Equilibria in symmetric games: Theory and Applications

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Abstract

This article presents a new approach to analyze the equilibrium set of symmetric, differentiable games by separating between multiple symmetric equilibria and asymmetric equilibria. This separation allows to investigate, for example, how various parameter constellations affect the scope for multiple symmetric or asymmetric equilibria, or how the equilibrium set depends on the nature of the strategies. The approach is particularly helpful in applications because (1) it allows to reduce the complexity of the uniqueness-problem to a two-player game, (2) boundary conditions are less critical compared to standard procedures, and (3) best-replies need not be everywhere differentiable. The usefulness of the separation approach is illustrated with several examples, including an application to asymmetric games and to a two-dimensional price-information game.

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1 Introduction

When does a symmetric game with an arbitrary, finite strategy space have asymmetric equilibria, multiple symmetric equilibria or a unique (symmetric) equilibrium? As an example, consider an oligopolistic firm that, confronted with $N-1$ competitors, must decide on the best way to sell its products. Such a decision may involve several critical aspects, e.g., which price to choose, how much to advertise or which quality level to produce. While setting up such a situation as an N-player game with a possibly multi-dimensional strategy space is fairly simple, solving the model *analytically* to obtain meaningful predictions about its possible outcomes – the equilibrium set – can be challenging, even in case of symmetric firms. Are there multiple outcomes or is the equilibrium unique? Do identical firms necessarily adopt the same actions in all equilibria, or could their behavior deviate from each other in an asymmetric equilibrium? Can specific details of the game influence whether asymmetric equilibria or multiple symmetric equilibria result? In the oligopoly example we could ask if and how certain aspects related to advertising and market demand (such as parameters of the advertising technology or the degree of substitutability) could affect the type of equilibria that emerge. Do all firms necessarily adopt the same advertising and pricing strategies, or could there be asymmetric "specialization" equilibria, where some firms set high prices but advertise only little, and others set low prices but advertise a lot?

To analyze such questions, standard approaches such as contraction mappings, univalence or Index theory can be insufficient as their applicability might be limited,[1] particularly in parametric applications. While these methods, if applicable, theoretically allow to decide whether or not there is a unique equilibrium, one typically cannot learn much more about the equilibrium set if there are multiple equilibria.[2] Moreover, even if these approaches are applicable, they might be hard to evaluate, especially in the presence of many players and a higher-dimensional strategy space, because they involve, at the very least, the evaluation of a determinant of a potentially very large and abstract matrix.

[1]See, e.g., Theocharis (1960) for a classical application of the contraction principle to Cournot, or Hefti (2015a) for a modern treatment of the contraction principle. See Gale and Nikaido (1965) for the univalence approach. E.g., Vives (1999) provides a game-theoretic textbook treatment of Index theory.

[2]This is demonstrated for the Index theorem in the online appendix (section A.1).

In this article I address the shortcomings of existing standard approaches by studying separately the possible occurrence of the two types of equilibria - symmetric and asymmetric equilibria - in these games. This approach yields separate conditions that rule out the possibility of asymmetric equilibria and multiple symmetric equilibria, respectively. These conditions are appealing, especially for applications, because of their simplicity (reduction to a two-player problem) and their applicability (e.g., boundary conditions are less problematic or best-replies may have non-differentiabilities). At a more theoretical level, my separation approach allows to investigate how the scope for multiple symmetric equilibria or asymmetric equilibria depends on the parameter constellations in a game, or on the general nature of the best-replies.

The usefulness of the separation approach is demonstrated with several applications. For example, I prove that a symmetric game with a two-dimensional strategy space (such as price and quality) can never possess strictly ordered asymmetric equilibria, where one player sets *both* a higher price and quality, if either price *or* quality is non-decreasing in the opponents' actions. Further, I show that sum-aggregative symmetric games with homogeneous revenues (such as contests) *naturally* have a unique symmetric equilibrium. In the well-studied Cournot model, the separation approach reveals that the classical assumption $c'' - P' > 0$ is key for uniqueness in the Cournot model *because* it rules out the possibility of asymmetric equilibria. Moreover, I analyze an oligopolistic model as suggested by this introduction, where firms compete in prices and must decide on their advertising intensities. This game nicely illustrates the shortcomings of Index theory – frequently regarded as the most general approach to uniqueness in "nice" games (e.g., Vives (1999)) – in applications, First, the Index theorem cannot be used due to violations of the respective boundary conditions, which occur naturally in this example.[3] Second, we could not learn much about the properties of the possible equilibria, even if the Index theorem were applicable. The separation approach on the other hand allows me to derive characteristic properties of equilibria both in the context of a parametrized example and at a more abstract level.

Analyzing the equilibrium set of a symmetric game may not only be interesting in itself, but

[3]Section 4.1 contains further violating examples.

matters also because we may learn more about the equilibrium set of asymmetric variations of that game. As an example, uniqueness of equilibria in a symmetric game is preserved under sufficiently small asymmetric variations of the symmetric game, provided that the symmetric equilibrium is regular (section A.4, online appendix). Moreover, I find a strong link between the inexistence of asymmetric equilibria in symmetric one-dimensional games and the equilibrium properties of certain asymmetric variations of those games (section 4.3).

Related literature While I am not aware of any contribution that studies existence and uniqueness of equilibria in general symmetric games, nor of a systematic separation between symmetric and asymmetric equilibria, the literature on globally supermodular games has focused on equilibrium existence and uniqueness in symmetric supermodular games.[4] In particular, it is a known result that in the class of symmetric globally supermodular games a symmetric equilibrium always exists, and if the symmetric equilibrium is unique, there cannot be any asymmetric equilibria. Moreover, globally supermodular games with a one-dimensional strategy space can never possess asymmetric equilibria.[5] While these are powerful results,[6] they depend on global strategic complementarity, while my approach also applies to games with non-monotonic best-replies (such as contests). Amir et al. (2010) study a symmetric one-dimensional two-player submodular game (decreasing best-replies). As the authors are interested only in asymmetric equilibria, the (single) symmetric equilibrium which such a game would naturally have (see section 3.1) is deliberately excluded by the assumption of a downward-jumping best-reply around the diagonal. The existence of asymmetric equilibria then is a consequence of supermodularity theory, which requires to revert the order of each player's action space. A known limitation of this trick is that the reversion-of-axis argument does not generalize to the N-player case (Vives, 1999). In contrast, my results on the (in)-existence of asymmetric equilibria are not restricted to $N = 2$ (but allow to study an N-player game as a two-player game) and do not presume submodularity. In the special case $N = 2$, I provide a characterization of asymmetric equilibria (section 6.5, online appendix). Finally, there are many articles where

[4]See Vives (2005) for a survey on supermodular games.

[5]A result which my approach can easily replicate, see section 3.2.

[6]These results hold under weaker conditions than the ones imposed by this article. For example, strategy spaces need only be compact lattices (Vives, 2005).

the symmetric version of a game is a separate part of the analysis. I discuss some of these contributions in the context of my applications.

The article is structured as follows. After introducing the notation, the separation approach is developed in section 3, and section 4 applies the approach to several examples.

2 Notation and assumptions

Consider a game of $N \geq 2$ players, indexed by $1, ..., N$. Let $x_g \equiv (x_{g1}, ..., x_{gk}) \in S(k)$ denote a strategy of player g, where $S \equiv S(k) = \times_{i=1}^{k} S_i$ with $S_i = \left[0, \bar{S}_i\right] \subset \mathbb{R}$, $\bar{S}_i > 0$. The interior of S_i is non-empty and denoted by $Int(S_i)$. All players have the same strategy space S. I only consider pure strategies. The vector $x_{-g} \in S^{N-1}$ is a strategy profile of player g's opponents. The payoff of g is represented by a function $\Pi^g(x) = \Pi^g\left(x_1, ..., x_g, ..., x_N\right) \equiv \Pi\left(x_g, x_{-g}\right)$. Unless stated otherwise, the following properties of $\Pi^g(\cdot)$ are assumed throughout this article:

- Symmetry: Payoff functions are permutation-invariant (Amir et al., 2008), meaning that for any permutation σ of $\{1, ..., N\}$ and any $x \in S^N$, payoff functions satisfy

$$\Pi^g\left(x_1, ..., x_N\right) = \Pi^{\sigma(g)}(x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(N)}).$$

- $\Pi\left(x_g, x_{-g}\right) \in C^2\left(O^N, \mathbb{R}\right)$, where $O \supset S$ is open in $\mathbb{R}^k$, and Π is strongly quasiconcave[7] in $x_g \in S$ for any $x_{-g} \in S^{(N-1)}$.

Let $\nabla\Pi^g(x)$ denote the gradient (a k-vector) of $\Pi(x_g, x_{-g})$ with respect to x_g, and $\nabla F(x) \equiv (\nabla\Pi^g(x))_{g=1}^{N}$ is the pseudogradient (a Nk vector, Rosen (1965)). The triple $\left(N, S(k)^N, \Pi\right)$ denotes a symmetric, differentiable k-dimensional N-player game, and the formulation "a game" in text refers to this triple.

[7]Strong quasiconcavity means that $z \cdot z = 1$ and $z \cdot \frac{\partial \Pi^g(x)}{\partial x_g} = 0$ imply $z \cdot \frac{\partial^2 \Pi^g(x)}{\partial x_g \partial x_g} z < 0$ (Avriel et al., 1981). The assumption of a strongly quasiconcave payoff function in own strategies is mainly for convenience, because it is a sufficient condition for the existence of a (possibly differentiable) best-reply function. However, many results require only differentiability of best replies (and do not otherwise hinge on quasiconcavity), and some results do not require that best-replies are everywhere differentiable.

Player g's best reply $\varphi^g(x_{-g})$ solves $\max_{x_g \in S} \Pi(x_g, x_{-g})$. The assumptions made assure that best-replies $\varphi(x_{-g}) \equiv \varphi^g(x_{-g})$ and the joint best-reply $\phi(x_1, ..., x_N) = (\varphi(x_{-1}), ..., \varphi(x_{-N}))$ are continuous functions. Moreover, φ^g is differentiable at x_{-g} if $\varphi^g(x_{-g}) \in Int(S)$. A (Nash) equilibrium is a fixed point (FP) $\phi(x^*) = x^*$, and the equilibrium is symmetric if $x_1^* = ... = x_N^*$. Any symmetric equilibrium $x^* \in S^N$ can be identified by its first projection $x_1^* \in S$.

To find *symmetric* equilibria, a simplified approach, called Symmetric Opponents Form Approach (SOFA) hereafter, is useful.[8] The SOFA takes an arbitrary indicative player ($g = 1$) and restricts all opponents to play the same strategies, i.e., $\bar{x}_{-g} = (\bar{x}, ..., \bar{x})$, where $\bar{x} \in S$. Let $\tilde{\Pi}(x_1, \bar{x}) \equiv \Pi^1(x_1, \bar{x}_{-1})$, with corresponding best-reply function $\tilde{\varphi}(\bar{x}) \equiv \varphi(\bar{x}_{-1})$. In this way, the SOFA reduces an N-player game to the structure of a two-player game. The derivative of $\tilde{\varphi}$ at $\bar{x}$ is denoted by $\partial\tilde{\varphi}(\bar{x})$.[9] The following result, which I include mainly for clarity and completeness, is a consequence of the above assumptions.

Proposition 1 *$x^* \in S^N$ is a symmetric equilibrium iff $x_1^* = \tilde{\varphi}(x_1^*)$. A symmetric game has a symmetric equilibrium, and the set of symmetric equilibria is compact.*

Proof: The first claim is obvious. Continuity and strong quasiconcavity of Π jointly with compactness and convexity of S yield continuity of $\tilde{\varphi}(\bar{x})$. Existence and compactness then follow from $\tilde{\varphi} \in C(S, S)$ and the Brouwer FP theorem. ■

3 The separation approach

Standard approaches to verify uniqueness are i) the contraction mapping approach, ii) the univalence approach and iii) the Index theorem approach. Obviously, these methods can be applied to symmetric games (see, e.g., Vives (1999)). Their shortcomings are that they may be restrictive, involve boundary conditions or require to calculate the determinant of possibly large matrices. Furthermore, we cannot use these methods to investigate, for example, what parameter constellations might cause a game to have multiple symmetric equilibria versus asymmetric equilibria. This observation is the starting point of the now proposed separation

[8]The SOFA has been applied, e.g., by Salop (1979), Grossman and Shapiro (1984), Dixit (1986) or Hefti (2015b) to find symmetric equilibria in specific games.

[9]Note that $\tilde{\varphi}$ inherits continuity and differentiability at interior solutions from φ^1.

approach. The main idea is to separate the analysis between the possibilities of symmetric and asymmetric equilibria. In both cases I am able to reduce the dimensionality of the respective problem from an N-player to the structure of a two-player game, albeit by a different set of arguments. Because symmetric equilibria are assured to exist, it is natural to begin with symmetric equilibria. For symmetric equilibria the essential simplification follows from the SOFA, where an application of Index theory to SOFA provides a powerful set of tools. With asymmetric equilibria I show, by exploiting and extending the Mean Value Theorem, that the symmetric geometry implied by asymmetric equilibria (they necessarily come in permuted pairs) imposes a slope condition on the best-reply function of the indicative player in a two-player version of the game, where the strategies of all other players are viewed as exogenous parameters. This slope condition provides a simple test to reject the possibility of asymmetric equilibria.

3.1 Multiple symmetric equilibria

To verify whether or not there are multiple symmetric equilibria, the Index theorem, applied to the SOFA, yields a powerful tool, especially because the SOFA version of the Index theorem may still be applicable even if the unrestricted version is not. Moreover, the SOFA index results indicate how to deal with cases where Index theory cannot be applied, e.g., because boundary conditions fail.[10] Finally, the SOFA index results allow for further exploration, e.g., about the relationship between stability and uniqueness of symmetric equilibria (see Hefti (2016a,b)).
Let $Cr^s = \{x_1 \in S : \nabla\tilde{\Pi}(x_1) = 0\}$ denote the set of critical points, where $\nabla\tilde{\Pi}(x_1)$ is the gradient of $\tilde{\Pi}(x_1, \bar{x})$ with respect to x_1, evaluated at $\bar{x} = x_1$. Further $\nabla\tilde{\Pi}(x_1) : S \to \mathbb{R}^k$, $x_1 \mapsto \nabla\tilde{\Pi}(x_1)$, is a C^1-vector field with corresponding $k \times k$ Jacobian $\tilde{J}(x_1)$. The index $I(x_1)$ of a zero of $\nabla\tilde{\Pi}$ is defined as $I(x_1) = +1$ if $Det(-\tilde{J}(x_1)) > 0$ and $I(x_1) = -1$ if $Det(-\tilde{J}(x_1)) < 0$. I call a symmetric game an *Index game* if i) $\nabla\tilde{\Pi}$ has only regular zeroes[11] and ii) $\nabla\tilde{\Pi}$ points inwards at the boundary of S.

[10] See section 4.1 for a one-dimensional, and section 4.2 for a two-dimensional application, where the symmetric Index theorem boundary conditions naturally fail.

[11] $Det(\tilde{J}(x_1)) \neq 0$ whenever x_1 is a zero of $\nabla\tilde{\Pi}$.

Theorem 1 *There is an odd number of symmetric equilibria in an Index game, and only interior symmetric equilibria exist. Moreover, the following four statements are equivalent. i)* $Det(-\tilde{J}(x_1)) > 0$ *if* $x_1 \in Cr^s$, *ii)* $Det\,(I - \partial\tilde{\varphi}(x_1)) > 0$ *if* $x_1 \in Cr^s$, *iii)* $\prod_{i=1}^{k}(1-\lambda_i) > 0$ *if* $x_1 \in Cr^s$, *where* λ_i *is an eigenvalue of* $\partial\tilde{\varphi}(x_1)$, *iv) there is only one symmetric equilibrium.*

Proof: Oddness, $x_1 \in Int(S)$ and i) $\Leftrightarrow$ iv) follow from the Index theorem (see, e.g., Vives (1999)). Decompose $\tilde{J}$ as $\tilde{J} = A + B$, where

$$A = \frac{\partial^2\tilde{\Pi}(x_1,\bar{x})}{\partial x_1 \partial x_1} \qquad B = \frac{\partial^2\tilde{\Pi}(x_1,\bar{x})}{\partial x_1 \partial \bar{x}},$$

both evaluated at $\bar{x} = x_1$. The Implicit Function Theorem (IFT) then asserts that $\partial\tilde{\varphi} = -A^{-1}B$, which shows that $Det(-\tilde{J}(x_1)) > 0 \Leftrightarrow Det\,(I - \partial\tilde{\varphi}(x_1)) > 0$, hence i) $\Leftrightarrow$ ii). Finally, iii) $\Leftrightarrow$ ii) because for any eigenvalue λ of $\partial\tilde{\varphi}(x_1)$ the number $(1-\lambda)$ is an eigenvalue of $I - \partial\tilde{\varphi}(x_1)$. ■

From the different conditions in theorem 1 several new conditions asserting that only one symmetric equilibrium exists can be derived (see section A.2 of the online appendix). Note that the dimensionality of the objects involved in theorem 1 is k rather than Nk. Moreover, regularity and the symmetric boundary conditions invoked in the definition of a symmetric Index game are weaker than the corresponding regularity and boundary conditions of the unrestricted vector field induced by ∇F. Thus, the Index conditions may be satisfied under $\nabla\tilde{\Pi}$ even if they are violated under ∇F. For example, the conventional Index theorem cannot be applied to the two-player game with FOC's $\nabla\Pi^i = -x_i - x_j$ and $S = [-1, 1]$, as there are *no* regular points. But as $\nabla\tilde{\Pi}(x_1) = -2x_1$ and $\tilde{J}(x_1) = -2$ the symmetric Index theorem (theorem 1) immediately tells us that $x = 0$ is the only symmetric equilibrium. If $k = 1$ then, by ii) of theorem 1, there is exactly one symmetric equilibrium if and only if $\tilde{\varphi}(\bar{x})$ crosses the 45°-degree line from above. This simple geometric insight provides a constructive way of showing that only one symmetric equilibrium exists even if theorem 1 cannot be applied (see sections 4.1 and 4.2 for examples).

3.2 Asymmetric equilibria

If $(x_1, ..., x_N)$ is an asymmetric equilibrium, then a permutation $\left(x_{\sigma(1)}, ..., x_{\sigma(N)}\right)$ gives a similar asymmetric equilibrium.[12] The main result of this section exploits this symmetry property. I first consider the case of a one-dimensional game, derive a sufficient condition (Theorem 2) for the inexistence of asymmetric equilibria under weak assumptions, and graphically illustrate the main idea behind the theorem. Theorem 3 generalizes this idea to the higher-dimensional case for everywhere differentiable best-reply functions. Theorem 4 further extends the result to best-replies with non-differentiabilities for $k = 2$, and corollary 2 highlights some central implications of theorem 4 for the nature of asymmetric equilibria in two-dimensional games.

One-dimensional case Since asymmetric equilibria come in permutations of each other, we can restrict attention to the strategies of two (arbitrary) players, and treat the strategies of all other players as an exogenous parameter vector.[13] Let $\varphi(x_2; X) \equiv \varphi^1(x_2; X)$, where $X \equiv (x_3, ..., x_N) \in S^{N-2}$. The derivative of $\varphi(\cdot; X)$ with respect to x_2 is denoted by $\partial\varphi(x_2; X)$. For given $X \in S^{N-2}$ let

$$T \equiv \{x_2 \in S : \varphi\,(x_2; X) \in Int(S), \varphi\,(x_2; X) \text{ not differentiable in } x_2\}\,.$$

We concentrate on one-dimensional symmetric games where every $x_2 \in T$ is locally isolated, which trivially includes $T = \emptyset$. Note that if Π satisfies the assumptions of section 2, and additionally it is known that $\varphi\left(S^{N-1}\right) \subset Int(S)$, then $T = \emptyset$.

Theorem 2 *Suppose that a one-dimensional symmetric game satisfies* $\varphi(x_{-1}) \in C\left(S^{N-1}, S\right)$ *and every* $x_2 \in T$ *is locally isolated for any given* $X \in S^{N-2}$. *This game has no asymmetric equilibria if*

$$x_2 \in Int(S)\backslash T, \varphi(x_2; X) \in Int(S) \Rightarrow \partial\varphi(x_2; X) > -1 \tag{1}$$

[12] The set of asymmetric equilibria which are permutations of each other forms an equivalence class within the set of all asymmetric equilibria.

[13] Note that it makes little sense to use the SOFA in the context of asymmetric equilibria because, by its construction, the SOFA could at best exclude asymmetric equilibria of the type where player one adopts strategy A and *all* other players adopt strategy $B \neq A$.

Proof: Appendix (6.1)

The geometric intuition behind theorem 2 can be depicted graphically for the case where $N = 2$ and $\varphi^{-1}(S) \subset Int(S)$. In its essence, it is an application of the Mean Value Theorem, and the idea is illustrated in figure 1. Suppose that the point $A = (x_1^a, x_2^a)$ corresponds to an asym-

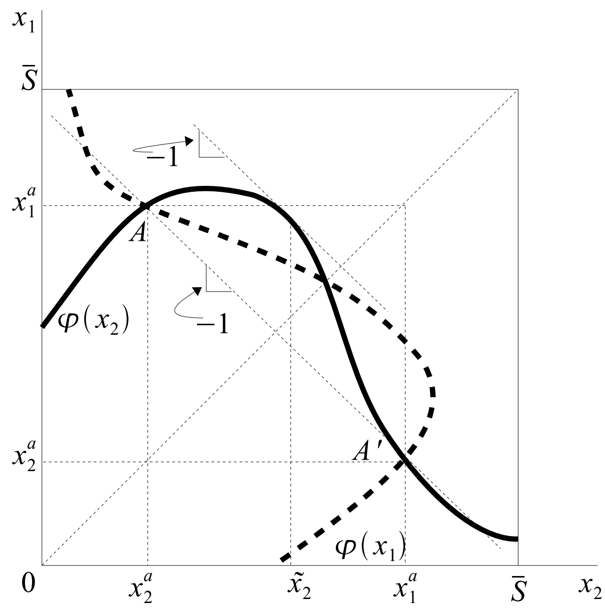


Figure 1: Theorem 2

metric equilibrium. By symmetry, its reflection, the point $A' = (x_2^a, x_1^a)$, also is an asymmetric equilibrium. Hence the line that connects A and A' must have a slope of -1. As $\varphi(x_2)$ remains in $Int(S)$, φ is differentiable on $Int(S)$. According to the Mean Value Theorem there is a point $\tilde{x}_2 \in (x_2^a, x_1^a)$ with $\partial\varphi(\tilde{x}_2) = -1$. Hence if in such a game $\partial\varphi(\tilde{x}_2) > -1$ for all $x_2 \in Int(S)$, then there cannot be any asymmetric equilibria.[14]

Theorem 2 applies, but is not limited, to games that satisfy the assumptions of section 2. For example, if condition (1) holds for a game with an only piecewise differentiable best-reply function, then this game has no asymmetric equilibria. Further, it should be noted that (1) also rules out asymmetric boundary equilibria, despite that we only need to evaluate the slope

[14]The general proof (see appendix) is complicated by the fact that $\varphi(x_2)$ is allowed to be on the boundary or not differentiable everywhere, which requires to extend the Mean Value Theorem appropriately (see lemmata 1 and 2, section 6.1). The "$>$" in (1) (rather than "$\neq$") comes from the fact that I allow for non-differentiabilities and boundary equilibria (see the proof of theorem 2 for details).

of φ at interior points. Moreover, the theorem imposes no restrictions on the shape of the best-reply function (up to condition (1)). For example, theorem 2 can be applied to games with non-monotonic behavior (such as contests, see section 4.1). Condition (1) is comparably simple, because it only needs information about the behavior of the reply-function in a two-player game (for various given values of X), while, e.g., the Index theorem would require to evaluate a $N \times N$-matrix. Finally, it should be mentioned that if additional information about $\varphi(x_{-1})$ is available, this can further restrict the relevant x_2-range in theorem 2. For example, it suffices to verify condition (1) only at points $x_2 \in Int(S) \cap \varphi(S^{N-1})$.

In applications, one typically calculates the slope of a best-reply function using the IFT. In particular, if $\varphi(x_2; X) \in Int(S)$ and $x_2 \in Int(S) \backslash T$, then

$$\partial\varphi(x_2; X) = -\frac{\Pi_{12}(x_1, x_2; X)}{\Pi_{11}(x_1, x_2; X)}, \qquad X \in S^{N-2}.$$

Therefore, (1) can be expressed in terms of the second partial derivatives of Π:

Corollary 1 *If in a one-dimensional symmetric game for all $x_1, x_2 \in Int(S)$ and any given $X \in S^{N-2}$ the condition*

$$\Pi_1(x_1, x_2; X) = 0, x_2 \notin T \quad \Rightarrow \quad \Pi_{11}(x_1, x_2; X) < \Pi_{12}(x_1, x_2; X) \tag{2}$$

is satisfied, then no asymmetric equilibria exist.

Higher-dimensional case I now generalize theorem 2 to the higher-dimensional case. For $k \geq 1$ the best-reply generally is a vector-valued function $\varphi(x_2; X) = (\varphi_1(\cdot), ..., \varphi_k(\cdot))$.

Theorem 3 *Let $k \geq 1$ and suppose that $\varphi(x_{-1})$ is everywhere differentiable. If for any k points $x_2^1, ..., x_2^k \in S$ and any given $X \in S^{N-2}$ the $k \times k$ matrix*

$$A = \begin{pmatrix} 1 + \frac{\partial\varphi_1(x_2^1;X)}{\partial x_{21}} & \frac{\partial\varphi_1(x_2^1;X)}{\partial x_{22}} & \cdots & \frac{\partial\varphi_1(x_2^1;X)}{\partial x_{2k}} \\ \frac{\partial\varphi_2(x_2^2;X)}{\partial x_{21}} & 1 + \frac{\partial\varphi_2(x_2^2;X)}{\partial x_{22}} & \cdots & \frac{\partial\varphi_2(x_2^2;X)}{\partial x_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\varphi_k(x_2^k;X)}{\partial x_{21}} & \cdots & \cdots & 1 + \frac{\partial\varphi_k(x_2^k;X)}{\partial x_{2k}} \end{pmatrix} \tag{3}$$

is non-singular ($Det(A) \neq 0$), then no asymmetric equilibria exist.

Proof: Appendix (6.2)

It is easy to see that for $k = 1$ and with the differentiability assumption, condition (1) implies the determinant condition (3) in theorem 3.

To obtain a better understanding of theorem 3 if $k > 1$, I discuss the case $k = 2$ in detail. Additionally, I show that the differentiability assumption can be weakened in the spirit of theorem 2. Let

$$(\alpha, \beta, \gamma, \delta) \equiv \left(\frac{\partial \varphi_1}{\partial x_{21}}, \frac{\partial \varphi_1}{\partial x_{22}}, \frac{\partial \varphi_2}{\partial x_{21}}, \frac{\partial \varphi_2}{\partial x_{22}} \right),$$

where all partial derivatives are evaluated at $(x_2; X)$.

Theorem 4 *Let $k = 2$ and suppose that $\varphi \in C\left(S^{N-1,}S\right)$, $\varphi\left(S^{N-1}\right) \subset Int(S)$, and $\varphi(x_{-1})$ is differentiable, except possibly for a set of isolated points. If for all $x_2, x_2' \in S$ and any given $X \in S^{N-2}$ the condition*

$$\alpha(x_2), \delta(x_2') > -1 \qquad (1 + \alpha(x_2))(1 + \delta(x_2')) > \beta(x_2)\gamma(x_2') \tag{4}$$

holds, then no asymmetric equilibria exist.

Proof: Appendix (6.3)

Comparing theorems 3 and 4 one easily sees that the second inequality of (4) implies the determinant condition in theorem 3. The additional slope conditions (first two inequalities in (4)) follow from the weakening of the differentiability assumption, similar to (1) in the one-dimensional case.

Condition (4) sheds light on the nature of asymmetric equilibria in symmetric two-dimensional games. The first two inequalties in (4) state that $\varphi_i(x_2; X)$ may not decrease too quickly in the i-th component strategy of player two, which is not surprising in light of (1), so suppose that $\alpha, \delta > -1$ everywhere. Then, the last inequality in (4) reveals that the cross-partial derivatives

β and γ crucially influence whether and what type of asymmetric equilibria may occur in the game. Suppose that $x^a = (x_1^a, x_2^a, ..., x_N^a)$ is an asymmetric equilibrium. I refer to x^a as a strictly ordered equilibrium if $x_g^a > x_h^a$, $g \neq h$, for any pair of strategies in x_a. I call an equilibrium with $x_{gi}^a > x_{hi}^a$ but $x_{gi'}^a < x_{hi'}^a$, $i, i' \in \{1, 2\}$ with $i \neq i'$ strictly unordered.

Corollary 2 *The following facts are satisfied under the presumptions of theorem 4:*

i) *If $\beta(x_2) \geq 0, \alpha(x_2) > -1$ or $\gamma(x_2) \geq 0, \delta(x_2) > -1$, for any $x_2 \in S$ and any $X \in S^{N-2}$, then there cannot be any strictly ordered equilibria.*

ii) *If $\beta(x_2) \leq 0, \alpha(x_2) > -1$ or $\gamma(x_2) \leq 0, \delta(x_2) > -1$, for any $x_2 \in S$ and any $X \in S^{N-2}$, then there cannot be any strictly unordered equilibria.*

iii) *Suppose that $\alpha(x_2), \delta(x_2) > -1$ for all $x_2 \in S$ and any $X \in S^{N-2}$. If additionally $\beta(x_2) \geq 0$ or $\gamma(x_2) \geq 0$ for any $x_2 \in S$ and any $X \in S^{N-2}$, then there cannot be any asymmetric equilibria with $x_g^a \geq x_h^a$. If instead $\beta(x_2) \leq 0$ or $\gamma(x_2) \leq 0$ there cannot be any asymmetric equilibria with $x_{gi}^a > x_{hi}^a$ but $x_{gi'} \leq x_{hi'}^a$.*

Proof: Appendix (6.4)

Corollary 2, iii), implies that two-dimensional games with weakly increasing best-replies can only have strictly unordered asymmetric equilibria, while games with weakly decreasing best-replies (and $\alpha, \delta > -1$) can only have strictly ordered asymmetric equilibria. Finally, a game with $\alpha, \delta > -1$ and *both* partially increasing and decreasing replies (e.g., $\beta \geq 0$ and $\gamma \leq 0$) can *never* have any asymmetric equilibria.

Theorem 4 is useful for applications because the IFT allows to express (4) in terms of the second partial derivatives of Π, similarly to corollary 1 in the one-dimensional case. If $\varphi(x_2; X) \in Int(S)$ then $\partial\varphi(x_2; X) = -H^{-1}B$, where

$$H = \frac{\partial^2 \Pi(x_1, x_2; X)}{\partial x_1 \partial x_1} \qquad B = \frac{\partial^2 \Pi(x_1, x_2; X)}{\partial x_1 \partial x_2}.$$

Moreover, it is possible to adapt theorem 4 to the case where $\varphi(x_{-1}) \in \partial S$ may occur, which is shown in the online appendix (section A.3). While the IFT remains the essential tool to calculate the slopes in applications with boundary solutions, it must be applied to an extended system of equations.[15]

From condition (4) one can derive further conditions to rule out asymmetric equilibria that may be useful for specific games. A compact way of expressing (4) is to say that, for any given $X \in S^{N-2}$, the matrix

$$\begin{pmatrix} 1+\alpha(x_2) & \beta(x_2) \\ \gamma(x_2{}') & 1+\delta(x_2{}') \end{pmatrix} = I + \underbrace{\begin{pmatrix} \partial\varphi_1(x_2) \\ \partial\varphi_2(x_2{}') \end{pmatrix}}_{\equiv A(x_2, x_2{}')} \tag{5}$$

has only positive principal minors for $x_2, x_2' \in S$. If for any (x_2, x_2') we have $\alpha(x_2), \delta(x_2') > -1$ and there is a matrix norm $\|\cdot\|$ such that $\|A(x_2, x_2')\| < 1$ (the spectral radius of $A(x_2, x_2')$ is less than one), then there cannot be any asymmetric equilibria. For example, if for any $x_1, x_2 \in S$ and any given $X \in S^{N-2}$ the local diagonal dominance condition

$$\Pi_1\left(x_1, x_2; X\right) = 0 \text{ or } \Pi_2\left(x_1, x_2; X\right) = 0 \Rightarrow |\Pi_{ii}| > \sum_{j \neq i, j \leq 4} |\Pi_{ij}| \qquad i = 1, 2$$

holds, then the game cannot have any asymmetric equilibria.

3.3 Summary

If the conditions of theorem 1 and of theorem 2 (for $k = 1$) or theorem 3 (for $k \geq 1$) are satisfied, then the game has a unique equilibrium, the symmetric equilibrium. Compared to the necessity of evaluating the determinant of a $Nk \times Nk$ matrix as required by the univalence or Index theorem, the separation approach enables us to reduce the dimensionality of the problem from Nk to k, and allows us to learn more about the nature of equilibria in particular games. This is generally not possible with standard approaches to uniqueness. For example, even if ∇F satisfies the Index conditions, and critical symmetric points have an algebraic index sum of $+1$, we may not conclude that there are no asymmetric equilibria, because there still could be an even number of asymmetric equilibria. Similarly, an index sum of -1 from critical

[15] Despite this complication, the central insights about the possibility of asymmetric equilibria as conveyed by theorem 4 and corollary 2 remain valid.

symmetric points does not necessarily imply the existence of multiple symmetric equilibria.[16] The separation approach provides use with a manageable set of tools that can be applied to non-Index games, which is demonstrated by several examples in the next section. In particular, theorems 2 and 4 can be used in non-Index games to rule out asymmetric equilibria, and even if $\nabla\tilde{\Pi}$ does not satisfy the Index conditions, we may still use the SOFA to rule out multiple symmetric equilibria.

4 Applications

The main objective of this section is to demonstrate the usefulness of the separation approach in various well-known examples. In section 4.1 I consider the important class of one-dimensional games with sum-aggregative payoffs, such as Cournot competition or Contests. Section 4.2 applies the separation approach to a two-dimensional price-information game. Section 4.3 reveals that there is a strong link between the inexistence of asymmetric equilibria in a symmetric game and certain properties of the equilibrium set of asymmetric variations of the symmetric game. This yields an additional reason for why having a specific set of tools to analyze the equilibrium set of symmetric games is useful.

4.1 One-dimensional sum-aggregative games

Several interesting games have the property that the strategies enter the payoff functions as a sum.[17] Payoff functions of such sum-aggregative games can be represented as $\Pi\left(x_g, x_{-g}\right) = \hat{\Pi}\left(x_g, Q\right)$, with $Q = \sum_{j=1}^{N} x_j$.[18]

Proposition 2 *Consider a sum-aggregative symmetric one-dimensional game.*

[16] See section A.1, online appendix, for what possibly could be inferred from Index theory.

[17] See, e.g., Corchon (1994), Cornes and Hartley (2005) or Jensen (2010) and the references therein. Recently, Martimort and Stole (2012) develop a method to study equilibrium aggregates, which also yields a proof of equilibrium existence.

[18] Note that, e.g., a game with payoff $\Pi(x_g, \sum f(x_j))$, where $f \in C^2(S, \mathbb{R})$ is strictly increasing, can be equivalently represented as a sum-aggregative game using the change of variable $e_j = f(x_j)$.

i) If for $(x_1, Q) \in Int(S) \times (0, \bar{S}N)$ *condition*

$$\hat{\Pi}_1(x_1, Q) + \hat{\Pi}_2(x_1, Q) = 0 \quad \Rightarrow \quad \hat{\Pi}_{11}(x_1, Q) + \hat{\Pi}_{12}(x_1, Q) < 0 \tag{6}$$

is satisfied, then no asymmetric equilibrium exists.

ii) A sum-aggregative symmetric Index game has only one symmetric equilibrium iff the following condition holds on Cr^s*:*

$$\hat{\Pi}_{11}(x_1, Nx_1) + (N+1)\hat{\Pi}_{12}(x_1, Nx_1) + N\hat{\Pi}_{22}(x_1, Nx_1) < 0 \tag{7}$$

Proof: (i) Use (2) of corollary 1 to obtain (6). (ii) Apply i) of theorem 1 to obtain (7). ■

Example 1: Cournot The symmetric Cournot model has $\hat{\Pi}(x_1, Q) = P(Q)x_1 - c(x_1)$, where Q is the aggregate quantity supplied, $P(Q)$ is inverse market demand and $c(x_1)$ are quantity costs. Presuming that the symmetric Index conditions are satisfied, there is exactly one symmetric Cournot equilibrium iff

$$P(Nx_1) + P'(Nx_1)x_1 - c'(x_1) = 0 \quad \Rightarrow \quad N\left(P'(Nx_1) + P''(Nx_1)x_1\right) < c''(x_1) - P'(Nx_1) \tag{8}$$

Moreover, from (6) we deduce that if $P' < c''$ is satisfied (whenever $P(Q) - c'(x_1) + P'(Q)x_1 = 0$), then the Cournot game has no asymmetric equilibria. Kolstad and Mathiesen (1987) derive general conditions of uniqueness for the (non-symmetric) Cournot game, imposing $P' < c''$ as an exogenous assumption. We learn from the separation approach that exactly this assumption rules out the possibility of asymmetric equilibria - and *therefore* is a natural precondition for uniqueness. It follows that non-uniqueness of equilibria in the symmetric Cournot model mainly arises from the possibility of multiple symmetric equilibria, and not from asymmetric equilibria. Notably, $P' < c''$ also rules out the possibility of asymmetric equilibria even if $\varphi(x_{-1}) \in \partial S$ or $\varphi(x_{-1})$ has kinks,[19] which is not unrealistic for a Cournot model with heterogeneous consumers. Similarly, the symmetric Index theorem can be applied to rule out multiple symmetric equilibria

[19] In such cases the Index theorem obviously is not applicable.

even if $\tilde{\varphi}$ has kinks, provided that the Index conditions are satisfied (i.e., symmetric kinks are not symmetric equilibria). If the uniqueness-condition[20] of Kolstad and Mathiesen (1987) is evaluated for the symmetric case under the assumption that $P' < c''$ we obtain *exactly* condition (8) ruling out multiple symmetric equilibria - which besides the simplicity of obtaining the result - nicely illustrates the generality of the separation approach.

Example 2: Contests Consider a sum-aggregative contest

$$\Pi = p\left(g(y_1), \sum_{j=1}^{N} g(y_j)\right)V - h(y_1),$$

where $V, g' > 0$, and $p(\cdot) \in [0,1]$ is a contest success function (Konrad, 2009). Note that, by a change of variables, such a contest may be represented as $\hat{\Pi} = p(x_1, Q)V - c(x_1)$, where $c(x_1) = h\left(g^{-1}(x_1)\right) \in C^2$. Using (6), (7) and assuming that the symmetric Index conditions are satisfied, we may conclude that such a contest has a unique symmetric equilibrium if at corresponding critical points:

$$(p_{11}\left(x_1, Q\right) + p_{12}(x_1, Q))V - c''(x_1) < 0 \tag{6'}$$

$$\left(p_{11}(x_1, Nx_1) + (N+1)p_{12}(x_1, Nx_1) + Np_{22}(x_1, Nx_1)\right)V - c''(x_1) < 0 \tag{7'}$$

Suppose that $c(0) = c'(0) = 0$ and

$$p(x_1, Q) = \begin{cases} \frac{1}{N+r} & x_1 = ... = x_N = 0 \\ \frac{1}{1+r} & x_1 > 0, x_2 = ... = x_N = 0 \\ f\left(\frac{x_1}{Q+r}\right) & \text{else} \end{cases}$$

where $r \geq 0$ is a noise parameter, $f \in C^2$ is strictly increasing, concave and $f'(0) > 0$. The best-reply $\varphi(x_{-1}) \in (0, \bar{S}]$ is continuous, and differentiable if $\varphi(x_{-1}) \in Int(S)$ whenever $x_2 > 0$. It can be verified that (6') is satisfied, meaning that there cannot be any asymmetric contest equilibria. Turning to symmetric equilibria, we note that $x_1 = 0$ can never be a best-reply

[20] Corollary 3.1, p. 687

to any $x_{-1} \in S^{N-1}$. While we cannot use the (symmetric) Index theorem because Π is not differentiable at the origin, it is straightforward to verify that this example satisfies (7') for respective interior points. As (7') implies that $\tilde{\varphi}(\bar{x})$ can cross the 45°-line at most once on $(0, \bar{S}]$, we conclude that there is a unique symmetric equilibrium $x_1^* \in (0, \bar{S}]$. If $f(z) = z$ then the previous example collapses to the often invoked Tullock success function. Hence uniqueness of equilibrium in Tullock contests (with noise) is easily and directly established by the separation approach.[21]

Example 3: Homogeneous revenue The Tullock contest with $r = 0$ is an important example, where revenues are homogeneous functions. Applying the separation approach to sum-aggregative games with general homogeneous revenues and strictly convex costs reveals that such games *naturally* have only one symmetric equilibrium, which also very likely is the unique equilibrium of the game. To see this, consider $\Pi(x) = \pi(x_1, \sum x_j) - c(x_1)$, where $\pi(x_1, \sum x_j)$ is homogeneous of degree $z < 1$ in $(x_1, ..., x_N)$, or equivalently $\pi(x_1, Q)$ is z-homogeneous in (x_1, Q).

Proposition 3 *Suppose that $\pi(x_1, Q)$ is homogeneous of degree $z < 1$ in (x_1, Q) and $c', c'' > 0$. Then there is only one symmetric equilibrium. If additionally $\pi_1 \geq 0$ and $\pi_{11} \leq 0$ for $x_1 > 0$, the symmetric equilibrium is unique.*

Proof: We start with the second claim. As $\pi_1(x_1, Q) \geq 0$ is $z - 1$-homogeneous the Euler-theorem and the sum-aggregative structure imply that $\pi_{11} + \frac{Q}{x_1}\pi_{12} \leq 0$ for $x_1 > 0$, which if $\pi_{11} \leq 0$ necessarily implies that $\pi_{11} + \pi_{12} \leq 0$. As $c''(x_1) > 0$ for $x_1 > 0$ there cannot be any asymmetric equilibria by (6). Turning to symmetric equilibria, as

$$\frac{\partial \Pi(x_1, \sum x_j)}{\partial x_1} \Big|_{x_j = x_1}$$

is $(z-1)$-homogeneous in x_1 we must have that $\nabla \tilde{\Pi}(x_1) = \omega x_1^{z-1} - c'(x_1)$, where $\omega > 0$ is a constant. Hence $\tilde{J}(x_1) = \omega(z-1)x_1^{z-2} - c''(x_1) < 0$ whenever $x_1 > 0$, which implies that $\tilde{\varphi}(\bar{x})$

[21] Cornes and Hartley (2005) prove uniqueness of equilibrium in linear Tullock contests (without noise) with help of share functions. The separation approach yields the same conclusion, but in a very simple way. Moreover, it follows that uniqueness of equilibrium in such contests is robust to noise or to a "concavication" of the success function.

can intersect the 45°-line at most once. Thus there cannot be multiple symmetric equilibria. ■

4.2 A two-dimensional Information-Pricing game

In this section I apply the separation approach to the two-dimensional information-pricing game as introduced by Grossman and Shapiro (1984). Each of two firms chooses its price p and the fraction a of consumers to be made aware of its product, taking $(\bar{p}, \bar{a})$ of its opponent as given. There is a measure of δ consumers, ex-ante unaware of both firms. Information (advertisement) is distributed randomly over the population, firms cannot discriminate between consumers, and products are imperfect substitutes. A firm's demand from consumers not aware of the other firm is $x(p)$, and $x(p, \bar{p})$ for consumers that receive ads from both firms. Assuming constant unit costs of production, the firm's expected profit is

$$\Pi(p, a) = a \left[(1 - \bar{a}) \underbrace{(p - c)x(p)}_{\equiv V(p)} + \bar{a} \underbrace{(p - c)x(p, \bar{p})}_{\equiv V(p, \bar{p})} \right] \delta - C(a) \equiv aV(p, \bar{p}, \bar{a})\,\delta - C(a) \qquad (9)$$

where $C(a)$ are information costs. I assume that $x(p) \geq x(p, \bar{p})$, i.e., for given prices $p, \bar{p}$ firm demand is never lower if a consumer is not aware of the competitor. Similarly, demand reacts more sensitively towards an unilateral price change in case of perfectly informed consumers $(x_p(p, \bar{p}) \leq x'(p))$, and (marginal) demand never decreases in the opponents price $(x_{\bar{p}}, x_{p,\bar{p}} \geq 0)$. Intuitively, these facts can be justified under free-trade, as perfectly informed consumers have two outside options (not consume or consume at the competitor's location) whereas unilaterally informed consumers just have one (not consume).

To be precise, the following formal assumptions are imposed: The function $V(p, \bar{p}, \bar{a}) \in C^2(S^2, \mathbb{R})$, where $S = [c, \hat{p}] \times [0, 1]$, is strongly quasiconcave in p, $V_{\bar{a}}, V_{p\bar{a}} \leq 0$, $V_{\bar{p}}, V_{p\bar{p}} \geq 0$ and $\hat{p} > c$ is the monopoly price. The cost function satisfies $C(0) = C'(0) = 0$ and $C'(a), C''(a) > 0$ for $a > 0$. Moreover, it is assumed that $(p, a) = \varphi(\bar{p}, \bar{a}) \in Int(S)$ for any $(\bar{p}, \bar{a}) \in [c, \hat{p}] \times (0, 1]$, and that there exists $p \in [c, \hat{p}]$: $V(p, c, 1) > 0$. The last assumption means that even under perfect information and marginal-cost pricing of the opponent, the firm can retain a strictly positive market demand for a price slightly above marginal costs, which is a typical feature of competition with imperfect substitutes.[22] A simple example for V is linear demand, e.g., derived from quadratic utility (LaFrance, 1985), where $x(p) = 1 - p$ and

$$x(p, \bar{p}) = \frac{1 - p + \gamma(\bar{p} - 1)}{1 - \gamma^2}.$$

The parameter $\gamma \in [0, 1/2]$ controls the degree of substitutability, and I set $c = 0$ for simplicity (hence $S_p = [0, 1/2]$). It is easy to see that this example verifies the above assumptions on V.

Because Π is continuous, V is strongly quasiconcave in p and $C'' > 0$, the best-reply $\varphi = (p, a)$ is a continuous function of $(\bar{p}, \bar{a})$, and it follows that at least one symmetric equilibrium exists.[23] Despite that (9) may look innocent, investigating the set of equilibria is not trivial. For example, the Index theorem cannot be used because the boundary conditions are naturally violated in this model.[24] Moreover, even if it were applicable, we would have to evaluate the determinant of a largely abstract 4×4-matrix.

I now use the separation approach to investigate the equilibrium set of this two-dimensional game. I first analyze the scope of asymmetric equilibria, and the turn to the possibility of multiple symmetric equilibria.

Asymmetric equilibria The assumptions made imply that $p'(\bar{p}) \geq 0$, $p'(\bar{a}) \leq 0$ and $a'(\bar{p}) \geq 0$.[25] Hence, by corollary 2, ii), we conclude that if asymmetric equilibria exist, these equilibria cannot be strictly unordered. Moreover, if additionally $a'(\bar{a}) > -1$ it follows from corollary 2, iii), that there cannot by *any* asymmetric equilibria, showing that $a'(\bar{a}) > -1$ is the crucial requirement to rule out asymmetric equilibria in this model. The condition $a'(\bar{a}) > -1$ holds if for $a, \bar{a} \in (0, 1)$ we have that:[26]

$$\frac{V(p) - V(p, \bar{p})}{(1 - \bar{a})V(p) + \bar{a}V(p, \bar{p})} < \frac{C''(a)}{C'(a)} \tag{10}$$

[22] Because $V(p, \bar{p}, \bar{a}) > 0$ for some $p > c$ is always feasible, $a = 0$ cannot be a part of a firm's best reply.
[23] See the proof of proposition 1.
[24] For example, $\Pi_a(c, 0) = V(c, \bar{p}, \bar{a})\delta - C'(0) = 0$, i.e., ∇F does not point inwards at $(c, 0, \bar{p}, \bar{a}) \in \partial S^2$.
[25] These are standard IFT results.
[26] Apply the IFT to the FOC pertaining to (9).

The LHS of (10) is maximal (for given p) if $(\bar{p}, \bar{a}) = (c, 1)$. Hence if $V(p)/V(p, c) < C''(a)/C'(a) + 1$, then (10) is satisfied. As illustrated below, we may possibly exploit the FOC pertaining to (9) to obtain a better estimate.

So far, the analysis has revealed two things about the scope of asymmetric equilibria. First, the fact that only (weakly) ordered asymmetric equilibria may exist (if any at all) is independent of scale effects. This can be seen as none of the above derivatives depends on the market size parameter δ, nor on unit production costs c, nor on multiplicative information cost parameters (if $C(a) = \theta c(a)$ then $\theta > 0$ plays no role). Second, (10) shows that asymmetric "specialization" equilibria could exist only if marginal costs are highly inelastic or monopoly rents exceed the duopoly rents by a relatively large amount (e.g., because products are strong substitutes). In such an equilibrium one firm could specialize on advertising (high (p, a)), earning quasi-monopoly rents from unilaterally informed consumers but incurring high advertising costs, whereas the other firm specializes in competition (low (p, a)), and thereby wins the fully informed consumers, but faces only little overall demand because of a small advertising campaign.

What is the scope for such asymmetric equilibria in our parametric example? Exploiting the linearity of the problem, it can be shown that

$$p(\bar{p}, \bar{a}) = \frac{1 - \gamma^2 - \gamma(1 - \bar{p} - \gamma)\bar{a}}{2 - 2\gamma^2(1 - \bar{a})} \in (\frac{1 - \gamma}{2}, \frac{1}{2})$$

for $\gamma, \bar{p} \in [0, 1/2]$ and $\bar{a} \in (0, 1]$. Using $\bar{p} = (1 - \gamma)/2$ and $\bar{a} = 1$ in the LHS of (10), this reveals that the LHS of (10) is smaller than $\gamma/(1 - \gamma - \gamma^2) \leq 2$. Hence, if $C''(a)/C'(a) \geq 2$ we can conclude that no asymmetric equilibrium exists. More specifically, for $C(a) = \theta a^\eta$, $\eta \geq 2$, no asymmetric equilibrium exists if competition is not too intense (if $\gamma \leq \sqrt{2} - 1$) or $\eta \geq 3$. In their analysis, Grossman and Shapiro (1984) use the particular advertising technology (CRIR) with $C(a) = Ln(1 - a)/Ln(1 - r)$, $r \in (0, 1)$. This cost function implies that $C''(a)/C'(a) = 1/1 - a \geq 1$. Hence if $\gamma \leq \sqrt{2} - 1$ and advertising technology follows the CRIR technology, there cannot be any asymmetric equilibria.

Symmetric equilibria Turning to symmetric equilibria, we calculate

$$\nabla\tilde{\Pi}(p,a) = \begin{pmatrix} aV_1(p,p,a)\delta \\ V(p,p,a)\delta - C'(a) \end{pmatrix} \tag{11}$$

Equation (11) shows that the Index theorem is not applicable, even if we restrict attention to symmetric equilibria, because $\nabla\tilde{\Pi}$ vanishes, e.g., at the corner point $(p,a) = (c,0)$. Whereas $(c,0)$ is a zero of (11), i.e., an equilibrium candidate, it obviously cannot constitute a symmetric equilibrium. While we cannot rely on the Index theorem to discuss the scope of multiple symmetric equilibria, (11) provides us with a guideline to prove uniqueness in a constructive way.

If $\nabla\tilde{\Pi}(p,a) = 0$ at an interior point (p,a), we have $Det(\tilde{J}(p,a)) > 0$ iff $V_{1p}(V_a\delta - C'') - V_{1a}V_p\delta > 0$. If the Index theorem were applicable, we could now conclude that if i) $V_1(p,p,a) = 0 \Rightarrow V_{1p}(p,p,a) < 0$ and ii) $V(p,p,a)\delta - C'(a) = 0 \Rightarrow V_p(p,p,a) > 0$, then there is exactly one symmetric equilibrium (p,a). I now show that these conditions imply this result without invoking the Index theorem. To see this, consider the pure symmetric pricing game, where each firm solves $\max_{p_i \in [c,\hat{p}]} aV(p_i,p_j,a)\delta$ for given $a > 0$. Then i) assures the existence of a single symmetric equilibrium $p = p(a) \in (c,\hat{p}]$, because $\tilde{p}(\bar{p};a)$ can reach the 45°-line just once.[27] Moreover, $p(0) = \hat{p}$, p is continuous in a and if $p(a) \in (c,\hat{p})$, then $p'(a) = -V_{1a}/V_{1p} \leq 0$. Next, consider the pure symmetric information game, where each firm solves $\max_{a_i \in [0,1]} a_iV(p,p,a_j)\delta - C(a_i)$. Then ii) implies the existence of a single symmetric equilibrium $a = a(p) \in [0,1]$, where $a(c) = 0$ and a is continuous in p. If $a(p) \in (0,1)$, then $a'(p) = V_p\delta/V_a\delta - C'' > 0$.

A symmetric equilibrium of the original information-pricing game is a FP of the mapping $(p(a), a(p))$, and because $p'(a) \leq 0$ but $a'(p) > 0$ hold at all respective interior points, the above analysis shows that there is exactly one such FP (see figure 2).

It is straightforward to check that the parametric example satisfies $p'(a) \leq 0$ and $a'(p) > 0$ at interior points, and therefore has only one symmetric equilibrium.

Proposition 4 *In the information-pricing game with linear demand there is a single symmetric*

[27] This holds, because $p = c$ cannot be an equilibrium by the assumptions made and $V_{1p}(p,p,a) = V_{11}(p,\bar{p},a) + V_{12}(p,\bar{p},a)$ at $\bar{p} = p$. Hence $V_{1p} < 0$ implies that $\tilde{p}'(p;a) < 1$ whenever $\tilde{p}(p,a) = p \in (c,\hat{p})$.

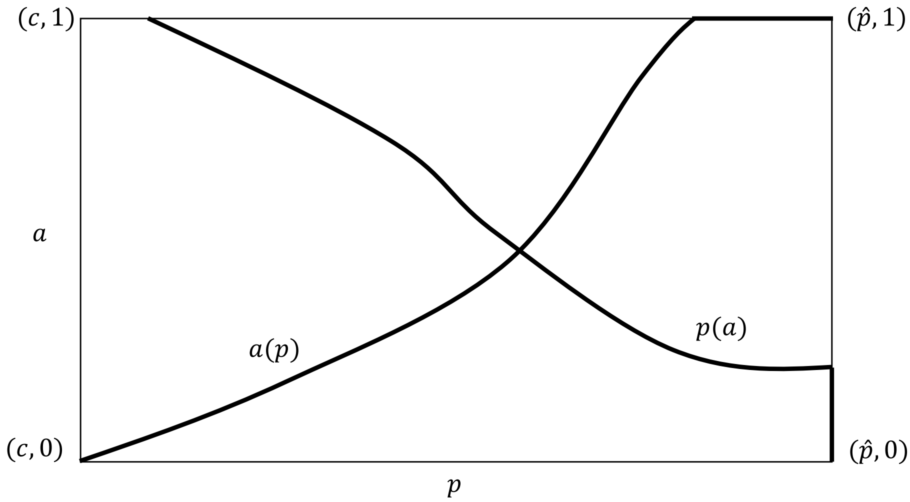


Figure 2: Inexistence of multiple symmetric equilibria

equilibrium. For $C(a) = \theta a^\eta$, $\eta \geq 2$, the symmetric equilibrium is even unique if information costs are sufficiently elastic ($\eta \geq 3$) or products are not too strong substitutes ($\gamma \leq \sqrt{2} - 1$).

4.3 Equilibria in asymmetric games

Let $c_g \in \mathcal{P}$ denote player g's parameter vector, where $\mathcal{P} \subset \mathbb{R}^m$ is a compact parameter space. $\Gamma(c) \equiv \left(N, S^N, \{\Pi^g(x, c_g)\}_{g=1}^N\right)$, $c \in \mathcal{P}^N$, is a game[28] with parameters $c_1, ..., c_N$. If $c_1 = c_2 = ... = c_N$ the game is symmetric. For now, we concentrate on one-dimensional games where the heterogeneity of the payoff-functions is restricted to the distribution of a single parameter. The following proposition shows that if for a game, where best-replies are increasing in the parameter $c \in [\underline{c}, \bar{c}]$, any underlying two-person symmetric game does not have an asymmetric equilibrium, then the strategies in every equilibrium of the asymmetric game are ordered exactly in the same way as the parameters c_i.

Proposition 5 *Suppose that $\varphi(x_{-1}, c)$ is increasing in c on $[\underline{c}, \bar{c}]$ and $\bar{c} \geq c_1 > c_2, ..., > c_N \geq \underline{c}$. If for any given $X \in S^{N-2}$ and any $c \in [\underline{c}, \bar{c}]$ the symmetric two-player game with payoffs $\Pi^j(x_1, x_2; X, c)$, $j = 1, 2$, has no asymmetric equilibria, then every equilibrium of the asymmetric game $\Gamma(c_1, ..., c_N)$ satisfies $x_1^* \geq x_2^* \geq, ..., \geq x_N^*$. Moreover, $x_1^* > x_2^* >, ..., > x_N^*$ results if $\varphi(x_{-1}, c)$ is strictly increasing in c on $[\underline{c}, \bar{c}]$.*

[28]In this section I assume that $\Pi^j(x, c)$ is twice continuously differentiable in (x, c) and strongly quasiconcave in x_j for any $c \in \mathcal{P}$.

The proof builds on a characterization result for asymmetric equilibria in two-player games (see appendix 6.5). If the game is decreasing in c, the inequalities of the equilibrium strategies are reverted. Proposition 5 tells us, e.g., that asymmetric games never possess symmetric equilibria if φ^j is strictly monotonic in c on $[\underline{c}, \bar{c}]$. Notably, we can use the simple slope condition of theorem 2 to exclude the possibility of equilibria that do not reflect the order of the parameters in c-monotonic asymmetric games.

As a simple illustration, reconsider the symmetric Cournot or Contest model (examples 1 and 2 in section 4.1) with a cost function of the form $c(x_i, \alpha)$, where $\alpha \in [\underline{\alpha}, \bar{\alpha}]$ is a parameter. Suppose that $c_{x\alpha}(x, \alpha) < 0$ for any $x > 0$, such that $\varphi(x_{-1}, \cdot)$ is strictly increasing. Because we know from section 4.1 that both *symmetric* games verify condition (6), there cannot be asymmetric equilibria in any symmetric two-player version of these games. Proposition 5 then assures that any asymmetric version of these games with $\bar{\alpha} = \alpha_1 > ... > \alpha_N = \underline{\alpha}$ can only have ordered equilibria $x_1^* >, ..., > x_N^*$, where the "cheapest technology does most". [29]

Proposition 5 extends to the case where c_j is a parameter vector in the natural way. If $c_1, ..., c_N$ are parameter vectors such that $\varphi(x_{-1}, c_g) \geq \varphi(x_{-1}, c_j)$, and the respective symmetric two-player games have no asymmetric equilibria for any of these parameter vectors, then $x_1 \geq x_2 \geq ... \geq x_N$ holds in any equilibrium of the asymmetric game.

5 Conclusion

Many strategic choices are of a multi-dimensional nature, and having a systematic methodology to get an analytical grasp on such problems is very useful. The separation approach developed in this article yields a comparably simple but powerful set of tools to examine the equilibrium set of symmetric games with a potentially higher-dimensional strategy space and many players, that may have eluded an analytical assessment so far, e.g., by the sheer formal complexity of the problem. The practical and theoretical usefulness of these tools was documented with several examples. The separation approach allows to study how the parameters of a game could possibly influence whether or not there are asymmetric equilibria or multiple symmetric

[29]See Hefti and Grossmann (2015) for a direct application of proposition 5 in case of a dynamic contest with heterogeneous participants.

equilibria. Analyzing the equilibrium set of a symmetric game may also shed light on the equilibria of certain asymmetric versions of the game, which was illustrated by means of a one-dimensional example. All in all, this article can provide valuable guidelines for a thorough equilibrium analysis of complex symmetric games in applied research in game theory, industrial economics and related fields.

6 Appendix

6.1 Proof of theorem 2

The proofs of theorems 2 and 4 build on the following lemmata.

Lemma 1 *Let $\psi \in C\left([t_0,t_1],[a,b]\right)$ with $\psi(t_0) \neq \psi(t_1)$. Suppose that the points in (t_0,t_1) where $\psi(t)$ is not differentiable are locally isolated. Then*

$$\begin{array}{ll} i) & \text{if } \psi(t_0) > \psi(t_1)\ \exists t' \in (t_0,t_1) \text{ such that } \psi'(t') \leq \frac{\psi(t_1)-\psi(t_0)}{t_1-t_0} \\ ii) & \text{if } \psi(t_0) < \psi(t_1)\ \exists t'' \in (t_0,t_1) \text{ such that } \psi'(t'') \geq \frac{\psi(t_1)-\psi(t_0)}{t_1-t_0} \end{array} \tag{12}$$

Proof: $\tilde{A} \subset (t_0,t_1)$ is the set of non-differentiable points of ψ and $A = \tilde{A} \cup \{t_0,t_1\}$. Let $\psi(t_0) > \psi(t_1)$, define

$$g(t) \equiv \frac{\psi(t_0)-\psi(t_1)}{t_0-t_1}(t-t_0)+\psi(t_0)$$

and $k(t) \equiv \psi(t) - g(t)$ for $t \in [t_0,t_1]$. Hence $k(t_0) = k(t_1) = 0$, k is continuous on $[t_0,t_1]$ and differentiable at t if $t \notin A$. Suppose that

$$\psi'(t) > \frac{\psi(t_1)-\psi(t_0)}{t_1-t_0}$$

holds, whenever $\psi(t)$ is differentiable. If $\tilde{A} = \emptyset$ then k is strictly increasing on $[t_0,t_1]$ by the Mean Value Theorem (MVT), which contradicts $k(t_0) = k(t_1)$. Hence suppose that $\tilde{A} \neq \emptyset$. Then, by local isolation, $\forall t \in \tilde{A}$ there is an interval $I_t = (t-\varepsilon_1, t+\varepsilon_2)$ such that k is differentiable on $I_t \backslash \{t\}$. One can choose $\varepsilon_2 > 0$ such that $t+\varepsilon_2 \in A$. Then the MVT and continuity of k at t imply k to be strictly increasing over I_t. As for any $t \in \tilde{A}\ \exists\ q(t) \in \mathbb{Q} \cap I_t$, the mapping

$q : \tilde{A} \to \mathbb{Q}$ is well-defined and injective, which shows that $\tilde{A}$ is countable. Hence there is a sequence (q_n) with $q_n \in \mathbb{Q} \cap (t_0, t_1)$ such that $q_n \to t_1$ and $k(q_{n+1}) > k(q_n)$, which implies that $k(t_1) > k(q_0)$ by the continuity of k. As $k(t_1) = 0$ we conclude that $k(q_0) < 0$. By the same reason there is a strictly decreasing sequence $\tilde{q}_n$, where $\tilde{q}_0 = q_0$, $\tilde{q}_n \to t_0$ and $k(\tilde{q}_{n+1}) < k(\tilde{q}_n)$. Then continuity and $k(\tilde{q}_0) < 0$ imply $k(t_0) < 0$, a contradiction. This proves i), and ii) follows from i) by setting $\rho(t) \equiv \psi(t_0 + t_1 - t)$. ■

Lemma 2 *Let $\psi \in C([t_0, t_1], [a, b])$ with $\psi(t_0) \neq \psi(t_1)$ and ψ differentiable on $\psi^{-1}((a, b))$ except possibly at a set of isolated points. Then* (12) *is satisfied.*

Proof: By the proof of lemma 1 it suffices to consider the case $\psi(t_0) > \psi(t_1)$. Hence $\psi(t_0) > a$ and $\psi(t_1) < b$. Let $T \equiv \psi^{-1}(\{a, b\}) \subset [t_0, t_1]$. If $T = \emptyset$ then the claim follows from lemma 1, so suppose that $T \neq \emptyset$. Note that T is a compact subset of $\mathbb{R}$, and let the min and max of T be denoted by $\underline{t}, \bar{t}$. The proof now is case-by-case. (I) $\psi(\underline{t}) = a$. Then ψ is continuous on $[t_0, \underline{t}]$ and differentiable on $(t_0, \underline{t})$ except possibly for a set of isolated points. Then, because of lemma 1, $\exists t \in (t_0, \underline{t})$ such that

$$\psi'(t) \leq \frac{a - \psi(t_0)}{\underline{t} - t_0} \leq \frac{\psi(t_1) - \psi(t_0)}{t_1 - t_0}.$$

(II) $\psi(\bar{t}) = b$. Then ψ is continuous on $[\bar{t}, t_1]$ and differentiable on $(\bar{t}, t_1)$ except possibly for a set of isolated points. Thus, by lemma 1, $\exists t \in (\bar{t}, t_1)$ such that

$$\psi'(t) \leq \frac{\psi(t_1) - b}{t_1 - \bar{t}} \leq \frac{\psi(t_1) - \psi(t_0)}{t_1 - t_0}.$$

(III) $\psi(\underline{t}) = b$ and $\psi(\bar{t}) = a$. Define $A \equiv \psi^{-1}(\{b\})$, which is a non-empty and compact set. Hence $\hat{t} = max\, A$ exists. Similarly, $B \equiv [\hat{t}, t_1] \cap \psi^{-1}(\{a\})$ also is non-empty and compact. Let $\check{t} = min\, B$. Hence ψ is continuous on $[\hat{t}, \check{t}]$ and differentiable on $(\hat{t}, \check{t})$ except possibly for a set of isolated points. Thus, by lemma 1, $\exists t \in (\hat{t}, \check{t})$ such that

$$\psi'(t) \leq \frac{a - b}{\hat{t} - \check{t}} \leq \frac{\psi(t_1) - \psi(t_0)}{t_1 - t_0}. \quad ■$$

Proof of theorem 2

Step 1: $N = 2$. Suppose that (x_1^a, x_2^a) is an asymmetric equilibrium. Then (x_2^a, x_1^a) is a different asymmetric equilibrium and $\varphi(x_2^a) = x_1^a$ and $\varphi(x_1^a) = x_2^a$. Let $\psi(t) \equiv \varphi\left(x_1^a + t(x_2^a - x_1^a)\right)$ for $t \in [0, 1]$. Then $\psi(0) = x_2^a$ and $\psi(1) = x_1^a$. Hence $\psi \in C\left([0,1], S\right)$, $\psi(0) \neq \psi(1)$ and $\psi(t)$ is differentiable whenever $\psi(t) \in Int(S)$ except possibly for a set of isolated points. If $\psi(0) > \psi(1)$, then lemma 2 and the chain rule imply that $\exists x_2 \in Int(S)$ such that $\varphi(x_2) \in Int(S)$, φ differentiable at x_2 and $\partial\varphi(x_2) \leq -1$. For $\psi(1) - \psi(0) > 0$ an identical conclusion follows. Step 2: $N > 2$. Suppose $(x_1^a, ..., x_N^a)$ is an asymmetric equilibrium, where we can assume $x_1^a \neq x_2^a$ without loss of generality. Take $X = (x_3^a, ..., x_N^a) \in S^{N-2}$ as an exogenously fixed parameter vector and suppose players $g = 1, 2$ play a two-player game, treating X as fixed. Then (x_1^a, x_2^a) as well as (x_2^a, x_1^a) must be asymmetric equilibria of this symmetric, parametrized two-player game. Thus, by step 1, if the N-player game has an asymmetric equilibrium $\exists\ X \in S^{N-2}$ and $x_2 \in Int(S)$ such that $\partial\varphi(x_2; X) \leq -1$, which completes the proof. ■

6.2 Proof of theorem 3

I prove the theorem for $N = 2$; the extension to $N > 2$ follows the same logic as in step 2 of the proof of theorem 2. Consider the two asymmetric equilibria (x_1^a, x_2^a) and (x_2^a, x_1^a). By the differentiability assumption made, $\psi_i(t) \equiv \varphi_i(x_1^a + t(x_2^a - x_1^a))$, $1 \leq i \leq k$, is differentiable on $(0, 1)$. Let $\Delta_i \equiv \varphi_i(x_1^a) - \varphi_i(x_2^a)$ and $\Delta \equiv (\Delta_1, ..., \Delta_k)$. Then the MVT, applied separately to each ψ_i, asserts the existence of k points $x_2^i \in Int(S)$, $1 \leq i \leq k$, such that $\tilde{A} \cdot \Delta = -\Delta$, where $\tilde{A}$ is a $k \times k$ matrix with entries $a_{ij} = \frac{\partial\varphi_i(x_2^i)}{\partial x_{2j}}$, $1 \leq i, j \leq k$. Equivalently, we get that $(I + \tilde{A}) \cdot \Delta = A \cdot \Delta = 0$ where A is the matrix (3). Consequently, if $Det(A) \neq 0$ at any points $x_2^1, ...x_2^k \in Int(S)$, we may conclude that there cannot be any asymmetric equilibria. ■

6.3 Proof of theorem 4

As before it suffices to let $N = 2$. Suppose (x_1^a, x_2^a) is an asymmetric equilibrium. Then (x_2^a, x_1^a) also is an asymmetric equilibrium and $\varphi(x_2^a) = x_1^a$, $\varphi(x_1^a) = x_2^a$. Define $\psi_i(t_i) \equiv \varphi_i\left(x_1^a + t_i(x_2^a - x_1^a)\right)$, where $i = 1, 2$ and $t_i \in [0, 1]$. Then $\psi_i(0) = \varphi_i(x_1^a)$ and $\psi_i(1) = \varphi_i(x_2^a)$. Note that $\psi_i(0) \neq \psi_i(1)$ for at least one i. Moreover, $\psi_i \in C\left([0, 1], S_i\right)$ and, according to the chain rule, if $\psi_i(t_i) \in Int(S_i)$ the function ψ_i is differentiable expect possibly for a set of isolated points by presupposition. Hence if $\varphi_i(x_1^a + t_i(x_2^a - x_1^a)) \in Int(S_i)$ and φ_i is differentiable at the point $x_1^a + t_i(x_2^a - x_1^a)$, then the chain rule implies:

$$\psi_i{}'(t_i) = \partial\varphi_i \left(x_1^a + t_i \left(x_2^a - x_1^a\right)\right) \cdot \begin{pmatrix} \psi_1(0) - \psi_1(1) \\ \psi_2(0) - \psi_2(1) \end{pmatrix} \tag{13}$$

The proof now is case-by-case. (I) $\psi_i(0) = \psi_i(1)$ for one i. Suppose that $\psi_1(0) = \psi_1(1)$ and hence $\psi_2(0) \neq \psi_2(1)$. Then, similar to step 1 of the proof of theorem 2, lemma 2 and (13) imply that $\exists\ x_2' \in S_1 \times Int(S_2)$ such that $\delta(x_2') \leq -1$ is satisfied. Similarly, if $\psi_2(0) = \psi_2(1)$ then $\alpha(x_2) \leq -1$ for some $x_2 \in Int(S_1) \times S_2$. Consequently, $\alpha(x_2), \delta(x_2) > -1$ for any $x_2 \in S$ where the respective derivative exist, rule out the possibility of asymmetric equilibria with a similar i-th projection, and henceforth assume this condition to be satisfied. Further, suppose that $\psi_i(0) \neq \psi_i(1)$ for $i = 1, 2$ and define $m \equiv (\psi_2(0) - \psi_2(1))/(\psi_1(0) - \psi_1(1))$. (II) $\psi_i(0) > \psi_i(1)$ or $\psi_i(0) < \psi_i(1)$, $i = 1, 2$, hence $m > 0$. Suppose that $\psi_i(0) > \psi_i(1)$. Then lemma 2 and (13) assert the existence of $x_2, x_2' \in S$ such that $\alpha(x_2) + m\beta(x_2) \leq -1$ and $\gamma(x_2')\frac{1}{m} + \delta(x_2') \leq -1$. Eliminating m gives $\beta(x_2)\gamma(x_2{}') \geq (1 + \alpha(x_2))\,(1 + \delta(x_2{}'))$. The same conclusion holds if $\psi_i(0) < \psi_i(1)$. (III) $\psi_1(0) < \psi_1(1)$ and $\psi_2(0) > \psi_2(1)$ (or opposite inequalities), hence $m < 0$. Then proceed as in (II) to obtain the same conclusion as in case (II). The above derivation implies that whenever (4) is satisfied, there cannot be any asymmetric equilibria. ■

6.4 Proof of corollary 2

Suppose that there is an asymmetric equilibrium $x^a = (x_1^a, x_2^a, ..., x_N^a)$ with $x_1^a > x_2^a$ but, e.g., $\beta \geq 0$ and $\alpha > -1$. Then by case (II) of the the proof of theorem 4 $\exists\ \tilde{x}_2 \in S$ such that $\alpha(\tilde{x}_2) + m\beta(\tilde{x}_2) \leq -1$ for some X. As $m > 0$ this implies that $\beta(\tilde{x}_2) < 0$, a contradiction. Hence there cannot be any strictly ordered equilibria, which proves i), and ii) is proved in the same way. If $\alpha, \delta > -1$ case (I) of the the proof of theorem 4 shows that there cannot be asymmetric equilibria, where two players choose the same component strategies, which proves iii). ■

6.5 Proof of proposition 5

The proof of proposition 5 requires the following lemma.

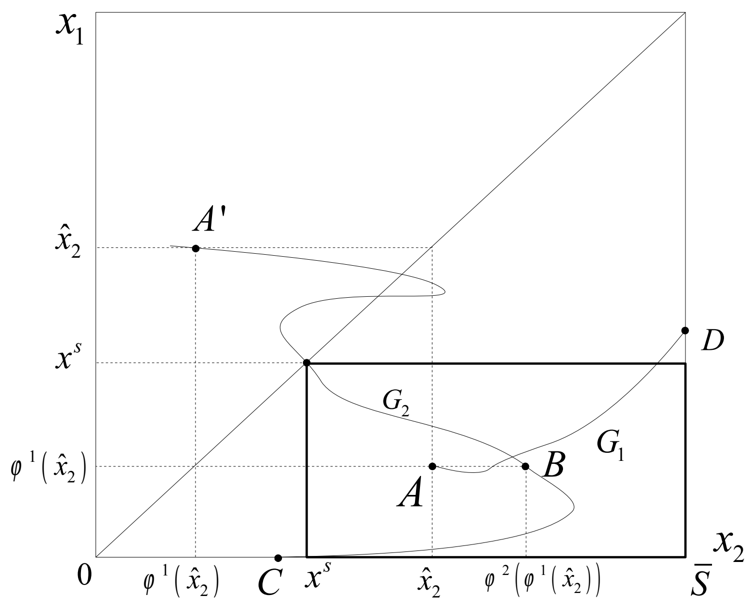
Lemma 3 (Characterization of asymmetric equilibria) *In a symmetric one-dimensional two-player game with $\varphi \in C(S,S)$ no asymmetric equilibria exist if and only if*

$$\varphi\left(\varphi(x)\right) < x \quad \forall x \in S : \varphi(x) < x \tag{14}$$

or equivalently

$$\varphi\left(\varphi(x)\right) > x \quad \forall x \in S : \varphi(x) > x \tag{14'}$$

Proof: I only prove the claim for (14), the claim for (14') is proved in the same way. "$\Rightarrow$" Suppose that (x_1, x_2) is an asymmetric equilibrium. By symmetry, we can assume that $x_1 < x_2$, i.e., $\varphi(x_2) < x_2$ but $\varphi\left(\varphi(x_2)\right) = x_2$, contradicting (14). "$\Leftarrow$" The proof of this direction naturally is more involved. Let $G_1 \equiv \{(x_1, x_2) \in S^2 : \varphi^1(x_2) = x_1\}$ and $G_2 \equiv \{(x_1, x_2) \in S^2 : \varphi^2(x_1) = x_2\}$ denote the graphs of the best-response functions of the two players. Further, $G_1(x_2) \equiv (\varphi^1(x_2), x_2)$ and $G_2(x_1) \equiv (x_1, \varphi^2(x_1))$ denote specific points on the graphs. The proof is by contraposition. Suppose $\exists\ \hat{x}_2$ such that $\varphi^1(\hat{x}_2) < \hat{x}_2$ but $\varphi^2\left(\varphi^1(\hat{x}_2)\right) \geq \hat{x}_2$. If $\varphi^2\left(\varphi^1(\hat{x}_2)\right) = \hat{x}_2$ then there is nothing to prove as $(\varphi^1(\hat{x}_2), \hat{x}_2)$ obviously is an asymmetric equilibrium, so suppose that $\varphi^2\left(\varphi^1(\hat{x}_2)\right) > \hat{x}_2$. Such a situation is illustrated in figure 6.5 with points $A = G_1(\hat{x}_2) \in G_1$ and $B = G_2\left(\varphi^1(\hat{x}_2)\right) \in G_2$. First, note that $G_2(0) \in \{0\} \times S$, as indicated by the point C. Next, note that, by symmetry, G_2 must pass through a point $A' = G_2(\hat{x}_2)$. By continuity of the best-response function there must be at least one symmetric equilibrium in the interval $(\varphi^1(\hat{x}_2), \varphi^2\left(\varphi^1(\hat{x}_2)\right))$. Let $x^s = min\left\{x_2 : \varphi^1(\hat{x}_2) \leq x_2 \leq \hat{x}_2, \varphi^1(x_2) = x_2\right\}$. Consider the rectangle $[0, x^s] \times \left[x^s, \bar{S}\right]$. By construction, (x^s, x^s) is the only symmetric equilibrium in this rectangle. Moreover, G_2 partitions this rectangle (because G_2 is continuous) and $G_1(\hat{x}_2)$ must lie in the lower partition ("beneath" G_2). But as $G_1(\bar{S}) \in S \times \left\{\bar{S}\right\}$ (indicated with D) and G_1 is contin-



uous there must be an $x^2 \in \left(\hat{x}^2, \bar{S}\right]$ such that $G_1(x_2) \in G_2$. Hence an asymmetric equilibrium exists. ■

In words, lemma 3 says that if player 1's reaction function lies below the graph of player 2's reaction function and $\varphi^1(x_2) < x_2$, then an asymmetric equilibrium must necessarily exist.

Proof of proposition 5:

By contradiction, suppose that the asymmetric game has an equilibrium with $x_j > x_g$, where $g < j$ (and thus $c_g > c_j$). Consequently, there exists X such that $\varphi^j(x_g; X, c_j) > x_g$ and $\varphi^g\left(\varphi^j(x_g; X, c_j); X, c_g\right) = x_g$. As best-replies are increasing on $[\underline{c}, \bar{c}]$ this implies that

$$\varphi^g\left(\varphi^j\left(x_g; X, c_j\right); X, c_g\right) \geq \varphi^g\left(\varphi^j\left(x_g; X, c_j\right); X, c_j\right)$$

Hence there exists x_g such that $\varphi^j(x_g; X, c_j) > x_g$ but $x_g \geq \varphi^g\left(\varphi^j\left(x_g; X, c_j\right); X, c_j\right)$, which in turn by (14') of lemma 3 implies that the symmetric two-player game with best reply function $\varphi(x; X, c_j)$ must have an asymmetric equilibrium, a contradiction. Hence $x_g \geq x_j$, and the result follows by induction. To prove the version for strictly increasing replies, suppose that the asymmetric game has an equilibrium with $x_g = x_j = x$. Thus there exists X such that $\varphi^g(x; X, c_g) = \varphi^j(x; X, c_j) = \varphi^g(x; X, c_j)$, contradicting $\varphi^g(x; X, c_g) > \varphi^g(x; X, c_j)$ as implied by strict monotonicity. ■

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