

A general solution method for moral hazard problems*

Rongzhu Ke[†] Christopher Thomas Ryan[‡]

August 2, 2017

Abstract

Principal-agent models are pervasive in theoretical and applied economics, but their analysis has largely been limited to the “first-order approach” (FOA) where incentive compatibility is replaced by a first-order condition. This paper presents a new approach to solving a wide class of principal-agent problems that satisfy the monotone likelihood ratio property but may fail to meet the requirements of the FOA. Our approach solves the problem via tackling a max-min-max formulation over agent actions, alternate best responses by the agent, and contracts.

Key Words: Principal-agent, moral hazard, solution method

JEL Code: D82, D86

1 Introduction

Moral hazard principal-agent problems are well-studied, but unresolved technical difficulties persist. An essential difficulty is finding a tractable method to deal with the incentive compatibility (IC) constraints that capture the strategic behavior of the agent. Incentive compatibility is challenging for at least two reasons. First, when the agent’s action space is continuous there are, in principle, infinitely-many IC constraints. Second, these constraints turn the principal’s decision into an optimization problem over a potentially nonconvex set. Much attention has been given to finding structure in special cases that overcome these issues. The *first-order approach* (FOA), where the IC constraints are replaced by the first-order condition of the agent’s problem (Jewitt (1988), Rogerson (1985)), applies when the agent’s objective function is concave in the agent’s action. Previous studies have proposed various sufficient conditions for the FOA to be valid (see, e.g., Conlon (2009), Jewitt (1988), Jung and Kim (2015), Kirkegaard (2017b), Rogerson (1985), Sinclair-Desgagné (1994)). Nonetheless, there remain natural settings where the FOA is invalid (see, for instance, Example 5 below).

When the FOA is invalid, more elaborate methods have been proposed.¹ Grossman and Hart (1983) explore settings where there are finitely many output scenarios and reduce incentive compatibility to a finite number of constraints. However, their method does not apply when the agent’s output takes on infinitely-many values. An alternate approach is due to Mirrlees (1999)

*We are indebted to the comments of two anonymous referees whose feedback greatly improved this paper.

[†]Department of Economics, Hong Kong Baptist University, E-mail: rongzhuke@hkbu.edu.hk

[‡]Booth School of Business, University of Chicago, E-mail: chris.ryan@chicagobooth.edu

¹Several authors have analyzed the moral hazard problem without the FOA in specialized settings. This includes Innes (1990), Wang and Hu (2016), and Kirkegaard (2017a).

31 (which originally appeared in 1975) and refined in [Mirrlees \(1986\)](#) and [Araujo and Moreira \(2001\)](#).
32 This method overcomes the limitations of the FOA by reintroducing a subset of IC constraints,
33 in addition to the first-order condition, to eliminate alternate best responses. These reintroduced
34 constraints — called *no-jump constraints* — isolate attention to contract-action pairs that are in-
35 centive compatible. The main difficulty in Mirrlees’s approach is in producing the required no-jump
36 constraints. There is a potential to reintroduce many — if not infinitely many — no-jump con-
37 straints. Moreover, a general method for generating these constraints is not known and brute force
38 enumeration is often difficult. [Araujo and Moreira \(2001\)](#) use second-order information to refine
39 the search, but the essential difficulties remain.

40 The procedure described in this paper systematically builds on Mirrlees’s approach. The prob-
41 lem of determining which no-jump constraints are needed is recast as a minimization problem that
42 identifies the hardest-to-satisfy no-jump constraint over the set of alternate best responses. This
43 makes the original problem equivalent to an optimization problem that involves three sequential
44 optimal decisions: maximizing over the contract, maximizing over the agent’s action, and minimiz-
45 ing over alternate best responses to that chosen action. We then propose a tractable relaxation to
46 this problem by changing the order of optimization to “max-min-max” where the former maximiza-
47 tion is over agent actions and the latter maximization is over contracts. The analytical benefits of
48 this new order are clear. The map that describes which optimal contracts support a given action
49 against deviation to a specific alternate best response has desirable topological properties explored
50 in [Section 3](#). We call this “max-min-max” relaxation the “sandwich” relaxation since the inner
51 minimization is “sandwiched” between two outer maximizations.

52 The main technical work of the paper is to uncover when the sandwich relaxation is tight.
53 This involves careful consideration of what utility can be guaranteed to the agent by an optimal
54 contract. In particular, if the individual rationality constraint is *not* binding, a family of sandwich
55 relaxations indexed by lower bounds on agent utility that are larger than the reservation utility
56 must be examined in order to find a relaxation that is tight. Constructing the appropriate bound,
57 and guaranteeing that the resulting relaxation is tight, is a main focus of our development. Our
58 development assumes monotonicity conditions on the output distribution; namely, the monotone
59 likelihood ratio property (MLRP) that is defined carefully in the main body of the text.

60 It should be noted that the MLRP assumption is common in the usual discussion of the FOA.
61 However, it is also well-known that the MLRP is *insufficient* to guarantee the validity of the FOA
62 ([Conlon 2009](#), [Grossman and Hart 1983](#), [Jewitt 1988](#), [Rogerson 1985](#)). We illustrate scenarios where
63 the sandwich approach is valid (that is, the sandwich relaxation is tight) but the FOA is invalid.
64 This is carefully discussed in [Section 5](#) where it is established that the sandwich approach ensures a
65 stationarity condition for a worst-case alternate best response that is stronger than the stationarity
66 condition in the FOA. This is due to the inner minimization over alternate best responses in the
67 sandwich approach that is absent from the FOA. However, when the FOA is valid, the sandwich
68 approach is also valid and both approaches result in the same optimal contract.

69 Finally, we comment here on some similarities with a related paper written by the authors. In
70 [Ke and Ryan \(2016\)](#), we consider a similar problem setting with similar assumptions. The main
71 focus of that paper is to establish an important structural result, namely to recover a monotonicity
72 result for optimal contracts under MLRP that holds even when the FOA is invalid. To that end,
73 that paper takes the approach of [Grossman and Hart \(1983\)](#) of taking the agent’s action as given and
74 finds structure on those optimal contracts that implement the given action. Consequently, [Ke and](#)
75 [Ryan \(2016\)](#) does not provide a general solution procedure for moral hazard problems, and instead

76 focuses on establishing structural properties of optimal contracts without explicitly constructing
77 such policies. By contrast, the current paper is focused on the full problem that allows the agent’s
78 action to respond optimally to an offered contract. Of course, this adds significant complication to
79 the analysis, hence the need for another paper. Indeed, consider the classical example of [Mirrlees](#)
80 (1999) that initiated discussion of the failure of the FOA. If a tight reservation utility and best
81 response are known, a first-order condition is easily shown to suffice. The failure of the FOA
82 is precisely its inability to identify a target action of the follower. See also our [Example 1](#) and
83 [Proposition 5](#) below for a related discussion.

84 A more subtle technical challenge here concerns questions of existence. The inner minimization
85 in the sandwich problem need not be attained, an issue that is precluded from the analysis of [Ke](#)
86 [and Ryan \(2016\)](#). There, a target best response a^* is specified and an assumption is made so that
87 an alternate and distinct best response \hat{a}^* exists. This assumption essentially rules out the validity
88 of the FOA. In other words, the analysis of [Ke and Ryan \(2016\)](#) does not apply to many problems
89 where the FOA is known to be valid. This is not a concern in that paper, since the goal there is to
90 devise the structure of optimal contracts, particularly monotonicity properties, which are already
91 known in the setting where the FOA is valid ([Rogerson 1985](#)). By contrast, the goal of this paper
92 is to develop a general procedure for solving moral hazard problems that satisfy the MLRP, and
93 thus should incorporate cases where the FOA additionally holds. [Section 5](#) provides more detail on
94 how this existence issue is connected to the FOA.

95 Although there are similarities in the development of both papers (the current paper and [Ke](#)
96 [and Ryan \(2016\)](#)) they can largely be read independently. [Ke and Ryan \(2016\)](#) does not reference
97 the current paper, and there are only several references to [Ke and Ryan \(2016\)](#) here, all of which
98 appear in the technical appendix.²

99 This paper is organized as follows. [Section 2](#) contains the model and reviews existing approaches
100 to solve the principal-agent problem. [Section 3](#) describes the sandwich relaxation and gives sufficient
101 conditions for the relaxation to be tight, given an appropriately chosen lower bound on agent utility.
102 [Section 4](#) describes the methodology to construct such lower bounds. [Section 5](#) discusses existence
103 and its connection the FOA. [Section 6](#) provides three examples that illustrate the mechanics of our
104 procedure. We consider a simplified moral hazard example throughout the paper to illuminate the
105 theory. Proofs of all technical results are contained in an appendix.

106 2 Model and existing approaches

107 2.1 Principal-agent model

108 We study the classic moral hazard principal-agent problem with a single task and single-dimensional
109 output. An agent chooses an action $a \in \mathbb{A}$ that is unobservable to a principal. This action influences
110 the random outcome $X \in \mathcal{X}$ through the probability density function $f(x, a)$ where x is an outcome
111 realization. The principal chooses a wage contract $w : \mathcal{X} \rightarrow [\underline{w}, \infty)$ where \underline{w} is an exogenously given
112 minimum wage. The value of output to the principal obeys the function $\pi : \mathcal{X} \rightarrow \mathbb{R}$.

113 Given an outcome realization $x \in \mathcal{X}$, the agent and principal derive the following utilities. The
114 agent’s utility under action a is separable in wage $w(x)$ and action cost $c(a)$. In particular, he

²We thank an anonymous referee for raising and shedding light on the question of existence during the review process of the paper. We also thank another anonymous referee for drawing attention to the similarities and distinctions between the current paper and [Ke and Ryan \(2016\)](#).

115 derives utility $u(w(x)) - c(a)$ where $u : [\underline{w}, \infty) \rightarrow \mathbb{R}$ and $c : \mathbb{A} \rightarrow \mathbb{R}$. The principal's utility for
 116 outcome x is a function of the net value $\pi(x) - w(x)$ and is denoted $v(\pi(x) - w(x))$ where $v : \mathbb{R} \rightarrow \mathbb{R}$.
 117 The agent's expected utility is $U(w, a) = \int u(w(x))f(x, a)dx - c(a)$ and the principal's expected
 118 utility is $V(w, a) = \int v(\pi(x) - w(x))f(x, a)dx$. The agent has an outside option worth utility \underline{U} .
 119 The principal faces the optimization problem:³

$$120 \quad \max_{w \geq \underline{w}, a \in \mathbb{A}} V(w, a) \quad (\text{P})$$

121
 122 subject to the following conditions

$$123 \quad U(w, a) \geq \underline{U} \quad (\text{IR})$$

$$124 \quad U(w, a) - U(w, \hat{a}) \geq 0 \quad \text{for all } \hat{a} \in \mathbb{A}, \quad (\text{IC})$$

125 where **(IR)** is the individual rationality constraint that guarantees participation of the agent by
 126 furnishing at least the reservation utility \underline{U} and **(IC)** are the incentive compatibility constraints
 127 that ensure the agent responds optimally.
 128

129 **Assumption 1.** The following hold:

130 (A1.1) the outcome set \mathcal{X} is an interval in \mathbb{R} and the action set is the bounded interval
 131 $\mathbb{A} \equiv [\underline{a}, \bar{a}]$,

132 (A1.2) the outcome X is a continuous random variable and $f(x, a)$ is continuous in x and
 133 twice continuously differentiable in $a \in \mathbb{A}$,

134 (A1.3) for $a, a' \in \mathbb{A}$ with $a \neq a'$, there exists a positive measure subset of \mathcal{X} such that
 135 $f(x, a) \neq f(x, a')$,

136 (A1.4) the support of $f(\cdot, a)$ does not depend on a , and hence (without loss of generality) we
 137 assume the support is \mathcal{X} for all a ,

138 (A1.5) w is a measurable function on \mathcal{X} ,

139 (A1.6) the value function π is increasing, continuous, and almost everywhere differentiable,

140 (A1.7) the expected value $\int \pi(x)f(x, a)dx$ of output is bounded for all a ,

141 (A1.8) the agent's cost function c is increasing and continuously differentiable in a ,

142 (A1.9) the agent's utility function u is continuously differentiable, increasing, and strictly
 143 concave, and

144 (A1.10) the principal's utility function v is continuously differentiable, increasing, and concave.

145 The above assumptions are standard, so we will not spend time to justify them here.

146 **Assumption 2.** We also make the following additional technical assumptions:

147 (A2.1) either $\lim_{y \rightarrow \infty} u(y) = \infty$ or $\lim_{y \rightarrow -\infty} v(y) = -\infty$, and

³The notation $w \geq \underline{w}$ is shorthand for expressing $w(x) \geq \underline{w}$ for almost all $x \in \mathcal{X}$.

148 (A2.2) the minimum wage \underline{w} , reservation utility \underline{U} , and least costly action \underline{a} are such that
 149 $u(\underline{w}) - c(\underline{a}) < \underline{U}$.

150 The two conditions in this assumption are required in the proof of Lemma 3 that uses a La-
 151 grangian duality method and ensures the existence of optimal dual solutions. Finally, to focus the
 152 scope of our paper we make one additional assumption.

153 **Assumption 3.** There exists an optimal solution to (P). Moreover, we assume the first-best
 154 contract is not optimal.

155 Existence is a challenging issue in its own right and not the focus of this paper. We are interested
 156 in how to construct an optimal solution when one is known to exist. Several existing papers pay
 157 careful attention to the issue of existence. For instance, Kadan et al. (2017) provide weak sufficient
 158 conditions that guarantee the existence of an optimal solution. Moreover, we may assume that
 159 the first-best contract is not optimal without loss of interest, since finding a first-best contract is a
 160 well-understood problem not worthy of additional consideration.

161 We use the following terminology and notation. Let $a^{BR}(w)$ denote the set of actions that
 162 satisfy the (IC) constraint for a given contract w . That is, $a^{BR}(w) \equiv \arg \max_{a'} U(w, a')$. Let \mathcal{F}
 163 denote the set of feasible solutions to (P). That is,

$$164 \quad \mathcal{F} \equiv \{(w, a) : w \geq \underline{w}, a \in a^{BR}(w), U(w, a) \geq \underline{U}\}.$$

165 Given an action a , contract w is said to *implement* a if $(w, a) \in \mathcal{F}$. An action a is *implementable*
 166 if there exists a w that implements a . Let $\text{val}(\ast)$ denote the optimal value of the optimization
 167 problem (\ast) . In particular, $\text{val}(\mathbf{P})$ denotes the optimal value of the original moral hazard problem
 168 (P). The single constraint in (IC) of the form
 169

$$170 \quad U(w, a) - U(w, \hat{a}) \geq 0, \quad (\text{NJ}(a, \hat{a}))$$

171 is called the *no-jump* constraint at \hat{a} given a .

172 2.2 Existing approaches

173 We discuss the approaches to solve (P) that appear in the literature and their limitations. The
 174 standard-bearer is the FOA, which replaces (IC) with first-order conditions. Every implementable
 175 action a is an optimizer of the agent's problem and so satisfies necessary optimality conditions for
 176 that problem. In particular, a satisfies the first-order necessary condition:

$$177 \quad U_a(w, a) = 0 \text{ if } a \in (\underline{a}, \bar{a}), U_a(w, a) \leq 0 \text{ if } a = \underline{a}, \text{ and } U_a(w, a) \geq 0 \text{ if } a = \bar{a} \quad (\text{FOC}(a))$$

178 where the subscripts denote partial derivatives. Replacing (IC) with (FOC(a)), problem (P) be-
 179 comes

$$180 \quad \max_{w \geq \underline{w}, a \in \mathbb{A}} \{V(w, a) : U(w, a) \geq \underline{U} \text{ and } (\text{FOC}(a))\}. \quad (\text{FOA})$$

181 When (FOA) and (P) have the same value (that is, $\text{val}(\mathbf{P}) = \text{val}(\text{FOA})$) and the solution (w, a) to
 182 (FOA) has a implemented by w , we say the FOA is *valid*. Otherwise, the FOA is *invalid*.

183 Following Mirrlees (1999), we consider a special (very simplified) case of the moral hazard
 184 model that facilitates a geometric understanding of the technical issues involved. We return to
 185 this example at several points throughout the paper to ground our intuition. Section 6 has three
 186 additional examples of more general moral hazard problems that provide additional insights.

187 **Example 1.** Suppose the principal chooses contract $z \in \mathbb{R}$ (following [Mirrlees \(1999\)](#)) we use z
188 to denote a single-dimensional contract instead of w) and the agent chooses an action $a \in [-2, 2]$
189 with reservation utility $\underline{U} = -2$. There is no lower bound on z . The principal obtains utility
190 $v(z, a) = za - 2a^2$ and the agent receives benefit $-za$, minus action cost $c(a) = (a^2 - 1)^2$, with
191 total agent utility

$$u(z, a) = -za - (a^2 - 1)^2.$$

193 The principal's problem is

$$\max_{(z,a)} \{v(z, a) : u(z, a) \geq -2 \text{ and } a \in \arg \max_{a'} u(z, a')\}. \quad (1)$$

195 If we use the FOA, the solutions are $(z, a) = (\frac{3}{2}, \frac{1}{2})$ and $(-\frac{3}{2}, -\frac{1}{2})$ which are not incentive compatible.
196 Thus, the FOA is invalid.

197 Since this problem is so simple we can solve it by inspection. We show that $(z, a) = \{(0, 1), (0, -1)\}$
198 is the set of optimal solutions to (1). Clearly, $a = \pm 1$ is a best response to $z = 0$, providing a utility
199 of -2 for the principal. To show that $z \neq 0$ is not an optimal choice for the principal, observe that
200 for a fixed z the agent's first-order condition yields

$$a(a^2 - 1) = -z/4 \quad (2)$$

203 where

$$\text{sgn}(a(a^2 - 1)) = \begin{cases} + & \text{if } a > 1 \text{ or } a \in (-1, 0) \\ - & \text{if } a < -1 \text{ or } a \in (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

206 Thus, from (2) if $z > 0$ then the optimal choice of a is either $a < -1$ or $a \in (0, 1)$ (the corner
207 solution $a = 2$ is not optimal since $\frac{\partial}{\partial a} u(z, 2) < 0$). Also, observe that $a \in (0, 1)$ cannot be optimal
208 since choosing action $-a$ instead only improves the agent's utility. Hence, an optimal response
209 to $z > 0$ must satisfy $a < -1$. However, this implies that $v(z, a) < -2$, and so $z > 0$ is not an
210 optimal choice of the principal (setting $z = 0$ gives the principal a utility of -2). Nearly identical
211 reasoning shows that $z < 0$ is also not an optimal choice for the principal. This verifies that
212 $(z^*, a^*) = \{(0, 1), (0, -1)\}$ are the optimal solutions to (1). ◀

213 To handle situations where the FOA is invalid, [Mirrlees \(1999\)](#) recognized that difficulties
214 arise when pairs (w, a) satisfy [\(FOC\(a\)\)](#) but w fails to implement a . To combat this, Mirrlees
215 reintroduced no-jump constraints from [\(IC\)](#). The resulting problem (cf. [Mirrlees \(1986\)](#)) is:

$$\max_{(w,a)} V(w, a) \quad (3a)$$

$$\text{subject to } U(w, a) \geq \underline{U}, \quad (3b)$$

$$U_a(w, a) = 0 \quad (3c)$$

$$U(w, a) - U(w, \hat{a}) \geq 0, \text{ for all } \hat{a} \text{ such that } U_a(w, \hat{a}) = 0 \quad (3d)$$

221 (where the complication of corner solutions is ignored for simplicity).⁴ If a candidate contract
222 violates a no-jump constraint in (3d) then an optimizing agent can improve his expected utility by

⁴If corner solutions are considered, (3c) is replaced by [\(FOC\(a\)\)](#) and instead of (3d), we have one no-jump constraint for every \hat{a} such that [\(FOC\(\$\hat{a}\$ \)\)](#) holds.

223 “jumping” to an alternate best response. The *best* choice of alternate action \hat{a}^* given w is included
 224 among the no-jump constraints, since such an \hat{a}^* satisfies the first-order condition $U_a(w, \hat{a}^*) = 0$.
 225 Hence if a candidate contract satisfies all no-jump constraints it must implement a^* . The practical
 226 challenge in applying Mirrlees’s approach is generating all of the necessary no-jump constraints.
 227 In principle, it requires knowing all of the stationary points to the agent’s problem for every
 228 feasible contract. This enumeration of policies may well be intractable, and no general procedure
 229 to systematically produce them is known. However, if additional information can guide the choice
 230 of no-jump constraints (such as having *a priori* knowledge of the optimal contract and its best
 231 responses) then Mirrlees approach can indeed recover the optimal contract. The following example
 232 demonstrates this approach and is in the spirit of how Mirrlees illustrated his method.

233 **Example 2** (Example 1 continued). If we know *a priori* the two best responses to an optimal
 234 contract, $\hat{a} = 1$ and $\hat{a} = -1$ (as determined in Example 1), we may solve (1) in the following
 235 manner:

$$236 \quad \max_{(z,a)} v(z, a)$$

237 subject to the first-order condition

$$238 \quad u_a(z, a) = -4a(a^2 - 1) - z = 0$$

239 and no-jump constraints

$$240 \quad u(z, a) - u(z, \hat{a}) \geq 0$$

241 for $\hat{a} \in \{1, -1\}$. According to (3), we should include many more no-jump constraints, but in fact we
 242 show these two are sufficient to determine the optimal solution. Figure 1 illustrates the constraint
 243 sets and optimal solutions.

244 We plot the first-order condition curve (blue line), the best response set (bold part of blue line)
 245 and the regions for the two constraints (the shaded regions in the graph):

$$246 \quad u(z, a) - u(z, 1) \geq 0$$

$$247 \quad u(z, a) - u(z, -1) \geq 0.$$

248 The region $\{(z, a) : u(z, a) - u(z, \hat{a}) \geq 0\}$ lies below the curve

$$249 \quad z = -(\hat{a} + a)(\hat{a}^2 + a^2 - 2)$$

250 for $a > \hat{a}$ and above the curve for $a < \hat{a}$. These constraints preclude the optimal solution of the
 251 FOA: $(z, a) = (\frac{3}{2}, \frac{1}{2})$ and $(-\frac{3}{2}, -\frac{1}{2})$. The only contract-action pairs that satisfy these conditions are
 252 $(z^*, a^*) = \{(0, 1), (0, -1)\}$, the optimal solutions to (1) (as established in Example 1). ◀

253 In our approach we show how, under additional monotonicity assumptions, reintroducing a
 254 single no-jump constraint is all that is required. Moreover, this single constraint can be found by
 255 solving an optimization problem in the alternate action \hat{a} . The next two sections describe and
 256 justify this procedure.

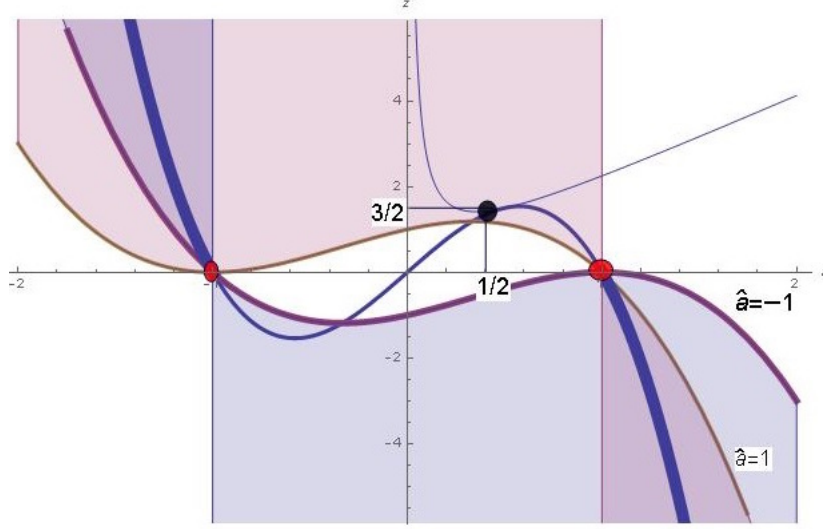


Figure 1: Figure for Example 2. The blue curve is the first-order condition curve, the light-blue region captures those points that satisfy $u(z, a) - u(z, -1) \geq 0$ and the light-red region captures those points that satisfy $u(z, a) - u(z, 1) \geq 0$.

257 3 The sandwich relaxation

258 We first introduce a family of restrictions of (P) that vary the right-hand side of the (IR) constraint
 259 (for reasons that will become clear later). Consider the parametric problem:

$$\begin{aligned}
 & \max_{w \geq \underline{w}, a \in \mathbb{A}} V(w, a) \\
 & \text{subject to } U(w, a) \geq b \tag{P|b} \\
 & U(w, a) - U(w, \hat{a}) \geq 0 \quad \text{for all } \hat{a} \in \mathbb{A}
 \end{aligned}$$

264 with parameter $b \geq \underline{U}$. The original problem (P) is precisely (P| \underline{U}). We restrict $b \geq \underline{U}$ so that
 265 $\text{val}(\text{P}|b) \leq \text{val}(\text{P})$ and a feasible solution of (P|b) remains feasible to (P). We restate (P|b) using
 266 an inner minimization over \hat{a} . Observe that (P|b) is equivalent to

$$\begin{aligned}
 & \max_{w \geq \underline{w}, a \in \mathbb{A}} V(w, a) \\
 & \text{subject to } U(w, a) \geq b \\
 & \inf_{\hat{a} \in \mathbb{A}} \{U(w, a) - U(w, \hat{a})\} \geq 0. \tag{4}
 \end{aligned}$$

271 To clarify the relationships between w , a , and \hat{a} , we pull the minimization operator out from
 272 the constraint (4) and behind the objective function. This requires handling the possibility that a
 273 choice of w does not implement the chosen a , in which case (4) is violated. We handle this issue as
 274 follows. Given $b \geq \underline{U}$, define the set

$$\mathcal{W}(\hat{a}, b) \equiv \{(w, a) : U(w, a) \geq b \text{ and } U(w, a) - U(w, \hat{a}) \geq 0\},$$

277 and the characteristic function

$$278 \quad V^I(w, a|\hat{a}, b) \equiv \begin{cases} V(w, a) & \text{if } (w, a) \in \mathcal{W}(\hat{a}, b) \\ -\infty & \text{otherwise.} \end{cases} \quad (5)$$

279

280 This is constructed so that if $V^I(w, a|\hat{a}, b)$ is maximized over (w, a) at a finite objective value then
 281 $(w, a) \in \mathcal{W}(\hat{a}, b)$. Similarly, if maximizing $\inf_{\hat{a} \in \mathbb{A}} V^I(w, a|\hat{a}, b)$ over (w, a) results in a finite objective
 282 value then we know (w, a) lies in $\mathcal{W}(\hat{a}, b)$ for all $\hat{a} \in \mathbb{A}$. This implies (w, a) is feasible to $(\mathbf{P}|b)$ and
 283 demonstrates the equivalence of $(\mathbf{P}|b)$ and the problem

$$284 \quad \max_{a \in \mathbb{A}} \max_{w \geq \underline{w}} \inf_{\hat{a} \in \mathbb{A}} V^I(w, a|\hat{a}, b). \quad (\text{Max-Max-Min}|b)$$

285 We explore what transpires when swapping the order of optimization in $(\text{Max-Max-Min}|b)$ so
 286 that \hat{a} is chosen *before* w . That is, we consider

$$287 \quad \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} \max_{w \geq \underline{w}} V^I(w, a|\hat{a}, b)$$

288 which is equivalent to

$$289 \quad \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} \max_{w \geq \underline{w}} \{V(w, a) : (w, a) \in \mathcal{W}(\hat{a}, b)\} \quad (\text{SAND}|b)$$

290 since an optimal choice of a precludes a subsequent optimal choice of \hat{a} that sets $\mathcal{W}(\hat{a}, b) = \emptyset$, imply-
 291 ing $V^I(w, a|\hat{a}, b) = V(w, a)$ when w is optimally chosen. We call $(\text{SAND}|b)$ the *sandwich problem*
 292 given bound b , where “sandwich” refers to the fact that the minimization over \hat{a} is sandwiched
 293 between two maximizations.

294 Our method allows for the nonexistence of a minimizer to the inner minimization over \hat{a} . On
 295 the other hand, the next lemma shows that the outer maximization over a always possesses a
 296 maximizer. This follows by establishing the upper semicontinuity of the value function over the
 297 inner two optimization problems.

298 **Lemma 1.** There exists a maximizer to the outer maximization problem in $(\text{SAND}|b)$.

299 Even when the inner minimization over \hat{a} does not exist we call (a^*, w^*) where $V(w^*, a^*) =$
 300 $\text{val}(\text{SAND}|b)$ an optimal solution to $(\text{SAND}|b)$. If the inner minimization is attained at an action
 301 \hat{a}^* then we can say (a^*, \hat{a}^*, w^*) is an optimal solution to $(\text{SAND}|b)$ without confusion.

302 **Lemma 2.** For every $b \geq \underline{U}$, $\text{val}(\mathbf{P}|b) \leq \text{val}(\text{SAND}|b)$. Moreover, if there exists an optimal solution
 303 (w^*, a^*) to (\mathbf{P}) such that $U(w^*, a^*) \geq b$ then $\text{val}(\mathbf{P}) \leq \text{val}(\text{SAND}|b)$.

304 From Lemma 2 we are justified in calling $(\text{SAND}|b)$ the *sandwich relaxation* of $(\mathbf{P}|b)$. There are
 305 two related benefits to studying the sandwich relaxation. First, changing the order of optimization
 306 from Max-Max-Min to Max-Min-Max improves analytical tractability. The map that describes
 307 which optimal contracts support a given action a against deviation to a specific alternate best
 308 response \hat{a} has desirable topological properties. These properties can be used to determine the
 309 “minimizing” alternative best response without resort to enumeration, as is required in the worst-
 310 case in Mirrlees’s approach. By contrast, to solve the original problem $(\text{Max-Max-Min}|b)$ one must
 311 work with the best-response set $a^{BR}(w)$ as a constraint for the inner maximization over w . The
 312 best-response set is notoriously ill-structured. More details are found in Section 3.1.

313 Second, if b satisfies a property called tightness-at-optimality (defined below), and other suffi-
 314 cient conditions are met, the sandwich relaxation is *equivalent* to (\mathbf{P}) . More details are found in
 315 Section 3.2.

3.1 Analytical benefit of changing the order of optimization

By changing the order of optimization, we solve for an optimal contract w given a choice of implementable action a and alternate best response \hat{a} . The resulting problem is:

$$\max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U(w, a) - U(w, \hat{a}) \geq 0\}. \quad (\text{SAND}|a, \hat{a}, b)$$

We show that this problem has a unique solution and provide necessary and sufficient optimality conditions.

The approach is to use Lagrangian duality. The Lagrangian function of (SAND| a, \hat{a}, b) is

$$\mathcal{L}(w, \lambda, \delta|a, \hat{a}, b) = V(w, a) + \lambda[U(w, a) - b] + \delta[U(w, a) - U(w, \hat{a})], \quad (6)$$

where $\lambda \geq 0$ and $\delta \geq 0$ are the multipliers for $U(w, a) \geq b$ and $U(w, a) - U(w, \hat{a}) \geq 0$, respectively. The Lagrangian dual is

$$\inf_{\lambda, \delta \geq 0} \max_{w \geq \underline{w}} \mathcal{L}(w, \lambda, \delta|a, \hat{a}, b). \quad (7)$$

Consider the inner maximization problem of (7) over w . By Assumption (A1.4) we can express the Lagrangian (6) as

$$\mathcal{L}(w, \lambda, \delta|a, \hat{a}, b) = \int L(w(x), \lambda, \delta|x, a, \hat{a}, b) f(x, a) dx$$

where $L(\cdot, \cdot, \cdot|x, a, \hat{a}, b)$ is a function from $\mathbb{R}^3 \rightarrow \mathbb{R}$ with

$$\begin{aligned} L(y, \lambda, \delta|x, a, \hat{a}, b) &= v(\pi(x) - y) + \lambda(u(y) - c(a) - b) + \delta \left[u(y) \left(1 - \frac{f(x, \hat{a})}{f(x, a)} \right) - c(a) + c(\hat{a}) \right] \\ &= v(\pi(x) - y) + \left[\lambda + \delta \left(1 - \frac{f(x, \hat{a})}{f(x, a)} \right) \right] u(y) - \lambda(c(a) + b) - \delta(c(a) - c(\hat{a})) \end{aligned} \quad (8)$$

where the ratio $1 - \frac{f(x, \hat{a})}{f(x, a)}$ results from factoring $f(x, a)$ from the terms involving u . This is possible since $f(\cdot, a)$ has the same support for all a .

The inner maximization of $\mathcal{L}(w, \lambda, \delta|a, \hat{a}, b)$ over w in (7) can be done pointwise via

$$\max_{y \geq \underline{w}} L(y, \lambda, \delta|x, a, \hat{a}, b) \quad (9)$$

for each x and setting $w(x) = y$ where y is an optimal solution to (9). Two cases can occur. If $\lambda + \delta \left(1 - \frac{f(x, \hat{a})}{f(x, a)} \right) \leq 0$ then $L(y, \lambda, \delta|x, a, \hat{a}, b)$ is a decreasing function of y by Assumptions (A1.9) and (A1.10). Hence, the unique optimal solution to (9) is $y = \underline{w}$.

On the other hand, if $\lambda + \delta \left(1 - \frac{f(x, \hat{a})}{f(x, a)} \right) > 0$ then $L(y, \lambda, \delta|x, \hat{a})$ is strictly concave in y (again by Assumptions (A1.9) and (A1.10)). If $\frac{\partial}{\partial y} L(\underline{w}, \lambda, \delta|x, a, \hat{a}, b) \leq 0$ then the corner solution $y = \underline{w}$ is optimal, otherwise there exists a unique y such that $\frac{\partial}{\partial y} L(y, \lambda, \delta|x, a, \hat{a}, b) = 0$ holds. In both cases (9) has a unique optimal solution $w(x)$. Hence, the optimal solution $w : \mathcal{X} \rightarrow \mathbb{R}$ to the inner maximization of (7) satisfies:

$$w(x) \begin{cases} \text{solves } \frac{\partial}{\partial y} L(w(x), \lambda, \delta|x, a, \hat{a}, b) = 0 & \text{if } \lambda + \delta \left(1 - \frac{f(x, \hat{a})}{f(x, a)} \right) > 0 \text{ and } \frac{\partial}{\partial y} L(\underline{w}, \lambda, \delta|x, a, \hat{a}, b) > 0 \\ = \underline{w} & \text{otherwise.} \end{cases}$$

351 Expressing the derivatives and dividing by $u'(w(x))$ (which is valid since $u' > 0$ by (A1.9)) yields

$$352 \quad w(x) \begin{cases} \text{solves } \frac{v'(\pi(x)-w(x))}{u'(w(x))} = \lambda + \delta \left(1 - \frac{f(x,\hat{a})}{f(x,a)}\right) & \text{if } \frac{v'(\pi(x)-w)}{u'(\underline{w})} < \lambda + \delta \left(1 - \frac{f(x,\hat{a})}{f(x,a)}\right) \\ = \underline{w} & \text{otherwise.} \end{cases} \quad (10)$$

354 Since v' and u' are both positive, the condition $\frac{v'(\pi(x)-w)}{u'(\underline{w})} < \lambda + \delta \left(1 - \frac{f(x,\hat{a})}{f(x,a)}\right)$ implies $\lambda +$
355 $\delta \left(1 - \frac{f(x,\hat{a})}{f(x,a)}\right) > 0$, handling both cases detailed above.

356 We have just shown that, given $(\lambda, \delta, a, \hat{a}, b)$, there is a unique choice w , denoted $w_{\lambda,\delta}(a, \hat{a}, b)$,
357 that satisfies (10). Such contracts are significant for our analysis and warrant a formal definition.

358 **Definition 1.** Any contract that satisfies (10) for some choice of $(\lambda, \delta, a, \hat{a}, b)$ is called a *generalized Mirrlees-Holmstrom (GMH) contract*. These contracts are generalized versions of Mirrlees-
359 Holmstrom contracts known for the special case of a binary action.
360

361 Observe that GMH contracts are continuous in x . There are five parameters $(\lambda, \delta, a, \hat{a}, b)$ in a
362 GMH contract. However, Lemma 3 below shows each GMH contract is a function of only three
363 parameters: a, \hat{a} and b .

364 **Lemma 3.** Suppose Assumptions 1–3 hold. For every (a, \hat{a}, b) with $\hat{a} \neq a$ there exists a *unique*
365 Lagrangian multipliers λ^* and δ^* and a *unique* contract w^* such that

- 366 (i) w^* satisfies (10) for λ^* and δ^* (in particular, w^* is a GMH contract),
367 (ii) strong duality between (SAND| a, \hat{a}, b) and (7) holds and, in particular, the complementary
368 slackness conditions

$$369 \quad \lambda^* \geq 0, U(w^*, a) - b \geq 0 \quad \text{and} \quad \lambda^*[U(w^*, a) - b] = 0, \quad (\text{ii-a})$$

$$370 \quad \delta^* \geq 0, U(w^*, a) - U(w^*, \hat{a}) \geq 0 \quad \text{and} \quad \delta^*[U(w^*, a) - U(w^*, \hat{a})] = 0, \quad (\text{ii-b})$$

372 are satisfied.

373 Moreover, the following additional properties hold:

- 374 (iii) $(\lambda^*, \delta^*) = (\lambda(a, \hat{a}, b), \delta(a, \hat{a}, b))$ is an upper semicontinuous function of (a, \hat{a}, b) and is contin-
375 uous and differentiable at any (a, \hat{a}, b) where $a \neq \hat{a}$.
376 (iv) $w^* = w_{\lambda(a,\hat{a},b),\delta(a,\hat{a},b)}(a, \hat{a}, b)$ is an upper semicontinuous function of (a, \hat{a}, b) and continuous
377 and differentiable at any (a, \hat{a}, b) where $a \neq \hat{a}$.

378 Lemma 3(iv) leaves open the possibility that there is a jump discontinuity when $a = \hat{a}$. As
379 an illustration, consider the case where the principal is risk-neutral and the FOA is valid. When
380 $\hat{a} > a$, the optimal solution to (SAND| a, \hat{a}, b) is the first best contract. However, as $\hat{a} - a \rightarrow 0^-$ we
381 have

$$382 \quad \lim_{\hat{a}-a \rightarrow 0^-} V(w_{\lambda(a,\hat{a},b),\delta(a,\hat{a},b)}(a, \hat{a}, b), a) = \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U_a(w, a) = 0\}$$

$$383 \quad < \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b\}.$$

384 Therefore, the value function is not continuous at that point.⁵

385 Lemma 3 provides insight into the inner “inf” of (SAND| b). Given an $a \in \mathbb{A}$, suppose the
 386 infimizing sequence \hat{a}^n to the inner “inf” converges to some a' . If $a' \neq a$ then, in fact, the infimum
 387 is attained by the continuity of w^* from Lemma 3(iv). An issue arises if $a' = a$ and the infimum is
 388 not attained, since this a point of discontinuity of w^* . The following result analyzes this scenario.
 389 We also refer the reader to Section 5 below, which provides additional discussion of this case.

390 **Lemma 4.** If the minimization of $\inf_{\hat{a}} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U(w, a) - U(w, \hat{a}) \geq 0\}$ is
 391 not attained, then

$$\inf_{\hat{a}} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U(w, a) - U(w, \hat{a}) \geq 0\} = \max_w \{V(w, a) : U(w, a) \geq b, (\text{FOC}(\mathbf{a}))\} \quad (11)$$

392 where (FOC(\mathbf{a})) is as defined in Section 2.2.

393 This result shows that when the infimum is not attained for a given action a , it suffices to take
 394 a “first-order approach” locally at a .

396 3.2 Tightness of the sandwich relaxation

397 The previous subsection provides a toolbox for analyzing the sandwich relaxation (SAND| b). How-
 398 ever, there remains the question of whether that relaxation is worth solving at all. In particular, we
 399 may ask whether there exists a b that makes (SAND| b) a *tight* relaxation; i.e., whether an optimal
 400 solution (a^*, w^*) to (SAND| b) yields an optimal solution (w^*, a^*) to (P). The following example
 401 illustrates a situation where such a choice is possible.

402 **Example 3** (Example 1 continued). We solve the sandwich relaxation (SAND|0) and show that
 403 (SAND|0) is a tight relaxation.⁶ That is, we solve:

$$\max_{a \in [-2, 2]} \inf_{\hat{a} \in [-2, 2]} \max_z \{v(z, a) : u(z, a) \geq 0 \text{ and } u(z, a) - u(z, \hat{a}) \geq 0\} \quad (12)$$

404 where

$$405 \quad v(z, a) = za - 2a^2 \text{ and } u(z, a) = -za - (a^2 - 1)^2.$$

406 We break up the outermost optimization (over a) across two subregions of $[-2, 0]$ and $[0, 2]$. The
 407 optimal value of (12) can be found by taking the larger of the two values across the two subregions.
 408 We consider $a \in [0, 2]$ first. In this case $v(z, a)$ is increasing in z and thus \hat{a} is chosen to minimize
 409 z . We show how z relates to the choice of a and \hat{a} . The $u(z, a) \geq 0$ constraint cannot be satisfied
 410 when $a = 0$ and so is equivalent to

$$411 \quad z \leq -\frac{(a^2 - 1)^2}{a}, \quad (13)$$

412 since dividing by $a \neq 0$ is legitimate. The no-jump constraint $u(z, a) - u(z, \hat{a}) \geq 0$ is equivalent to

$$413 \quad z \begin{cases} \geq -(\hat{a} + a)(\hat{a}^2 + a^2 - 2) & \text{for } \hat{a} > a \\ \leq -(\hat{a} + a)(\hat{a}^2 + a^2 - 2) & \text{for } \hat{a} < a \\ \in (-\infty, \infty) & \text{for } \hat{a} = a. \end{cases} \quad (14)$$

⁵We thank an anonymous referee for alerting us to this observation.

⁶In fact, one can show that setting $b = \underline{U} = -2$ does not give rise to a tight relaxation. For details see the discussion following (38) below.

417 Clearly, $\hat{a} = a$ will never be chosen in the inner minimization over \hat{a} in (12) since it cannot prevent
 418 sending $z \rightarrow \infty$, when the goal is to minimize z . When $\hat{a} > a$ observe that

$$\begin{aligned}
 419 \quad & \inf_{\hat{a} > a} -(\hat{a} + a)(\hat{a}^2 + a^2 - 2) \\
 420 \quad & = \begin{cases} 4a - 4a^3 & \text{for } 1/\sqrt{3} \leq a \leq 2 \\ \frac{4}{27}(9a - 5a^3) + \frac{4}{27}\sqrt{2}\sqrt{(3 - a^2)^3} & \text{for } 0 \leq a \leq 1/\sqrt{3}. \end{cases} \quad (15)
 \end{aligned}$$

421 When $a \in [0, 1)$ one can verify that

$$422 \quad \inf_{\hat{a} > a} -(\hat{a} + a)(\hat{a}^2 + a^2 - 2) > 0 > -\frac{(a^2-1)^2}{a}$$

423 using (15). By (14) this implies $z > \frac{(a^2-1)^2}{a}$ when $\hat{a} > a$, violating (13). Hence, when $a \in [0, 1)$, the
 424 inner minimization over \hat{a} in (12) will choose $\hat{a} > a$ and thus make a choice of z infeasible. This
 425 drives the value of the inner minimization over \hat{a} to $-\infty$. This, in turn, implies that $a \in [0, 1)$ will
 426 never be chosen in the outer maximization, and so we may restrict attention to $a \in [1, 2]$.

427 When $a \in [1, 2]$ we return to (14) and consider the two cases: (i) $\hat{a} > a$ and (ii) $\hat{a} < a$. In case
 428 (i) note that

$$429 \quad \inf_{\hat{a} > a} -(\hat{a} + a)(\hat{a}^2 + a^2 - 2) = 4a - 4a^3 \leq -\frac{(a^2-1)^2}{a},$$

430 when $a \in [1, 2]$ and so from (13)–(15) we have

$$431 \quad 4a - 4a^3 \leq z \leq -\frac{(a^2-1)^2}{a}. \quad (16)$$

432 However in case (ii) we have from (13) and (14) that

$$433 \quad z \leq \min \left\{ \frac{(a^2-1)^2}{a}, \inf_{\hat{a} < a} -(\hat{a} + a)(\hat{a}^2 + a^2 - 2) \right\}. \quad (17)$$

434 Note that

$$435 \quad \inf_{\hat{a} < a} -(\hat{a} + a)(\hat{a}^2 + a^2 - 2) = 4a - 4a^3 \text{ for } 1 \leq a \leq 2 \quad (18)$$

436 and $4a - 4a^3 < -\frac{(a^2-1)^2}{a}$ when $a \in [1, 2]$. Observe that the infimum is not attained since the only
 437 real solution to $-(\hat{a} + a)(\hat{a}^2 + a^2 - 2) = 4a - 4a^3$ when $a \in [1, 2]$ is $\hat{a} = a$. Lemma 4 applies and
 438 yields

$$439 \quad z^*(a) = 4a - 4a^3 \quad (19)$$

440 via (18). Since the principal's utility $v(z^*(a), a)$ is decreasing in $a \in [1, 2]$, we obtain the solution
 441 $a^* = 1$ and the optimal choice of z^* is thus $z^*(1) = 0$. One can see this graphically in Figure 2.⁷

442 We return to the case where $a \in [-2, 0]$. Nearly identical reasoning (with care to adjust negative
 443 signs) shows $a^* = -1$ and, again, the optimal choice of z is $z^*(1) = 0$. Hence, the overall problem
 444 (12) gives rise to *two* optimal choices of (z^*, a^*) , namely $(0, 1)$ and $(0, -1)$. However, this is precisely
 445 the optimal solution to the original problem (1), as shown by inspection in Example 1. ◀

⁷This example has the special structure that the FOA applies *locally*. That is, given an a , the optimal choice of z is uniquely determined by the first-order condition. The classic example in Mirrlees (1999) also has this property.

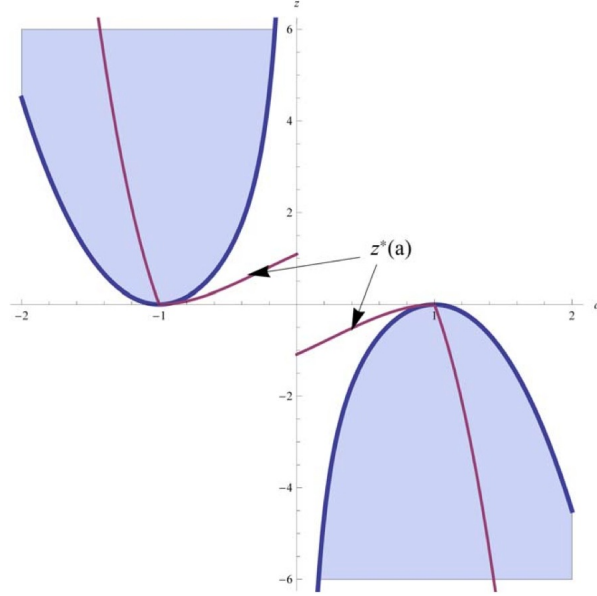


Figure 2: Figure for Example 3. The blue curve and region are those (z, a) that satisfy the constraint $U(z, a) \geq 0$. The red curve are those (z, a) that satisfy the inner maximization over z given by (19). Observe that the optimal solution in the region $a \in [0, 2]$ is $(z, a) = (0, 1)$ since the principal's utility is increasing in z .

449 Note that by choosing b correctly in the above example we were able to arrive at the first-order
 450 condition curve $U_a(z, a) = 0$ used in Mirrlees's approach. This underscores that we do not need
 451 to *explicitly* include the FOC in our definition of the sandwich relaxation. This issue is taken up
 452 more carefully in Section 5. Comparing Figure 1 and Figure 2 we see that the (IR) is not needed
 453 to specify the optimal contract in Figure 1, but is needed (with an adjusted right-hand side) when
 454 using the sandwich relaxation in Figure 2. However, the first-order condition curve does not appear
 455 in Figure 2 to characterize the optimal contract.

456 Of course, the question remains as to whether there always exists a b such that (SAND| b) is a
 457 tight relation of (P), and if so, how to determine it. We make the following definition.

458 **Definition 2.** We say $b \geq \underline{U}$ is *tight-at-optimality* (or simply tight) if there exists an optimal
 459 solution (w^*, a^*) to (P) such that $b = U(w^*, a^*)$.

460 At least one such b exists by Assumption 3. The main result of this section is to show that for
 461 such a b , the sandwich relaxation (SAND| b) is tight under certain conditions. The key assumption
 462 is that f satisfies the *monotone likelihood ratio property* (MLRP) where for any a , $\frac{\partial \log f(\cdot, a)}{\partial a}$ is
 463 nondecreasing. This property is well-known in the literature (see Holmstrom (1979), Rogerson
 464 (1985) and others).

465 **Assumption 4.** The output distribution f satisfies the MLRP condition.

466 The following is the key technical result of the paper.

467 **Theorem 1.** Suppose Assumptions 1–4 hold. If b is tight-at-optimality then (SAND| b) is a tight
468 relaxation; that is, $\text{val}(\text{SAND}|b) = \text{val}(\mathbf{P})$ and if $(a^\#, \hat{a}^\#, w^\#)$ is an optimal solution to (SAND| b)
469 then $(w^\#, a^\#)$ is an optimal solution to (P). If the infimum in (SAND| b) is not attained, and
470 $(a^\#, w^\#)$ is an optimal solution to the inner and outer maximization in (SAND| b), then $(w^\#, a^\#)$
471 is an optimal solution to (P).

472 The proof of Theorem 1 is involved and relies on several nontrivial, but largely technical,
473 intermediate results. Full details are found in the appendices, along with further discussion. We
474 note that Lemma 4 is essential for the case where the infimum is not attained.

475 For the sake of developing intuition regarding the proof of Theorem 1, we consider here the
476 special case where \mathcal{X} is a singleton and the inner infimum is attained. Of course, the single-
477 outcome case is not a difficult problem to solve and provides little economic intuition, but it does
478 highlight some important features of the more general argument that we discuss below.

479 When \mathcal{X} is a singleton, contracts w are characterized by a single number $z = w(x_0)$ (following the
480 notation of Example 2 and Mirrlees (1999)) and so $U(w, a) = u(z) - c(a)$ and $V(w, a) = v(\pi(x_0) - z)$.
481 For consistency we denote the minimum wage by \underline{z} (as opposed to \underline{w}).

482 *Proof of Theorem 1 for a single-dimensional contract.* Since b is tight-at-optimality, there exists an
483 optimal solution (z^*, a^*) of (P) such that $b = U(z^*, a^*)$. Let $(a^\#, \hat{a}^\#, z^\#)$ be an optimal solution to
484 (SAND| b).

485 There are two cases to consider.

486 *Case 1:* $U(z^\#, a^\#) = b$.

487 By Lemma 2 we know $\text{val}(\mathbf{P}) \leq \text{val}(\text{SAND}|b)$. It suffices to argue that $\text{val}(\text{SAND}|b) \leq \text{val}(\mathbf{P})$.
488 By the optimality of $(a^\#, \hat{a}^\#, z^\#)$ in (SAND| b) we know

$$489 V(z^\#, a^\#) = \inf_{\hat{a} \in \mathbb{A}} \max_{z \geq \underline{z}} \left\{ V(z, a^\#) : U(z, a^\#) \geq b, U(z, a^\#) - U(z, \hat{a}) \geq 0 \right\}. \quad (20)$$

491 Let \hat{a}' be a best response to $z^\#$. Then from the minimization over \hat{a} in (20) we have

$$492 V(z^\#, a^\#) \leq \max_{z \geq \underline{z}} \left\{ V(z, a^\#) : U(z, a^\#) \geq b, U(z, a^\#) - U(z, \hat{a}') \geq 0 \right\}. \quad (21)$$

494 Suppose (21) holds with equality. Since V is decreasing in z (under Assumption (A1.10)) and
495 the feasible region is single-dimensional, the optimal solution to the right-hand side problem is
496 unique and therefore $z^\#$ must be that unique optimal solution under the equality assumption. This
497 implies $z^\#$ is feasible to the right-hand side problem and so $U(z^\#, a^\#) \geq U(z^\#, \hat{a}')$. Since \hat{a}' is
498 a best response to $z^\#$, $a^\#$ is also. This implies that $(z^\#, a^\#)$ is a feasible solution to (P). Thus,
499 $\text{val}(\text{SAND}|b) \leq \text{val}(\mathbf{P})$, establishing the result.

500 Hence, it remains to argue that (21) is satisfied with equality. Suppose otherwise that

$$501 V(z^\#, a^\#) < \max_{z \geq \underline{z}} \left\{ V(z, a^\#) : U(z, a^\#) \geq b, U(z, a^\#) - U(z, \hat{a}') \geq 0 \right\}. \quad (22)$$

503 There must exist a z' in the argmax of right-hand side such that $V(z^\#, a^\#) < V(z', a^\#)$. Since V
504 is strictly decreasing in z , this implies $z^\# > z'$. However, since U is increasing in z this further
505 implies that $U(z', a^\#) < U(z^\#, a^\#) = b$ (where the equality holds under the assumption of Case
506 1). That is, $U(z', a^\#) < b$, contradicting the feasibility of z' to (SAND| b).

507 *Case 2:* $U(z^\#, a^\#) > b$.

508 This requires the following intermediate lemma, whose proof is in the appendix:

509 **Lemma 5.** Let $(a^\#, z^\#)$ be an optimal solution to the single-dimensional version of $(\text{SAND}|b)$ with
510 $U(z^\#, a^\#) > b$ (in particular, the infimum in $(\text{SAND}|b)$ need not be attained). Then there exists
511 an $\epsilon > 0$ such that the perturbed problem $(\text{SAND}|b + \epsilon)$ also has an optimal solution $(a_\epsilon^\#, z_\epsilon^\#)$ with
512 $U(z_\epsilon^\#, a_\epsilon^\#) = b + \epsilon$ and the same optimal value; that is, $V(z_\epsilon^\#, a_\epsilon^\#) = V(z^\#, a^\#) = \text{val}(\text{SAND}|b)$.

513 The proof of this lemma relies on strong duality and the fact that if a constraint is slack, the dual
514 multiplier on that constraint is 0 by complementary slackness. A small perturbation of the right-
515 hand side of a slack constraint does not impact the optimal value. This argument is standard (see
516 for instance, Bertsekas (1999)) in the absence of the inner minimization problem $\inf_{\hat{a}}$ in $(\text{SAND}|b)$.
517 With the inner minimization, the proof becomes nontrivial.

518 Returning to our proof of Case 2, by Lemma 5 there exists an $\epsilon > 0$ and an optimal solution
519 $(a_\epsilon^\#, z_\epsilon^\#)$ to $(\text{SAND}|b + \epsilon)$ where $U(z_\epsilon^\#, a_\epsilon^\#) = b + \epsilon$ and $\text{val}(\text{SAND}|b + \epsilon) = \text{val}(\text{SAND}|b)$. Apply the
520 logic of Case 1 to the problem $(\text{SAND}|b + \epsilon)$ and conclude that $\text{val}(\text{SAND}|b + \epsilon) = \text{val}(\mathbf{P})$. Hence,
521 since $\text{val}(\text{SAND}|b + \epsilon) = \text{val}(\text{SAND}|b)$, $(\text{SAND}|b)$ is a tight-relaxation of (\mathbf{P}) . \square

522 We provide here some intuition behind Theorem 1 in the single-outcome setting. For a given
523 target action a^* , we can think of the contracting problem as a sequential game where the principal
524 chooses z and the agent chooses \hat{a} . The original (\mathbf{IC}) constraint is equivalent to the situation that
525 the principal chooses z first, followed by the agent's choice of \hat{a} . So the optimal choice of z should
526 take all possible \hat{a} into consideration. The agent has a second-mover advantage. Now consider a
527 change in the order of decisions and let the agent chooses \hat{a} first, with the principal choosing z in
528 response. In this case, the principal has a second-mover advantage since the principal need not
529 consider all possible \hat{a} . This provides intuition behind the bound in Lemma 2. However, if the
530 agent utility bound b is tight given a^* , the principal cannot gain an advantage by moving second.
531 No choice of contract by the principal can drive the agent's utility down any further. Since the
532 principal and agent have a direct conflict of interest over the direction of z , this means the principal
533 cannot improve her utility. In other words, the order of decisions does not matter when b is tight
534 and so the sandwich problem provides a tight relaxation.

535 This argument relies on the fact that w is unidimensional. In the multidimensional case, we
536 parameterize the payment function through a unidimensional z using a variational argument. As
537 long as a conflict of interest exists, we obtain a similar intuition and result. An analogous result to
538 Lemma 5 is also leveraged in the argument.

539 We remark that Assumption 4 is not used in the proof of Theorem 1 for the singleton case.
540 However, Assumption 4 is essential for continuous outcome sets. The MLRP is essential for showing
541 that optimal solutions to sandwich relaxations are, in fact, GMH contracts as defined in Section 3.1.
542 In particular, monotonicity of the output function greatly simplifies the first-order conditions of
543 (\mathbf{P}) to reduce them to the necessary and sufficient conditions of (10).

544 Of course, there remains the question of how to find a tight b . The simplest case is when the
545 reservation utility \underline{U} itself is tight. The following gives a sufficient condition for this to hold.

546 **Proposition 1.** Suppose Assumption 1–3 hold. Then the reservation utility \underline{U} is tight-at-optimality
547 if there exist an optimal solution w^* to (\mathbf{P}) and an $\delta > 0$ such that $w^*(x) > \underline{w} + \delta$ for almost all
548 $x \in \mathcal{X}$.

549 A main task of the next section is to provide a systematic approach to finding a b that is tight-
550 at-optimality even when the conditions of Proposition 1 fail to hold. We should remark that it is
551 not uncommon in the FOA literature to focus on the case where the limited liability constraint

552 is not binding (see Rogerson (1985) and Jewitt et al. (2008)). If that convention is taken here,
 553 Proposition 1 is useful in determining optimal contracts.

554 4 The sandwich procedure

555 The remaining steps to systematically solve (P) are (i) finding a b that is tight-at-optimality and
 556 (ii) determining a systematic way to solve (SAND| b). We approach both tasks concurrently us-
 557 ing what we call the *sandwich procedure*. The basic logic of the procedure is to use backwards
 558 induction, leveraging Lemma 3 and the GMH structure (see Definition 1) of optimal solutions to
 559 (SAND| a, \hat{a}, b). The structure of these optimal solutions is used to compute a tight b by solving a
 560 carefully designed optimization problem.

561

THE SANDWICH PROCEDURE

562

563 Step 1 CHARACTERIZE CONTRACT: Characterize an optimal solution to the inner maximization
 564 in (SAND| b):

$$565 \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U(w, a) - U(w, \hat{a}) \geq 0\} \quad (\text{SAND}|a, \hat{a}, b)$$

566 as a function of $a \in \mathbb{A}$, $\hat{a} \in \mathbb{A}$ and $b \geq \underline{U}$ where $\hat{a} \neq a$. Denote the resulting optimal contract
 567 by $w(a, \hat{a}, b)$.

568 Step 2 CHARACTERIZE ACTIONS: Determine optimal solutions to the outer maximization and
 569 minimization

$$570 \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} V(w(a, \hat{a}, b), a) \quad (23)$$

571 as functions of b . If a minimizer $\hat{a}(a, b)$ exists, find $a(b) \in \operatorname{argmax}_{a \in \mathbb{A}} V(w(a, \hat{a}(a, b), b), a)$ and
 572 set $w(b) = w(a(b), \hat{a}(a(b), b), b)$. If a minimizer does not exist, solve

$$573 \max_{a \in \mathbb{A}} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, (\text{FOC}(a))\},$$

574

575 which uses (11) from Lemma 4. Call the resulting solution $(a(b), w(b))$.

576 Step 3 COMPUTE A TIGHT BOUND: Solve the one-dimensional optimization problem:

$$577 b^* \equiv \min \left\{ \operatorname{argmin}_{b \geq \underline{U}} \left\{ V(w(b), a(b)) - \max_{a \in a^{BR}(w(b))} V(w(b), a) \right\} \right\}. \quad (24)$$

578 Let $a^* \equiv a(b^*)$, $\hat{a}^* \equiv \hat{a}(a^*, b^*)$ (when it exists), and $w^* \equiv w(b^*)$.

579

580 **Proposition 2.** For a given b , let $a(b)$, $\hat{a}(a(b), b)$ (if it exists) and $w(b)$ be as defined at the end
 581 of Step 2 of the sandwich procedure. Then $(a(b), \hat{a}(a(b), b), w(b))$ is an optimal solution to the
 582 sandwich relaxation (SAND| b). If $\hat{a}(a(b), b)$ does not exist then $(a(b), w(b))$ (as defined in Step 2)
 583 solves (SAND| b).

584 The proof is essentially by definition and thus omitted. However, to *guarantee* the tractability
 585 of each step we must make Assumptions 1–4. These same conditions ensure that (SAND| b) is, in
 586 fact, a tight relaxation.

587 **Theorem 2.** Suppose Assumption 1–4 hold and let b^* , a^* , and w^* be as defined in Step 3 of the
588 sandwich procedure. Then b^* is tight-at-optimality, (w^*, a^*) is an optimal solution to (P), and
589 $\text{val}(\text{SAND}|b^*) = \text{val}(\text{P})$.

590 Note that if a given b is known to be tight-at-optimality through some independent means, Step
591 3 of the procedure can be avoided. A special case of this is when the reservation utility \underline{U} itself
592 is tight-at-optimality. Proposition 1 gives a sufficient conditions for this to hold. In practice, one
593 may try to solve $(\text{SAND}|\underline{U})$ and if the resulting optimal solution $(a^\#, w^\#)$ is implementable (i.e.
594 $w^\#$ implements $a^\#$) then the complete sandwich procedure can be avoided.

595 In the remainder of this subsection we provide lemmas that justify each step of the sandwich
596 procedure. This culminates in a proof of Theorem 2 that is relatively straightforward given the
597 work up to that point. In the final subsection we note that even when all of Assumption 1–4 do
598 not hold, we can sometimes use the sandwich procedure to construct an optimal contract. We use
599 our motivating example to illustrate how this can be done.

600 4.1 Analysis of Step 1

601 We undertake an analysis of this step under Assumptions 1–3 following from Lemma 3 in Section 3.1.
602 The optimal contract $w(a, \hat{a}, b)$ sought in Step 1 is precisely the unique optimal contract guaranteed
603 by Lemma 3(i). That lemma also guarantees that $w(a, \hat{a}, b)$ is a well-behaved function of (a, \hat{a}, b) .

604 Indeed, by strong duality (Lemma 3(ii)), the optimal value of $(\text{SAND}|a, \hat{a}, b)$ is

$$605 \quad \text{val}(\text{SAND}|a, \hat{a}, b) = \inf_{\lambda, \delta \geq 0} \max_{w \geq \underline{w}} \mathcal{L}(w, \lambda, \delta|a, \hat{a}, b) = \mathcal{L}^*(a, \hat{a}|b)$$

606 where

$$607 \quad \mathcal{L}^*(a, \hat{a}|b) \equiv \mathcal{L}(w(a, \hat{a}, b), \lambda(a, \hat{a}, b), \delta(a, \hat{a}, b)|a, \hat{a}, b) \quad (25)$$

608 is called the *optimized Lagrangian* for the sandwich relaxation. The following result, a straightfor-
609 ward consequence of the Theorem of Maximum and Lemma 3, shows that the optimized Lagrangian
610 has useful structure for Step 2 of the procedure.

611 **Lemma 6.** The optimized Lagrangian $\mathcal{L}^*(a, \hat{a}|b)$ is upper semicontinuous. Moreover, it is contin-
612 uous and differentiable when $a \neq \hat{a}$.

613 4.2 Analysis of Step 2

614 The case where the inner infimum is not attained is sufficiently handled by Lemma 4 and existing
615 knowledge of the FOA. Here we examine the case where the inner infimum is attained and provide
616 necessary optimality conditions for a and \hat{a} to optimize $(\text{SAND}|b)$ given the contract $w(a, \hat{a}, b)$ and
617 its associated dual multipliers $\lambda(a, \hat{a}, b)$ and $\delta(a, \hat{a}, b)$. In particular, we solve (23) in Step 2 by
618 solving:

$$619 \quad \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} \mathcal{L}^*(a, \hat{a}|b) \quad (26)$$

620 using the definition of the optimized Lagrangian \mathcal{L}^* in (25). The optimal solution to the outer
621 optimization exists since \mathbb{A} is compact and \mathcal{L}^* is upper semicontinuous (via Lemma 6). Moreover, by
622 the differentiability properties of \mathcal{L} (when $\hat{a} \neq a$) we can obtain the following optimality conditions
623 for solutions of (26).

624 **Lemma 7.** Suppose a^* and \hat{a}^* solve (26) for a given $b \geq \underline{U}$ with $\hat{a}^* \neq a^*$. The following hold:

625 (i) for an interior solution $\hat{a}^* \in (\underline{a}, \bar{a})$,

626
$$\frac{\partial}{\partial \hat{a}} \mathcal{L}^*(a^*, \hat{a}^* | b) = -\delta^*(a^*, \hat{a}^*, b) U_a(w(a^*, \hat{a}^*, b), \hat{a}^*) = 0,$$

628 and $U_a(w(a^*, \hat{a}^*, b), \hat{a}^*) \geq 0$ (≤ 0) for $\hat{a}^* = \bar{a}$ ($\hat{a}^* = \underline{a}$);

629 (ii) for an interior solution $a^* \in (\underline{a}, \bar{a})$, the right derivative is

630
$$\frac{\partial}{\partial a^+} \min_{\hat{a} \in \mathbb{A}} \mathcal{L}^*(a^*, \hat{a}^* | b) \leq 0,$$

631

632 and left derivative is

633
$$\frac{\partial}{\partial a^-} \min_{\hat{a} \in \mathbb{A}} \mathcal{L}^*(a^*, \hat{a}^* | b) \geq 0,$$

634

635 and $\frac{\partial}{\partial \hat{a}} \mathcal{L}^*(a^*, \hat{a}^* | b) \leq 0$ (≥ 0) for $a^* = \underline{a}$ ($a^* = \bar{a}$).

636 Note that the conditions for a^* and \hat{a}^* are not symmetric in (i) and (ii) above. This is because
637 a^* is a function of \hat{a}^* and so has weaker topological properties to leverage for first-order conditions.

638 4.3 Analysis of Step 3

639 To work with (24) we re-express it in a slightly different way. Note that $V(w(b), a(b)) = \text{val}(\text{SAND}|b)$
640 via Proposition 2. We also denote the optimization problem in the second term inside the ‘‘argmin’’
641 of (24) as $(P|w(b))$:

642
$$\max_{a \in a^{\text{BR}}(w(b))} V(w(b), a). \tag{P|w(b)}$$

643

644 Thus, we can re-express (24) as:

645
$$b^* \equiv \min \{ \text{argmin}_{b \geq \underline{U}} \{ \text{val}(\text{SAND}|b) - \text{val}(P|w(b)) \} \}. \tag{27}$$

646 Note that $(P|w(b))$ is a restriction of $(P|b)$ and so $\text{val}(P|w(b)) \leq \text{val}(P|b) \leq \text{val}(\text{SAND}|b)$ and
647 all three values are decreasing in b . Also from Assumption 3, there exists an optimal solution
648 (w^*, a^*) to (P) and so there exists a b (namely, $b^* = U(w^*, a^*)$) such that all three problems
649 share the same optimal value. Hence, we must have $\min_{b \geq \underline{U}} (\text{val}(\text{SAND}|b) - \text{val}(P|w(b))) = 0$
650 and so b^* is the first time where $\text{val}(\text{SAND}|b) = \text{val}(P|w(b))$, forcing $\text{val}(\text{SAND}|b) = \text{val}(P|b)$
651 and implying b^* is tight-at-optimality. See Figure 3. We make this argument formally in the
652 proof of the following lemma, which also shows that b^* is well-defined in the sense that the set
653 $\text{argmin}_{b \geq \underline{U}} \{ \text{val}(\text{SAND}|b) - \text{val}(P|w(b)) \}$ has a minimum.

654 **Lemma 8.** If Assumptions 1–4 hold then there exists a real number b^* that satisfies (24). Fur-
655 thermore, b^* is tight-at-optimality.

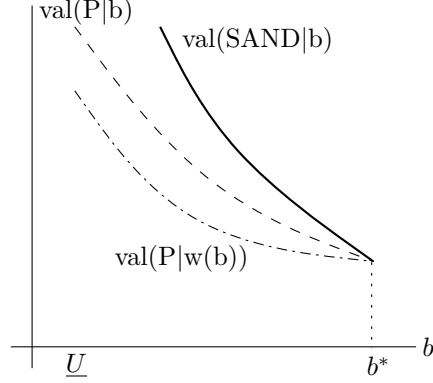


Figure 3: An illustration of Step 3 of the sandwich procedure.

656 4.4 Overall verification of the procedure

657 We are now ready to prove Theorem 2. The proof is a straightforward application of the lemmas
658 of this section.

659 *Proof of Theorem 2.* By Lemma 8 there exists a b^* that satisfies (24) and is tight-at-optimality.
660 Hence, by Theorem 1, $\text{val}(\text{SAND}|b^*) = \text{val}(\mathbf{P})$ and every optimal solution $(w(b^*), a(b^*))$ to $(\text{SAND}|b^*)$
661 is optimal to (\mathbf{P}) . Note that we need not require that the infimum is attained. However, when \hat{a}^*
662 is attained with $\hat{a}^* \neq a^*$, the GMH contract $w(a(b^*), \hat{a}(b^*), b^*)$ resulting from Lemma 3 is precisely
663 one such optimal contract where $a(b^*)$ and $\hat{a}(b^*)$ satisfy the optimality conditions of Lemma 7. \square

664 4.5 An illustrative example

665 Our motivating example serves to illustrate the steps of the sandwich procedure and how to work
666 with (24), even when Theorem 2 does not apply.

667 **Example 4** (Example 1 continued). Recall, our problem is to solve

$$668 \max_{(z,a)} \{v(z, a) : u(z, a) \geq -2 \text{ and } a \in \arg \max_{a'} u(z, a')\},$$

669 where $v(z, a) = za - 2a^2$ and $u(z, a) = -za - (a^2 - 1)^2$. We apply each step of the procedure
670 and determine an optimal contract. There is some overlap of analysis from Example 3, but our
671 approach here is more systematic and follows the reasoning of the sandwich procedure.

672 *Step 1. Characterize Contract.*

673 First, we characterize the optimal solutions $z(a, \hat{a}, b)$ of

$$674 \max_z \{v(z, a) : u(z, a) \geq b, u(z, a) - u(z, \hat{a}) \geq 0\}, \quad (28)$$

675 where $a \in [0, 2]$. The case where $a \in [-2, 0]$ is symmetric and analogous reasoning holds throughout.
676 Observe that $v(z, a)$ is increasing in z for fixed a and \hat{a} and so (28) can be solved by simply
677 maximizing z . The constraints on z are (from $u(z, a) \geq b$):

$$678 z \leq Q(a, b) \quad (29)$$

680 when $a \neq 0$, where $Q(a, b) \equiv -\frac{b+(a^2-1)^2}{a}$, and (from $u(z, a) - u(z, \hat{a}) \geq 0$):

$$681 \quad z \begin{cases} \geq R(a, \hat{a}) & \text{if } \hat{a} > a \\ \leq R(a, \hat{a}) & \text{if } \hat{a} < a \\ \in (-\infty, \infty) & \text{if } \hat{a} = a, \end{cases} \quad (30)$$

682

683 where $R(a, \hat{a}) \equiv -(\hat{a} + a)(\hat{a}^2 + a^2 - 2)$. Maximizing z subject to (29) and (30) yields:

$$684 \quad z(a, \hat{a}, b) = \begin{cases} \min \{Q(a, b), R(a, \hat{a})\} & \text{if } (a \neq 0) \wedge (\hat{a} < a) \\ Q(a, b) & \text{if } (a \neq 0) \wedge ((\hat{a} = a) \vee ((\hat{a} > a) \wedge [Q(a, b) \geq R(a, \hat{a})])) \\ R(a, \hat{a}) & \text{if } (a = 0) \wedge (b \leq -1) \wedge (\hat{a} < a), \\ +\infty & \text{if } (a = 0) \wedge (b \leq -1) \wedge (\hat{a} \geq a) \\ -\infty & \text{if } (a \neq 0) \wedge (\hat{a} > a) \wedge [R(a, \hat{a}) > Q(a, b)] \\ -\infty & \text{if } (a = 0) \wedge (b > -1) \end{cases}$$

685

686 where \wedge is the logical “and” and \vee is the logical “or”. The value $+\infty$ comes from the fact that
687 $u(z, a) \geq b$ does not constrain z when $a = 0$ and (30) does not constrain z when $\hat{a} = a$. Hence,
688 the value of z can be driven to $+\infty$. The value $-\infty$ comes from two cases that we separate for
689 clarity. In the first case, $z \leq Q(a, b)$ and $z \geq R(a, \hat{a})$ with $R(a, \hat{a}, b) > Q(a, b)$, leaving no choice
690 for z and thus we set $z = -\infty$ to denote the maximizer of an empty set. In the second case, $a = 0$
691 and $b > 1$ so the constraint $u(z, a) \geq 0$ is assuredly violated and so again $z = -\infty$. The case
692 where $z(a, \hat{a}, b) = R(a, \hat{a}, b)$ comes from the fact (29) does not constrain z when $a = 0$ as long as
693 $u(z, 0) = -1 \geq b$. Since $\hat{a} < a$, z is driven to the upper bound $R(a, \hat{a}, b)$ from (30).

694 *Step 2. Characterize Actions.*

695 The next step is to solve

$$696 \quad \inf_{\hat{a} \in [-2, 2]} v(z(a, \hat{a}, b), a). \quad (31)$$

697

698 As noted in Example 3, this infimum may not be attained and so we work with the possibility that
699 no $\hat{a}(a, b)$ exists. For fixed a , $v(z(a, \hat{a}, b), a)$ is an increasing function of $z(a, \hat{a}, b)$ and so \hat{a} should
700 be chosen to minimize $z(a, \hat{a}, b)$. This eliminates the case where $z(a, \hat{a}, b) = +\infty$. A key step is to
701 remove the dependence of $R(a, \hat{a}, b)$ on \hat{a} through optimizing. To this end, we define:

$$702 \quad R^\uparrow(a) \equiv \sup_{\hat{a} > a} R(a, \hat{a}), \text{ and}$$

$$703 \quad R^\downarrow(a) \equiv \inf_{\hat{a} < a} R(a, \hat{a}).$$

704

705 Since \hat{a} is chosen to minimize $z(a, \hat{a}, b)$ we have:

$$706 \quad z(a, b) \equiv \begin{cases} \min \{Q(a, b), R^\downarrow(a)\} & \text{if } (a \neq 0) \wedge [R^\uparrow(a) \leq Q(a, b)] \\ R^\downarrow(0) & \text{if } (a = 0) \wedge (b \leq -1) \\ -\infty & \text{if } (a = 0) \wedge (b > -1) \\ -\infty & \text{if } (a \neq 0) \wedge [R^\uparrow(a) > Q(a, b)]. \end{cases} \quad (32)$$

707

708 If it exists, we may set

$$709 \quad \hat{a}(a, b) = \begin{cases} \hat{a}^\uparrow(a) & \text{if } (a \neq 0) \wedge [R^\uparrow(a) > Q(a, b)] \\ \hat{a}^\downarrow(a) & \text{otherwise,} \end{cases}$$

710
711 where

$$712 \quad \hat{a}^\uparrow(a) \in \operatorname{argmax}_{\hat{a} > a} R(a, \hat{a}), \text{ and}$$

$$713 \quad \hat{a}^\downarrow(a) \in \operatorname{argmin}_{\hat{a} < a} R(a, \hat{a})$$

715 if they exist. The rest of the development is not contingent on the existence of $\hat{a}(a, b)$, $\hat{a}^\uparrow(a)$, and
716 $\hat{a}^\downarrow(a)$. In the case where the infimum is not attained, Lemma 4 can be used to determine $w(b)$
717 given $a(b)$ directly. Whether the infimum is attained or not depends on b , but does not impact the
718 analysis that follows, which simply works with the values $R^\uparrow(a)$ and $R^\downarrow(a)$.

719 Finally, we choose $a(b)$ to maximize $v(z(a, b), a)$. We first examine the choice of b . If b is
720 such that $\inf_a (R^\uparrow(a) - Q(a, b)) > 0$ then $z(a, b) = -\infty$ and so $v(z(a, b), a)$ is $-\infty$, no matter the
721 choice of a . Moreover, since $Q(a, b)$ is decreasing in b , any larger b will also not be chosen. Let
722 $\bar{b} := \inf_{b \geq -2} \{\inf_a (R^\uparrow(a) - Q(a, b)) > 0\}$. As discussed, any $b > \bar{b}$ will not be chosen. To compute \bar{b}
723 we can use the expressions:

$$724 \quad R^\uparrow(a) = \begin{cases} 4a(1 - a^2) & \text{if } 1/\sqrt{3} \leq a \leq 2 \\ \frac{4}{27}(9a - 5a^3 + \sqrt{2}(3 - a^2)^{3/2}) & \text{if } 0 \leq a \leq 1/\sqrt{3} \end{cases}$$

$$725 \quad R^\downarrow(a) = \begin{cases} 4a(1 - a^2) & \text{if } 1 \leq a \leq 2 \\ -\frac{4}{27}(9a - 5a^3 + \sqrt{2}(3 - a^2)^{3/2}) & \text{if } 0 \leq a \leq 1. \end{cases}$$

727 The reader may verify that \bar{b} is finite and strictly greater than 0. We can write an expression for
728 $a(b)$ as follows:

$$729 \quad a(b) \begin{cases} = 0 & \text{if } -2 \leq b \leq -1 \\ = a^\uparrow(b) & \text{if } -1 \leq b < \bar{b} \\ \in [0, 2] & \text{if } b \geq \bar{b}, \end{cases} \quad (33)$$

731 where $a^\uparrow(b)$ is an optimal solution to

$$732 \quad \max_{a \in (0, 2]} \min \{Q(a, b), R^\downarrow(a)\} a - 2a^2 \quad (34)$$

$$733 \quad \text{s.t. } R^\uparrow(a) \leq Q(a, b). \quad (35)$$

735 Our expression for $a(b)$ in (33) follows since if $b \leq -1$ then $v(z(a, b), a) < 0$ if $a > 0$ because we are
736 in the first case of (32) and $\min \{Q(a, b), R^\downarrow(a)\} < 0$. Hence $a(b) = 0$ since $v(z(a, b), a) = 0$. When
737 $-1 \leq b < \bar{b}$ we cannot set $a = 0$, otherwise $z(a, b) = -\infty$ and the problem is infeasible. The only
738 other option is the first case of (32) where $a(b)$ solves (34). Finally, when $b \geq \bar{b}$ then $z(a, b) = -\infty$
739 from (32) and so the choice of a is irrelevant.

740 With $a(b)$ as defined above we may write

$$741 \quad z(b) \equiv z(a(b), b) = \begin{cases} R^\downarrow(0) & \text{if } -2 \leq b \leq -1 \\ \min \{Q(a^\uparrow(b), b), R^\downarrow(a^\uparrow(b))\} & \text{if } -1 \leq b < \bar{b} \\ -\infty & \text{if } b \geq \bar{b} \end{cases}$$

742

743 and finally

$$744 \quad \text{val}(\text{SAND}|b) = v(z(b), b) = \begin{cases} 0 & \text{if } -2 \leq b \leq -1 \\ z(b)a^\uparrow(b) - 2(a^\uparrow(b))^2 & \text{if } -1 \leq b < \bar{b} \\ -\infty & \text{if } b \geq \bar{b}. \end{cases} \quad (36)$$

745

746 Since the original problem is feasible we can eliminate $b \geq \bar{b}$ from consideration. In (36) we now
 747 have the first term in the “inner” minimization of (24) for determining b^* . The second term can
 748 be expressed:

$$749 \quad \max_{a \in a^{BR}(z(b))} v(z(b), a). \quad (37)$$

750

751 We claim that $b = 0$ solves (24) in Step 3 of the sandwich procedure. To see this, we make the
 752 following observation:

$$753 \quad b < 0 \text{ implies } a(b) < 1 \text{ and } z(b) < 0. \quad (38)$$

754

755 This follows by observing that when $b < 0$ there are two cases: $b \leq -1$ and $-1 < b < 0$. When $b \leq$
 756 -1 then $a(b) = 0$ and $z(b) = R^\downarrow(0) < 0$. When $-1 < b < 0$ observe that $\min\{Q(a, b), R^\downarrow(a)\} < 0$
 757 for all $a \in (0, 2]$ and so $z(b) < 0$ and the objective function in (34) is decreasing in a implying the
 758 constraint in (34) is tight; that is, $R^\uparrow(a) = Q(a, b)$. The reader may verify that this implies $a < 1$
 759 and so $a(b) = a^\uparrow(b) < 1$. This yields (38).

760 Returning to (37), suppose $b < 0$. Consider the set $a^{BR}(z(b))$ when (from (38)) $z(b) < 0$.
 761 Taking the derivative of $u(z, a)$ with respect to a when $a \leq 1$ yields:

$$762 \quad \frac{\partial}{\partial a} u(z(b), a) = -z(b) - 4a(a^2 - 1) > 0$$

763

764 and so any $a \leq 1$ cannot be a best response to $z(b)$. This implies $a(b)$ (which is greater than 1 from
 765 (38)) is not a best response to $z(b)$ and hence

$$766 \quad \text{val}(\text{SAND}|b) > \max_{a \in a^{BR}(z(b))} v(z(b), a) \quad (39)$$

767

768 when $b < 0$. In Example 3 we showed (SAND| b) when $b = 0$ is a tight-relaxation. In particular,
 769 this means $(z(0), a(0))$ is an optimal solution to (P) and thus $a(0)$ is a best response to $z(0)$. Thus,

$$770 \quad \text{val}(\text{SAND}|0) = \max_{a \in a^{BR}(z(0))} v(z(0), a)$$

771

772 and so $b = 0$ is in the “argmin” in (24). Since (39) holds for any $b < 0$ this implies that $b^* = 0$. ◀

773 5 Non-existence of the inner minimization and the relationship 774 with the FOA

775 In this section we remark on a few connections between the sandwich approach and the FOA. We
 776 show how this relationship is connected to the issue of non-existence of a minimizer to the inner
 777 minimization in the definition of (SAND| b). We have already remarked (and Example 5 below

778 confirms) that our procedure applies when the FOA is invalid. However, there is more to say about
 779 the connection between these two approaches.

780 The astute reader will have noticed that (SAND| b) does not include the first-order constraint
 781 (FOC(a)) common to both the FOA and Mirrlees's approach. The fact that the (FOC(a)) is not
 782 present is connected to how we have handled the agent's optimization problem via (4), and how
 783 this optimization was pulled into the objective in (Max-Max-Min| b). Indeed, the minimization over
 784 the alternate best response included in (Max-Max-Min| b) and (SAND| b) can be understood as our
 785 way for accounting for the optimality of the agent's best response. In this perspective, first-order
 786 conditions are not explicitly necessary in the formulation, they are implied when the sandwich
 787 approach is valid.

788 We have already discussed the case when the inner minimization over \hat{a} in (SAND| b) is not
 789 attained in Lemma 4, where the sandwich problem is equivalent to one with a local station-
 790 arity condition. In the case where the inner minimization is attained for some $\hat{a}^* \neq a^*$ and
 791 the first-best contract is not optimal (the remaining case), we recover first-order conditions via
 792 Lemma 7 when \hat{a}^* is an interior point. In this case, $-\delta^*(a^*, \hat{a}^*, b)U_a(w(a^*, \hat{a}^*, b), \hat{a}^*) = 0$ and since
 793 $\delta^*(a^*, \hat{a}^*, b) = 0$ would imply the first-best contract is optimal, contradicting Assumption 3. We
 794 conclude that $U_a(w(a^*, \hat{a}^*, b), \hat{a}^*) = 0$. This implies the first-order condition holds for \hat{a}^* . Since
 795 $U(w(a^*, \hat{a}^*, b), a^*) \geq U(w(a^*, \hat{a}^*, b), \hat{a}^*)$ from the no-jump constraint in (SAND| b), this further im-
 796 plies $U_a(w(a^*, \hat{a}^*, b), a^*) = 0$ must also be satisfied since a^* will also be a best response (here we
 797 have assumed for simplicity that a^* is an interior point).

798 We examine this phenomenon from a more basic perspective. Suppose the sandwich approach is
 799 valid (for instance, because b is tight-at-optimality) and sandwich relaxation (SAND| b) has optimal
 800 solution (a^*, \hat{a}^*, w^*) . Moreover, suppose (i) the Lagrangian multiplier $\delta(a^*, \hat{a}^*, b)$ from Lemma 3
 801 is strictly positive and (ii) $\hat{a}^* < a^*$. Condition (ii) is reasonable since typically an alternate best
 802 response is to deviate to a lower effort level, not a higher effort level. Recall that cost is assumed
 803 to be nondecreasing in (A1.8). In a special case we can show this formally.

804 **Proposition 3.** If the principal is risk neutral and the FOA is not valid then there exists an
 805 alternate best response \hat{a} such that $\hat{a} < a^*$.

806 In other words, with a risk neutral principal, unless the FOA is valid, the agent will have a
 807 best-response “shirking” action. Observe that this assumption does not require any monotonicity
 808 assumptions on the output distribution f .

809 Given this scenario, we have the following equivalence

$$\begin{aligned}
 810 \quad \text{val}(\text{SAND}|b) &= \inf_{\hat{a} \in \mathbb{A}} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b, U(w, a^*) - U(w, \hat{a}) \geq 0\} \\
 811 &= \inf_{\hat{a} \leq a^*} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b, U(w, a^*) - U(w, \hat{a}) \geq 0\}. \\
 812
 \end{aligned}$$

813 To understand the above equivalence, we note that the “ \leq ” direction is always true since the right-
 814 hand side has an additional restriction on the minimization, but $\hat{a} = \hat{a}^* \leq a^*$ attains the minimum
 815 that is achieved by the left-hand side problem.

816 The right-hand side problem above is equivalent to

$$817 \quad \inf_{\hat{a} \leq a^*} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b, \frac{U(w, a^*) - U(w, \hat{a})}{a^* - \hat{a}} \geq 0\}$$

818 since $a^* - \hat{a} \geq 0$ in the range of choices for \hat{a} . Since $U(w, a)$ is differentiable in a , by the mean-value
 819 theorem, there exists an $\tilde{a} \in [\hat{a}, a^*]$ such that $\frac{U(w, a^*) - U(w, \hat{a})}{a^* - \hat{a}} = U_a(w, \tilde{a})$. Therefore, we have the

820 equivalence

$$\begin{aligned}
821 \quad \text{val}(\text{SAND}|b) &= \inf_{\tilde{a} \leq a^*} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b, \frac{U(w, a^*) - U(w, \tilde{a})}{a^* - \tilde{a}} \geq 0\} \\
822 &= \max_{w \geq \underline{w}} \inf_{\tilde{a} \leq a^*} \{V(w, a^*) : U(w, a^*) \geq b, \frac{U(w, a^*) - U(w, \tilde{a})}{a^* - \tilde{a}} \geq 0\} \\
823 &= \max_{w \geq \underline{w}} \inf_{\tilde{a} \leq a^*} \{V(w, a^*) : U(w, a^*) \geq b, U_a(w, \tilde{a}) \geq 0\} \\
824 &\leq \inf_{\tilde{a} \leq a^*} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b, U_a(w, \tilde{a}) \geq 0\} \tag{40} \\
825 &\leq \max_{a \in \mathbb{A}} \inf_{\tilde{a} \leq a} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U_a(w, \tilde{a}) \geq 0\} \\
826 &\leq \max_{a \in \mathbb{A}} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U_a(w, a) \geq 0\} \\
827 &= \text{val}(\text{FOA}). \\
828
\end{aligned}$$

829 The second equality follows from the tightness of b , the third equality uses the main-value theorem,
830 and the first inequality is simply the min-max inequality. Note that the constraint $U_a(w, \tilde{a}) \geq 0$
831 usually is binding for the problem $\max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b, U_a(w, \tilde{a}) \geq 0\}$, particularly if
832 the principal is risk-neutral (Jewitt 1988, Rogerson 1985). Then

$$\begin{aligned}
833 \quad &\inf_{\tilde{a} \leq a^*} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b, U_a(w, \tilde{a}) \geq 0\} \\
834 &= \inf_{\tilde{a} \leq a^*} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b, U_a(w, \tilde{a}) = 0\}, \\
835
\end{aligned}$$

836 which means that the sandwich relaxation must satisfy the stationary condition $U_a(w, \tilde{a}) = 0$ as a
837 constraint. Note that in the FOA, \tilde{a} must be taken as a^* and so is a weaker requirement.

838 Note that even when the sandwich approach is not valid, (40) reveals that it is a stronger
839 relaxation than the FOA. Indeed, the FOA requires $U_a(w, a) = 0$ whereas the sandwich approach
840 requires $U_a(w, \tilde{a}) = 0$ where \tilde{a} is a minimizer. The latter is a more stringent condition to satisfy.

841 These observations provide an interpretation of the sandwich relaxation as a strengthening of
842 the FOA, where we are required to satisfy an additional first-order condition over a worst-case
843 choice of alternate best response.

844 There remains the question of how the sandwich procedure proceeds when the FOA is, in fact,
845 valid. The next result shows that the two approaches are compatible in this case.

846 **Proposition 4.** When the FOA is valid, $\text{val}(\text{SAND}|\underline{U}) = \text{val}(\text{FOA}) = \text{val}(\text{P})$. That is, both the
847 sandwich approach and the FOA both recover the optimal contract of the original problem.

848 Observe that the validity of the FOA implies that the starting reservation utility \underline{U} is tight-at-
849 optimality. The next result reveals a partial converse in the case where the infimum in (SAND| b) is
850 not attained. We emphasize that the MLRP assumption is needed to establish the following result,
851 which we pull out of a proof of an earlier result stated and proven in the appendix.

852 **Proposition 5.** Suppose b is tight optimality and the sandwich problem (SAND| b) has solution
853 (a^*, w^*) where the inner minimization does not have a solution. Then, given the action a^* and with
854 modified (IR) constraint $U(w, a^*) \geq b$, the FOA is valid. That is,

$$855 \quad \text{val}(\text{P}) = \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b \text{ and } (\text{FOC}(a^*))\} \tag{41}$$

856 and the optimal solution to the right-hand side implements a^* .

6 Additional examples

In this section we provide three additional examples that further illustrate the sandwich procedure. The first example is one where the FOA is invalid but nonetheless satisfies Assumptions 1–4 and so is amenable to the sandwich procedure.

Example 5. Consider the following principal-agent problem. The distribution of output X is exponential with $f(x, a) = \frac{1}{a}e^{-x/a}$, for $x \in \mathcal{X} = \mathbb{R}_+$ and $a \in \mathbb{A} := [\frac{1}{10}, \frac{1}{2}]$. The principal is risk-neutral (and so $v(y) = y$), the value of output is $\pi(x) = x$, the agent’s utility is $u(y) = 2\sqrt{y}$, the agent’s cost of effort $c(a) = 1 - (a - \frac{1}{2})^2$, and the outside reservation utility is $\underline{U} = 0$. The minimum wage $\underline{w} = \frac{1}{16}$. It is straightforward to check that Assumptions 1 and 2 are satisfied. Existence of an optimal solution is guaranteed by Kadan et al. (2017) and so Assumption 3 is also satisfied. Finally, the monotonicity conditions in Assumption 4 hold trivially for f . This means that Theorems 1 and 2 apply.

Note also that the FOA is invalid. To see this, using the first-order condition $U_a(w, a) = 0$ to replace the original IC constraint, the resulting solution is $a^{\text{foa}} = \frac{1}{2}$ and $w^{\text{foa}}(x) = \frac{1}{4}$. Clearly, $w^{\text{foa}}(x)$ is a constant function and under $w^{\text{foa}}(x)$, the agent’s optimal choice is $a = \frac{1}{10}$, not $a^{\text{foa}} = \frac{1}{2}$. Hence the FOA is invalid.

Now we apply the sandwich procedure to derive an explicit solution. We start with $b = \underline{U}$ and show that Proposition 1 holds.

Step 1. Characterize Contract.

According to Lemma 3 the unique optimal contract to (SAND| $a, \hat{a}, \underline{U}$) is of the form

$$w_{\lambda, \delta}(a, \hat{a}, \underline{U}) = \left[\lambda + \delta \left(1 - \frac{f(x, \hat{a})}{f(x, a)} \right) \right]^2,$$

assuming that $w(x) > \underline{w}$ for all x (we verify this is the case below). Plugging the above contract into the two constraints $U(w_{\lambda, \delta}(a, \hat{a}, \underline{U}), a) = \underline{U}$ and $U(w_{\lambda, \delta}(a, \hat{a}, \underline{U}), a) = U(w_{\lambda, \delta}(a, \hat{a}, \underline{U}), \hat{a})$, we find

$$\begin{aligned} \lambda(a, \hat{a}, \underline{U}) &= \frac{1}{2}(1 - (a - 1/2)^2) \\ \delta(a, \hat{a}, \underline{U}) &= \frac{(2a - \hat{a})\hat{a}(a + \hat{a} - 1)}{2(a - \hat{a})^2}. \end{aligned}$$

Step 2. Characterize Actions.

We plug $w_{\lambda(a, \hat{a}, \underline{U}), \delta(a, \hat{a}, \underline{U})}(a, \hat{a}, \underline{U})$ from Step 1 into the principal’s utility function to obtain the optimized Lagrangian from (25)

$$\mathcal{L}^*(a, \hat{a}|\underline{U}) = a - \frac{1}{4}[1 - (a - 1/2)^2]^2 - \frac{1}{4}(2a - \hat{a})\hat{a}(a + \hat{a} - 1)^2.$$

Now we solve the max-min problem in (26) where $\mathcal{L}^*(a, \hat{a}|\underline{U})$ is a fourth order polynomial equation of \hat{a} with first-order condition

$$\frac{\partial}{\partial \hat{a}} \mathcal{L}^*(a, \hat{a}|\underline{U}) = \frac{1}{4}(a + \hat{a} - 1)[\hat{a}(a + \hat{a} - 1) - (2a - \hat{a})(a + \hat{a} - 1) - 2(2a - \hat{a})\hat{a}] = 0.$$

This yields three solutions, $\hat{a} = a - 1$, $\hat{a} = \frac{1}{2} \left(a + \frac{1}{2} - \sqrt{3a^2 - a + \frac{1}{4}} \right)$ and $\hat{a} = \frac{1}{2} \left(a + \frac{1}{2} + \sqrt{3a^2 - a + \frac{1}{4}} \right)$. Since $\hat{a} \in [1/10, 1/2]$, the only feasible interior minimizer is

$$\hat{a}(a, \underline{U}) = \frac{1}{2} \left(a + \frac{1}{2} - \sqrt{3a^2 - a + \frac{1}{4}} \right).$$

894 Plugging the $\hat{a}(a, \underline{U})$ into \mathcal{L}^* , we can solve the outer maximization problem in (26) over a , which
 895 yields $a^* = \frac{1}{2}$ and hence $\hat{a}^* = \frac{1}{4}(2 - \sqrt{2})$. This implies

$$896 \quad w^*(x) = \left[\frac{1}{2} + \frac{1}{16} \left(1 - \frac{f(x, \frac{1}{4}(2 - \sqrt{2}))}{f(x, 1/2)} \right) \right]^2 = \left[\frac{1}{2} + \frac{1}{16} (1 - (2 + \sqrt{2})e^{-2x(1 + \sqrt{2})}) \right]^2 > \frac{1}{16}.$$

897 Hence, Proposition 1 implies that \underline{U} is tight-at-optimality and so by Theorem 1 we have found
 898 an optimal contract. \blacktriangleleft

899 Second, the equivalence of the sandwich approach and the FOA when the FOA is valid (from
 900 Proposition 4) is illustrated by examining the classical example of Holmstrom (1979).

901 **Example 6.** The distribution of output X is exponential with $f(x, a) = \frac{1}{a}e^{-x/a}$, for $x \in \mathcal{X} = \mathbb{R}_+$
 902 and $a \in \mathbb{A} := [0, \bar{a}]$. The principal is risk-neutral (and so $v(y) = y$), the value of output is $\pi(x) = x$,
 903 the agent's utility is $u(y) = 2\sqrt{y}$, the agent's cost of effort $c(a) = a^2$, minimum wage $\underline{w} = 0$, and
 904 the outside reservation utility is $\underline{U} \geq 7^{-2/3}$.⁸

905 Holmstrom (1979) showed that the FOA applies to this problem. Now we apply the sandwich
 906 procedure to derive an explicit solution.

907 *Step 1. Characterize Contract.*

908 According to Lemma 3, the unique optimal contract to (SAND| a, \hat{a}, b) is of the form

$$909 \quad w_{\lambda, \delta}(a, \hat{a}, \underline{U}) = \left[\lambda + \delta \left(1 - \frac{f(x, \hat{a})}{f(x, a)} \right) \right]^2,$$

910 assuming that $w(x) > \underline{w}$ for all x (we verify this is the case below). Plugging the above contract
 911 into the two constraints $U(w_{\lambda, \delta}(a, \hat{a}, \underline{U}), a) = \underline{U}$ and $U(w_{\lambda, \delta}(a, \hat{a}, \underline{U}), a) = U(w_{\lambda, \delta}(a, \hat{a}, \underline{U}), \hat{a})$ yields

$$912 \quad \begin{aligned} \lambda(a, \hat{a}, \underline{U}) &= \frac{1}{2}(a^2 + \underline{U}) \\ 913 \quad \delta(a, \hat{a}, \underline{U}) &= \max\left\{0, \frac{(2a - \hat{a})\hat{a}(a^2 - \hat{a}^2)}{2(a - \hat{a})^2}\right\} = \max\left\{0, \frac{(2a - \hat{a})\hat{a}(a + \hat{a})}{2(a - \hat{a})}\right\}. \end{aligned}$$

914 *Step 2. Characterize Actions.*

915 We plug $w_{\lambda(a, \hat{a}, \underline{U}), \delta(a, \hat{a}, \underline{U})}(a, \hat{a}, \underline{U})$ from Step 1 into the principal's utility function to obtain the
 916 optimized Lagrangian from (25)

$$917 \quad \mathcal{L}^*(a, \hat{a}|\underline{U}) = \begin{cases} a - \frac{1}{4}(a^2 + \underline{U})^2 - \frac{1}{4}(2a - \hat{a})\hat{a}(a + \hat{a})^2 & \text{if } \frac{(2a - \hat{a})\hat{a}(a + \hat{a})}{2(a - \hat{a})} > 0 \\ 918 \quad a - \frac{1}{4}(a^2 + \underline{U})^2 & \text{if } \frac{(2a - \hat{a})\hat{a}(a + \hat{a})}{2(a - \hat{a})} \leq 0. \end{cases}$$

919 Now we solve the max-min problem in (26) where $\mathcal{L}^*(a, \hat{a}|\underline{U})$ is a fourth order polynomial
 920 equation of \hat{a} with first-order condition

$$921 \quad \frac{\partial}{\partial \hat{a}} \mathcal{L}^*(a, \hat{a}|\underline{U}) = -(a + \hat{a})(a^2 + 2a\hat{a} - 2\hat{a}^2) = 0.$$

922 This yields two solutions, $\hat{a} = (1 - \sqrt{3})a/2$, and $(1 + \sqrt{3})a/2$. Since $a > 0$, $\hat{a} = (1 - \sqrt{3})a/2$ is not
 923 feasible. It is not optimal to choose $\hat{a} \geq 2a$ as a minimizer, and so $-\frac{1}{4}(2a - \hat{a})\hat{a}(a + \hat{a})^2 \geq 0$. Also

⁸This number is chosen to ensure that the minimum wage constraint is strictly satisfied at the optimal solution, as explicitly assumed in Holmstrom (1979). For example, $\underline{U} = 0$ may lead a positive probability that the contract equals \underline{w} .

924 $a \leq \hat{a} < 2a$ is not optimal, since with this choice, $\mathcal{L}^*(a, \hat{a}|\underline{U}) = a - \frac{1}{4}(a^2 + \underline{U})^2$. So the minimizer
 925 should be taken in $0 \leq \hat{a} < a$, where $-(a + \hat{a})(a^2 + 2a\hat{a} - 2\hat{a}^2)$ is decreasing in \hat{a} . Therefore, the
 926 infimum is not attained and we have

$$927 \quad \inf_{\hat{a}} \mathcal{L}^*(a, \hat{a}|\underline{U}) = a - \frac{1}{4}(a^2 + \underline{U})^2 - a^4.$$

928 This yields a solution $a^*(\underline{U})$ that is specified by the first-order condition of the above optimization
 929 problem

$$930 \quad 1 - 5a^3 - 2a\underline{U} = 0,$$

931 where we may assume $\underline{U} \geq 7^{-2/3}$ so that

$$932 \quad w^*(x = 0) = \frac{1}{2}(a^{*2} + \underline{U}) - a^{*2} = \frac{1}{2}(\underline{U} - a^*(\underline{U})^2) \geq 0.$$

933 By L'Hôpital's rule, we have

$$934 \quad \lim_{\hat{a} \rightarrow a} \delta(a, \hat{a}, \underline{U}) \left(1 - \frac{f(x, \hat{a})}{f(x, a)}\right) \rightarrow a^3 \left(\frac{x-a}{a^2}\right) = a(x-a),$$

935 so the optimal GMH contract according to the sandwich procedure is

$$936 \quad w^*(x) = \frac{1}{2}a^{*2} + a^*(x - a^{*2}).$$

937 The resulting solution is consistent with the FOA solution, where the resulting Lagrangian multi-
 938 plier for the first-order condition is $\mu(a) = a^3$ (Holmstrom 1979) and the principal's value function
 939 is exactly the same:

$$940 \quad V(w^{foa}(a), a) = a - \lambda(a)^2 - \mu(a)^2 \mathbb{E} \left(\frac{\partial \log f(X, a)}{\partial a} \right)^2 = a - \frac{1}{4}(a^2 + \underline{U})^2 - a^4.$$

941 This completes the example. ◀

942 Note the similarity in the set-ups of Examples 5 and 6. The first can be seen as a relatively
 943 minor variation on the second, and yet the FOA approach fails in the first but holds in the second.
 944 In both cases the sandwich procedure applies. This illustrates, in a concrete way, aspects of the
 945 rigidity of the FOA and the robustness of the sandwich approach.

946 In our final example, we solve an adjustment of the problem proposed by Araujo and Moreira
 947 (2001), who show that the FOA fails but nonetheless construct an optimal solution by solving
 948 a nonlinear optimization problem with 20 constraints using Kuhn-Tucker conditions. Although
 949 this problem fails the conditions of Theorem 2 (it fails Assumption (A1.1) since there are only
 950 two outcomes), we can nonetheless use our approach (specifically Lemma 2 and Proposition 2) to
 951 construct an optimal contract. We remark that this example has the nice feature that all best
 952 responses are interior to the interval of actions $\mathbb{A} = [-1, 1]$, in contrast to all previous examples.
 953 Moreover, there are multiple alternate best responses. As can be seen below, and in relation to
 954 remarks in Section 5, stationarity conditions at these interior points are implicitly recovered via
 955 the sandwich approach.

956 **Example 7.** The principal has expected utility $V(w, a) = \sum_{i=1}^2 p_i(a)(x_i - w_i)$, where $p_1(a) = a^2$,
 957 $p_2(a) = 1 - a^2$ for $a \in [-1, 1]$ and with two possible outcomes $x_1 = 1$ and $x_2 = 3/4$, denoting
 958 $w_i = w(x_i)$ for $i = 1, 2$. The minimum wage is $\underline{w} = 0$. The agent's expected utility is $U(w, a) =$

959 $\sum_{i=1}^2 p_i(a)\sqrt{w_i} - 2a^2(1 - 2a^2 + \frac{4}{3}a^4)$ with reservation utility $\underline{U} = 0$. We apply **Step 1** and **Step 2** of
 960 the sandwich procedure.

961 *Step 1. Characterize Contract.*

962 The first-order conditions (10) imply that an optimal solution (SAND| $a, \hat{a}, 0$) must satisfy:

$$963 \quad w_i^* = w^*(x_i) = \frac{1}{4} \left[\lambda + \delta \left(1 - \frac{p_i(\hat{a})}{p_i(a)} \right) \right]^2 \text{ for } i = 1, 2, \quad (42)$$

964 assuming that $w_i^* \geq \underline{w}$ for $i = 1, 2$ (we check below that this is the case) for some choice of λ and δ .
 965 To characterize these λ and δ we plug the above contract into the two constraints of (SAND| a, \hat{a}, b),
 966 $U(w^*, a) = 0$ and $U(w^*, a) = U(w^*, \hat{a})$, to find

$$967 \quad \lambda(a, \hat{a}, 0) = 4a^2(1 - 2a^2 + \frac{4}{3}a^4) \text{ and } \delta(a, \hat{a}, 0) = \frac{4a^2(1-a^2)[3+4a^4+4\hat{a}^4+4a^2\hat{a}^2-6(\hat{a}^2+a^2)]}{3(a^2-\hat{a}^2)}. \quad (43)$$

968 *Step 2. Characterize Actions.*

969 We solve (26), where

$$\begin{aligned} 970 \quad \mathcal{L}^*(a, \hat{a}|0) &= \sum_{i=1}^2 p_i(a)(x_i - w(a, \hat{a}, 0)_i) \\ 971 &= \sum_{i=1}^2 p_i(a)x_i - \frac{1}{4}\lambda(a, \hat{a}, 0)^2 - \frac{1}{4} \frac{\delta(a, \hat{a}, 0)^2}{\sum_{i=1}^2 \left(1 - \frac{p_i(\hat{a})}{p_i(a)}\right)^2 p_i(a)} \\ 972 &= a^2 + \frac{3}{4}(1 - a^2) - \frac{4}{9}a^4(3 - 6a^2 + 4a^4)^2 - \frac{4}{9}a^2(1 - a^2)[3 + 4a^4 + 4\hat{a}^4 + 4a^2\hat{a}^2 - 6(\hat{a}^2 + a^2)]^2 \end{aligned}$$

973 by leveraging Lemma 7. Note that only the last term $t(a, \hat{a}) \equiv [3 + 4a^4 + 4\hat{a}^4 + 4a^2\hat{a}^2 - 6(\hat{a}^2 + a^2)]^2$
 974 in the last line of the above expression involves \hat{a} . By taking the first-order condition with respect
 975 to \hat{a} , we obtain three solutions

$$976 \quad \hat{a} = 0, \hat{a} = \frac{\sqrt{3-2a^2}}{2}, \hat{a} = -\frac{\sqrt{3-2a^2}}{2}.$$

977 One can verify that for any $a \in [-1, 1]$,

$$978 \quad t(a, 0) = (3 - 6a^2 + 4a^4)^2 < \frac{9}{16}(1 - 2a^2)^4 = t(a, \frac{\sqrt{3-2a^2}}{2}) = t(a, -\frac{\sqrt{3-2a^2}}{2}).$$

979 Therefore, the unique minimizer of $\mathcal{L}^*(a, \hat{a}|0)$ over \hat{a} is $\hat{a}^*(a) \equiv 0$. Then,

$$980 \quad \mathcal{L}^*(a, 0|0) = a^2 + \frac{3}{4}(1 - a^2) - \frac{4}{9}a^4(3 - 6a^2 + 4a^4)^2 - \frac{4}{9}a^2(1 - a^2)[3 + 4a^4 - 6a^2]^2$$

981 has a maximum at $a^* = \frac{\sqrt{3}}{2}$ (there are three maximizers, $a^* = -\frac{\sqrt{3}}{2}$ and $a^* = 0$, all interior to \mathbb{A} ,
 982 we just pick $a^* = \frac{\sqrt{3}}{2}$). This completes the sandwich procedure and we have produced an optimal
 983 solution to (SAND|0) of the form (a^*, \hat{a}^*, w^*) where $a^* = \frac{\sqrt{3}}{2}$, $\hat{a}^* = 0$ and $w_1^* = 1$ and $w_2^* = 0$ (using
 984 the fact $\lambda(\frac{\sqrt{3}}{2}, 0, 0) = \frac{3}{4}$ and $\delta(\frac{\sqrt{3}}{2}, 0, 0) = \frac{1}{4}$). Note, in particular, that $w_i^* \geq \underline{w} = 0$ for $i = 1, 2$.

985 Second, we show that (w^*, a^*) is feasible to (P). It suffices to show that a^* is a best response to
 986 w^* . The agent's expected utility under the contract $w^* = w(a, \hat{a}, 0)$ and taking action \tilde{a} is (using
 987 (42) and (43))

$$988 \quad U(w^*, \tilde{a}) = \frac{4}{3}(a^2 - \tilde{a}^2)(\tilde{a}^2 - \hat{a}^2)(2a^2 + 2\hat{a}^2 + 2\tilde{a}^2 - 3).$$

989 Given $a^* = \frac{\sqrt{3}}{2}$ and $\hat{a}^* = 0$, $U(w^*, \tilde{a})$ is indeed maximized at $\tilde{a} = \pm \frac{\sqrt{3}}{2}$ and $\tilde{a} = 0$. This shows that
990 a^* is a best response to w^* and hence (w^*, a^*) is feasible to (P).

991 Finally, by Lemma 2 we know $\text{val}(\text{SAND}|0) \geq \text{val}(\text{P})$ and this implies (w^*, a^*) achieves the best
992 possible principal utility in (P). We conclude that w^* is an optimal contract. However, one can
993 check that the FOA is not valid. The solution to (FOA) will yield $a^{foa} = 0.798$, which cannot be
994 implemented by the corresponding w^{foa} . Details are suppressed. ◀

995 7 Conclusion

996 We provide a general method to solve moral hazard problems when output is a continuous random
997 variable with a distribution that satisfies the MLRP (Assumption 4). This involves solving a
998 tractable relaxation of the original problem using a bound on agent utility derived from our proposed
999 procedure.

1000 We do admit that, in general, Step 3 of the sandwich procedure may be *a priori* difficult unless
1001 sufficient structural information is known about the set $a^{\text{BR}}(w(b))$. However, as the examples in
1002 this paper illustrate, this may not be an issue in sufficiently well-behaved cases.

1003 Indeed, Proposition 1 is helpful in this regard. If one initially assumes $b = \underline{U}$, and the resulting
1004 optimal contract can be shown to strictly satisfy the limited liability constraint, there is no need
1005 for the Step 3 of the sandwich procedure. This method is on display in Example 5. As mentioned
1006 in the paper, there is precedence in the moral hazard literature to restrict analysis to cases where
1007 the limited liability constraint is guaranteed not to bind.

1008 Finding additional criteria for the (IR) constraint to be tight is an important area for further
1009 investigation. Moreover, finding other scenarios where (24) is tractable is also of interest. Exam-
1010 ples 4–7 show that the basic framework of our approach can help solve problems that may not
1011 satisfy all the assumptions used in our theorems.

1012 References

- 1013 A. Araujo and H. Moreira. A general Lagrangian approach for non-concave moral hazard problems.
1014 *Journal of Mathematical Economics*, 35(1):17–39, 2001.
- 1015 D.P. Bertsekas. *Nonlinear Programming*. Athena Scientific, 1999.
- 1016 J.R. Conlon. Two new conditions supporting the first-order approach to multisignal principal–agent
1017 problems. *Econometrica*, 77(1):249–78, 2009.
- 1018 S. Dempe. *Foundations of Bilevel Programming*. Springer, 2002.
- 1019 S.J. Grossman and O.D. Hart. An analysis of the principal-agent problem. *Econometrica*, 51(1):
1020 7–45, 1983.
- 1021 B. Holmstrom. Moral hazard and observability. *The Bell Journal of Economics*, 10(1):74–91, 1979.
- 1022 R.D. Innes. Limited liability and incentive contracting with ex-ante action choices. *Journal of*
1023 *Economic Theory*, 52(1):45–67, 1990.
- 1024 I. Jewitt. Justifying the first-order approach to principal-agent problems. *Econometrica*, 56(5):
1025 1177–90, 1988.

- 1026 I. Jewitt, O. Kadan, and J.M. Swinkels. Moral hazard with bounded payments. *Journal of Economic*
1027 *Theory*, 143(1):59–82, 2008.
- 1028 J.Y. Jung and S.K. Kim. Information space conditions for the first-order approach in agency
1029 problems. *Journal of Economic Theory*, 160:243–279, 2015.
- 1030 O. Kadan, Reny P.J., and J.M. Swinkels. Existence of optimal mechanisms in principal-agent
1031 problems. *Econometrica*, 85(1):769–823, 2017.
- 1032 R. Ke and C.T. Ryan. Monotonicity of optimal contracts without the first-order approach. *Working*
1033 *paper*, 2016.
- 1034 R. Kirkegaard. Moral hazard and the spanning condition without the first-order approach. *Games*
1035 *and Economic Behavior*, 102:373–387, 2017a.
- 1036 R. Kirkegaard. A unifying approach to incentive compatibility in moral hazard problems. *Theo-*
1037 *retical Economics*, 12(1):25–51, 2017b.
- 1038 L.S. Lasdon. *Optimization Theory for Large Systems*. Dover Publications, 2011.
- 1039 J.A. Mirrlees. The theory of optimal taxation. *Handbook of Mathematical Economics*, 3:1197–1249,
1040 1986.
- 1041 J.A. Mirrlees. The theory of moral hazard and unobservable behaviour: Part I. *The Review of*
1042 *Economic Studies*, 66(1):3–21, 1999.
- 1043 W.P. Rogerson. The first-order approach to principal-agent problems. *Econometrica*, 53(6):1357–
1044 67, 1985.
- 1045 B. Sinclair-Desgagné. The first-order approach to multi-signal principal-agent problems. *Econo-*
1046 *metrica*, 62(2):459–65, 1994.
- 1047 W. Wang and S. Hu. Moral hazard with limited liability: Incentive contracting under monotone
1048 ratio assumptions. *Working paper*, 2016.

1049 **A Appendix: Proofs**

1050 **A.1 Proof of Lemma 1**

1051 We set the notation $V^*(a, \hat{a}) = \max_{w \geq \underline{w}} \{V(w, a) : (w, a) \in \mathcal{W}(\hat{a}, b)\}$ and $V^*(a) = \inf_{\hat{a}} V^*(a, \hat{a})$.
1052 The result follows by establishing the following claim:

1053 **Claim 1.** $V^*(a)$ is upper semicontinuous in a .

1054 Indeed, if $V^*(a)$ is upper semicontinuous then, since \mathbb{A} is compact, an outer maximizer a exists
1055 by the Weierstrass Theorem.

1056 We now establish the claim. By definition of upper semicontinuity, we want to show that for
1057 any constant $\alpha \in \mathbb{R}$, $\{a | V^*(a) < \alpha\}$ is open, where α is independent of a . This shows there exists
1058 an $\epsilon > 0$ such that for all $a' \in \mathcal{N}_\epsilon(a)$, $V^*(a') < \alpha$, where $\mathcal{N}_\epsilon(a)$ is an open neighborhood of a . Now

1059 we pick any $a_0 \in \{a | V^*(a) < \alpha\}$. Note that $\inf_{\hat{a}} V(a_0, \hat{a}) < \alpha$ implies that there exists some \hat{a}_0
 1060 such that

$$1061 \quad V(a_0, \hat{a}_0) < \alpha.$$

1062 On the other hand, since $V(a, \hat{a})$ is upper-semicontinuous by the Theorem of Maximum, the set

$$1063 \quad \{(a, \hat{a}) | V(a, \hat{a}) < \alpha\}$$

1064 is open. Therefore, there exists an $\epsilon > 0$ such that $V(a', \hat{a}') < \alpha$ for any $(a', \hat{a}') \in \mathcal{B}_\epsilon(a_0, \hat{a}_0)$ where
 1065 $\mathcal{B}_\epsilon(a_0, \hat{a}_0)$ is an the open ball in \mathbb{R}^2 centered at (a_0, \hat{a}_0) with radius ϵ . Thus, we can find an open
 1066 neighborhood $\mathcal{N}_{\epsilon_1}(a_0)$ of a_0 and $\mathcal{N}_{\epsilon_2}(\hat{a}_0)$ of \hat{a}_0 such that

$$1067 \quad \mathcal{N}_{\epsilon_1}(a_0) \times \mathcal{N}_{\epsilon_2}(\hat{a}_0) \subseteq \mathcal{B}_\epsilon(a_0, \hat{a}_0).$$

1068 Therefore, we have $V(a', \hat{a}') < \alpha$ for any $a' \in \mathcal{N}_{\epsilon_1}(a_0)$ and $\hat{a}' \in \mathcal{N}_{\epsilon_2}(\hat{a}_0)$. As a result, for any,
 1069 $a' \in \mathcal{N}_{\epsilon_1}(a_0)$, we have

$$1070 \quad V^*(a') = \inf_{\hat{a}} V(a', \hat{a}) \leq V(a', \hat{a}') < \alpha,$$

1071 for a given $\hat{a}' \in \mathcal{N}_{\epsilon_2}(\hat{a}_0)$, which shows that $\{a | V^*(a) < \alpha\}$ is open. We thus obtain the desired
 1072 upper-semicontinuity of $\inf_{\hat{a}} V(a, \hat{a})$.

1073 A.2 Proof of Lemma 2

1074 Observe that

$$\begin{aligned} 1075 \quad \text{val}(\mathbf{P}|b) &= \text{val}(\text{Max-Max-Min}|b) \\ 1076 \quad &= \max_{a \in \mathbb{A}} \max_{w \geq w} \inf_{\hat{a} \in \mathbb{A}} V^I(w, a | \hat{a}, b) \\ 1077 \quad &\leq \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} \max_{w \geq w} V^I(w, a | \hat{a}, b) \\ 1078 \quad &= \text{val}(\text{SAND}|b), \end{aligned}$$

1080 where the inequality follows by the min-max inequality. If there exists an optimal solution (w^*, a^*)
 1081 to (\mathbf{P}) such that $U(w^*, a^*) \geq b$ (and thus is also a feasible solution to $(\mathbf{P}|b)$) then $\text{val}(\mathbf{P}) \leq \text{val}(\mathbf{P}|b)$.
 1082 However, we already argued in the main text that $\text{val}(\mathbf{P}) \geq \text{val}(\mathbf{P}|b)$. This implies $\text{val}(\mathbf{P}) = \text{val}(\mathbf{P}|b)$
 1083 and so the above inequality implies $\text{val}(\mathbf{P}) \leq \text{val}(\text{SAND}|b)$. \square

1084 A.3 Proof of Lemma 3

1085 The proof of (i) and (ii) is analogous to the proof of Theorem 3.5 in Ke and Ryan (2016). In both
 1086 cases, a, \hat{a} and b are fixed constants. The difference here is that the no-jump constraint defining
 1087 $(\text{SAND}|b)$ is an inequality, while in Ke and Ryan (2016) the no-jump constraint is an equality.
 1088 However, this only changes the complementary slackness properties, as detailed in the lemma.
 1089 Moreover, in Ke and Ryan (2016) we need not entertain the case where $\hat{a} = a$. Fortunately, the
 1090 case where $\hat{a} = a$ is straightforward since then $(\text{SAND}|a, \hat{a}, b)$ is solved by the first-best contract,
 1091 which is unique. Further details are omitted.

1092 The proof of (iii) and (iv) is standard by applying the Theorem of Maximum. Details are
 1093 omitted.

1094 Assumption 2 is required in the proof of Theorem 3.5 in Ke and Ryan (2016), and that is also
 1095 why it is required here. \square

1096 **A.4 Proof of Lemma 4**

1097 For convenience we denote

1098
$$V^*(a, \hat{a}|b) = \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U(w, a) - U(w, \hat{a}) \geq 0\}.$$

1099 If $\inf_{\hat{a}} V^*(a, \hat{a}|b)$ is not attained, it must be that the infimizing sequence converges to a (for more
1100 details on this argument see the discussion following Lemma 3 in the main text of the paper). We
1101 can decompose the minimization problem as

1102
$$\inf_{\hat{a}} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U(w, a) - U(w, \hat{a}) \geq 0\} = \inf_{\hat{a} \leq a} \{ \inf_{w \geq \underline{w}} V^*(a, \hat{a}|b), \inf_{\hat{a} \geq a} V^*(a, \hat{a}|b) \}.$$

1103

1104 **Case 1.** $\inf_{\hat{a} \leq a} V^*(a, \hat{a}|b) = \inf_{\hat{a}} V^*(a, \hat{a}|b)$.

1105 We begin by observing that if $\inf_{\hat{a} \leq a} V^*(a, \hat{a}|b)$ has an infimizing sequence that does not converge
1106 to a , then by the supposition of non-existence, we must have

1107
$$\inf_{\hat{a} \leq a} V^*(a, \hat{a}|b) > \inf_{\hat{a}} V^*(a, \hat{a}|b).$$

1108 In this case, we will switch to consider $\inf_{\hat{a} \geq a} V^*(a, \hat{a}|b)$, which is discussed in Case 2 below.

1109 By the mean-value theorem, there exists an $\tilde{a} \in [\hat{a}, a]$ such that $\frac{U(w, a) - U(w, \hat{a})}{a - \hat{a}} = U_a(w, \tilde{a})$.
1110 Therefore, we have the equivalence

1111
$$\begin{aligned} \inf_{\hat{a} \leq a} V^*(a, \hat{a}|b) &= \inf_{\hat{a} \leq a} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a) \geq b, \frac{U(w, a) - U(w, \hat{a})}{a - \hat{a}} \geq 0\} \\ &= \lim_{\hat{a} \rightarrow a^-} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a) \geq b, \frac{U(w, a) - U(w, \hat{a})}{a - \hat{a}} \geq 0\} \\ &= \lim_{\tilde{a} \rightarrow a^-} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U_a(w, \tilde{a}) \geq 0\}. \end{aligned} \quad (44)$$

1115 Note that $\max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U_a(w, \tilde{a}) \geq 0\}$ is continuous in \tilde{a} (since U is continuously
1116 differentiable in a) and, as mentioned above, the infimizing sequence converges to a and so a
1117 minimizer exists to (44), yielding

1118
$$\inf_{\hat{a} \leq a} V^*(a, \hat{a}|b) = \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U_a(w, a) \geq 0\}.$$

1119 It remains to show that the constraint $U_a(w, a) \geq 0$ is binding for any $a \in \text{int}\mathbb{A}$ and slack only if
1120 $a = \bar{a}$. Suppose that the constraint in the above problem is slack at the optimum, i.e., $U_a(w, a) > 0$,
1121 then the Lagrangian multiplier is zero, and we have

1122
$$\max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U_a(w, a) \geq 0\} = \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b\}.$$

1123 This means $w^{fb}(a|b)$ solves $\max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U_a(w, a) \geq 0\}$, where $w^{fb}(a|b)$ is the
1124 first-best contract. Equivalently, we have

1125
$$\inf_{\hat{a}} V^*(a, \hat{a}|b) = \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b\}. \quad (45)$$

1126 We now claim that $w^{fb}(a|b)$ implements a . Continuing from (45), let $\hat{a}' \in a^{BR}(w^{fb}(a|b))$, we have

1127
$$\inf_{\hat{a}} V^*(a, \hat{a}|b) \leq V^*(a, \hat{a}'|b) \leq \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b\} = \inf_{\hat{a}} V^*(a, \hat{a}|b),$$

1128 where the first inequality is by the definition of minimization and the second inequality is straight-
 1129 forward by withdrawing a constraint from the maximization problem. Therefore, all inequalities
 1130 become equalities and $w^{fb}(a|b)$ satisfies the no-jump constraint $U(w, a) - U(w, \hat{a}) \geq 0$. This implies
 1131 $a \in a^{BR}(w^{fb}(a|b))$. Therefore, for any $a \in \text{int}\mathbb{A}$, $U_a(w^{fb}(a|b), a) = 0$, and $U_a(w^{fb}(a|b), a) > 0$ only
 1132 occurs when $a = \bar{a}$, where $w^{fb}(\bar{a}|b)$ implements \bar{a} . This completes Case 1.

1133 **Case 2.** $\inf_{\hat{a} \geq a} V^*(a, \hat{a}|b) = \inf_{\hat{a}} V^*(a, \hat{a}|b)$
 1134 In this case, we have the equivalence

$$\begin{aligned}
 1135 \quad \inf_{\hat{a} \geq a} V^*(a, \hat{a}|b) &= \lim_{\hat{a} \rightarrow a^+} \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a) \geq b, \frac{U(w, a) - U(w, \hat{a})}{a - \hat{a}} \leq 0\} \\
 1136 \quad &= \lim_{\hat{a} \rightarrow a^+} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b, U_a(w, \hat{a}) \leq 0\}. \tag{46} \\
 1137
 \end{aligned}$$

1138 The rest of argument is quite similar to Case 1 and thus omitted. Combining these two cases, we
 1139 have the desired conclusion.

1140 A.5 Proof of Lemma 5

1141 We require the following lemma:

1142 **Lemma 9** (Theorem 6 in Section 8.5 of Lasdon (2011)). Consider a maximization problem

$$\begin{aligned}
 1143 \quad &\max_y \{f(y) : g(y) \geq 0\} \\
 1144
 \end{aligned}$$

1145 where $f : \mathbb{Y} \rightarrow \mathbb{R}$, and $g : \mathbb{Y} \rightarrow \mathbb{R}^k$ for some compact subset $\mathbb{Y} \subset \mathbb{R}^n$. Assume that both f and g
 1146 are continuous and differentiable. If the Lagrangian $L(y, \alpha) = f(y) + \alpha \cdot g(y)$ is strictly concave in
 1147 y , then

$$\begin{aligned}
 1148 \quad &\max_y \{f(y) : g(y) \geq 0\} = \inf_{\alpha \geq 0} \max_y L(y, \alpha) \\
 1149
 \end{aligned}$$

1150 where we assume the maximum of $L(y, \alpha)$ over y exists for any given α .

1151 *Proof of Lemma 5.* When the infimum in (SAND|b) is not attained or attained at $a^\#$, the result
 1152 follows a standard application of duality theory via Lemma 9, due to Lemma 4.

1153 We now consider the case where the infimum is attained. Let (a^*, \hat{a}^*, z^*) be an optimal solution
 1154 (SAND|b); that is,

$$\begin{aligned}
 1155 \quad V(z^*, a^*) &= \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} \max_{z \geq \underline{z}} \{V(z, a) : U(z, a) \geq b, U(z, a) - U(z, \hat{a}) \geq 0\}.
 \end{aligned}$$

1156 Given a^* , consider the Lagrangian dual of the inner maximization problem over z ; that is,

$$\begin{aligned}
 1157 \quad \mathcal{L}(z, \lambda, \delta | a^*, \hat{a}^*, b) &= V(z, a^*) + \lambda[U(z, a^*) - b] + \delta[U(z, a^*) - U(z, \hat{a}^*)].
 \end{aligned}$$

1158 Note that \mathcal{L} is strictly concave in z since $V(z, a^*) = v(\pi(x_0) - z)$ is concave and $U(z, a^*) = u(z)$
 1159 is strictly concave in z and the term involving δ is a function only of a since $U(z, a^*) - U(z, \hat{a}^*) =$
 1160 $u(z) - c(a^*) - (u(z) - c(\hat{a}^*)) = c(\hat{a}^*) - c(a^*)$. Lemma 9 implies

$$\begin{aligned}
 1161 \quad \inf_{\hat{a} \in \mathbb{A}} \max_{z \geq \underline{z}} \{V(z, a^*) : U(z, a^*) \geq d, U(z, a^*) - U(z, \hat{a}) \geq 0\} &= \inf_{\hat{a} \in \mathbb{A}} \inf_{\lambda, \delta \geq 0} \max_{z \geq \underline{z}} \mathcal{L}(z, \lambda, \delta | a^*, \hat{a}, d) \tag{47} \\
 1162
 \end{aligned}$$

1163 for all $d \in [b, b + \epsilon)$. We now consider three cases. We show the first two cases do not occur, leaving
 1164 only the third case where we can establish the result. The cases consider how perturbing b can
 1165 effect the primal and dual problems in (47).

1166 *Case 1.* The set $\cap_{\hat{a} \in \mathbb{A}} \{z : U(z, a^*) \geq b + \epsilon, U(z, a^*) - U(z, \hat{a}) \geq 0\}$ is empty, for any arbitrarily
 1167 small $\epsilon > 0$.

1168 Here, the Lagrangian multiplier

$$1169 \quad \lambda(a^*, \hat{a}_\epsilon^*) \in \arg \inf_{\lambda, \delta \geq 0} \max_{z \geq \underline{z}} \mathcal{L}(z, \lambda, \delta | a^*, \hat{a}, b + \epsilon)$$

1170 is unbounded, where $\hat{a}_\epsilon^* \in \arg \min_{\hat{a}} \inf_{\lambda \geq 0, \delta \geq 0} \max_{z \geq \underline{z}} L(z, \lambda, \delta | a^*, \hat{a}, b + \epsilon)$. Also, $U(z_\epsilon^*, a^*) < b + \epsilon$
 1171 for any z_ϵ^* such that

$$1172 \quad \mathcal{L}(z_\epsilon^*, \lambda(a^*, \hat{a}_\epsilon^*), \delta(a^*, \hat{a}_\epsilon^*) | a^*, \hat{a}_\epsilon^*, b + \epsilon) = \inf_{\lambda \geq 0} \inf_{\delta \geq 0} \max_{z \geq \underline{z}} \mathcal{L}(z, \lambda, \delta | a^*, \hat{a}, b + \epsilon).$$

1173 Therefore, we choose a sequence $\epsilon_n = \frac{\epsilon}{n}$, and we have

$$1174 \quad U(z_{\epsilon_n}^*, a^*) - b - \epsilon_n < 0,$$

1175 where $z_{\epsilon_n}^*$ is a sequence such that

$$1176 \quad V(z_{\epsilon_n}^*, a^*) = \inf_{\hat{a} \in \mathbb{A}} \inf_{\lambda \geq 0, \delta \geq 0} \max_{z \geq \underline{z}} L(z, \lambda, \delta | a^*, \hat{a}, b + \epsilon_n).$$

1177 Note that $(z_\epsilon^*, a_\epsilon^*, \hat{a}_\epsilon^*)$ is upper hemicontinuous in ϵ by the Theorem of Maximum. Then as
 1178 $n \rightarrow \infty$, the limit $(z_0^*, a_0^*, \hat{a}_0^*; \lambda(a_0^*, \hat{a}_0^*), \delta(a_0^*, \hat{a}_0^*))$ is a solution to the problem without perturbation
 1179 ($\epsilon = 0$). Without loss of generality, we choose

$$1180 \quad (z^*, a^*, \hat{a}^*; \lambda(a^*, \hat{a}^*), \delta(a^*, \hat{a}^*)) = (z_0^*, a_0^*, \hat{a}_0^*; \lambda(a_0^*, \hat{a}_0^*), \delta(a_0^*, \hat{a}_0^*)).$$

1181 Then, passing to the limit (taking a subsequence if necessary), $z_{\epsilon_n}^* \rightarrow z^*$ and we have

$$1182 \quad \lim_{n \rightarrow \infty} [U(z_{\epsilon_n}^*, a^*) - b - \epsilon_n] = U(z^*, a^*) - b \leq 0,$$

1183 which contradicts of the supposition $U(z^*, a^*) > b$.

1184 *Case 2.* The set $\cap_{\hat{a} \in \mathbb{A}} \{z : U(z, a^*) \geq b + \epsilon, U(z, a^*) - U(z, \hat{a}) \geq 0\}$ is nonempty and $\lambda(a^*, \hat{a}_\epsilon^*) > 0$,
 1185 for any $\epsilon > 0$.

1186 Note that $\lambda(a^*, \hat{a}_\epsilon^*) > 0$ implies the constraint $U(z_\epsilon^*, a^*) \geq b + \epsilon$ is binding given strong duality.
 1187 We choose a sequence $\epsilon_n = \frac{\epsilon}{n}$. Passing to the limit (taking a subsequence if necessary), $z_{\epsilon_n}^* \rightarrow z^*$
 1188 and we have

$$1189 \quad 0 = \lim_{n \rightarrow \infty} [U(z_{\epsilon_n}^*, a^*) - b - \epsilon_n] = U(z^*, a^*) - b,$$

1190 which contradicts with the supposition $U(z^*, a^*) > b$.

1191 *Case 3.* The set $\cap_{\hat{a} \in \mathbb{A}} \{z : U(z, a^*) \geq U^* + \epsilon, U(z, a^*) - U(z, \hat{a}) \geq 0\}$ is nonempty and $\lambda(a^*, \hat{a}_\epsilon^*) = 0$,
 1192 for some arbitrarily small $\epsilon > 0$.

1193 Given $\lambda(a^*, \hat{a}_\epsilon^*) = 0$, we have

$$\begin{aligned} 1194 \quad V(z_\epsilon^*, a^*) &= \max_z V(z, a^*) + \lambda(a^*, \hat{a}_\epsilon^*)(U(z, a^*) - b - \epsilon) + \delta(a^*, \hat{a}_\epsilon^*)(U(z, a^*) - U(z, \hat{a}_\epsilon^*)) \\ 1195 &= \max_z V(z, a^*) + \lambda(a^*, \hat{a}_\epsilon^*)(U(z, a^*) - b) + \delta(a^*, \hat{a}_\epsilon^*)(U(z, a^*) - u(z, \hat{a}_\epsilon^*)) \\ 1196 &\geq \inf_{\hat{a}} \inf_{\lambda, \delta \geq 0} \max_z V(z, a^*) + \lambda(U(z, a^*) - b) + \delta(U(z, a^*) - U(z, \hat{a}_\epsilon^*)) \\ 1197 &= V(z^*, a^*). \end{aligned}$$

1198 We already know $V(z^*, a^*) \geq V(z_\epsilon^*, a^*)$ since $\epsilon > 0$. Therefore, we have shown $V(z_\epsilon^*, a^*) =$
 1199 $V(z^*, a^*)$, as required.

1200 The above argument shows that as we increase b to $b + \epsilon$, we can find a new optimal contract
 1201 that does not change the objective value. This can be repeated until we find a sufficiently large ϵ
 1202 such that $U(z_\epsilon^*, a_\epsilon^*) = b + \epsilon$. \square

1203 A.6 Proof of Theorem 1

1204 There are two cases to consider. The first is when the inner “inf” in (SAND| b) is not attained.
 1205 This is handled by the following lemma.

1206 **Lemma 10.** Suppose b is tight-at-optimality and the sandwich problem (SAND| b) has solution
 1207 (a^*, w^*) where the inner minimization does not have a solution. Then, given the action a^* and with
 1208 modified (IR) constraint $U(w, a^*) \geq b$, the FOA is valid. That is,

$$1209 \quad \text{val(P)} = \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b \text{ and } (\text{FOC}(a^*))\} = \text{val(SAND|}b\text{)}. \quad (48)$$

1211 *Proof.* We first argue that $a^{BR}(w(b))$ is not a singleton. Suppose there exists an $\hat{a}^* \neq a^*$ such that
 1212 the GMH contract $w(a^*, \hat{a}^*, b)$ implements a^* (see Proposition 6 and also Remark 4.17 in Ke and
 1213 Ryan (2016)), i.e., $V(w(a^*, \hat{a}^*, b), a^*) = \text{val}(P)$. Note that for any $\hat{a} \in \mathbb{A}$,

$$1214 \quad \text{val(SAND|}a^*, \hat{a}, b\text{)} \geq \max_{(w, a^*)} \{V(w, a^*) : U(w, a^*) \geq U^*, a^* \in a^{BR}(w)\}.$$

1215 Therefore, \hat{a}^* is the solution to the inner minimization problem

$$1216 \quad \hat{a}^* \in \arg \min_{\hat{a}} V^*(a^*, \hat{a}|b),$$

1217 which contradicts the supposition of non-existence. Therefore, the best response set $a^{BR}(w(b))$
 1218 must be a singleton, i.e., a^* is the unique best response at the optimal solution. In this case,
 1219 according to Mirrlees (1999), all no-jump constraints are slack at optimality and the FOA is valid
 1220 (up to the modified IR constraint $U(w, a^*) \geq b$).

1221 Finally, by Lemma 4, we know that $\text{val(SAND|}b\text{)}$ is equal to the value of the FOA with modified
 1222 IR constraint $U(w, a^*) \geq b$. \square

1223 We now return to the case where the infimum in (SAND| b) is attained. The proof proceeds in
 1224 two stages. In the first stage we examine a subclass of problems where the agent’s action a is given.
 1225 In the second stage we illustrate how to determine the right choice for a .

1226 **Remark 1.** We remark that the analysis of the first stage of the proof is drawn from results in Ke
 1227 and Ryan (2016). In that paper it is assumed that an action a^* is given and is implemented by an
 1228 optimal contract w^* such that $U(w^*, a^*) = \underline{U}$. In this setting, the assumption that $U(w^*, a^*) = \underline{U}$
 1229 is without loss of interest, since we assume that a^* and w^* are given and so \underline{U} can be defined as
 1230 $U(w^*, a^*)$. The assumption that $U(w^*, a^*) = \underline{U}$ is critical in Section 4 of Ke and Ryan (2016). See
 1231 Remark 4.16 of that paper for further discussion of this point. This is an important difference with
 1232 our current analysis. Here we no longer assume that a target a^* is given and so we cannot assume
 1233 without loss of generality that $U(w^*, a^*) = \underline{U}$. Indeed, uncovering a method to find w^* and a^* is
 1234 the focus of this paper.

1235 Accordingly, the analysis here proceeds in a different manner than Ke and Ryan (2016). First,
1236 Ke and Ryan (2016) considers a simpler version of (Min-Max| a, b') where the no-jump constraint
1237 is an equality. This is sufficient in that setting because they do not need to further analyze this
1238 problem to determine a^* , it is simply given. This oversimplifies the current development. Moreover,
1239 Stage 2 is not needed to analyze the situation in Ke and Ryan (2016). The added complexity of
1240 Stage 2 arises precisely because the optimal action for the agent and the utility delivered to the
1241 agent at optimality are both *a priori* unknown. ◀

1242 A.6.1 Analysis of Stage 1

1243 Define the following intermediate problem, which is the parametric problem (P| b') with $b' \geq \underline{U}$ and
1244 the agent's action a fixed:

$$\begin{aligned}
1245 & \max_{w \geq \underline{w}} V(w, a) \\
1246 & \text{subject to } U(w, a) \geq b' & (\text{P}|a, b') \\
1247 & U(w, a) - U(w, \hat{a}) \geq 0 & \text{for all } \hat{a} \in \mathbb{A}.
\end{aligned}$$

1249 We isolate attention to where the above problem is feasible; that is, a is an implementable action
1250 that delivers at least utility b' to the agent. Note we need not take b' equal to the b that is tight-
1251 at-optimality provided in the hypothesis of the theorem. It is an arbitrary $b' \geq \underline{U}$ with the above
1252 property.

1253 We can define the related problem

$$1254 \quad \inf_{\hat{a} \in \mathbb{A}} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b', U(w, a) - U(w, \hat{a}) \geq 0\}. \quad (\text{Min-Max}|a, b')$$

1255 We denote an optimal solution to (Min-Max| a, b') by $\hat{a}(a, b')$ and $w_{a, b'}$.

1256 Note that (P| a, b') is analogous to (P| b) and (Min-Max| a, b') is analogous to (SAND| b), however
1257 with a given. The key result is an implication of Theorem 4.15 in Ke and Ryan (2016) carefully
1258 adapted to this setting.

1259 **Proposition 6.** Suppose Assumptions 1–4 hold. Let a be an implementable action and let $b' =$
1260 $U(w_{a, \underline{U}}, a)$ where $w_{a, \underline{U}}$ is an optimal solution to (P| a, \underline{U}). Then $w_{a, b'}$ is equal to $w_{a, b'}$, an optimal
1261 solution to (Min-Max| a, b'). In particular, $w_{a, b'}$ is a GMH contract that implements a , $U(w_{a, b'}, a) =$
1262 b' and $\hat{a}(a, b')$ is an alternate best response to $w_{a, b'}$. Moreover, the Lagrange multipliers $\lambda(a, b')$
1263 and $\delta(a, b')$ in problem (SAND| $a, \hat{a}(a, b'), b'$) are both positive.

1264 *Proof.* The proof mimics the development in Section 4 of Ke and Ryan (2016) with two key dif-
1265 ferences. First, Ke and Ryan (2016) does not work with problem (Min-Max| a, b'), instead with a
1266 relaxed problem where \hat{a} is given.⁹ Moreover, the relaxed problem (P| \hat{a}) in Ke and Ryan (2016)
1267 was defined where the no-jump constraint was an equality. This suffices there because the target
1268 action a^* is given. We need more flexibility here, and hence to follow the logic of Ke and Ryan
1269 (2016) we must establish the following claims.

1270 **Claim 2.** Let $(w_{a, b'}, \hat{a}(a, b'))$ be an optimal solution to (Min-Max| a, b'), then

$$1271 \quad U(w_{a, b'}, a) - U(w_{a, b'}, \hat{a}(a, b')) = 0. \quad (49)$$

⁹In that paper, determining the optimal choice of \hat{a}^* , see the definition of \hat{a}^* in (4.31) of Ke and Ryan (2016).

1273 *Proof.* We argue that the Lagrangian multiplier δ^* in Lemma 3 applied to $(\text{SAND}|a, \hat{a}(a, b'), b')$ is
 1274 strictly greater than zero. Then complementary slackness (Lemma 3(ii-b)) implies (49) holds.

1275 Suppose $\delta^* = 0$. This implies that w_{a^*} is the first best contract, denoted $w^{fb}(b')$. We want to
 1276 show a^* is implemented by $w^{fb}(b')$. This, in turn, implies that the first-best contract is optimal,
 1277 contradicting Assumption 3. Let $\hat{a}' \in a^{BR}(w^{fb}(b'))$ and observe

$$\begin{aligned}
 1278 \quad \text{val}(\text{SAND}|a^*, \hat{a}(a^*), b') &= V(w^{fb}(b'), a^*) \\
 1279 &= \inf_{\hat{a} \in \mathbb{A}} \inf_{\lambda, \delta} \max_{w \geq \underline{w}} \mathcal{L}(w, \lambda, \delta | a^*, \hat{a}, b') \\
 1280 &\leq \inf_{\lambda, \delta} \max_{w \geq \underline{w}} \mathcal{L}(w, \lambda, \delta | a^*, \hat{a}', b') \\
 1281 &= \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b', U(w, a^*) - U(w, \hat{a}') \geq 0\} \\
 1282 &\leq \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq b'\} \\
 1283 &= V(w^{fb}(b'), a^*),
 \end{aligned} \tag{50}$$

1284 where the second equality is by strong duality, the first inequality is by the definition of a minimizer,
 1285 the third equality is again by strong duality, and the final inequality follows since we have relaxed
 1286 a constraint. Therefore, all inequalities in the above expression are equalities.

1287 If $U(w^{fb}(b'), a^*) = U(w^{fb}(b'), \hat{a}')$ then a^* is a best response to $w^{fb}(b')$ and we are done. Oth-
 1288 erwise from (50) we must assume $\delta(a^*, \hat{a}') = 0$. This follows by the uniqueness of Lagrangian
 1289 multipliers (Lemma 3). Therefore, $w^{fb}(b')$ is the solution to $\arg \max_{w \geq \underline{w}} \{V(w, a^*) : U(w, a^*) \geq$
 1290 $b', U(w, a^*) - U(w, \hat{a}') \geq 0\}$ and $U(w^{fb}(b'), a^*) - U(w^{fb}(b'), \hat{a}') \geq 0$ is satisfied. Since $\hat{a}' \in$
 1291 $a^{BR}(w^{fb}(b'))$, we have $a^* \in a^{BR}(w^{fb}(b'))$ as desired. \square

1292 The next two claims are adapted from Ke and Ryan (2016). To state them we need some
 1293 additional definitions. We let

$$1294 \quad T(x) \equiv \frac{v'(\pi(x) - w^*(x))}{u'(w^*(x))} \tag{51}$$

1295 and

$$1297 \quad R(x) \equiv 1 - \frac{f(x, \hat{a}(a, b'))}{f(x, a)}. \tag{52}$$

1298 Let

$$1300 \quad \mathcal{X}_{\underline{w}}^* = \{x \in \mathcal{X} : w^*(x) = \underline{w}\}. \tag{53}$$

1302 We say two functions φ and ψ with shared domain \mathcal{X} are *comonotone on the set* $S \subseteq \mathcal{X}$ if φ
 1303 and ψ are either both nonincreasing or both nondecreasing in S . If φ and ψ are comonotone on all
 1304 of \mathcal{X} we simply say that φ and ψ are *comonotone*.

1305 **Claim 3.** If both $T(x)$ and $R(x)$ are comonotone functions of x on $\mathcal{X} \setminus \mathcal{X}_{\underline{w}}^*$ then w^* is equal to $w_{a, b'}$.
 1306 Moreover, the Lagrangian multipliers λ and δ associated with the dual of $(\text{SAND}|a, \hat{a}(a, b'), b')$ are
 1307 strictly positive.

1308 *Proof.* This is Corollary 4.13 of Ke and Ryan (2016) setting \underline{U} in that paper to b' . The condition
 1309 that a is an implementable action and $b' = U(w^{a, \underline{U}}, a)$ where $w^{a, \underline{U}}$ is an optimal solution to $(\text{P}|a, \underline{U})$
 1310 is required for this proof to hold. \square

1311 The next result is to establish how our assumptions on the output distribution (Assumption 4)
 1312 guarantee comonotonicity.

1313 **Claim 4.** If Assumptions 1–4 hold then $T(x)$ and $R(x)$ are comonotone on $\mathcal{X} \setminus \mathcal{X}_{\underline{w}}^*$.

1314 *Proof.* This is Lemma 4.14 of Ke and Ryan (2016). Note that the condition that a be an im-
 1315 plementable action and $b' = U(w^{a, \underline{U}}, a)$ where $w^{a, \underline{U}}$ is an optimal solution to $(P|a, \underline{U})$ is required
 1316 for this proof to hold. Moreover, this also requires Claim 2, where the equality of the no-jump
 1317 constraint is used to establish (C.14) in Ke and Ryan (2016). \square

1318 Putting the last two claims together yields Proposition 6. \square

1319 An easy implication of the above proposition is that

$$1320 \quad \text{val}(\text{Min-Max}|a, b') = \text{val}(P|a, b')$$

1322 whenever a is implementable and delivers the agent utility b' in optimality. This will prove to be a
 1323 useful result in the rest of the proof of Theorem 1. It remains to determine the right implementable
 1324 a . This is the task of Stage 2.

1325 A.6.2 Analysis of Stage 2

1326 Recall that we are working with a specific $b = U(w^*, a^*)$ where (w^*, a^*) is an optimal solution
 1327 to (P) (guaranteed to exist by Assumption 3). The goal of the rest of the proof is to show that
 1328 $\text{val}(P) = \text{val}(\text{SAND}|b)$.

1329 We divide this stage of the proof into two further substages. The first substage (Stage 2.1)
 1330 shows the equivalence between the original problem and a variational max-min-max problem. This
 1331 intermediate variational problem allows us to leverage the single-dimensional reasoning on display
 1332 in the proof of Theorem 1 in the single-outcome case in the main body of the paper.

1333 The second substage (Stage 2.2) shows the equivalence between this variational max-min-max
 1334 and the sandwich problem $(\text{SAND}|b)$.

1335 *Stage 2.1.* We lighten the notation of Stage 1, and let w_a denote an optimal solution to $(\text{Min-Max}|a, b)$
 1336 with optimal alternate best response $\hat{a}(a)$ when b is our target agent utility. We construct a varia-
 1337 tional problem based on w_a as follows. Given $z \in [-1, 1]$, define a set of variations

$$1338 \quad \mathcal{H}(a, z) \equiv \{h \leq \bar{h}(x) : h(x) = 0 \text{ if } w_a(x) = \underline{w} \text{ and } w_a + zh \geq \underline{w} \text{ otherwise}\}$$

1339 where $\bar{h}(x) > w_a(x)$ is sufficiently large, but also where $\int \bar{h}(x)f(x, a)dx < K < \infty$ for some K . We
 1340 add an additional restriction

$$1341 \quad \mathcal{M}(a, z) = \{h \in \mathcal{H}(a, z) : \int v'(\pi(x) - w_a(x))h(x)f(x, a)dx \geq 0, \int u'(w_a(x))h(x)f(x, a)dx \geq 0\}.$$

1342 If $h \in \mathcal{M}(a, z)$ then it is not plausible for both the principal and agent to be strictly better off
 1343 under the variational problem as compared to the original problem. They have a direct conflict of
 1344 interest in z . This puts us into a situation analogous to the single-outcome case.

1345 We now show the following equivalence:

$$1346 \quad \text{val}(P) = \text{val}(\text{Var}|b) \tag{54}$$

1347 where $(\text{Var}|b)$ is the variational optimization problem

$$1348 \quad \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} \max_{z \in [-1,1]} \max_{h \in \mathcal{M}(a,z)} \{V(w_a + zh, a) : (w_a, a) \in \mathcal{W}(\hat{a}, b)\}. \quad (\text{Var}|b)$$

1349 The “ \leq ” direction of (54) is straightforward since

$$\begin{aligned} 1350 \quad & \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} \max_{z \in [-1,1]} \max_{h \in \mathcal{M}(a,z)} \{V(w_a + zh, a) : (w_a, a) \in \mathcal{W}(\hat{a}, b)\} \\ 1351 \quad & \geq \inf_{\hat{a} \in \mathbb{A}} \max_{z \in [-1,1]} \max_{h \in \mathcal{M}(a^*, z)} \{V(w_{a^*} + zh, a^*) : (w_{a^*} + zh, a^*) \in \mathcal{W}(\hat{a}, b)\} \\ 1352 \quad & \geq V(w_{a^*}, a^*) = \text{val}(\mathbf{P}), \end{aligned}$$

1354 where the first inequality follows since the optimal action a^* is a feasible choice for a in the outer-
1355 maximization, the second inequality follows by taking $z = 0$, and the final equality holds from
1356 Proposition 6.

1357 It remains to consider the “ \geq ” direction of (54). The following claim is analogous Lemma 9 in
1358 the proof of Lemma 5.

1359 **Claim 5.** Given any \hat{a} and a , strong duality holds for $(\text{Var}|b)$. That is, for a given $z \in [-1, 1]$

$$\begin{aligned} 1360 \quad & \max_{h \in \mathcal{M}(a,z)} \{V(w_a + zh, a) : (w_a + zh, a) \in \mathcal{W}(\hat{a}, b)\} \\ 1361 \quad & = \inf_{\lambda, \delta, \gamma \geq 0} \max_{h \in \mathcal{H}(a,z)} \mathcal{L}^h(zh, \lambda, \delta, \gamma | a, \hat{a}, b) \end{aligned} \quad (55)$$

1363 where

$$\begin{aligned} 1364 \quad \mathcal{L}^h(zh, \lambda, \delta, \gamma | a, \hat{a}, b) &= V(w_a + zh, a) + \lambda[U(w_a + zh, a) - b] + \delta[U(w_a + zh, a) - U(w_a + zh, \hat{a})] \\ 1365 \quad &+ \text{sgn}(z)\gamma_1 \int v'(\pi(x) - w_a(x))zh(x)f(x, a)dx + \text{sgn}(z)\gamma_2 \int u'(w_a(x))zh(x)f(x, a)dx \\ 1366 \end{aligned}$$

1367 is the Lagrangian function (which combines the choice of z and h into the product zh since this is
1368 how z and h appear in both the objective and constraints), and $\lambda \geq 0$, $\delta \geq 0$ and $\gamma = (\gamma_1, \gamma_2) \geq 0$
1369 are the Lagrangian multipliers for the remaining constraints defining $\mathcal{M}(a, z)$. Moreover, given
1370 that $h^*(\cdot|z)$ solves (55) as a function of z , complementary slackness holds for the optimal choice of
1371 $z \in \text{argmax}_{z \in [-1,1]} \max_{h \in \mathcal{M}(a,z)} \{V(w_a + zh, a) : (w_a + zh, a) \in \mathcal{W}(\hat{a}, b)\}$; that is,

$$\begin{aligned} 1372 \quad & \lambda[U(w_a + zh^*(\cdot|z), a) - b] = 0, \lambda \geq 0, U(w_a + zh^*(\cdot|z), a) - b \geq 0 \\ 1373 \quad & \delta[U(w_a + zh, a) - U(w_a + zh^*(\cdot|z), \hat{a})] = 0, \delta \geq 0, U(w_a + zh^*(\cdot|z), a) \geq U(w_a + zh^*(\cdot|z), \hat{a}) \\ 1374 \quad & \gamma_1 \int v'(\pi(x) - w_a(x))h^*(x|z)f(x, a)dx = 0, \gamma_1 \geq 0, \int v'(\pi(x) - w_a(x))h^*(x|z)f(x, a)dx \geq 0 \\ 1375 \quad & \gamma_2 \int u'(w_a(x))h^*(x|z)f(x, a)dx = 0, \gamma_2 \geq 0, \int u'(w_a(x))h^*(x|z)f(x, a)dx \geq 0. \end{aligned}$$

1377 *Proof.* By weak duality the “ \leq ” direction of (55) is immediate. It remains to consider the “ \geq ” direc-
1378 tion. For every λ, δ and γ , $\max_{h \in \mathcal{H}(a,z)} \mathcal{L}^h(zh, \lambda, \delta, \gamma | a, \hat{a}, b)$ is convex in $(\lambda, \delta, \gamma)$. Let $((zh)^*, \lambda^*, \delta^*, \gamma^*)$
1379 denote an optimal solution to the right-hand side of (55). To establish strong duality, we want
1380 show a complementary slackness condition with $(\lambda^*, \delta^*, \gamma^*)$.

1381 The optimization of $\mathcal{L}^h(zh, \lambda, \delta, \gamma|a, \hat{a}, b)$ over zh can be done in a pointwise manner similar to
 1382 how we approached (SAND $|a, \hat{a}, b$). Given z , by the concavity and monotonicity of v and u , the
 1383 optimal solution $h(x|z)$ to $\max_{h \in \mathcal{H}(a, \hat{a}, z)} \mathcal{L}^h(zh, \lambda, \delta, \gamma|a, \hat{a}, b)$ must satisfy the following necessary
 1384 and sufficient conditions:

1385 (i) when $z \geq 0$, $zh(x|z)$ satisfies:

$$1386 \left\{ \begin{array}{l} \frac{v'(\pi(x)-w_a(x)-zh(x|z))}{u'(w_a(x)+zh(x|z))} \\ \quad = [\lambda + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)})] + \frac{\gamma_1 v'(\pi-w_a) + \gamma_2 u'(w_a)}{zu'(w_a(x)+zh(x|z))} \\ h(x|z) = 0 \\ h(x|z) = \bar{h}(x) \end{array} \right. \begin{array}{l} \text{if } \frac{v'(\pi(x)-w_a(x))}{u'(w_a(x))} (1 - \frac{\gamma_1}{z}) < \lambda + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)}) + \frac{\gamma_2}{z} \\ \leq \frac{v'(\pi(x)-w_a(x)-z\bar{h}(x))}{u'(w_a(x)+z\bar{h}(x))} - \frac{\gamma_1 v'(\pi-w_a) + \gamma_2 u'(w_a)}{zu'(w_a(x)+z\bar{h}(x))} \\ \text{if } \frac{v'(\pi(x)-w_a(x))}{u'(w_a(x))} (1 - \frac{\gamma_1}{z}) \geq \lambda + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)}) + \frac{\gamma_2}{z} \\ \lambda + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)}) + \frac{\gamma_2}{z} \\ \text{if } > \frac{v'(\pi(x)-w_a(x)-z\bar{h}(x))}{u'(w_a(x)+z\bar{h}(x))} - \frac{\gamma_1 v'(\pi-w_a) + \gamma_2 u'(w_a)}{zu'(w_a(x)+z\bar{h}(x))} \end{array}$$

1387 (ii) when $z \leq 0$, $zh(x|z)$ satisfies:

$$1388 \left\{ \begin{array}{l} \frac{v'(\pi(x)-w_a(x)-zh(x|z))}{u'(w_a(x)+zh(x|z))} \\ \quad = [\lambda + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)})] + \frac{\gamma_1 v'(\pi-w_a) + \gamma_2 u'(w_a)}{u'(w_a(x)+zh(x|z))} \\ h(x|z) = 0 \\ h(x|z) = \bar{h}(x) \end{array} \right. \begin{array}{l} \text{if } \frac{v'(\pi(x)-w_a(x))}{u'(w_a(x))} (1 - \frac{\gamma_1}{z}) > \lambda + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)}) + \frac{\gamma_2}{z} \\ \geq \frac{v'(\pi(x)-w_a(x)-z\bar{h}(x))}{u'(w_a(x)+z\bar{h}(x))} - \frac{\gamma_1 v'(\pi-w_a) + \gamma_2 u'(w_a)}{zu'(w_a(x)+z\bar{h}(x))} \\ \text{if } \frac{v'(\pi(x)-w_a(x))}{u'(w_a(x))} (1 - \frac{\gamma_1}{z}) \leq \lambda + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)}) + \frac{\gamma_2}{z} \\ \lambda + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)}) + \frac{\gamma_2}{z} \\ \text{if } < \frac{v'(\pi(x)-w_a(x)-z\bar{h}(x))}{u'(w_a(x)+z\bar{h}(x))} - \frac{\gamma_1 v'(\pi-w_a) + \gamma_2 u'(w_a)}{zu'(w_a(x)+z\bar{h}(x))}. \end{array}$$

1389 Now we show that, given z , we have the strong duality

$$1390 \max_{h \in \mathcal{M}(a, z)} \{V(w_a + zh, a) : (w_a + zh, a) \in \mathcal{W}(\hat{a}, b)\} = \inf_{\lambda, \delta, \gamma \geq 0} \max_{h \in \mathcal{H}(a, z)} \tilde{\mathcal{L}}^h(zh, \lambda, \delta, \gamma|a, \hat{a}, b)$$

1391 where the Lagrangian is

$$1392 \tilde{\mathcal{L}}^h(zh, \lambda, \delta, \gamma|a, \hat{a}, b) = V(w_a + zh, a) + \lambda[U(w_a + zh, a) - U^*] + \delta[U(w_a + zh, a) - U(w_a + zh, \hat{a})] \\ 1393 + \gamma_1 \int v'(\pi(x) - w_a(x))h(x)f(x, a)dx + \gamma_2 \int u'(w_a(x))h(x)f(x, a)dx.$$

1394 This result follows the uniqueness of $h(x|z)$ as the maximizer of $\tilde{\mathcal{L}}^h(zh, \lambda, \delta, \gamma|a, \hat{a}, b)$ over h .
 1395 Therefore, the Lagrangian dual function $\psi(\lambda, \delta, \gamma|z) = \max_{h \in \mathcal{H}(a, z)} \mathcal{L}^h(zh, \lambda, \delta, \gamma|a, \hat{a}, b)$ is continu-
 1396 ous, differentiable, and convex in $(\lambda, \delta, \gamma)$. This allows us to establish strong duality using similar
 1397 reasoning as in the proof of Lemma 3.

1398 Let z^* denote the optimal choice of z . We discuss the case where $z^* > 0$. The case where $z^* < 0$
 1399 is similar and thus is omitted. In this case the constraint

$$1400 \int v'(\pi(x) - w_a(x))h(x)f(x, a)dx \geq 0$$

1401 is equivalent to $\int v'(\pi(x) - w_a(x))zh(x)f(x, a)dx \geq 0$ and $\int u'(w_a(x))h(x)f(x, a)dx$ is equivalent
 1402 to $\int u'(w_a)zhf(x, a)dx \geq 0$. Since $h(x|z)$ is uniquely determined, it is continuous in z . Let

$$1403 h^*(x|z^*) \in \arg \max_{h \in \mathcal{H}(a, z)} \{V(w_a + z^*h, a) : (w_a + z^*h, a) \in \mathcal{W}(\hat{a}, b)\} \\ 1404$$

1405 be the unique solution to the problem given z^* . Note that $\int v'(\pi(x) - w_a(x))h^*(x|z^*)f(x, a)dx > 0$
 1406 and $\int u'(w_a(x))h^*(x|z^*)f(x, a)dx > \frac{1}{z} \int (u(w_a(x) + z^*h^*(x|z)) - u(w_a(x)))f(x, a)dx \geq 0$ and

$$\begin{aligned}
 1407 \quad & - \int v'(\pi(x) - w_a(x) - z^*h(x|z^*))h^*(x|z^*)f(x, a)dx \\
 1408 \quad & < - \int v'(\pi(x) - w_a(x))h^*(x|z^*)f(x, a)dx \\
 1409 \quad & < 0.
 \end{aligned}$$

1410 Then, there must exist Lagrange multipliers $(\lambda^o, \delta^o, \gamma^o)$ such that

$$\begin{aligned}
 1411 \quad 0 &= \frac{\partial}{\partial z} \mathcal{L}^h(z^*h^*(x|z^*), \lambda^o, \delta^o, \gamma^o | a, \hat{a}, b) \\
 1412 &= \int \left(\begin{aligned} & -v'(\pi(x) - w_a(x) - z^*h^*(x|z^*)) + [\lambda^o + \delta^o(1 - \frac{f(x, \hat{a})}{f(x, a)})]u'(w_a(x) + z^*h^*(x|z^*)) \\ & + \gamma_1^o v'(\pi(x) - w_a(x)) + \gamma_2^o u'(w_a(x)) \end{aligned} \right) h(x|z^*)f(x, a)dx
 \end{aligned}$$

1413 and $(\lambda^o, \delta^o, \gamma^o)$ satisfies the complementarity slackness condition. \square

1414 The above claim is used to establish another important technical result. The proof is completely
 1415 analogous to the proof of Lemma 5 in the single-outcome case and thus omitted.

1416 **Claim 6.** Let $(a^*, \hat{a}^*, z^*, h^*)$ be an optimal solution to $(\text{Var}|b)$ such that $U(w_{a^*} + z^*h^*, a^*) > b$.
 1417 Then there exists an $\epsilon > 0$ and optimal solution $(a_\epsilon^*, \hat{a}_\epsilon^*, z^*, h_\epsilon^*)$ such that $U(w_{a_\epsilon^*} + z^*h_\epsilon^*, a_\epsilon^*) = b + \epsilon$
 1418 and $V(w_{a^*} + z^*h^*, a^*) = V(w_{a_\epsilon^*} + z^*h_\epsilon^*, a_\epsilon^*)$.

1419 Via Claim 6 there exists a $b^* \geq b$ and an optimal solution $(\tilde{a}^*, \hat{a}^*, z^*, h^*)$ to $(\text{Var}|b)$ such that
 1420 $\text{val}(\text{Var}|b) = \text{val}(\text{Var}|b^*)$ and $U(w_{\tilde{a}^*} + z^*h^*, \tilde{a}^*) = b^*$. It then suffices to argue that \tilde{a}^* is imple-
 1421 mentable (and thus feasible to (\mathbf{P})), thus satisfying (54).

1422 To establish implementability, we let $\hat{a}' \in a^{BR}(w_{\tilde{a}^*} + z^*h^*)$ and claim

$$\begin{aligned}
 1423 \quad & V(w_{\tilde{a}^*} + z^*h^*, \tilde{a}^*) \\
 1424 \quad &= \max_{z \in [-1, 1]} \max_{h \in \mathcal{M}(\tilde{a}^*, z)} \{V(w_{\tilde{a}^*} + z^*h^* + zh, \tilde{a}^*) : (w_{\tilde{a}^*} + z^*h^* + zh, \tilde{a}^*) \in \mathcal{W}(\hat{a}', b^*)\},
 \end{aligned} \tag{56}$$

1425 where

$$\tilde{\mathcal{M}}(\tilde{a}^*, z) = \left\{ h \in \tilde{\mathcal{H}}(\tilde{a}^*, z) : \begin{aligned} & \int v'(\pi(x) - w_{\tilde{a}^*}(x) - z^*h^*(x))h(x)f(x, \tilde{a}^*)dx \geq 0, \\ & \int u'(w_{\tilde{a}^*}(x) + z^*h^*(x))h(x)f(x, \tilde{a}^*)dx \geq 0 \end{aligned} \right\}$$

1426 and

$$1427 \quad \tilde{\mathcal{H}}(a, z) \equiv \{h \leq \bar{h}(x) : h(x) = 0 \text{ if } w_a(x) + z^*h^*(x) + zh(x) = \underline{w} \text{ and } w_a + z^*h^* + zh \geq \underline{w} \text{ otherwise}\}.$$

1431 If (56) holds then \tilde{a}^* is indeed implementable since $zh = 0$ is a solution to the right-hand
 1432 side problem. The condition in the right-hand side that $(w_{\tilde{a}^*} + z^*h^*, \tilde{a}^*) \in \mathcal{W}(\hat{a}', b^*)$ implies
 1433 $U(w_{\tilde{a}^*} + z^*h^*, \tilde{a}^*) \geq U(w_{\tilde{a}^*} + z^*h^*, \hat{a}')$ and so \tilde{a}^* itself must be a best response to $w_{\tilde{a}^*} + z^*h^*$.

1434 To establish (56), note that “ \leq ” follows immediately since $(\tilde{a}^*, \hat{a}^*, z^*, h^*)$ solves the left-hand
 1435 side of (54), whereas in the right-hand side of (56), a particular \hat{a} is chosen (namely \hat{a}') and there
 1436 is an additional degree of freedom zh . Next, suppose that

$$\begin{aligned}
 1437 \quad & V(w_{\tilde{a}^*} + z^*h^*, \tilde{a}^*) \\
 1438 \quad & < \max_{z \in [-1, 1]} \max_{h \in \tilde{\mathcal{M}}(\tilde{a}^*, z)} \{V(w_{\tilde{a}^*} + z^*h^* + zh, \tilde{a}^*) : (w_{\tilde{a}^*} + z^*h^* + zh, \tilde{a}^*) \in \mathcal{W}(\hat{a}', b^*)\}
 \end{aligned} \tag{57}$$

1439 and derive a contradiction.

1440 Let $(z^{*'}, h^{*'})$ denote an optimal solution to $\max_{z \in [-1, 1]} \max_{h \in \tilde{\mathcal{M}}(\tilde{a}^*, z)} \{V(w_{\tilde{a}^*} + z^* h^* + zh, \tilde{a}^*) :$
 1441 $(w_{\tilde{a}^*} + z^* h^* + zh, \tilde{a}^*) \in \mathcal{W}(\hat{a}', b^*)\}$. If (57) holds, $V(w_{\tilde{a}^*} + z^* h^*, \tilde{a}^*) < V(w_{\tilde{a}^*} + z^* h^* + z^{*'} h^{*'}, \tilde{a}^*)$
 1442 and thus

$$1443 \quad 0 < V(w_{\tilde{a}^*} + z^* h^* + z^{*'} h^{*'}, \tilde{a}^*) - V(w_{\tilde{a}^*} + z^* h^*, \tilde{a}^*) \\ 1444 \quad \leq -z^{*'} \int h^{*'}(x) v'(\pi(x) - w_{\tilde{a}^*}(x) - z^* h^*(x)) f(x, \tilde{a}^*) dx,$$

1445 since v is concave. Note that $\int h^{*'} v'(\pi - w_{\tilde{a}^*} - z^* h^*) f(x, \tilde{a}^*) dx = 0$ will generate the contradiction
 1446 $0 < 0$. This further implies $z^{*'} \leq 0$ since $\int h^{*'} v'(\pi - w_{\tilde{a}^*} - z^* h^*) f(x, \tilde{a}^*) dx \geq 0$ by design of the
 1447 variation set $\tilde{\mathcal{M}}(\tilde{a}^*, z)$. This, in turn, implies $b^* = U(w_{\tilde{a}^*} + z^* h^*, \tilde{a}^*) > U(w_{\tilde{a}^*} + z^* h^* + z^{*'} h^{*'}, \tilde{a}^*)$
 1448 since u is concave and $\int h^{*'} u'(w_{\tilde{a}^*} + z^* h^*) f(x, \tilde{a}^*) dx \geq 0$ by design of the variation set $\tilde{\mathcal{M}}(\tilde{a}^*, z)$:

$$1449 \quad U(w_{\tilde{a}^*} + z^* h^* + z^{*'} h^{*'}, \tilde{a}^*) - U(w_{\tilde{a}^*} + z^* h^*, \tilde{a}^*) \\ 1450 \quad = \int [u(w_{\tilde{a}^*}(x) + z^* h^*(x) + z^{*'} h^{*'}(x)) - u(w_{\tilde{a}^*}(x) + z^* h^*(x))] f(x, \tilde{a}^*) dx \\ 1451 \quad < \int z^{*'} h^{*'}(x) u'(w_{\tilde{a}^*}(x) + z^* h^*(x)) f(x, \tilde{a}^*) dx \\ 1452 \quad \leq 0.$$

1453 This is a contradiction, since the constraint $(w_{\tilde{a}^*} + z^* h^* + z^{*'} h^{*'}, \tilde{a}^*) \in \mathcal{W}(\hat{a}', b^*)$ implies $U(w_{\tilde{a}^*} +$
 1454 $z^* h^* + z^{*'} h^{*'}, \tilde{a}^*) \geq b^*$. This completes Stage 2.1.

1455 *Stage 2.2:* It remains to show

$$1456 \quad \text{val}(\mathbf{Var}|b) = \text{val}(\mathbf{SAND}|b). \quad (58)$$

1457 Combined with (54), this shows $\text{val}(\mathbf{P}) = \text{val}(\mathbf{SAND}|b)$, finishing the proof. The direction

$$1458 \quad \text{val}(\mathbf{Var}|b) = \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} \max_{z \in [-1, 1]} \max_{h \in \mathcal{M}(a, z)} \{V(w_a + zh, a) : (w_a + zh, a) \in \mathcal{W}(\hat{a}, b)\} \\ 1459 \quad \leq \max_{a \in \mathbb{A}} \inf_{\hat{a} \in \mathbb{A}} \max_{w \geq \underline{w}} \{V(w, a) : (w, a) \in \mathcal{W}(\hat{a}, b)\} = \text{val}(\mathbf{SAND}|b) \\ 1460$$

1461 follows immediately. It remains to show the “ \geq ” direction of (58).

1462 Let $(a^\#, \hat{a}^\#, w_{a^\#})$ be an optimal solution to $(\mathbf{SAND}|b)$ that delivers utility $b' \geq b$ to the agent.
 1463 That is, the constraint $U(w, a) = b'$ is binding in $(\mathbf{SAND}|b')$. We have

$$1464 \quad \text{val}(\mathbf{Var}|b) \geq \inf_{\hat{a} \in \mathbb{A}} \max_{z \in [-1, 1]} \max_{h \in \mathcal{M}(a^\#, z)} \{V(w_{a^\#} + zh, a^\#) : (w_{a^\#} + zh, a^\#) \in \mathcal{W}(\hat{a}, b)\} \quad (59)$$

$$1465 \quad \geq \inf_{\hat{a} \in \mathbb{A}} \max_{z \in [-1, 1]} \max_{h \in \mathcal{M}(a^\#, z)} \{V(w_{a^\#} + zh, a^\#) : (w_{a^\#} + zh, a^\#) \in \mathcal{W}(\hat{a}, b')\} \quad (60)$$

$$1466 \quad = \max_{z \in [-1, 1]} \max_{h \in \mathcal{M}(a^\#, z)} \{V(w_{a^\#} + zh, a^\#) : (w_{a^\#} + zh, a^\#) \in \mathcal{W}(\hat{a}^0, b')\} \quad (61) \\ 1467$$

1468 where \hat{a}^0 is any action in the argmin of the right-hand side of (60). If such an action does not
 1469 exist we use a first-order condition following Lemma 4. The details of this case are analogous
 1470 and thus omitted. Let $(z^\#, h^\#)$ be in the argmax of the right-hand side of (61). It suffices
 1471 to show that $\text{val}(\mathbf{SAND}|b)$ is equal to the value of the right-hand side of (61). Observe that
 1472 $\text{val}(\mathbf{SAND}|b) = \text{val}(\mathbf{SAND}|b')$ and so in the sequel we work with b' without loss.

1473 We argue this in two further substages. First, we argue that (i) $\text{val}(61) = \text{val}(\text{Min-Max}|a^\#, b^\#)$
1474 where $b^\# = U(w_{a^\#} + z^\#h^\#, a^\#) \geq b'$. For this we use Proposition 6 of Stage 1. Second, we
1475 argue that, in fact (ii) $b' = b^\#$. In this case, $\text{val}(\text{Min-Max}|a^\#, b^\#) = \text{val}(\text{Min-Max}|a^\#, b') =$
1476 $\text{val}(\text{SAND}|b')$ since $(a^\#, \hat{a}^\#, w^\#)$ is an optimal solution to $(\text{SAND}|b')$. From (i) this implies $\text{val}(61) =$
1477 $\text{val}(\text{SAND}|b')$. In light of (59)–(61), and the fact $\text{val}(\text{SAND}|b) = \text{val}(\text{SAND}|b')$, this implies
1478 $\text{val}(\text{Var}|b) \geq \text{val}(\text{SAND}|b)$ and completes the proof. It remains to establish (i) and (ii) in Stages
1479 2.2.1 and 2.2.2 respectively.

1480 *Stage 2.2.1:* (i) $\text{val}(61) = \text{val}(\text{Min-Max}|a^\#, b^\#)$.

1481 Using similar arguments as in Stage 2.1 we can conclude that $a^\#$ is implemented by $w_{a^\#} + z^\#h^\#$,
1482 using the fact that $U(w_{a^\#} + z^\#h^\#, a^\#) = b^\#$ to construct a contradiction.

1483 Given that $w_{a^\#} + z^\#h^\#$ implements $a^\#$ and delivers utility $b^\#$ to the agent, we can apply
1484 Proposition 6 to construct an optimal contract $w_{a^\#, b^\#}$ to $(P|a^\#, b^\#)$ with alternate best response
1485 $\hat{a}(a^\#, b^\#)$. We then claim the following:

$$1486 \quad V(w_{a^\#, b^\#}, a^\#) = \text{val}(61). \quad (62)$$

1488 To establish this, we show that $h = w_{a^\#, b^\#} - w_{a^\#}$ belongs to $\mathcal{M}(a^\#, z)$ for $z = 1$. Clearly $w_{a^\#} + h =$
1489 $w_{a^\#, b^\#} \geq \underline{w}$ is satisfied, and $w_{a^\#, b^\#} - w_{a^\#} \leq \bar{h}(x)$ by defining K appropriately large (recall its size
1490 was previously left unspecified). Next, we use the concavity of v to see

$$1491 \quad \int [w_{a^\#, b^\#}(x) - w_{a^\#}(x)]v'(\pi(x) - w_{a^\#}(x))f(x, a^\#)dx$$

$$1492 \quad \geq \int [v(\pi(x) - w_{a^\#}(x)) - v(\pi(x) - w_{a^\#, b^\#}(x))]f(x, a^\#)dx$$

$$1493 \quad = \text{val}(\text{SAND}|b) - V(w_{a^\#, b^\#}, a^\#)$$

$$1494 \quad \geq \text{val}(\text{SAND}|b) - V(w_{a^\#, b'}, a^\#)$$

$$1495 \quad = 0,$$

1496 since $V(w_{a^\#, b}, a^\#)$ is decreasing in b and using the fact that $b^\# \geq b'$. Next, we note

$$1497 \quad \int [w_{a^\#, b^\#}(x) - w_{a^\#}(x)]u'(w_{a^\#}(x))f(x, a^\#)dx$$

$$1498 \quad \geq \int [u(w_{a^\#, b^\#}(x)) - u(w_{a^\#}(x))]f(x, a^\#)dx$$

$$1499 \quad = b^\# - b'$$

$$1500 \quad \geq 0$$

1501 by the concavity of u . This shows $h = w_{a^\#, b^\#} - w_{a^\#} \in \mathcal{M}(a^\#, z)$ for $z = 1$. Letting $zh =$
1502 $w_{a^\#, b^\#} - w_{a^\#}$, it is immediate that $w_{a^\#} + zh = w_{a^\#, b^\#} \in \mathcal{W}(\hat{a}^0, b)$. Indeed, $U(w_{a^\#, b^\#}, a^\#) = b^\# \geq b'$
1503 and $U(w_{a^\#, b^\#}, a^\#) - U(w_{a^\#, b^\#}, \hat{a}^0) \geq 0$, since $a^\#$ is implemented by $w_{a^\#, b^\#}$. This implies that
1504 $zh = w_{a^\#, b^\#} - w_{a^\#}$ is feasible choice to (61) and so

$$1505 \quad \text{val}(61) \geq V(w_{a^\#, b^\#}, a^\#).$$

1506 Similarly, since $w_{a^\#} + z^\#h^\#$ is a feasible solution to $(P|a^\#, b^\#)$ (and $w_{a^\#, b^\#}$ is an optimal solution)
1507 so we get the reverse direction of the above and conclude

$$1508 \quad V(w_{a^\#} + z^\#h^\#, a^\#) = V(w_{a^\#, b^\#}, a^\#).$$

1509 This yields (62) and completes Stage 2.2.1. This implies that $\hat{a}(a^\#, b^\#)$ can be chosen as \hat{a}^0 .

1510 *Stage 2.2.2:* (ii) $b' = b^\#$.

1511 It suffices to show $U(w_{a^\#} + z^\# h^\#, a^\#) = b'$. To do so, we leverage the Lagrangian dual in (55)
 1512 and argue the Lagrangian multiplier $\lambda_{z^\#}$ for constraint $U(w_{a^\#} + z^\# h^\#, a^\#) \geq b'$ is strictly positive.
 1513 Then by complementary slackness this implies $U(w_{a^\#} + z^\# h^\#, a^\#) = b'$, as required.

1514 Note that $V(w_{a^\#} + z^\# h^\#, a^\#) < V(w_{a^\#}, a^\#)$, (otherwise this already establishes the “ \geq ” di-
 1515 rection of (54)) and so we have $z^\# > 0$, again using a concavity argument as above. Then $z^\# h^\#$ is
 1516 uniquely determined by the first-order condition (i) in Claim 5.

1517 Suppose $U(w_{a^\#} + z^\# h^\#, a^\#) > b'$, then we have $\lambda_{z^\#} = 0$. Then $h^\# = w_{a^\#, b^\#} - w_{a^\#} \neq 0$ implies
 1518 $\int [w_{a^\#, b^\#}(x) - w_{a^\#}(x)] u'(w_{a^\#}(x)) f(x, a^\#) dx > 0$ and thus $\gamma_2^* = 0$. Moreover, $\text{val}(\text{SAND}|b) >$
 1519 $V(w_{a^\#, b^\#}, a^\#)$ implies $\int [w_{a^\#, b^\#}(x) - w_{a^\#}(x)] v'(\pi(x) - w_{a^\#}(x)) f(x, a^\#) dx > 0$, which yields $\gamma_1^* = 0$.
 1520 Therefore, the first-order condition for $w_{a^\#, b^\#}$ becomes

$$1521 \frac{v'(\pi(x) - w_{a^\#, b^\#}(x))}{u'(w_{a^\#, b^\#}(x))} = \lambda_{z^\#} + \delta_{z^\#} \left(1 - \frac{f(x, \hat{a}^0)}{f(x, a^\#)}\right) = \delta_{z^\#} \left(1 - \frac{f(x, \hat{a}^0)}{f(x, a^\#)}\right), \text{ whenever } w(a^\#, \hat{a}^0, b^\#) > \underline{w}, \quad (63)$$

1522 where $\lambda_{z^\#}$ and $\delta_{z^\#}$ are the Lagrangian multipliers for the variation problem given $z^\#$. In the
 1523 case where $\hat{a}_0 \rightarrow a^\#$, Lemma 4 applies and the same structure as (63) holds with the second
 1524 term equal to $\delta_{z^\#} \frac{f_a(x, a^\#)}{f(x, a^\#)}$. The argument for this case is equivalent and so we omit it. However,
 1525 from Proposition 6, we know there is positive Lagrangian multiplier $\lambda(a^\#, b^\#)$ for optimal contract
 1526 $w_{a^\#, b^\#}$. By (63), and the fact $w_{a^\#, b^\#}$ is a GMH contract, we have:

$$1527 \frac{v'(\pi(x) - w_{a^\#, b^\#}(x))}{u'(w_{a^\#, b^\#}(x))} = \delta_{z^\#} \left(1 - \frac{f(x, \hat{a}^0)}{f(x, a^\#)}\right) = \lambda(a^\#, b^\#) + \delta(a^\#, b^\#) \left(1 - \frac{f(x, \hat{a}^0)}{f(x, a^\#)}\right)$$

1529 for all x such that $w_{a^\#, b^\#}(x) > \underline{w}$. However, if $\left(1 - \frac{f(x, \hat{a}^0)}{f(x, a^\#)}\right)$ is not a constant for almost all
 1530 x the above equalities cannot hold since $\lambda(a^\#, b^\#) > 0$. This contradicts the supposition that
 1531 $U(w_{a^\#} + z^\# h^\#, a^\#) > b'$ and $\lambda_{z^\#} = 0$.

1532 It only remains to consider the case where $\left(1 - \frac{f(x, \hat{a}^0)}{f(x, a^\#)}\right)$ is a constant for almost all x such that
 1533 $w_{a^\#, b^\#}(x) > \underline{w}$. In this case, by the continuity of $\frac{v'(\pi(x) - w_{a^\#, b^\#}(x))}{u'(w_{a^\#, b^\#}(x))}$ in x ($w_{a^\#, b^\#}$ is continuous in x
 1534 because it is a GMH contract), we know $\frac{v'(\pi(x) - w_{a^\#, b^\#}(x))}{u'(w_{a^\#, b^\#}(x))}$ is a constant and thus characterizes the
 1535 first best contract $w(a^\#, b^\#) = w^{fb}$. Then $w_{a^\#, b^\#}$ implements $a^\#$ and $U(w_{a^\#} + z^\# h^\#, a^\#) = b'$.
 1536 This completes Stage 2.2.2.

1537 Stage 2.2, Stage 2, and Theorem 1 now follow.

1538 A.7 Proof of Proposition 1

1539 It suffices to prove the (IR) constraint is binding in (P). Our proof that (IR) is binding is inspired
 1540 by the proof of Proposition 2 in Grossman and Hart (1983), but adapted to a setting where there
 1541 are infinitely many (rather than a finite number of) outcomes.

1542 Suppose to the contrary that (w^*, a^*) is an optimal contract where (IR) is not binding; i.e.,

$$1543 U(w^*, a^*) = \underline{U} + \gamma, \quad (64)$$

1545 where $\gamma > 0$. We construct a feasible contract that implements a^* but makes the principal better
 1546 off, revealing a contradiction.

1547 Under the assumption of the theorem, there exists a $\delta > 0$ such that $w^*(x) > \underline{w} + \delta$ for almost
 1548 all x . Since u is continuous and increasing, for $\epsilon > 0$ sufficiently small there exists a contract w^ϵ
 1549 such that

$$1550 \qquad w^\epsilon(x) \geq \underline{w} \qquad (65)$$

1552 and

$$1553 \qquad u(w^\epsilon(x)) = u(w^*(x)) - \epsilon. \qquad (66)$$

1555 Observe that for all $a \in \mathbb{A}$

$$\begin{aligned} 1556 \qquad U(w^\epsilon, a) &= \int u(w^\epsilon(x))f(x, a)dx - c(a) \\ 1557 \qquad &= \int (u(w^*(x)) - \epsilon)f(x, a)dx - c(a) \\ 1558 \qquad &= \int u(w^*(x))f(x, a)dx - \epsilon \int f(x, a)dx - c(a) \\ 1559 \qquad &= U(w^*, a) - \epsilon, \end{aligned} \qquad (67)$$

1561 where the first equality is by the definition of U , the second equality is by definition of w^ϵ , the third
 1562 equality is by the linearity of the integral, and the fourth equality collects terms to form $U(w^*, a)$
 1563 and uses the fact $\int f(x, a)dx = 1$ since f is a probability density function.

1564 We are now ready to show there exists an $\epsilon > 0$ such that (w^ϵ, a^*) is a feasible solution to **(P)**.
 1565 We already know that w^ϵ satisfies the limited liability constraint for sufficiently small ϵ by (65).
 1566 We now argue **(IR)** and **(IC)** also hold. For individual rationality observe:

$$\begin{aligned} 1567 \qquad U(w^\epsilon, a^*) &= U(w^*, a^*) - \epsilon \\ 1568 \qquad &= \underline{U} + \gamma - \epsilon \\ 1569 \qquad &\geq \underline{U} \quad \text{if } \epsilon < \gamma, \end{aligned}$$

1571 where the first equality follows from (67) and the second equality uses (64). Since (65) holds for
 1572 arbitrarily small ϵ , the condition that $\epsilon < \gamma$ can easily be granted.

1573 Finally, for incentive compatibility observe that for all $a \in \mathbb{A}$:

$$\begin{aligned} 1574 \qquad U(w^\epsilon, a^*) - U(w^\epsilon, a) &= [U(w^*, a^*) - \epsilon] - [U(w^*, a) - \epsilon] \\ 1575 \qquad &= U(w^*, a^*) - U(w^*, a) - \epsilon + \epsilon \\ 1576 \qquad &\geq 0, \end{aligned}$$

1578 where the first equality holds from (67) (noting that ϵ is uniform in a). Hence, we conclude that
 1579 (w^ϵ, a^*) is a feasible solution to **(P)**. Since u is an increasing function, (66) implies $w^\epsilon(x) < w^*(x)$
 1580 for all x . Hence, $V(w, a)$ is a decreasing function of w and $w^\epsilon(x) < w^*(x)$, which contradicts the
 1581 optimality of (w^*, a^*) to **(P)**.

1582 **A.8 Proof of Lemma 7**

1583 For part (i), since

$$1584 \quad \inf_{\hat{a} \in \mathbb{A}} \inf_{\lambda, \delta \geq 0} \max_{w \geq \underline{w}} \mathcal{L}(w, \lambda, \delta | a, \hat{a}, b) = \inf_{\lambda, \delta \geq 0} \inf_{\hat{a} \in \mathbb{A}} \max_{w \geq \underline{w}} \mathcal{L}(w, \lambda, \delta | a, \hat{a}, b)$$

1585 the desired result follows from the Envelope Theorem. For part (ii), note that $\inf_{\hat{a}} \mathcal{L}^*(a, \hat{a} | b)$ is
 1586 continuous and directionally differentiable in a (see e.g., Corollary 4.4 of Dempe (2002)). Since a^*
 1587 is a maximum, $\frac{\partial}{\partial a^+} (\inf_{\hat{a}} \mathcal{L}^*(a^*, \hat{a} | b)) \leq 0$ and $\frac{\partial}{\partial a^-} (\inf_{\hat{a}} \mathcal{L}^*(a^*, \hat{a} | b)) \geq 0$.

1588 **A.9 Proof of Lemma 8**

1589 Let b^* be as defined in (24). First, our goal is to show that b^* is tight-at-optimality, assuming
 1590 that it exists (we return to existence later in the proof). We first show that $b^* \leq U(w^*, a^*)$ for
 1591 all optimal solutions (w^*, a^*) to the original problem (P). Let $U^* = U(w^*, a^*)$ for some arbitrary
 1592 optimal solution (w^*, a^*) and we show $b^* \leq U^*$ by arguing U^* is in the “argmin” in (24). Our goal
 1593 is thus to show

$$1594 \quad U^* \in \operatorname{argmin}_{b \geq \underline{U}} \{ \operatorname{val}(\text{SAND} | b) - (P | w(b)) \}. \quad (68)$$

1596 First, observe that

$$1597 \quad \operatorname{val}(P | w(b)) \leq \operatorname{val}(P | b), \quad (69)$$

1599 where $(P | b)$ is defined at the beginning of Section 3. This follows since $(P | w(b))$ considers a problem
 1600 with a fixed contract $w(b)$ that delivers utility at least b to the agent, whereas $(P | b)$ is an unrestricted
 1601 version of this problem. Moreover, from Lemma 2 we know

$$1602 \quad \operatorname{val}(P | b) \leq \operatorname{val}(\text{SAND} | b). \quad (70)$$

1604 Putting (69) and (70) together implies

$$1605 \quad \min_{b \geq \underline{U}} \{ \operatorname{val}(\text{SAND} | b) - \operatorname{val}(P | w(b)) \} \geq 0.$$

1607 With this inequality in hand, we argue that U^* satisfies

$$1608 \quad \operatorname{val}(\text{SAND} | U^*) - \operatorname{val}(P | w(U^*)) = 0, \quad (71)$$

1610 implying our target condition (68). Note this will also imply that the inner “argmin” in (24) gives
 1611 a minimum value of

$$1612 \quad \min_{b \geq \underline{U}} \{ \operatorname{val}(\text{SAND} | b) - \operatorname{val}(P | w(b)) \} = 0. \quad (72)$$

1614 By Theorem 1, we know (w^*, a^*) is an optimal solution to (P). Also, by Proposition 2, $(w(U^*), a(U^*))$
 1615 is an optimal solution to (P). Note, however that $(w(U^*), a(U^*))$ is also an optimal solution to
 1616 $(P | w(U^*))$, since feasibility of $(w(U^*), a(U^*))$ to (P) implies $a(U^*) \in a^{\text{BR}}(w(U^*))$. This, in turn,
 1617 implies $\operatorname{val}(P | w(U^*)) = \operatorname{val}(\text{SAND} | U^*)$ since, as we have just argued, both values are equal to
 1618 $\operatorname{val}(P)$. This establishes (71) and hence we can conclude (68). This shows $b^* \leq U^*$, since b^* is the

1619 least element in $\operatorname{argmin}_{b \geq \underline{U}} \{\operatorname{val}(\text{SAND}|b) - (P|w(b))\}$. For any tight U^* , since $\operatorname{val}(\text{SAND}|b)$ is a
 1620 weakly decreasing function of b and $\operatorname{val}(\mathbf{P}) = \operatorname{val}(\text{SAND}|U^*)$ for any tight U^* , we have

$$1621 \qquad \qquad \qquad \operatorname{val}(\text{SAND}|b^*) \geq \operatorname{val}(\mathbf{P}). \qquad (73)$$

1623 Also, by definition (assuming b^* exists), b^* is in the “argmin” in (24) and so from (72) we know
 1624 $\operatorname{val}(P|w(b^*)) = \operatorname{val}(\text{SAND}|b^*)$. However, since $\operatorname{val}(P|w(b^*)) \leq \operatorname{val}(\mathbf{P})$, from (73) we can conclude
 1625 that $\operatorname{val}(\text{SAND}|b^*) = \operatorname{val}(\mathbf{P})$. In particular, this means that $(w(b^*), a(b^*))$ is an optimal solution to
 1626 (P). Moreover, from Proposition 6, we know $U(w(b^*), a(b^*)) = b^*$. Thus, b^* is tight-at-optimality.

1627 We now show that such a b^* , in fact, exists. Let

$$1628 \qquad \qquad \qquad \hat{b} = \inf \{b \in [\underline{U}, \infty) : \operatorname{val}(\text{SAND}|b) - \operatorname{val}(P|w(b)) = 0\}. \qquad (74)$$

1630 For ease of notation let $s(b) = \operatorname{val}(\text{SAND}|b)$ and $t(b) = \operatorname{val}(P|w(b))$. Let B denote the set
 1631 $\{b \in [\underline{U}, \infty) : s(b) = t(b)\}$ and thus \hat{b} is the infimum of B . The goal is to show $\hat{b} \in B$ and hence
 1632 $\hat{b} = b^*$ as defined in (24) using (72). This is achieved by showing B is closed and bounded below.
 1633 Clearly B is bounded below by \underline{U} . It remains to show closedness. We consider the topological
 1634 structure of $s(b)$ and $t(b)$. By the Theorem of , $s(b)$ is a continuous function of b . Also by the
 1635 Theorem of Maximum, $w(b)$ is continuous and $a^{\text{BR}}(b)$ is upper hemicontinuous and so $t(b)$ is up-
 1636 per semicontinuous. To show B is closed, consider a sequence b_n in B converging to \bar{b} . Since s
 1637 is continuous function of b , $\lim_{n \rightarrow \infty} s(b_n) = s(\bar{b})$. Also, since t is upper semicontinuous we have
 1638 $\lim_{n \rightarrow \infty} t(b_n) \geq t(\bar{b})$. However, since $t(b) \leq s(b)$ for all b (by (69)) we know $t(\bar{b}) \leq s(\bar{b})$. Conversely,
 1639 since $s(b_n) = t(b_n)$, we have $\lim_{n \rightarrow \infty} t(b_n) = \lim_{n \rightarrow \infty} s(b_n) = s(\bar{b})$ and so $s(\bar{b}) \leq t(\bar{b})$. This implies
 1640 $s(\bar{b}) = t(\bar{b})$, which establishes that B is closed. This completes the proof.

1641 A.10 Proof of Proposition 3

1642 Suppose that for all alternate best responses \hat{a} we have $\hat{a} \geq a$. Observe that when w is a constant
 1643 function (the same wage for all outputs x), we know that all no-jump constraints

$$1644 \qquad \qquad \qquad U(w, a^*) - U(w, \hat{a}) \geq 0$$

1646 are redundant. Indeed,

$$1647 \qquad \qquad \qquad U(w, a) - U(w, \hat{a}) = c(\hat{a}) - c(a) \geq 0$$

1649 since $\hat{a} \geq a$ and c is an increasing function. Next, observe that when the principal is risk neutral
 1650 the first-best contract is a constant contract. This implies that this constant first-best contract is
 1651 feasible to (P) and thus optimal, yielding a contradiction.

1652 A.11 Proof of Proposition 4

1653 We now claim that $\operatorname{val}(\text{SAND}|\underline{U}) = \operatorname{val}(\text{FOA})$. First we argue that

$$1654 \qquad \qquad \qquad \operatorname{val}(\text{SAND}|\underline{U}) \geq \operatorname{val}(\text{FOA}). \qquad (75)$$

1656 When the first-order approach is valid we have $\operatorname{val}(\text{FOA}) = \operatorname{val}(\mathbf{P})$. Moreover, by Lemma 2 we also
 1657 know $\operatorname{val}(\text{SAND}|\underline{U}) \geq \operatorname{val}(\mathbf{P})$. Putting these together implies (75).

1658 We now turn to showing the reverse inequality of (75); that is,

$$1659 \quad \text{val}(\text{SAND}|\underline{U}) \leq \text{val}(\text{FOA}). \quad (76)$$

1661 By similar reasoning to the proof of Lemma 3, the Lagrangian approach also applies to (FOA) and
 1662 strong duality holds for (FOA) and its Lagrangian dual (see also Jewitt et al. (2008) for a proof
 1663 of a setting with certain boundedness assumptions). Let μ^* be the corresponding multiplier for
 1664 constraint (FOC(a)) in problem (FOA). Let $(a^\#, \hat{a}^\#, w^\#)$ be an optimal solution to (SAND| \underline{U}).

1665 If $\mu^* = 0$, then (SAND| \underline{U}) has a smaller value than (FOA) by strong duality. This yields (76).

1666 We are left to consider the case where $\mu^* \neq 0$. Suppose $a^\#$ is not a corner solution (similar
 1667 arguments apply to the corner solution case). If $\mu^* > 0$, we choose some \hat{a} to approach $a^\#$ from
 1668 below. If $\mu^* < 0$, we choose \hat{a} to approach $a^\#$ from above. Note that the solution $\hat{a}^\#$ is a global
 1669 minimum (given the choices of the other variables) and so for very small $\epsilon = a^\# - \hat{a}$, for \hat{a} sufficiently
 1670 close to $a^\#$, we have:

$$1671 \quad \begin{aligned} \text{val}(\text{SAND}|\underline{U}) &= \inf_{\hat{a}} \inf_{(\lambda, \delta)} \max_{w \geq \underline{w}} \mathcal{L}(w, \lambda, \delta | a^\#, \hat{a}, \underline{U}) = \inf_{(\lambda, \delta)} \inf_{\hat{a}} \max_{w \geq \underline{w}} \mathcal{L}(w, \lambda, \delta | a^\#, \hat{a}, \underline{U}) \\ 1672 &\leq \inf_{(\lambda, \delta)} \max_{w \geq \underline{w}} \{V(w, a^\#) + \lambda[U(w, a^\#) - \underline{U}] + \delta \epsilon U_a(w, a^\#) + o(\epsilon)\}. \end{aligned} \quad (77)$$

1674 The first equality follows by strong duality of (SAND| $a^\#, \hat{a}, \underline{U}$) with its dual (via Lemma 3). The
 1675 inequality follows from the mean value theorem. Since \hat{a} approaches $a^\#$ in the direction we chose,
 1676 we have

$$1677 \quad \begin{aligned} &\inf_{(\lambda, \delta)} \max_{w \geq \underline{w}} V(w, a^\#) + \lambda[U(w, a^\#) - \underline{U}] + \delta \epsilon U_a(w, a^\#) \\ 1678 &= \inf_{\lambda} \inf_{\mu \in \mathbb{R}} \max_{w \geq \underline{w}} V(w, a^\#) + \lambda[U(w, a^\#) - \underline{U}] + \mu U_a(w, a^\#) \\ 1679 &\leq \max_{a \in \mathbb{A}} \inf_{\lambda} \inf_{\mu \in \mathbb{R}} \max_{w \geq \underline{w}} V(w, a) + \lambda[U(w, a) - \underline{U}] + \mu U_a(w, a) = \text{val}(\text{FOA}), \end{aligned}$$

1681 where we simply redefine $\delta \epsilon = \mu$, without loss of generality. Note that the right-hand side is the
 1682 statement of the Lagrangian dual of (FOA), and so by strong duality of (FOA) and (77) this implies
 1683 (76). Combined with (75), this implies $\text{val}(\text{SAND}|\underline{U}) = \text{val}(\text{FOA})$, as required.