## Appendices B, C, and D to

"Optimal Deadlines for Agreements"

## Ettore Damiano Li, Hao Wing Suen

published in Theoretical Economics

## Appendix B. Stochastic Deadlines

Proposition B1. Suppose that $T=\infty$ and $\epsilon>0$. There exists a symmetric equilibrium in which the high types always persist; the low types with belief $\gamma$ concede with a flow rate equal to $\epsilon(\alpha-\gamma) /(2(1-$ $\left.\gamma_{*}\right) \gamma$ ) if $\gamma(t) \in(0, \min \{1, \alpha\})$, concede with probability one if $\gamma=0$ and persist if $\gamma \in[\min \{1, \alpha\}, 1)$; and the belief $\gamma(t)$ of the low types solves (16) with the initial value $\gamma_{0}$ if $\gamma_{0}<\min \{1, \alpha\}$, and is equal to $\gamma_{0}$ if $\gamma_{0} \in[\min \{1, \alpha\}, 1)$.

Proof. First, we derive the differential equation (16) for the equilibrium belief evolution. Note that the expected payoff of the low types from conceding is still given by (2). The payoff from persisting becomes

$$
\begin{aligned}
& \gamma(t) x(t) \mathrm{d} t\left(v_{L}+\beta\right)+(\gamma(t)(1-x(t) \mathrm{d} t)+(1-\gamma(t)))(1-\epsilon \mathrm{d} t)\left(-\kappa \mathrm{d} t+\mathcal{U}_{L}(t+\mathrm{d} t)\right) \\
& \quad+\epsilon \mathrm{d} t\left((1-\gamma(t)) \frac{v_{H}+v_{L}+\beta}{2}+\gamma(t)(1-x(t) \mathrm{d} t) \frac{2 v_{L}+\beta}{2}\right)
\end{aligned}
$$

where $x(t)$ denotes the flow rate of concession by the low types. Equating the two payoff expressions and using the same Bayes' rule as in the proof of Lemma 1 immediately give us (16). The corresponding flow rate of concession is

$$
x(t)=\frac{\epsilon(\alpha-\gamma(t))}{2\left(1-\gamma_{*}\right) \gamma(t)} .
$$

For the case of $\gamma_{0} \in(0, \min \{1, \alpha\})$, it suffices to verify that the equilibrium payoff of the high types is at least as large as the payoff from deviating to conceding, which is equal to $v_{L}$ regardless of $\epsilon$. The differential equation for the value function of the high types is

$$
U_{H}^{\prime}(\gamma)=-\frac{\left(\alpha-\gamma_{*}\right) \beta+\left(1-\gamma_{*}\right)\left(v_{H}-v_{L}+\beta\right)}{(1-\gamma)(\alpha-\gamma)}+\frac{\alpha-\gamma+2\left(1-\gamma_{*}\right) \gamma}{\gamma(1-\gamma)(\alpha-\gamma)}\left(v_{H}+\beta-U_{H}(\gamma)\right),
$$

with the boundary condition $U_{H}(0)=v_{H}+\beta$. The solution to this differential equation is

$$
U_{H}(\gamma)=v_{H}+\beta-\left(1-\frac{1-\gamma}{\gamma} \frac{K(\gamma)}{2\left(1-\gamma_{*}\right)}\right) \frac{\left(\alpha-\gamma_{*}\right) \beta+\left(1-\gamma_{*}\right)\left(v_{H}-v_{L}+\beta\right)}{\left(\alpha-\gamma_{*}\right)+\left(1-\gamma_{*}\right)}
$$

where

$$
K(\gamma) \equiv \alpha-\alpha\left(\frac{\alpha(1-\gamma)}{\alpha-\gamma}\right)^{2 \epsilon \beta /(2 \kappa-\epsilon \beta)}
$$

Note that $K(\gamma)>0$ for all $\gamma \in(0, \alpha)$, regardless of whether $\alpha$ is greater or less than one. Since

$$
\frac{\left(\alpha-\gamma_{*}\right) \beta+\left(1-\gamma_{*}\right)\left(v_{H}-v_{L}+\beta\right)}{\left(\alpha-\gamma_{*}\right)+\left(1-\gamma_{*}\right)} \leq v_{H}-v_{L}+\beta,
$$

it follows immediately from Assumption 1 that $U_{H}(\gamma) \geq \underline{v}_{L}$ for all $\gamma$.
For the case of $\gamma_{0} \in[\min \{1, \alpha\}, 1)$, in equilibrium the game ends with exogenous exit, with a terminal payoff of $\left(v_{H}+v_{L}+\beta\right) / 2$ to the high types and

$$
\gamma \frac{2 v_{L}+\beta}{2}+(1-\gamma) \frac{v_{H}+v_{L}+\beta}{2}
$$

to the low types. Further, the exogenous exit time follows an exponential distribution with parameter $\epsilon$, and hence the expected duration of the game is $1 / \epsilon$. Thus, the equilibrium expected payoff loss from delay is $\kappa / \epsilon$ for both the high and low types. If the low types deviate to conceding, the expected payoff is

$$
\gamma \beta+(1-\gamma) v_{H}<\gamma \frac{2 v_{L}+\beta}{2}+(1-\gamma) \frac{v_{H}+v_{L}+\beta}{2}-\frac{\kappa}{\epsilon},
$$

because $\gamma<\alpha$. For the high types, the expected payoff from concession is $v_{L}$, which is lower than the equilibrium payoff because $v_{H}-v_{L}+\beta>2 \kappa / \epsilon$, by Assumption 1 and by the assumption that $\alpha<1$.

Proposition B2. Suppose that $T=\infty$. For any $\gamma_{0}>\gamma_{*}$, the optimal exogenous exit rate is either zero or infinity.

Proof. It suffices to establish that $U_{H}\left(\gamma_{0}\right)$ for the case $\gamma_{0}<\min \{1, \alpha\}$ is decreasing in $\epsilon$ for $\gamma_{0}>\gamma_{*}$.
It is convenient to use the fact that $\lim _{\gamma_{0} \rightarrow 0} K\left(\gamma_{0}\right)=0$ to write

$$
K\left(\gamma_{0}\right)=\int_{0}^{\gamma_{0}} k(\gamma) \mathrm{d} \gamma,
$$

where

$$
k(\gamma)=\frac{2\left(1-\gamma_{*}\right)}{(1-\gamma)^{2}}\left(\frac{\alpha(1-\gamma)}{\alpha-\gamma}\right)^{(2 \kappa+\epsilon \beta) /(2 \kappa-\epsilon \beta)} .
$$

The term $K\left(\gamma_{0}\right)\left(1-\gamma_{0}\right) / \gamma_{0}$ is decreasing in $\gamma_{0}$ because its derivative is

$$
\begin{aligned}
& \frac{1-\gamma_{0}}{\gamma_{0}} k\left(\gamma_{0}\right)-\frac{1}{\gamma_{0}^{2}} K\left(\gamma_{0}\right) \\
= & -\frac{\alpha}{\gamma_{0}^{2}}\left(1-\left(\frac{\alpha\left(1-\gamma_{0}\right)}{\alpha-\gamma_{0}}\right)^{2 \epsilon \beta /(2 \kappa-\epsilon \beta)}\left(\frac{2\left(1-\gamma_{*}\right) \gamma_{0}}{\alpha-\gamma_{0}}+1\right)\right) \\
= & -\frac{\alpha}{\gamma_{0}^{2}} \int_{0}^{\gamma_{0}} 2\left(1-\gamma_{*}\right)\left(\frac{\alpha(1-\gamma)}{\alpha-\gamma}\right)^{2 \epsilon \beta /(2 \kappa-\epsilon \beta)} \frac{\gamma\left(\left(\alpha-\gamma_{*}\right)+\left(1-\gamma_{*}\right)\right)}{(\alpha-\gamma)^{2}(1-\gamma)} \mathrm{d} \gamma,
\end{aligned}
$$

which is negative as $\alpha>\gamma_{*}$. Now, since $\lim _{\gamma_{0} \rightarrow 0} K\left(\gamma_{0}\right)=0$, and thus

$$
\lim _{\gamma_{0} \rightarrow 0} \frac{K\left(\gamma_{0}\right)}{\gamma_{0}}=\lim _{\gamma_{0} \rightarrow 0} k\left(\gamma_{0}\right)=2\left(1-\gamma_{*}\right),
$$

we have

$$
\frac{1-\gamma_{0}}{\gamma_{0}} \frac{K\left(\gamma_{0}\right)}{2\left(1-\gamma_{*}\right)}<1
$$

for all $\gamma_{0}>0$. Because the coefficient on $K\left(\gamma_{0}\right)$ in the $U_{H}\left(\gamma_{0}\right)$ function is increasing in $\epsilon$, a sufficient condition for $U_{H}\left(\gamma_{0}\right)$ to be decreasing in $\epsilon$ is that $K\left(\gamma_{0}\right)$ is increasing in $\alpha$. A sufficient condition for the latter is that $\ln k\left(\gamma_{0}\right)$ is increasing in $\alpha$, or

$$
-\ln \left(\frac{\alpha\left(1-\gamma_{0}\right)}{\alpha-\gamma_{0}}\right)+\frac{(\alpha-1) \gamma_{0}}{\alpha\left(1-\gamma_{0}\right)} \frac{\left(\alpha-\gamma_{*}\right)+\left(1-\gamma_{*}\right)}{2\left(1-\gamma_{*}\right)}>0 .
$$

Since the above is equal to zero at $\gamma_{0}=0$, it is sufficient if its derivative with respect to $\gamma_{0}$ is strictly positive. This derivative is given by

$$
\left(\frac{\alpha-1}{\alpha-\gamma_{0}}\right)^{2}\left(\frac{1}{1-\gamma_{0}}-\frac{1}{2\left(1-\gamma_{*}\right)}\right) .
$$

Therefore, $U_{H}\left(\gamma_{0}\right)$ decreases with $\epsilon$ so long as $\gamma_{0}>\gamma_{*}$.

## Appendix C. Deadline Penalties

Proposition C1. Suppose that $T<\infty$, and $\lambda \in(0, \beta / 2)$. There is a symmetric equilibrium in which the high types always persist; the strategy of the low types at time $t$ with any belief $\gamma$ is such that: (i) if $t=T$, concede with probability one if $\gamma \leq \underline{\gamma}_{*}$, with probability zero if $\gamma \geq \bar{\gamma}_{*}$, and with probability $Y(\gamma)$ if $\gamma \in\left(\underline{\gamma}_{*}, \bar{\gamma}_{*}\right)$; (ii) if $T-t \in(0, B(\gamma)]$, persist; and (iii) if $T-t>B(\gamma)$, concede at a flow rate $\kappa /(\beta \gamma)$ if $\gamma>0$ and with probability one if $\gamma=0$.

Proof. Case (i) is already established in the text. Here we provide explicit formulas that will be used in the rest of Appendix D. The two critical beliefs in the deadline game are

$$
\begin{aligned}
& \underline{\gamma}_{*} \equiv \frac{v_{H}-v_{L}-\beta+2 \lambda}{v_{H}-v_{L}+4 \lambda}, \\
& \bar{\gamma}_{*} \equiv \frac{v_{H}-v_{L}-\beta+2 \lambda}{v_{H}-v_{L}} .
\end{aligned}
$$

The equilibrium probability of concession by the low types for $\gamma \in\left(\underline{\gamma}_{*}, \bar{\gamma}_{*}\right)$, given in equation (18), is

$$
Y(\gamma)=\frac{v_{H}-v_{L}-\beta+2 \lambda-\left(v_{H}-v_{L}\right) \gamma}{4 \lambda \gamma} .
$$

The equilibrium payoff function for the low types in the deadline game, given by (19), is

$$
U_{L}^{0}\left(\gamma_{0}\right)= \begin{cases}\gamma_{0}\left(v_{L}+\beta / 2-\lambda\right)+\left(1-\gamma_{0}\right) v_{H} & \text { if } \gamma_{0} \in\left[0, \underline{\gamma}_{*}\right) \\ \gamma_{0} v_{L}+\left(1-\gamma_{0}\right) v_{H}+\gamma_{0} Y\left(\gamma_{0}\right)(\beta / 2-\lambda) & \text { if } \gamma_{0} \in\left[\underline{\gamma}_{*}, \bar{\gamma}_{*}\right] \\ \gamma_{0}\left(v_{L}+\beta / 2-\lambda\right)+\left(1-\gamma_{0}\right)\left(\left(v_{H}+v_{L}+\beta\right) / 2-\lambda\right) & \text { if } \gamma_{0} \in\left(\bar{\gamma}_{*}, 1\right)\end{cases}
$$

and for the high types is given by

$$
U_{H}^{0}(\gamma)= \begin{cases}v_{H}+\beta & \text { if } \gamma \in\left[0, \underline{\gamma}_{*}\right) \\ Y(\gamma)\left(v_{H}+\beta\right)+(1-Y(\gamma))\left(\left(v_{H}+v_{L}+\beta\right) / 2-\lambda\right) & \text { if } \gamma \in\left[\underline{\gamma}_{*}, \bar{\gamma}_{*}\right] \\ \left(v_{H}+v_{L}+\beta\right) / 2-\lambda & \text { if } \gamma \in\left(\bar{\gamma}_{*}, 1\right)\end{cases}
$$

For case (ii), the equilibrium payoff to the low types at any time $t^{\prime} \in[t, T)$ from persisting throughout the game is given by

$$
\gamma\left(\tilde{Y}(\gamma)\left(v_{L}+\beta\right)+(1-\tilde{Y}(\gamma))\left(\frac{2 v_{L}+\beta}{2}-\lambda\right)\right)+(1-\gamma)\left(\frac{v_{H}+v_{L}+\beta}{2}-\lambda\right)-\kappa\left(T-t^{\prime}\right),
$$

where $\tilde{Y}(\gamma)$ is 1 for $\gamma \leq \underline{\gamma}_{*}, 0$ for $\gamma \geq \bar{\gamma}_{*}$ and $Y(\gamma)$ otherwise. It is easy to show that if $t^{\prime}=t$ and $T-t=B(\gamma)$, the above is equal to $U_{L}(\gamma)$, the deviation payoff to a low type from conceding at time $t^{\prime}$ given the equilibrium strategy of the low type opponent. Thus, there is no incentive for the low types to deviate for any time $t^{\prime} \in[t, T)$. For the high types, at any $t^{\prime} \in[t, T]$ the equilibrium payoff from persisting is

$$
\tilde{Y}(\gamma)\left(v_{H}+\beta\right)+(1-\tilde{Y}(\gamma))\left(\frac{v_{H}+v_{L}+\beta}{2}-\lambda\right)-\kappa\left(T-t^{\prime}\right)
$$

The payoff from conceding right away is $v_{L}$. It is optimal for the high types to persist if

$$
\tilde{Y}(\gamma)\left(v_{H}-v_{L}+\beta\right)+(1-\tilde{Y}(\gamma))\left(\frac{v_{H}-v_{L}+\beta}{2}-\lambda\right) \geq \kappa T
$$

We have just argued that the low types weakly prefer persisting until the deadline followed by conceding with probability $\tilde{Y}(\gamma)$ to conceding immediately. Since $\tilde{Y}(\gamma)>0$ for $\gamma<\bar{\gamma}_{*}$, the equilibrium condition of the low types implies that

$$
\gamma \tilde{Y}(\gamma)\left(\frac{2 v_{L}+\beta}{2}-\lambda\right)+\gamma(1-\tilde{Y}(\gamma)) v_{L}+(1-\gamma) v_{H}-\kappa(T-t) \geq U_{L}(\gamma)
$$

or

$$
\gamma \tilde{Y}(\gamma)\left(\frac{\beta}{2}-\lambda\right) \geq \kappa(T-t)
$$

By Assumption 1 and the assumption that $\lambda \leq \beta / 2$, we have

$$
\tilde{Y}(\gamma)\left(v_{H}-v_{L}+\beta\right)+(1-\tilde{Y}(\gamma))\left(\frac{v_{H}-v_{L}+\beta}{2}-\lambda\right)>\frac{v_{H}-v_{L}+\beta}{2}-\lambda>\gamma \tilde{Y}(\gamma)\left(\frac{\beta}{2}-\lambda\right),
$$

and thus the equilibrium condition of the high types is satisfied. For the case of $\gamma \geq \bar{\gamma}_{*}$ we have $\tilde{Y}(\gamma)=0$, and the equilibrium condition of the low types is

$$
\gamma\left(\frac{2 v_{L}+\beta}{2}-\lambda\right)+(1-\gamma)\left(\frac{v_{H}+v_{L}+\beta}{2}-\lambda\right)-\kappa(T-t) \geq \gamma v_{L}+(1-\gamma) v_{H}
$$

which implies

$$
\gamma\left(\frac{\beta}{2}-\lambda\right)>\kappa(T-t)
$$

Thus, the equilibrium condition of the high types is satisfied.
For case (iii), for any initial belief $\gamma_{0}$, either $T>D\left(\gamma_{0}\right)$, in which case the proof is the same as the case of no deadlines in Section 3, or otherwise on the equilibrium path there is a unique time $S\left(T ; \gamma_{0}\right)=S$ satisfying

$$
T-S=B\left(g\left(S ; \gamma_{0}\right)\right) .
$$

By construction, the low types are indifferent between conceding and persisting for all $t \in[0, S)$, so there is no profitable deviation before $t=S$. Further, by construction, the equilibrium payoff to the low types at $t=S$ is

$$
\begin{aligned}
\mathcal{U}_{L}(S)= & g\left(S ; \gamma_{0}\right)\left(\tilde{Y}\left(\gamma_{0}\right)\left(v_{L}+\beta\right)+\left(1-\tilde{Y}\left(\gamma_{0}\right)\right) \frac{2 v_{L}+\beta}{2}-\lambda\right) \\
& +\left(1-g\left(S ; \gamma_{0}\right)\right)\left(\frac{v_{H}+v_{L}+\beta}{2}-\lambda\right)-\kappa(T-S) .
\end{aligned}
$$

Thus, by the argument for cases (i) and (ii) above there is no profitable deviation for the low types after $t=S$ either. For the high types, given the arguments for cases (i) and (ii), it suffices to show that there is no profitable deviation before $t=S$. The equilibrium payoff function $U_{H}(\gamma)$ at any $\gamma=g\left(t ; \gamma_{0}\right)$ for $t<S$ is given by the solution to the differential equation (6) with the boundary condition that

$$
\begin{aligned}
U_{H}\left(g\left(S ; \gamma_{0}\right)\right)=\tilde{Y} & \left(g\left(S ; \gamma_{0}\right)\right)\left(v_{H}+\beta\right)-\kappa(T-S) \\
& \quad\left(1-\tilde{Y}\left(g\left(S ; \gamma_{0}\right)\right)\right)\left(\frac{v_{H}+v_{L}+\beta}{2}-\lambda\right) .
\end{aligned}
$$

The claim that it is optimal for the high types to persist at all $t<S$ follows from identical arguments as in the proof of Proposition 2.

Proposition C2. Suppose that $\lambda \in(0, \beta / 2)$. There exist thresholds $\underline{\gamma}$ and $\bar{\gamma}$, with $\underline{\gamma}_{*}<\underline{\gamma}<\bar{\gamma}_{*}<\bar{\gamma}<$ 1 , such that the optimal deadline for any initial belief $\gamma_{0}$ is $\underline{D}_{*}\left(\gamma_{0}\right)$ if $\gamma_{0} \in(\underline{\gamma}, \bar{\gamma})$, and is zero otherwise.

Proof. We first verify that the welfare effects are positive in Regions I and II but negative in Region III in Figure 2.

In Region II, the phase-switch time $S$ is defined by the indifference condition for the low types at the boundary $B$ :

$$
\kappa(T-S)=g\left(S ; \gamma_{0}\right) Y\left(g\left(S ; \gamma_{0}\right)\right)\left(\frac{\beta}{2}-\lambda\right)
$$

Taking derivative with respect to $T$, and using the definition of $Y$ in equation (18), we obtain:

$$
\frac{\partial S}{\partial T}=\frac{8 \lambda \beta}{8 \lambda \beta+\left(1-g\left(S ; \gamma_{0}\right)\right)\left(v_{H}-v_{L}\right)(\beta-2 \lambda)}
$$

Now, an explicit calculation of $\partial U_{H}\left(\gamma_{0}\right) / \partial T$ given in equation (20) yields:

$$
\begin{aligned}
\frac{\partial U_{H}\left(\gamma_{0}\right)}{\partial T}= & \frac{\kappa}{8 \lambda \beta g\left(S ; \gamma_{0}\right)}\left((\beta+2 \lambda)\left(v_{H}-v_{L}+\beta+2 \lambda\right)\right. \\
& \left.\quad+\left(v_{H}-v_{L}\right)(\beta-2 \lambda)\left(\bar{\gamma}_{*}-g\left(S ; \gamma_{0}\right)\right)\right) \frac{\partial S}{\partial T}-\kappa .
\end{aligned}
$$

Since $\partial S / \partial T>0$, by Assumption 1 the above expression is greater than:

$$
\kappa \frac{(\beta+2 \lambda)^{2}+\left(v_{H}-v_{L}\right)(\beta-2 \lambda)\left(\bar{\gamma}_{*}-g\left(S ; \gamma_{0}\right)\right)}{g\left(S ; \gamma_{0}\right)\left(8 \lambda \beta+\left(1-g\left(S ; \gamma_{0}\right)\right)\left(v_{H}-v_{L}\right)(\beta-2 \lambda)\right)}-\kappa,
$$

which is equal to $\kappa / g\left(S ; \gamma_{0}\right)-\kappa>0$.
In Region I, the phase-switch time $S$ is defined by the indifference condition:

$$
\kappa(T-S)=\frac{g\left(S ; \gamma_{0}\right)-\gamma_{*}}{2\left(1-\gamma_{*}\right)} \beta-\lambda .
$$

Take derivative respect to $T$ to get

$$
\frac{\partial S}{\partial T}=\frac{2\left(1-\gamma_{*}\right)}{1-2 \gamma_{*}+g\left(S ; \gamma_{0}\right)}
$$

Furthermore, by Assumption 1,

$$
v_{H}+\beta-\mathcal{U}_{H}(S)=\frac{v_{H}-v_{L}+\beta}{2}+\kappa(T-S)+\lambda>\frac{\beta}{2} \frac{1-2 \gamma_{*}+g\left(S ; \gamma_{0}\right)}{1-\gamma_{*}} .
$$

Finally, since $x(S)=\kappa /\left(\beta g\left(S ; \gamma_{0}\right)\right)$, we have

$$
\frac{\partial U_{H}\left(\gamma_{0}\right)}{\partial T}=-\kappa+x(S)\left(v_{H}+\beta-\mathcal{U}_{H}(S)\right) \frac{\partial S}{\partial T}>0
$$

In Region III, the phase-switch time $S$ is defined by:

$$
\kappa(T-S)=g\left(S ; \gamma_{0}\right)\left(\frac{\beta}{2}-\lambda\right) .
$$

Take derivative respect to $T$ to get

$$
\frac{\partial S}{\partial T}=\frac{2\left(1-\gamma_{*}\right)}{2\left(1-\gamma_{*}\right)-\left(1-g\left(S ; \gamma_{0}\right)\right)\left(1-\bar{\gamma}_{*}\right)}
$$

Furthermore,

$$
v_{H}+\beta-\mathcal{U}_{H}(S)=\kappa(T-S)=\frac{g\left(S ; \gamma_{0}\right)}{2} \frac{1-\bar{\gamma}_{*}}{1-\gamma_{*}} \beta
$$

Therefore,

$$
\begin{aligned}
\frac{\partial U_{H}\left(\gamma_{0}\right)}{\partial T} & =-\kappa+x(S)\left(v_{H}+\beta-\mathcal{U}_{H}(S)\right) \frac{\partial S}{\partial T} \\
& =-\kappa+\frac{\kappa\left(1-\bar{\gamma}_{*}\right)}{2\left(1-\gamma_{*}\right)-\left(1-g\left(S ; \gamma_{0}\right)\right)\left(1-\bar{\gamma}_{*}\right)} \\
& \leq \frac{2 \kappa\left(\gamma_{*}-\bar{\gamma}_{*}\right)}{2\left(1-\gamma_{*}\right)-\left(1-\bar{\gamma}_{*}\right)},
\end{aligned}
$$

which is negative.
The remainder of the proof is to compare the value of ex ante welfare $U_{L}\left(\gamma_{0}\right)$ at the two local maxima of zero and $\underline{D}_{*}\left(\gamma_{0}\right)$ for $\gamma_{0}>\underline{\gamma}_{*}$.

Under the deadline $T=\underline{D}_{*}\left(\gamma_{0}\right)$, the payoff to the low types is simply $U_{L}^{*}\left(\gamma_{0}\right)=U_{L}\left(\gamma_{0}\right)$ as in (3). To compute the payoff to the high types, we solve the differential equation (6) with the boundary condition

$$
U_{H}\left(\underline{\gamma}_{*}\right)=v_{H}+\beta-\kappa B\left(\underline{\gamma}_{*}\right)
$$

This gives the payoff to the high types when the deadline is $T=\underline{D}_{*}\left(\gamma_{0}\right)$ :

$$
\begin{gathered}
U_{H}^{*}\left(\gamma_{0}\right)=v_{H}+\beta-\frac{1-\gamma_{0}}{\gamma_{0}}\left(\ln \left(\frac{1-\gamma_{0}}{1-\underline{\gamma}_{*}}\right)+\frac{\gamma_{0}-\underline{\gamma}_{*}}{\left(1-\gamma_{0}\right)\left(1-\underline{\gamma}_{*}\right)}\right) \beta \\
-\frac{1-\gamma_{0}}{\gamma_{0}} \frac{\gamma_{*}^{2}}{1-\underline{\gamma}_{*}}\left(\frac{\beta}{2}-\lambda\right) .
\end{gathered}
$$

The difference in ex ante welfare $U_{L}^{*}\left(\gamma_{0}\right)-U_{L}^{0}\left(\gamma_{0}\right)$ is

$$
\frac{1}{2-\gamma_{0}}\left(U_{L}^{*}\left(\gamma_{0}\right)-U_{L}^{0}\left(\gamma_{0}\right)\right)+\frac{1-\gamma_{0}}{2-\gamma_{0}}\left(U_{H}^{*}\left(\gamma_{0}\right)-U_{H}^{0}\left(\gamma_{0}\right)\right) \equiv \frac{1}{2\left(2-\gamma_{0}\right)} \Delta\left(\gamma_{0}\right)
$$

Since $Y\left(\underline{\gamma}_{*}\right)=1$, we have

$$
\Delta\left(\underline{\gamma}_{*}\right)=-\underline{\gamma}_{*}(\beta-2 \lambda)-\underline{\gamma}_{*}\left(1-\underline{\gamma}_{*}\right)(\beta-2 \lambda)<0
$$

Since $Y\left(\bar{\gamma}_{*}\right)=0$, we have

$$
\begin{aligned}
\Delta\left(\bar{\gamma}_{*}\right)= & \left(1-\bar{\gamma}_{*}\right)\left(v_{H}-v_{L}+\beta-2 \lambda\right)-\frac{2\left(1-\bar{\gamma}_{*}\right)^{2}}{\bar{\gamma}_{*}} \frac{\hat{\gamma}_{*}^{2}}{1-\underline{\gamma}_{*}}(\beta-2 \lambda) \\
& -\frac{2\left(1-\bar{\gamma}_{*}\right)^{2}}{\bar{\gamma}_{*}}\left(\ln \left(\frac{1-\bar{\gamma}_{*}}{1-\underline{\gamma}_{*}}\right)+\frac{\bar{\gamma}_{*}-\hat{\gamma}_{*}}{\left(1-\bar{\gamma}_{*}\right)\left(1-\underline{\gamma}_{*}\right.}\right) \beta .
\end{aligned}
$$

Using Assumption 1, we can show that

$$
\Delta\left(\bar{\gamma}_{*}\right) \geq \frac{1-\bar{\gamma}_{*}}{\beta+2 \lambda}\left(\left(1-\underline{\gamma}_{*}\right)(\beta-2 \lambda)^{2}+8\left(1-\bar{\gamma}_{*}\right) \lambda \beta\right)>0
$$

Thus, there exists a $\underline{\gamma} \in\left(\underline{\gamma}_{*}, \bar{\gamma}_{*}\right)$ such that $\Delta(\underline{\gamma})=0$. Taking derivatives of $\Delta\left(\gamma_{0}\right)$ with respect to $\gamma_{0} \in\left(\underline{\gamma}_{*}, \bar{\gamma}_{*}\right)$ and evaluating at $\underline{\gamma}$ using $\Delta(\underline{\gamma})=0$ yield

$$
\begin{array}{ll}
\Delta^{\prime}(\underline{\gamma})= & \frac{\bar{\gamma}_{*}\left(1-\underline{\gamma}_{*}\right)}{\gamma_{( }\left(\bar{\gamma}_{*}-\gamma_{*}\right)}\left(v_{H}-v_{L}+\beta+2 \lambda\right)+\frac{\underline{\gamma}_{*}\left(2 \underline{\gamma}-\bar{\gamma}_{*}(1+\underline{\gamma})\right)}{\underline{\gamma}(1-\gamma)\left(\bar{\gamma}_{*}-\underline{\gamma}_{*}\right)}(\beta-2 \lambda)-2 \beta \\
> & \frac{1-\underline{\gamma}_{*}}{\bar{\gamma}_{*}-\bar{\gamma}_{*}}\left(v_{H}-v_{L}+\beta+2 \lambda\right)+\frac{2 \underline{\gamma}_{*}-\bar{\gamma}_{*}\left(1+\underline{\gamma}_{*}\right)}{\left(1-\gamma_{*}\right)\left(\bar{\gamma}_{*}-\underline{\gamma}_{*}\right)}(\beta-2 \lambda)-2 \beta \\
>\quad & \frac{\left(1-\underline{\gamma}_{*}\right) \bar{\gamma}_{*}}{\bar{\gamma}_{*}-\underline{\gamma}_{*}}(\beta+2 \lambda)+\frac{2 \hat{\gamma}_{*}-\bar{\gamma}_{*}\left(1+\underline{\gamma}_{*}\right)}{\left(1-\underline{\gamma}_{*}\right)\left(\bar{\gamma}_{*}-\underline{\gamma}_{*}\right)}(\beta-2 \lambda)-2 \beta,
\end{array}
$$

where the first inequality follows because the first term in the expression is decreasing in $\underline{\gamma}$ while the second term is increasing in $\underline{\gamma}$, and the second inequality uses Assumption 1 and the assumption that $\lambda<\beta / 2$. The above can be shown to be equal to

$$
\frac{\beta-2 \lambda}{2}\left(\frac{v_{H}-v_{L}-\beta}{\lambda}\left(\frac{v_{H}-v_{L}-\beta}{\beta+2 \lambda}+\frac{3}{2}\right)+\frac{\beta-2 \lambda}{\beta+2 \lambda}+\frac{\beta-2 \lambda}{\lambda}\right),
$$

which is positive because $\lambda<\beta / 2$. As a result, $\underline{\gamma}$ is unique, with $\Delta\left(\gamma_{0}\right)>0$ if $\gamma_{0} \in\left(\underline{\gamma}, \bar{\gamma}_{*}\right)$, and the opposite holding if $\gamma_{0} \in\left(\underline{\gamma}_{*}, \underline{\gamma}\right)$.

At the other end, we have

$$
\lim _{\gamma_{0} \rightarrow 1} \Delta\left(\gamma_{0}\right)=-(\beta-2 \lambda)<0 .
$$

Thus, there exists a $\bar{\gamma} \in\left(\bar{\gamma}_{*}, 1\right)$ such that $\Delta(\bar{\gamma})=0$. The derivative of $\Delta\left(\gamma_{0}\right)$ with respect to $\gamma_{0} \in\left(\bar{\gamma}_{*}, 1\right)$ is given by

$$
\begin{aligned}
\Delta^{\prime}\left(\gamma_{0}\right)=- & 2\left(v_{H}-v_{L}+2 \lambda\right)-3 \beta+\frac{\left(1-\gamma_{0}^{2}\right) \underline{\gamma}_{*}^{2}}{\gamma_{0}^{2}\left(1-\underline{\gamma}_{*}\right.}(\beta-2 \lambda) \\
& +\frac{2\left(1-\gamma_{0}^{2}\right)}{\gamma_{0}^{2}}\left(\ln \left(\frac{1-\gamma_{0}}{1-\underline{\gamma}_{*}}\right)+\frac{\gamma_{0}-\underline{\gamma}_{*}}{\left(1-\gamma_{0}\right)\left(1-\underline{\gamma}_{*}\right)}\right) \beta .
\end{aligned}
$$

As in the case of $\lambda=0$, the sum of the last two terms in the above expression is increasing in $\gamma_{0}$ and approaches $4 \beta$ as $\gamma_{0}$ approaches 1 . Thus,

$$
\Delta^{\prime}\left(\gamma_{0}\right)<-2\left(v_{H}-v_{L}+\lambda\right)+\beta<0
$$

because $\lambda<\beta / 2$. It follows that $\bar{\gamma}$ is uniquely defined in $\left(\bar{\gamma}_{*}, 1\right)$, and $\Delta\left(\gamma_{0}\right)>0$ for $\gamma_{0} \in\left(\bar{\gamma}_{*}, \bar{\gamma}\right)$ and the opposite holds for $\gamma_{0} \in(\bar{\gamma}, 1)$.

## Appendix D. Discounting

Proposition D1. Let $T$ be finite. There exists a symmetric equilibrium in which the high types always persist, and the strategy $(y(t), x(t))$ and the belief $\gamma(t)$ of the low types are such that:

$$
\begin{aligned}
& \begin{cases}y(t)=0, x(t)=r U_{L}(\gamma(t)) /(\beta \gamma(t)), \gamma(t)=g\left(t ; \gamma_{0}\right) & \text { if } T-t>B\left(g\left(t ; \gamma_{0}\right)\right), t<D\left(\gamma_{0}\right), \\
y(t)=0, x(t)=0, \gamma(t)=g\left(S\left(T ; \gamma_{0}\right) ; \gamma_{0}\right) & \text { if } B\left(g\left(t ; \gamma_{0}\right)\right) \geq T-t>0, t<D\left(\gamma_{0}\right), \\
y(t)=1, \gamma(t)=0 & \text { if } T>t \geq D\left(\gamma_{0}\right) ;\end{cases} \\
& \begin{cases}y(T)=0, \gamma(T)=g\left(S\left(T ; \gamma_{0}\right) ; \gamma_{0}\right) & \text { if } g\left(S\left(T ; \gamma_{0}\right) ; \gamma_{0}\right)>\gamma_{*}, \\
y(T)=2 U_{L}\left(\gamma_{*}\right)\left(e^{r\left(T-S\left(T ; \gamma_{0}\right)\right)}-1\right) /\left(\beta \gamma_{*}\right), \gamma(T)=\gamma_{*} & \text { if } g\left(S\left(T ; \gamma_{0}\right) ; \gamma_{0}\right)=\gamma_{*}, \\
y(T)=1, \gamma(T)=g\left(S\left(T ; \gamma_{0}\right) ; \gamma_{0}\right) & \text { if } g\left(S\left(T ; \gamma_{0}\right) ; \gamma_{0}\right)<\gamma_{*} .\end{cases}
\end{aligned}
$$

Proof. Case (i): $T \geq D\left(\gamma_{0}\right)$. Following the same steps as in the proof of Proposition 1, we only need to show that it is optimal for the high types to always persist for $t<D\left(\gamma_{0}\right)$. Using $U_{H}(0)=v_{H}+\beta$ as the boundary condition, we can solve the differential equation (23) and obtain

$$
U_{H}(\gamma)=\frac{v_{H}+\beta}{v_{L}+\beta} \frac{U_{L}(\gamma)}{\gamma}\left(1-\left(\frac{(1-\gamma) v_{H}}{U_{L}(\gamma)}\right)^{\left(v_{L}+\beta\right) / v_{L}}\right) .
$$

We claim that $U_{H}(\gamma)$ is decreasing. The derivative $U_{H}^{\prime}(\gamma)$ is

$$
\frac{-\left(v_{H}+\beta\right)}{\left(v_{L}+\beta\right)(1-\gamma) \gamma^{2}}\left((1-\gamma) v_{H}-\left(\frac{(1-\gamma) v_{H}}{U_{L}(\gamma)}\right)^{\left(v_{L}+\beta\right) / v_{L}}\left(\gamma\left(v_{L}+\beta\right)+(1-\gamma) v_{H}\right)\right) .
$$

Thus, $U_{H}^{\prime}(\gamma) \leq 0$ if and only if

$$
(1-\gamma) v_{H}\left(1+\frac{\gamma v_{L}}{(1-\gamma) v_{H}}\right)^{\left(v_{L}+\beta\right) / v_{L}} \geq \gamma\left(v_{L}+\beta\right)+(1-\gamma) v_{H},
$$

which is true because the left-hand-side is greater than or equal to

$$
(1-\gamma) v_{H}\left(1+\frac{v_{L}+\beta}{v_{L}} \frac{\gamma v_{L}}{(1-\gamma) v_{H}}\right)=\gamma\left(v_{L}+\beta\right)+(1-\gamma) v_{H} .
$$

We now have

$$
U_{H}(\gamma) \geq U_{H}(1)=\frac{\left(v_{H}+\beta\right) v_{L}}{v_{L}+\beta}>v_{L}
$$

implying that it is optimal for the high types to persist for any $t<D\left(\gamma_{0}\right)$.
Case (ii): $T \leq B\left(\gamma_{0}\right)$. Following the proof in case (i) of Proposition 2, it is enough to observe that for the high types, at any $t \leq T$, persisting for the rest of the game yields

$$
\begin{aligned}
\left(y\left(v_{H}+\beta\right)+(1-y) \frac{v_{H}+v_{L}+\beta}{2}\right) e^{-r(T-t)} & \geq \frac{v_{H}+v_{L}+\beta}{2} e^{-r B\left(\gamma_{0}\right)} \\
& \geq \frac{v_{H}+v_{L}+\beta}{2} e^{-r B(1)} \geq v_{L} .
\end{aligned}
$$

Case (iii): $T \in\left(B\left(\gamma_{0}\right), D\left(\gamma_{0}\right)\right)$. Following the proof in case (ii) of Proposition 2, we note that for the high types, at any $t<S\left(T ; \gamma_{0}\right)$ and corresponding belief $\gamma=g\left(t ; \gamma_{0}\right)$ of the low types, the equilibrium payoff $U_{H}(\gamma)$ is given by the following solution to the differential equation (23):

$$
U_{H}(\gamma)=\frac{v_{H}+\beta}{v_{L}+\beta} \frac{U_{L}(\gamma)}{\gamma}\left(1-C\left(\frac{(1-\gamma) v_{H}}{U_{L}(\gamma)}\right)^{\left(v_{L}+\beta\right) / v_{L}}\right),
$$

where $C$ is a constant determined by the boundary condition:

$$
U_{H}\left(g\left(S ; \gamma_{0}\right)\right)=\left(y\left(v_{H}+\beta\right)+(1-y) \frac{v_{H}+v_{L}+\beta}{2}\right) e^{-r(T-S)}
$$

We need to show that $U_{H}(\gamma) \geq v_{L}$, which is equivalent to:

$$
\begin{aligned}
& \left(1-\frac{v_{L}\left(v_{L}+\beta\right)}{v_{H}+\beta} \frac{\gamma}{U_{L}(\gamma)}\right)\left(\frac{(1-\gamma) v_{H}}{U_{L}(\gamma)}\right)^{-\left(v_{L}+\beta\right) / v_{L}} \\
\geq & C=\left(1-\frac{v_{L}\left(v_{L}+\beta\right)}{v_{H}+\beta} \frac{g\left(S ; \gamma_{0}\right)}{U_{L}\left(g\left(S ; \gamma_{0}\right)\right)} \frac{U_{H}\left(g\left(S ; \gamma_{0}\right)\right)}{v_{L}}\right)\left(\frac{\left(1-g\left(S ; \gamma_{0}\right)\right) v_{H}}{U_{L}\left(g\left(S ; \gamma_{0}\right)\right)}\right)^{-\left(v_{L}+\beta\right) / v_{L}} .
\end{aligned}
$$

The left-hand-side of the above is increasing in $\gamma$ because its derivative is equal to

$$
\begin{aligned}
& \left(\frac{(1-\gamma) v_{H}}{U_{L}(\gamma)}\right)^{-\left(v_{L}+\beta\right) / v_{L}} \frac{v_{L}+\beta}{(1-\gamma) L_{L}(\gamma)}\left(1-\frac{v_{L}}{v_{H}+\beta} \frac{\gamma\left(v_{L}+\beta\right)+(1-\gamma) v_{H}}{\gamma v_{L}+(1-\gamma) v_{H}}\right) \\
\geq & \left(\frac{(1-\gamma) v_{H}}{U_{L}(\gamma)}\right)^{-\left(v_{L}+\beta\right) / v_{L}} \frac{v_{L}+\beta}{(1-\gamma) U_{L}(\gamma)}\left(1-\frac{v_{L}}{v_{H}+\beta} \frac{v_{L}+\beta}{v_{L}}\right) \geq 0 .
\end{aligned}
$$

Thus, the left-hand-side attains a minimum at $\gamma=g\left(S ; \gamma_{0}\right)$. Therefore it is greater than

$$
\begin{aligned}
& \left(1-\frac{v_{L}\left(v_{L}+\beta\right)}{v_{H}+\beta} \frac{g\left(S ; \gamma_{0}\right)}{U_{L}\left(g\left(S ; \gamma_{0}\right)\right)}\right)\left(\frac{\left(1-g\left(S ; \gamma_{0}\right)\right) v_{H}}{U_{L}\left(g\left(S ; \gamma_{0}\right)\right)}\right)^{-\left(v_{L}+\beta\right) / v_{L}} \\
\geq & \left(1-\frac{v_{L}\left(v_{L}+\beta\right)}{v_{H}+\beta} \frac{g\left(S ; \gamma_{0}\right)}{\left.U_{L}\left(g\left(S ; \gamma_{0}\right)\right)\right)} \frac{U_{H}\left(g\left(S ; \gamma_{0}\right)\right)}{v_{L}}\right)\left(\frac{\left(1-g\left(S ; \gamma_{0}\right)\right) v_{H}}{U_{L}\left(g\left(S ; \gamma_{0}\right)\right)}\right)^{-\left(v_{L}+\beta\right) / v_{L}},
\end{aligned}
$$

where the last inequality follows because $U_{H}\left(g\left(S ; \gamma_{0}\right)\right) \geq v_{L}$ by case (ii).
Proposition D2. There exists a $\bar{\gamma} \in\left(\gamma_{*}, 1\right)$ such that the length of the deadline $T$ that maximizes $U^{T}\left(\gamma_{0}\right)$ is $\underline{D}_{*}\left(\gamma_{0}\right)$ if $\gamma_{0} \in\left(\gamma_{*}, \bar{\gamma}\right)$, and is 0 otherwise.

Proof. The first part of the proof is the welfare analysis of a marginal extension of deadline in the regions corresponding to those marked in Figure 1. Clearly, the analysis in Regions IV, V, and VI is identical to that for the case of additive delay cost.

In Region II, where $T \in\left[\bar{D}_{*}\left(\gamma_{0}\right), \underline{D}_{*}\left(\gamma_{0}\right)\right)$, the effect of lengthening the deadline is to make the low types persist longer after the phase switch, but concede with a larger probability when the deadline arrives. Since the behavior of the players during the concession phase does not depend on $T$, the phase-switch time $S\left(T ; \gamma_{0}\right)$ is also independent of $T$. Once the negotiation enters the persistence phase, the low types persist from time $S\left(T ; \gamma_{0}\right)$ through $T$, and then concede with probability $y(T)=$ $2 U_{L}\left(\gamma_{*}\right)\left(e^{r\left(T-S\left(T ; \gamma_{0}\right)\right)}-1\right) /\left(\beta \gamma_{*}\right)$. The payoff to the high type at the deadline is

$$
U_{H}^{0}\left(\gamma_{*} ; y(T)\right)=y(T)\left(v_{H}+\beta\right)+(1-y(T)) \frac{v_{H}+v_{L}+\beta}{2} .
$$

Lengthening the deadline increases the delay for the high types, but also increases their chance of getting their favorite decision rather than a coin toss. The net effect on the welfare of the high types is

$$
\begin{aligned}
\frac{\partial U_{H}\left(\gamma_{0}\right)}{\partial T} & =e^{-r(T-S)}\left(\frac{\partial U_{H}^{0}\left(\gamma_{*} ; y(T)\right)}{\partial T}-r U_{H}^{0}\left(\gamma_{*} ; y(T)\right)\right) \\
& =\frac{e^{-r(T-S)}}{2}\left(\left(v_{H}-v_{L}+\beta\right) \frac{\partial y(T)}{\partial T}-r\left(\left(v_{H}+v_{L}+\beta\right)+y(T)\left(v_{H}-v_{L}+\beta\right)\right)\right) \\
& =\frac{r e^{-r(T-S)}}{2}\left(\left(v_{H}-v_{L}+\beta\right) \frac{2 U_{L}\left(\gamma_{*}\right)}{\gamma_{*} \beta}-\left(v_{H}+v_{L}+\beta\right)\right) \\
& =\frac{r e^{-r(T-S)}}{2}\left(\left(v_{H}-v_{L}+\beta\right) \frac{2 v_{L}}{\beta}+2 v_{H}+\frac{2 v_{H}}{v_{H}-v_{L}} \beta-\left(v_{H}+v_{L}+\beta\right)\right) \geq 0 .
\end{aligned}
$$

Next, consider Region I where $T \in\left[B\left(\gamma_{0}\right), \bar{D}_{*}\left(\gamma_{0}\right)\right)$. From the deadline play of the low types, the payoff to the high types at $t=S\left(T ; \gamma_{0}\right)$ is

$$
\mathcal{U}_{H}\left(S\left(T ; \gamma_{0}\right)\right)=\frac{v_{H}+v_{L}+\beta}{2} e^{-r\left(T-S\left(T ; \gamma_{0}\right)\right)} .
$$

Lengthening the deadline affects the welfare of the high types by changing the boundary value $\mathcal{U}_{H}\left(S\left(T ; \gamma_{0}\right)\right)$ directly and by prolonging the concession phase through increasing $S\left(T ; \gamma_{0}\right)$. The overall effect is

$$
\frac{\partial U_{H}\left(\gamma_{0}\right)}{\partial T}=-r \mathcal{U}_{H}\left(S\left(T ; \gamma_{0}\right)\right)+x\left(S\left(T ; \gamma_{0}\right)\right)\left(v_{H}+\beta-\mathcal{U}_{H}\left(S\left(T ; \gamma_{0}\right)\right)\right) \frac{\partial S\left(T ; \gamma_{0}\right)}{\partial T} .
$$

The loss of a longer deadline is $r \mathcal{U}_{H}\left(S\left(T ; \gamma_{0}\right)\right)$, while the gain is the increased length of the concession phase times the flow rate of concession times the value of the resulting improvement in the decision. The phase-switch time $S$ is defined by the indifference condition:

$$
T-S=\frac{1}{r} \ln \frac{U_{L}^{0}(g)}{U_{L}(g)}
$$

where we write $g=g\left(S\left(T ; \gamma_{0}\right) ; \gamma_{0}\right)$ to economize on notation. Taking derivative respect to $T$, and using the fact that $\dot{g}=-(1-g) r U_{L}(g) / \beta$, we obtain:

$$
\frac{\partial S}{\partial T}=\frac{2 \beta U_{L}^{0}(g)}{\left(\left(v_{L}+\beta\right)^{2}-v_{H} v_{L}\right)(1-g)+\left(\left(v_{L}+\beta\right)^{2}-v_{L}^{2}\right) g}
$$

Therefore, $\partial \mathcal{U}_{H}(0) / \partial T$ is equal to:

$$
\begin{aligned}
& \frac{r U_{L}(g)}{g \beta}\left(v_{H}+\beta-\frac{v_{H}+v_{L}+\beta}{2} \frac{U_{L}(g)}{U_{L}(g)}\right) \frac{\partial S}{\partial T}-\frac{v_{H}+v_{L}+\beta}{2} \frac{r U_{L}(g)}{U_{L}(g)} \\
\geq & -\frac{r\left(v_{H}+\beta\right)\left(2 v_{L}+\beta\right)}{2\left(v_{L}+\beta\right)} \frac{U_{L}(g)}{U_{L}^{0}(g)}+\frac{r U_{L}(g)}{g \beta} \frac{\left(v_{H}+\beta\right) U_{L}^{0}(g)}{\left(v_{L}+\beta\right) U_{L}^{0}(g)} \\
= & \frac{\left.r\left(v_{H}+\beta\right)\right)_{L}(g)}{2\left(v_{L}+\beta\right) U_{L}^{0}(g)} \frac{1-g}{g}\left(v_{H}+v_{L}+\beta\right) \geq 0 .
\end{aligned}
$$

Finally, consider Region III where $T \in\left[B\left(\gamma_{0}\right), D\left(\gamma_{0}\right)\right)$ for $\gamma_{0}<\gamma_{*}$ or $T \in\left[\underline{D}_{*}\left(\gamma_{0}\right), D\left(\gamma_{0}\right)\right)$ for $\gamma_{0} \geq \gamma_{*}$. The analysis is similar to Region I, except that the boundary value becomes

$$
\mathcal{U}_{H}\left(S\left(T ; \gamma_{0}\right)\right)=\left(v_{H}+\beta\right) e^{-r\left(T-S\left(T ; \gamma_{0}\right)\right)}
$$

Take derivative of the phase-switch time $S$ with respect to $T$ to get:

$$
\frac{\partial S}{\partial T}=\frac{2 U_{L}^{0}(g)}{g\left(2 v_{L}+\beta\right)+(1+g) v_{H}}
$$

Furthermore, $\partial \mathcal{U}_{H}(0) / \partial T$ is equal to

$$
\begin{aligned}
& -r\left(v_{H}+\beta\right) \frac{U_{L}(g)}{U_{L}^{0}(g)}+r\left(v_{H}+\beta\right) \frac{U_{L}(g)}{g \beta}\left(1-\frac{U_{L}(g)}{U_{L}^{0}(g)}\right) \frac{2 U_{L}^{0}(g)}{g\left(2 v_{L}+\beta\right)+(1-g) v_{H}} \\
= & r\left(v_{H}+\beta\right) \frac{U_{L}(g)}{U_{L}^{0}(g)}\left(-1+\frac{U_{L}^{0}(g)}{g\left(2 v_{L}+\beta\right)+(1-g) v_{H}}\right) \\
= & -r \frac{v_{H}+\beta}{2} \frac{U_{L}(g)}{U_{L}^{0}(g)} \frac{g\left(2 v_{L}+\beta\right)}{g\left(2 v_{L}+\beta\right)+(1-g) v_{H}} \leq 0 .
\end{aligned}
$$

The second part of the proof is to compare the value of $U^{T}\left(\gamma_{0}\right)$ at the two local maxima $T=0$ and $T=\underline{D}_{*}\left(\gamma_{0}\right)$ for $\gamma_{0}>\gamma_{*}$. Under $T=\underline{D}_{*}\left(\gamma_{0}\right)$, we have $U_{L}^{*}\left(\gamma_{0}\right)=\gamma_{0} v_{L}+\left(1-\gamma_{0}\right) v_{H}$, and solving (23) with the boundary condition $U_{H}^{T}\left(\gamma_{*}\right)=\left(v_{H}+\beta\right) e^{-r B\left(\gamma_{*}\right)}$, we obtain

$$
U_{H}^{*}\left(\gamma_{0}\right)=\frac{v_{H}+\beta}{v_{L}+\beta} \frac{U_{L}\left(\gamma_{0}\right)}{\gamma_{0}}\left(1-\left(1-\frac{\gamma_{*}\left(v_{L}+\beta\right)}{U_{L}^{0}\left(\gamma_{*}\right)}\right)\left(\frac{\left(1-\gamma_{0}\right) /\left(1-\gamma_{*}\right)}{U_{L}\left(\gamma_{0}\right) / U_{L}\left(\gamma_{*}\right)}\right)^{\left(v_{L}+\beta\right) / v_{L}}\right)
$$

For $\gamma_{0}=\gamma_{*}$, we have

$$
\begin{aligned}
U_{H}^{*}\left(\gamma_{*}\right)-U_{H}^{0}\left(\gamma_{*}\right) & =\left(v_{H}+\beta\right) \frac{U_{L}\left(\gamma_{*}\right)}{U_{L}^{0}\left(\gamma_{*}\right)}-\frac{v_{H}+v_{L}+\beta}{2} \\
& =\frac{1}{2 U_{L}^{0}\left(\gamma_{*}\right)}\left(U_{L}\left(\gamma_{*}\right)\left(2\left(v_{H}+\beta\right)-\left(v_{H}+v_{L}+\beta\right)\right)-\gamma_{*} \beta \frac{v_{H}+v_{L}+\beta}{2}\right) \\
& \left.=\frac{\gamma_{*} \beta}{4 U_{L}^{0}\left(\gamma_{*}\right)}\left(\left(\frac{v_{L}}{\beta}+\frac{v_{H}}{v_{H}-v_{L}-\beta}\right) 2\left(v_{H}-v_{L}+\beta\right)\right)-\left(v_{H}+v_{L}+\beta\right)\right) \\
& \left.>\frac{\gamma_{*} \beta}{4 U_{L}^{0}\left(\gamma_{*}\right)}\left(\left(\frac{v_{L}}{\beta}+1\right) 2\left(v_{H}-v_{L}+\beta\right)\right)-\left(v_{H}+v_{L}+\beta\right)\right) \\
& >\frac{\gamma_{*} \beta}{4 U_{L}^{0}\left(\gamma_{*}\right)}\left(2\left(v_{H}-v_{L}+\beta\right)+3 v_{L}-\left(v_{H}+v_{L}+\beta\right)\right)>0 .
\end{aligned}
$$

Therefore,

$$
\lim _{\gamma_{0} \downarrow \gamma_{*}} U^{*}\left(\gamma_{0}\right)-U^{0}\left(\gamma_{0}\right)=\frac{1-\gamma_{*}}{2-\gamma_{*}}\left(U_{H}^{*}\left(\gamma_{*}\right)-U_{H}^{0}\left(\gamma_{*}\right)\right)>0 .
$$

Furthermore,

$$
\lim _{\gamma_{0} \uparrow 1} U^{*}\left(\gamma_{0}\right)-U^{0}\left(\gamma_{0}\right)=U_{L}^{*}(1)-U_{L}^{0}(1)=-\frac{\beta}{2}<0
$$

Therefore, there exists $\bar{\gamma} \in\left(\gamma_{*}, 1\right)$ such that $U^{*}(\bar{\gamma})-U^{0}(\bar{\gamma})=0$.
Finally, note that the derivative of $U^{*}\left(\gamma_{0}\right)-U^{0}\left(\gamma_{0}\right)$ is

$$
\frac{U^{*}\left(\gamma_{0}\right)-U^{0}\left(\gamma_{0}\right)}{2-\gamma_{0}}-\frac{U_{H}^{*}\left(\gamma_{0}\right)-U_{H}^{0}\left(\gamma_{0}\right)}{2-\gamma_{0}}-\frac{v_{H}-v_{L}}{2\left(2-\gamma_{0}\right)}+\frac{1-\gamma_{0}}{2-\gamma_{0}} \frac{\partial U_{H}^{*}\left(\gamma_{0}\right)}{\partial \gamma_{0}} .
$$

When the first term is equal to zero, we must have $U_{H}^{*}(\bar{\gamma})>U_{H}^{0}(\bar{\gamma})$. We show that $U_{H}^{*}\left(\gamma_{0}\right)$ is decreasing, and hence $U^{*}\left(\gamma_{0}\right)-U^{0}\left(\gamma_{0}\right)$ is decreasing when it is equal to zero. The derivative of $U_{H}^{*}\left(\gamma_{0}\right)$ has the same sign as

$$
C_{*}-\frac{\left(1-\gamma_{0}\right) v_{H}}{\gamma_{0}\left(v_{L}+\beta\right)+\left(1-\gamma_{0}\right) v_{H}}\left(\frac{U_{L}\left(\gamma_{0}\right)}{\left(1-\gamma_{0}\right) v_{H}}\right)^{\left(v_{L}+\beta\right) / v_{L}}
$$

where

$$
C_{*}=\frac{2\left(1-\gamma_{*}\right) v_{H}-\gamma_{*} \beta}{\gamma_{*}\left(2 v_{L}+\beta\right)+2\left(1-\gamma_{*}\right) v_{H}}\left(\frac{U_{L}\left(\gamma_{*}\right)}{\left(1-\gamma_{*}\right) v_{H}}\right)^{\left(v_{L}+\beta\right) / v_{L}} .
$$

It can be shown that the second term above is increasing in $\gamma_{0}$, and is therefore greater than or equal to

$$
\frac{\left(1-\gamma_{*}\right) v_{H}}{\gamma_{*}\left(v_{L}+\beta\right)+\left(1-\gamma_{*}\right) v_{H}}\left(\frac{U_{L}\left(\gamma_{*}\right)}{\left(1-\gamma_{*}\right) v_{H}}\right)^{\left(v_{L}+\beta\right) / v_{L}}>C_{*} .
$$

We have shown that $U^{*}\left(\gamma_{0}\right)-U^{0}\left(\gamma_{0}\right)$ is decreasing when it is equal to zero. This implies that $\bar{\gamma}$ is unique and is such that $U_{H}^{*}\left(\gamma_{0}\right)>U_{H}^{0}\left(\gamma_{0}\right)$ if and only if $\gamma_{0} \in\left(\gamma_{*}, \bar{\gamma}\right)$.

