Supplementary Material

Supplement to "Slow persuasion"

(*Theoretical Economics*, Vol. 18, No. 1, January 2023, 129–162)

Matteo Escudé
Department of Economics and Finance, LUISS

Ludvig Sinander
Department of Economics and Nuffield College, University of Oxford

Appendix H: Solving for the value function

In this Appendix, we explain how to solve for the sender's value function using Proposition 1. We detail in particular how the value is computed in the two-action example, allowing us to draw Figure 1.

Partition $C \setminus D$ into maximal intervals $(R_k)_{k=1}^K$. (In the two-action example, the maximal intervals are $[0, 2/3)$ and $(2/3, 1]$.) Fix a continuity interval R_k. The homogeneous part (without the u) of the differential equation (∂) has general solution $AH_1 - BH_2$ for constants $A, B \in \mathbf{R}$, where

$$H_1(p) := p^{\xi}(1-p)^{1-\xi}, \qquad \xi := 1/2 + \sqrt{1/4 + 2r\sigma^2/\overline{\lambda}} \quad \text{and} \quad H_2(p) := H_1(1-p).$$

A particular solution may be obtained from formula (6.2) in Coddington (1961, ch. 3). Things are easier when the sender has expected-utility preferences, so that $f(a, \cdot)$ is affine, as u itself is then a particular solution. This is the case in the two-action example, and in the three-action example below. In the expected-utility case, the value function is given on each maximal interval R_k of $C \setminus D$ as

$$v(p) = u(p) + A_{R_k} H_1(p) - B_{R_k} H_2(p) \quad \text{for all } p \in R_k,$$

where the constants $(A_{R_k}, B_{R_k})_{k=1}^K$ are the unique ones that ensure that the properties in Proposition 1 are satisfied: the boundary condition $v = u$ on $\{0, 1\}$, the continuity of v on D, and smooth pasting on $C \cup D$.

H.1 *The two-action example (Section 4.1)*

Here, $D = \{2/3\}$, and C contains $[0, 2/3)$ and may or may not contain $[2/3, 1]$. In either case,

$$v(p) = \begin{cases} A_{[0,2/3)} H_1(p) - B_{[0,2/3)} H_2(p) & \text{for } p \in \big[0, 1/2\big) \\ \alpha p - \beta + A_{(2/3,1]} H_1(p) - B_{(2/3,1]} H_2(p) & \text{for } p \in (2/3, 1\big], \end{cases}$$

Matteo Escudé: mescude@luiss.it
Ludvig Sinander: ludvig.sinander@economics.ox.ac.uk

 https://doi.org/10.3982/TE5175

where $\alpha = 3/2$ and $\beta = 1/2$.[1] The boundary conditions require that $B_{[0,2/3)} = 0$ and $A_{(2/3,1]} = 0$. Continuity of v at 2/3 requires that

$$A_{[0,2/3)} H_1(2/3) = \alpha(2/3) - \beta - B_{(2/3,1]} H_2(2/3).$$

If $\overline{\lambda}$ is sufficiently high, then $2/3 \in C$, in which case smooth pasting must hold at 2/3:

$$A_{[0,2/3)} H_1'(2/3) = \alpha - B_{(2/3,1]} H_2'(2/3).$$

Thus the constants are uniquely pinned down.

If $\overline{\lambda}$ is low, then $2/3 \notin C$, in which case $v = u$ on $[2/3, 1]$. Thus $B_{(2/3,1]} = 0$, whence $A_{[0,2/3)}$ is pinned down by the continuity condition.

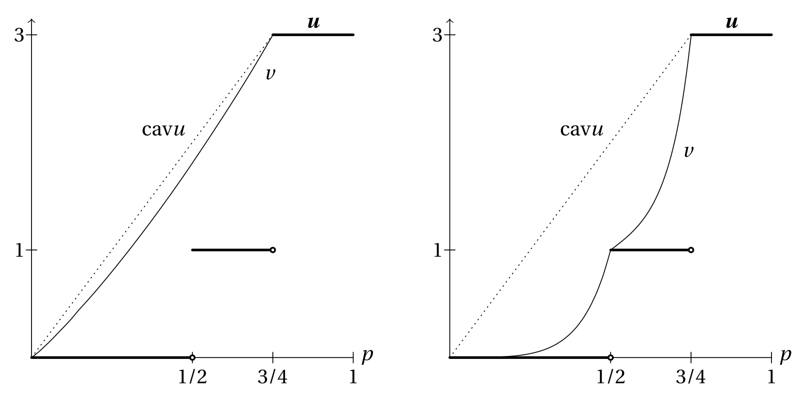
To determine which case applies for a given value of $\overline{\lambda}$, calculate $A_{[0,2/3)}$ assuming that the first case applies. If

$$A_{[0,2/3)} H_1(2/3) \geq u(1/2) = 1/2,$$

then the first case does indeed apply; if not, then not.

H.2 *A three-action example*

Consider the flow payoff u depicted in Figure 2. The underlying model has actions $\mathcal{A} = \{0, 1, 3\}$, flow payoff $f_S(a, p) = a$ for the sender, and payoffs $f_D(0, p) = 0$, $f_D(1, p) = 2p - 1$ and $f_D(3, p) = \frac{14}{3} p - 3$ for the decision maker. Figure 2 depicts the concave envelope, as well as the sender's value function for high and low values of $\overline{\lambda}$.



(a) Value v for high $\overline{\lambda}$. (b) Value v for low $\overline{\lambda}$.

FIGURE 2. Three-action example.

[1] If C contains $[2/3, 1]$, then the expression for $p \in (2/3, 1]$ holds since (HJB) must be satisfied in the classical sense by Proposition 1. If not, then Proposition 1 requires that $v = u$, which amounts to setting $A_{(2/3,1]} = B_{(2/3,1]} = 0$.

Clearly, C contains $[0, 1/2)$ and $(1/2, 3/4)$, and does not contain $[3/4, 1]$. Thus the value function off D is

$$v(p) = \begin{cases} A_{[0,1/2)}H_1(p) - B_{[0,1/2)}H_2(p) & \text{for } p \in [0, 1/2) \\ \ell + A_{(1/2,3/4)}H_1(p) - B_{(1/2,3/4)}H_2(p) & \text{for } p \in (1/2, 3/4) \\ h, & \end{cases}$$

where $\ell = 1$ and $h = 3$. The boundary condition at $p = 0$ again requires that $B_{[0,1/2)} = 0$. Continuity of v at $1/2$ and at $3/4$ requires that

$$A_{[0,1/2)}H_1(1/2) = \ell + A_{(1/2,3/4)}H_1(1/2) - B_{(1/2,3/4)}H_2(1/2)$$

and

$$\ell + A_{(1/2,3/4)}H_1(3/4) - B_{(1/2,3/4)}H_2(3/4) = h.$$

These are two equations in three unknowns.

If $\overline{\lambda}$ is sufficiently high that $1/2 \in C$, then smooth pasting must hold at $1/2$, giving us the third equation

$$A_{[0,1/2)}H_1'(1/2) = A_{(1/2,3/4)}H_1'(1/2) - B_{(1/2,3/4)}H_2'(1/2).$$

If $\overline{\lambda}$ is sufficiently low that $1/2 \notin C$, then $v(1/2) = u(1/2) = \ell$. We thus obtain a third equation from the requirement that v be continuous at $1/2$:

$$A_{[0,1/2)}H_1(1/2) = \ell.$$

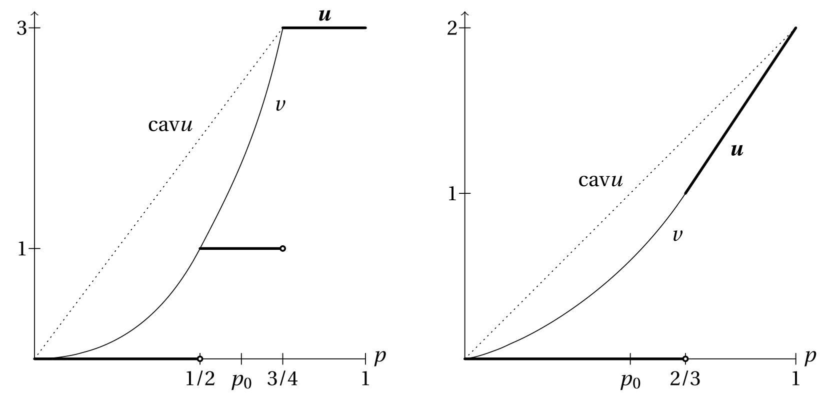
To discern which case applies, compute $A_{[0,1/2)}$ assuming that the first (information arrives fast) case applies. If $A_{[0,1/2)}H_1(1/2) \geq \ell$, then the fast-arrival case does indeed apply; if not, then not.

Appendix I: Generic uniqueness of long-run beliefs

We claimed in Section 5.1 that provided $v(p_0) > u(p_0)$, generically, all best replies of the sender induce the same long-run beliefs (namely, the beliefs $\{p^-, p^+\}$ defined in Corollary 3).

To see how uniqueness can fail, consider the three-action example in supplemental Appendix H.2. Figure 3a depicts the knife-edge case in which $\overline{\lambda}$ is such that the fast-information value function with the convex-flat shape in Figure 2a touches u at $1/2$.[2] In this case, the sender is indifferent between providing and not providing information at $1/2$, and strictly prefers to do so on $(0, 1/2)$ and $(0, 3/4)$. The best reply $\Lambda^\star$ from Corollary 2 stops at $1/2$, inducing the long-run beliefs $\{p^-, p^+\} = \{1/2, 3/4\}$ from Corollary 3. But since the sender is indifferent at $1/2$, she also has a best reply that provides information at $1/2$, which induces long-run beliefs $\{0, 3/4\}$.

[2] We thank Jeff Ely for pointing out this scenario.



(a) Three-action example from Figure 2. (b) Two-action example from Section 4.1.

FIGURE 3. Knife-edge cases in which long-run beliefs are not unique.

This scenario is nongeneric in the sense that slightly increasing $\overline{\lambda}$ puts us back in Figure 2a, where the sender strictly prefers to provide information at full tilt at $1/2$, whereas slightly decreasing $\overline{\lambda}$ puts us in Figure 2b, where she strictly prefers to stop at $1/2$.

Similarly, Figure 3b depicts the case in Section 4.1 in which $\overline{\lambda}$ has exactly the value needed for the fast-information value function with the convex shape in Figure 1a to just touch u at $2/3$. In this example, there is more multiplicity: the sender is indifferent on $[1/2, 1]$, so has best replies that induce any mean-p_0 distribution of long-run beliefs supported on $\{0\} \cup [2/3, 1]$. (The best reply $\Lambda^\star$ induces the beliefs $\{0, 2/3\}$.) Again, perturbing $\overline{\lambda}$ makes the sender's preference strict at $2/3$, so that long-run induced beliefs are unique (either $\{0, 2/3\}$ or $\{0, 1\}$).

The nongenericity of multiplicity in these examples is a general phenomenon. Multiplicity occurs for some prior p_0 with $v(p_0) > u(p_0)$ precisely if the sender is indifferent between stopping and continuing at some $p \in (0, 1)$ and weakly prefers to continue on a neighborhood of p. In such cases, her preference becomes strict when $\overline{\lambda}$ is perturbed slightly.

APPENDIX J: PIECEWISE CONTINUITY IS MERELY TIE-BREAKING

We asserted in Section 2.3 that provided the decision-maker's flow payoff f_D is nondegenerate in a mild sense, it is without loss of optimality for her to restrict attention to piecewise continuous Markov strategies $A : [0, 1] \to \Delta(\mathcal{A})$.

To justify this claim, begin by recalling from Section 3 that the decision maker best-replies to a Markov strategy of the sender by myopically maximizing $f_D(a, p)$ at each p. Fix two actions $a, a' \in \mathcal{A}$, and write

$$\psi(p) := f_D(a, p) - f_D\big(a', p\big)$$

for their payoff difference. Say that ψ *strictly up-crosses* at $p \in (0, 1)$ if and only if $\psi(p) = 0$ and for any $\varepsilon > 0$, there are $p' \in (p - \varepsilon, p)$ and $p'' \in (p, p + \varepsilon)$ such that $\psi(p') < 0 < \psi(p'')$, *strictly down-crosses* if the reverse inequalities hold, and simply *strictly crosses* if either is the case. Write $K \subseteq (0, 1)$ for the set on which ψ strictly crosses. We claim that given some weak nondegeneracy condition on f_D, the crossing set K is discrete, so that the decision maker strictly prefers to switch actions only on a discrete set. (It suffices to consider only two arbitrary actions $a, a' \in \mathcal{A}$ because $\mathcal{A}$ is finite.)

To see what can go wrong, suppose that $f_D(a, p) = 0$ and that $p \mapsto f_D(a', p)$ is a typical path of a standard Brownian motion. Then ψ is continuous, but the strict crossing set K is nonempty with no isolated points (see, e.g., Theorem 9.6 in Karatzas and Shreve (1991, ch. 2)). This preference dithers maniacally, wishing to switch actions back and forth extremely frequently.

As a first pass, observe that if ψ is monotone, or more generally if ψ or $-\psi$ has the single-crossing property ($\psi(p) \geq (>)\, 0$ implies $\psi(p') \geq (>)\, 0$ for $p < p'$), then K is empty or a singleton, so certainly discrete. These assumptions are satisfied by expected-utility preferences.

A weak nondegeneracy condition that suffices is *local single-crossing*: for each $p \in K$, we have either $\psi \geq 0$ or $\psi \leq 0$ on a left-neighborhood of p, and similarly on a right-neighborhood. Then each $p \in K$ is manifestly the unique strict crossing of ψ on a neighborhood, hence isolated. A sufficient condition for this is *local monotonicity*: for each $p \in K$, we have $\psi(p - \varepsilon) \leq 0 \leq \psi(p + \varepsilon)$ for all sufficiently small $\varepsilon > 0$, or the reverse inequality.

Appendix K: A very brief introduction to viscosity solutions

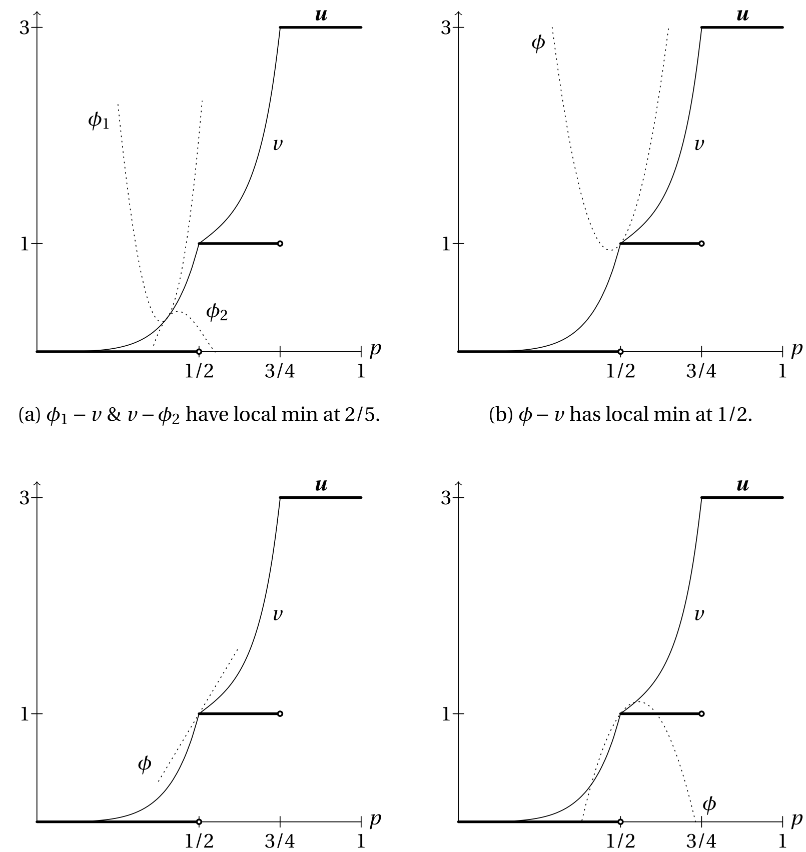
Crandall (1997), Katzourakis (2015) and Crandall, Ishii, and Lions (1992) provide overviews of the theory of viscosity solutions of second-order differential equations. Moll (2020), Evans (2010, ch. 10), Calder (2018) and Bressan (2011) give easier treatments that deal mostly with first-order equations.

The general idea of viscosity solutions is as follows. If w is a viscosity solution of (HJB), then it must satisfy (HJB) in the classical sense on any neighborhood on which w'' exists and is continuous. If w'' does not exist at $p \in [0, 1]$, we require instead that (HJB) hold with the appropriate inequality when $w''(p)$ is replaced by $\phi''(p)$ for some twice continuously differentiable local approximation ϕ to w at p. (The formal definition was given in Section 4.2.)

K.1 *Illustration of the definition*

Consider the three-action example from supplemental Appendix H.2 (Figure 2). Write $\mathcal{C}^2$ for the set of twice continuously differentiable functions $(0, 1) \to \mathbf{R}$. Begin by observing that v is continuous, hence upper and lower semicontinuous.

Consider a p in whose vicinity v is twice continuously differentiable, for example, $p = 2/5$. We may easily find $\phi_1, \phi_2 \in \mathcal{C}^2$ such that $\phi_1 - v$ and $v - \phi_2$ are locally minimized at p, as in Figure 4a. But in particular, we may choose $\phi \in \mathcal{C}^2$ to coincide with v on a neighborhood of p. Then $\phi - v$ and $v - \phi$ are *both* locally minimized at p, and



(a) $\phi_1 - v$ & $v - \phi_2$ have local min at 2/5.

(b) $\phi - v$ has local min at 1/2.

(c) $\phi - v$ has local min at 1/2, $\phi''(1/2) = 0$.

(d) $v - \phi$ does not have local min at 1/2.

FIGURE 4. Functions $\phi \in \mathcal{C}^2$ that approximate v locally.

$\phi''(p) = v''(p)$. Since v is a viscosity subsolution (supersolution) by Theorem 1, and $u(p) = u^\star(p) = u_\star(p)$, it follows that

$$v(p) \leq (\geq)\ u(p) + \overline{\lambda} \frac{p^2(1-p)^2}{2r\sigma^2} \max\{0, v''(p)\}.$$

So (HJB) must be satisfied in the classical sense at p.

Next, consider a point at which v'' is undefined, for example, $p = 1/2$. There are many $\phi \in \mathcal{C}^2$ such that $\phi - v$ has a local minimum at p; an example is depicted in Figure 4b. Since v is a viscosity subsolution of (HJB) and $u^\star(p) = u(p)$, we must have

$$v(p) \leq u(p) + \overline{\lambda} \frac{p^2(1-p)^2}{2r\sigma^2} \max\{0, \phi''(p)\}$$

for any such ϕ. In fact, ϕ can be chosen so that $\phi''(p) \leq 0$: the ϕ depicted in Figure 4c is affine, so has $\phi''(p) = 0$. The subsolution condition therefore requires precisely that

$$v(p) \leq \inf_{\substack{\phi \in \mathcal{C}^2: \\ \phi - v \text{ loc. min. at } p}} \left\{ u(p) + \overline{\lambda} \frac{p^2(1-p)^2}{2r\sigma^2} \max\{0, \phi''(p)\} \right\} = u(p),$$

which holds (with equality, in fact).

By contrast, there are no $\phi \in \mathcal{C}^2$ such that $v - \phi$ has a local minimum at p; a (failed) attempt to find such a ϕ is drawn in Figure 4d. The fact that v is a viscosity supersolution of (HJB) therefore has no bite at $p = 1/2$.

K.2 *Some properties of viscosity solutions*

There are other nonclassical notions of "solution" of a differential equation, most importantly distributional solutions (e.g., Evans (2010, chs. 5–9)). But for many differential equations, including HJB equations, viscosity solutions are the appropriate notion. The chief reasons are twofold: viscosity solutions exist, and they satisfy a comparison principle.

Begin with existence. Many HJB equations, including ours, fail to have a classical solution. Many also fail to have nonclassical solutions of, for example, the distributional variety. By contrast, HJB equations always have a viscosity solution.

The other virtue of viscosity solutions is that they satisfy a comparison principle (also called a "maximum principle") of the following kind: if $\underline{w}$ is a subsolution on (a, b), $\overline{w}$ is a supersolution on (a, b), and $\underline{w} \leq \overline{w}$ on $\{a, b\}$, then $\underline{w} \leq \overline{w}$ on (a, b). (See Crandall, Ishii, and Lions (1992, Theorem 3.3).) Classical sub- and supersolutions also satisfy a comparison principle, but other nonclassical notions of "solution" do not.

The comparison principle may be used to obtain uniqueness results; a standard one is that the HJB equation has at most one viscosity solution with the right boundary conditions satisfying a linear-growth condition. It follows that the value function is the unique solution with the right boundary conditions and linear growth; see Fleming and Soner (2006, ch. V). We use the comparison principle in this manner in the proofs of Lemmata 1 and 3 (Appendices B and D).

The comparison principle may also be used to establish the continuity of solutions, and thus of the value function. In particular, suppose that we have shown that the upper (lower) semicontinuous envelope $v^\star$ ($v_\star$) of the value v is a subsolution (supersolution) of the HJB equation, and that $v_\star = v^\star$ on $\{0, 1\}$. (We do precisely this in the proof of Theorem 1 in Appendix A.) A comparison principle then yields $v^\star \leq v_\star$, which since $v_\star \leq v \leq v^\star$ implies that v is itself a viscosity solution, hence continuous.

In our proof of Theorem 1 (Appendix A), we eschew this approach in favor of a direct proof that v is continuous. We do this because we are not aware of a comparison principle that applies assuming only piecewise continuity of u. The closest result that we know of is Theorem 3.3 in Soravia (2006), which would be applicable under the additional hypotheses that u has only *finitely* many discontinuities and satisfies $u(p) \in [u(p-) \wedge u(p+), u(p-) \vee u(p+)]$ at every $p \in (0, 1)$.

References

Bressan, Alberto (2011), "Viscosity solutions of Hamilton–Jacobi equations and optimal control problems." Lecture notes. https://www.math.psu.edu/bressan/PSPDF/HJ-lnotes.pdf. [5]

Calder, Jeff (2018). "Lecture notes on viscosity solutions." Lecture notes. http://www-users.math.umn.edu/~jwcalder/8590F18/viscosity_solutions.pdf. [5]

Coddington, Earl A. (1961), *An Introduction to Ordinary Differential Equations*. Dover, New York, NY. [1]

Crandall, Michael G. (1997), "Viscosity solutions: A primer." In *Viscosity Solutions and Applications* (Italo Capuzzo Dolcetta and Pierre-Louis Lions, eds.), 1–43, Springer, Berlin. [5]

Crandall, Michael G., Hitoshi Ishii, and Pierre-Louis Lions (1992), "User's guide to viscosity solutions of second order partial differential equations." *Bulletin of the American Mathematical Society. New Series*, 27, 31–53. [5, 7]

Evans, Lawrence C. (2010), *Partial Differential Equations*, second edition. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI. [5, 7]

Fleming, Wendell H. and H. Mete Soner (2006), *Controlled Markov Processes and Viscosity Solutions*, second edition. Springer, New York, NY. [7]

Karatzas, Ioannis and Steven E. Shreve (1991), *Brownian Motion and Stochastic Calculus*, second edition. Graduate Texts in Mathematics. Springer, New York, NY. [5]

Katzourakis, Nikos (2015), *An Introduction to Viscosity Solutions for Fully Nonlinear PDE With Applications to Calculus of Variations in L^∞*. SpringerBriefs in Mathematics. Springer, Cham, CH. [5]

Moll, Benjamin (2020), "Viscosity solutions for dummies (including economists)." https://benjaminmoll.com/wp-content/uploads/2019/07/viscosity_slides.pdf. [5]

Soravia, Pierpaolo (2006), "Uniqueness results for fully nonlinear degenerate elliptic equations with discontinuous coefficients." *Communications on Pure & Applied Analysis*, 5, 213–240. [7]

Co-editor Marina Halac handled this manuscript.

Manuscript received 12 January, 2022; final version accepted 2 April, 2022; available online 6 April, 2022.