Friends and enemies: A model of signed network formation

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I propose a simple model of *signed* network formation, where agents make friends to extract payoffs from weaker enemies. The model thereby accounts for the interplay between friendship and alliance on one hand and enmity and antagonism on the other. Nash equilibrium configurations are such that either everyone is friends with everyone or agents can be partitioned into different sets, where agents within the same set are friends and agents in different sets are enemies. Any strong Nash equilibrium must be such that a single agent is in an antagonistic relationship with everyone else. Furthermore, I show that Nash equilibria cannot be Pareto ranked. This paper offers a game-theoretic foundation for a large body of work on signed networks, called structural balance theory, which has been studied in sociology, social psychology, bullying, international relations, and applied physics. The paper also contributes to the literature on contests and economics of conflict.

Keywords. Signed network formation, structural balance, contest success function, bullying, economics of conflict, international relations.

JEL classification. D74, D85, F51.

1. Introduction

In much of the economics literature on networks, links have a positive meaning and are commonly interpreted as friendship, collaboration, or transmission of information. In many contexts, however, links may also be associated with negative sentiments, such as antagonism, coercion, or even outright conflict. The aim of this paper is to shed light on the interplay between these two forces by way of a game-theoretic model of signed network formation.

The study of signed networks, consisting of positive and negative links, has a long tradition in sociology and social psychology, dating back to Heider's (1946) seminal contribution on "cognitive dissonance." The essential idea is that positively connected individuals tend to match their attitudes relative to third agents. That is, triads are expected to either consist of three positive, or one positive and two negative links. Cartwright and Harary (1956) proved that these local properties, which they coined as structural balance, yield sharp predictions globally.[1] In particular, the only network configurations that are structurally balanced are such that either all agents are friends or there exist two distinct sets, also called cliques, where agents in the same set are friends and agents in different sets sustain antagonistic relationships. Davis (1967) showed that, when allowing for triads of three negative links, "weakly balanced" graphs may consist of multiple cliques. Structural balance has also been employed in the study of international relations, as discussed in more detail below, and has more recently gained increased interest in the physics and mathematics literature.[2]

This paper provides a game-theoretic foundation for structural balance. One of the central features of my model is that stronger agents are able to obtain payoffs at the expense of weaker enemies and that the strength of agents is determined endogenously. That is, my model may be interpreted as one of bullying, where agents gang up on others for their own benefit. Incentives to bully may be present in different contexts and the exercise of power, as well as the particular mechanism at work, may take different forms. What is common to the situations I have in mind is that there is little scope for agents to avoid being bullied, that having more and stronger friends or allies is advantageous in a given antagonistic relationship, and that payoff extraction is embedded in pairwise connections.

Bullying has been studied extensively in interpersonal relationships and is defined as "negative actions which may be physical or verbal, have hostile intent [...] and involve a power differential" (O'Connell et al. 1999). Studies range from bullying behavior among children and youth in schools to bullying at the workplace, in prisons, and in the army.[3] Bullying is considered to be a group phenomenon and its driving force is thought to be a quest for high status and dominance within the peer group.[4] The act of bullying is associated with prestige in terms of perceived status and popularity, while victims are typically those of low status.[5] Differences in power (where power is interpreted as the ability to force one's will on others) and status differences (differences in popularity) are understood to be key aspects of bullying.[6] Furthermore, a crucial source of power differentials in the bully–victim relationship are peers backing the aggressor, while victims may also seek the support of others.[7]

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[1]For a good introduction to the literature on structural balance, see Easley and Kleinberg (2010).

[2]See, for example, Antal et al. (2005), Marvel et al. (2011), and Marvel et al. (2009).

[3]See Björkqvist et al. (1996), Veenstra et al. (2005), Bowling and Beehr (2006), Hoel and Rayner (1997), Nielsen et al. (2008), Ireland (1999 and 2005), Ireland et al. (2007), and Østvik and Rudmin (2001).

[4]See Pellegrini (2002), Peets and Salmivalli (2008), and South and Wood (2006).

[5]See, for example, Lindenberg et al. (2009). Note, however, that perceived popularity is more strongly linked to perceptions of power than social preference (Vaillancourt et al., 2003). That is, individuals, who are perceived as popular are not necessarily well liked.

[6]See Vaillancourt et al. (2003).

[7]See O'Connell et al. (1999), Boulton and Fox (2006), and Hodges et al. (1999).

Interpersonal bullying has also been studied in a network context. Huttunen et al. (1997) analyze the structure of peer networks and bullying in schools, while Cairns et al. (1988) conduct a study with a similar focus among adolescents. Both papers find that bullies belong to social clusters and are supported in their activities by their peer group, while victims and children that side with victims form separate clusters. These results are much in line with my model predictions. A recent paper by Huitsing et al. (2012) explicitly studies structural balance properties in the context of bullying. The authors find structural balance in general like and bullying relationships. Children who were nominated as bullies by the same victims tend to like each other, while children who are victimized by the same bullies also often like each other. Note, however, that Huitsing et al. (2012) do not find structural balance for general like and general dislike relationships. I take this as an indication that a tendency toward structural balance properties is, in fact, driven by bullying motives.[8]

Incentives of agents to gang up on others are not confined to interpersonal relations, but may also be present in other contexts. An area where the study of structural balance has been applied fruitfully, and where incentives to bully are thought to be important, is international relations. Military alliances enhance a country's military weight, either via direct logistic and military support or via intelligence and spillovers in terms of the use of military machinery. Note that this may not only be valuable in military or diplomatic disputes, but also has implications for the distribution of the gains from trade.[9] One of the earliest applications of structural balance to international relations is Harary (1961), who examines the rapid shifts of relationships among nations in the Middle Eastern crisis of 1956 and observes a strong tendency toward balance. Moore (1979) also employs structural balance when explaining the "United States' somewhat surprising support of Pakistan" in the conflict over Bangladesh's separation from Pakistan in 1972. Another case in point is provided by Antal et al. (2006). They link the formation of alliances in the 19th century—ultimately leading up to WWI—to structural balance. While the above examples provide interesting anecdotal evidence, it is worth nothing that my simple model abstracts from many important issues that are relevant for international relations, such as trade, aggressive vs. defensive military alliances, complexity or coordination cost of alliances, and a distinction between open conflict and diplomatic disputes. A brief discussion concerning international relations is provided toward the end of the paper, together with extensions of the setup.

This paper presents a parsimonious model of signed network formation, which provides a game-theoretic foundation for structural balance and builds on agents' incentives to bully and gang up on each other. Players can either extend a friendly (positive) or an antagonistic (negative) link to each of the remaining agents. A reciprocated positive link constitutes a *friendship* or *alliance*. Extending a negative link is thought of as

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[8]More generally, research in sociology has examined the evolution of signed network relations. Doreian and Mrvar (1996) and Doreian and Krackhardt (2001) are two such empirical studies. In both cases a movement toward balance is apparent.

[9]Riddell (1988) records a positive relationship between the United States' military power on the one hand and the terms of trade and the profit rate of domestic firms on the other hand. See also Brito and Intriligator (1978), who state that "the distribution of the gains from trade is a function not only of markets and initial resources, as in the classical theory of international trade, but also of the power of the nations involved."
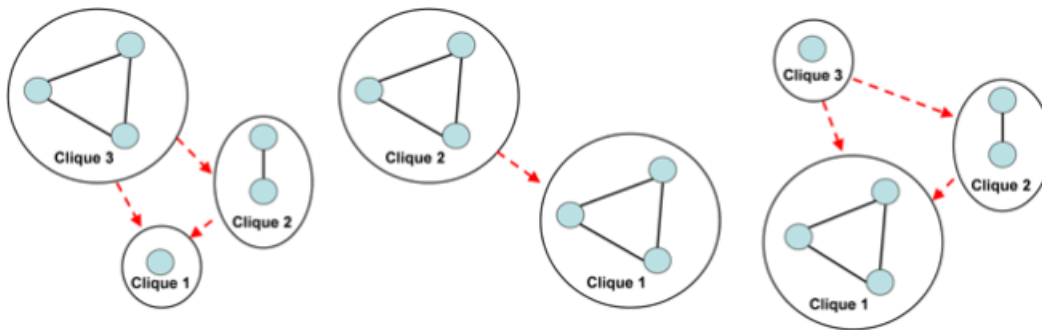
FIGURE 1. Three examples of Nash equilibrium networks.

picking a fight (or argument) and triggers an *antagonistic* or *enemy* relationship. I assume that picking a fight is costly (this cost may be arbitrarily small) and I also allow for a cost of conflict, which is incurred by both agents involved in an antagonistic relationship. An agent's strength is determined endogenously. More precisely, the strength of an agent is given by the agent's intrinsic strength plus the intrinsic strength of the agent's friends or allies. Note that, unless otherwise stated, I mean by strength an agent's ex post strength, i.e., the strength of an agent given a particular network. An important attribute of the model is that the payoff a stronger agent can extract from a weaker enemy is strictly increasing in own strength and strictly decreasing in the respective enemy's strength. This is what drives stronger agents to match their strategies relative to weaker agents. Essentially, by coordinating on who to bully, the bullied agent is weaker and extraction payoffs are therefore higher.

The first main finding is that every Nash equilibrium obeys (weak) structural balance. That is, Nash equilibrium configurations are such that either all links are positive or agents can be divided into two or more distinct sets, where agents within the same set are friends and agents in different sets are enemies. Furthermore, stronger agents extend negative links to weaker agents.

The above result is illustrated via the three examples depicted in Figure 1. Agents in the same clique are circled and a solid line represents a positive link. All negative links across cliques are summarized by a single dashed line. The direction of the link indicates who extends the negative link and, therefore, who is bullying or extracting payoffs from whom. Note that when agents are homogeneous with respect to their intrinsic strength, then only the graph on the very left can be sustained as a Nash equilibrium. In this case, strength is determined by the number of friends and, for negative links to arise in equilibrium, cliques must be of different sizes.

I characterize pure strategy Nash equilibria for a general class of payoff functions, which map the respective strengths of agents into an extraction payoff under an antagonistic relationship. The characterization, together with my existence results, allows for agents who differ in their intrinsic strength, while the remainder of the paper focuses on the homogeneous agents case. Sufficient conditions for at most two cliques to arise in any Nash equilibrium and for the existence of Nash equilibria with multiple cliques are

derived. I thereby differentiate the case of strong structural balance (which allows for at most two cliques) from weak structural balance (which allows for multiple cliques).

Multiplicity of equilibria is addressed by characterizing the set of strong Nash equilibria. These are shown to be such that one agent is the enemy of everyone else. That is, if a strong Nash equilibrium exists, then it yields a unique prediction. I provide simple conditions for the existence of a strong Nash equilibrium. Note next that my results are largely driven by agents' incentives to coordinate their linking behavior relative to weaker agents. To highlight the role of these incentives, I show that if the extraction value is assumed to be a constant and coordination incentives are, therefore, not present, then Nash equilibria exist that are not structurally balanced. Finally, I analyze welfare properties of the model. The configuration with the highest sum of payoffs is the network where all links are positive, as this minimizes the cost of negative links. I then show the somewhat surprising result that Nash equilibria cannot be Pareto ranked.

This paper relates to the economics of conflict literature, where conflict is modeled in terms of a contest success function (see Tullock 1967, 1980, and Hirshleifer 1989) and an agent's probability of winning is determined by the resources available for arming. Part of this research focuses on coalition formation. See, for example, Wärneryd (1998) and Esteban and Sákovics (2003).[10] Note that in these models of distributional conflict, agents typically obtain identical outcomes. In contrast, payoffs may differ across agents in my setup.[11] Jordan (2006) considers coalitional games, where wealth must be allocated among agents and opportunities to pillage by varying coalitions are determined by a power function, which at least partly depends on the wealth of the members of a coalition. Any coalition can take all the wealth of any other coalition that is less powerful at zero cost. The farsighted core allocations, where no coalitions are formed and no acts of pillage occur, depend on the properties of the power function. In the specification that may be considered as most closely related to my model, i.e., when the power to pillage mainly depends on the size of a coalition (and the number of agents is odd), then there exists a stable set such that aggregate wealth is divided among a minimal majority. A similar equilibrium exists in my model if payoff extraction is very efficient (and agents are homogeneous). Another important strand of the economics of conflict literature studies production and appropriation. To relate to this body of work, an extension that allows for costly production is provided in Appendix B, which is available in a supplementary file on the journal website, http://econtheory.org/supp/1937/supplement.pdf. I assume (as, for example, Garfinkel 1990, Hirshleifer 1995, and Skaperdas 1992) that production enters into a common pool, which is subject to appropriation. A simple condition is provided such that again all Nash equilibria are structurally balanced. However, efficiency results are now shown to depend on the curvature of the cost function.

This paper contributes to the literature on networks in economics. See, for example, Myerson (1977), Aumann and Myerson (1988), Bala and Goyal (2000), and Jackson and Wolinsky (1996). Recent papers that feature contest success functions in a network

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[10]Good overviews of this literature are Garfinkel and Skaperdas (2007) and Bloch (2012).

[11]See Rietzke and Roberson (2013) for a setting in which two potential allies face a fixed, common enemy.

setting are Goyal and Vigier (2014), König et al. (2015), and Jackson and Nei (2015), all of which have a different focus. Goyal and Vigier (2014) study a design problem and ask how to optimally structure networks so that they are robust to attacks in the face of an adversary. König et al. (2015) model a setting where agents are embedded in a network of bilateral alliances and conflicts, and bring their model to the data. The authors are concerned with conflict intensity on a fixed network and do not consider endogenous alliance formation and conflict. Note that the extension presented in Appendix B may be viewed as a variant of König et al. (2015), where, instead of assuming that the network of alliances and conflicts is fixed and agents choose their fighting effort endogenously, fighting effort is now assumed to be fixed, while the network of alliances and conflicts is endogenous (and production is introduced). Jackson and Nei (2015) aim to understand the relationship between trade and war among countries. In their model, agents create bilateral military alliances, which allow agents to coordinate attacks against other agents, who in turn may be defended by their allies. The authors establish that, in the absence of trade, there is no stable network such that no country has an incentive to start a war. The authors then shown that trade may restore stability and prevent the outbreak of war. Jackson and Nei (2015) and my paper study very different, yet complementary, questions. More specifically, while Jackson and Nei (2015) are concerned with conditions for equilibria such that no conflict arises, I am concerned with commonly observed network patterns in the presence of antagonism and conflict. To the best of my knowledge, my paper presents the first game-theoretic model of *signed* network formation.[12] The remaining parts of the paper are organized as follows: Section 2 introduces the model, Section 3 presents the results formally, Section 4 provides a discussion of applications, and Section 5 concludes. All proofs of the main part of the paper are relegated to Appendix A. Appendix B (the supplementary file) introduces production.

2. Model

Let $N = \{1, 2, \ldots, n\}$ be the set of agents, with $n \geq 3$. A strategy for an agent $i \in N$ is defined as a row vector $\mathbf{g}_i = (g_{i,1}, g_{i,2}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,n})$, where $g_{i,j} \in \{-1, 1\}$ for each $j \in N\backslash\{i\}$. Agent i is said to extend a positive link to j if $g_{i,j} = 1$ and a negative link if $g_{i,j} = -1$. The set of strategies of i is defined by G_i and the strategy space is defined by $G = G_1 \times \cdots \times G_n$. The resulting network of relationships is written as $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n)$. Define the undirected network $\bar{\mathbf{g}}$ in the following way: The link between agents i and j is positive in the undirected network $\bar{\mathbf{g}}$ if both directed links are positive, so that $\bar{g}_{i,j} = 1$ if $g_{i,j} = g_{j,i} = 1$; the link in the undirected network is negative if at least one of the two undirected links is negative, so that $\bar{g}_{i,j} = -1$ if either $g_{i,j} = -1$ or $g_{j,i} = -1$ (or both). Given a network $\mathbf{g}$, $\mathbf{g} + g_{i,j}^+$ and $\mathbf{g} + g_{i,j}^-$ have the following interpretation. If $g_{i,j} = -1$ in $\mathbf{g}$, then $\mathbf{g} + g_{i,j}^+$ changes the directed link $g_{i,j} = -1$ into $g_{i,j} = 1$, while if $g_{i,j} = 1$ in $\mathbf{g}$, then $\mathbf{g} + g_{i,j}^+ = \mathbf{g}$. Similarly, if $g_{i,j} = 1$ in $\mathbf{g}$, then $\mathbf{g} + g_{i,j}^-$ changes the directed link $g_{i,j} = 1$ into $g_{i,j} = -1$, while if $g_{i,j} = -1$ in $\mathbf{g}$, then $\mathbf{g} + g_{i,j}^- = \mathbf{g}$. We sometimes use the summation sign to denote multiple link changes.

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[12]Note that a variant of the model, presented in a previous working paper version Hiller (2013), features positive, negative, *and* zero links in equilibrium. I provide a short discussion toward the end of the paper.

Define the following sets: $N_i^+(\mathbf{g}) = \{j \in N \mid \bar{g}_{i,j} = 1\}$ is the set of agents to which agent i reciprocates a positive link and, therefore, $\bar{g}_{i,j} = 1$ in the undirected network $\bar{\mathbf{g}}$; $N_i^-(\mathbf{g}) = \{j \in N \mid \bar{g}_{i,j} = -1\}$ is the set of agents such that i extends and/or receives a negative link and, therefore, $\bar{g}_{i,j} = -1$; $N_i^{e-}(\mathbf{g}) = \{j \in N \mid g_{i,j} = -1\}$ is the set of agents to which agent i extends a negative link. Note that $N_i^{e-}(\mathbf{g}) \subseteq N_i^-(\mathbf{g})$. Denote the cardinalities $c_i(\mathbf{g}) = |N_i^-(\mathbf{g})|$ and $e_i(\mathbf{g}) = |N_i^{e-}(\mathbf{g})|$. Strength or power of an agent is determined endogenously and consists of an agent i's intrinsic strength, $\lambda_i > 0$, and the strength of all agents that reciprocate agent i's positive link. The strength of agent i in network $\mathbf{g}$ is then given by $n_i(\mathbf{g}) = \lambda_i + \sum_{j \in N_i^+(\mathbf{g})} \lambda_j$. Note that we refer to $n_i(\mathbf{g})$ as an agent's strength in network $\mathbf{g}$ and specifically state it when referring to an agent's intrinsic strength, λ_i. When considering homogeneous agents, we assume for simplicity that $\lambda_i = 1$ $\forall i$ and, therefore, $n_i(\mathbf{g}) = |N_i^+(\mathbf{g})| + 1$.

Links are interpreted in the following way: A reciprocated positive link, i.e., a positive link in the undirected network, $\bar{\mathbf{g}}$, establishes a friendship or alliance, which is assumed to contribute to the agents' strengths. Extending a negative link initiates an antagonistic relationship and may be thought of as picking a fight, where picking a fight incurs a cost of $\varepsilon > 0$. Furthermore, we also allow for a cost of conflict, $\kappa \geq 0$. Note that for a pair of agents to enter into an antagonistic or conflict relationship, it is sufficient that only one agent extends a negative link. That is, once at least one negative link is extended, then both agents are assumed to engage in conflict (and to incur the cost of conflict, κ).[13] Gross payoffs from a negative link are determined by a function $f(n_i(\mathbf{g}), n_j(\mathbf{g}))$, which depends on the agents' respective strengths $n_i(\mathbf{g})$ and $n_j(\mathbf{g})$. We present this function in detail below.

The utility of player i under strategy profile $\mathbf{g}$ is given by

$$u_i(\mathbf{g}) = \sum_{j \in N_i^-(\mathbf{g})} f\big(n_i(\mathbf{g}), n_j(\mathbf{g})\big) - e_i(\mathbf{g})\varepsilon - c_i(\mathbf{g})\kappa.$$

To simplify notation, we sometimes write n_i for $n_i(\mathbf{g})$ and $f(n_i, n_j)$ for $f(n_i(\mathbf{g}), n_j(\mathbf{g}))$. Note first that we only sum over agents with whom a negative link is sustained. That is, direct payoffs from a reciprocated positive link are zero. The purpose of a friendship or alliance is, therefore, to increase the respective agents' strengths, thereby increasing payoffs on any negative links, and to avoid the cost associated with antagonistic links. This brings out the tension between friendship/alliance and antagonism in the starkest manner.[14] The value an agent extracts through an antagonistic link is assumed to be another agent's loss and, therefore, $f(n_i, n_j) + f(n_j, n_i) = 0$ $\forall n_i, n_j$. If a pair of agents of equal strength is negatively connected, then no payoff extraction takes place, so that

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[13]Note that our equilibrium characterization also goes through when allowing for stronger agents to incur a different cost of conflict than weaker agents. However, for notational simplicity and clarity, we present the case when cost of conflict is symmetric.

[14]Alternatively, one could specify a model where each pair of agents divides a fixed payoff, which is shared equally if agents are friends, while the shares are determined by the respective agents' strengths in the case of antagonistic links. The equilibrium characterization goes through for this specification as well. Normalizing makes the exposition clearer, as we only need to sum over negative links.

$f(n_i, n_j) = 0 \; \forall n_i, n_j : n_i = n_j$. We assume that f is strictly increasing in n_i. This implies, together with $f(n_i, n_j) + f(n_j, n_i) = 0 \; \forall n_i, n_j$, that f is strictly decreasing in n_j. Finally, note that the total linking cost of an agent i in network $\mathbf{g}$ is determined by $e_i(\mathbf{g})$ extended negative links, which each incurs a cost of $\varepsilon > 0$, and $c_i(\mathbf{g})$ undirected negative links, which each incurs a cost of $\kappa \geq 0$. To allow for the possibility of negative links in equilibrium, we assume that $f(\sum_{j \in N \setminus \{k\}} \lambda_j, \lambda_k) > \varepsilon + \kappa$, where k is such that $\lambda_k \leq \lambda_j \; \forall j \in N$.

The two most prominent contest success functions (henceforth CSFs) in the economics of conflict literature are the CSF in ratio and the CSF in difference form (for a detailed account, see Hirshleifer 1989). Normalizing these functions by subtracting $\frac{1}{2}$ yields functions that satisfy our requirements on f. Note that in both cases the function is parametrized by ϕ, where a higher value of ϕ is favorable for the stronger agent.

DEFINITION 1.   *The normalized CSF in ratio form is*

$$(1) \quad f(n_i, n_j) = \frac{n_i^\phi}{n_i^\phi + n_j^\phi} - \frac{1}{2}$$

and in difference form[15]

$$(2) \quad f(n_i, n_j) = \frac{1}{1 + e^{\phi(n_j - n_i)}} - \frac{1}{2}.$$

The equilibrium concept used is pure strategy Nash equilibrium. A strategy profile $\mathbf{g}^*$ is a pure strategy Nash equilibrium (NE) if and only if

$$u_i(\mathbf{g}_i^*, \mathbf{g}_{-i}^*) \geq u_i(\mathbf{g}_i, \mathbf{g}_{-i}^*) \quad \forall \mathbf{g}_i \in G_i, \forall i \in N.$$

We denote agent i's deviation strategy from a given network $\mathbf{g}$ with $\mathbf{g}_i'$ and the network after a proposed deviation is denoted with $\mathbf{g}'$. Social welfare is defined as the sum of individual payoffs. For any network $\mathbf{g}$, social welfare is given by $W(\mathbf{g}) = \sum_{i \in N} u_i(\mathbf{g})$. A network $\hat{\mathbf{g}}$ is socially efficient if and only if $W(\hat{\mathbf{g}}) \geq W(\mathbf{g}) \; \forall \mathbf{g} \in G$.

3. ANALYSIS

3.1 *Equilibrium characterization*

This section provides a characterization of pure strategy Nash equilibria. We show that either all links are positive or agents can be partitioned into maximal cliques of different

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[15]Note that for $\phi > 0$, $\frac{n_i^\phi}{n_i^\phi + n_j^\phi} \in (0, 1)$ holds $\forall n_i, n_j \in \mathbb{R}_+$ and, since $\frac{n_i^\phi}{n_i^\phi + n_j^\phi} - \frac{1}{2}$ can be written as $\frac{n_i^\phi}{n_i^\phi + n_j^\phi} \cdot \frac{1}{2} +$ $(1 - \frac{n_i^\phi}{n_i^\phi + n_j^\phi}) \cdot (-\frac{1}{2})$, we can interpret the gross payoffs of an antagonistic link under the normalized CSF as a bilateral zero-sum contest over a payoff of $\frac{1}{2}$. In the case of a discrete either–or competition, $\frac{n_i^\phi}{n_i^\phi + n_j^\phi}$ may be thought of as the probability of winning the bilateral contest. The normalized CSF in difference form can be interpreted in the same way. is

Figure 2. Coordination Incentives.

(ex post) strength, where agents in stronger cliques extract payoffs from agents in weaker cliques.[16]

To illustrate where and how our model assumptions come into play, we briefly comment on some of the main arguments of the equilibrium characterization. Note first that a negative link is never reciprocated by a negative link in any Nash equilibrium, since extending a positive link instead increases payoffs by ε (Lemma 1 in Appendix A). Furthermore, in any Nash equilibrium, there does not exist an agent that extends a negative link to a weakly stronger agent. By extending a positive link instead, the agent not only increases payoffs from that link (by at least $\varepsilon + \kappa$), but the agent's strength also increases, which in turn increases payoffs on any remaining negative links (Lemma 2 and Step 1 of Proposition 1 in Appendix A). Therefore, agents of equal strength are positively connected and stronger agents determine the sign of links relative to weaker agents.

For the following discussion, it is convenient to rank agents by their strength in network $\mathbf{g}$, $n_i(\mathbf{g})$. We denote the set of the strongest agents with $P^m(\mathbf{g})$, the set of the second strongest agents with $P^{m-1}(\mathbf{g})$, and proceed in this way until the set of weakest agents, $P^1(\mathbf{g})$. From the previous paragraph we know that all agents in $P^m(\mathbf{g})$ must be positively connected. Our equilibrium characterization is driven by incentives of agents to coordinate their strategies relative to weaker agents. These incentives are due to our assumption that $f(n_i, n_j)$ is strictly decreasing in n_j. We illustrate this by way of a simple example. Assume that $\mathbf{g}$ is a NE. Take a pair of agents i and j in $P^m(\mathbf{g})$, and assume that i and j display identical linking behavior to all remaining agents, with the exception of agents k and l, with $k, l \notin P^m(\mathbf{g})$.

Such a configuration is depicted on the left hand side of Figure 2, where we focus only on the relevant links between agents i, j, k, and l. Note that agent i is positively connected with agent l and negatively connected with agent k, while the opposite is true for agent j. Without loss of generality, assume that $u_j(\mathbf{g}) \geq u_i(\mathbf{g})$ and consider a deviation where agent i imitates agent j's strategy. That is, after proposed deviation, agent i extends a negative link to l and a positive link to k, as depicted on the right hand side of Figure 2. After proposed deviation, $n_i(\mathbf{g}') = n_j(\mathbf{g})$ holds, since i and j are linked to each other and share the same friends in $\mathbf{g}'$. However, l has fewer friends after proposed deviation and, since $f(n_i, n_j)$ is decreasing in the second argument, i extracts more from l in $\mathbf{g}'$ than j in $\mathbf{g}$, while payoffs from any other links remain unchanged. Therefore, $u_i(\mathbf{g}') > u_j(\mathbf{g})$ and proposed deviation is profitable. The same intuition carries over to

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[16]Formally, a *clique* is a set of agents $C(\mathbf{g}) \subseteq N$, such that $\bar{g}_{i,j} = 1 \ \forall i, j \in C(\mathbf{g})$. A clique is *maximal* and denoted with $C^m(\mathbf{g})$ if, for any $l \notin C^m(\mathbf{g})$, $C^m(\mathbf{g}) \cup \{l\}$ is not a clique.

strategies that differ in more complicated ways. By iteratively using similar imitation strategies, one can then show that in any NE, agents in $P^m(\mathbf{g})$ must extend negative links to all agents not in $P^m(\mathbf{g})$. Finally, we use an induction argument to show that all agents in $P^{m-x}(\mathbf{g})$ with $x \in \mathbb{N}$ extend negative links to agents in cliques with weaker agents. Our first main result is summarized in Proposition 1.

PROPOSITION 1 (Heterogeneous agents). *In any NE, if $n_i(\mathbf{g}) = n_j(\mathbf{g})$, then $\bar{g}_{i,j} = 1$, and if $n_i(\mathbf{g}) \neq n_j(\mathbf{g})$, then $\bar{g}_{i,j} = -1$.*

Proposition 2 provides existence results. We only present the analysis for homogeneous agents here. However, one can easily extend the analysis for heterogeneous agents, as is briefly discussed at the end of this section. We first show that there always exists an equilibrium where everyone extends positive links to everyone else. In this case, payoffs are zero for each agent and a deviation consists of extending negative links to some subset of the remaining $n - 1$ agents. Note that after a deviation, the deviating agent is either the weakest agent or at most as strong as one other agent (if the deviation consists of extending a single negative link). Therefore, gross payoffs from each negative link are at most zero, while each negative link incurs a cost of $\varepsilon + \kappa > 0$. Therefore, no profitable deviation exists. We then show that there also always exists an equilibrium such that $n - 1$ agents form a clique and extract payoffs from a single agent. The intuition is similar to the previous argument, but we now need to also account for deviations where one of the $n - 1$ agents extends a positive link to the agent that is an enemy of everyone else, while extending negative links to some subset of the remaining agents. To see why networks different from the ones mentioned above may not be Nash equilibria, consider a network with two cliques: one of size $n - 2$ and the other of size 2. An agent in the larger clique may now find it profitable to extend a positive link to one of the agents in the smaller clique, forgoing the extraction payoff of this agent, so as to increase the payoff from the remaining negative link.

PROPOSITION 2 (Homogeneous agents). *Any network $\mathbf{g}$ such that (i) all agents are friends and (ii) $n - 1$ agents are friends of each other and one agent is an enemy of everyone else (and Lemmas 1 and 2 hold) is a NE.*

Next we comment briefly and informally on the case of heterogeneous agents. If f is continuous and agents are sufficiently homogeneous, then, for any ε and κ, a Nash equilibrium exists such that all links are positive. The intuition is that if f is continuous and agents are sufficiently homogeneous, then the gross payoff from a negative link in a deviation is either negative, zero, or arbitrarily close to zero. Therefore, for given ε and κ, again no profitable deviation exists. For the equilibrium where $n - 1$ agents gang up on a single agent, assume that the single agent is of (weakly) lowest intrinsic strength. One can then show that assuming the function f to be continuous and agents to be sufficiently homogeneous is again sufficient for this type of equilibrium to exist.[17]

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[17]Formal proofs for the heterogeneous agent case are available upon request.
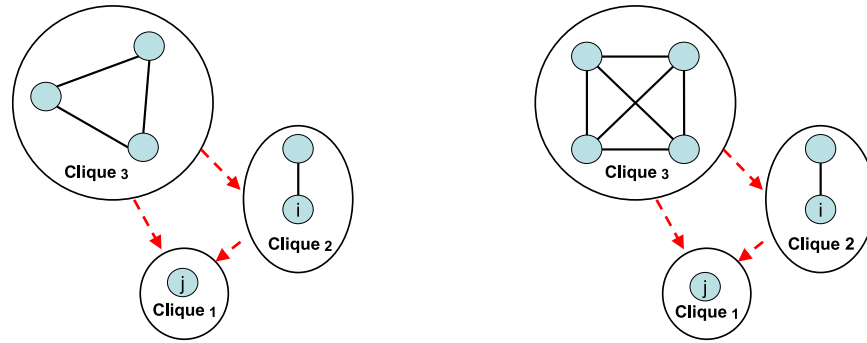
FIGURE 3. Two examples illustrating Proposition 3.

3.2 *Strong versus weak structural balance*

For the remaining part of the paper, we assume that agents are homogeneous. Next we offer two results regarding the number of cliques. Proposition 3 presents a condition for the existence of Nash equilibria that display multiple cliques, while Proposition 4 provides a condition for any Nash equilibrium to display at most two cliques. We thereby disentangle the case when all Nash equilibria obey structural balance in its strong form (no triads of negative links and at most two cliques) from the case when Nash equilibria allow for weak structural balance (triads of negative links and multiple cliques). Propositions 3 and 4 translate into simple requirements for both of the contest success functions considered.

Before introducing Proposition 3, we define $\delta = f(n-1,1) - \min_{2 \leq x \leq n-1} f(x, x-1)$. Note that $f(n-1,1)$ is the maximum extraction value that can be obtained from a single negative link with n agents and δ may therefore be interpreted as a measure of the effectiveness of the conflict technology. That is, if δ is small, then conflict is effective in the sense that having just one more friend than the respective enemy yields an extraction payoff that is close to $f(n-1,1)$. Note also that Proposition 3 implies that the relative size of groups is at least geometrically increasing with common ratio 2.

PROPOSITION 3 (Homogeneous agents). *If δ is sufficiently small, then $\mathbf{g}$ is a NE if and only if either $g_{i,j} = 1 \ \forall i, j$ or $\sum_{i=1}^{j-1} |P^i(\mathbf{g})| < |P^j(\mathbf{g})|$ holds $\forall P^j(\mathbf{g})$ (together with Lemma 1, Lemma 2, and Proposition 1).*

We illustrate Proposition 3 by way of two examples. Assume δ to be arbitrarily close to zero. Consider first the network structure depicted on the left hand side of Figure 3. This configuration is not an equilibrium, since agent i in clique 2 can profitably deviate by extending a positive link to agent j in clique 1. Agent i forgoes the payoff from the link with j, which is arbitrarily close to $f(n-1,1) - (\varepsilon + \kappa) > 0$. However, since after proposed deviation agent i has the same strength as agents in clique 3, i's payoffs increases by arbitrarily close to three times $f(n-1,1)$ and proposed deviation is therefore profitable. The same argument cannot be made for the example on the right hand side of Figure 3. After extending a positive links to agent j in clique 1, agent i in clique 2 is still weaker

than agents in clique 3. Agent i thereby forgoes a payoff of arbitrarily close to $f(n-1,1) - (\varepsilon + \kappa)$, while gaining arbitrarily little from links with agents in clique 3. Such a deviation is therefore not profitable for δ sufficiently small. One can show that relative group size must increase at least geometrically to preclude these types of deviations. Finally, agents in the largest clique do not find it profitable to deviate by extending positive links, as they forgo payoffs arbitrarily close to $f(n-1,1) - (\varepsilon + \kappa)$, while gaining arbitrarily little from any remaining negative links. Proposition 3 directly extends to the aforementioned contest success functions. To see this, note that δ is arbitrarily small for ϕ sufficiently large.

COROLLARY 1. *If f is the normalized CSF in ratio form or the normalized CSF in difference form, then for ϕ sufficiently large, $\mathbf{g}$ is a NE if and only if the conditions of Proposition 3 hold.*

Proposition 4 provides sufficient conditions such that there are at most two cliques in any Nash equilibrium. We assume that f is twice differentiable. Essentially, the conditions ensure that an agent in the second smallest clique always finds it profitable to extend a positive link to an agent in the smallest clique.

PROPOSITION 4 (Homogeneous agents). *If $\partial^2 f(n_i, n_j)/\partial n_i \partial n_j$ and $\partial^2 f(n_i, n_j)/\partial^2 n_i$ are sufficiently close to zero $\forall n_i, n_j$ and $\partial^2 f(n_i, n_j)/\partial^2 n_i \leq 0$ $\forall n_i \geq n_j$, then there are at most two maximal cliques in any Nash equilibrium.*

Corollary 2 shows that Proposition 4 also extends to both of the contest success functions. More precisely, if ϕ is sufficiently small, then both contest success functions satisfy the conditions of Proposition 4 and therefore any Nash equilibrium displays at most two cliques. The intuition is that, for ϕ sufficiently small, the contest success functions become almost linear in n_i and the derivative with respect to n_i depends arbitrarily little on n_j. That is, what drives this result is not that the conflict technology is very inefficient, but rather that, for ϕ sufficiently small, the contest success functions approach a linear function in the relevant range. To contrast this with the result in Proposition 3, note that, for ϕ sufficiently large, both contest success functions approach a step function.

COROLLARY 2. *If f is the normalized CSF in ratio form or the normalized CSF in difference form, and ϕ is sufficiently small, then there are at most two maximal cliques in any Nash equilibrium.*

3.3 *Strong Nash equilibrium*

In this section we show that any strong Nash equilibrium must be such that $n-1$ agents gang up on a single agent. Note that strong Nash equilibrium allows for deviations of any subset of agents and a deviation is considered profitable if payoffs strictly increase for all deviating agents. A formal definition is provided below.

DEFINITION 2. A Nash equilibrium $\mathbf{g} \in G$ is a strong Nash equilibrium if there is no $J \subseteq N$ and $\mathbf{g}' \in G$ such that (i) $\mathbf{g}'_i = \mathbf{g}_i$ $\forall i \notin J$ and (ii) $u_i(\mathbf{g}') > u_i(\mathbf{g})$ $\forall i \in J$.

The following example illustrates our result. Assume there are two cliques such that the smaller clique consists of three agents and the larger clique consists of four agents. Consider a deviation where three of the agents in the larger clique and one agent in the smaller clique deviate to create a new maximal clique. There are then three cliques, where the smallest is of size 1, the second smallest is of size 2, and the largest clique is of size 4. The deviation proposed is profitable for all deviating agents. A similar argument can be made for any Nash equilibrium, other than the Nash equilibrium in which $n-1$ agents gang up on a single agent.

PROPOSITION 5 (Homogeneous agents). *In any strong Nash equilibrium there are two maximal cliques and* $|P^1(\mathbf{g})| = 1$.

A strong Nash equilibrium may not always exist. However, a simple sufficient condition for the existence of a strong Nash equilibrium is that $\varepsilon + \kappa$ is sufficiently close to $f(n-1, 1)$.[18] Below we provide two simple examples, which both display three types of Nash equilibria, but only in the first example does a strong Nash equilibrium exists.

EXAMPLE 1 (Strong Nash equilibrium). Assume $n = 5$ and that f is such that $f(4, 1) = 7$, $f(4, 2) = f(3, 1) = 3$, $f(3, 2) = 2$, and $f(4, 3) = f(2, 1) = 1$ with $\varepsilon = 1$ and $\kappa = 0$. There are then three types of Nash equilibria: (i) all agents are friends, (ii) one clique of size 3 and one clique of size 2, and (iii) one clique of size 4 and one singleton agent. The latter Nash equilibrium is a strong Nash equilibrium. To see this, note that for any deviation of at least two agents, at least one of the agents must be from the clique of size 4. Note also that an agent with one reciprocated positive link can at most obtain a payoff of 3 in any network, while the equilibrium payoff of an agent in the clique of size 4 is 6. Therefore, after any profitable deviation, a deviating agent from the clique of size 4 must have at least two reciprocated positive links. One can then show that the largest relevant deviation payoff for an agent from this clique is 5. Therefore, no profitable deviation exists. ◊

EXAMPLE 2 (Strong Nash equilibrium). Assume that $f(4, 1) = 4$, while the remaining payoffs and parameter values are as in Example 1. The set of Nash equilibria is the same as in Example 1. However, the configuration with a clique of size 4 and a singleton agent is not a strong Nash equilibrium and, therefore, no strong Nash equilibrium exists. To see this, note that the equilibrium payoff of an agent in the clique of size 4 is 3, while a deviation in which three agents gang up on the remaining agent in the clique of size 4 yields a deviation payoff of 4. ◊

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[18]Note that if $\varepsilon + \kappa$ is sufficiently close to $f(n-1, 1)$, then the only Nash equilibria are such that either all agents are positively connected or $n-1$ agents extract payoffs from a single agent. One can show that a different, more elaborate sufficient condition for the existence of a strong Nash equilibrium is given by $f(n-1, 1) - \varepsilon \geq \max_{x \in [1, \frac{n-1}{2}) \cap \mathbb{N}} x \cdot (f(n-1-x, x) - \varepsilon - \kappa) + f(n-1-x, 1)$. The proof is available upon request. Note further that the latter condition is satisfied in Example 1, but not in Example 2.

3.4 *The role of coordination incentives (constant extraction values)*

Our equilibrium characterization provides sufficient conditions for all pure strategy Nash equilibria to be structurally balanced. However, it is also interesting to understand the conditions on f under which we may obtain Nash equilibria that are not structurally balanced. In the following discussion, we uphold our assumptions, with one exception. We assume that $f(n_i, n_j) = z > 0$ holds $\forall n_i, n_j$: $n_i > n_j$. That is, the extraction value is constant. Furthermore, we assume that $z > \varepsilon + \kappa$. In Proposition 6, we show that there then exist Nash equilibria that are not structurally balanced. Note that with a constant extraction value, there are no incentives for stronger agents to coordinate their actions relative to weaker agents. This result thereby highlights the relevance of coordination incentives for obtaining structurally balanced structures. Note also that in the network configuration considered in Proposition 6, balanced and unbalanced triads coexist.

PROPOSITION 6 (Homogeneous agents). *If $f(n_i, n_j) = z > 0$ holds $\forall n_i, n_j$: $n_i > n_j$, then for any $n \geq 4$, there exists a n' with $|n' - n| \leq 2$ such that a Nash equilibrium exists that is not structurally balanced.*

3.5 *Efficiency*

This section analyzes welfare properties of our model. It is easy to see that the sum of payoffs is largest if all links are positive, as this minimizes the cost of negative links. In Proposition 7 we show that Nash equilibria cannot be Pareto ranked. More specifically, when comparing a Nash equilibrium, $\mathbf{g}$, to another Nash equilibrium, $\tilde{\mathbf{g}}$, then there always exists an agent who is strictly worse off in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. For the following discussion, it is convenient to define the set of agents who have strictly fewer friends in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, which we denote with $S(\mathbf{g} \to \tilde{\mathbf{g}})$, and to define the set of agents who have weakly more friends in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, which we denote with $B(\mathbf{g} \to \tilde{\mathbf{g}})$. To simplify notation we sometimes just write S and B. Formal definitions, together with additional definitions and notation used in the proof, can be found in Appendix A.

   To build intuition for our result, we briefly discuss some of the relevant cases, without commenting on all of them here. Note first that if all agents have strictly fewer positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, so that $S(\mathbf{g} \to \tilde{\mathbf{g}}) = N$, then the number of negative links is strictly larger in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. Therefore, the sum of payoffs is strictly smaller in $\tilde{\mathbf{g}}$ and at least one agent is strictly worse off in $\tilde{\mathbf{g}}$. If all agents have strictly more positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, so that $B(\mathbf{g} \to \tilde{\mathbf{g}}) = N$, then one can show that there exists an agent i with a deviation payoff in $\mathbf{g}$ that is strictly larger than i's equilibrium payoff in $\tilde{\mathbf{g}}$. Therefore, agent i is strictly worse off in $\tilde{\mathbf{g}}$. If $\tilde{\mathbf{g}}$ is such that $S(\mathbf{g} \to \tilde{\mathbf{g}}) \neq \varnothing$ and $B(\mathbf{g} \to \tilde{\mathbf{g}}) \neq \varnothing$, then we can focus on the set S. We first show that payoffs of agents in S must decrease from their links with agents in B. Then, for agents in S to be at least as well off in $\tilde{\mathbf{g}}$ as in $\mathbf{g}$, it must be the case that there are fewer negative links among agents in S in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. However, for agents in S to display more negative links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, it must also be the case that for each negative link less than among agents in S, there must be at least two more negative links that agents in B extend to agents in S in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. We show that this more than outweighs any benefits from fewer negative links among agents in S. Therefore, there exists at least one agent in S that is strictly worse off in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$.

PROPOSITION 7 (Homogeneous agents). *Nash equilibria cannot be Pareto ranked.*

4. DISCUSSION OF EXTENSIONS AND APPLICATIONS

The focus of this paper is to provide a simple game-theoretic model for structural balance. However, the framework lends itself to study many other, potentially interesting questions. A particularly promising area is international relations and international conflict. Jackson and Nei (2015) argue that the study of multilateral conflicts is, in fact, a network problem. The authors find that multilateral wars between 1823 and 2003 never involved conflict with fully allied coalitions (i.e., cliques) of more than two countries fighting each other. In our current model specification, fully connected cliques arise because we do not incorporate zero (or neutral) links and we assume that positive links are costless. However, one can easily allow for costly alliances and for positive, zero, and negative links. In this setting one can show that there exist equilibrium configurations involving conflict that do not consist of fully allied coalitions and that again all triads are structurally balanced (for a detailed account, see Hiller 2013). Note further that our present model, as well as the literature on structural balance, only considers one type of negative link. In particular, we do not allow for the choice between fighting back and giving in, which may be of interest for applications in the economics of conflict. These types of considerations can be addressed in our framework and we have solved a simple extension of the model where we distinguish between fighting back and giving in. Fighting back is assumed to be costly, but, relative to giving in, it decreases the value that the stronger agent is able to extract from the weaker. Again simple conditions can be derived such that all equilibria are structurally balanced.[19]

In the current model, extending a negative link triggers costly conflict and agents cannot bargain to avoid this cost. However, one can show that an alternative setup with strictly positive conflict cost and Nash bargaining is analytically equivalent to assuming a conflict cost of zero in the original specification presented here.[20] This reading of the model may be a starting point for the study of applications related to corruption and payment of bribes. More generally, Fearon (1995) points out that asymmetric information, lack of commitment, and dynamic motives may prevent agents from reaching bargaining outcomes and all of these considerations are interesting in our context.

There are many other questions that may be worthwhile to study in our framework. For example, the existing theoretical and empirical literature on the formation of coalitions, states, and alliances typically models a trade-off between economies of scale and

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[19] For both of the extensions described, we use bilateral equilibrium Goyal and Vega-Redondo (2007) as our notion of strategic stability. Bilateral equilibrium allows pairs of agents to deviate and refines Nash equilibrium and pairwise stability. Furthermore, we make an additional concavity assumption (which is satisfied for both contest success functions for $0 < \phi < 1$). A bilateral equilibrium may not always exist. However, simple sufficient conditions for the existence of a bilateral equilibrium can be obtained.

[20] Assume $\kappa > 0$ and that agents can engage in bilateral bargaining in isolation. More precisely, assume that once a negative link is extended, then agents may either engage in costly conflict or reach an agreement over the payoffs from the negative link, thereby avoiding the cost of conflict, while linking behavior is assumed to be fixed. Calculating the Nash bargaining solution yields payoffs from negative links as if $\kappa = 0$. That is, the analysis is then equivalent to assuming $\kappa = 0$ in our model.

cost from being part of a larger and more heterogeneous group (see, for example, Alesina and Spolare 2005). Allowing for investment into arming may lead to incentives to free ride among allies, potentially favoring smaller groups (see Olson 1965, but also Esteban and Ray 2001). Furthermore, combining the incentives of arming and production may yield novel insights into the trade-off between "guns and butter."

Finally, note that the current model specification may be interpreted as individual level conflict and value extraction. However, one may also be interested in conflict or value extraction that operates at a group level. In this vein, consider a setup where agents create positive links and agents in a positively connected component are able to coordinate and optimally choose their negative links relative to other components (possibly in a second stage of the game). Payoffs of agents in a given component may now depend on the relative strength of the component, which in turn may depend on the particular pattern of negative links across components.[21] A model along these lines may provide new insights regarding the formation of alliances and conflict. If payoffs are evenly distributed within components and negative links among agents in the same component weaken a component's strength, then negative links will only exist across components and again all equilibrium configurations are structurally balanced.

5. Conclusion

This paper presents, to the best of my knowledge, the first game-theoretic model of *signed* network formation. Agents enter positive (friendship or alliance) and negative (antagonism or conflict) relationships. Stronger agents extract payoffs from weaker enemies and an agent's power is determined endogenously. There are three main insights to be drawn. First, the model shows how in this context self-interested behavior yields the following sharp structural predictions under Nash equilibrium. Either everyone is friends with everyone or agents can be partitioned into distinct sets, also called cliques, such that agents within the same set are friends and agents in different sets are enemies. This provides a game-theoretic foundation for structural balance theory and mirrors findings on signed networks obtained in sociology, social psychology, bullying, international relations, and applied physics. Second, cliques are asymmetric in terms of strength. This stands in contrast to models of coalition formation in the economics of conflict in two dimensions. Groups are typically shown to be symmetric in this literature, while our model allows for asymmetric structures. Furthermore, agents then also obtain identical payoffs, while our setup allows for payoff differences across agents. Third, the game-theoretic approach permits us to provide answers to novel questions, which could previously not be addressed. We provide conditions that disentangle strong and weak structural balance. For the contest success functions in difference and ratio form these imply that if payoff extraction is sufficiently effective, then there exist Nash equilibria with multiple cliques, while if payoff extraction is sufficiently ineffective, then there are at most two cliques in equilibrium. The only strong Nash equilibria are such

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[21]For example, assume that the ability of an agent to stay operational in the light of attacks from agents in other components depends on the structure of the respective components, while the strength (and total payoffs) of a component depends on the network of operational agents only.

that one agent is enemies of everyone else. Furthermore, Nash equilibria are shown to not be Pareto ranked.

Appendix A

Before characterizing the set of Nash equilibria, we present two auxiliary lemmas. Lemma 1 shows that there does not exist a Nash equilibrium such that a negative link is reciprocated by a negative link. In Lemma 2, we show that in any Nash equilibrium, for all negative links in place in the undirected network $\bar{\mathbf{g}}$, it must be the stronger agent that extends the directed negative link.

Lemma 1. *In any NE, there does not exist a pair of agents i and j such that $g_{i,j} = g_{j,i} = -1$.*

Proof. Assume there exists a pair of agents i and j that extend negative links to each other. Assume without loss of generality that agent i deviates by extending a positive link to j. The sets of i's friends and enemies are not altered by the deviation and payoffs accruing from any agent other than j remain the same. However, payoffs from the link with j strictly increase by $\varepsilon > 0$ and proposed deviation is therefore profitable.   □

Lemma 2. *In any NE, if $\bar{g}_{i,j} = -1$ with $n_i(\mathbf{g}) < n_j(\mathbf{g})$, then $g_{i,j} = 1$.*

Proof. Note first that, from Lemma 1, we know that a negative link is not reciprocated in any Nash equilibrium. Assume next, and contrary to the above, that $\bar{g}_{i,j} = -1$ with $g_{i,j} = -1$, $g_{j,i} = 1$, and $n_i(\mathbf{g}) < n_j(\mathbf{g})$. Then i can profitably deviate with deviation strategy $\mathbf{g}_i + g_{i,j}^+$, which yields $\bar{g}'_{i,j} = 1$. Payoffs strictly increase for i from the link with j. To see this, note that the payoff in $\mathbf{g}$ is given by $f(n_i(\mathbf{g}), n_j(\mathbf{g})) - (\varepsilon + \kappa)$, which is negative for $n_i(\mathbf{g}) < n_j(\mathbf{g})$, while the payoff is zero after proposed deviation. Furthermore, by reciprocating a positive link from j, agent i increases his strength and payoffs from any remaining negative links also increase. Proposed deviation is therefore profitable.   □

Proof of Proposition 1. We start with the first part of the statement.

Step 1. In any NE, if $n_i(\mathbf{g}) = n_j(\mathbf{g})$, then $\bar{g}_{i,j} = 1$.

Assume, to the contrary, that there exists a pair of agents i and j, such that $n_i(\mathbf{g}) = n_j(\mathbf{g})$ and $\bar{g}_{i,j} = -1$. From Lemma 1, we know that a negative link is not reciprocated in any Nash equilibrium. Assume without loss of generality (w.l.o.g.) that i extends the negative link, i.e., $g_{i,j} = -1$ and $g_{j,i} = 1$. Note that since $n_i(\mathbf{g}) = n_j(\mathbf{g})$, payoffs for agent i from the link $\bar{g}_{i,j} = -1$ are then given by $-(\varepsilon + \kappa)$. Agent i can profitably deviate by extending a positive link to j. Payoffs from the link increase by $\varepsilon + \kappa > 0$, while payoffs on any remaining negative links also increase, since $n_i(\mathbf{g}') > n_i(\mathbf{g})$. Note further that if $n_i(\mathbf{g}) = n_j(\mathbf{g}) \ \forall i, j \in N$, then all links must be positive.

The remaining part of the proof uses an induction argument and we start by proving the base case in three steps (Steps 2–4).

Base Case (*Steps* 2–4). *In any NE, $\bar{g}_{i,j} = 1 \ \forall i, j \in P^m(\mathbf{g})$ and $\bar{g}_{i,k} = -1 \ \forall i \in P^m(\mathbf{g})$ and $\forall k \notin P^m(\mathbf{g})$.*

Step 2. In any NE, $N_i^+(\mathbf{g}) \backslash \{j\} = N_j^+(\mathbf{g}) \backslash \{i\}$ and $N_i^-(\mathbf{g}) = N_j^-(\mathbf{g}) \ \forall i, j \in P^m(\mathbf{g})$.

This statement holds trivially for $|P^m(\mathbf{g})| = 1$. Assume $|P^m(\mathbf{g})| \geq 2$ and, contrary to the above, that $\exists i, j \in P^m(\mathbf{g}) : N_i^+(\mathbf{g}) \backslash \{j\} \neq N_j^+(\mathbf{g}) \backslash \{i\}$ (and therefore $N_i^-(\mathbf{g}) \neq N_j^-(\mathbf{g})$). That is, there exists a pair of agents $i, j \in P^m(\mathbf{g})$, such that their respective sets of friends and enemies are different. Note that from the first part of the proof, we know that $\bar{g}_{i,j} = 1 \ \forall i, j \in P^m(\mathbf{g})$. From Lemma 2 we further know that agents not in $P^m(\mathbf{g})$ extend positive links to all agents in $P^m(\mathbf{g})$. That is, $g_{k,i} = 1 \ \forall k, i$ such that $k \notin P^m(\mathbf{g})$ and $i \in P^m(\mathbf{g})$. Therefore, for a pair of agents i and j to exist such that $i, j \in P^m(\mathbf{g})$ and $N_i^+(\mathbf{g}) \backslash \{j\} \neq N_j^+(\mathbf{g}) \backslash \{i\}$ (and therefore $N_i^-(\mathbf{g}) \neq N_j^-(\mathbf{g})$), it must be that i and j play different strategies relative to third agents, which we denote with $\mathbf{g}_{i \backslash j} \neq \mathbf{g}_{j \backslash i}$. Without loss of generality, assume $u_j(\mathbf{g}) \geq u_i(\mathbf{g})$. We show that i can then profitably deviate by imitating j's strategy (while keeping the positive link to j), so that $\mathbf{g}'_{i \backslash j} = \mathbf{g}_{j \backslash i}$. More specifically, we show that $u_i(\mathbf{g}') > u_j(\mathbf{g}) \geq u_i(\mathbf{g})$ holds. There are two types of agents to consider when comparing payoffs of i *after* proposed deviation, $u_i(\mathbf{g}')$, with payoffs of j *prior* to it, $u_j(\mathbf{g})$: first, agents k that are i and j's enemies prior to the deviation, $k \in N_j^-(\mathbf{g}) \cap N_i^-(\mathbf{g})$; second, agents l that are j's enemies, but i's friends prior to the deviation, $l \in N_j^-(\mathbf{g}) \cap N_i^+(\mathbf{g})$. We start by showing that i's payoffs from links with agents $k \in N_j^-(\mathbf{g}) \cap N_i^-(\mathbf{g})$ in $\mathbf{g}'$ are the same as the payoffs that j obtains from these agents in $\mathbf{g}$. Note first that since i is imitating j's strategy, $n_i(\mathbf{g}') = n_j(\mathbf{g})$. To see this, note that $n_j(\mathbf{g}) = \lambda_j + \lambda_i + \sum_{k \in N_j^+(\mathbf{g}) \backslash \{i\}} \lambda_k$, while $n_i(\mathbf{g}') = \lambda_i + \lambda_j + \sum_{k \in N_i^+(\mathbf{g}') \backslash \{j\}} \lambda_k = \lambda_i + \lambda_j + \sum_{k \in N_i^+(\mathbf{g}) \backslash \{i\}} \lambda_k$. Because i does not change his linking behavior relative to agents $k \in N_j^-(\mathbf{g}) \cap N_i^-(\mathbf{g})$ in $\mathbf{g}'$, $n_k(\mathbf{g}) = n_k(\mathbf{g}')$ also holds. Therefore, i's payoffs from all links with agents $k \in N_j^-(\mathbf{g}) \cap N_i^-(\mathbf{g})$ are the same for i in $\mathbf{g}'$ as for j in $\mathbf{g}$. Next, we consider payoffs from agents that are j's enemies, but i's friends prior to the deviation, i.e., payoffs from agents $l \in N_j^-(\mathbf{g}) \cap N_i^+(\mathbf{g})$. From $\mathbf{g}_{i \backslash j} \neq \mathbf{g}_{j \backslash i}$ and $i, j \in P^m(\mathbf{g})$ we know that at least one such agent l exists. Since i extends a positive link to l in $\mathbf{g}$ and a negative link in $\mathbf{g}'$, $n_l(\mathbf{g}') < n_l(\mathbf{g})$ holds. Recall from the above that when i imitates j's strategy, then $n_i(\mathbf{g}') = n_j(\mathbf{g})$. Therefore, agent i's payoffs from agents $l \in N_j^-(\mathbf{g}) \cap N_i^+(\mathbf{g})$ in $\mathbf{g}'$ are strictly larger than j's payoffs from these agents in $\mathbf{g}$. That is, $u_i(\mathbf{g}') > u_j(\mathbf{g}) \geq u_i(\mathbf{g})$ holds and proposed deviation is profitable.

Step 3. In any NE, $\bar{g}_{i,k} = -1 \ \forall i \in P^m(\mathbf{g})$ *and* $\forall k \in P^{m-1}(\mathbf{g})$.

Assume to the contrary that there exists an agent $k \in P^{m-1}(\mathbf{g})$ such that $\bar{g}_{i,k} = 1$ for some $i \in P^m(\mathbf{g})$. From Step 2, we know that then $\bar{g}_{i,k} = 1 \ \forall i \in P^m(\mathbf{g})$. Furthermore, from Lemma 1, we know that then $\bar{g}_{j,k} = 1$ must hold $\forall j, k \in P^{m-1}(\mathbf{g})$. By Lemma 2, we further know that $g_{h,k} = 1 \ \forall h \in P^{m-x}(\mathbf{g})$ and $\forall x \in \mathbb{N}: x \geq 2$. Therefore, $g_{l,k} = 1 \ \forall l \in N \backslash \{k\}$. We discern two cases. If $\mathbf{g}_{k \backslash i} \neq \mathbf{g}_{i \backslash k}$, then we can again use the same argument as in Step 2 to show that either k or i (or both) can strictly increase payoffs by imitating the respective other agent's strategy. If $\mathbf{g}_{k \backslash i} = \mathbf{g}_{i \backslash k}$, then we reach an immediate contradiction, as then $n_k(\mathbf{g}) = n_i(\mathbf{g})$ for $k \in P^{m-1}(\mathbf{g})$ and $i \in P^m(\mathbf{g})$.

Step 4. In any NE, $\bar{g}_{i,k} = -1 \ \forall i \in P^m(\mathbf{g})$ *and* $\forall k \notin P^m(\mathbf{g})$.

If there are only two sets of agents with different numbers of friends, $P^m(\mathbf{g})$ and $P^{m-1}(\mathbf{g})$, then we are done by Step 3. Assume that there are at least three such sets and that there exists a pair of agents $i \in P(\mathbf{g})$ and $k \in P^{m-2}(\mathbf{g})$ such that $\bar{g}_{i,k} = 1$. From Step 2, we know that then $\bar{g}_{i,k} = 1 \ \forall i \in P(\mathbf{g})$. Assume first that $u_k(\mathbf{g}) \leq u_i(\mathbf{g})$. Recall that $\bar{g}_{i,l} = -1 \ \forall i, l$ such that $i \in P^m(\mathbf{g})$ and $l \in P^{m-1}(\mathbf{g})$. Next we construct a deviation strategy

for agent k such that the sets of friends and enemies of k in $\mathbf{g}'$ are the same as those of i in $\mathbf{g}$. If $g_{l,k} = 1$ for $l \in P^{m-1}(\mathbf{g})$, then $g'_{k,l} = -1$, while if $g_{l,k} = -1$ for $l \in P^{m-1}(\mathbf{g})$, then $g'_{k,l} = 1$ (and therefore $\bar{g}_{k,l} = -1 \; \forall l \in P^{m-1}(\mathbf{g})$). For all remaining agents, assume agent k imitates agent i's strategy in $\mathbf{g}$. From Step 1, Step 2, and Lemma 2, we know that $g_{l,k} = 1 \; \forall l \notin P^{m-1}(\mathbf{g})$. Therefore, if $\bar{g}_{i,l} = 1$, then $\bar{g}_{k,l} = 1$ and if $\bar{g}_{i,l} = -1$, then $\bar{g}_{k,l} = -1$ $\forall l \notin P^{m-1}(\mathbf{g})$. That is, after proposed deviation $N_i^+(\mathbf{g})\backslash\{k\} = N_k^+(\mathbf{g}')\backslash\{i\}$, $N_i^-(\mathbf{g}) = N_k^-(\mathbf{g}')$ and $n_i(\mathbf{g}) = n_k(\mathbf{g}')$ hold. We can now employ an argument similar to the one used in Step 2. Agent k obtains the same payoffs in $\mathbf{g}'$ as i in $\mathbf{g}$ from all agents $l \in N_i^-(\mathbf{g}) \cap N_k^-(\mathbf{g})$ and obtains strictly larger payoffs in $\mathbf{g}'$ than i in $\mathbf{g}$ from all agents $l \in N_i^-(\mathbf{g}) \cap N_k^+(\mathbf{g})$. If there exists an agent $l \in P^{m-1}(\mathbf{g})$ with $\bar{g}_{l,k} = 1$, then $l \in N_i^-(\mathbf{g}) \cap N_k^+(\mathbf{g})$ and proposed deviation is profitable. Assume next that such an agent does not exist. That is, $g_{l,k} = -1$ $\forall l \in P^{m-1}(\mathbf{g})$. But then linking costs of agent k are lower in $\mathbf{g}'$ than for agent i in $\mathbf{g}$ (since $g'_{k,l} = 1$ and $g_{i,l} = -1 \; \forall l \in P^{m-1}(\mathbf{g})$), while gross payoffs are the same for agent k in $\mathbf{g}'$ as for i in $\mathbf{g}$. Again, $u_k(\mathbf{g}') > u_i(\mathbf{g})$ and proposed deviation is profitable. Assume next that $u_k(\mathbf{g}) > u_i(\mathbf{g})$. Consider the following deviation. Agent i extends positive links to agents in $P^{m-1}(\mathbf{g})$ and imitates k's linking behavior to all remaining agents. Note that i's payoffs are zero in $\mathbf{g}'$ from all links with agents in $P^{m-1}(\mathbf{g})$, while they are at most zero for agent k in $\mathbf{g}$ (and negative for k if the link between k and an agent in $P^{m-1}(\mathbf{g})$ is negative in $\mathbf{g}$). Note further that $n_i(\mathbf{g}') \geq n_k(\mathbf{g})$ and $n_l(\mathbf{g}') \leq n_l(\mathbf{g}) \; \forall l \in N_i^-(\mathbf{g}')$. Therefore, $u_i(\mathbf{g}') \geq u_k(\mathbf{g}) > u_i(\mathbf{g})$ and proposed deviation is profitable. If there are more than three sets of agents with different numbers of friends, then we can use the above argument iteratively to show that $\bar{g}_{i,k} = -1 \; \forall i \in P^m(\mathbf{g})$ and $\forall k \notin P^m(\mathbf{g})$.

Define the super set $\tilde{P}^r(\mathbf{g}) = \bigcup_{i=m-r}^{i=m} P^i(\mathbf{g})$. Note that $\tilde{P}^0(\mathbf{g}) = P^m(\mathbf{g})$.

Inductive Step. In any NE, if $\bar{g}_{i,j} = 1 \; \forall i, j \in P^{m-x}(\mathbf{g})$, $\bar{g}_{i,k} = -1 \; \forall i \in P^{m-x}(\mathbf{g})$, *and* $\forall k \notin P^{m-x}(\mathbf{g})$ *holds* $\forall x \in \mathbb{N} : 0 \leq x \leq r$, *then* $\bar{g}_{i,j} = 1 \; \forall i, j \in P^{m-(r+1)}(\mathbf{g})$, $\bar{g}_{i,k} = -1 \; \forall i \in P^{m-(r+1)}(\mathbf{g})$, *and* $\forall k \notin P^{m-(r+1)}(\mathbf{g})$.

In Step 4 we showed that $\bar{g}_{i,j} = 1 \; \forall i, j \in P^m(\mathbf{g})$ and $\bar{g}_{i,k} = -1 \; \forall i \in P^m(\mathbf{g})$ and $\forall k \notin P^m(\mathbf{g})$. Assume the statement holds for all sets $P^{m-x}(\mathbf{g})$ with $x \in \mathbb{N} : 0 \leq x \leq r$. From Lemma 2, we know that $g_{i,k} = -1$ and $g_{k,i} = 1 \; \forall i \in \tilde{P}^r(\mathbf{g})$ and $\forall k \notin \tilde{P}^r(\mathbf{g})$, while from Lemma 1, we know that in any Nash equilibrium there does not exist a pair of agents i and k such that $g_{i,k} = g_{k,i} = -1$. We can now use an argument analogous to that used in Step 2, Step 3, and Step 4 of the base case, relabeling $P^m(\mathbf{g})$ with $P^{m-(r+1)}(\mathbf{g})$, $P^{m-1}(\mathbf{g})$ with $P^{m-(r+2)}(\mathbf{g})$, and $P^{m-2}(\mathbf{g})$ with $P^{m-(r+3)}(\mathbf{g})$, to establish the above result. □

PROOF OF PROPOSITION 2. We start by proving the first statement. Note that if all agents are friends, then payoffs are zero for all agents. A deviation for agent i then consists of extending negative links to some subset of $N\backslash\{i\}$. Note that in any deviation of agent i, $n_i(\mathbf{g}') \leq n_j(\mathbf{g}')$ and, therefore, $f(n_i(\mathbf{g}'), n_j(\mathbf{g}')) \leq 0$ holds $\forall j \in N_i^-(\mathbf{g}')$. Furthermore, $e_i(\mathbf{g}')\varepsilon > 0$ and, therefore, no profitable deviation exists. Next we prove the second statement. Denote with k the agent that is in an antagonistic relationship with everyone else. First we check for deviations by k. From Lemma 2, we know that for a negative link in the undirected network, $\bar{g}_{i,k} = -1$, it must be the weaker agent extending the positive link. From $n_k(\mathbf{g}) < n_i(\mathbf{g}) \; \forall i \in N\backslash\{k\}$, it then follows that $g_{k,i} = 1 \; \forall i \in N\backslash\{k\}$. A deviation for agent k therefore consists of extending negative links to some subset of $N\backslash\{k\}$. For

each agent to which k extends a negative link in a deviation, payoffs decrease by ε and no such deviation is profitable. Next we consider an agent i different from k. There are three types of deviations to consider. First, i extends a positive link to k. Agent i's payoff from the link with k decreases strictly, since $n_k(\mathbf{g}) < n_i(\mathbf{g})$ and $f(n-1,1) > \varepsilon + \kappa$, while payoffs from all other links remain zero in $\mathbf{g}'$. Proposed deviation is therefore not profitable. Second, i extends negative links to some subset of $N \backslash \{k, i\}$. Note that each of these negative links incurs a negative payoff, by the argument used above. Furthermore, agent i's payoffs from the link with k decrease since $n_i(\mathbf{g}') < n_i(\mathbf{g})$. Proposed deviation is therefore not profitable. Third, consider a combination of the two deviations above. Assume first that i extends a positive link to k and *one* negative link to some $j \in N \backslash \{k, i\}$. Agent i's payoffs remain constant if $n = 3$, as then $n_j(\mathbf{g}') = n_k(\mathbf{g}) = 1$. For $n \geq 4$, i obtains strictly lower payoffs. To see this, note that $n_i(\mathbf{g}') = n_i(\mathbf{g}) = n - 1$, while $n_j(\mathbf{g}') = n - 2$ and $n_k(\mathbf{g}) = 1$. Therefore, $n_j(\mathbf{g}') > n_k(\mathbf{g})$ holds and proposed deviation is not profitable. Extending more than one negative link to $N \backslash \{k, i\}$ yields even lower payoffs, by an argument analogous to that used to prove the first statement.   □

PROOF OF PROPOSITION 3. From Proposition 2 we know that the network $\mathbf{g}$ such that $g_{i,j} = 1 \; \forall i, j$ is always a Nash equilibrium when agents are homogenous. Assume next, and contrary to Proposition 3, that there exists a Nash equilibrium $\mathbf{g}$ such that not all undirected links are positive and $\sum_{i=1}^{j-1} |P^i(\mathbf{g})| \geq |P^j(\mathbf{g})|$ holds for some $P^j(\mathbf{g})$. Consider a deviation where $k \in P^{j-1}(\mathbf{g})$ extends positive links to a subset of agents in $\bigcup_{i=1}^{i=j-2} P^i(\mathbf{g})$, such that $n_k(\mathbf{g}') = n_j(\mathbf{g})$ for $j \in P^j(\mathbf{g})$. This is feasible since $\sum_{i=1}^{j-1} |P^i(\mathbf{g})| \geq |P^j(\mathbf{g})|$. Denote with x the number of agents to which k needs to extend a positive link such that $n_k(\mathbf{g}') = n_j(\mathbf{g})$. Note first that $|P^{j-1}(\mathbf{g})| + x = |P^j(\mathbf{g})|$ with $x < |P^j(\mathbf{g})|$. For δ sufficiently small, k forgoes payoffs of arbitrarily close to $f(n-1,1) - (\varepsilon + \kappa)$ from each of the x agents to which k extends a positive link. However, k increases payoffs by arbitrarily close to $f(n-1,1)$ from the $|P^j(\mathbf{g})|$ agents in $P^j(\mathbf{g})$. Furthermore, since $n_k(\mathbf{g}') > n_k(\mathbf{g})$, payoffs from any remaining negative links are larger in $\mathbf{g}'$ than in $\mathbf{g}$. From $|P^j(\mathbf{g})| > x$ it then follows that proposed deviation is profitable. Assume next that $\sum_{i=1}^{j-1} |P^i(\mathbf{g})| < |P^j(\mathbf{g})|$ holds. Note that in this case no agent can obtain at least as many friends as agents in any of the larger cliques by extending positive links to agents in smaller cliques. The only relevant deviations then consist of extending positive links to weaker agents (by an argument analogous to Proposition 2). However, payoffs from any such deviation are negative for δ sufficiently small. To see this, note that since $f(n-1,1) > \varepsilon + \kappa$ holds, $f(n-1,1) - \delta - (\varepsilon + \kappa) > 0$ also holds for δ sufficiently small. A deviating agent therefore forgoes payoffs of at least $f(n-1,1) - \delta - (\varepsilon + \kappa) > 0$ for each positive link extended to agents in smaller cliques, while payoffs on each remaining negative link increase by at most δ. For δ sufficiently small, proposed deviation is not profitable.   □

PROOF OF COROLLARY 1. For the contest success function in ratio form, $\min_{2 \leq x \leq n-1} f(x, x-1)$ is given by $2^\phi / (2^\phi + 1^\phi)$ and, therefore, δ is given by

$$\delta = \frac{(n-1)^\phi}{(n-1)^\phi + 1^\phi} - \frac{2^\phi}{2^\phi + 1^\phi}.$$

Note that δ is decreasing and continuous in ϕ, and

$$\lim_{\phi \to \infty} \frac{(n-1)^\phi}{(n-1)^\phi + 1^\phi} - \frac{2^\phi}{2^\phi + 1^\phi} = 0.$$

That is, for ϕ sufficiently high, δ is arbitrarily close to zero and Proposition 3 applies. For the contest success function in difference form, $\min_{2 \le x \le n-1} f(x, x-1)$ is given by $1/(1 + e^{-\phi})$ and δ is, therefore, given by

$$\delta = \frac{1}{1 + e^{-n\phi}} - \frac{1}{1 + e^{-\phi}}.$$

Note that δ is again decreasing and continuous in ϕ, and

$$\lim_{\phi \to \infty} \frac{1}{1 + e^{-\phi n}} - \frac{1}{1 + e^{-\phi}} = 0.$$

That is, for ϕ sufficiently high, δ is arbitrarily close to zero and Proposition 3 again applies. $\square$

PROOF OF PROPOSITION 4. Assume, contrary to the above discussion, that $\mathbf{g}$ is such that there are at least three cliques. We consider a deviation in which an agent $i \in P^2(\mathbf{g})$ extends a positive links to an agent $j \in P^1(\mathbf{g})$. We start by analyzing marginal payoffs accruing from links to agents in the three smallest cliques, where we denote $|P^1(\mathbf{g})| = x$, $|P^2(\mathbf{g})| = y$, and $|P^3(\mathbf{g})| = z$ with $1 \le x < y < z$. Proposed deviation is profitable for any $\varepsilon > 0$ and $\kappa \ge 0$ if the following condition holds:

$$(x-1) \cdot \big(f(y+1, x) - f(y, x)\big) + z \cdot \big(f(y+1, z) - f(y, z)\big) + \varepsilon > f(y, x).$$

Note first that from $\partial^2 f(n_i, n_j)/\partial^2 n_i \le 0 \; \forall n_i \ge n_j$ we know that the inequality $(y - x) \cdot (f(x+1, x) - f(x, x)) \ge f(y, x)$ holds. A sufficient condition for the above condition to hold and a deviation to be profitable is therefore given by

$$z \cdot \big(f(y+1, z) - f(y, z)\big) + \varepsilon > (y - x) \cdot \big(f(x+1, x) - f(x, x)\big).$$

Note that for $\partial^2 f(n_i, n_j)/\partial n_i \partial n_j$ and $\partial^2 f(n_i, n_j)/\partial^2 n_i$ sufficiently close to zero $\forall n_i, n_j$, the terms $f(y+1, z) - f(y, z)$ and $f(x+1, x) - f(x, x)$ are arbitrarily close to each other. The above inequality then holds, since $\varepsilon > 0$ and $z > y - x$. Note that for any additional clique(s) with $|P_l(\mathbf{g})| = l > z$, proposed deviation yields strictly larger marginal payoffs, as agent i now also increases payoffs from any negative links with agents $k \in P_l(\mathbf{g})$. $\square$

PROOF OF COROLLARY 2. Note first that for both contest success functions, for ϕ sufficiently low, $\partial^2 f(n_i, n_j)/\partial^2 n_i < 0$ holds $\forall n_i, n_j$. For the contest success function in ratio form, the partial cross-derivative is given by

$$\frac{\partial^2 f(n_i, n_j)}{\partial n_i \partial n_j} = \frac{2 n_i^{\phi-1} n_j^{\phi-1} \big(n_i^\phi - 2n_j^\phi\big) \phi^2}{\big(n_i^\phi - 2n_j^\phi\big)^3}$$

and the second derivative with respect to n_i is given by

$$\frac{\partial^2 f(n_i, n_j)}{\partial^2 n_i} = \frac{n_i^{\phi-2}(\phi-1)\phi}{n_i^\phi + 2n_j^\phi} + \frac{2n_i^{3\phi-2}\phi^2}{n_i^\phi + 2n_j^\phi} - \frac{n_i^{2\phi-2}\phi^2}{\left(n_i^\phi + 2n_j^\phi\right)^2} - \frac{n_i^{2\phi-2}(2\phi-1)}{\left(n_i^\phi + 2n_j^\phi\right)^2}.$$

The corresponding limits when ϕ approaches zero are given by $\lim_{\phi\to 0} \partial^2 f(n_i, n_j)/$ $\partial n_i \partial n_j = 0$ and $\lim_{\phi\to 0} \partial^2 f(n_i, n_j)/\partial^2 n_i = 0$. Both $\partial^2 f(n_i, n_j)/\partial n_i \partial n_j$ and $\partial^2 f(n_i, n_j)/\partial^2 n_i$ are continuous functions in ϕ and, for ϕ sufficiently small, the conditions provided in Proposition 4 are therefore satisfied. For the normalized contest success function in difference form, the corresponding expressions are given by

$$\frac{\partial^2 f(n_i, n_j)}{\partial n_i \partial n_j} = \frac{e^{\phi(n_i+n_j)}\left(e^{\phi n_i} - e^{\phi n_j}\right)\phi^2}{\left(\left(e^{\phi n_i} - e^{\phi n_j}\right)\right)^3}$$

and

$$\frac{\partial^2 f(n_i, n_j)}{\partial^2 n_i} = \frac{2e^{2(n_j-n_i)\phi}\phi^2}{\left(1 + e^{(n_j-n_i)\phi}\right)^3} - \frac{e^{(n_j-n_i)\phi}\phi^2}{\left(1 + e^{(n_j-n_i)\phi}\right)^2}.$$

The corresponding limits when ϕ approaches zero are given by $\lim_{\phi\to 0} \partial^2 f(n_i, n_j)/$ $\partial n_i \partial n_j = 0$ and $\lim_{\phi\to 0} \partial^2 f(n_i, n_j)/\partial^2 n_i = 0$. Again $\partial^2 f(n_i, n_j)/\partial n_i \partial n_j$ and $\partial^2 f(n_i, n_j)/\partial^2 n_i$ are continuous functions in ϕ and, for ϕ sufficiently small, the conditions provided in Proposition 4 are therefore satisfied.[22]   □

We present the auxiliary Lemma 3 and Lemma 4, which are useful to show Proposition 5 and Proposition 7.

LEMMA 3. *In any NE* **g**, *if* $g_{i,j} = -1$, *then* i *obtains strictly positive payoffs from the link with* j.

PROOF. Note first that the only type of Nash equilibrium network such that there exists an agent i with $|N_i^-(\mathbf{g})| = 1$ is the one with two cliques and $|P^1(\mathbf{g})| = 1$. That is, **g** is such that $n - 1$ agents gang up on a singleton agent. The payoff from a negative link is strictly positive, due to the assumption that $f(n-1, 1) > \varepsilon + \kappa$. Assume next that **g** is a Nash equilibrium that is different from the previous case. Assume without loss of generality that agent i extends a negative link to agent j, so that $g_{i,j} = -1$. Assume further that the payoff from $g_{i,j} = -1$ is zero, i.e., $f(n_i(\mathbf{g}), n_j(\mathbf{g})) - (\varepsilon + \kappa) = 0$. Agent i can then profitably deviate by extending a positive link to j. Payoffs from the link remain zero, while payoffs from any negative links with agents k strictly increase since $n_i(\mathbf{g}') > n_i(\mathbf{g})$ (by Lemma 1). At least one such agent k exists, since we assumed that **g** is different from a Nash equilibrium where $n - 1$ agents extend a negative link to a singleton agent.   □

LEMMA 4. *In any NE* **g**, *if* $n_i(\mathbf{g}) = n_j(\mathbf{g})$, *then* $u_i(\mathbf{g}) = u_j(\mathbf{g})$, *and if* $n_i(\mathbf{g}) > n_j(\mathbf{g})$, *then* $u_i(\mathbf{g}) > u_j(\mathbf{g})$.

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[22]The calculations were executed in Mathematica.

PROOF. If $n_i(\mathbf{g}) = n_j(\mathbf{g})$, then i and j belong to the same clique in $\mathbf{g}$ and $N_i^-(\mathbf{g}) = N_j^-(\mathbf{g})$ by Proposition 1. Since $n_i(\mathbf{g}) = n_j(\mathbf{g})$, payoffs from links with any third agent k are the same for i and j in $\mathbf{g}$, and therefore $u_i(\mathbf{g}) = u_j(\mathbf{g})$. If $n_i(\mathbf{g}) > n_j(\mathbf{g})$, then i belongs to a clique of larger size than j. Note first that then payoffs are strictly larger for i than for agent j from negative links to any agent k with $n_k(\mathbf{g}) \neq n_i(\mathbf{g})$ and $n_k(\mathbf{g}) \neq n_j(\mathbf{g})$. Payoffs are zero for i's positive links to all agents k (different from i) with $n_k(\mathbf{g}) = n_i(\mathbf{g})$, while they are strictly negative for agent j's negative links with agents k with $n_k(\mathbf{g}) = n_i(\mathbf{g})$. By Lemma 3 agent i's payoffs are strictly positive from any negative links to agents k (different from j) with $n_k(\mathbf{g}) = n_j(\mathbf{g})$, while they are zero for agent j's positive links with agents k with $n_k(\mathbf{g}) = n_j(\mathbf{g})$. Finally, i's payoff from the negative link with j is strictly positive (by Lemma 3), while the payoff is strictly negative for j. $\square$

PROOF OF PROPOSITION 5. Note first that the Nash equilibrium $\mathbf{g}$, such that $\bar{g}_{i,j} = 1$ $\forall i, j \in N$, is not a strong Nash equilibrium. Since $f(n-1, 1) > \varepsilon + \kappa$, any subset of agents J with $|J| = n - 1$ can strictly increase payoffs for all agents in J by extending negative links to agent $k \notin J$ (so that $g'_{i,k} = -1 \ \forall i \in J$). Consider next a Nash equilibrium $\mathbf{g}$, which displays two cliques and the smaller clique, $P^1(\mathbf{g})$, consists of more than one agent, i.e., $|P^1(\mathbf{g})| > 1$. Consider a deviation of an agent $i \in P^1(\mathbf{g})$ and $|P^2(\mathbf{g})| - 1$ agents in $P^2(\mathbf{g})$. Denote with k the agent such that $k \in P^2(\mathbf{g})$ and $k \notin J$. The following deviation is profitable: $\bar{g}'_{i,j} = 1 \ \forall i, j \in J$ and $g'_{j,k} = -1 \ \forall j \in J \setminus \{i\}$. To see this, note that after proposed deviation there are three cliques, with $|P^1(\mathbf{g}')| = 1$, $|P^2(\mathbf{g}')| = |P^1(\mathbf{g})| - 1$, and $|P^3(\mathbf{g}')| = |P^2(\mathbf{g})|$. Since $f(n_i, n_j)$ is decreasing in n_j, an agent $j \in P^3(\mathbf{g}') \setminus \{i\}$ obtains strictly larger payoffs in $\mathbf{g}'$ than agents $j \in P^2(\mathbf{g})$ in $\mathbf{g}$. Therefore, $u_j(\mathbf{g}') > u_j(\mathbf{g})$ holds for all agents $j \in J \setminus \{i\}$. Note next that since $g_{k,i} = g'_{k,i} = -1$, we also know that $u_i(\mathbf{g}') = u_j(\mathbf{g}') + \varepsilon$ holds for $i \in J \cap P^1(\mathbf{g})$ and $\forall j \in J \setminus \{i\}$. Therefore, $u_i(\mathbf{g}') > u_j(\mathbf{g}') > u_j(\mathbf{g}) > u_i(\mathbf{g})$ holds, where the last inequality follows from Lemma 4. That is, $u_j(\mathbf{g}') > u_j(\mathbf{g}) \ \forall j \in J$. Assume next that there are three or more cliques in $\mathbf{g}$. A profitable deviation exists by the above argument, where we take J to consist of $i \in P^{m-1}(\mathbf{g})$ and $|P^m(\mathbf{g})| - 1$ agents in $P^m(\mathbf{g})$. $\square$

PROOF OF PROPOSITION 6. Define $n' = 2 + 4k$, where $k = \{1, 2, 3, \ldots\}$. For any $n \geq 4$, pick a k such that $|n' - n| \leq 2$. For $n \geq 4$, such a k and corresponding n' always exists. To show our result, first partition the set of agents N into two sets $P^1(\mathbf{g})$ and $P^2(\mathbf{g})$, such that $|P^1(\mathbf{g})| = 2k$ and $|P^2(\mathbf{g})| = 2k + 2$. Partition the set $P^1(\mathbf{g})$ further into two subsets, denoted with $P_1^1(\mathbf{g})$ and $P_2^1(\mathbf{g})$, such that $|P_1^1(\mathbf{g})| = |P_2^1(\mathbf{g})| = k$. Likewise, partition the set $P^2(\mathbf{g})$ into two subsets such that $|P_1^2(\mathbf{g})| = |P_2^2(\mathbf{g})| = k + 1$. Consider the following network $\mathbf{g}$. Assume that $\bar{g}_{i,j} = 1 \ \forall i, j \in P^1(\mathbf{g})$ and $\bar{g}_{i,j} = 1 \ \forall i, j \in P^2(\mathbf{g})$. Furthermore, assume that $\bar{g}_{i,j} = 1 \ \forall i, j$ such that $i \in P_1^1(\mathbf{g})$, $j \in P_1^2(\mathbf{g})$, and $\bar{g}_{i,j} = 1 \ \forall i, j$ such that $i \in P_2^1(\mathbf{g})$ and $j \in P_2^2(\mathbf{g})$, while $g_{i,j} = -1$ and $g_{j,i} = 1 \ \forall i, j$ such that $i \in P_1^2(\mathbf{g})$, $j \in P_2^1(\mathbf{g})$, $g_{i,j} = -1$, and $g_{j,i} = 1 \ \forall i, j$ such that $i \in P_2^2(\mathbf{g})$ and $j \in P_1^1(\mathbf{g})$. To see that $\mathbf{g}$ is not structurally balanced, consider the triplet of agents i, j, and l such that $i \in P_1^2(\mathbf{g})$, $j \in P_2^2(\mathbf{g})$, and $l \in P_1^1(\mathbf{g})$. Then $\bar{g}_{i,j} = 1$ and $\bar{g}_{i,l} = 1$, but $\bar{g}_{j,l} = -1$ and $\mathbf{g}$ is therefore not structurally balanced. To see that $\mathbf{g}$ is a Nash equilibrium, note first that $n_i(\mathbf{g}) = 3k + 2 \ \forall i \in P^2(\mathbf{g})$, while $n_j(\mathbf{g}) = 3k + 1 \ \forall j \in P^1(\mathbf{g})$. Note also that for any agent $i \in P^1(\mathbf{g})$, $g_{i,j} = 1 \ \forall j \in N$ and a deviation for an agent $i \in P^1(\mathbf{g})$ therefore consists of extending negative links to some subset of $N \setminus \{i\}$. Since

$i \in P^1(\mathbf{g})$ cannot extract payoffs from any agent by extending negative links and $\varepsilon > 0$, no profitable deviation exists. Consider next an agent $i \in P^2(\mathbf{g})$. Note first that for any deviation by agent i such that $n_i(\mathbf{g}') \geq n_i(\mathbf{g})$, $u_i(\mathbf{g}') \leq u_i(\mathbf{g})$ must hold. To see this, note that agent i's equilibrium payoffs are given by $u_i(\mathbf{g}) = (n - n_i(\mathbf{g})) \cdot (z - (\varepsilon + \kappa))$, where $n - n_i(\mathbf{g})$ is the number of agent i's negative links in $\mathbf{g}$. Note that the payoff from a negative link is at most $z - (\varepsilon + \kappa)$. Therefore, if $n_i(\mathbf{g}') \geq n_i(\mathbf{g})$ holds, $u_i(\mathbf{g}') \leq u_i(\mathbf{g})$ must also hold and no deviation with $n_i(\mathbf{g}') \geq n_i(\mathbf{g})$ is profitable. Consider next a deviation by agent i such that $n_i(\mathbf{g}') < n_i(\mathbf{g})$. We distinguish two cases. Assume first that $n_i(\mathbf{g}') \leq n_i(\mathbf{g}) - 2$. Note that then $n_i(\mathbf{g}') \leq 3k - 1$ holds, while $n_j(\mathbf{g}') \geq 3k + 1 \; \forall j \in P^2(\mathbf{g}) \setminus \{i\}$ and $n_j(\mathbf{g}') \geq 3k$ $\forall j \in P^1(\mathbf{g})$. That is, after proposed deviation, agent i is the weakest agent in $\mathbf{g}'$. Therefore, $u_i(\mathbf{g}') < 0 < u_i(\mathbf{g})$, where the last inequality follows from $z > \varepsilon + \kappa$. Assume next that $n_i(\mathbf{g}') = n_i(\mathbf{g}) - 1$. Consider any deviation $\mathbf{g}'_i$ in which $g'_{i,j} = -1$ for some agent $j \in P^2(\mathbf{g}) \setminus \{i\}$. Note that payoffs for i from $g'_{i,j} = -1$ are negative (since then $n_i(\mathbf{g}') = n_j(\mathbf{g}')$) and a deviation such that $\mathbf{g}'_i + g^+_{i,j}$ yields strictly larger payoffs. We therefore only need to consider deviations in which agent i alters his linking behavior only relative to agents $j \in P^1(\mathbf{g})$. Furthermore, since $n_i(\mathbf{g}') = n_i(\mathbf{g}) - 1$, we only need to consider deviations where agent i extends $k - 1$ positive and $k + 1$ negative links to agents $j \in P^1(\mathbf{g})$. Note next that since agent i extends $k + 1$ negative links in $\mathbf{g}'$ and k negative links in $\mathbf{g}$, there must be at least one agent $j \in P^1(\mathbf{g})$ such that $g_{i,j} = g'_{i,j} = -1$ and, therefore, $n_i(\mathbf{g}') = n_j(\mathbf{g}')$. Payoffs from this link are given by $-(\varepsilon + \kappa)$. Therefore, $u_i(\mathbf{g}') \leq k \cdot z - (k + 1)(\varepsilon + \kappa)$. Note next that since $n - n_i(\mathbf{g}) = k$, we can write the equilibrium payoffs of agent i in $\mathbf{g}$ as $u_i(\mathbf{g}) = k \cdot z - k \cdot (\varepsilon + \kappa)$. Proposed deviation is therefore not profitable and $\mathbf{g}$ is a Nash equilibrium. $\qquad \square$

Before proving Proposition 7, we introduce auxiliary Lemma 5 and denote with cl($\mathbf{g}$) the number of maximal cliques in a Nash equilibrium $\mathbf{g}$.

LEMMA 5. *If there exists a pair of Nash equilibria, $\tilde{\mathbf{g}}$ and $\mathbf{g}$, such that $n_i(\tilde{\mathbf{g}}) \geq n_i(\mathbf{g}) \; \forall i \in N$ and there exists an agent j with $n_j(\tilde{\mathbf{g}}) > n_j(\mathbf{g})$, then the number of maximal cliques in $\tilde{\mathbf{g}}$, cl($\tilde{\mathbf{g}}$), is strictly smaller than the number of maximal cliques in $\mathbf{g}$, cl($\mathbf{g}$). Furthermore, $|P^{m-(r-1)}(\tilde{\mathbf{g}})| \geq |P^{m-(r-1)}(\mathbf{g})|$ for all $r \in \{1, \ldots, \text{cl}(\tilde{\mathbf{g}})\}$ and $|P^{m-(r-1)}(\tilde{\mathbf{g}})| > |P^{m-(r-1)}(\mathbf{g})|$ for some $r \in \{1, \ldots, \text{cl}(\tilde{\mathbf{g}})\}$.*

PROOF. Note that in any equilibrium network, $\mathbf{g}$, the cardinality of a set $P^x(\mathbf{g})$ with $i \in P^x(\mathbf{g})$, is given by $|P^x(\mathbf{g})| = n_i(\mathbf{g})$. The second part of the statement therefore follows directly from $n_i(\tilde{\mathbf{g}}) \geq n_i(\mathbf{g}) \; \forall i \in N$ and $n_j(\tilde{\mathbf{g}}) > n_j(\mathbf{g})$ for some agent j. For the first part of the statement, assume to the contrary that $n_i(\tilde{\mathbf{g}}) \geq n_i(\mathbf{g}) \; \forall i \in N$, that there exists an agent j with $n_j(\tilde{\mathbf{g}}) > n_j(\mathbf{g})$, and that the number of cliques in $\tilde{\mathbf{g}}$, cl($\tilde{\mathbf{g}}$), is weakly larger than the number of cliques in $\mathbf{g}$, cl($\mathbf{g}$). That is, assume cl($\tilde{\mathbf{g}}$) $\geq$ cl($\mathbf{g}$). Rank cliques by their size, with clique 1 being the smallest and cliques cl($\mathbf{g}$) and cl($\tilde{\mathbf{g}}$) being the largest cliques in $\mathbf{g}$ and $\tilde{\mathbf{g}}$, respectively. But then, since cl($\tilde{\mathbf{g}}$) $\geq$ cl($\mathbf{g}$) holds, together with $|P^{m-(r-1)}(\tilde{\mathbf{g}})| \geq |P^{m-(r-1)}(\mathbf{g})|$ for all cliques with $|P^{m-(r-1)}(\tilde{\mathbf{g}})| > |P^{m-(r-1)}(\mathbf{g})|$ for some clique, $\sum_{r=1}^{\text{cl}(\tilde{\mathbf{g}})} |P^{m-(r-1)}(\tilde{\mathbf{g}})| > \sum_{r=1}^{\text{cl}(\mathbf{g})} |P^{m-(r-1)}(\mathbf{g})|$ also holds. We have reached a contradiction, since summing up over all cliques yields n in $\tilde{\mathbf{g}}$ and in $\mathbf{g}$. $\qquad \square$

Before proving [Proposition 3](), we formally define Pareto efficiency and isomorphic networks. Essentially, two networks are isomorphic if they are the same up to a relabeling of agents.

DEFINITION 3. A network $\tilde{\mathbf{g}}$ Pareto dominates another network $\mathbf{g}$, written as $\tilde{\mathbf{g}} \succ \mathbf{g}$, if and only if $u_i(\tilde{\mathbf{g}}) \geq u_i(\mathbf{g}) \ \forall i \in N$ and $\exists i : u_i(\tilde{\mathbf{g}}) > u_i(\mathbf{g})$.

DEFINITION 4. The networks $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are isomorphic, if there exists a bijection $f : N \to \tilde{N}$, such that $g_{i,j} = 1$ if and only if $\tilde{g}_{f(i),f(j)} = 1$ and $g_{i,j} = -1$ if and only if $\tilde{g}_{f(i),f(j)} = -1$.

We introduce some additional notation and definitions that are useful for proving [Proposition 7](). When comparing two Nash equilibria, $\mathbf{g}$ and $\tilde{\mathbf{g}}$, partition the set of agents as follows: $B(\mathbf{g} \to \tilde{\mathbf{g}}) = \{i \in N \mid n_i(\tilde{\mathbf{g}}) \geq n_i(\mathbf{g})\}$ is the set of agents with weakly more positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, while $S(\mathbf{g} \to \tilde{\mathbf{g}}) = \{i \in N \mid n_i(\tilde{\mathbf{g}}) < n_i(\mathbf{g})\}$ is the set of agents with strictly fewer positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. For notational simplicity, we sometimes write B for $B(\mathbf{g} \to \tilde{\mathbf{g}})$ and S for $S(\mathbf{g} \to \tilde{\mathbf{g}})$. Furthermore, denote with $S_i^-(\tilde{\mathbf{g}}) = \{j \in S \mid \bar{\tilde{g}}_{i,j} = -1\}$ the set of all agents to which agent i sustains an antagonistic relationship within the set S in $\tilde{\mathbf{g}}$. Correspondingly, define with $B_i^-(\tilde{\mathbf{g}}) = \{j \in B \mid \bar{\tilde{g}}_{i,j} = -1\}$ the set of all agents to which agent i sustains an antagonistic relationship within the set B in $\tilde{\mathbf{g}}$. The set of agents to which agent i extends a negative link within the set S in $\tilde{\mathbf{g}}$ is denoted with $S_i^{e-}(\tilde{\mathbf{g}}) = \{j \in S \mid \tilde{g}_{i,j} = -1\}$. Likewise, define $B_i^{e-}(\tilde{\mathbf{g}}) = \{j \in B \mid \tilde{g}_{i,j} = -1\}$ as the set of agents to which agent i extends a negative link within the set B in $\tilde{\mathbf{g}}$. Denote the cardinalities $c_i^S(\tilde{\mathbf{g}}) = |S_i^-(\tilde{\mathbf{g}})|$, $e_i^S(\tilde{\mathbf{g}}) = |S_i^{e-}(\tilde{\mathbf{g}})|$, $c_i^B(\tilde{\mathbf{g}}) = |B_i^-(\tilde{\mathbf{g}})|$, and $e_i^B(\tilde{\mathbf{g}}) = |B_i^{e-}(\tilde{\mathbf{g}})|$.

Denote with $u_{SS}(\tilde{\mathbf{g}})$ the sum of net payoffs that agents $i \in S$ incur from other agents within S in $\tilde{\mathbf{g}}$. Note that since $\sum_{i \in S} \sum_{j \in S \setminus \{i\}} f(n_i(\tilde{\mathbf{g}}), n_j(\tilde{\mathbf{g}})) = 0$, we can express $u_{SS}(\tilde{\mathbf{g}})$ in terms of linking cost. Denote with $l_S^-(\tilde{\mathbf{g}}) = \frac{1}{2} \sum_{i \in S} c_i^S(\tilde{\mathbf{g}})$ the total number of undirected negative links of agents S within the set S in $\tilde{\mathbf{g}}$. Similarly, denote with $l_S^-(\mathbf{g}) = \frac{1}{2} \sum_{i \in S} c_i^S(\mathbf{g})$ the total number of undirected negative links of agents S within the set S in $\mathbf{g}$. Since each negative link within S incurs a total cost of $\varepsilon + 2\kappa$, $u_{SS}(\tilde{\mathbf{g}})$ can be written as $u_{SS}(\tilde{\mathbf{g}}) = -(\varepsilon + 2\kappa) \cdot l_S^-(\tilde{\mathbf{g}})$. Denote with $u_{SB}(\tilde{\mathbf{g}})$ the sum of net payoffs that agents $i \in S$ incur from links with agents in B in $\tilde{\mathbf{g}}$, which is defined as $u_{SB}(\tilde{\mathbf{g}}) = \sum_{i \in S} \sum_{j \in B} f(n_i(\tilde{\mathbf{g}}), n_j(\tilde{\mathbf{g}})) - \sum_{i \in S}(e_i^B(\tilde{\mathbf{g}})\varepsilon + c_i^B(\tilde{\mathbf{g}})\kappa)$. Note that $\sum_{i \in S} u_i(\tilde{\mathbf{g}}) = u_{SS}(\tilde{\mathbf{g}}) + u_{SB}(\tilde{\mathbf{g}})$. We write $\tilde{\mathbf{g}} \not\succ \mathbf{g}$ to indicate that the Nash equilibrium $\tilde{\mathbf{g}}$ does not Pareto dominate $\mathbf{g}$.

PROOF OF PROPOSITION 7. Assume without loss of generality that $\tilde{\mathbf{g}} \succ \mathbf{g}$. That is, $u_i(\tilde{\mathbf{g}}) \geq u_i(\mathbf{g}) \ \forall i \in N$ and there exists an agent i with $u_i(\tilde{\mathbf{g}}) > u_i(\mathbf{g})$. Assume first that $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are isomorphic. If $n_i(\tilde{\mathbf{g}}) = n_i(\mathbf{g}) \ \forall i \in N$, then $u_i(\tilde{\mathbf{g}}) = u_i(\mathbf{g}) \ \forall i \in N$ and $\tilde{\mathbf{g}} \not\succ \mathbf{g}$. If there exists an agent with $n_i(\tilde{\mathbf{g}}) \neq n_i(\mathbf{g})$, then there also exists an agent such that $n_j(\tilde{\mathbf{g}}) < n_j(\mathbf{g})$. Then $\tilde{\mathbf{g}} \not\succ \mathbf{g}$ follows directly from [Lemma 4]() and the assumption that $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are isomorphic. Assume next that $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are not isomorphic. We distinguish three cases.

Case 1: $B(\mathbf{g} \to \tilde{\mathbf{g}}) \neq \varnothing$ and $S(\mathbf{g} \to \tilde{\mathbf{g}}) \neq \varnothing$. We show that $\sum_{i \in S} u_i(\tilde{\mathbf{g}}) < \sum_{i \in S} u_i(\mathbf{g})$ and, therefore, there exists an agent i such that $u_i(\tilde{\mathbf{g}}) < u_i(\mathbf{g})$ and $\tilde{\mathbf{g}} \not\succ \mathbf{g}$.

Subcase 1. Assume first that $l_S^-(\tilde{\mathbf{g}}) \geq l_S^-(\mathbf{g})$, i.e., there are at least as many negative links among agents in S in $\tilde{\mathbf{g}}$ as in $\mathbf{g}$. Note that from $l_S^-(\tilde{\mathbf{g}}) \geq l_S^-(\mathbf{g})$ it follows immediately

that $u_{SS}(\tilde{\mathbf{g}}) \leq u_{SS}(\mathbf{g})$. Next we show that $u_{SB}(\tilde{\mathbf{g}}) \leq u_{SB}(\mathbf{g})$ holds. Recall that agents in S have strictly fewer positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, while agents in B have weakly more positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. There are four different constellations between a pair of agents $i \in S$ and $j \in B$ to consider. First, $\tilde{\bar{g}}_{i,j} = \bar{g}_{i,j} = 1$. Payoffs from links with these agents are zero in both Nash equilibrium networks. Second, $\tilde{\bar{g}}_{i,j} = \bar{g}_{i,j} = -1$. Payoffs are strictly lower for agent i from the link to j in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. To see this, note that i has strictly fewer positive links in $\tilde{\mathbf{g}}$, while j has weakly more positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. Third, $\tilde{\bar{g}}_{i,j} = 1$ and $\bar{g}_{i,j} = -1$. Again payoffs from i's link with j are strictly lower in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. To see this, note that since $\tilde{\bar{g}}_{i,j} = 1$, i and j are in the same clique in $\tilde{\mathbf{g}}$, while i and j are in different cliques in $\mathbf{g}$. Since i has strictly fewer positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$ and since j has weakly more positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, i extracts payoffs from j in $\mathbf{g}$. In contrast, payoffs of the link between i and j are zero in $\tilde{\mathbf{g}}$. That is, payoffs from i's link with j are strictly positive in $\mathbf{g}$ (by Lemma 3) and zero in $\tilde{\mathbf{g}}$. Fourth, $\tilde{\bar{g}}_{i,j} = -1$ and $\bar{g}_{i,j} = 1$. Then i and j are in the same clique in $\mathbf{g}$, while j is in a larger clique than i in $\tilde{\mathbf{g}}$. That is, i's payoffs from the link with j are zero in $\mathbf{g}$, while they are strictly negative in $\tilde{\mathbf{g}}$. Therefore, $u_{SB}(\tilde{\mathbf{g}}) \leq u_{SB}(\mathbf{g})$ holds. Assume first that $l_S(\tilde{\mathbf{g}}) > l_S(\mathbf{g})$. Then $u_{SS}(\tilde{\mathbf{g}}) < u_{SS}(\mathbf{g})$ holds and, therefore, $\sum_{i \in S} u_i(\tilde{\mathbf{g}}) < \sum_{i \in S} u_i(\mathbf{g})$ and $\tilde{\mathbf{g}} \not\succ \mathbf{g}$. Assume next that $l_S(\tilde{\mathbf{g}}) = l_S(\mathbf{g})$ and, therefore, $u_{SS}(\tilde{\mathbf{g}}) = u_{SS}(\mathbf{g})$ holds. Note that by the construction of S, $\sum_{i \in S} n_i(\tilde{\mathbf{g}}) < \sum_{i \in S} n_i(\mathbf{g})$ must hold. But then, if $l_S(\tilde{\mathbf{g}}) = l_S(\mathbf{g})$, for $\sum_{i \in S} n_i(\tilde{\mathbf{g}}) < \sum_{i \in S} n_i(\mathbf{g})$ to hold, there must be at least one pair of agents $i \in S$ and $j \in B$ such that $\bar{g}_{i,j} = 1$ and $\tilde{\bar{g}}_{i,j} = -1$. From the above we know that payoffs for agent i from this link are strictly lower in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, while payoffs are weakly lower for all remaining links between agents in S and B in $\tilde{\mathbf{g}}$. Therefore, $u_{SB}(\tilde{\mathbf{g}}) < u_{SB}(\mathbf{g})$ and again $\sum_{i \in S} u_i(\tilde{\mathbf{g}}) < \sum_{i \in S} u_i(\mathbf{g})$ holds. Therefore, $\tilde{\mathbf{g}} \not\succ \mathbf{g}$.

Subcase 2. Assume that $l_S^-(\tilde{\mathbf{g}}) < l_S^-(\mathbf{g})$, i.e., there are strictly fewer negative links among agents in S in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. Denote with k the difference in negative links in $\mathbf{g}$ and $\tilde{\mathbf{g}}$ among agents in S, $k = l_S^-(\mathbf{g}) - l_S^-(\tilde{\mathbf{g}})$, and note that $u_{SS}(\mathbf{g}) + k \cdot (\varepsilon + 2\kappa) = u_{SS}(\tilde{\mathbf{g}})$. For $\sum_{i \in S} n_i(\tilde{\mathbf{g}}) < \sum_{i \in S} n_i(\mathbf{g})$ to hold, there must be at least $2k + 1$ links such that $i \in S$, $j \in B$, $\bar{g}_{i,j} = 1$, and $\tilde{\bar{g}}_{i,j} = -1$. Note also that since $i \in S$, $j \in B$, and $\bar{g}_{i,j} = 1$, it must be that $\tilde{g}_{j,i} = -1$. That is, it must be that agents $j \in B$ extends the negative link in $\tilde{\mathbf{g}}$. From Lemma 3 we know that what $j \in B$ extracts from $i \in S$ in $\tilde{g}_{i,j} = -1$ is strictly larger than $\varepsilon + \kappa$, while $i \in S$ still incurs the cost of κ. Therefore, the at least $2k + 1$ links such that $\bar{g}_{i,j} = 1$ and $\tilde{\bar{g}}_{i,j} = -1$ with $i \in S$ and $j \in B$ more than outweigh $k \cdot (\varepsilon + 2\kappa)$. Since payoffs are weakly lower for all remaining links between agents in S and B in $\tilde{\mathbf{g}}$ (by the argument presented in Case 1, Subcase 1), $\sum_{i \in S} u_i(\tilde{\mathbf{g}}) < \sum_{i \in S} u_i(\mathbf{g})$ holds and, therefore, $\tilde{\mathbf{g}} \not\succ \mathbf{g}$.

Case 2: $B(\mathbf{g} \to \tilde{\mathbf{g}}) = \varnothing$ and $S(\mathbf{g} \to \tilde{\mathbf{g}}) = N$. Note that then the number of negative links is strictly larger in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$, and since gross payoffs sum to zero, $\sum_{i \in N} u_i(\tilde{\mathbf{g}}) < \sum_{i \in N} u_i(\mathbf{g})$ and, therefore, $\tilde{\mathbf{g}} \not\succ \mathbf{g}$.

Case 3: $B(\mathbf{g} \to \tilde{\mathbf{g}}) = N$ and $S(\mathbf{g} \to \tilde{\mathbf{g}}) = \varnothing$. Note first that if $n_i(\tilde{\mathbf{g}}) = n_i(\mathbf{g}) \ \forall i \in N$, then $\mathbf{g}$ and $\tilde{\mathbf{g}}$ are isomorphic and, therefore, $\tilde{\mathbf{g}} \not\succ \mathbf{g}$, as shown above. Consider next the case where $n_i(\tilde{\mathbf{g}}) \geq n_i(\mathbf{g}) \ \forall i \in N$ and $n_j(\tilde{\mathbf{g}}) > n_j(\mathbf{g})$ for some j. From Lemma 5 we know that there are then strictly fewer cliques in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. We discern two subcases.

Subcase 1. There exists an agent i such that $n_i(\tilde{\mathbf{g}}) = n_i(\mathbf{g})$. Note that then not all links in $\tilde{\mathbf{g}}$ are positive and at least two cliques exist. Consider an agent j with $n_j(\tilde{\mathbf{g}}) > n_j(\mathbf{g})$. We next show that payoffs from i's link with j are strictly lower in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. We distinguish

three cases. First, assume j is in the same clique as i in $\tilde{\mathbf{g}}$. Then i and j are in different cliques in $\mathbf{g}$ and $n_i(\mathbf{g}) > n_j(\mathbf{g})$. Agent i obtains a payoff of zero from the positive link with j in $\tilde{\mathbf{g}}$, while agent i obtains a strictly negative payoff from the negative link with j in $\mathbf{g}$ (by Lemma 3). That is, i's payoffs from the link with j are strictly lower in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. Second, assume agents i and j are in different cliques in both $\tilde{\mathbf{g}}$ and $\mathbf{g}$. Then the link between i and j is negative in $\tilde{\mathbf{g}}$ and in $\mathbf{g}$. Since $n_j(\tilde{\mathbf{g}}) > n_j(\mathbf{g})$ and $n_i(\tilde{\mathbf{g}}) = n_i(\mathbf{g})$, i's payoffs from the link with j are strictly lower in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$. Third, if agent j is in a different clique than i in $\tilde{\mathbf{g}}$, and i and j are in the same clique in $\mathbf{g}$, then $n_j(\tilde{\mathbf{g}}) > n_i(\mathbf{g})$ and i's payoffs from the negative link with j are negative in $\tilde{\mathbf{g}}$, while the payoffs from the positive link in $\mathbf{g}$ are zero. All remaining agents have weakly more positive links in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$ and i's payoffs from these links are weakly lower in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$ (by the argument presented in Case 1, Subcase 1). Therefore, i's payoffs are strictly lower in $\tilde{\mathbf{g}}$ than in $\mathbf{g}$ and $\tilde{\mathbf{g}} \not\succ \mathbf{g}$.

Subcase 2: $n_i(\tilde{\mathbf{g}}) > n_i(\mathbf{g})\ \forall i \in N$. Note first that if $\tilde{\mathbf{g}}$ is such that all links are positive, then agents that are in the largest clique in $\mathbf{g}$ obtain strictly lower payoffs in $\tilde{\mathbf{g}}$. To see this, note that by Lemma 3, payoffs from a negative link are strictly positive for the stronger agent and agents in the largest clique therefore obtain strictly positive payoffs in $\mathbf{g}$ (while payoffs are zero for all agents if all links are positive in $\tilde{\mathbf{g}}$). Therefore, $\tilde{\mathbf{g}} \not\succ \mathbf{g}$.

Next, assume that there are at least two cliques in $\tilde{\mathbf{g}}$. Pick the minimum r such that $|P^1(\tilde{\mathbf{g}})| \leq |\bigcup_{k=1}^{k=r} P^k(\mathbf{g})|$. We distinguish two cases. Assume first that there exists an agent j such that $j \in P^1(\tilde{\mathbf{g}})$ and $j \in P^r(\mathbf{g})$. Consider the deviation strategy of agent j in $\mathbf{g}$, $\mathbf{g}_j + \sum_{i \in \bigcup_{k=1}^{k=r-1} P^k(\mathbf{g})} g_{j,i}^+$, and denote the resulting network after proposed deviation with $\mathbf{g}'$. Next we show that $u_j(\mathbf{g}') > u_j(\tilde{\mathbf{g}})$. Consider any ranking of agents by their strength in ascending order in $\tilde{\mathbf{g}}$, so that $n^1(\tilde{\mathbf{g}}) \leq n^2(\tilde{\mathbf{g}}) \leq \cdots \leq n^{n-1}(\tilde{\mathbf{g}}) \leq n^n(\tilde{\mathbf{g}})$. Likewise, consider any ranking of agents by their strength in ascending order in $\mathbf{g}$, so that $n^1(\mathbf{g}) \leq n^2(\mathbf{g}) \leq \cdots \leq n^{n-1}(\mathbf{g}) \leq n^n(\mathbf{g})$. Note that from $n_i(\tilde{\mathbf{g}}) > n_i(\mathbf{g})\ \forall i \in N$, we know that $n^x(\tilde{\mathbf{g}}) > n^x(\mathbf{g})$ holds for all $x \in \mathbb{N}: 1 \leq x \leq n$. From $|P^1(\tilde{\mathbf{g}})| \leq |\bigcup_{k=1}^{k=r} P^k(\mathbf{g})|$, we know that $n_j(\mathbf{g}') \geq n_j(\mathbf{g})$. If $|N_j^-(\mathbf{g}')| = \varnothing$, then $u_j(\mathbf{g}') = 0 > u_i(\tilde{\mathbf{g}})$, where the second inequality follows from $j \in P^1(\tilde{\mathbf{g}})$ and that there are at least two cliques in $\tilde{\mathbf{g}}$. If $|N_j^-(\mathbf{g}')| \neq \varnothing$, then note that the $n - |P^1(\tilde{\mathbf{g}})|$ strongest agents in $\tilde{\mathbf{g}}$ extend negative links to $j \in P^1(\tilde{\mathbf{g}})$ in $\tilde{\mathbf{g}}$, while the x strongest agents in $\mathbf{g}'$ extend negative links to $j \in P^r(\mathbf{g})$ in $\mathbf{g}'$, with $x \in \mathbb{N}: 1 \leq x \leq n - |P^1(\tilde{\mathbf{g}})|$. Since $n^x(\tilde{\mathbf{g}}) > n^x(\mathbf{g})$ holds for all $x \in \mathbb{N}: 1 \leq x \leq n$, we know that what agents in $N_j^-(\tilde{\mathbf{g}})$ extract from j in $\tilde{\mathbf{g}}$ is strictly larger than what agents in $N_j^-(\mathbf{g}')$ extract from j in $\mathbf{g}'$ and $u_j(\mathbf{g}') > u_j(\tilde{\mathbf{g}})$. Note next that, for $\mathbf{g}$ to be a Nash equilibrium, it must be that no profitable deviation exists and, therefore, $u_j(\mathbf{g}) \geq u_j(\mathbf{g}')$. That is, $u_j(\mathbf{g}) \geq u_j(\mathbf{g}') > u_j(\tilde{\mathbf{g}})$ holds and $\tilde{\mathbf{g}} \not\succ \mathbf{g}$. Assume next that there does not exists an agent j such that $j \in P^1(\tilde{\mathbf{g}})$ and $j \in P^r(\mathbf{g})$. Since we chose r to be the minimum r such that $|P^1(\tilde{\mathbf{g}})| \leq |\bigcup_{k=r}^{k=r} P^k(\mathbf{g})|$ holds, there then exists an agent l such that $l \in P^1(\tilde{\mathbf{g}})$ and $l \notin \bigcup_{k=1}^{k=r} P^k(\mathbf{g})$. Note that $l \notin \bigcup_{k=1}^{k=r} P^k(\mathbf{g})$ implies that $n_l(\mathbf{g}) > n_j(\mathbf{g})$ for any $j \in P^r(\mathbf{g})$. From Lemma 4 we know that $u_l(\mathbf{g}) > u_j(\mathbf{g})\ \forall j \in P^r(\mathbf{g})$ and, therefore, $u_l(\mathbf{g}) > u_l(\tilde{\mathbf{g}})$ holds. That is, $\tilde{\mathbf{g}} \not\succ \mathbf{g}$.   □

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