

[2] The result with a similar flavor for environments without transfers also appeared in Pycia (2012).

[3] As explained later, we also require an additional technical condition.

associated with this system of subsets. The nerve theorem helps to map the geometrical properties of the set to the geometrical properties of the nerve. This result could be of special interest to economists working in the areas of mechanism design, social choice, network theory, and operations research.

Related Literature. The cyclic monotonicity condition was introduced by Rockafellar (1966) to characterize the subdifferentials of convex functions. For mechanism design applications, Rochet (1987) was the first to show that in quasilinear environments an allocation rule can be implemented if and only if it is cyclically monotone. He also drew a parallel between the cyclic monotonicity condition and the strong axiom of revealed preferences (see also Brown and Calsamiglia 2007, Makowski and Ostroy 2013). Saks and Yu (2005) simplified the characterization of implementable allocation rules by establishing that any monotone function is cyclically monotone on convex domains with a finite set of outcomes (see also Bikhchandani et al. 2006).[4] Importantly, Ashlagi et al. (2010) showed that Saks and Yu's characterization cannot be extended beyond domains with convex closure if the equivalence is required to hold for all finite-valued randomized allocation rules. For an infinite set of outcomes, Müller et al. (2007), Archer and Kleinberg (2014), and Carbajal and Müller (2015, 2017) provided various additional conditions to guarantee that the cyclical monotonicity condition is satisfied.

For nonconvex domains, the literature is scarce. For single-peaked preferences, Mishra et al. (2014) showed the equivalence of monotone and cyclically monotone conditions. Vohra (2011) provided an inspiring example of a simple domain with two objects and the agent's valuation of a bundle of objects equals the maximum value of objects in the bundle. For the setting, he established that any monotone function is also cyclically monotone. This is an example of a domain where every valuation satisfies the gross substitutes condition. Ever since, it has been an open question as to whether the equivalence between the monotone and cyclically monotone conditions can be extended to the whole domain of gross substitutes.[5]

The gross substitutes condition was introduced by Kelso and Crawford (1982) in the context of labor matching markets. Sun and Yang (2006, 2009) and Shioura and Yang (2015) extended the gross substitutes condition to allow for some complementarities and multiple objects of the same type, a domain they referred to as generalized gross substitutes and complements. They also designed a dynamic auction for efficiently allocating the objects to the agents. Our most general results in Section 4 apply to the latter domain.

Our main theorem exploits the local-to-global condition that relates local incentive compatibility to global incentive compatibility constraints in convex and nonconvex domains. This condition is closely connected to the decomposition monotonicity

[4]Jehiel et al. (1999) also contains the proof a geometric lemma that is the main step in Saks and Yu (2005). See also Cuff et al. (2012) and Edelman and Weymark (forthcoming) for the cases when every monotone function is cyclically monotone.

[5]The only progress in that direction was made in a concurrent paper by Agarwal and Roy (2019), who extended Vohra's (2011) example to the case of an arbitrary number of objects.

Mishra and Roy (2013) also showed that the nonnegativity of any three-cycle is sufficient for implementability in dichotomous domains. The conditions of Agarwal and Roy (2015, 2017) also apply formally to nonconvex domains.

condition first proposed by Müller et al. (2007) to study Bayesian incentive compatible allocation rules on convex domains.[6] In a related paper, Archer and Kleinberg (2014) considered convex domains and showed that if a function with a finite or infinite set of outcomes is locally monotone and its loop is integral over every sufficiently small triangle vanishes, then it is also incentive compatible. Carroll (2012) also thoroughly studied local and global incentive compatibility constraints. He showed that local incentive compatibility always implies global incentive compatibility for convex domains *with transferable utility*, the single-peaked preference domain, and the single-crossing domain *without transferable utility*.[7] Though Carroll (2012) did not study nonconvex domains with transferable utilities, one of his geometric characterizations has proved to be very useful for our purposes (see Section 3).

One of our results (Proposition 1) is also closely related to the Helmholtz decomposition of Jiang et al. (2011). Candogan et al. (2011) used these techniques to decompose any finite game into potential, harmonic, and nonstrategic components. In a recent paper, Caradonna (2020) also used the decomposition to analyze when the weak axiom of revealed preferences implies the rationalizability of choice functions.

The paper proceeds as follows. Section 2 introduces notation and definitions. Section 3 presents our main results. We use these results in Section 4 to study functions defined on the domain of gross substitutes and the domain of generalized gross substitutes and complements. Section 5 concludes the paper.

2. Notation and definitions

We begin by introducing some notation and definitions. Then we motivate them from the perspective of mechanism design. Consider a domain $T \subseteq \mathbb{R}^N$, a finite set $A \subset \mathbb{R}^N$ for $N \geq 1$, and some function $f : T \to A$. The vector product of $t \in T$ and $a \in A$ is denoted as both $t \cdot a$ and ta. We consider two monotonicity conditions.

Definition 1. Function $f : T \to A$ is *monotone* if for all $t, t' \in T$,

$$t\big(f(t) - f(t')\big) + t'\big(f(t') - f(t)\big) \geq 0. \tag{1}$$

This is a generalization of the one-dimensional monotonicity condition to multidimensional settings. We use the term "monotone function" following Rockafellar (1966). Some recent papers also call such functions weakly monotone (see, e.g., Bikhchandani et al. 2006). Our second and more demanding condition is defined as follows.

Definition 2. Function $f : T \to A$ is *cyclically monotone* if for any integer M and any points $t^0, t^1, \dots, t^M = t^0$ in T,

$$\sum_{k=0}^{M-1} t^k\big(f(t^k) - f(t^{k+1})\big) \geq 0. \tag{2}$$

[6]See also Berger et al. (2009, 2017). The condition is also related to the reverse triangle inequality in Mishra et al. (2014).

[7]See also Gibbard (1977), Mishra et al. (2016), Pycia and Ünver (2010), and Sato (2013) for related results.

As we mentioned in the Introduction, both definitions could be conveniently interpreted using graph theory. Consider a weighted graph with points $t \in T$ being vertices and directed edges from any point t to any other point t'. With each edge, we associate weight $t(f(t) - f(t'))$. Hence, if f is cyclically monotone, then the weight of any cycle has to be nonnegative. If f is monotone, then the above condition is restricted to cycles of length 2. Note that to check whether f is monotone, we need to verify only inequality (1). At the same time, we need to verify a system of inequalities (2) for all integers M to check whether f is cyclically monotone. The latter is a much more demanding task.

For any $f : T \to A$, we also consider a cover of T by a finite number of subsets. To define these subsets for any ordered pair $a, b \in A$, we define lower bound

$$\ell_{ab} = \inf_{t \in T : f(t) = a} t(a - b).$$

Using the lower bounds, we construct a cover $\{T_a^f\}_{a \in A}$ of set T, where for each $a \in A$,

$$T_a^f = \{t \in T : t(a - b) \geq \ell_{ab} \ \forall b \in A\}.$$

Note that T_a^f depends on the choice of function f and $T = \bigcup_{a \in A} T_a^f$. In addition, if outcome $a \notin f(T)$, then we have $\ell_{ab} = +\infty$ and T_a^f is the empty set. For all other outcomes, each set T_a^f is nonempty and contains the set of points that leads to outcome a, i.e., $\{t \in T, f(t) = a\} \subseteq T_a^f$.

In our analysis, we study functions that lead to path-connected subsets T_a^f and functions that are defined on a simply connected domain T. Set T_a^f is *path-connected* if it is nonempty and any two points $x \in T_a^f$ and $y \in T_a^f$ can be connected with a continuous curve lying inside T_a^f. A domain T is *simply connected* if it is path-connected and any loop in T can be continuously contracted to a point.[8] For example, a triangle without an interior is not simply connected (see Example 1). At the same time, any set with a point that can be connected to each of the set's other points with a line segment within the set is simply connected. Such a set is called *star-shaped* (or *star-convex*).

For a closed line segment connecting two points $x, y \in \mathbb{R}^N$, we use the standard notation $[x, y] = \{z \in \mathbb{R}^N : z = \alpha x + (1 - \alpha)y, \alpha \in [0, 1]\}$. We also employ $[x, y)$, $(x, y]$, and (x, y) throughout the paper, depending on whether the boundary points are included. Finally, the following property is helpful in our analysis.

Definition 3. Function $f : T \to A$ satisfies the *local-to-global* condition if for any two outcomes $a, b \in f(T)$ with $T_a^f \cap T_b^f = \varnothing$, there exists a path $\{a \equiv a_0, \dots, a_M \equiv b\}$ such that $T_{a_m}^f \cap T_{a_{m+1}}^f \neq \varnothing$, $m = 0, \dots, M-1$, and $\ell_{ab} \geq \sum_{m=0}^{M-1} \ell_{a_m a_{m+1}}$.

The local-to-global condition can be most accurately interpreted through the prism of mechanism design and we slightly postpone its discussion to the end of this section.

[8]A formal definition of domain T being simply connected is as follows. Let S^1 denote a circle (in $\mathbb{R}^2$). Then, for any continuous function (a loop) $\gamma : S^1 \to T$, there must exist a continuous function $F : [0, 1] \times S^1 \to T$ such that for all $s \in [0, 1]$, $F(0, s) \equiv \gamma(s)$ and $F(1, s) \equiv t_0$ for some $t_0 \in T$.

Here, we mention only one of our results related to the condition. At first glance, verifying the local-to-global condition might require a significant effort. However, we show that it is not the case. We present a sufficient geometric property that ensures that a monotone function f satisfies the local-to-global condition (see Lemma A1). In particular, the geometric property requires that for any $a, b \in f(T)$, any $x \in T_a^f$, there should exist type $y \in T_b^f$ such that $[x, y] \subset T$. In Section 4, we show that the geometric property is satisfied for any monotone function defined on some important economic domains.

Mechanism design. For mechanism design applications, one could think of N as the number of outcomes and think of T as the set of agent types. Type $t \in T$ can be interpreted as a vector of agent's valuations for all possible outcomes. With each outcome, we associate an indicator $a \in \{0, 1\}^N$ that has one component equal to 1 and all other components equal to 0. The union of these indicators is then a finite set $A \subset \mathbb{R}^N$. The agent's utility from $a \in A$ can conveniently be written then as $u(t, a, p) = t \cdot a - p$, where p is the agent's payment.

We consider direct mechanisms characterized by two functions: an allocation rule, $f : T \to A$, mapping an agent's reported type to the set of possible outcomes, and a payment rule, $p : T \to \mathbb{R}$, mapping an agent's reported type to the set of real numbers. We consider only deterministic allocation rules and do not allow randomizations over outcomes.[9] We can then write the agent's utility as

$$tf(t') - p(t'),$$

where t' and t refer to the agent's reported and true types, respectively. We call allocation rule f *implementable* if there exists a payment rule p such that mechanism (f, p) is incentive compatible; that is, if it satisfies the constraints

$$tf(t) - p(t) \geq tf(t') - p(t') \quad \forall t, t' \in T.$$

Rochet (1987) proved an important result that characterizes the set of implementable allocations as stated in the following theorem.

Rochet's Theorem (1987). *An allocation rule is implementable if and only if it is cyclically monotone.*

Though the cyclic monotonicity condition characterizes the set of all implementable allocation rules, this condition is often tedious to verify. Remarkably, Saks and Yu (2005) showed that for convex domains it is enough to check that only two cycles are nonnegative. Saks and Yu's (2005) characterization and its modifications have been successfully used in several important applications (see Lavi and Swamy 2009, Mishra and Roy 2013, Carbajal and Mu'alem 2020, Shi et al. 2018). One of our main results extends Saks and Yu's (2005) result to important nonconvex domains, including the domain of gross substitutes (see Section 4).

Before proceeding to our main results, we discuss the interpretation of the lower bounds ℓ_{ab} and the local-to-global condition (Definition 3). Let us consider all agent

[9]See Ashlagi et al. (2010) for the study of randomized mechanisms.

types that lead to outcome $a \in A$. Lower bound ℓ_{ab} then corresponds to the lowest benefit from revealing its true type compared to lying when lying leads to outcome $b \in A$ (excluding transfers). For a monotone allocation, set T_a^f almost coincides with the set of types that lead to outcome $a \in A$ (up to the boundary points).

The local-to-global condition can then be interpreted as a condition that ensures that local incentive compatibility implies global incentive compatibility (see Archer and Kleinberg 2014, Carroll 2012). Lower bound ℓ_{ab} is the lowest benefit from revealing true type compared to lying when lying leads to outcome b. Hence, $-\ell_{ab}$ can be regarded as the maximum gains from lying. We interpret $-\ell_{ab}$, $T_a^f \cap T_b^f = \varnothing$, as the gains from global deviations and interpret $-\ell_{ab}$, $T_a^f \cap T_b^f \neq \varnothing$, as the gains from local deviations. Then the local-to-global condition ensures that the gains from global deviations are smaller than the total gains from deviations along the path connecting $t \in T_a^f$ and some type in T_b^f; i.e., *local incentive compatibility* implies *global incentive compatibility*. Similar conditions are considered in previous literature (see Müller et al. 2007, Mishra et al. 2014). The main difference is that our condition is also applicable to nonconvex domains.

3. Main result

The main result of this section, Theorem 1, provides a set of conditions on a domain T, a set A, and a function $f : T \to A$ that ensure that if f is monotone, it is also cyclically monotone. These conditions are simplified in Corollary 1. The result of Corollary 1 is then used to analyze monotone demand functions on the domain of gross substitutes and the domain of generalized gross substitutes and complements in Section 4.

Theorem 1 (Main result). *Consider a domain $T \subset \mathbb{R}^N$, a finite set $A \subset \mathbb{R}^N$, and a function $f : T \to A$. Suppose that*

(i) T is simply connected

(ii) T_a^f is either path-connected or empty for each $a \in A$

(iii) f satisfies the local-to-global condition.

Then if f is monotone, it is also cyclically monotone.

Discussion. The *simply connected* condition is satisfied for most economically relevant models.[10] It ensures that domain T does not contain "holes" of a certain type. The condition on sets T_a^f is technical. For instance, any star-shaped set is path-connected. To check that a monotone function f satisfies the local-to-global condition, we show that it is enough for f to satisfy the following geometric property: for any $a, b \in f(T)$ and any $x \in T_a^f$, there should exist type $y \in T_b^f$ such that $[x, y] \subset T$ (see Lemma A1). This condition was originally proposed by Carroll (2012) to show that local incentive compatibility implies global incentive compatibility in single-peaked preferences settings *without transfers*.

[10]One exception is a circular domain in monopolistic competition models (Salop 1979).

Proof of Theorem 1. First, we establish that it is sufficient to prove the statement for f such that $f(T) = A$. Indeed, if we prove the statement under this assumption, then for an arbitrary $f : T \to A$ that satisfies the theorem conditions, we put $A' = f(T)$ and apply the established result for $f : T \to A'$. Hence, without loss of generality, we assume from this point on that $f(T) = A$ and, hence, $T_a^f \neq \varnothing$ for each $a \in A$.

The proof of Theorem 1 is based on graph theory and algebraic topology. We associate two graphs with set A. The first graph Γ is the complete directed graph with vertices corresponding to each outcome in A and directed edges connecting every ordered pair of vertices. To distinguish vertices from outcomes, for each $a, b, c \in A$, we denote the corresponding vertices in Γ by Gothic letters $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$, respectively, and denote the set of vertices by A_0. Each directed edge $\mathfrak{a} \to \mathfrak{b}$ in Γ has weight equal to ℓ_{ab}.

The second graph Γ^n is a subgraph of Γ with the same set of vertices A_0, but directed edges connecting only adjacent outcomes; i.e., $\mathfrak{a}$ and $\mathfrak{b}$ are connected (by both directed edges) in Γ^n if and only if $T_a^f \cap T_b^f \neq \varnothing$. The subgraph's directed edges still have weight ℓ_{ab}. We call graph Γ^n a *neighborhood subgraph.*

We first notice that Definitions 1 and 2 can be reformulated using the weights of cycles in Γ.[11] We say that Γ is cyclically monotone if any M-cycle $\mathfrak{a}_0 \to \mathfrak{a}_1 \to \cdots \mathfrak{a}_{M-1} \to \mathfrak{a}_M \equiv \mathfrak{a}_0$ with $\mathfrak{a}_i \in A_0$ for $i = 0, \dots, M-1$ and $M \geq 2$ has nonnegative weight, i.e., $\sum_{m=0}^{M-1} \ell_{a_m a_{m+1}} \geq 0$. We also say that Γ is monotone if any 2-cycle has nonnegative weight. It is straightforward to verify that f is monotone (cyclically monotone) if and only if Γ is monotone (cyclically monotone) (e.g., Heydenreich et al. 2009). Therefore, to prove that f is cyclically monotone, it is enough to establish that Γ is cyclically monotone.

To prove that Γ is cyclically monotone, we first establish that all cycles in the neighborhood subgraph Γ^n have exactly zero weight using the conditions that domain T is *simply connected* and sets T_a^f are *path-connected* for $a \in A$. Then we show that the *local-to-global* condition implies that all cycles in Γ have nonnegative weight.

Proposition 1. *If conditions (i) and (ii) are satisfied and f is monotone, then any cycle in the neighborhood subgraph Γ^n has exactly zero weight.*

Proof. We first establish two simple facts about subgraph Γ^n. Consider some directed edge $\mathfrak{a} \to \mathfrak{b}$ in Γ^n. As f is monotone, graph Γ is also monotone and $\ell_{ba} + \ell_{ab} \geq 0$. At the same time, as $\mathfrak{a} \to \mathfrak{b}$ is in Γ^n, there exists $t \in T_a^f \cap T_b^f$, and $t(a-b) \geq \ell_{ab}$ and $t(b-a) \geq \ell_{ba}$ by definition of T_a^f and T_b^f. Hence, $\ell_{ab} + \ell_{ba} \leq 0$. Overall, we have $\ell_{ba} + \ell_{ab} = 0$. Hence, we obtain the following fact.

Fact 1. *If $T_a^f \cap T_b^f \neq \varnothing$, then $\ell_{ab} = -\ell_{ba}$ and $\ell_{ab} = t(a-b)$ for any $t \in T_a^f \cap T_b^f$.*

Second, let us show that if $T_a^f \cap T_b^f \cap T_c^f \neq \varnothing$, then the 3-cycle $\mathfrak{a} \to \mathfrak{b} \to \mathfrak{c} \to \mathfrak{a}$ in Γ^n has 0 weight. Indeed, for $t \in T_a^f \cap T_b^f \cap T_c^f$, we have $\ell_{ab} = t(a-b)$, $\ell_{bc} = t(b-c)$ and $\ell_{ca} = t(c-a)$ by Fact 1. Hence, $\ell_{ab} + \ell_{bc} + \ell_{ca} = 0$. This result is summarized below.

[11] The weight of a cycle (or any path) in a graph is the sum of the weights of its directed edges.

FACT 2. *If* $T_a^f \cap T_b^f \cap T_c^f \neq \varnothing$, *then* $\ell_{ab} + \ell_{bc} + \ell_{ca} = 0$.

We now investigate whether it is possible to "pave" any cycle in Γ^n by triplets that satisfy Fact 2. For example, if $T_a^f \cap T_b^f \cap T_c^f \neq \varnothing$ and $T_a^f \cap T_{b'}^f \cap T_c^f \neq \varnothing$, then 4-cycle $\mathfrak{a} \to \mathfrak{b} \to \mathfrak{c} \to \mathfrak{b}' \to \mathfrak{a}$ has zero weight. Indeed, Facts 1 and 2 imply that

$$\ell_{ab} + \ell_{bc} + \ell_{cb'} + \ell_{b'a} = (\ell_{ab} + \ell_{bc} + \ell_{ca}) + (\ell_{ac} + \ell_{cb'} + \ell_{b'a}) = 0.$$

For the general case, we consider a construction in topology called the *nerve* of a cover $\bigcup_{a \in A} T_a^f$.[12] Nerve $\mathcal{N} = (A_0, A_1, A_2, \dots)$ of the cover $T = \bigcup_{a \in A} T_a^f$ is formally composed of vertices A_0, edges A_1, triangles A_2, and their k-dimensional counterparts A_k defined as follows:

- Set A_0 consists of *vertices* $\mathfrak{a}$ corresponding to sets $T_a^f \neq \varnothing$, $a \in A$.
- Set A_1 consists of unordered pairs $\{a, b\}$ (where a and b are different), such that $T_a^f \cap T_b^f \neq \varnothing$. The elements of A_1 are called *edges* in $\mathcal{N}$.
- Set A_2 consists of unordered triples $\{a, b, c\}$ (where a, b, and c are different) such that $T_a^f \cap T_b^f \cap T_c^f \neq \varnothing$. The elements of A_2 are called *triangles* in $\mathcal{N}$.
- Set A_k for $k \geq 2$ is defined similarly. The elements of A_k for any k are generally called *simplices* in $\mathcal{N}$.

Nerve $\mathcal{N}$ is usually identified with its geometrical realization, which is a polytope $P(\mathcal{N})$ in $\mathbb{R}^{A_0}$, where the elements of A_0 form a basis and linear space $\mathbb{R}^{A_0}$ consists of formal sums $\sum_{\mathfrak{a} \in A_0} x_{\mathfrak{a}} \cdot \mathfrak{a}$ with $x_{\mathfrak{a}} \in \mathbb{R}$. The vertices of $P(\mathcal{N})$ are the endpoints of the basis vectors in $\mathbb{R}^{A_0}$. Two vertices $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbb{R}^{A_0}$ are connected by a segment in $P(\mathcal{N})$ if and only if $T_a^f \cap T_b^f \neq \varnothing$ (i.e., $\{a, b\} \in A_1$). Three vertices $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ are the extreme points of a triangle face in $P(\mathcal{N})$ if and only if $T_a^f \cap T_b^f \cap T_c^f \neq \varnothing$ (i.e., $\{a, b, c\} \in A_2$), etc. Overall, $P(\mathcal{N})$ is contained in the standard simplex $\{\sum_{\mathfrak{a} \in A_0} x_{\mathfrak{a}} = 1, x_{\mathfrak{a}} \geq 0\}$.

At this point, we are able to explain the main idea of the proof. We first show that the geometrical properties of domain T imply that polytope $P(\mathcal{N})$ has a special structure. In particular, Lemma 1 below establishes that conditions (i) and (ii) of Theorem 1 imply that $P(\mathcal{N})$ is simply connected. We then prove that $P(\mathcal{N})$ being simply connected ensures that any cycle in Γ^n can be "paved" by triples with zero weight.[13]

To relate the geometrical properties of domain T to that of polytope $P(\mathcal{N})$, we use a variation of the nerve theorem from algebraic topology. The nerve theorem has multiple versions (see, e.g., Björner 1995). The classical one requires each set T_a^f, $a \in A$, and each possible intersection $T_{a_0}^f \cap T_{a_1}^f \cap \dots \cap T_{a_M}^f$ to be either empty or contractible. If these conditions are satisfied, the nerve theorem says, roughly speaking, that the geometrical properties of T and $P(\mathcal{N})$ coincide. We need to establish, however, a weaker conclusion

[12]The definition of the nerve goes back to Alexandroff (1928).

[13]The latter step is related to Theorem 4 in Jiang et al. (2011). However, they consider a more restrictive setting where $\{\mathfrak{a}, \mathfrak{b}\}, \{\mathfrak{b}, \mathfrak{c}\}, \{\mathfrak{c}, \mathfrak{a}\} \in A_1$ implies $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\} \in A_2$. This assumption is natural in their setting, but might not be satisfied in our environment.

so that T being simply connected implies that $P(\mathcal{N})$ is simply connected. Hence, we use a weaker requirement on sets T_a^f and no requirement on their intersections.

We state and prove the formal result in Lemma 1 below. The result is new and does not follow from the existing versions of the nerve theorem; hence, it requires a separate proof. As the proof is technical, we postpone it to the Appendix.

Lemma 1. *Let $\mathcal{N}$ be the nerve of the cover $T = \bigcup_{a\in A} T_a^f$. If conditions (i) and (ii) are satisfied, then $P(\mathcal{N})$ is simply connected.*

We now show that any cycle in Γ^n can be paved by triplets with zero weight. For this purpose, we consider the following algebraic construction. Let us enumerate the elements of finite set A in some way. We write $a < b$ if a comes before b. Consider now the linear space

$$\mathbb{R}^{A_1} = \left\{ \sum_{\substack{\{a,b\}\in A_1 \\ a<b}} x_{\mathfrak{ab}} \cdot \mathfrak{ab} \text{ where } x_{\mathfrak{ab}} \in \mathbb{R} \right\}.$$

We use the order on A to avoid counting $\mathfrak{ab}$ and $\mathfrak{ba}$ twice. For $\mathfrak{ba}$, we then write $\mathfrak{ba} = (-1)\cdot \mathfrak{ab} = -\mathfrak{ab} \in \mathbb{R}^{A_1}$. Any path $p = (\mathfrak{a}_0 \to \mathfrak{a}_1 \to \cdots \to \mathfrak{a}_M)$ in Γ^n has then a representative in $\mathbb{R}^{A_1}$:

$$r(p) = \mathfrak{a}_0\mathfrak{a}_1 + \mathfrak{a}_1\mathfrak{a}_2 + \cdots + \mathfrak{a}_{M-1}\mathfrak{a}_M \in \mathbb{R}^{A_1}.$$

Having an order on A is important here, as p consists of directed edges in Γ^n, and an order on A allows us to distinguish directed edges $\mathfrak{a} \to \mathfrak{b}$ and $\mathfrak{b} \to \mathfrak{a}$ in $\mathbb{R}^{A_1}$. Indeed, $r(\mathfrak{a} \to \mathfrak{b}) = \mathfrak{ab}$ and $r(\mathfrak{b} \to \mathfrak{a}) = \mathfrak{ba} = -\mathfrak{ab}$.

Slightly abusing the notation, we now define a linear function $\ell : \mathbb{R}^{A_1} \to \mathbb{R}$ as

$$\ell\Bigg(\sum_{\substack{\{a,b\}\in A_1 \\ a<b}} x_{\mathfrak{ab}} \cdot \mathfrak{ab} \Bigg) \stackrel{\text{def}}{=} \sum_{\substack{\{a,b\}\in A_1 \\ a<b}} x_{\mathfrak{ab}} \ell_{ab}.$$

Function ℓ measures the weight of any path p in Γ^n. Indeed, if $a_i < a_{i+1}$, then $\ell(\mathfrak{a}_i\mathfrak{a}_{i+1}) = \ell_{a_i a_{i+1}}$, and if $a_i > a_{i+1}$, then Fact 1 implies $\ell(\mathfrak{a}_i\mathfrak{a}_{i+1}) = \ell(-\mathfrak{a}_{i+1}\mathfrak{a}_i) = -\ell_{a_{i+1}a_i} = \ell_{a_i a_{i+1}}$. Therefore,

$$\ell\big(r(p)\big) = \ell(\mathfrak{a}_0\mathfrak{a}_1) + \ell(\mathfrak{a}_1\mathfrak{a}_2) + \cdots + \ell(\mathfrak{a}_{M-1}\mathfrak{a}_M) = \ell_{a_0a_1} + \ell_{a_1a_2} + \cdots + \ell_{a_{M-1}a_M}.$$

Using the above definitions, we need to prove that if path $p = (\mathfrak{a}_0 \to \mathfrak{a}_1 \to \cdots \to \mathfrak{a}_M)$ in Γ^n is a cycle (i.e., $\mathfrak{a}_M = \mathfrak{a}_0$), then $\ell(r(p)) = 0$. We reformulate the last statement using a linear map $\partial_1 : \mathbb{R}^{A_1} \to \mathbb{R}^{A_0}$ (called a *boundary operator*):

$$\partial_1\Bigg(\sum_{\substack{\{a,b\}\in A_1 \\ a<b}} x_{\mathfrak{ab}} \cdot \mathfrak{ab} \Bigg) \stackrel{\text{def}}{=} \sum_{\substack{\{a,b\}\in A_1 \\ a<b}} x_{\mathfrak{ab}} \cdot (\mathfrak{b} - \mathfrak{a}).$$

Operator ∂_1 maps edge $\mathfrak{ab}$ in $\mathcal{N}$ to the difference between its tail and head $\mathfrak{b}-\mathfrak{a}$. Hence, if path p is a cycle in Γ^n, we must have $\partial_1(r(p)) = 0$ or $r(p) \in \ker \partial_1 \subset \mathbb{R}^{A_1}$. It is then sufficient to show that ℓ vanishes on any $r(p) \in \ker \partial_1$.

To establish the latter result, we use Fact 2. According to Fact 2, function ℓ vanishes on the boundary of any triangle in A_2. In addition, function ℓ is linear. Hence, to show that any cycle in Γ^n has zero weight, it is sufficient to pave the cycle with triangles in A_2.

To give an exact algebraic meaning to the idea of paving, consider the linear space $\mathbb{R}^{A_2}$ (defined as a linear space of formal sums $\sum_{\substack{\{a,b,c\}\in A_2\\ a<b<c}} x_{\mathfrak{abc}}\mathfrak{abc}$ with notation $\mathfrak{abc} = \mathfrak{bca} = \mathfrak{cab} = -\mathfrak{acb} = -\mathfrak{cba} = -\mathfrak{bac}$ for $a < b < c$) and a linear map $\partial_2 : \mathbb{R}^{A_2} \to \mathbb{R}^{A_1}$ defined as

$$\partial_2\Bigg(\sum_{\substack{\{a,b,c\}\in A_2\\ a<b<c}} x_{\mathfrak{abc}} \cdot \mathfrak{abc}\Bigg) = \sum_{\substack{\{a,b,c\}\in A_2\\ a<b<c}} x_{\mathfrak{abc}}(\mathfrak{ab} + \mathfrak{bc} + \mathfrak{ca}).$$

Operator ∂_2 maps a triangle $\mathfrak{abc}$ to the sum of its boundary edges $\mathfrak{ab} + \mathfrak{bc} + \mathfrak{ca}$. For example, for $a < c < b$, we have $\partial_2(\mathfrak{abc}) = \partial_2(-\mathfrak{acb}) = -(\mathfrak{ac} + \mathfrak{cb} + \mathfrak{ba}) = \mathfrak{ab} + \mathfrak{bc} + \mathfrak{ca}$.

Hence, for cycle p, if we have $r(p) \in \operatorname{Im} \partial_2$, then p can be paved by triangles in A_2. More precisely, Fact 2 allows us to prove that ℓ vanishes on image $\operatorname{Im} \partial_2$. Indeed, for any $\{a, b, c\} \in A_2$, we have $\ell(\partial_2(\mathfrak{abc})) = \ell(\mathfrak{ab} + \mathfrak{bc} + \mathfrak{ca}) = 0$. As maps ℓ and ∂_2 are linear, ℓ also vanishes on $\operatorname{Im} \partial_2$.

We know that $r(p) \in \ker \partial_1$ for any cycle p. Hence, it remains to show that $\ker \partial_1 = \operatorname{Im} \partial_2$. It is easy to see that $\operatorname{Im} \partial_2 \subset \ker \partial_1$, as $\partial_1(\partial_2(\mathfrak{abc})) = \partial_1(\mathfrak{ab} + \mathfrak{bc} + \mathfrak{ca}) = (\mathfrak{b} - \mathfrak{a}) + (\mathfrak{c} - \mathfrak{b}) + (\mathfrak{a} - \mathfrak{c}) = 0$ for any $\{a, b, c\} \in A_2$. To measure the difference between $\ker \partial_1$ and $\operatorname{Im} \partial_2$, we consider the first homology of $\mathcal{N}$ defined as

$$H_1(\mathcal{N}, \mathbb{R}) = \ker \partial_1 / \operatorname{Im} \partial_2.$$

Note that $H_1(\mathcal{N}, \mathbb{R})$ is a linear space with $\dim H_1(\mathcal{N}, \mathbb{R}) = \dim \ker \partial_1 - \dim \operatorname{Im} \partial_2$. Hence, $\operatorname{Im} \partial_2 = \ker \partial_1$ if and only if $H_1(\mathcal{N}, \mathbb{R}) = 0$. The latter is guaranteed by the Hurewicz theorem, which ensures that if $P(\mathcal{N})$ is simply connected, then $H_1(\mathcal{N}, \mathbb{R}) = 0$ (see Hatcher 2001). Hence, $\ker \partial_1 = \operatorname{Im} \partial_2$ by Lemma 1. In particular, $r(p) \in \operatorname{Im} \partial_2$ for any cycle p in Γ^n and the weight of any cycle in Γ^n is 0. □

Finally, to establish the statement of Theorem 1, we show that any cycle in Γ has nonnegative weight. Consider some cycle $\mathfrak{a}_0 \to \cdots \mathfrak{a}_{M-1} \to \mathfrak{a}_0$ in Γ. The local-to-global condition (iii) then implies that for each $j = 0, \dots, M-1$, there exists a path $\mathfrak{a}_j \equiv \mathfrak{a}_j^0 \to \cdots \mathfrak{a}_j^{M(j)} \equiv \mathfrak{a}_{j+1}$ in Γ^n such that $\ell_{\mathfrak{a}_j\mathfrak{a}_{j+1}} \geq \sum_{m=0}^{M(j)-1} \ell_{\mathfrak{a}_j^m \mathfrak{a}_j^{m+1}}$. Therefore,

$$\sum_{j=0}^{M-1} \ell_{\mathfrak{a}_j\mathfrak{a}_{j+1}} \geq \sum_{j=0}^{M-1} \sum_{m=0}^{M(j)-1} \ell_{\mathfrak{a}_j^m \mathfrak{a}_j^{m+1}} = 0.$$

This implies that Γ is cyclically monotone. Hence, f is also cyclically monotone. □

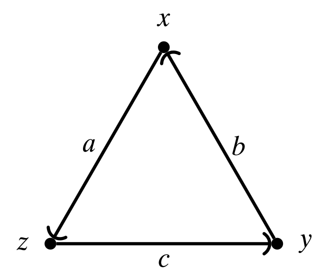

FIGURE 1. An example of a function that is monotone, but not cyclically monotone.

We now illustrate that the conditions of Theorem 1 are indispensable. For this purpose, we consider the following example.

EXAMPLE 1. Consider a domain that is the boundary of a triangle with vertices $x = (0, 1, -1)$, $y = (-1, 0, 1)$, and $z = (1, -1, 0)$. Its sides are $[x, z)$, $[z, y)$, and $[y, x)$ (see Figure 1). Assume that $A = \{a, b, c\}$, where $a = (1, 0, 0)$, $b = (0, 1, 0)$, and $c = (0, 0, 1)$. Function f is defined as

$$f(t) = \begin{cases} a & \text{if } t \in [x, z), \\ b & \text{if } t \in (x, y], \\ c & \text{if } t \in [z, y). \end{cases}$$

A calculation shows that $\ell_{ab} = -\ell_{ba} = \ell_{bc} = -\ell_{cb} = \ell_{ca} = -\ell_{ac} = -1$; hence, f is monotone. There is a negative cycle, $\ell_{ab} + \ell_{bc} + \ell_{ca} = -3$; hence, f is not cyclically monotone. ♢

Example 1 presents a domain $T = [x, z) \cup [z, y) \cup [y, x)$ that is not simply connected and an allocation rule that is monotone, but not cyclically monotone.[14] Note the local-to-global condition is automatically satisfied for the allocation rule of Example 1 as each pair of outcome sets intersects. To present an example of a function and a domain that violates the local-to-global condition, cut a piece from the end of side $[x, z)$. We obtain simply connected domain $T' = [x, z') \cup [z, y) \cup [y, x)$ with $z' = (1/2, 0, -1/2)$. If we keep the same allocation rule on the remaining parts of the domain, lower bounds ℓ do not change. At the same time, sets T_a^f and T_b^f cease to be neighbors. Hence, lower bound ℓ_{ab} should satisfy the local-to-global condition. However, it is not the case, as $-1 = \ell_{ab} \leq \ell_{ac} + \ell_{cb} = 2$. Finally, we illustrate the importance of sets T_a^f, $a \in A$, being path-connected. Let us take a piece out of the middle of side $[x, z)$ leading to simply connected domain $T'' = [x, x') \cup (z', z) \cup [z, y) \cup [y, x)$, $x' = (1/4, 1/2, -3/4)$. If we keep the same allocation rule, we obtain an example of an allocation rule that satisfies the local-to-global condition, but with set T_a^f not path-connected. We again observe a domain and a monotone function that is not cyclically monotone.

[14]Note that domain $T = [x, z) \cup [z, y) \cup [y, x)$ is not simply connected, as it is a loop that cannot be continuously contracted within the domain to a point.

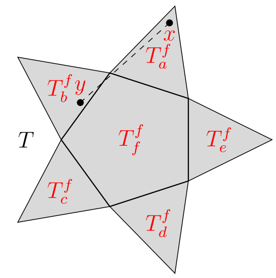

Figure 2. An example of a star-shaped domain T and outcome sets corresponding to $f : T \to \{a, b, c, d, e, f\}$ that satisfy the conditions of Corollary 1.

We now establish a corollary to Theorem 1 that provides easy to check sufficient conditions on a domain T, a finite set A, and a monotone function $f : T \to A$ that guarantees f is also cyclically monotone.

Corollary 1. *Suppose domain T is star-shaped and $f : T \to A$ is a monotone allocation rule that satisfies the following property: for every $a \in f(T)$, there exists $t \in T_a^f$ such that $[s, t] \subset T$ for all $s \in T$. Then f is cyclically monotone.*

A star-shaped domain and sets T_a^f, $a \in A$, that satisfy the conditions of Corollary 1 are illustrated in Figure 2. In the figure, domain T is star-shaped because any point in T_f^f can be connected to any point in T with a line segment. Moreover, for any $a, b \in A$ and for any $x \in T_a^f$, there exists $y \in T_b^f$ such that line segment $[x, y]$ lies in T.

We now present a simple example of a star-shaped domain and a monotone allocation rule that is not cyclically monotone. The example also illustrates that the additional condition on sets T_a^f in Corollary 1 cannot be dropped.

Example 2. Let us consider a modification of Example 1. We consider the domain consisting of the union of two segments $[x, z]$ and $[x, y]$. This is a star-shaped domain, as x can be connected to any point in the domain with a line segment. There are three alternatives $A = \{a, b, c\}$ and allocation f' that coincides with allocation f in Example 1 on domain $[x, z] \cup [x, y]$. Note that the only pre-image of alternative c is point z. The definition of lower bounds then implies that ℓ_{ab}, ℓ_{ba}, ℓ_{ac}, and ℓ_{bc} remain unchanged. A direct calculation also shows that $\ell_{ca} = z_c - z_a = -1$ and $\ell_{cb} = z_c - z_b = 1$. Hence, f' is monotone, but there is a negative cycle, $\ell_{ab} + \ell_{bc} + \ell_{ca} = -3$. ♢

Remark 1. The result of Saks and Yu (2005) follows straightforwardly from Corollary 1, as its conditions are trivially satisfied for convex domains.[15] However, Corollary 1 can be applied to more general settings when a domain can be represented as a union of convex sets C_i, $T = \bigcup_{i=1}^I C_i$, that have nonempty intersection $T_{\text{core}} = \bigcap_{i=1}^I C_i$. In such a case, any point in T_{core} can be connected to any point of the domain with a line segment.

[15] In the working paper version (Kushnir and Lokutsievskiy 2019), we also show how the result for single-peaked preferences by Mishra et al. (2014) follows from Theorem 1.

To ensure that the conditions of Corollary 1 are satisfied for a given monotone function, it remains to show that each nonempty set T_a^f, $a \in A$, contains a point in T_{core}.

We use the conditions of Corollary 1 and the idea of Remark 1 to study the relationship between monotone and cyclically monotone functions on the domain of gross substitutes and the domain of generalized gross substitutes and complements in the next section.

4. Gross substitutes and complements

In this section, we apply our main results to study functions choosing among possible object bundles defined on two important economic domains: the domain of gross substitutes and the domain of generalized gross substitutes and complements. The concept of gross substitutes provides a sufficient condition that ensures the existence of Walrasian equilibria in economies with indivisible objects. The domain of gross substitutes has been explored extensively in the matching, auction, equilibrium, and algorithmic literatures (see, e.g., Paes Leme (2017) and Murota (2016) for extensive surveys). The domain of generalized gross substitutes and complements is a generalization of the first domain that also allows for multiple objects of the same type and some complementarities across objects (see Shioura and Yang 2015).

To define these domains, we consider a finite set of objects E and $n = |E|$. The set of possible object bundles then equals 2^E and $N = 2^{|E|}$. For each bundle $S \subseteq E$, we denote the agent's value as $t(S)$. Hence, vector t is an element of $\mathbb{R}^N$. Valuation t is called *modular* if $t(S) = \sum_{e \in S} t(e)$ for all $S \subseteq E$. Finally, we define the demand correspondence for any price $p \in \mathbb{R}^n$ as

$$D(t, p) = \arg\max_{S \subseteq E} \left\{ t(S) - \sum_{e \in S} p(e) \right\}.$$

We now consider the following condition (see Kelso and Crawford 1982).

Definition 4. Valuation t satisfies the gross substitutes (GS) condition if, for any price $p \in \mathbb{R}^n$ and any $S \in D(t, p)$, if p' is a price vector with $p' \geq p$, then there exists $S' \in D(t, p')$ such that $\{e \in S : p(e) = p'(e)\} \subseteq S'$. The domain of all valuations satisfying the GS condition is denoted by T^{gs}.

In other words, an increase in the price of some goods does not cause a decrease in the demand for other goods. Reijnierse et al. (2002) and Fujishige and Yang (2003) show that the GS condition can be formulated purely in terms of inequalities on the agent's values. The following example presents their characterization for the case of three objects.

Example 3. For $|E| = 3$, $t \in T^{gs}$ if and only if for all distinct $i, j, k \in E$, we have

$$t(\{i, j\} \cup \{k\}) + t(\{k\}) \leq t(\{i, k\}) + t(\{j, k\}),$$

$$t(\{i,j\}) + t(\varnothing) \le t(\{i\}) + t(\{j\}),$$
$$t(\{i,j\}) + t(\{k\}) \le \max\{t(\{i,k\}) + t(\{j\}), t(\{j,k\}) + t(\{i\})\}. \qquad \Diamond$$

Example 3 illustrates that domain T^{gs} is not convex (because of the third set of inequalities). In general, domain T^{gs} consists of several convex polytopes and, therefore, can be quite complex.

For the domain of gross substitutes, it is natural to consider functions $f : T^{gs} \to 2^E$. However, 2^E is not a subset of $\mathbb{R}^N$, and, hence, neither Definitions 1 and 2 in Section 2 nor our results in Section 3 formally apply. To accommodate the subtlety, with each set $S \subseteq E$ we associate an indicator $\alpha(S) \in \{0,1\}^N$, with a component corresponding to set S equal to 1 and all other components equal to 0. Hence, for any $t \in T^{gs}$, we have $t(S) = t \cdot \alpha(S)$. We denote then the union of these indicators as $A \subset \mathbb{R}^N$ and the constructed one-to-one function as $\alpha : 2^E \to A$.[16]

This construction allows us to consider functions $f : T^{gs} \to 2^E$ within our framework. Define $\tilde{f} = \alpha \circ f$. Then $f : T^{gs} \to 2^E$ is monotone (or cyclically monotone) if and only if $\tilde{f} : T^{gs} \to A$ is monotone (or cyclically monotone) according to Definitions 1 and 2. We can then apply the result of Corollary 1 to obtain the following result.

Theorem 2 (Gross substitutes). *If a function $f : T^{gs} \to 2^E$ is monotone, then it is cyclically monotone.*

Proof. Let us consider a monotone function $f : T^{gs} \to 2^E$. Function f is monotone if and only if the associated $\tilde{f} = \alpha \circ f$ is monotone. We show that every monotone $\tilde{f} : T^{gs} \to A$ satisfies the conditions of Corollary 1. For this purpose, we establish that domain T^{gs} satisfies two important properties:

(i) Any *modular* valuation m belongs to T^{gs}.

(ii) For any modular m and $\beta \in [0,1]$, if $t \in T^{gs}$, then $\beta t + (1-\beta)m \in T^{gs}$.

The first property is well known (e.g., Paes Leme 2017) and follows from the definition of demand correspondence. Indeed, for any modular m and $p \in \mathbb{R}^n$, we have

$$S \in D(m,p) \quad \Leftrightarrow \quad \{e \in E : m(e) > p(e)\} \subseteq S \subseteq \{e \in E : m(e) \ge p(e)\}.$$

Therefore, an increase in the price of some goods does not cause a decrease in demand for other goods.

To verify the second property, for any modular valuation $m \in \mathbb{R}^N$, we denote $\tilde{m} \in \mathbb{R}^n$ such that $\tilde{m}(e) = m(e)$ for each $e \in E$. For any price $p \in \mathbb{R}^n$ and any $\beta \in (0,1]$, we then have

$$D(\beta t + (1-\beta)m, p) = \arg\max_{S \subseteq E} \left\{ \beta t(S) + (1-\beta)m(S) - \sum_{e \in S} p(e) \right\}$$
$$= D(t, p/\beta - \tilde{m}(1-\beta)/\beta).$$

[16]This construction is similar to the one used in our mechanism design interpretation (see the Introduction).

Therefore, if demand $D(t, p)$ satisfies the GS condition for any $p \in \mathbb{R}^n$, so does $D(\beta t + (1-\beta)m, p)$. For $\beta = 0$, the second property follows from the first one.

Properties (i) and (ii) stated above imply that domain T^{gs} is star-shaped. To check the requirement of Corollary 1 on sets $T_a^f, a \in A$, we need to establish that every nonempty set T_a^f contains a point that can be connected to any point in T^{gs} with a line segment within T^{gs}. Using the one-to-one correspondence $\alpha : 2^E \to A$, consider $S \subseteq E$ such that $a = \alpha(S)$. We then have

$$T_a^f = \{t \in T^{gs} : t(S) - t(G) \geq \ell_{SG}, \forall G \subset E\},$$

where $\ell_{SG} = \ell_{\alpha(S)\alpha(G)}$ for all $G \subset E$. If T_a^f is nonempty, we have $\ell_{SG} < +\infty$ for all $G \subset E$. Consider a modular valuation

$$m(e) = \begin{cases} M & \text{if } e \in S, \\ -M & \text{if } e \notin S. \end{cases}$$

For sufficiently large $M > 0$, such modular valuation m satisfies inequalities $m(S) - m(G) \geq \ell_{SG}$ for all $G \subset E$. Therefore, m lies in T_a^f. Hence, property (ii) implies that each nonempty T_a^f contains a point that can be connected with a line segment to any point in T^{gs}. Overall, every monotone function $\tilde{f} : T^{gs} \to A$ satisfies the conditions of Corollary 1. Hence, it is cyclically monotone. Therefore, every monotone function $f : T^{gs} \to 2^E$ is cyclically monotone. □

The proof of Theorem 2 has a nice geometric interpretation. As we mentioned before Example 3, Reijnierse et al. (2002) and Fujishige and Yang (2003) showed that domain T^{gs} can be represented as a union of convex polytopes. We established that the intersection of these convex polytopes is nonempty and contains all modular valuations and each nonempty T_a^f contains a modular valuation. Hence, all conditions of Corollary 1 are satisfied (see Remark 1).

We now extend the above result to the domain of *generalized gross substitutes and complements* (GGSC). To define the GGSC condition, consider a finite set of object types E and $n = |E|$. There can be several objects of each type, and we denote the bundle of available objects as $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}_+^E$, where ω_e denotes the available number of objects of type $e \in E$. The types can be divided into two classes $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \varnothing$. We also denote E_j^c as the complement of set E_j, $j = 1, 2$. The objects are substitutes within each class and complements across the classes. For example, E_1 could be considered the set of left shoes and E_2 the set of right shoes. A more practical example concerns the allocation of spectrum licenses. There are two geographic regions and radio spectra. Radio spectrum licenses are substitutes within each region, but complements across regions.

Denote the set of feasible object bundles as $\Omega = \{z \in \mathbb{Z}_+^n : z \leq \omega\}$ and $N = |\Omega|$. Each agent valuation is then $t : \Omega \to \mathbb{R}$. A vector $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ indicates the price for each type. For each price and agent valuation, we consider demand correspondence

$$D(p, t) = \arg\max_{z \in \Omega} \{t(z) - p \cdot z\}.$$

Denote $\chi_e \in \mathbb{R}^n$ as the vector with all zeros except one on the place corresponding to object type $e \in E$ and $\chi_0 = 0$. We say also that $C \subseteq \mathbb{Z}^n$ is *a discrete convex set* if it contains all integer vectors in its convex hull. Shioura and Yang (2015) introduced the following definition.

Definition 5. Valuation t satisfies the generalized gross substitutes and complements (GGSC) condition if the following conditions hold:

(i) For any price $p \in \mathbb{R}^n$, D(p,t) is a discrete convex set.

(ii) For any price $p \in \mathbb{R}^n$ and any $e \in E_j$, $j = 1, 2$, $\delta > 0$, and $z \in D(p, t)$, there exists $z' \in D(p + \delta\chi_e, t)$ such that

$$(\forall l \in E_j \backslash \{e\}) z'_l \geq z_l, \quad (\forall l \in E_j^c) z'_l \leq z_l,$$
$$\sum_{l \in E_j} z_l - \sum_{l \in E_j^c} z_l \geq \sum_{l \in E_j} z'_l - \sum_{l \in E_j^c} z'_l.$$

The domain of all the valuations satisfying the GGSC condition is denoted by T^{ggsc}.

The GGSC condition states that the objects of each type in each set E_j are substitutes, but that goods across the two sets E_1 and E_2 are complements. In particular, let us assume that bundle z is demanded for price vector p. If the price of type $k \in E_j$ is increased, then demand for objects of the other types in E_j will not decrease whereas demand for objects of each type in the other group E_j^c will not increase. In addition, the difference in demand between the two groups at the new prices should not exceed the difference at the old prices.

Note that when there is only one object of each type $\omega = (1, \ldots, 1)$, domain T^{ggsc} coincides with the domain of gross substitutes and complements introduced by Sun and Yang (2006, 2009). Additionally, if $\omega = (1, \ldots, 1)$ and either $E_1 = \varnothing$ or $E_2 = \varnothing$, domain T^{ggsc} coincides with the domain of gross substitutes T^{gs} (see Shioura and Yang 2015).

Furthermore, any function $f : T^{ggsc} \to \Omega$ can be put into our environment in a similar way. For any $z \in \Omega$, we again associate an indicator $\alpha(z) \in \{0, 1\}^N$ with a component corresponding to z equal to 1 and all other components equal to 0. Therefore, $t(z) = t \cdot \alpha(z)$. Denote the union of all these indicators as $A_\Omega \subset \mathbb{R}^N$ and $\tilde{f} = \alpha \circ f : T^{ggsc} \to A_\Omega$. So f is called monotone (or cyclically monotone) if $\tilde{f}$ is monotone (or cyclically monotone). We then establish the following result.

Theorem 3 (Generalized gross substitutes and complements). *If a function $f : T^{ggsc} \to \Omega$ is monotone, then it is cyclically monotone.*

Theorem 3 subsumes Theorem 2. At the same time, Theorem 3 requires a more involved proof that uses the characterization of domain T^{ggsc} in terms of the GM concave functions introduced in Shioura and Yang (2015). We postpone the details until the Appendix.

5. Conclusion

In this paper, we provide sufficient conditions for a monotone function with a finite set of outcomes to be cyclically monotone. Using these conditions, we established that for the domain of gross substitutes and the domain of generalized gross substitutes and complements, any monotone function that chooses among possible object bundles is cyclically monotone.

The relationship between the monotone and cyclically monotone conditions has implications beyond mechanism design. In revealed preference theory, Chambers and Echenique (2018) use it to establish that a demand function is strongly rationalizable with a quasilinear utility if and only if it satisfies a continuity condition and the law of demand (i.e., the negative of the demand function is monotone) (see also Amir et al. 2017). In producer choice theory, the working paper version (Kushnir and Lokutsievskiy 2019) shows that any weakly rationalizable supply functions with a finite range that is positive homogeneous of degree zero is characterized by the law of supply (i.e., the supply function is monotone). We also explain how the reduction of cyclic monotonicity to the requirement of an allocation being monotone is helpful in solving spatial allocation problems.

Finally, we want to highlight a limitation of our approach. Our approach is confined to settings with a finite set of outcomes, because our main building block—the nerve theorem—does not hold when the set of outcomes is infinite (see Björner 1995). For those interested in an infinite set of outcomes, Carbajal and Müller (2015, 2017) and Archer and Kleinberg (2014) provide conditions when a monotone function is cyclically monotone.[17] Understanding when these conditions are applicable to various convex and nonconvex domains is an important direction for future research. This extension will have invaluable implications for some identification problems in econometrics (e.g., Shi et al. 2018).

Appendix

A.1 *Proof of Lemma 1*

We call a subset of T closed if it is an intersection of T and some closed subset of $\mathbb{R}^N$. This is a standard convention when one considers the closed subsets of some set.

Let us first prove that $P(\mathcal{N})$ is path-connected. For this purpose, consider the union of all edges in $P(\mathcal{N})$ that we call 1-skeleton $P^1(\mathcal{N})$. We show that $P^1(\mathcal{N})$ is path-connected. Indeed, if we were able to decompose A into two nonintersecting sets $A = A' \cup A''$ such that for any $a' \in A'$ and $a'' \in A''$ vertices $\mathfrak{a}'$ and $\mathfrak{a}''$ are not connected in $P^1(\mathcal{N})$, then sets $T' = \bigcup_{a' \in A'} T^f_{a'}$ and $T'' = \bigcup_{a'' \in A''} T^f_{a''}$ would not intersect. This contradicts set T being path-connected, as $T = T' \cup T''$ and both T' and T'' are closed subsets of T. Hence, $P^1(\mathcal{N})$ is path-connected, which implies that $P(\mathcal{N})$ is also path-connected.

Now, we prove that $P(\mathcal{N})$ is simply connected. The proof is based on the following carrier theorem, which is a standard tool to prove nerve-type theorems (see Nagórko 2007, Björner 1995). In Nagórko (2007), the carrier theorem is proved under very general assumptions. We adopt his statements to our setting.

[17] See also Berger et al. (2017).

Definition A1. Let $X = \bigcup_{a \in A} X_a$ and $Y = \bigcup_{b \in B} Y_b$, where A and B are some sets of indices. A *carrier* is a map $\mathcal{C} : A \to B$ such that if $\bigcap_{a \in A'} X_a \neq \varnothing$ for some $A' \subset A$, then $\bigcap_{b \in \mathcal{C}(A')} Y_b \neq \varnothing$. We say that a map $f : X' \to Y$ defined on a closed subset $X' \subset X$ is *carried* by $\mathcal{C}$ if $f(X_a \cap X') \subset Y_{\mathcal{C}(a)}$ for all $a \in A$.

Definition A2. A topological space Z is an *absolute extensor* for a topological space W if each continuous map from a closed subset of W into Z extends over the entire W.[18]

For example, two-point set $\{0, 1\}$ is not an absolute extensor for interval $[0, 1]$, and any space Z is an absolute extensor for $\{0, 1\}$, as any map $\{0, 1\} \to Z$ is continuous.

Theorem A1 (Carrier theorem, Nagórko (2007)). *Let $X = \bigcup_{a \in A} X_a \subset \mathbb{R}^n$ and $Y = \bigcup_{b \in B} Y_b \subset \mathbb{R}^m$, and let $\mathcal{C} : A \to B$ be a carrier. If A and B are finite, X_a is a closed subset of X for each $a \in A$, and for any nonempty $B' \subset B$, $\bigcap_{b \in B'} Y_b$ is an absolute extensor for X, then there exists a continuous map $f : X \to Y$ carried by $\mathcal{C}$.*

To use the carrier theorem, we need some covers of two spaces. We already have cover $T = \bigcup_{a \in A} T_a^f$, where T_a^f are closed subsets of T (since each T_a^f is an intersection of T with a collection of closed half-spaces). The second space is the geometric realization of nerve $P(\mathcal{N})$, and we consider its cover by barycentric stars that can be constructed as follows. For any simplex $\sigma \in \mathcal{N}$, we denote the corresponding face center of mass by $B(\sigma) \in P(\mathcal{N})$, which is also called barycenter $B(\sigma) = 1/|\sigma| \sum_{\mathfrak{a} \in \sigma} \mathfrak{a}$. For a given vertex $\mathfrak{a} \in A_0$, we also consider new simplices with vertices in barycenters $B(\sigma_1), \ldots, B(\sigma_r)$ and such that $\mathfrak{a} \in \sigma_1 \subset \cdots \subset \sigma_r \in \mathcal{N}$ (including the case $\sigma_1 = \{\mathfrak{a}\}$). The union of all such simplices is called the barycentric star of $\mathfrak{a}$ and is denoted

$$\operatorname{bst} \mathfrak{a} = \bigcup_{a \in \sigma_1 \subset \cdots \subset \sigma_r \in \mathcal{N}} \operatorname{conv}\{B(\sigma_1), \ldots, B(\sigma_k)\}.$$

Barycentric stars are closed star-shaped sets and $P(\mathcal{N}) = \bigcup_{\mathfrak{a} \in A_0} \operatorname{bst} \mathfrak{a}$.

Definition A3. A cover $X = \bigcup_{a \in A} X_a$ with A being finite and X_a being closed subsets of X is called *regular* for metric spaces if, for any nonempty $A' \subset A$ set, $\bigcap_{a \in A'} X_a$ is an absolute extensor for any metric space.

Lemma 3.2 in Nagórko (2007) shows that cover $P(\mathcal{N}) = \bigcup_{\mathfrak{a} \in A_0} \operatorname{bst} \mathfrak{a}$ is a regular cover for metric spaces. Therefore, for any nonempty $A'_0 \subset A_0$, $\bigcap_{\mathfrak{a} \in A'_0} \operatorname{bst} \mathfrak{a}$ are absolute extensors for T as $T \subset \mathbb{R}^N$.

Consider a carrier $\mathcal{C} : A \to A_0$ for covers $T = \bigcup_{a \in A} T_a^f$ and $P(\mathcal{N}) = \bigcup_{\mathfrak{a} \in A_0} \operatorname{bst} \mathfrak{a}$ that sends each $a \in A$ to the corresponding $\mathfrak{a} \in A_0$. Note that both $\mathcal{C}$ and $\mathcal{C}^{-1}$ by the definition of the nerve (see Lemma 3.2 in Nagórko (2007)). Moreover, for any nonempty $A'_0 \subset A_0$, $\bigcap_{\mathfrak{a} \in A'_0} \operatorname{bst} \mathfrak{a}$ is an absolute extensor for T as shown above. By the carrier theorem, there exists a continuous map $\kappa : T \to P(\mathcal{N})$ carried by $\mathcal{C}$.

[18]The definition is given for arbitrary topological spaces. Any subset of $\mathbb{R}^n$ is a metric space and any metric space is a topological space.

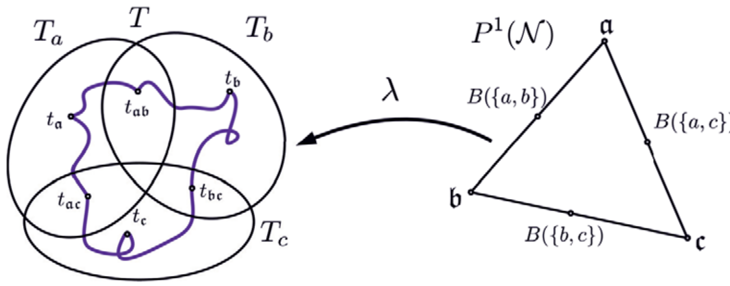

FIGURE 3. Map $\lambda : P^1(\mathcal{N}) \to T$ carried by $\mathcal{C}^{-1}$, where $\mathcal{C} : A \to A_0$ is a carrier for covers $T = \bigcup_{a\in A} T_a^f$ and $P(\mathcal{N}) = \bigcup_{\mathfrak{a}\in A_0} \operatorname{bst} \mathfrak{a}$.

We construct a map $\lambda : P^1(\mathcal{N}) \to T$ carried by $\mathcal{C}^{-1}$ (see Figure 3). For any $\mathfrak{a} \in A_0$, we pick a point $t_\mathfrak{a}$ in T_a^f, and for any $\{\mathfrak{a}, \mathfrak{b}\} \in A_1$, we pick a point $t_{\mathfrak{ab}}$ in $T_a^f \cap T_b^f$. Recall that the definition of nerve $\mathcal{N}$ implies that $T_a^f \neq \varnothing$ for $\mathfrak{a} \in A_0$ and $T_a^f \cap T_b^f \neq \varnothing$ for $\{\mathfrak{a}, \mathfrak{b}\} \in A_1$. We define λ by sending each half of edge $[\mathfrak{a}, B(\{\mathfrak{a}, \mathfrak{b}\})]$ into a continuous path connecting $t_\mathfrak{a}$ and $t_{\mathfrak{ab}}$ (note that such a path exists as T_a^f is path-connected). This construction implies that for any $x \in P^1(\mathcal{N}) \cap \operatorname{bst} \mathfrak{a} = \bigcup_{b\in A}[\mathfrak{a}, B(\{a, b\})]$, we have $\lambda(x) \in T_a^f$. Hence, λ is carried by $\mathcal{C}^{-1}$.

Recall now some definitions from topology. A continuous map from circle S^1 to Y is called a loop in Y, i.e., $\eta : S^1 \to Y$. Loop η is called trivial if there exists $y_0 \in Y$ such that $\eta(S^1) \equiv y_0$. Two continuous maps $f_0, f_1 : X \to Y$ are homotopic if there exists a continuous map $F : [0, 1] \times X \to Y$ such that $F(0, x) \equiv f_0(x)$ and $F(1, x) \equiv f_1(x)$. Hence, Y is simply connected if it is path-connected and any loop in Y is homotopic to some trivial loop.

Using continuous maps $\kappa : T \to P(\mathcal{N})$ and $\lambda : P^1(\mathcal{N}) \to T$, we can now prove that $P(N)$ is simply connected. Consider a loop γ in $P(\mathcal{N})$. By the cellular approximation theorem (see Hatcher 2001), γ is homotopic to a loop γ' in 1-skeleton $P^1(\mathcal{N})$, $\gamma' : S^1 \to P^1(\mathcal{N})$. We consider a loop $\lambda \circ \gamma'$ in T, $\lambda \circ \gamma' : S^1 \to T$. Loop $\lambda \circ \gamma'$ is homotopic to a trivial loop in T, as T is simply connected. Hence, there exists a continuous map $F[0, 1] \times S^1 \to T$ such that $F(0, s) \equiv \lambda(\gamma'(s))$ and $F(1, s) \equiv \text{const}$. We claim that $\gamma'' = \kappa \circ \lambda \circ \gamma'$ is homotopic to a trivial loop in $P(\mathcal{N})$. Indeed, $\kappa \circ F$ is a homotopy contracting γ'' to a point, since $\kappa(F(0, s)) \equiv \gamma''(s)$ and $\kappa(F(1, s)) \equiv \text{const}$. To establish that γ is homotopic to a trivial loop in $P(\mathcal{N})$, it remains to prove that γ' and γ'' are homotopic, as the homotopic property is transitive.

To prove that γ' and γ'' are homotopic, we first show that they have the following property: for any $s \in S^1$, there exists $\mathfrak{a} \in A_0$ such that $\gamma'(s), \gamma''(s) \in \operatorname{bst} \mathfrak{a}$. Nagórko (2007) calls such loops $\{\operatorname{bst} \mathfrak{a}\}_{\mathfrak{a}\in A_0}$-close. Indeed, for any $s \in S^1$, there exists $\mathfrak{a} \in A_0$ such that $\gamma'(s) \in \operatorname{bst} \mathfrak{a}$, as $P(\mathcal{N}) = \bigcup_{\mathfrak{a}\in A_0} \operatorname{bst} \mathfrak{a}$ is a cover. Since λ is carried by $\mathcal{C}^{-1}$, we also must have $\lambda(\gamma'(\mathfrak{a})) \in T_a^f$. Since κ is carried by $\mathcal{C}$, we also have $\gamma''(s) = \kappa(\lambda(\gamma'(s))) \in \operatorname{bst} \mathfrak{a}$. In addition, cover $P(\mathcal{N}) = \bigcup_{\mathfrak{a}\in A_0} \operatorname{bst} \mathfrak{a}$ is regular for metric space (see Lemma 3.1 in Nagórko (2007)) and $S^1 \times [0, 1]$ is a metric space. Therefore, Corollary 2.1 in Nagórko (2007) implies that γ' and γ'' are homotopic.

A.2 *Proof of Corollary 1*

We first note that any star-shaped domain is simply connected. Indeed, take some point $x \in T$ that can be connected with a line segment to any point in T. Such x is called *a base point.* Hence, any loop in T can be continuously contracted to x. Hence, any star-shaped domain is simply connected.

Since any nonempty T_a^f contains a base point of T, the line segment connecting the base point and a point in T_a^f lies in T. The line segment also lies in T_a^f as T_a^f is an intersection of T with some half-spaces. Hence, T_a^f is path-connected.

Finally, we establish that any monotone function f that complies with the conditions of the corollary also satisfies the local-to-global condition. In particular, if every nonempty T_a^f contains a base point of T, function f satisfies the following geometric property: for any $a, b \in f(T)$ and for any $x \in T_a^f$, there exists $y \in T_b^f$ such that line segment $[x, y]$ lies within T. The following lemma establishes that this geometric property ensures that every monotone f also satisfies the local-to-global condition.

Lemma A1. *Consider a domain $T \subset \mathbb{R}^N$, a finite set $A \subset \mathbb{R}^N$, and $f : T \to A$. Suppose that f is monotone, and for any $a, b \in f(T)$ and $x \in T_a^f$, there exists $y \in T_b^f$ such that line segment $[x, y]$ lies within T. Then f satisfies the local-to-global condition.*

Proof. Consider outcomes $a, b \in f(T)$ with $T_a^f \cap T_b^f = \varnothing$. Take some $x \in T_a^f$ and $y \in T_b^f$ such that line segment $[x, y]$ lies within T. Denote the intersection of closed half-spaces as $\widetilde{T}_q^f = \{t \in \mathbb{R}^N : t(q - c) \geq \ell_{qc}, \forall c \in A\}$. Note $T_q^f = T \cap \widetilde{T}_q^f$. Since any set $\widetilde{T}_q^f$ is closed and convex for any $q \in A$, intersection $[x, y] \cap T_q^f = [x, y] \cap \widetilde{T}_q^f$ is either a closed interval, a point, or an empty set. We claim that the following choices are possible:

(i) A path $\{a \equiv a_0, \dots, a_M \equiv b\}$ such that $T_{a_m}^f \cap T_{a_{m+1}}^f \neq \varnothing$, $m = 0, \dots, M-1$, and $[x, y] \cap T_{a_m}^f \neq \varnothing$.

(ii) Points $z_m \in [x, y] \cap T_{a_m}^f$ such that $z_{m+1} - z_m \neq 0$ and the vectors $z_{m+1} - z_m$ and $x - y$ are co-directed for any $m = 1, \dots, M-1$.

This can be done in the following way. We put $a_0 = a$ and $z_0 = x$. Then we denote the right end of interval $[x, y] \cap T_{a_0}^f$ by z_1. Point z_1 must belong to some set T_q^f, $q \neq a_0$. We put $a_1 = q$. The right end of interval $[x, y] \cap T_{a_1}^f$ we denote by z_2. Point z_2 must belong to some set $T_{q'}^f$, $q' \neq a_0, a_1$. We put $a_2 = q'$. We repeat the process until we cover the whole interval $[x, y]$. We finish in a finite number of steps as set A is finite and we pick different points at each step. Finally, we eliminate those z_m and a_m for which $z_{m-1} = z_m$ and update the numeration of a_m and z_m, preserving the order. Note that $T_{a_m}^f \cap T_{a_{m+1}}^f \neq \varnothing$ because both sets contain z_{m+1}.

For each $z_m \in [x, y] \cap T_{a_m}$, $m = 1, \dots, M$, we could write

$$x(a - b) = \sum_{m=0}^{M-1} x(a_m - a_{m+1})$$

$$= x(a_0 - a_1) + \sum_{m=1}^{M-1}(x - z_m)(a_m - a_{m+1}) + \sum_{m=1}^{M-1} z_m(a_m - a_{m+1}).$$

Since all z_m belong to the same interval $[x, y]$ and $z_m \neq z_{m+1}$, there exists λ_m such that $x - z_m = \lambda_m(z_m - z_{m+1})$. Moreover, $\lambda_m > 0$ by the choice of z_m. As $z_m \in T^f_{a_m}$ and $z_{m+1} \in T^f_{a_{m+1}}$, f being monotone implies

$$\begin{aligned}(x - z_m)(a_m - a_{m+1}) &= \lambda_m\big(z_m(a_m - a_{m+1}) + z_{m+1}(a_{m+1} - a_m)\big)\\ &\geq \lambda_m(\ell_{a_m a_{m+1}} + \ell_{a_{m+1} a_m}) \geq 0.\end{aligned}$$

Taking into account that $x(a_0 - a_1) \geq \ell_{a_0 a_1}$ and $z_m(a_m - a_{m+1}) \geq \ell_{a_m a_{m+1}}$, we obtain $x(a - b) \geq \sum_{m=0}^{M-1} \ell_{a_m a_{m+1}}$. Hence, $\ell_{ab} \geq \sum_{m=0}^{M-1} \ell_{a_m a_{m+1}}$, where $T^f_{a_m} \cap T^f_{a_{m+1}} \neq \varnothing$ for each $m = 1, \dots, M-1$. □

All three conditions of Theorem 1 are satisfied. Hence, f is cyclically monotone.

A.3 *Proof of Theorem 3*

Similarly to how the domain of gross substitutes can be characterized in terms of inequalities on the agent's values (see Example 3; Reijnierse et al. 2002, Fujishige and Yang 2003), Theorem 3.3 in Shioura and Yang (2015) shows that $t \in T^{ggsc}$ if and only if it is GM-concave.

To define *GM-concave* valuations, let $U : \mathbb{R}^n \to \mathbb{R}^n$ be a diagonal matrix $U = \operatorname{diag}(1, \dots, 1, -1, \dots, -1)$ that contains 1 as the first $|E_1|$ elements and -1 as the remaining $|E_2|$ elements. Denote $\operatorname{supp}(z) = \{e : z_e > 0\}$ for $z \in \mathbb{Z}^n$. A valuation $t : \Omega \to \mathbb{R}$ is called GM-concave if

$$\begin{aligned}&\forall z, z' \in \Omega\ \forall e \in \operatorname{supp}\big(U(z - z')\big)\ \exists l \in \operatorname{supp}\big(U(z' - z)\big) \cup 0 :\\ &\quad t(z) + t(z') \leq t\big(z - U(\chi_e - \chi_l)\big) + t\big(z' + U(\chi_e - \chi_l)\big).\end{aligned} \tag{3}$$

In fact, each inequality (3) determines a half-space in $\mathbb{R}^N$. Hence, the set of all GM-concave functions (and, hence, domain T^{ggsc}) is the union of some convex polytopes.

Now we construct a generalization of modular valuations. Let $g_e : \{0, 1, \dots, \omega_e\} \to \mathbb{R}$ be arbitrary concave functions for $1 \leq e \leq n$. Put

$$m : \Omega \to \mathbb{R}, \quad m(z) = \sum_{e=1}^{n} g_e(z_e) \quad \text{for } z \in \Omega.$$

Note that valuation m is linear in object types, but concave in the number of objects. So we call such valuations *modular-concave*. We show below that these valuations satisfy inequalities (3) for *each* $e \in \operatorname{supp}(U(z - z'))$ and *each* $l \in \operatorname{supp}(U(z' - z)) \cup 0$. Hence, they belong to the intersection of the convex polytopes shaping domain T^{ggsc}. This helps us to establish that the conditions of Corollary 1 are satisfied (see also Remark 1).

Let us show that each modular-concave valuation satisfies inequalities (3) for each $e \in \text{supp}(U(z - z'))$ and each $l \in \text{supp}(U(z' - z)) \cup 0$. Using the definition of modular-concave valuation, we obtain that condition (3) is equivalent to

$$g_e(z_e) + g_e(z'_e) + g_l(z_l) + g_l(z'_l)$$
$$\leq g_e(z_e - U\chi_e) + g_e(z'_e + U\chi_e) + g_l(z_l + U\chi_l) + g_l(z'_l + U\chi_l).$$

Hence, it is sufficient to prove separately the following inequalities for each $e \in \text{supp}(U(z - z'))$ and each $l \in \text{supp}(U(z' - z)) \cup 0$:

$$g_e(z_e) + g_e(z'_e) \leq g_e(z_e - U\chi_e) + g_e(z'_e + U\chi_e), \tag{4}$$

$$g_l(z_l) + g_l(z'_l) \leq g_e(z_l + U\chi_l) + g_l(z'_l - U\chi_l). \tag{5}$$

We begin with inequality (4). If $e \in E_1$, then $z_e > z'_e$ and (4) becomes

$$g_e(z_e) - g_e(z_e - 1) \leq g_e(z'_e + 1) - g_e(z'_e),$$

which follows from the concavity of g_e. If $e \in E_2$, then $z_e < z'_e$ and (4) becomes

$$g_e(z_e + 1) - g_e(z_e) \geq g_e(z'_e) - g_e(z'_e - 1),$$

which follows from the concavity of g_e. Hence, (4) holds for each $e \in \text{supp}(U(z - w))$.

Now we proceed to inequality (5). If $l = 0$, then (5) is obviously satisfied as equality. If $l \in E_1$, then $z_l < z'_l$ and (5) becomes

$$g_l(z_l + 1) - g_l(z_l) \geq g_l(z'_l) - g_l(z'_l - 1),$$

which follows from the concavity of g_l. Finally, if $l \in E_2$, then $z_l > z'_l$ and (5) becomes

$$g_l(z_l) - g_l(z_l - 1) \leq g_l(z'_l + 1) - g_l(z'_l),$$

which follows from the concavity of g_l. Hence, (5) holds for each $l \in \text{supp}(U(z' - z)) \cup 0$.

Now we use modular-concave valuations to show that any monotone function $\tilde{f} : T^{ggsc} \to A_\Omega$ satisfies the conditions of Corollary 1. First, let us show that any modular-concave valuation m can be connected with an arbitrary $t \in T^{ggsc}$ by a segment line within T^{ggsc}. In other words, we need to show that for any $\beta \in [0, 1]$, $(1 - \beta)t + \beta m \in T^{ggsc}$. Fix arbitrary $z, w \in \Omega$ and $e \in \text{supp}(U(z - w))$. Since $1 - \beta \geq 0$, valuation $(1 - \beta)t$ satisfies (3) for *some* $l = l_0 \in \text{supp}(U(z' - z)) \cup 0$. At the same time, valuation βm is modular-concave and satisfies (3) for *each* $l \in \text{supp}(U(z' - z)) \cup 0$, as we showed above. Therefore, $(1 - \beta)t + \beta m$ satisfies (3) for $l = l_0$ and, hence, $(1 - \beta)t + \beta m \in T^{ggsc}$. Therefore, domain T^{ggsc} is star-shaped.

It remains to establish that any nonempty set T^f_a contains a modular-concave valuation. Using the one-to-one correspondence $\alpha : \Omega \to A_\Omega$, consider $z \in \Omega$ such that $a = \alpha(z)$. We then have

$$T^f_a = \{t \in T^{ggsc} : t(z) - t(z') \geq \ell_{zz'} \; \forall z' \in \Omega\},$$

where $\ell_{zz'} \equiv \ell_{\alpha(z)\alpha(z')}$ for all $z' \in \Omega$. If T_a^f is nonempty, then $\ell_{zz'} < +\infty$ for all $z' \in \Omega$. For a given $z \in \Omega$, consider concave functions

$$g_e(i) = -M|z_e - i|$$

for $e = 1, \dots, n$, $i = 1, \dots, \omega_e$, and $M > 0$. These functions define a modular-concave valuation m^*. Let us show that $m^* \in T_a^f$ for sufficiently large $M > 0$. Indeed, for any $z' \in \Omega$, we have

$$m^*(z) - m^*(z') = \sum_{e=1}^{n} \big(g_e(z_e) - g_e(z'_e)\big) = M \sum_{e=1}^{n} |z_e - z'_e| \geq \ell_{zz'}.$$

For $z' = z$, the above inequality is satisfied as $\ell_{zz} \equiv \ell_{aa} = 0$. For $z' \neq z$, the above inequalities are satisfied as $\ell_{zz'} < +\infty$ and M is large enough.

Therefore, m^* lies in T_a^f. Hence, T_a^f contains a valuation that can be connected with a line segment to any valuation in T^{ggsc}. Overall, any monotone function $\tilde{f} : T^{ggsc} \to A_\Omega$ satisfies the conditions of Corollary 1. Hence, it is cyclically monotone. Therefore, every monotone function $f : T^{ggsc} \to \Omega$ is cyclically monotone.

References

Agarwal, Satish and Souvik Roy (2019), "Sufficiency of weak-monotonicity for implementability in max domains." Indian Statistical Institute. [855]

Alexandroff, Paul (1928), "Über den allgemeinen Dimensionsbegriff und seine Beziehungen zur elementaren." *Mathematische Annalen*, 98, 617–635. [861]

Amir, Rabah, Philip Erickson, and Jim Jin (2017), "On the microeconomic foundations of linear demand for differentiated products." *Journal of Economic Theory*, 169, 641–665. [870]

Archer, Aaron and Robert Kleinberg (2014), "Truthful germs are contagious: A local-to-global characterization of truthfulness." *Games and Economic Behavior*, 86, 340–366. [855, 856, 859, 870]

Ashlagi, Itai, Mark Braverman, Avinatan Hassidim, and Dov Monderer (2010), "Monotonicity and implementability." *Econometrica*, 78, 1749–1772. [854, 855, 858]

Ausubel, Lawrence M. and Paul R. Milgrom (2002), "Ascending auctions with package bidding." *Frontiers of Theoretical Economics*, 1, 1–42. [854]

Berger, André, Rudolf Müller, and Seyed Hossein Naeemi (2009), "Characterizing incentive compatibility for convex valuations." In *International Symposium on Algorithmic Game Theory*, 24–35. [856]

Berger, André, Rudolf Müller, and Seyed Hossein Naeemi (2017), "Characterizing implementable allocation rules in multi-dimensional environments." *Social Choice and Welfare*, 48, 367–383. [856, 870]

Bikhchandani, Sushil, Shurojit Chatterji, Ron Lavi, Ahuva Mu'alem, Noam Nisan, and Arunava Sen (2006), "Weak monotonicity characterizes deterministic dominant-strategy implementation." *Econometrica*, 74, 1109–1132. [854, 855, 856]

Björner, Anders (1995), "Topological methods." In *Handbook of Combinatorics*, volume 2, 1819–1872, North Holland, Amsterdam. [854, 861, 870]

Brown, Donald J. and Caterina Calsamiglia (2007), "The nonparametric approach to applied welfare analysis." *Economic Theory*, 31, 183–188. [855]

Candogan, Ozan, Ishai Menache, Asuman Ozdaglar, and Pablo A. Parrilo (2011), "Flows and decompositions of games: Harmonic and potential games." *Mathematics of Operations Research*, 36, 474–503. [856]

Caradonna, Peter (2020), "How strong is the weak axiom?" Unpublished paper, Department of Economics, Georgetown University. [856]

Carbajal, Juan Carlos and Ahuva Mu'alem (2020), "Selling mechanisms for a financially constrained buyer." *Games and Economic Behavior*, 124. [854, 858]

Carbajal, Juan Carlos and Rudolf Müller (2015), "Implementability under monotonic transformations in differences." *Journal of Economic Theory*, 160, 114–131. [855, 870]

Carbajal, Juan Carlos and Rudolf Müller (2017), "Monotonicity and revenue equivalence domains by monotonic transformations in differences." *Journal of Mathematical Economics*, 70, 29–35. [855, 870]

Carroll, Gabriel (2012), "When are local incentive constraints sufficient?" *Econometrica*, 80, 661–686. [856, 859]

Chambers, Christopher P. and Federico Echenique (2018), "A characterization of combinatorial demand." *Mathematics of Operations Research*, 43, 222–227. [854, 870]

Cuff, Katherine, Sunghoon Hong, Jesse A. Schwartz, Quan Wen, and John A. Weymark (2012), "Dominant strategy implementation with a convex product space of valuations." *Social Choice and Welfare*, 39, 567–597. [855]

Edelman, Paul and John Weymark (2020), "Dominant strategy implementability and zero length cycles." *Economic Theory*, 1–30. [855]

Fujishige, Satoru and Zaifu Yang (2003), "A note on Kelso and Crawford's Gross substitutes condition." *Mathematics of Operations Research*, 28, 463–469. [866, 868, 874]

Gibbard, Allan (1977), "Manipulation of schemes that mix voting with chance." *Econometrica*, 45, 665–681. [856]

Gul, Faruk and Ennio Stacchetti (1999), "Walrasian equilibrium with Gross substitutes." *Journal of Economic Theory*, 87, 95–124. [854]

Hatcher, Allen (2001), *Algebraic Topology*. Cambridge University Press, Cambridge, UK. [863, 872]

Heydenreich, Birgit, Rudolf Müller, Marc Uetz, and Rakesh V. Vohra (2009), "Characterization of revenue equivalence." *Econometrica*, 77, 307–316. [860]

Jehiel, Philippe, Benny Moldovanu, and Ennio Stacchetti (1999), "Multidimensional mechanism design for auctions with externalities." *Journal Economic Theory*, 85, 258–293. [855]

Jiang, Xiaoye, Lek-Heng Lim, Yuan Yao, and Yinyu Ye (2011), "Statistical ranking and combinatorial Hodge theory." *Mathematical Programming*, 127, 203–244. [856, 861]

Kelso, Alexander S. and Vincent P. Crawford (1982), "Job matching, coalition formation, and Gross substitutes." *Econometrica*, 50, 1483–1504. [855, 866]

Kushnir, Alexey and Lev Lokutsievskiy (2019), "On the equivalence of weak- and cyclic-monotonicity." Unpublished paper, Carnegie Mellon University and Steklov Mathematical Institute of Russian Academy of Sciences, SSRN 3422846. [854, 865, 870]

Lavi, Ron and Chaitanya Swamy (2009), "Truthful mechanism design for multidimensional scheduling via cycle monotonicity." *Games and Economic Behavior*, 67, 99–124. [854, 858]

Makowski, Louis and Joseph M. Ostroy (2013), "From revealed preference to preference revelation." *Journal of Mathematical Economics*, 49, 71–81. [855]

Mishra, Debasis, Anup Pramanik, and Souvik Roy (2014), "Multidimensional mechanism design in single peaked type spaces." *Journal of Economic Theory*, 153, 103–116. [855, 856, 859, 865]

Mishra, Debasis, Anup Pramanik, and Souvik Roy (2016), "Local incentive compatibility with transfers." *Games and Economic Behavior*, 100, 149–165. [856]

Mishra, Debasis and Souvik Roy (2013), "Implementation in multidimensional dichotomous domains." *Theoretical Economics*, 8, 431–466. [854, 855, 858]

Müller, Rudolf, Andrés Perea, and Sascha Wolf (2007), "Weak monotonicity and Bayes–Nash incentive compatibility." *Games and Economic Behavior*, 61, 344–358. [855, 856, 859]

Murota, Kazuo (2016), "Discrete convex analysis: A tool for economics and game theory." *Journal of Mechanism and Institution Design*, 1, 151–273. [866]

Myerson, Roger B. (1981), "Optimal auction design." *Mathematics of Operations Research*, 6, 58–73. [853]

Nagórko, Andrzej (2007), "Carrier and nerve theorems in the extension theory." *Proceedings of the American Mathematical Society*, 135, 551–558. [870, 871, 872]

Paes Leme, Renato (2017), "Gross substitutability: An algorithmic survey." *Games and Economic Behavior*, 106, 294–316. [854, 866, 867]

Pycia, Marek (2012), "Stability and preference alignment in matching and coalition formation." *Econometrica*, 80, 323–362. [854]

Pycia, Marek and M. Utku Ünver (2010), "Decomposing random mechanisms." Technical report, University of California, Los Angeles. [856]

Reijnierse, Hans, Anita van Gellekom, and Jos A. M. Potters (2002), "Verifying Gross substitutability." *Economic Theory*, 20, 767–776. [866, 868, 874]

Rochet, Jean-Charles (1987), "A necessary and sufficient condition for rationalizability in a quasi-linear context." *Journal of Mathematical Economics*, 16, 191–200. [853, 855, 858]

Rockafellar, Ralph Tyrrell (1966), "Characterization of the subdifferentials of convex functions." *Pacific Journal of Mathematics*, 17, 497–510. [855, 856]

Roth, Alvin E. (1991), "A natural experiment in the organization of entry-level labor markets: Regional markets for new physicians and surgeons in the United Kingdom." *American Economic Review*, 415–440. [854]

Saks, Michael and Lan Yu (2005), "Weak monotonicity suffices for truthfulness on convex domains." In *Proceedings of 6th ACM Conference on Electronic Commerce*, 286–293, ACM Press, New York. [854, 855, 858, 865]

Salop, Steven C. (1979), "Monopolistic competition with outside goods." *The Bell Journal of Economics*, 10, 141–156. [859]

Sato, Shin (2013), "A sufficient condition for the equivalence of strategy-proofness and nonmanipulability by preferences adjacent to the sincere one." *Journal of Economic Theory*, 148, 259–278. [856]

Shi, Xiaoxia, Matthew Shum, and Wei Song (2018), "Estimating semi-parametric panel multinomial choice models using cyclic monotonicity." *Econometrica*, 86, 737–761. [858, 870]

Shioura, Akiyoshi and Zaifu Yang (2015), "Equilibrium, auction, and generalized Gross substitutes and complements." *Journal of the Operations Research Society of Japan*, 58, 410–435. [854, 855, 866, 869, 874]

Sun, Ning and Zaifu Yang (2006), "Equilibria and indivisibilities: Gross substitutes and complements." *Econometrica*, 74, 1385–1402. [854, 855, 869]

Sun, Ning and Zaifu Yang (2009), "A double-track adjustment process for discrete markets with substitutes and complements." *Econometrica*, 77, 933–952. [855, 869]

Vohra, Rakesh V. (2011), *Mechanism Design: A Linear Programming Approach*. Cambridge University Press, Cambridge. [855]

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