Supplementary Material

Supplement to “Bottleneck links, essential intermediaries, and competing paths of diffusion in networks”

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S1. Limits to indirect appropriability in large random networks

We present a set of results suggesting that protection of intellectual property is necessary for providing sellers with incentives to create the good in many markets. We show that in sufficiently “dense” networks, the effects of competition between sellers of the original good and buyers of copies are extreme and eliminate indirect appropriability. If creating the prototype requires large investments, sellers do not have incentives to produce it even when production enhances welfare. Then prohibiting reproduction of the good is socially optimal.

Theorem 1 implies that buyer b receives the good for free from seller s in state S if and only if $b \nsim_{G(S)} s$. Since $b \nsim_{G(S)} s$ whenever $b \nsim_G s$, seller s obtains zero profit from trading with buyer b if $b \nsim_G s$. The latter condition is equivalent to the fact that removing the link bs from network G does not disconnect the network. If b is linked to any other neighbor of s in G, then the network obtained by removing the link bs from G is connected, so seller s must trade with buyer b at zero price. Hence, if G is sufficiently “clustered,” in the sense that neighbors of s tend to be neighbors with each other,[1] then s is unable to extract any profits from his neighbors. Furthermore, if seller s has at least two links in G and the network obtained by removing node s (and its links) from G is connected, then the network obtained by removing any link of s from G is also connected, so s obtains zero total profit in state S. Another immediate observation is that if there exists a cycle in G that contains all nodes—conventionally called a *Hamiltonian cycle*—then for any $S \in \mathcal{S}$, all equivalence classes of $\sim_{G(S)}$ are singletons, and Theorem 1 implies that no seller makes profits in state S. Intuitively, the previous two statements suggest that sellers are unable to generate any profits if G is “sufficiently connected.” We established the following result.

Proposition S.1. *Fix a seller configuration $S \in \mathcal{S}$ in the network G.*

(i) If every neighbor of seller $s \in S$ in G is linked in G to at least one other neighbor of s, then s makes no profit in state S.

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[1]This principle, known as *triadic closure*, was popularized by the work of Granovetter (1973).

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(ii) If seller $s \in S$ has at least two links in G and the network obtained by removing node s from G is connected, then s makes no profit in state S.

(iii) If G contains a Hamiltonian cycle, then no seller earns any profit in state S.

One can asymptotically estimate probabilities related to connectivity in the context of large random networks. We focus on the well known random graph model of Erdos and Renyi (1959),[2] for which the relevant asymptotic results are readily available. Our exposition of theorems here relies on the monograph of Bollobas (2001). A (Erdos–Renyi) *random graph* with parameters (n, q) is defined by the probability distribution over networks with a fixed set of n nodes in which each link is present independently with probability q or, alternatively, by the random variable $\mathbf{G}_{n,q}$ that has this distribution. In what follows, let ω be any function of n such that $\omega(n) \to \infty$ as $n \to \infty$.

Theorem 7.3 in Bollobas (2001) implies that if $q_n \geq q^C(n) := (\log n + \omega(n))/n$ for all n (where $\omega(n) \to \infty$ as $n \to \infty$), then the probability that the random graph $\mathbf{G}_{n,q_n}$ is connected converges to 1 as $n \to \infty$.[3] A rough interpretation of this result is that random networks with n nodes and an average degree slightly greater than $\log n$ are asymptotically connected for large n. Fix a seller s who belongs to $\mathbf{G}_{n,q_n}$ with $q_n \geq q^C(n)$ for all n. Given the link independence assumption embedded in the definition of random graphs, the network $\mathbf{G}'_{n-1,q_n}$ obtained by removing node s from $\mathbf{G}_{n,q_n}$ is a random graph with parameters $(n-1, q_n)$. Applying the result above for the sequence $(\mathbf{G}'_{n-1,q_n})_{n\geq 2}$ (with a simple adjustment in the corresponding function ω), we conclude that $\mathbf{G}'_{n-1,q_n}$ is connected with limit probability 1 as $n \to \infty$. Since $q_n \geq q^C(n)$ for all n, the probability that s has at least two links in $\mathbf{G}_{n,q_n}$ converges to 1 as $n \to \infty$. The second part of Proposition S.1 then implies that seller s gets zero profit in $\mathbf{G}_{n,q_n}$ with limit probability 1 as $n \to \infty$.

Similarly, Theorem 8.9 from Bollobas (2001) states that if $q_n \geq q^H(n) =: (\log n + \log\log n + \omega(n))/n$ for all n, then the probability that the random graph $\mathbf{G}_{n,q_n}$ contains a Hamiltonian cycle converges to 1 as $n \to \infty$.[4] Thus, a relatively small increase in the average degree of $\mathbf{G}_{n,q_n}$ by the amount $\log\log n$ over the threshold $\log n$ needed for $\mathbf{G}_{n,q_n}$ to be asymptotically connected generates a clear instance of connectedness—the existence of a Hamiltonian cycle. Based on the third part of Proposition S.1, we conclude that if $q_n \geq q^H(n)$ for all n, then all sellers make zero profits in $\mathbf{G}_{n,q_n}$ with limit probability 1 as $n \to \infty$. The next result summarizes our findings related to random graphs.

PROPOSITION S.2. *Consider a sequence of random networks $(\mathbf{G}_{n,q_n})_{n\geq 1}$ and a function ω such that* $\lim_{n\to\infty} \omega(n) = \infty$.

[2] This first article of Erdos and Renyi on the topic considered a variation of the model presented here based on random graphs with fixed numbers of edges, but followup work developed parallel results for the two versions of the model.

[3] It is remarkable that, as Bollobas explains, the threshold function q^C is sharp in the following sense: if, alternatively, $q_n \leq (\log n - \omega(n))/n$ for all n, then $\mathbf{G}_{n,q_n}$ has an isolated node and is thus not connected, with limit probability 1 as $n \to \infty$.

[4] Analogous to the remark from footnote 3, Bollobas argues the threshold function q^H is sharp: if $q_n \leq (\log n + \log\log n - \omega(n))/n$ for all n instead, then the probability that at least one node has fewer than two neighbors in $\mathbf{G}_{n,q_n}$, and hence $\mathbf{G}_{n,q_n}$ does not contain any Hamiltonian cycle, converges to 1 as $n \to \infty$.

(i) If $q_n \geq (\log n + \omega(n))/n$ *for all* $n \geq 1$*, then any particular seller who belongs to all networks in the sequence earns 0 profit in* $\mathbf{G}_{n,q_n}$ *with limit probability 1 as* $n \to \infty$*.*

(ii) If $q_n \geq (\log n + \log\log n + \omega(n))/n$ *for all* $n \geq 1$*, then all sellers in* $\mathbf{G}_{n,q_n}$ *obtain 0 profits with limit probability 1 as* $n \to \infty$*.*

Both parts of Proposition S.2 apply when the number of sellers changes arbitrarily with the size of the network. Versions of this result in which only the network of buyers is random and sellers are linked to several buyers can be derived using the same ideas.

The negative effects of competition on seller profits may be more pronounced in large networks observed in applications than the Erdos–Renyi model suggests. Empirical research provides extensive evidence that social and economic networks are highly clustered.[5] For such networks, the first part of Proposition S.1 implies that it is difficult for sellers to earn high profits when reproduction and resale are allowed. While the refinement of the bargaining solution favors trade and generates extreme competition, clustering represents an obstacle to indirect appropriability even for solutions that do not survive the refinement. Indeed, under all solutions, the existence of a link between a pair of a seller's neighbors implies that the seller cannot extract any profits from one of the two neighbors, a point echoed by Ali et al. (2020).

S2. The cases of pure intermediation and no intermediation

The competitive and monopolistic forces driving market outcomes in the present model are similar to those arising in the non-cooperative intermediation game of Manea (2018), in which a single unit of a *nonreplicable* indivisible good is sequentially traded between linked intermediaries in a network until a player consumes it. Mirroring the assumption from the information selling game that the division of gains from trade between sellers and buyers is determined by Nash bargaining with weights $(p, 1-p)$, when the player holding the good selects a buyer for bargaining in the intermediation game, the holder makes an offer with probability p and the buyer makes an offer with probability $1-p$. Pricing in the information selling game hinges on competition among sellers, whereas pricing in the intermediation game is determined by competition among buyers.

When the good is not replicable, there is one initial seller and only one of the traders can consume it. Thus, to understand the strategic differences between the two models, it is natural to consider a network G with an initial seller s and a buyer b in which all traders except b have zero intrinsic value for the good. In this setting, all traders different from b and s serve pure intermediation roles. Nevertheless, we argue that the bargaining power of intermediaries depends significantly on the replicability of the good.

In both models, if there is a unique path from s to b in the network G, and this path has length k, then the good is resold along the path at prices $(p^k v_b, p^{k-1} v_b, \ldots, p v_b)$; in

[5] See Jackson (2008) and Easley and Kleinberg (2010) for references. In social networks, clustering captures the idea that individuals who have common friends are more likely to be friends with each other. Another expression of this phenomenon, highlighted by the random graph model of Jackson and Rogers (2007), is that individuals are typically friends with the friends of their friends.

the absence of any competition among traders, either model boils down to a sequence of bilateral bargaining problems. However, the two models generate different price dynamics in the more interesting case in which there are competing intermediation paths between s and b. In that case, since $b \nsim_G s$ and thus $b \nsim_{G(\{s\})} s$, buyer b can receive the good only following a sequence of trades in which the dealer d of his equivalence class under $\sim_{G(\{s\})}$ acquires it, and trade subsequently proceeds along the unique intermediation chain between d and b. Again, due to lack of competition in transactions within d's equivalence class, either model predicts prices $(p^k v_b, p^{k-1} v_b, \dots, p v_b)$ for the k intermediaries transmitting the good from d to b.

In contrast, prices along the multiple trading paths from seller s to dealer d diverge substantially in the two models. In the information selling game, all prices over any trading path between s and d are 0, reflecting competition between seller s and buyers who sell replicas of the good. In the intermediation game, prices along the equilibrium trading path take the form $(p^{l+k} v_b, \dots, p^{l+k} v_b, p^{l+k-1} v_b, \dots, p^{l+k-1} v_b, \dots, p^{k+1} v_b, \dots, p^{k+1} v_b)$ for some l. Prices are constant over segments of the path where multiple intermediaries with maximal resale values compete for the good and decline by a factor of p at stages where competition is insufficient; intermediaries acquiring the good in the latter scenario make positive profits. While seller s is unable to indirectly appropriate part of buyer b's value in the information selling game, he obtains a share p^{l+k} of b's value in the intermediation game. For an illustration, in the network from Figure S1, the path of prices is $(0, 0, p v_b)$ in the information selling game and $(p^2 v_b, p^2 v_b, p v_b)$ in the intermediation game.

Another "network" that highlights the role of replicability is one in which intermediation is unnecessary. Suppose that seller s is linked directly to $n \geq 2$ buyers $(b_i)_{i=1}^n$ who do not have other links as illustrated in Figure S2. Assume that $v_{b_1} \geq v_{b_2} \geq \cdots \geq v_{b_n} > 0$ and $v_{b_2} \geq p v_{b_1}$. Then, in the bargaining game of Manea (2018), seller s can trade with a single buyer and exploits the competition between buyers b_1 and b_2 to extract a profit of v_{b_2} from buyer b_1. In the information selling game, seller s supplies a copy of the good to each buyer b_i at price $p v_{b_i}$. Clearly, for $p \geq 1/2$, we have that $\sum_{i=1}^n p v_{b_i} \geq v_{b_2}$, so the seller obtains higher profits in the information selling game. However, for p close to 0, we have $v_{b_2} > \sum_{i=1}^n p v_{b_i}$, and the seller is better off in the bargaining game with unit supply. Transitioning between the two games in this network is equivalent to increasing the supply of the good from 1 to n units. Increasing the supply eliminates competition among buyers and reduces the price the seller can charge to buyer b_1 from v_{b_2} to

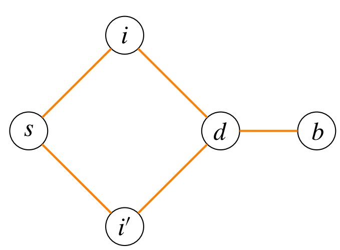

Figure S1. Diverging price dynamics in the two models.

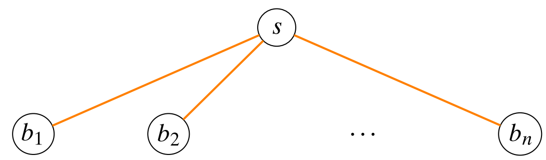

Figure S2. No intermediation.

pv_{b_1}, but allows the seller to extract a surplus of pv_{b_i} from every other buyer b_i. This effect adds a bargaining theory dimension to the trade-off between price and quantity in standard monopoly pricing.

S3. Proof of Theorem 3

For a general network H, let $H \setminus ij$ denote the network obtained by deleting the link ij from H (which is identical to H if $ij \notin H$). Fix a connected network G with $ij \in G$ and let $G' = G \setminus ij$. When the network G' is not connected, the proof relies on applications of earlier results to the connected components of G'. For every seller configuration $S \in \mathcal{S}$, let $G'(S)$ denote the network derived from G' in the same fashion $G(S)$ is derived from G. Note that $G'(S) = G(S)$ if $i, j \in S$ and $G'(S) = G(S) \setminus ij$ otherwise. We use the notation $C'_k(S)$ for the equivalence class of k under $\sim_{G'(S)}$, use $u'_k(S)$ for the payoff of player k in network G' in state S, and use $\delta'(k, l)$ for the distance between nodes k and l in network G'. Fix a seller configuration $S \in \mathcal{S}$ and assume that $\{i, j\} \not\subseteq S$, so $G'(S) = G(S) \setminus ij$.

Suppose that ij is a bottleneck link. As argued in Section 8, the condition $\{i, j\} \not\subseteq S$ implies that $i \sim_{G(S)} j$. Then the link ij represents the unique path between i and j in $G(S)$, which implies that it is also the unique path connecting i and j in G. Since $G' = G \setminus ij$ and $G'(S) = G(S) \setminus ij$, both G' and $G'(S)$ are disconnected. Each of G' and $G'(S)$ must have exactly two connected components, which separate i from j, because G and $G(S)$ are connected. Furthermore, the partition of (nondummy) players into the two components is identical for the two networks. Since $G'(S)$ is disconnected and all sellers in S are linked with one another in $G'(S)$, information does not reach all players in G'.

The relation $i \sim_{G(S)} j$ implies that there is no cycle in $G(S)$ that contains link ij. Then every link that is part of a cycle in $G(S)$ is also part of a cycle in $G'(S)$. It follows that the forests derived by eliminating cycles from $G(S)$ and $G'(S)$ satisfy $\mathcal{F}(G'(S)) = \mathcal{F}(G(S)) \setminus ij$. The removal of link ij from the forest $\mathcal{F}(G(S))$ breaks up the connected component of $\mathcal{F}(G(S))$ containing i and j into two components, and does not affect other components. Therefore, $C_i(S) = C'_i(S) \cup C'_j(S)$ with $C'_i(S) \cap C'_j(S) = \varnothing$ and $C'_k(S) = C_k(S)$ for all $k \nsim_{G(S)} i$. Theorem 1 implies that sellers outside $C_i(S)$ obtain the same profits in G and G'. If $d(S, C_i(S))$ is a seller, Theorem 1, along with $C'_{d(S,C_i(S))}(S) \subset C_i(S)$, implies that $d(S, C_i(S))$'s profit is lower in G' than in G (strictly lower if $v_b > 0$ for all $b \in N \setminus \underline{S}$).

To investigate the effects of ij's removal from G on information diffusion and buyer payoffs, suppose without loss of generality that $d(S, C_i(S)) \in C'_i(S)$ (it is possible that $d(S, C_i(S)) = i$). Then i and $d(S, C_i(S))$ are in the same connected component of G', which is different from j's component. There is a path in G from a seller in S to $d(S, C_i(S))$ that does not contain any other node from $C_i(S)$ and, in particular, does not contain the link ij. Hence, $d(S, C_i(S))$ is in the same connected component as a seller in $G'(S)$. Since sellers are linked to one another in $G'(S)$, all nodes in S must be in the same connected component of $G'(S)$ as i and $d(S, C_i(S))$. This implies that the good cannot reach the players in j's connected component in G' (this component is a superset of $C'_j(S)$; it can be a strict superset formed by the union of $C'_j(S)$ and some of the sets $C_k(S)$ with $k \nsim_{G(S)} i$). Hence, players in j's connected component in G' obtain zero payoffs in G'.

Consider now a buyer b from i's connected component in G'. As $j \notin S \cup b$, we have $G'(S \cup b) = G(S \cup b) \setminus ij$. Since the links in $G'(S \cup b) \setminus G'(S)$ connect only nodes in the set $S \cup \{b, 0\}$, which is disjoint from j's connected component in $G'(S)$, it must be that $G'(S \cup b)$ and $G'(S)$ have identical connected components. Thus, i and j are in distinct components of $G'(S \cup b)$, which means that the link ij constitutes the only path in $G(S \cup b)$ between i and j, and, hence, $i \sim_{G(S \cup b)} j$. Arguments analogous to those above then show that $\mathcal{F}(G'(S \cup b)) = \mathcal{F}(G(S \cup b)) \setminus ij$. If $b \notin C_i(S)$, then $b \notin C_i(S \cup b)$, which implies that $C'_b(S \cup b) = C_b(S \cup b)$. If $b \in C_i(S)$, then we have that $C'_b(S \cup b) \subseteq C_b(S \cup b)$, with strict inclusion if $b \in \{i, d(S, C_i(S))\}$. Note that b is a dealer for $C'_b(S)$ in state S if and only if b is a dealer for $C_b(S)$ in state S. Theorem 1 then implies that all buyers in i's connected component in G' that do not belong to $C_i(S)$ obtain the same payoffs in G and G', while buyers in $C'_i(S)$ have weakly lower payoffs in G' than in G (with i and $d(S, C_i(S))$ having strictly lower payoffs in G' if $v_b > 0$ for all $b \in N \setminus \underline{S}$).

Suppose next that ij is a redundant link, i.e., $i \nsim_{G(\underline{S})} j$. Then we also have that $i \nsim_{G(S)} j$, so there exists a path between i and j in $G(S)$ that does not involve link ij. Since $G'(S) = G(S) \setminus ij$ and $G(S)$ is connected, the path is contained in $G'(S)$, and $G'(S)$ is also connected. This means that every buyer is connected to a seller by a path in G', so information reaches all buyers eventually. The removal of link ij leads to a weak expansion in each player's equivalence class in $G(S)$. For a proof, fix a player $k \in N$. Since $i \nsim_{G(S)} j$, it cannot be that both i and j belong to $C_k(S)$. By Lemma 1, every pair of nodes in $C_k(S)$ is connected by a unique path in $G(S)$, which necessarily contains only nodes in $C_k(S)$ and, thus, excludes link ij. As $G'(S) = G(S) \setminus ij$, every pair of nodes in $C_k(S)$ is connected by a unique path in $G'(S)$ as well. Hence, all nodes in $C_k(S)$ are in the same equivalence class of $\sim_{G'(S)}$, i.e., $C_k(S) \subseteq C'_k(S)$. Theorem 1 implies that every seller's payoff is weakly higher in G' than in G. Indeed, for all $s \in S$, $C_s(S) \subseteq C'_s(S)$ implies that

$$u_s(S) = \sum_{k \in C_s(S) \setminus s} p^{\delta(k,s)} v_k \leq \sum_{k \in C'_s(S) \setminus s} p^{\delta'(k,s)} v_k = u'_s(S).$$

The inequality above relies on the fact that $\delta(k, s) = \delta'(k, s)$ for all $k \in C_s(S)$. This follows from the observation that there is a single path in G between s and any node $k \in C_s(S)$, which does not include the link ij and, hence, constitutes the unique path between s and k in G'. We have established that the payoffs of all sellers weakly increase when the redundant link ij is removed from G.

Similarly, for every buyer b, we have $i \nsim_{G(S \cup b)} j$, so $C_b(S \cup b) \subseteq C'_b(S \cup b)$. Suppose that b is not the dealer for $C_b(S)$ in state S. Then b is not the dealer for $C'_b(S)$ in state S

either. For a proof by contradiction, assume that there is a path in G' from a seller in S to b that does not contain any node from $C'_b(S)$ except for b. The path also lies in G because $G' = G \setminus ij$. Since $C_b(S) \subseteq C'_b(S)$, the path does not contain any node from $C_b(S)$ other than b. Then b should be the dealer for $C_b(S)$ in state S, a contradiction. Theorem 1, along with the condition $C_b(S \cup b) \subseteq C'_b(S \cup b)$ and the equality $\delta(b,k) = \delta'(b,k)$ for $k \in C_b(S \cup b)$, implies that

$$\begin{aligned} u_b(S) &= (1-p)\left(v_b + \sum_{k \in C_b(S \cup b)\setminus b} p^{\delta(b,k)} v_k\right) \\ &\leq (1-p)\left(v_b + \sum_{k \in C'_b(S \cup b)\setminus b} p^{\delta'(b,k)} v_k\right) = u'_b(S). \end{aligned}$$

This proves that nondealer buyers weakly benefit from the removal of the redundant link ij from G.

We finally prove that if i is a dealer buyer in state S, then i is hurt by the deletion of the redundant link ij if and only if ij is a pivotal link for i in state S.

Suppose first that i has exactly two potential suppliers, j and some other node k, in state S in the network G. Then in the network $G \setminus ij$, player i can receive the good only from neighbor k. Theorem 2 implies that player i is not a dealer in state S in $G \setminus ij$. We next prove that the set of players for whom i is an essential intermediary in state S cannot expand when the link ij is removed from the network.

Assume that i is an essential intermediary for player k in state S in the network $G \setminus ij$. The following statements must be true: (a) there is a unique path in $G \setminus ij$ between i and k, and (b) every path in $G \setminus ij$ from a node in S to k passes through i. We show that i is also an essential intermediary for player k in state S in the network G. If this were not the case, then either there exist multiple paths in G between i and k or there is a path in G from a node in S to k that does not pass through i.

In the first case, (a) implies that one of the paths from i to k in G includes the link ij; this path contains a subpath between j and k in $G \setminus ij$. As j is a potential supplier for i in state S, there should be a path from S to j that does not contain the link ij and, thus, does not pass through i. By pasting the two paths (and removing potential overlap), we obtain a path in $G \setminus ij$ from S to k that does not go through i, a contradiction with (b).

In the second case, the path in G from a node in S to k that does not pass through i necessarily excludes the link ij and, thus, must lie in $G \setminus ij$. This implies the existence of a path in $G \setminus ij$ from a node in S to k not containing i, a contradiction with (b).

We have established that if ij is a pivotal link for i in state S, then the removal of the link ij from the network leads to the loss of dealer status for i and does not expand the set of buyers for whom i provides essential intermediation in the network $G \setminus ij$. Theorem 1 and Lemma 2 then imply that buyer i's payoff is strictly lower in $G \setminus ij$ than in G if $v_b > 0$ for all $b \in N \setminus \underline{S}$.

We are left to consider the case in which ij is not a pivotal link for dealer i in state S. In this case, Theorem 2 implies that i has at least two potential suppliers k and k' different from j in state S. By definition, there exist paths in G from nodes in S to k and k' that do not contain node i and, thus, exclude the link ij; these paths lie in $G \setminus ij$. It follows that k and k' are potential suppliers for i in state S in the network $G \setminus ij$. From Theorem 2, we infer that player i maintains his dealer role in state S in the network $G \setminus ij$. Since i's equivalence class in $G(S \cup i)$ can only expand when the redundant link ij is removed, Theorem 1 implies that buyer i's profit weakly increases after the removal of link ij. $\square$

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