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Ergodic Markov equilibrium with incomplete markets and short sales

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This paper studies recursive exchange economies with short sales. Agents maximize discounted expected utility. The asset structure is general and includes real securities, infinite-lived stocks, options, and other derivatives. The main result shows the existence of a competitive equilibrium process that is stationary and has an invariant ergodic measure. Ergodicity is required in finance for time series analysis of structural asset pricing models. This equilibrium property is difficult to obtain when heterogeneous agents can accumulate debt over time. Bounded marginal utility is shown to be a key condition for ergodicity in this setting.

Keywords. General equilibrium, incomplete markets, recursive, Markov, stationary, ergodic, existence.

JEL classification. D52, D80, D90, G10.

1. Introduction

Contemporaneous works by Magill and Quinzii (1994), Hernández and Santos (1996), and Levine and Zame (1996) prove the existence of a sequential equilibrium for a broad class of infinite-horizon exchange economies with incomplete financial markets. In Magill and Quinzii (1994), for instance, agents trade one-period numeraire securities in zero net supply and their debt paths are restricted by three alternative criteria: (i) personalized transversality conditions, (ii) implicit debt constraints, or (iii) an explicit uniform debt ceiling that never binds. They show that the equilibrium concepts implied by each of these three criteria coincide, given their stated assumptions. Similar equivalence results appear in Hernández and Santos (1996) and Levine and Zame (1996). These papers are celebrated because they present different ways to rule out Ponzi schemes without introducing additional market imperfections into the economy. Among the alternatives, models with explicit debt constraints became usual in macroeconomics and finance.

In another seminal paper, Duffie et al. (1994) analyze a class of recursive exchange economies with incomplete financial markets. Agents maximize expected discounted utility and trade numeraire assets of infinite maturity in positive net supply. Short sales

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are not allowed and agents cannot accumulate debt. The authors prove the existence of a spotless sequential equilibrium that can be represented by a stationary Markov transition probability. Moreover, they show that if one introduces a weak form of sunspot, then the stationary Markov equilibrium has an invariant ergodic measure.

This paper extends the ergodic analysis to recursive economies in which agents can accumulate debt over time. The setup accommodates one-period numeraire assets traded in zero net supply, infinite-lived real assets in positive net supply, and complex securities in zero net supply. Complex securities represent forward contracts, options, and other derivatives whose payoffs depend on the prices of other assets. Agents can short-sell all assets. As in Radner (1972), trades in securities whose payoffs depend on prices are restricted by exogenous short-sale constraints.[1] Moreover, as in Magill and Quinzii (1994), individual total debt is restricted by an explicit debt ceiling that never binds in equilibrium. In other words, agents can accumulate debt without restriction by selling one-period numeraire securities. They can also short-sell a limited amount of infinite-lived stocks, forward contracts, options, and derivatives.

This general asset structure is adopted so as to make the results useful for financial economists. In a stationary Markov equilibrium process with initial states drawn from the invariant ergodic measure μ, the time series distribution of the state vector (which includes asset prices) asymptotically converges to μ. Since ergodicity is not empirically testable, this type of existence result is our best justification for using asymptotic theory in financial time series.[2]

Ergodicity is not a simple object in economies where heterogeneous agents are allowed to hold debt. For instance, consider the standard setup with complete markets, one consumption good, time-separable preferences with different discount factors, and continuously increasing and concave Bernoulli utilities. As pointed out by Ramsey (1928), any Pareto optimal allocation in this setting is such that agents with the lowest discount factor gradually accumulate the entire wealth of the economy, and the consumption levels of all other agents gradually converge to zero over time.[3] When marginal utilities are unbounded, the Pareto optimal consumption of the less patient agents approaches zero but never reaches that level. This type of process does not have an invariant ergodic measure and thus, from the first welfare theorem, there is no ergodic equilibrium in this context. The main result in this paper shows that bounded marginal utility is a sufficient condition for the existence of an ergodic Markov equilibrium.

The remainder of the paper is organized as follows. Section 2 describes a general class of recursive exchange economies, defines the competitive equilibrium concept,

[1] Hart (1975) shows that a competitive equilibrium might not exist when asset payoffs depend on endogenous prices and portfolios are not subject to short-sale constraints. Typically, the set of economies without an equilibrium is not generic. For instance, Magill and Quinzii (1996) and Hernández and Santos (1996) present generic existence results for economies with long-lived assets and debt constraints that never bind.

[2] Tests for stationarity of a time series are important and useful, but they do not assure the existence of an invariant ergodic measure, as required in standard time series econometrics.

[3] This result was studied in many alternative settings. Becker (1980), Rader (1981), and Bewley (1982) address this topic in environments without uncertainty. Blume and Easley (2006) analyze the problem under uncertainty.

derives a uniform debt ceiling that never binds, and constructs bounds for equilibrium portfolios and prices. Section 3 proves the existence of two related Markov equilibrium concepts. Concluding remarks appear in Section 4. The Appendix is reserved for a technical step of the main proof.

2. Model

Consider a class of pure exchange economies with uncertainty, countably infinite periods $t \in \mathbb{T} \equiv \{0, 1, \dots\}$, multiple consumption goods $l \in \mathbb{L} \equiv \{1, \dots, L\}$, and long-lived heterogeneous agents $i \in \mathbb{I} \equiv \{1, \dots, I\}$, where $L \geq 1$ and $I \geq 1$. Uncertainty is represented by a probability space $(\Omega, \mathcal{B}_\Omega, \nu)$, where $\Omega \equiv [0, 1]$, $\mathcal{B}_\Omega$ is the Borel σ-algebra on Ω, and ν is the Lebesgue measure on $(\Omega, \mathcal{B}_\Omega)$. Each $\omega \in \Omega$ determines a sequence of fundamental shocks $\{s_t(\omega)\}_{t\in\mathbb{T}}$, where $s_t(\omega)$ takes $S \geq 1$ possible values in the set $\mathbb{S} \equiv \{1, \dots, S\}$. The sequence $\{s_t(\omega)\}_{t\in\mathbb{T}}$ follows a time-homogeneous Markov process with transition $P(s_{t+1}|s_t) > 0$.

The possibility of extrinsic (sunspot) uncertainty is represented by independent and uniformly distributed random variables $\eta_t : \Omega \to [0, 1]$ for $t \in \mathbb{T}$. The economy's information structure is given by a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$, where $\mathcal{F}_t$ is the σ-algebra generated by $(s_0(\omega), \dots, s_t(\omega), \eta_0(\omega), \dots, \eta_t(\omega))$. The period-$t$ entry of each stochastic process in this economy is $\mathcal{F}_t$-measurable. A process is said to be spotless when its period-t entry is also measurable with respect to the σ-algebra generated by $(s_0(\omega), \dots, s_t(\omega))$, that is, when it does not vary with the history of sunspots $(\eta_0(\omega), \dots, \eta_t(\omega))$.

For notational convenience, the decision node index (t, ω) is omitted throughout the paper. The time index t is used when the context requires. Otherwise, the subscripts $+1$ and -1, respectively, indicate the next-period and previous-period realization of the underlying random variable. The same letter is used to represent the random variable and its respective realizations. For instance, s_{+1} might represent the random variable $s_{t+1}(\omega)$ or a particular realization $s_{t+1} \in \mathbb{S}$. The distinction is clear from the context.

In each period t, individual endowments are determined by time-invariant functions $e_i : \mathbb{S} \to \mathbb{R}^L_+$ with the property that, for every i and s, $e_{i,1,s} > 0$ and $\sum_{i\in\mathbb{I}} e_{i,s} \in \mathbb{R}^L_{++}$. Agent i's preference is numerically represented by a time-separable discounted expected utility function U_i. For any $\mathbb{R}^L_+$-valued stochastic process $\mathbf{x}_i \equiv \{x_{i,t}\}_{t\in\mathbb{T}}$ on $(\Omega, \mathcal{B}_\Omega, \nu)$, define

$$U_i(\mathbf{x}_i) \equiv E\left[\sum_{t\in\mathbb{T}} \beta_i^t u_i(x_{i,t}) \middle| \mathcal{F}_0\right],$$

where $\beta_i \in (0, 1)$ is an agent-specific discount factor, $x_{i,t} \in \mathbb{R}^L_+$ is agent i's random consumption in period t, and $u_i : \mathbb{X} \to \mathbb{R}$ is a continuous, nondecreasing, and concave Bernoulli utility function that is increasing in its first entry and whose domain $\mathbb{X}$ is an open set containing $\mathbb{R}^L_+$.

Remark 1. The assumptions on u_i imply that the supergradient correspondence

$$\partial u_i(x_i) \equiv \{d_i \in \mathbb{R}^L_+ : u_i(\tilde{x}_i) \leq u_i(x_i) + d_i \cdot (\tilde{x}_i - x_i), \forall \tilde{x}_i \in \mathbb{X}\}$$

is upper hemicontinuous, nonempty, and compact valued at every $x_i \in \mathbb{R}^L_+$. (Marginal utilities are bounded on the boundary of $\mathbb{R}^L_+$, since u_i is defined on an open set $\mathbb{X} \supset \mathbb{R}^L_+$.)

Markets open in each decision node (t, ω) to trade the $L > 0$ consumption goods at prices $p \in \mathbb{R}^L_+$. Good 1 is the numeraire and its price is normalized to 1 in all nodes. Markets also trade the following three types of assets.

1. *One-period numeraire assets in zero net supply*. There are $J^a \geq 1$ one-period numeraire assets traded in zero net supply. In each period, their payoffs are contingent on the current realization of the exogenous shock $s \in \mathbb{S}$. Payoffs are measured in units of good 1 and represented by (nonnull) linearly independent vectors $a_j \in \mathbb{R}^S_+$ for $j \in \mathbb{J}^a \equiv \{1, \dots, J^a\}$. For each decision node, let $\theta^a_i \in \mathbb{R}^{J^a}$ and $q^a \in \mathbb{R}^{J^a}_+$ represent agent i's portfolio and the market prices for these J^a securities. There is no explicit short-sale constraint on these assets.

2. *Infinite-lived real assets in positive net supply*. There are also $J^b \geq 0$ infinite-lived real assets traded in positive net supply. These assets pay dividends in each period—measured in units of the L consumption goods—according to vectors $A_{j,s} \in \mathbb{R}^L_+$ for $s \in \mathbb{S}$ and $j \in \mathbb{J}^b \equiv \{J^a + 1, \dots, J^b\}$. Their total supply is normalized to 1. For each decision node, $\theta^b_i \in \mathbb{R}^{J^b}$ and $q^b \in \mathbb{R}^{J^b}_+$ represent agent i's portfolio and the market prices for these J^b securities. There is a vector of short-sale limits $\bar{\theta}^b \in \mathbb{R}^{J^b}_+$ such that
$$\theta^b_i \geq -\bar{\theta}^b. \tag{1}$$

3. *Complex securities in zero net supply*. The third class of assets in this economy comprises $J^c \geq 0$ complex financial securities traded in zero net supply. These assets pay in units of good 1 depending on the exogenous shock $s \in \mathbb{S}$ and prices (p, q^a, q^b). Formally, their payoffs are defined by continuous functions $r_{j,s} : \mathbb{R}^{L+J^a+J^b}_+ \to \mathbb{R}_+$ for each $s \in \mathbb{S}$ and $j \in \mathbb{J}^c \equiv \{J^a + J^b + 1, \dots, J\}$. The term $r_{j,s}(p, q^a, q^b)$ represents the payoff of security j in nodes in which the shock s and the prices (p, q^a, q^b) are realized. For each decision node, let $\theta^c_i \in \mathbb{R}^{J^c}$ and $q^c \in \mathbb{R}^{J^c}_+$ represent agent i's portfolio and the market prices for these J^c securities. There is a vector of short-sale limits $\bar{\theta}^c \in \mathbb{R}^{J^c}_{++}$ such that
$$\theta^c_i \geq -\bar{\theta}^c. \tag{2}$$

Short-sale and debt constraints. Conditions (1) and (2) are short-sale constraints on assets whose payoff depends on equilibrium prices (namely, infinite-lived real assets and complex securities). This follows well established results in the general equilibrium literature and is necessary for the existence of a competitive equilibrium (see footnote 1). It must be emphasized, however, that no short-sale constraint is imposed on trades of the one-period numeraire assets. Agents are free to accumulate debt on these assets and, therefore, a restriction on total debt is needed to preclude Ponzi schemes. Define a uniform debt ceiling $M > 0$ such that
$$q \cdot \theta_i \geq -M, \tag{3}$$

where $q \equiv (q^a, q^b, q^c) \in \mathbb{R}^J_+$, $\theta_i \equiv (\theta_i^a, \theta_i^b, \theta_i^c) \in \mathbb{R}^J$, and $J \equiv J^a + J^b + J^c$. The value of M is chosen later such that inequality (3) never binds in equilibrium.

Individual problem and market clearing. In each node (t, ω), agents choose their personal consumption bundle and portfolio $(x_i, \theta_i) \in \mathbb{R}^L_+ \times \mathbb{R}^J$, taking as given (i) their previous-period portfolio $\theta_{i,-1} \in \mathbb{R}^J$, (ii) the current prices $(p, q) \in \mathbb{R}^{L+J}_+$, and (iii) the stochastic process describing the future prices. The choices (x_i, θ_i) must be $\mathcal{F}_t$-measurable, that is, they must coincide for all nodes (t, ω) belonging to the same information set in $\mathcal{F}_t$. Moreover, agents' choices must ν-almost surely (ν-a.s.) satisfy constraints (1)–(3) and the budget inequality

$$p \cdot (x_i - e_{i,s}) + q \cdot \theta_i \leq W_s(\theta_{i,-1}, p, q), \tag{4}$$

where

$$W_s(\theta_{i,-1}, p, q) \equiv \sum_{j \in \mathbb{J}^a} a_{j,s}\theta^a_{i,j,-1} + \sum_{j \in \mathbb{J}^b} (p \cdot A_{j,s} + q^b_j)\theta^b_{i,j,-1} + \sum_{j \in \mathbb{J}^c} r_{j,s}(p, q^a, q^b)\theta^c_{i,j,-1}. \tag{5}$$

This economy is characterized by

$$\mathcal{E} \equiv \{\mathbb{I}, \mathbb{T}, (\Omega, \mathcal{B}_\Omega, \nu), \{\mathcal{F}_t\}_{t \in \mathbb{T}}, \mathbb{S}, P, (U_i, e_i)_{i \in \mathbb{I}}, a, A, r, \bar{\theta}^b, \bar{\theta}^c, M\}.$$

Markets are said to clear when the feasibility conditions

$$\sum_{i \in \mathbb{I}} (x_i - e_{i,s}) = \sum_{j \in \mathbb{J}^b} A_{j,s} \tag{6}$$

and

$$\sum_{i \in \mathbb{I}} \theta_i \equiv \sum_{i \in \mathbb{I}} (\theta_i^a, \theta_i^b, \theta_i^c) = (\mathbf{0}, \mathbf{1}, \mathbf{0}) \tag{7}$$

hold ν-almost surely.[4] Equation (6) states that the aggregate consumption equals the aggregate endowment plus the amount of goods delivered by assets in positive net supply. Equation (7) is the financial market clearing condition.

Remark 2. Conditions (1)–(4), (6), and (7) are required to hold ν-a.s. since the framework allows for sunspots. The sets $\{\omega \in \Omega : (s_0(\omega), \dots, s_t(\omega)) = (s_0, \dots, s_t)\}$ in $\mathcal{F}_t$ have positive probability under ν. They constitute the relevant information in a spotless scenario.

2.1 *Competitive equilibrium*

A competitive equilibrium is represented in this setting by a stochastic sequence $\{z_t\}_{t \in \mathbb{T}}$ on $(\Omega, \mathcal{B}_\Omega, \nu)$. Each element z_t—or equivalently $z \equiv (s, \theta_{-1}, x, \theta, p, q)$—is a random variable consisting of the exogenous state $s \in \mathbb{S}$, the previous-period portfolios $\theta_{-1} \in \mathbb{R}^{IJ}$, and the current individual choices and market prices $(x, \theta, p, q) \in \mathbb{R}^{IL}_+ \times \mathbb{R}^{IJ} \times \mathbb{R}^{L+J}_+$.

[4]The vector $(\mathbf{0}, \mathbf{1}, \mathbf{0})$ has zeros in its first J^a entries, ones in the following J^b entries, and zeros in the final J^c entries.

A stochastic sequence $\{z_t\}_{t\in\mathbb{T}}$ on $(\Omega, \mathcal{B}_\Omega, \nu)$ is said to be *consistent* for an economy $\mathcal{E}$ if, for every $t \in \mathbb{T}$, (i) z_t is $\mathcal{F}_t$-measurable, (ii) the marginal probability distribution of s_{t+1} conditional on $(z_0, \dots, z_t)$ is ν-a.s. given by $P(\cdot|s_t)$, and (iii) the vector θ_{-1} in state z_{t+1} is ν-a.s. equal to θ in state z_t.

For a given consistent process $\{z_t\}_{t\in\mathbb{T}}$, a policy $\{\tilde{x}_{i,t}, \tilde{\theta}_{i,t}\}_{t\in\mathbb{T}}$ on $(\Omega, \mathcal{B}_\Omega, \nu)$ is *budget feasible* if $(\tilde{x}_{i,t}, \tilde{\theta}_{i,t})$ is $\mathcal{F}_t$-measurable and ν-a.s. satisfies (1)–(4), for every $t \in \mathbb{T}$. For a given consistent $\{z_t\}_{t\in\mathbb{T}}$, a budget-feasible policy $\{\tilde{x}_{i,t}, \tilde{\theta}_{i,t}\}_{t\in\mathbb{T}}$ is *individually optimal* if there is no other budget-feasible policy $\{x'_{i,t}, \theta'_{i,t}\}_{t\in\mathbb{T}}$ such that $U_i(\mathbf{x}'_i) > U_i(\tilde{\mathbf{x}}_i)$.

Definition 1. A *competitive equilibrium* is a consistent stochastic process $\{z_t\}_{t\in\mathbb{T}}$ with the properties that (i) the policy $\{x_{i,t}, \theta_{i,t}\}_{t\in\mathbb{T}}$ is individually optimal for all $i \in \mathbb{I}$ and (ii) the feasibility constraints (6) and (7) hold ν-almost surely for all $t \in \mathbb{T}$.

Remark 3. There is an initial condition $(\hat{s}, \hat{\theta}_{-1}) \in \mathbb{S} \times \mathbb{R}^{IJ}$ associated with the period-zero element of each equilibrium process. Some papers set this condition in the description of the economy. In recursive environments, the initial condition is part of the equilibrium definition.

Remark 4. Take a consistent process $\{z_t\}_{t\in\mathbb{T}}$. Since preferences have a time-separable discounted expected utility representation, a plan $\{\tilde{x}_{i,t}, \tilde{\theta}_{i,t}\}_{t\in\mathbb{T}}$ is individually optimal for $\{z_t\}_{t\in\mathbb{T}}$ if and only if $\{\tilde{x}_{i,t}, \tilde{\theta}_{i,t}\}_{t=\tau}^{\infty}$ is individually optimal for $\{z_t\}_{t=\tau}^{\infty}$, for every $\tau \geq 1$.

2.2 *A uniform debt ceiling that never binds*

The feasibility condition (6) implies that equilibrium consumption vectors lie ν-a.s. in $K_i \equiv \{x_i \in \mathbb{R}_+^L : \max(x_i) \leq \bar{x}\}$, where

$$\bar{x} \equiv \max_{(l,s)\in\mathbb{L}\times\mathbb{S}} 2\left(\sum_{i\in\mathbb{I}} e_{i,l,s} + \sum_{j\in\mathbb{J}^b} A_{j,l,s}\right) > 0.$$

Agents are not restricted to choose consumption levels in K_i, but they do so in equilibrium. This feature is used throughout the paper.

The assumptions on utility functions and endowments guarantee the existence of a uniform lower bound on impatience, in the sense of Magill and Quinzii (1994), Hernández and Santos (1996), and Levine and Zame (1996). Formally, there is a $\rho \in (0, 1)$ such that

$$u_i(x_{i,0} + (1, 0, \dots, 0)) + E\left[\sum_{t\geq 1} \beta_i^t u_i(\rho x_{i,t}) \middle| \mathcal{F}_0\right] > U_i(\mathbf{x}_i) \tag{8}$$

for any $i \in \mathbb{I}$ and any $\mathbb{R}_+^L$-valued process $\mathbf{x}_i \equiv \{x_{i,t}\}_{t=0}^{\infty}$ on $(\Omega, \mathcal{B}_\Omega, \nu)$ that is ν-a.s. uniformly bounded by $\bar{x}$. For the remainder of the paper, fix a $\rho \in (0, 1)$ satisfying condition (8).

LEMMA 1. *In any competitive equilibrium, the values of individual portfolios are ν-a.s. uniformly bounded by*

$$-\frac{I}{1-\rho} \leq q \cdot \theta_i \leq \frac{1}{1-\rho}. \tag{9}$$

PROOF. The proof is adapted from Magill and Quinzii (1994, p. 873). The second inequality holds in equilibrium. If $q \cdot \theta_i > 1/(1-\rho)$ with positive probability, then agent i would be willing to modify the current-period portfolio to $\rho\theta_i$, and the consumption and asset holdings in all future nodes to $\rho(x_{i,+\tau}, \theta_{i,+\tau})$, where $\tau \geq 1$. This modification is budget feasible and it frees $(1-\rho)q \cdot \theta_i > 1$ units of account. This is enough to purchase one unit of good 1 in the present (recall that $p_1 = 1$). It follows from (8) that this modification would be desired by agent i, contradicting the individual optimality of the original equilibrium process.

To derive the first inequality in (9), notice from (7) that $\sum_{i \in \mathbb{I}} q \cdot \theta_i = q^b \cdot \mathbf{1} \geq 0$ ν-almost surely. Thus, the second inequality in (9) implies

$$q \cdot \theta_i \geq -\sum_{\hat{\imath} \neq i} q \cdot \theta_{\hat{\imath}} \geq -\frac{I}{1-\rho} \quad (\nu\text{-a.s.}).$$

□

The debt limit $-I/(1-\rho)$ does not depend on the previous-period portfolios or equilibrium future prices. An explicit debt ceiling greater than $I/(1-\rho)$ never binds in any competitive equilibrium. Hereafter fix $M > I/(1-\rho)$.

2.3 *Uniform bounds on portfolios and prices*

Let us now define bounds for portfolios and prices associated with any equilibrium process.

LEMMA 2. *Prices are ν-a.s. uniformly bounded in any competitive equilibrium.*

PROOF. Recall that u_i is increasing in good 1, and $\partial u_i(x_i)$ is upper hemicontinuous, nonempty, and compact valued at every x_i in $\mathbb{R}^L_+$. Therefore, (i) there is a positive upper bound $\delta_{\max} > 0$ for the set of feasible marginal utilities $\{\max(d_i) : d_i \in \partial u_i(x_i)$ for some $(i, x_i) \in \mathbb{I} \times K_i\}$ and (ii) there is a positive lower bound $\delta_{1,\min} > 0$ for the set of feasible marginal utilities with respect to good 1 $\{\text{proj}_1(d_i) : d_i \in \partial u_i(x_i)$ for some $(i, x_i) \in \mathbb{I} \times K_i\}$, where $\text{proj}_1(\cdot)$ is the projection of a vector into its first entry.

It can be shown that $\max(p)$ is ν-a.s. uniformly bounded by $\delta_{\max}/\delta_{1,\min}$ in any competitive equilibrium process. The feasibility condition (6) implies that, for each good l, there is at least one agent consuming strictly positive amounts of it. Concavity of u_i implies that if $p_l > \delta_{\max}/\delta_{1,\min}$ with positive probability, this agent would prefer to reduce the consumption of good l and increase the consumption of good 1 in these nodes at the rate of $p_l > \delta_{\max}/\delta_{1,\min}$ units of good 1 per unit of good l. This modification in the consumption plan is budget feasible since $p_1 = 1$, which contradicts the individual optimality of the equilibrium process.

The second step constructs an upper bound for q^a. Define $a_{\max} \equiv \max\{a_{j,s} \in \mathbb{R}_+ : (j,s) \in \mathbb{J}^a \times \mathbb{S}\}$ and $e_{1,\min} \equiv \min\{e_{i,1,s} \in \mathbb{R}_+ : (i,s) \in \mathbb{I} \times \mathbb{S}\}$. In any equilibrium process, one must ν-a.s. have $\max(q^a) \leq a_{\max}/((1-\rho)e_{1,\min})$. To prove this, suppose there is an asset $j \in \mathbb{J}^a$ such that $q^a_j > a_{\max}/((1-\rho)e_{1,\min})$ with positive probability. Notice from (7) that there is always some agent i with $q \cdot \theta_i \geq 0$. Take this agent, modify the current-period portfolio to $\rho\theta_i$, and modify the consumption and asset holdings in all future nodes to $\rho(x_{i,+\tau}, \theta_{i,+\tau})$, where $\tau \geq 1$. The vector $(x_i, \rho\theta_i)$ ν-a.s. satisfies the budget inequalities (1)–(4). Moreover, this modification frees the income $(1-\rho)p_{+\tau} \cdot e_{i,s_{+\tau}}$ in each future node (i.e., when $\tau \geq 1$). The extra income in the next-period nodes (i.e., when $\tau = 1$) allows agent i to rebalance the modified portfolio $\rho\theta_i$ by selling $\min((1-\rho)e_{1,\min}/a_{\max}, M/q_j)$ units of asset j. Since $q \cdot \theta_i \geq 0$, the debt constraint (3) still holds after this transaction. Moreover, since $M > I/(1-\rho) > 1$, this allows one to consume an additional unit of good 1 in the present (i.e., when $\tau = 0$). It then follows from (8) that this contradicts the individual optimality of the original equilibrium process.

Third, take the second inequality in (9) to obtain $q \cdot \sum_{i\in\mathbb{I}} \theta_i \leq I/(1-\rho)$ ν-almost surely. It follows from (7) that $q \cdot (\mathbf{0}, \mathbf{1}, \mathbf{0}) = \sum_{j\in\mathbb{J}^b} q^b_j \leq I/(1-\rho)$ ν-almost surely. Since $q^b \in \mathbb{R}^{J^b}_+$, this inequality defines a uniform upper bound for the prices of infinite-lived assets.

From the previous steps, equilibrium (p, q^a, q^b) lies ν-a.s. in a compact set, say $\Delta' \subset \mathbb{R}^{L+J^a+J^b}_+$. When $J^c > 0$, the payoff of each complex asset is ν-a.s. uniformly bounded by $r_{\max} \equiv \max\{r_{j,s}(p, q^a, q^b) : (j, s, p, q^a, q^b) \in \mathbb{J}^c \times \mathbb{S} \times \Delta'\}$. This maximum is well defined since $r_{j,s}$ is continuous and $\mathbb{J}^c \times \mathbb{S} \times \Delta'$ is nonempty and compact. Therefore, the second step of this proof can be replicated to show that $\max(q^c)$ is ν-a.s. uniformly bounded by $\max(r_{\max}/((1-\rho)e_{1,\min}), 1/((1-\rho)\min(\bar{\theta}^c)))$.[5] □

Lemma 3. *Individual portfolios are ν-a.s. uniformly bounded in any competitive equilibrium.*

Proof. The short-sale constraints (1) and (2) restrict equilibrium portfolios of infinite-lived and complex securities. For every $i \in \mathbb{I}$, the following inequalities hold ν-almost surely: $-\bar{\theta}^b \leq \theta^b_i \leq \mathbf{1} + I\bar{\theta}^b$ and $-\bar{\theta}^c \leq \theta^c_i \leq I\bar{\theta}^c$. Let us then adapt the argument used in Magill and Quinzii (1994, p. 873) to derive equilibrium bounds for θ^a_i.

The budget inequality (4) holds with equality (ν-a.s.), since u_i is increasing in its first entry. Define

$$\begin{aligned} &H(s, \theta^b_{-1}, \theta^c_{-1}, x, p, q^a, q^b) \\ &\qquad \equiv p \cdot (x_i - e_{i,s}) - \sum_{j\in\mathbb{J}^b} (p \cdot A_{j,s} + q^b_j)\theta^b_{i,j,-1} - \sum_{j\in\mathbb{J}^c} r_{j,s}(p, q^a, q^b)\theta^c_{i,j,-1}, \end{aligned}$$

and notice from (4), (5), and (9) that

$$H(\cdot) - \frac{I}{1-\rho} \leq a_s \cdot \theta^a_{i,-1} \leq H(\cdot) + \frac{1}{1-\rho} \quad (\nu\text{-a.s.}), \tag{10}$$

[5]The short-sale constraint (2) is nonbinding after individual i's portfolio is scaled down to $\rho\theta_i$. When adapting the argument used in the second step, agent i is able to sell $\min((1-\rho)e_{1,\min}/r_{\max}, M/q_j, (1-\rho)\bar{\theta}^c_j)$ units of each asset $j \in \mathbb{J}^c$.

where $a_s \equiv (a_{1,s}, \dots, a_{J^a,s})$ and $a_s \cdot \theta^a_{i,-1} = \sum_{j\in\mathbb{J}^a} a_{j,s}\theta^a_{i,j,-1}$.

There is $\bar{H} > 0$ such that $-\bar{H} \leq H(\cdot) \leq \bar{H}$ ν-almost surely. This is thanks to the following facts: (i) $\mathbb{S}$ is finite; (ii) $(\theta^b_{-1}, \theta^c_{-1}, x, p, q^a, q^b)$ is ν-a.s. uniformly bounded; (iii) $r_{j,s}(p, q^a, q^b)$ is continuous. Since (10) holds in every period, each entry of the $S \times 1$ vector $[a_s \cdot \theta^a_i]_{s\in\mathbb{S}}$ is ν-a.s. uniformly bounded from below by $-\bar{H} - I/(1-\rho)$ and from above by $\bar{H} + 1/(1-\rho)$.

The $S \times J^a$ matrix represented by $[a_s]_{s\in\mathbb{S}}$ has full column rank, since these payoffs are linearly independent. For each $y \in \mathbb{R}^S$, define $\theta^a_i(y)$ as the unique solution for the linear system

$$[a_s \cdot \theta^a_i]_{s\in\mathbb{S}} = y.$$

The function $\theta^a_i(y)$ is continuous in y and can be explicitly derived through Cramer's formula. Therefore, by taking y in $[-\bar{H} - I/(1-\rho), \bar{H} + 1/(1-\rho)]^S$, one can find upper and lower bounds for θ^a_i.[6] □

3. Ergodic Markov equilibrium

The equilibrium analysis follows the structure formulated in Duffie et al. (1994), where the main elements are the state space $\mathbb{Z}$ and the expectations correspondence g. Each state variable $z \equiv (s, \theta_{-1}, x, \theta, p, q)$ describes the current endogenous and exogenous variables and is a sufficient statistic for the future evolution of the model.

Recall from Section 2.2 that equilibrium consumption levels lie ν-a.s. in a compact set $K \equiv K_1 \times \cdots \times K_I$. Lemma 2 states that equilibrium prices lie ν-a.s. in a compact set, say $\Delta \subset \mathbb{R}^{L+J}_+$. Moreover, it follows from Lemma 3 that there is a compact set $\Theta_i \subset \mathbb{R}^J$ such that, in equilibrium, individual portfolio choices lie ν-a.s. in its interior. Take $\Theta \equiv \Theta_1 \times \cdots \times \Theta_I$ and let Θ^- be a copy of Θ. One can then restrict z to take values in a nonempty state space $\mathbb{Z}$ that embeds these equilibrium properties and the economy's feasibility constraints, namely,

$$\mathbb{Z} \equiv \{z \in \mathbb{S} \times \Theta^- \times K \times \Theta \times \Delta : (6)\text{–}(7)\}.$$

Markov equilibria is represented by transition probabilities over subsets of $\mathbb{Z}$. For any arbitrary subset $Z \subset \mathbb{Z}$, let $\mathcal{B}_Z$ be the Borel σ-algebra over Z and let $\mathcal{P}_Z$ be the set of probability measures on $(Z, \mathcal{B}_Z)$. Any transition $\Pi : Z \to \mathcal{P}_Z$ can be represented in the form $(x_{+1}, \theta_{+1}, p_{+1}, q_{+1}) = f(s_{+1}, \eta_{+1}, z)$, where f is a Borel measurable function and η_{+1} follows an independent and identically distributed (i.i.d.) uniform distribution in $[0, 1]$.[7] Thus, Markov transitions can be used to define stochastic processes in the original information structure $(\Omega, \mathcal{B}_\Omega, \nu, \{\mathcal{F}_t\}_{t\in\mathbb{T}})$.

The second main element of the analysis is the expectations correspondence $g : \mathbb{Z} \to \mathcal{P}_\mathbb{Z}$ defined as follows. For each current state $z \in \mathbb{Z}$, let $g(z)$ be a (possibly empty) set of probability measures over the next-period state variable z_{+1} such that conditions (a) and (b) are satisfied.

[6] This reasoning does not apply for the other securities because the rank of their payoff matrix depends on endogenous prices.

[7] Since $\mathbb{S}$ is finite, this follows from Lemma 3.22 in Kallenberg (2001, p. 56).

(a) The set $g(z)$ is empty unless the variables in z are such that inequalities (1)–(3) hold and condition (4) is satisfied with equality for every agent i in $\mathbb{I}$.

(b) A measure $\pi \in g(z)$ if and only if

(b.1) there is a function $h: \mathbb{S} \to \Theta^- \times K \times \Theta \times \Delta$ such that the support of π is the graph of h

(b.2) the marginal of π on $\mathbb{S}$ is $P(\cdot|s)$

(b.3) the marginal of π on Θ^- degenerates to θ

(b.4) the next-period states $\{z_{s_{+1}} : s_{+1} \in \mathbb{S}\}$ in the support of π are such that

(b.4.1) the budget-constraint inequalities (1)–(3) hold and condition (4) is satisfied with equality, for every agent $i \in \mathbb{I}$

(b.4.2) for each $i \in \mathbb{I}$, there exist $(d_i, \lambda_i) \in \partial u_i(x_i) \times \mathbb{R}_+$ and $(d_{i,s_{+1}}, \lambda_{i,s_{+1}}) \in \partial u_i(x_{i,s_{+1}}) \times \mathbb{R}_+$, for every $s_{+1} \in \mathbb{S}$, such that the following conditions hold:

$$\lambda_i p_l \geq d_{i,l}$$

with equality if $x_{i,l} > 0$, for each $l \in \mathbb{L}$;

$$\lambda_{i,s_{+1}} p_{l,s_{+1}} \geq \beta_i d_{i,l,s_{+1}} P(s_{+1}|s)$$

with equality if $x_{i,l,s_{+1}} > 0$, for each $(l, s_{+1}) \in \mathbb{L} \times \mathbb{S}$;

$$\lambda_i q_j^a = \sum_{s_{+1} \in \mathbb{S}} \lambda_{i,s_{+1}} p_{1,s_{+1}} a_{j,s_{+1}}$$

for each $j \in \mathbb{J}^a \equiv \{1, \ldots, J^a\}$;

$$\lambda_i q_j^b \geq \sum_{s_{+1} \in \mathbb{S}} \lambda_{i,s_{+1}} (p_{s_{+1}} \cdot A_{j,s_{+1}} + q^b_{j,s_{+1}})$$

with equality if $\theta^b_{i,j} > -\bar{\theta}^b_j$, for each $j \in \mathbb{J}^b \equiv \{J^a + 1, \ldots, J^b\}$; and

$$\lambda_i q_j^c \geq \sum_{s_{+1} \in \mathbb{S}} \lambda_{i,s_{+1}} r_{j,s_{+1}} (p_{s_{+1}}, q^a_{s_{+1}}, q^b_{s_{+1}})$$

with equality if $\theta^c_{i,j} > -\bar{\theta}^c_j$, for each $j \in \mathbb{J}^c \equiv \{J^a + J^b + 1, \ldots, J^c\}$.

Condition (a) states that $g(z)$ is empty for states z that are not consistent with budget feasibility. The requirement that (4) holds with equality—found in condition (a) and also in (b.4.1)—accounts for the complementary slackness condition associated with this restriction in the individual optimization problem.

Condition (b) defines which measures are included in $g(z)$ when this set is not empty.[8] First, conditions (b.1) and (b.2) require that all measures in $g(z)$ have a finite

[8] The set $g(z)$ will still be empty if no measure satisfies those conditions.

support with S mass points and that their marginal probabilities on $\mathbb{S}$ are identical to the exogenous-shock probability. As a consequence, any transition selected from g is spotless. Condition (b.3) requires all measures in $g(z)$ to be consistent with the law of motion for asset holdings. Finally, (b.4) requires that the choices embedded in (z, π) satisfy the next-period budget constraints and are dynamically optimal.[9]

Let us now define and prove the existence of the first Markov equilibrium concept used in this paper.

Definition 2 (Stationary Markov equilibrium). A stationary Markov equilibrium for an economy $\mathcal{E}$ is a pair (Z, Π), where Z is a Borel measurable subset of $\mathbb{Z}$ and $\Pi : Z \to \mathcal{P}_Z$ is a transitional probability such that $\Pi_z \in g(z)$, for every z in Z.

A stationary Markov equilibrium consists of a set of states and a law of motion such that the current realization of z determines the future stochastic equilibrium path. This concept encompasses the competitive notion of equilibrium, since conditions (a) and (b) in the definition of g imply that any stationary Markov $\mathbb{Z}$-valued process $\{z_t\}_{t=0}^{\infty}$ with transition Π is consistent and individually optimal. Moreover, given conditions (b.1) and (b.2), any stationary Markov equilibrium for g is spotless.

Lemma 4. *The correspondence $g(z)$ has a closed graph.*

Proof. Take two convergent sequences $\{z_n\}_{n=0}^{\infty} \to z$ and $\{\pi_n\}_{n=0}^{\infty} \to \pi$ such that $z_n \in \mathbb{Z}$ and $\pi_n \in g(z_n)$ for all n. Let us show that $\pi \in g(z)$ by noticing the following facts. First, inequalities (1)–(3) hold and condition (4) is satisfied with equality for every agent i and every element z_n of the sequence $\{z_n\}_{n=0}^{\infty}$. These properties are preserved in the limit and then condition (a) holds for z.

Second, for every n, there is $h_n : \mathbb{S} \to \Theta^- \times K \times \Theta \times \Delta$ such that the support of π_n is the graph of h_n. Entries in the image of h_n are bounded. Therefore, there is $h \equiv \lim_{n\to\infty} h_n$ such that the support of π is the graph of h; i.e., π satisfies condition (b.1).

The marginal of π_n on $\mathbb{S}$ is $P(\cdot|s)$ and the marginal of π_n on Θ^- degenerates to θ_n. Then the limit probability π also satisfies conditions (b.2) and (b.3). Moreover, the conditions listed in (b.4) are preserved in the limit—since ∂u_i is upper hemicontinuous, nonempty, and compact valued on $\mathbb{R}_+^L$—and then hold for $(z, \pi) \equiv \lim(z_n, \pi_n)$. □

Proposition 1. *There exists a (spotless) stationary Markov equilibrium (Z, Π) for any economy $\mathcal{E}$ such that $M > I/(1-\rho)$.*

Proof. The proof has two steps. First, define a T-horizon equilibrium to be a competitive equilibrium for an economy in which time is restricted to lie in $\{0, \dots, T\} \subset \mathbb{T}$. It can be shown that there exists a spotless T-horizon equilibrium for a given set of initial portfolios and any finite $T > 0$. Appendix formalizes and proves this statement. The

[9]Dynamic optimality follows from the Kuhn–Tucker sufficient conditions for finite-dimensional nonsmooth optimization problems—see Balder (2010)—coupled with standard arguments in dynamic programming.

proof relies on standard arguments but is presented because complex securities—whose payoffs depend on the prices of other assets—are not usual in the general equilibrium literature.

One must notice from Appendix that the states derived from any T-horizon equilibrium process lie in the compact set $\mathbb{Z}$ for every $T > 0$. To conclude this proof, one must find a closed subset of $\mathbb{Z}$ that is self-justified for g; see Duffie et al. (1994, Proposition 1.1, p. 748). Formally, a self-justified set is a nonempty Borel measurable subset $Z \subset \mathbb{Z}$ such that $g(z) \cap \mathcal{P}_Z$ is nonempty for all $z \in Z$. It can be derived by following the steps used to prove Theorem 1.2 in Duffie et al. (1994). Since that proof is constructive, it is worth presenting a version here.

Define $Z_0 \equiv \mathbb{Z}$ and $Z_n \equiv \{z \in Z_{n-1} : \exists \pi \in g(z) \text{ with } \sup_{\mathcal{A}\subset Z_{n-1}} \pi(\mathcal{A}) = 1\}$ for every integer $n > 0$, where $\sup_{\mathcal{A}\subset Z_{n-1}} \pi(\mathcal{A})$ is the supremum of π taken over Borel measurable subsets of Z_{n-1}. Notice that $Z_n \subset Z_{n-1}$. Moreover, Z_n is nonempty for all $n \geq 0$. (To see this, take $n > 0$ and consider the period-0 states of every n-horizon equilibrium process. Clearly, conditions (a)–(b.3) in the definition of g hold for these states. Notice that $J^a \geq 1$, condition (3) never binds, and q^a is strictly positive in all nonterminal nodes.[10] The Slater condition is then satisfied and the Kuhn–Tucker conditions in (b.4) necessarily hold for these period-0 states.[11] To conclude the nonemptiness argument, notice that any period-1 state of an n-horizon equilibrium process lies in Z_{n-1}.[12])

Now, let $\bar{Z}_n$ be the closure of Z_n and let $Z \equiv \bigcap_{n=0}^{\infty} \bar{Z}_n$. Notice that Z is nonempty and compact since it is the intersection of a nested sequence of nonempty compact sets. One can then find a measure $\pi \in g(z)$ for each $z \in Z$. Take a sequence $\{z_n\}_{n=0}^{\infty} \to z$ such that $z_n \in Z_n$. For each n, take $\pi_n \in g(z_n)$ such that $\sup_{\mathcal{A}\subset Z_{n-1}} \pi(\mathcal{A}) = 1$ (which is possible since $z_n \in Z_n$). Since $\pi_n \in \mathcal{P}_{\bar{Z}_{n-1}}$ and $\{\mathcal{P}_{\bar{Z}_n}\}_{n=0}^{\infty}$ is a descending sequence of compact sets with intersection $\mathcal{P}_Z$, there is a measure $\pi \equiv \lim \pi_n$ that lies in $\mathcal{P}_Z$. This limit measure π must lie in $g(z)$ since, according to Lemma 4, this correspondence has a closed graph. $\square$

It is important to associate a stationary Markov equilibrium transition with an invariant ergodic measure μ. This pair defines a new equilibrium concept with the property that if the initial state is drawn with distribution μ, the distribution of future realizations of the system is also μ. This is the analogue of the deterministic notion of a steady state and it is also the key property behind consistency results in financial time series.

Definition 3 (Invariant ergodic measure). An invariant ergodic probability measure for a transition (Z, Π) is a measure $\mu \in \mathcal{P}_Z$ such that (i) $\mu(\mathcal{A}) \equiv \int_Z \Pi_z(\mathcal{A})\, d\mu(z)$ for any measurable set $\mathcal{A} \subset Z$ and (ii) either $\mu(\mathcal{A}) = 1$ or $\mu(\mathcal{A}) = 0$ for any measurable set $\mathcal{A} \subset Z$ such that $\Pi_z \in \mathcal{P}_{\mathcal{A}}$ for μ-almost every $z \in \mathcal{A}$.

[10] Recall that numeraire assets pay in units of good 1 and preferences are monotonically increasing in this good.

[11] See Balder (2010) for the Kuhn–Tucker necessary conditions for finite-dimensional nonsmooth optimization problems.

[12] This is immediate for $n = 1$, and it holds for $n > 1$ because the period-1 states of every n-horizon equilibrium process also satisfy conditions (a) and (b).

Definition 4 (Ergodic Markov equilibrium). An ergodic Markov equilibrium for $\mathcal{E}$ is a stationary Markov equilibrium (Z, Π) with an invariant ergodic measure $\mu \in \mathcal{P}_Z$.

Proposition 2. *There exists a (conditionally spotless) ergodic Markov equilibrium* (Z, Π, μ) *for any economy* $\mathcal{E}$ *such that* $M > I/(1-\rho)$.

Proof. Let Z be a compact self-justified set for g (see the proof of Proposition 1). Define the expectations correspondence $G: Z \to \mathcal{P}_Z$ as the closure of the convex hull of $g(z) \cap \mathcal{P}_Z$. The result follows then from Corollary 1.1 (p. 751) and Proposition 1.3 (p. 757) in Duffie et al. (1994). □

The term "conditionally spotless" deserves an explanation. According to Corollary 1.1 in Duffie et al. (1994, p. 751), ergodicity can be obtained when the expectations correspondence is convex valued. The correspondence g is not convex valued. By taking G as the closure of the convex hull of g, one allows randomizations over spotless equilibrium transitions. As argued before, any transition $\Pi: Z \to \mathcal{P}_Z$ can be written as $(x_{+1}, \theta_{+1}, p_{+1}, q_{+1}) = f(s_{+1}, \eta_{+1}, z)$, where f is a Borel measurable function and η_{+1} follows an i.i.d. uniform distribution in $[0, 1]$. The variable η_{+1} is interpreted as the sunspot. In our case, the budget restrictions and individual optimality conditions hold for each realization of η_{+1}. This is as if agents observed the sunspot variable η_{+1} before making their decisions in each node. Thus, the ergodic Markov equilibrium is spotless conditional on the realization of η_{+1}.

4. Concluding remarks

Individual initial debts cannot be arbitrary in economies with short sales; otherwise, the set of budget-feasible allocations might be empty. Lemma 5 in Appendix shows the existence of a T-horizon equilibrium for a set of initial portfolios in which agents are born with no debt. If financial trades occur in equilibrium, there will be alternative portfolio vectors that can be drawn in the initial state for each given equilibrium process. In fact, for a given stationary equilibrium (Z, Π), any state drawn from Z can be an initial condition. The only particularity to be noticed is that the set of possible initial conditions depends on the stochastic process describing the equilibrium prices. The choice $\hat{\theta}_{i,-1} = (\mathbf{0}, \mathbf{1}/I, \mathbf{0})$ is the simplest available for computation. However, just as in Duffie et al. (1994), there is no guarantee that an ergodic equilibrium process converges to its invariant ergodic measure μ unless the process is initially drawn from μ.

Another important issue to be addressed is whether the assumptions on marginal utilities can be relaxed. Recursive models usually assume that the Bernoulli utility (and marginal utility) functions are unbounded from below. In many models, this assumption guarantees that the equilibrium consumption process is uniformly bounded away from zero. This implies that the supergradient correspondence is upper hemicontinuous, nonempty, and compact valued in the subset of the consumption set where the equilibrium process lies. This is then used to guarantee the closed graph property of the expectations correspondence.[13]

However, this logic is not valid in environments with short sales and without default. According to Beker and Chattopadhyay (2010), individual consumption may get arbitrarily close to zero in this context, even when the Bernoulli utility functions are unbounded.

In fact, ergodicity is not always achievable in recursive economies with short sales and unbounded marginal utilities. Blume and Easley (2006) analyze an environment with a single consumption good, heterogeneous agents, and uniformly bounded aggregated endowment. They show that in any Pareto optimal allocation, the consumption of the agent with the highest discount factor asymptotically converges to the aggregate endowment, while the consumption of each other agent asymptotically converges to zero (see Lemma 1′, p. 951, in that paper).

Consider the case in our basic framework where $L = 1$, $\beta_1 > \max(\beta_2, \dots, \beta_I)$, and $J^a = S$ (complete markets). The first welfare theorem implies that in any ergodic Markov equilibrium (Z, Π, μ), the invariant ergodic measure μ must reproduce the asymptotic features of the Pareto optimal allocation. This implies that the marginal of μ on the consumption entries of the less-patient agents $(i > 1)$ must degenerate to zero. However, zero consumption can be avoided by this agent, since $M > I/(1-\rho)$ and the debt constraint (3) never binds for $z \in Z$. Therefore, this allocation is not individually optimal when marginal utilities are unbounded around zero consumption levels and no ergodic Markov equilibrium exists in this case.

Appendix: T-horizon equilibrium

A T-horizon economy is a version of the original economy in which time is restricted to lie in $\{0, 1, \dots, T\} \subset \mathbb{T}$. All concepts from the original economy extend directly by replacing the set $\mathbb{T}$ by $\{0, 1, \dots, T\}$.

A $\mathbb{Z}$-valued stochastic process $\{z_t\}_{t=0}^{T}$ is said to be spotless if z_t is measurable with respect to the σ-algebra generated by $(s_0(\omega), \dots, s_t(\omega))$ for every $t \in \{0, 1, \dots, T\}$. Since $\mathbb{S}$ is finite, any T-horizon spotless process can be written as an N-dimensional vector, where $N \geq 2$ is the number of possible history of shocks $\varpi_t = (s_0, \dots, s_t)$ for $t \leq T$.[14] Then a spotless T-horizon competitive equilibrium can be expressed by a vector $(\mathbf{x}, \boldsymbol{\theta}, \mathbf{p}, \mathbf{q}) \in \mathbb{R}_+^{ILN} \times \mathbb{R}^{IJN} \times \mathbb{R}_+^{(L+J)N}$ such that $(\mathbf{x}_i, \boldsymbol{\theta}_i)$ is individually optimal, for every agent i, and the feasibility constraints (6) and (7) are attained in every node (t, ϖ_t).

Lemma 5. *There exists a spotless T-horizon competitive equilibrium for any economy $\mathcal{E}$ such that $M > I/(1-\rho)$.*

[13] This reasoning was used by Duffie et al. (1994) in a scenario with a single consumption good and without short sales. The same argument is valid in economies with multiple commodities and uniformly bounded aggregate endowments; see Braido (2008) for an analysis in a multigood economy with short sales and default.

[14] By fixing an initial shock $s_0 = \hat{s}$ and taking $T \geq 1$, one has $N = (S^{T+1} - 1)/(S - 1)$ when $S > 1$ and has $N = T + 1$ when $S = 1$.

Proof. Fix an arbitrary initial shock $s_0 = \hat{s} \in \mathbb{S}$ and an initial portfolio such that $(\hat{\theta}^a_{-1}, \hat{\theta}^c_{-1}) = (\mathbf{0}, \mathbf{0})$ and $\min(\hat{\theta}^b_{-1}) \geq 0$ subject to $\sum_{i\in\mathbb{I}} \hat{\theta}^b_{i,-1} = \mathbf{1}$. Take Δ, $K = K_1 \times \cdots \times K_I$ and $\Theta = \Theta_1 \times \cdots \times \Theta_I$ as defined in Section 3. Let agent i's truncated budget set be given by

$$B_i(\mathbf{p}, \mathbf{q}) = \{(\mathbf{x}_i, \boldsymbol{\theta}_i) \in K_i^N \times \Theta_i^N : (1)\text{–}(4), \forall (t, \varpi_t)\}.$$

Define the correspondence $\Psi : K^N \times \Theta^N \times \Delta^N \to K^N \times \Theta^N \times \Delta^N$ to be such that $(\mathbf{x}', \boldsymbol{\theta}', \mathbf{p}', \mathbf{q}') \in \Psi(\mathbf{x}, \boldsymbol{\theta}, \mathbf{p}, \mathbf{q})$ if and only if

$$(\mathbf{x}'_i, \boldsymbol{\theta}'_i) \in \underset{(\tilde{\mathbf{x}}_i, \tilde{\boldsymbol{\theta}}_i) \in B_i(\mathbf{p},\mathbf{q})}{\arg\max}\ E\left[\sum_{t=0}^{T} \beta_i^t u_i(\tilde{x}_{i,t,\varpi_t}) \middle| \mathcal{F}_0\right], \quad \forall i \in \mathbb{I} \tag{11}$$

and

$$\begin{aligned}(\mathbf{p}', \mathbf{q}') \in \underset{(\tilde{\mathbf{p}},\tilde{\mathbf{q}}) \in \Delta^N}{\arg\max} \sum_{t,\varpi_t} \Bigg\{ \tilde{p}_{t,\varpi_t} \cdot \Bigg[\sum_{i\in\mathbb{I}} (x_{i,t,\varpi_t} - e_{i,t,\varpi_t}) - \sum_{j\in\mathbb{J}^b} A_{j,s}\Bigg] \\ + \tilde{q}_{t,\varpi_t} \cdot \Bigg[\sum_{i\in\mathbb{I}} \theta_{i,t,\varpi_t} - (\mathbf{0}, \mathbf{1}, \mathbf{0})\Bigg]\Bigg\}.\end{aligned} \tag{12}$$

The correspondence Ψ is upper hemicontinuous, nonempty, compact, and convex valued. The first three properties follow from Berge's theorem of the maximum since: (i) condition (11) defines a maximization problem in which the objective function is continuous, and the budget correspondence $B_i : \Delta^N \to K_i^N \times \Theta_i^N$ is nonempty, compact valued, and continuous on Δ^N,[15] and (ii) condition (12) defines a linear optimization problem over a nonempty compact set. Moreover, Ψ is convex valued since (i) the objective function in (11) is concave and B_i is a convex-valued correspondence,[16] and (ii) the objective function in (12) is linear and Δ^N is convex.

It follows from Kakutani's fixed-point theorem that there exists some vector $(\mathbf{x}^{**}, \boldsymbol{\theta}^{**}, \mathbf{p}^*, \mathbf{q}^*) \in \Psi(\mathbf{x}^{**}, \boldsymbol{\theta}^{**}, \mathbf{p}^*, \mathbf{q}^*)$. Since u_i is increasing in good 1, condition (11) implies that the budget equation (4) holds with equality at the fixed point. One can then modify the fixed-point allocation and portfolio as follows, to make markets clear in all nodes.

Take the initial note $(0, \varpi_0)$ and recall that $\varpi_0 = \hat{s}$ and $\sum_{i\in\mathbb{I}} \hat{\theta}_{-1} = (\mathbf{0}, \mathbf{1}, \mathbf{0})$. This implies

$$p^*_{0,\varpi_0} \cdot \Bigg[\sum_{i\in\mathbb{I}} (x^{**}_{i,0,\varpi_0} - e_{i,\hat{s}}) - \sum_{j\in\mathbb{J}^b} A_{j,\hat{s}}\Bigg] + q^*_{0,\varpi_0} \cdot \Bigg[\sum_{i\in\mathbb{I}} \theta^{**}_{i,0,\varpi_0} - (\mathbf{0}, \mathbf{1}, \mathbf{0})\Bigg] = 0.$$

Since $(\mathbf{x}^{**}, \boldsymbol{\theta}^{**}, \mathbf{p}^*, \mathbf{q}^*)$ satisfies condition (12), one must have $\gamma_{x,0,\varpi_0} \equiv \sum_{i\in\mathbb{I}} (x^{**}_{i,0,\varpi_0} - e_{i,\hat{s}}) - \sum_{j\in\mathbb{J}^b} A_{j,\hat{s}} \leq 0$ and $\gamma_{\theta,0,\varpi_0} \equiv \sum_{i\in\mathbb{I}} \theta^{**}_{i,0,\varpi_0} \leq (\mathbf{0}, \mathbf{1}, \mathbf{0})$, where these inequalities can only be strict in entries associated with a corresponding zero price. Let us then modify

[15] The correspondence B_i is lower hemicontinuous thanks to $\hat{\theta}$ being nonnegative and $p_s \cdot e_{i,s} \geq 1 e_{i,1,s} > 0$ for all i and s.

[16] Although $r_{j,s}(\cdot)$ need not be linear, $B_i(\mathbf{p}, \mathbf{q})$ is a convex set for each given $(\mathbf{p}, \mathbf{q})$.

the fixed-point allocation and portfolio in this node as

$$x^*_{i,0,\varpi_0} = \begin{cases} x^{**}_{i,0,\varpi_0} - \gamma_{x,0,\varpi_0} & \text{for } i = 1 \\ x^{**}_{i,0,\varpi_0} & \text{for } i \neq 1 \end{cases}$$

and

$$\theta^*_{i,0,\varpi_0} = \begin{cases} \theta^{**}_{i,0,\varpi_0} - \gamma_{\theta,0,\varpi_0} & \text{for } i = 1 \\ \theta^{**}_{i,0,\varpi_0} & \text{for } i \neq 1. \end{cases}$$

Since u_i is increasing in good 1, the price of an asset is only zero when the value of its payoff is also zero in all subsequent nodes. Therefore, $W_s(\theta^{**}_{i,0,\varpi_0}, p^*_{1\varpi_1}, q^*_{\varpi_1}) = W_s(\theta^*_{i,0,\varpi_0}, p^*_{1,\varpi_1}, q^*_{1,\varpi_1})$ for all $\varpi_1 = (\hat{s}, s_1)$. Moreover, by construction, $\sum_{i\in\mathbb{I}} \hat{\theta}_{0,\varpi_0} = (\mathbf{0}, \mathbf{1}, \mathbf{0})$. One can then replace $\theta^{**}_{i,0,\varpi_0}$ by $\theta^*_{i,0,\varpi_0}$ and repeat the procedure described in the previous paragraph for all subsequent nodes $(1, \varpi_1)$. By using this algorithm period by period, it can be shown that $(\mathbf{x}^*, \boldsymbol{\theta}^*)$ satisfies the feasibility conditions (6) and (7) for all (t, ϖ_t).

It follows from (11) that $(\mathbf{x}^{**}, \boldsymbol{\theta}^{**})$ is individually optimal on the truncated budget set $B_i(\mathbf{p}^*, \mathbf{q}^*)$. Since U_i is continuous and concave, $\max(\mathbf{x}_i^{**}) < \bar{x}$, and $\boldsymbol{\theta}_i^{**}$ lies in the interior of Θ^N, then $(\mathbf{x}_i^{**}, \boldsymbol{\theta}_i^{**})$ also maximizes U_i on the untruncated budget set $\{(\mathbf{x}_i, \boldsymbol{\theta}_i) \in \mathbb{R}_+^{LN} \times \mathbb{R}^{JN} : (1)\text{–}(4), \forall (t, \varpi_t)\}$, evaluated at prices $(\mathbf{p}^*, \mathbf{q}^*)$. To conclude the proof, notice that $(\mathbf{x}^*, \boldsymbol{\theta}^*)$ also maximizes U_i on the untruncated budget set, since this vector differs from $(\mathbf{x}^{**}, \boldsymbol{\theta}^{**})$ only by positive amounts added in entries associated with a corresponding zero price. Thus, $(\mathbf{x}^*, \boldsymbol{\theta}^*, \mathbf{p}^*, \mathbf{q}^*)$ is a spotless T-horizon competitive equilibrium for $\mathcal{E}$. □

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