Full surplus extraction and within-period ex post implementation in dynamic environments*

Shunya Noda†

May 1, 2018

Abstract

We study full surplus extraction and implementation in dynamic environments. We exploit intertemporal correlations of agents' types to construct within-period ex post incentive compatible mechanisms. First, we formulate one-shot environments, in which a single agent has a hidden type and the planner observes a public signal about the agent's type after a type-contingent allocation is chosen. We propose necessary and sufficient conditions for full surplus extraction (*strong detectability*) and for implementability of the targeted allocation rule (*weak detectability*) in this one-shot problem. We decompose the general dynamic problem into one-shot problems, and obtain sufficient conditions for surplus extraction and implementation.

JEL Classification: C73, D47, D82, D86
Keywords: dynamic mechanism design, within-period ex post implementation, revenue maximization, full surplus extraction

---

*I have been supported by the Funai Overseas Scholarship, the E. K. Potter fellowship, the JSPS research fellowship (DC1), and the JSPS KAKENHI Grant Number 26J10076. I am thankful to Fuhito Kojima, Hitoshi Matsushima and Takuo Sugaya for their guidance and encouragement. I am also grateful to Alex Bloedel, Gabriel Carroll, Matthew Jackson, Michihiro Kandori, Toshihiro Matsumura, Yuri Matsumura, Delong Meng, Daisuke Oyama, Julia Salmi, Dan Sasaki, Ilya Segal, Wataru Tamura, Juuso Toikka, Yuichi Yamamoto, and all the seminar participants at the 2013 Econometric Society Summer School in Seoul, the University of Tokyo, the spring meeting of the 2014 Japanese Economic Association, Stanford University, the University of Electro-Communications, and the 2016 North American meeting and Asian meeting of the Econometric Society for helpful comments. My thanks also go to the co-editor, Johannes Hörner, and anonymous referees. All remaining errors are my own.

†Department of Economics, Stanford University, 579 Serra Mall, Stanford, CA 94305-6072, USA. E-mail: shunya.noda[at]gmail.com

1 Introduction

This paper investigates the possibility of full surplus extraction and the implementability of general allocation rules in dynamic environments in which the agents may have interdependent values and their hidden types evolve over time. For such environments, we establish a way to use correlations among agents' types to induce truthful revelation of type realizations.

For *static* problems, Crémer and McLean (1985, 1988) prove that full surplus extraction is possible whenever beliefs are *convex independent*: for each agent i and for each agent i's type ($\theta^i \in \Theta^i$), his belief about the other agents' types (θ^{-i}) that is associated with θ^i is not in the convex hull of the beliefs about θ^{-i} that are associated with the other types of agent i ($\Theta^i \setminus \{\theta^i\}$). Under this condition, the planner can detect agents' private types without leaving information rent; therefore, full surplus extraction is achievable. Their convex-independence condition is generically satisfied in static environments.

However, many real-world problems are *dynamic*. We consider the following dynamic environment. In each period t, each agent $i \in \mathcal{I}$ privately observes his type, θ^i_t. Hence, the state in period t is the profile of agents' types in t, $\theta_t \equiv (\theta^i_t)_{i \in \mathcal{I}}$. The planner needs to collect information about the state θ_t in order to decide an allocation. The state in the next period, θ_{t+1}, depends on the state as well as on the allocation decision in t.

The goal of this paper is to provide (reasonably tight) sufficient conditions for full surplus extraction and implementation in dynamic environments, by extending the convex-independence condition of Crémer and McLean. We allow interdependent values, with which implementation of an efficient allocation rule itself is not trivial (see, e.g., Jehiel and Moldovanu (2001)). We require the mechanisms to be *within-period ex post incentive compatible* (wp-EPIC); that is, truthtelling would remain optimal if agents observed all the private information up to the current reports (including the other agents' current types), as long as the other agents make truthful reports from this point on.

We require wp-EPIC for three reasons. First, wp-EPIC is desirable because truthtelling constitutes a perfect Bayesian equilibrium under every assumption about the observability of the current states. Second, wp-EPIC seems the strongest incentive compatibility notion that we can hope for in our setting.[1] Third, wp-EPIC is satisfied by the private-value benchmarks, proposed by Bergemann and Välimäki (2010), Cavallo, Parkes, and Singh (2010) and Athey and Segal (2013).

To satisfy wp-EPIC, we cannot use the intraperiod correlation of agents' types (i.e., the correlation between θ_t^i and θ_t^{-i}) because we must incentivize agent i to make a truthful report even when he were to observe θ_t^{-i}. However, we can instead use the *future types* as *ex post signals* to construct a payment rule that provides an incentive for truthtelling. No one knows the realization of future types at the timing of the report. Therefore, incentive payments contingent on future types are useful for constructing wp-EPIC mechanisms.

We start by formulating the *one-shot problem*, which concentrates on a reporting problem of a single agent (say, i) in a single period (say, t). The reported θ_t^i determines a type-contingent allocation. An ex post signal, which may be correlated with the realization of θ_t^i, is then publicly observed. Realizations of ex post signals stand for realizations of the state profiles in the next period (θ_{t+1}). For such a one-shot problem, we study the condition on the correlation between private types and ex post signals that enables the planner to construct a truthful (one-shot) mechanism. We propose two necessary and sufficient conditions—the *strong-detectability condition* and the *weak-detectability condition*. (The precise definitions and statements are in Section 4.)

1. *Strong detectability* is the necessary and sufficient condition for the following: (i) the targeted allocation rule is implementable with arbitrary valuations over allocations, and (ii) the planner can provide *arbitrary* equilibrium payoffs for each type (Lemma 1). Under strong detectability, each agent's payoff can be set to zero, leaving him with no information rent.

2. *Weak detectability* is the necessary and sufficient condition for the targeted allocation rule to be implementable with arbitrary valuations over allocations (Lemma 2). It is useful for implementing an efficient allocation rule that maximizes the social surplus.[2]

---

[1] For example, Athey and Segal (2013) state that "requiring the mechanism to be robust to observation of future types would be too strong for the dynamic setting, even with a single agent" (p. 2472).

[2] For static environments, Aoyagi (1998) shows that if each agent has a different belief whenever his type is different, then any allocation rules can be implemented by a Bayesian incentive compatible mechanism.

Weak detectability is weaker than strong detectability: strong detectability implies weak detectability. In the online appendix, we also show that weak detectability is generically satisfied under a weaker condition (about dimensionality of the signal space) than strong detectability is.

Next, we decompose the general dynamic problem into one-shot problems to apply Lemmas 1 or 2 and obtain one-shot mechanisms. Then, we combine the one-shot mechanisms to construct a dynamic mechanism.

First, we specify θ_{t+1}^{-i} as ex post signals of θ_t^i (Subsection 5.1). The other agents' future type θ_{t+1}^{-i} is a tractable ex post signal of θ_t^i because an incentive payment for θ_t^i does not influence the reporting problem of his future types as long as it is independent of agent i's own future type (θ_{t+1}^i). We show that (i) if strong detectability is satisfied in the *initial* period, we don't have to leave any information rent (Proposition 1); and (ii) if weak detectability is satisfied for *all* periods, we can implement a targeted allocation rule (Proposition 2). When both conditions are satisfied, full surplus extraction is guaranteed.

In Subsection 5.2, we weaken the sufficient conditions further. Even when the correlation between θ_t^i and θ_{t+1}^{-i} does not satisfy either strong or weak detectability, the correlation between θ_t^i and $\theta_{t+1} = (\theta_{t+1}^i, \theta_{t+1}^{-i})$ may satisfy these detectability conditions. Recall that strong detectability guarantees that we can provide arbitrary continuation payoffs. Therefore, if strong detectability in period $t+1$ is satisfied, the planner can use the continuation payoff at $t+1$ as a "contingent incentive payment" to induce a truthful report of θ_t^i. In this case, we can use $\theta_{t+1} = (\theta_{t+1}^i, \theta_{t+1}^{-i})$ rather than θ_{t+1}^{-i} as the ex post signal. This yields our weakest assumptions, which are used in our main theorems (Theorems 1 and 2).

2 Related Literature

Assuming private values, Bergemann and Välimäki (2010), Cavallo et al. (2010), and Athey and Segal (2013) construct dynamic versions of Vickrey–Clarke–Groves (VCG) mechanisms, which implement an efficient allocation rule. They use distinct assumptions

---

Weak detectability is different from his condition for implementation since in our environment, (i) the signal distribution also depends on the selected allocation, and (ii) we do not have to give a strong incentive for truthtelling over misreporting that does not change the resultant allocation.

and their mechanisms display distinct properties, but all three mechanisms are wp-EPIC under some assumptions, including private values. Our formulation of dynamic environments is close to theirs. However, we construct a dynamic version of Crémer-McLean mechanism; thus, we assume neither private values nor efficiency of the targeted allocation rule, while we impose detectability conditions on the state transition. Furthermore, our mechanism satisfies wp-EPIC, same as dynamic VCG mechanisms they establish.

Mezzetti (2007) and Obara (2008) study full surplus extraction using ex post signals. Mezzetti (2004, 2007) examines a static single-unit auction problem, in which agents' valuations are interdependent, while types are independent. He establishes that the planner can implement an efficient allocation (Mezzetti (2004)) and extract full surplus (Mezzetti (2007)) under a wide variety of settings in which she can use a payment rule that depends on the agents' realized payoffs. For these static problems with ex post signals (which correspond to realized payoffs), our Lemmas 1 and 2 provide necessary and sufficient conditions for surplus extraction and implementability when the ex post signals need not be realized payoffs.[3] Obara (2008) studies a two-stage allocation problem in which agents privately choose actions before their payoff-relevant types are realized, and derives the necessary and sufficient condition for efficiency to be implementable, without leaving information rents.

Independent of our work, Liu (2017) also analyzes the implementation of an efficient allocation rule in an interdependent-value setting using the intertemporal correlation of agents' types. He provides a condition, essentially equivalent to strong detectability, under which the planner can align individual and collective social incentives, as in the canonical VCG mechanism. In contrast, we show that *weak* detectability is crucial for implementing an efficient allocation rule, rather than *strong* detectability; in this sense, our condition for implementability is weaker than Liu's. Our contribution relative to Liu's is discussed

---

[3] More recently, Nath, Zoeter, Narahari, and Dance (2015) and He and Li (2016) extend Mezzetti (2004) to implement the efficient allocation rule in dynamic environments. In their settings, the valuations are interdependent, types evolve independently, and agents observe their actual flow valuations after allocations. In this environment, Nath et al. (2015) develop an efficient dynamic mechanism in which truthtelling is strictly wp-EPIC. He and Li (2016) show that a within-period budget balance can also be achieved with interim incentive compatibility (which requires that truthful strategies constitute a perfect Bayesian equilibrium) in such environments. To accommodate our model the assumption that the agents can recognize the realized flow valuation (or signals about that) between periods in our model, we can redefine the time horizon as $2T$, and let θ_t represent the true payoff characteristics if t is even and the valuations realized if t is odd.

further in Remark 3 and Subsection 6.2.[4]

It is well known that (i) full surplus extraction in static mechanism design with monetary transfers, and (ii) folk theorems in repeated games without monetary transfers, are closely related; when the discount factor is sufficiently large, we can treat continuation payoffs as monetary transfers, as seen in Fudenberg and Levine (1994) and Fudenberg, Levine, and Maskin (1994). Recently, Hörner, Takahashi, and Vieille (2015) show that this relationship is readily generalized to mechanism design and dynamic (stochastic) Bayesian games. Now that this paper provides a dynamic mechanism to extract the full surplus in a wide rage of environments. Applying Hörner et al. (2015)'s method to replace the monetary transfer with the continuation payoff would allow us to prove a new folk theorem in dynamic Bayesian games. In particular, if strong detectability holds for every T periods and weak detectability holds for every period, then by applying the method of Hörner et al. (2015) in every T periods, we can construct an efficient equilibrium in dynamic Bayesian games without monetary transfers.[5]

3 Environment: the Original Problem

Consider an environment with a finite set of agents, indexed by $i \in \mathcal{I} = \{1, 2, \cdots, I\}$ where $I \geq 2$. For now, we focus on a finite horizon, where time is indexed by $t \in \mathcal{T} = \{0, 1, \cdots, T, T+1\}$ and where $T \in \mathbb{Z}_+$. We can extend the results to an infinite horizon, under some additional assumptions (see Subsection 6.3 and the online appendix). In period t, agent i observes his private state (or type) $\theta_t^i \in \Theta_t^i$, and the planner can directly observe $\theta_t^0 \in \Theta_t^0$, where Θ_t^i is assumed to be finite for all $i \in \mathcal{I} \cup \{0\}$ and $t \in \mathcal{T}$.[6] Hence, the state space in t, $\Theta_t = \times_{i=0}^{I} \Theta_t^i$, is also finite. After θ_t is realized, the allocation $x_t \in X_t$ (where X_t is also assumed to be finite), and the transfer $(y_t^1, \cdots, y_t^I) \in \mathbb{R}^I$ are determined based on the mechanism to which the planner commits ex ante.

---

[4] In contrast to this paper, which focuses on finite type spaces, Liu also studies a way to implement the targeted allocation rule when the type space is infinite.

[5] To be more precise, we need some additional conditions for satisfying within-period budget balance, because this is a condition required for the mechanism to be convertible into a dynamic Bayesian game.

[6] Superscripts denote the names of the agent and subscripts denote time periods.

Agent i wants to maximize the expectation of his payoff,

$$\sum_{t=0}^{T+1} \delta^t \left[v_t^i(x_t, \theta_t) + y_t^i \right],$$

which is determined by the sequence of state profiles $\theta_{0:T+1} \equiv (\theta_0, \theta_1, \cdots, \theta_{T+1}) \in \Theta_{0:T+1} \equiv \times_{t=0}^{T+1} \Theta_t$, allocations $x_{0:T+1}$, and agent i's monetary transfers $y_{0:T+1}^i$. Throughout this paper, $z_{t:s} \equiv (z_t, \cdots, z_s)$ ($Z_{t:s} \equiv \times_{k=t}^{s} Z_k$) denotes the sequence of variables (sets) z_k (Z_k) from period t to period s. The discount factor $\delta \in (0, 1]$ is common and $v_t^i : X_t \times \Theta_t \to \mathbb{R}$ is agent i's flow valuation function in t. We assume that the flow valuation functions are Markov, in the sense that v_t^i does not depend on $\theta_{0:t-1}$.

We assume that agents do not face an allocation problem in period $T+1$, but that they receive additional signals θ_{T+1} for the type realizations until T, $\theta_{0:T}$. Formally, we assume $|X_{T+1}| = 1$, and $v_{T+1}^i(x_{T+1}, \theta_{T+1}) = 0$ for all $i \in \mathcal{I}$, $\theta_{T+1} \in \Theta_{T+1}$.[7]

The type distribution in period 0 is given by $\mu_0 \in \Delta(\Theta_0)$, and subsequent states evolve according to the transition probability function $\mu_t : X_{t-1} \times \Theta_{t-1} \to \Delta(\Theta_t)$. For simplicity, we assume that μ_t has *full support*, that is, $\mu_0(\theta_0) > 0$ for all $\theta_0 \in \Theta_0$, and $\mu_t(\theta_t; x_{t-1}, \theta_{t-1}) > 0$ for all $(x_{t-1}, \theta_{t-1}, \theta_t) \in X_{t-1} \times \Theta_{t-1} \times \Theta_t$. We call $(\Theta_t, \mu_t)_{t=0}^{T+1}$ the *information structure*. The roles of those assumptions are discussed in the online appendix.

We focus on *direct mechanisms,* in which agent i reports his state θ_t^i in period t. $(\chi_t, \psi_t)_{t=0}^{T+1}$ denotes the mechanism where $\chi_t : \Theta_t \to X_t$ is the allocation rule in period t, and $\psi_t = (\psi_t^1, \cdots, \psi_t^I)$ where $\psi_t^i : \Theta_{0:t} \to \mathbb{R}$ is agent i's payment rule in period t. Slightly abusing terminology, we also call $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ a mechanism. We concentrate on Markov allocation rules, i.e., we assume that χ_t is determined by the report in period t, θ_t, but is not affected by the reported type profile until $t-1$, $\theta_{0:t-1}$. There exists an efficient Markov allocation rule since we assume that neither the flow valuation function v_t^i nor the transition probability function μ_t is affected by $\theta_{0:t-1}$. In the online appendix, we discuss

---

[7] This assumption simplifies the analysis *for the last period* ($t = T$), in which we cannot make use of the ex post signals to induce truthtelling. Except for the last period, our mechanism does not rely on this assumption. To guarantee incentive compatibility in the last period, we can alternatively assume the existence of a (static) ex post incentive compatible mechanism in $T+1$ (which can leave arbitrarily large information rents). For all the mechanisms presented in this paper, agent i's payment in period $T+1$ does not depend on his own type in $T+1$, θ_{T+1}^i; thus, (static) ex post incentive compatibility in $T+1$ is not affected by the incentive scheme for $t = 0, 1, \cdots, T$.

how our results are generalized to the case of non-Markov allocation rules.

We define $V_t^i(\cdot; (\chi_k)_{k=0}^{T+1}) : \Theta_t \to \mathbb{R}$ as agent i's expected present value (hereafter, EPV) from valuations by the following:

$$V_t^i(\theta_t; (\chi_k)_{k=0}^{T+1}) \equiv \mathbb{E}\left[\sum_{s=t}^{T+1} \delta^{s-t} v_s^i(\chi_s(\theta_s), \theta_s) \middle| (\chi_k)_{k=0}^{T+1}, \theta_t \right].$$

Recall that once we specify $(\Theta_s, \mu_s, \chi_s)_{s=0}^{T+1}$ and θ_t, the probability that θ_s realizes for $s \geq t$ is determined. Given allocation rule $(\chi_t)_{t=0}^{T+1}$, the expected social welfare is $\mathbb{E}\left[\sum_{i\in\mathcal{I}} V_0^i(\theta_0; (\chi_t)_{t=0}^{T+1})\right]$. An allocation rule $(\chi_t)_{t=0}^{T+1}$ is *efficient* if it maximizes $\mathbb{E}\left[\sum_{i\in\mathcal{I}} V_0^i(\theta_0; (\chi_t)_{t=0}^{T+1})\right]$.

Similarly, we define agent i's EPV from payments $\Psi_t^i(\cdot; (\chi_k)_{k=0}^{T+1}) : \Theta_{0:t} \to \mathbb{R}$ by

$$\Psi_t^i(\hat{\theta}_{0:t-1}, \theta_t; (\chi_k)_{k=0}^{T+1}) \equiv \mathbb{E}\left[\sum_{s=t}^{T+1} \delta^{s-t} \psi_s^i(\hat{\theta}_{0:t-1}, \theta_t, \theta_{t+1:s}) \middle| (\chi_k)_{k=0}^{T+1}, \theta_t \right].$$

Since the state transition is Markov, conditional on θ_t, Ψ_t^i is independent of realizations of $\theta_{0:t-1}$. However, Ψ_t^i depends on the reported $\hat{\theta}_{0:t-1}$ because we don't assume that ψ_t^i is Markov.

In this paper, we sometimes decompose the transfer rule ψ_t^i into several parts, e.g., $\psi_t^i(\theta_{0:t}) = g_t^i(\theta_{0:t}) + \phi_t^i(\theta_{0:t})$. By analogy with the relationship between ψ and Ψ, we represent the EPVs of the parts of payment rules g, ϕ by the capital letters G, Φ, respectively. For notational convenience, when we write EPV terms such as $V_t^i(\theta_t; (\chi_k)_{k=0}^{T+1})$, $\Psi_t^i(\theta_{0:t}; (\chi_k)_{k=0}^{T+1})$, we drop $(\chi_t)_{t=0}^{T+1}$, and simply write $V_t^i(\theta_t), \Psi_t^i(\theta_{0:t})$.

We require a dynamic version of ex post incentive compatibility. In dynamic environments, there are many ways to model what agent i learns about the past reports and realized type profiles of the other agents, $(\hat{\theta}_{0:t-1}^{-i}, \theta_{0:t}^{-i})$. Here, we take a conservative approach: we construct mechanisms in which truthful reporting of agent i's type, θ_t^i, is optimal even if he observed all of the past reports $\hat{\theta}_{0:t-1}$ as well as the current type profile θ_t (including the types of the other agents).[8] We do not exploit an agent's information about the other agents' types. Instead, we construct mechanisms that are robust against the leakage of the other agents' private information. We also require truthtelling on and

---

[8] Note that conditional on the realization of the current type profile θ_t, agent i's expected payoff from period t is independent of the realizations of $\theta_{0:t-1}$ and the past allocation $x_{0:t-1}$.

off the equilibrium path, i.e., truthful reporting maximizes each agent's payoff as long as other agents tell the truth from this point on. This notion of incentive compatibility is called *within-period ex post incentive compatibility* (wp-EPIC). Wp-EPIC is the incentive-compatibility notion that the dynamic versions of VCG mechanisms (Bergemann and Välimäki (2010), Cavallo et al. (2010), and Athey and Segal (2013)) satisfy.

Definition 1 (wp-EPIC). $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ is *within-period ex post incentive compatible (wp-EPIC) for agent i at $(\theta_{0:t-1}, \theta_t^i, \theta_t^{-i}) \in \Theta_{0:t}$* if, for all $\hat{\theta}_t^i \in \Theta_t^i$,

$$\begin{aligned}
&V_t^i(\theta_t^i, \theta_t^{-i}) + \Psi_t^i(\theta_{0:t-1}, \theta_t^i, \theta_t^{-i}) \\
&\geq v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t^i, \theta_t^{-i}) + \psi_t^i(\theta_{0:t-1}, \hat{\theta}_t^i, \theta_t^{-i}) \\
&\quad + \delta \cdot \mathbb{E}\left[V_{t+1}^i(\theta_{t+1}) + \Psi_{t+1}^i(\theta_{0:t-1}, \hat{\theta}_t^i, \theta_t^{-i}, \theta_{t+1}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t^i, \theta_t^{-i} \right].
\end{aligned} \tag{1}$$

$(\chi_t, \psi_t^i)_{t=0}^{T+1}$ is *wp-EPIC for agent i* if it is wp-EPIC for agent i for every t and $(\theta_{0:t-1}, \theta_t) \in \Theta_{0:t}$. A mechanism $(\chi_t, \psi_t)_{t=0}^{T+1}$ is *wp-EPIC* if for all $i \in \mathcal{I}$, $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ is wp-EPIC for i.

We define the *no-information-rent property* and *full surplus extraction* as follows:

Definition 2 (No Information Rent). $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ *leaves no information rent for agent i* if

$$V_0^i(\theta_0; (\chi_t)_{t=0}^{T+1}) + \Psi_0^i(\theta_0; (\chi_t)_{t=0}^{T+1}) = 0,$$

for all $\theta_0 \in \Theta_0$.

Definition 3 (Full Surplus Extraction). $(\chi_t, \psi_t)_{t=0}^{T+1}$ *extracts the full surplus* if (i) the allocation rule $(\chi_t)_{t=0}^{T+1}$ is efficient, and (ii) for each $i \in \mathcal{I}$, $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ leaves no information rent.

Here, we assume that each agent's *outside option* in period 0 is zero for all $\theta_0 \in \Theta_0$.[9] Hence, $-\Psi_0^i(\theta_0; (\chi_t)_{t=0}^{T+1}) = V_0^i(\theta_0; (\chi_t)_{t=0}^{T+1})$ is the largest period-0 expected revenue collected from agent i when the allocation rule $(\chi_t)_{t=0}^{T+1}$ is implemented. It is natural to

---

[9] We set the outside option to zero for simplicity's sake. We can still achieve full surplus extraction even if the outside option depends on the state profile, θ_t.

assume that the planner maximizes the ex ante expected revenue from the agents, since the planner commits to a mechanism $(\chi_t, \psi_t)_{t=0}^{T+1}$ ex ante.[10]

We do not impose participation constraints for $t \geq 1$. Since we consider a finite-horizon problem with finite types, for all $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ there exists the worst-case EPV for agent i, namely, $M^i \equiv \min_{t,\theta_{0:t}} [V_t^i(\theta_t) + \Psi_t^i(\theta_{0:t})]$. When this is negative, agent i leaves the mechanism once such $\theta_{0:t}$ realizes. However, consider a modified mechanism $(\chi_t, \bar{\psi}_t^i)_{t=0}^{T+1}$ defined by

$$
\begin{aligned}
\bar{\psi}_0^i(\theta_0) &\equiv \psi_0^i(\theta_0) + M^i, \\
\bar{\psi}_t^i(\theta_{0:t}) &\equiv \psi_t^i(\theta_{0:t}) \quad \text{for all } t \in \{1, \cdots, T\}, \text{ and} \\
\bar{\psi}_{T+1}^i(\theta_{0:T+1}) &\equiv \psi_{T+1}^i(\theta_{0:T+1}) - \delta^{-(T+1)} M^i.
\end{aligned}
$$

Then, $\bar{\Psi}_0^i(\theta_0) = \Psi_0^i(\theta_0)$ holds in period 0. Furthermore, $\bar{\Psi}_t^i(\theta_{0:t}) = \Psi_t^i(\theta_{0:t}) - \delta^{-t} M^i$ holds for all $t \geq 1$; thus, $V_t^i(\theta_t) + \bar{\Psi}_t^i(\theta_{0:t}) \geq 0$ for all $t \geq 1$. Intuitively, the planner additionally requires a deposit to make sure that agents do not leave the mechanism until it terminates. The deposit changes neither the agents' EPV in the initial period nor the planner's revenue because the deposit will be paid back with appropriate interest in the last period, as long as agents stay in. Using this "deposit scheme," we can satisfy participation constraints for $t \geq 1$ without increasing the rent.

4 Necessary and Sufficient Conditions for the One-Shot Problem

4.1 Formulation

To explain our main results for the original problem, we introduce the following *one-shot problem*, which consists of two stages, and is characterized by (u^i, δ) and $(X, \Theta^i, \chi, S, \pi)$.

1. A single agent, say, agent i, observes his private type $\theta^i \in \Theta^i$. He makes a type report to the planner, $\hat{\theta}^i \in \Theta^i$. The planner chooses an allocation, $x = \chi(\hat{\theta}^i)$ according to

---

[10] The impossibility of satisfying the participation constraints with equality at every node $\theta_{0:t}$ (instead of only in period 0) is explained in the online appendix.

a committed allocation rule, $\chi : \Theta^i \to X$.

2. An ex post signal $s \in S$ (where S is assumed to be finite) is drawn according to $\pi : X \times \Theta^i \to \Delta(S)$, which depends on the allocation and agent i's true type. According to payment rule, $p^i : \Theta^i \times S \to \mathbb{R}$, agent i receives a monetary transfer. Agent i's payoff is $u^i(\chi(\hat{\theta}^i), \theta^i) + \delta p^i(\hat{\theta}^i, s)$, where $u^i : X \times \Theta^i \to \mathbb{R}$ denotes the agent's valuation.

We call $(X, \Theta^i, \chi, S, \pi)$ the *signal structure*. Importantly, the planner has to choose an allocation when the agent reports $\hat{\theta}^i$, while the payment can also depend on the realization of the ex post signal s. Just like Crémer and McLean (1988), we exploit the correlation between θ^i and s to induce truthtelling. However, in contrast to Crémer and McLean (1988), in our model, (i) the signal is observable only *after* the allocation is determined, and (ii) its distribution also depends on the allocation.

There are two ways to interpret this one-shot problem.

1. When $T = 0$, the original problem can be decomposed into $\sum_{i \in \mathcal{I}} |\Theta_0^{-i}|$ one-shot problems. Each one-shot problem is identical to the reporting problem of θ_0^i, given for each $(i, \theta_0^{-i}) \in \mathcal{I} \times \Theta_0^{-i}$. The other agents' types θ_0^{-i} cannot be used as a signal for achieving wp-EPIC, because agent i must tell the truth even when he observes θ_0^{-i}. The only available ex post signal to induce the truthtelling of θ_0^i is θ_1^{-i}. Thus, by choosing $X \equiv X_0$, $\Theta^i \equiv \Theta_0^i$, $\chi \equiv \chi_0$, $S \equiv \Theta_1^{-i}$, $\pi \equiv \mu_1^{-i}(\cdot, \cdot, \theta_0^{-i})$, $u^i \equiv v_0^i(\cdot, \cdot, \theta_0^{-i})$ and $p^i \equiv \psi_1^i(\cdot, \theta_0^{-i})$, the one-shot problem becomes equivalent to agent i's reporting problem of θ_0^i, given that θ_0^{-i} is realized.

2. For the general original problem, we can still use θ_{t+1}^{-i} as an ex post signal to solve the reporting problem of θ_t^i (given an agent i and a particular sequence of type reports $\hat{\theta}_{0:t-1}$ and a type profile of the other agents θ_t^{-i}). Hence, defining $S \equiv \Theta_{t+1}^{-i}$ and $\pi \equiv \mu_{t+1}^{-i}(\cdot, \cdot, \theta_t^{-i})$ and applying the results for one-shot problems, we can derive a (loose) sufficient condition for full surplus extraction and implementation of an allocation rule (Subsection 5.1). Furthermore, under a certain condition (described later), we can also use θ_{t+1}^i as an ex post signal to induce the truthtelling of θ_t^i. As an extreme case, we can even take $S \equiv \Theta_{t+1}$ and $\pi \equiv \mu_{t+1}(\cdot, \cdot, \theta_t^{-i})$, which yields a

weaker sufficient condition than the case of $S \equiv \Theta_{t+1}^{-i}$ and $\pi \equiv \mu_{t+1}^{-i}(\cdot, \cdot, \theta_t^{-i})$. See Subsection 5.2.

Remark 1. While θ^{-i} is dropped from the notation, we are not assuming private values. When we apply the result from the one-shot problem the original problem, we can choose a different u^i for each θ_t^{-i}, which allows us to model interdependency of the valuation function. Similarly, since we can choose a different payment rule p^i for each history $(\theta_{0:t-1}, \theta_t^{-i})$, the payment rule does not need to be Markov either.

4.2 Extraction

First, we study the condition on $(X, \Theta^i, \chi, S, \pi)$ that guarantees that for all u^i, there exists p^i such that truthtelling is induced with *arbitrary* expected payoffs. Then, in particular, we can provide a zero expected payoff to each agent for all θ^i; i.e., there are no information rent left in the one-shot problem.

Definition 4 (Strong Detectability)**.** Θ^i is *strongly detectable with* $(X, \Theta^i, \chi, S, \pi)$ if, for all $\theta^i \in \Theta^i$,

$$\pi(\chi(\theta^i), \theta^i) \notin \operatorname{co}\left(\left\{\pi(\chi(\theta^i), \hat{\theta}^i)\right\}_{\hat{\theta}^i \in \Theta^i \setminus \{\theta^i\}}\right). \tag{2}$$

Parallel to Crémer and McLean (1988)'s convex-independence condition, strong detectability is the necessary and sufficient condition for the existence of a lottery $\lambda : \Theta^i \times S \to \mathbb{R}$ that provides (i) a zero expected payoff when the agent tells the truth, and (ii) a negative expected payoff when the agent misreports. The construction of λ is described in the online appendix.[11] Using this lottery, we can punish any misreport. The existence of such lotteries is necessary and sufficient for truthtelling, while providing arbitrary expected payoffs.

Lemma 1. *The following are equivalent:*

1. Θ^i *is strongly detectable with* $(X, \Theta^i, \chi, S, \pi)$.

2. *For all* $\delta \in (0, 1]$, $u^i : X \times \Theta^i \to \mathbb{R}$, *and* $U^i : \Theta^i \to \mathbb{R}$, *there exists* $p^i : \Theta^i \times S \to \mathbb{R}$ *such that*

$$U^i(\theta^i) = u^i(\chi(\theta^i), \theta^i) + \delta \cdot \mathbb{E}\left[p^i(\theta^i, s) \middle| \chi(\theta^i), \theta^i\right], \tag{3}$$

---

[11] See the proof of Lemma 3.

for all $\theta^i \in \Theta^i$, *and*

$$U^i(\theta^i) \geq u^i(\chi(\hat{\theta}^i), \theta^i) + \delta \cdot \mathbb{E}\left[p^i(\hat{\theta}^i, s) \middle| \chi(\hat{\theta}^i), \theta^i \right], \tag{4}$$

for all $(\theta^i, \hat{\theta}^i) \in \Theta^i \times \Theta^i$.

All proofs are in Appendix. As shown in the proof, when strong detectability is violated, we can always find (u^i, δ) such that $U^i(\theta^i) = 0$ for all θ^i cannot be achieved when (3) and (4) are satisfied.

4.3 Implementation

Next, we consider the condition on $(X, \Theta^i, \chi, S, \pi)$ that guarantees that for all u^i, the planner can induce truthtelling for *some* payoffs.

Definition 5 (Weak detectability)**.** Θ^i is *weakly detectable* with $(X, \Theta^i, \chi, S, \pi)$ if, for all non-empty $\bar{\Theta}^i \subset \Theta^i$, there exists $\bar{\theta}^i \in \bar{\Theta}^i$ such that

$$\pi(\chi(\bar{\theta}^i), \bar{\theta}^i) \notin \text{co}\left(\left\{ \pi(\chi(\bar{\theta}^i), \hat{\theta}^i) \right\}_{\hat{\theta}^i \in \bar{\Theta}^i \text{ s.t. } \chi(\hat{\theta}^i) \neq \chi(\bar{\theta}^i)} \right). \tag{5}$$

Since we do not have to achieve *arbitrary* payoffs, weak detectability is weaker than strong detectability. More precisely, if Θ^i is strongly detectable with $(X, \Theta^i, \chi, S, \pi)$, then Θ^i is also weakly detectable with $(X, \Theta^i, \chi, S, \pi)$. This is because (i) the convex hull that appears in the definition of strong detectability includes the convex hull of weak detectability as a subset; and (ii) while strong detectability requires that, *for all* $\theta^i \in \Theta^i$, $\pi(\chi(\theta^i), \theta^i)$ is not in the (larger) convex hull, weak detectability only requires that (for every $\bar{\Theta}^i \subset \Theta^i$) *there exists* $\bar{\theta}^i \in \bar{\Theta}^i \subset \Theta^i$ such that $\pi(\chi(\bar{\theta}^i), \bar{\theta}^i)$ is not in the (smaller) convex hull. In the online appendix, we further (i) prove that weak detectability is generic under a weaker condition than strong detectability, and (ii) show some numerical simulations which help understand to what extent weak detectability is more likely to be satisfied than strong detectability.

Weak detectability is necessary and sufficient to implement χ with arbitrary (u^i, δ).

Lemma 2. *The following are equivalent:*

1. *Θ^i is weakly detectable with $(X, \Theta^i, \chi, S, \pi)$.*

2. *For all $\delta \in (0,1]$ and $u^i : X \times \Theta^i \to \mathbb{R}$, there exist $U^i : \Theta^i \to \mathbb{R}$ and $p^i : \Theta^i \times S \to \mathbb{R}$ such that*

$$U^i(\theta^i) = u^i(\chi(\theta^i), \theta^i) + \delta \cdot \mathbb{E}\left[p^i(\theta^i, s)\middle|\, \chi(\theta^i), \theta^i\right] \tag{3, revisited}$$

for all $\theta^i \in \Theta^i$, and

$$U^i(\theta^i) \geq u^i(\chi(\hat{\theta}^i), \theta^i) + \delta \cdot \mathbb{E}\left[p^i(\hat{\theta}^i, s)\middle|\, \chi(\hat{\theta}^i), \theta^i\right] \tag{4, revisited}$$

for all $(\theta^i, \hat{\theta}^i) \in \Theta^i \times \Theta^i$.

Whereas Lemma 1 says that we can induce truthtelling while providing *for all* on-path payoff functions $U^i : \Theta^i \to \mathbb{R}$ (with strong detectability), Lemma 2 only says that *there exists* $U^i : \Theta^i \to \mathbb{R}$. Hence, while weak detectability implies wp-EPIC, it guarantees neither full surplus extraction nor flexible control of on-path expected payoffs.

The following two examples illustrate the sufficiency and necessity of weak detectability.

Example 1 (Sufficiency)**.** Assume that $X = \{l, r\}$; $\Theta^i = \{L, R_1, R_2\}$; $\chi(L) = l$; $\chi(R_1) = \chi(R_2) = r$; and $\pi(l, L) = \pi(l, R_1) = \pi(l, R_2)$; but $\pi(r, L) \neq \pi(r, R_1) = \pi(r, R_2)$. In this example, Θ^i is not strongly detectable with $(X, \Theta^i, \chi, S, \pi)$, for two reasons: (i) $\pi(r, R_1) = \pi(r, R_2)$ implies violations of (2) with $\theta^i = R_1, R_2$; and (ii) $\pi(l, L) = \pi(l, R_1) = \pi(l, R_2)$ implies a violation of (2) with $\theta^i = L$. However, weak detectability is satisfied. To see this, if we take $\bar{\Theta}^i$ such that $\{R_1, R_2\} \cap \bar{\Theta}^i \neq \emptyset$, then we can choose either $\bar{\theta}^i = R_1$ or R_2 to show (5) (the convex hull becomes either $\{\pi(r, L)\}$ or $\emptyset$). Otherwise, $\bar{\Theta}^i = \{L\}$ and (5) is trivially satisfied.

How can we induce truthtelling? First, recall that in order to induce truthtelling, we do not have to distinguish the reports of R_1 and R_2 because these reports lead to the same allocation, r. If we set $p^i(R_1, s) = p^i(R_2, s)$ for all s, then the reports of R_1 and R_2 result in an identical allocation and payment; thus, the agent becomes indifferent between reporting R_1 and R_2. Accordingly, he has a (weak) incentive for truthtelling. Hereafter, we regard the type reports of R_1 and R_2 as the identical report, say, R.[12]

Even after R_1 and R_2 are clustered, strong detectability is still not satisfied because $\chi(L) = l$ and $\pi(l, L) = \pi(l, R)$ imply a violation of (2). On the other hand, if agent i reports $\hat{\theta}^i = R$, the allocation r is chosen, and $\pi(r, L) \neq \pi(r, R)$. Therefore,

- R can pretend to be L: If the agent reports L when his true type is R, the signal distribution is $\pi(l, R) = \pi(l, L)$. The planner cannot statistically identify the agent's true type.

- L cannot pretend to be R: When L reports R, the resultant signal distribution is different from the one generated when R reports R—i.e., $\pi(r, L) \neq \pi(r, R)$. Hence, the planner can statistically identify this deviation.

Formally, we can construct a lottery such that, when R is reported (and $\chi(R) = r$ is chosen), it follows that (i) if i's true type is R (i.e., the expectation is taken with respect to $\pi(r, R)$), the lottery's expected value is zero, and (ii) if i's true type is L (i.e., the expectation is taken with respect to $\pi(r, L)$), the lottery's expected value is negative. Using this lottery as a part of the payment rule when R is reported, we can provide arbitrarily strong punishment to prevent L from reporting R. Since L cannot pretend to be R, we can induce R's truthful report by giving a "constant" subsidy (independent of s) when the agent reports R. Here, the planner needs to distribute a subsidy (so weak detectability does not guarantee that U^i can be controlled arbitrarily), but truthtelling can be induced with arbitrary valuation functions.

More generally, when weak detectability is satisfied, the planner can construct a weak order of the agent's types and a set of lotteries that enable the planner to punish the agent's "upward misreport" (i.e., the agent would be punished if he pretended to be of a higher type) without changing each agent's on-path payoffs. Furthermore, according to the constructed order, types are equivalent only if they lead to an identical allocation and

---

[12] To be more precise, we must still require agent i to distinguish between R_1 and R_2 because the payments of *the other agents* may be different. However, the reports of R_1 and R_2 lead to the same allocation and payment for agent i; as a result, when we consider agent i's problem, we do not have to distinguish between them. Once agent i has an incentive to report $R \equiv \{R_1, R_2\}$, agent i is indifferent between reporting R_1 and R_2, i.e., he has a weak incentive for truthtelling.

lottery (e.g. R_1 and R_2 of Example 1 are equivalent and lead to the same allocation and payment).

When ex post signals are absent, an allocation rule χ is implementable if and only if along with the endowed valuation function u^i, χ satisfies the *cycle-monotonicity condition* of Rochet (1987): for all finite cycles $\theta^i_{(0)}, \theta^i_{(1)}, \cdots, \theta^i_{(K)}, \theta^i_{(K+1)} = \theta^i_{(0)}$ in Θ^i, we have

$$\sum_{k=1}^{K+1} \left\{ u^i(\chi(\theta^i_{(k)}), \theta^i_{(k)}) - u^i(\chi(\theta^i_{(k)}), \theta^i_{(k-1)}) \right\} \geq 0. \tag{6}$$

Weak detectability guarantees that we can artificially generate cycle monotonicity from the signal structure. Formally, weak detectability ensures the existence of a lottery $\lambda : \Theta^i \times S \to \mathbb{R}$ such that for all finite cycles $\theta^i_{(0)}, \cdots, \theta^i_{(K)}, \theta^i_{(K+1)} = \theta^i_{(0)}$ in Θ^i, we have

$$\sum_{k=1}^{K+1} \left\{ \begin{array}{c} \left(u^i(\chi(\theta^i_{(k)}), \theta^i_{(k)}) + \mathbb{E}\left[\lambda(\theta^i_{(k)}, s) \middle| \chi(\theta^i_{(k)}), \theta^i_{(k)} \right] \right) \\ - \left(u^i(\chi(\theta^i_{(k)}), \theta^i_{(k-1)}) + \mathbb{E}\left[\lambda(\theta^i_{(k)}, s) \middle| \chi(\theta^i_{(k)}), \theta^i_{(k-1)} \right] \right) \end{array} \right\} \geq 0. \tag{7}$$

If all types in a cycle are equivalent (with respect to the constructed order), then they lead to the same allocation and lottery; thus, (7) is trivially satisfied with equality. Otherwise, the cycle contains at least one upward misreport (i.e., there exists k such that $\theta^i_{(k)}$ is a higher type than $\theta^i_{(k-1)}$). Weak detectability enables the planner to punish such an upward misreport to satisfy (7). Accordingly, we can implement χ as if it satisfies cycle monotonicity.

Weak detectability is not only sufficient but also necessary for signal structures to generate such a lottery. Accordingly, if weak detectability is not satisfied and χ does not satisfy cycle monotonicity with respect to the valuation function u^i, truthtelling may not be induced. Example 2 illustrates this fact.

Example 2 (Necessity)**.** We assume $X = \{a, b, c\}$, $\Theta^i = \{A, B, C\}$, $\chi(A) = a$, $\chi(B) = b$, and $\chi(C) = c$. Furthermore, we assume:

$$\pi(a, A) \in \mathrm{co}\left(\{\pi(a, C), \pi(a, B)\}\right),$$
$$\pi(b, B) \in \mathrm{co}\left(\{\pi(b, A), \pi(b, C)\}\right),$$
$$\pi(c, C) \in \mathrm{co}\left(\{\pi(c, B), \pi(c, A)\}\right).$$

Clearly, taking $\bar{\Theta}^i = \Theta^i$ produces a violation of weak detectability. We assume that $\delta = 1$ and $u^i(\chi(\theta^i), \theta^i) = 0$ and $u^i(x, \theta^i) = 1$ for $x \neq \chi(\theta^i)$. Note that χ does not satisfy cycle monotonicity with respect to u^i.[13]

Towards a contradiction, suppose that there exists p^i that satisfies (3) and (4). Since $\pi(a, A) \in \mathrm{co}\left(\{\pi(a, B), \pi(a, C)\}\right)$, there exists $\alpha \in [0, 1]$ such that

$$\pi(a, A) = \alpha\pi(a, B) + (1 - \alpha)\pi(a, C).$$

Regarding $p^i(A) : S \to \mathbb{R}$ as a $|S|$-dimensional vector, and multiplying it from the left, we have

$$p^i(A) \cdot \pi(a, A) = \alpha p^i(A) \cdot \pi(a, B) + (1 - \alpha)p^i(A) \cdot \pi(a, C);$$

or, equivalently,

$$\mathbb{E}\left[p^i(A, s)\middle|\, a, A\right] = \alpha\mathbb{E}\left[p^i(A, s)\middle|\, a, B\right] + (1 - \alpha)\mathbb{E}\left[p^i(A, s)\middle|\, a, C\right]. \tag{8}$$

(8) indicates that either $\mathbb{E}\left[p^i(A, s)|\, a, B\right] \geq \mathbb{E}\left[p^i(A, s)|\, a, A\right]$ holds or $\mathbb{E}\left[p^i(A, s)|\, a, C\right] \geq \mathbb{E}\left[p^i(A, s)|\, a, A\right]$ does. Otherwise, $\alpha\mathbb{E}\left[p^i(A, s)|\, a, B\right] + (1-\alpha)\mathbb{E}\left[p^i(A, s)|\, a, C\right] < \mathbb{E}\left[p^i(A, s)|\, a, A\right]$, which contradicts (8).

Without loss of generality, we assume $\mathbb{E}\left[p^i(A, s)|\, a, B\right] \geq \mathbb{E}\left[p^i(A, s)|\, a, A\right]$. For B to make a truthful report against misreporting A, the following must hold:

$$U^i(B) = 0 + \mathbb{E}\left[p^i(B, s)\middle|\, b, B\right] \geq 1 + \mathbb{E}\left[p^i(A, s)\middle|\, a, B\right].$$

---

[13] Such an allocation rule cannot be efficient under private values. However, without the assumption of private values, the flow valuation function of the other agents could be affected by θ^i. In that case, if the other agents strongly preferred such an allocation rule, then this allocation rule would maximize social welfare. See the online appendix.

When this is taken together with the fact that $U^i(A) = 0 + \mathbb{E}\left[p^i(A,s)\,|\,a, A\right]$, we obtain that $U^i(B) > U^i(A)$ is necessary.

Applying the above argument to B, we obtain either $\mathbb{E}\left[p^i(B,s)\,|\,b, A\right] \geq \mathbb{E}\left[p^i(B,s)\,|\,b, B\right]$ or $\mathbb{E}\left[p^i(B,s)\,|\,b, C\right] \geq \mathbb{E}\left[p^i(B,s)\,|\,b, B\right]$. If the former inequality holds, we obtain $U^i(A) > U^i(B)$, which contradicts $U^i(A) < U^i(B)$. If the latter inequality holds, we have $U^i(C) > U^i(B)$ $(> U^i(A))$. However, applying the above argument to C, we obtain either $U^i(A) > U^i(C)$ or $U^i(B) > U^i(C)$. In every case, there is a contradiction. Hence, there is no p^i and U^i that satisfy (3) and (4).

Generalizing the argument in Examples 1 and 2, we can prove that weak detectability is the necessary and sufficient condition for the implementability of the targeted allocation rule χ of the one-shot problem.

5 Sufficient Conditions for the Original Problem

5.1 Without Backup: a Basic but Loose Sufficient Condition

We now construct a dynamic mechanism for the original problem (defined in Section 3). It is instructive to begin by constructing simpler mechanisms from stronger conditions. To consider the reporting problem of θ^i_t, we can always use θ^{-i}_{t+1} as an ex post signal for the realization of θ^i_t because agent i's incentive for reporting after period $t+1$ is not disturbed by such payments.[14] In this subsection, we describe a sufficient condition that relies only on the correlation between θ^i_t and θ^{-i}_{t+1}.

First, we formulate the one-shot problem for detecting θ^i_t (*for each* θ^{-i}_t). The planner chooses an allocation in period t from X_t. The type space of agent i is trivially Θ^i_t. The allocation rule in the one-shot problem is $\chi_t(\cdot;\theta^{-i}_t) : \Theta^i_t \to X_t$. In this subsection, we specify the set of ex post signals as Θ^{-i}_{t+1}. Given θ^{-i}_t, θ^{-i}_{t+1}'s (marginal) distribution, conditional on (x_t, θ^i_t), is $\mu^{-i}_{t+1}(\cdot;\cdot,\theta^{-i}_t) : X_t \times \Theta^i_t \to \Delta(\Theta^{-i}_{t+1})$, where

$$\mu^{-i}_{t+1}(\theta^{-i}_{t+1}; x_t, \theta_t) = \sum_{\theta^i_{t+1} \in \Theta^i_{t+1}} \mu_{t+1}(\theta^i_{t+1}, \theta^{-i}_{t+1}; x_t, \theta_t).$$

---

[14] Note that this property is not always guaranteed when we also use θ^i_{t+1} as an ex post signal for θ^i_t.

Hence, the signal structure for detecting θ_t^i given θ_t^{-i} is:

$$\bar{\Gamma}_t^i(\theta_t^{-i}) \equiv (X_t, \Theta_t^i, \chi_t(\cdot;\theta_t^{-i}), \Theta_{t+1}^{-i}, \mu_{t+1}^{-i}(\cdot,\cdot;\theta_t^{-i})).$$

Given that there exists a wp-EPIC mechanism $(\chi_t, g_t)_{t=0}^{T+1}$, under what conditions can we modify it to satisfy the no-information-rent property? Since we consider period-0 full surplus extraction (i.e., to exploit all the expected payoffs from participation in period 0), it suffices to incentivize truthful reporting of types in the initial period.

Proposition 1. *Given an allocation rule $(\chi_t)_{t=0}^{T+1}$, suppose that for all $i \in \mathcal{I}$ and $\theta_0^{-i} \in \Theta_0^{-i}$, Θ_0^i is strongly detectable with $\bar{\Gamma}_0^i(\theta_0^{-i})$. Suppose also that there exists a payment rule $(g_t)_{t=0}^{T+1}$ such that the mechanism $(\chi_t, g_t)_{t=0}^{T+1}$ is wp-EPIC. Then, there exists a mechanism $(\chi_t, \psi_t)_{t=0}^{T+1}$ that is wp-EPIC and leaves no information rent.*

To construct $(\chi_t, \psi_t)_{t=0}^{T+1}$, (i) for $t = 0$, we define $\psi_0 \equiv 0$; and (ii) for $t = 2, \cdots, T+1$, we fix some $\bar{\theta}_0 \in \bar{\Theta}_0$ arbitrarily, and define $\psi_t^i(\theta_{1:t}) \equiv g_t^i(\bar{\theta}_0, \theta_{1:t})$ for all $(i, \theta_{1:t}) \in \mathcal{I} \times \Theta_{1:t}$. This makes ψ_t^i for $t \geq 2$ independent of the report in period 0. To obtain ψ_1^i, for each $(i, \theta_0^{-i}) \in \mathcal{I} \times \Theta_0^{-i}$, we apply Lemma 1 where we set

$$\begin{aligned} u^i(x_0, \theta_0^i; \theta_0^{-i}) &\equiv v_0^i(x_0, \theta_0) + \delta \mathbb{E}\left[V_1^i(\theta_1) + G_1^i(\bar{\theta}_0, \theta_1) \middle| x_0, \theta_0 \right] \\ U^i(\theta_0^i; \theta_0^{-i}) &\equiv 0 \end{aligned} \tag{9}$$

to obtain $p^i(\cdot,\cdot;\theta_0^{-i}) : \Theta_0^i \times \Theta_1^{-i} \to \mathbb{R}$ that satisfies (3) and (4). Define

$$\psi_1^i(\theta_0^i, \theta_0^{-i}, \theta_1) \equiv p^i(\theta_0^i, \theta_1^{-i}; \theta_0^{-i}) + g_1^i(\bar{\theta}_0, \theta_1).$$

Importantly, $p^i(\cdot,\cdot;\theta_0^{-i})$ is independent of agent i's own report in period 1, θ_1^i.

This mechanism, $(\chi_t, \psi_t)_{t=0}^{T+1}$, leaves no information rent. From

$$0 \equiv U^i(\theta_0) = V_0^i(\theta_0) + \delta \cdot \mathbb{E}\left[G_1^i(\bar{\theta}_0, \theta_1) + p^i(\theta_0^i, \theta_1^{-i}; \theta_0^{-i}) \middle| x_0, \theta_0 \right]$$

and

$$
\begin{aligned}
\Psi_0^i(\theta_0) &= \delta \cdot \mathbb{E}\left[\Psi_1^i(\theta_0, \theta_1) \middle| x_0, \theta_0 \right], \\
&= \delta \cdot \mathbb{E}\left[G_1^i(\bar{\theta}_0, \theta_1) + p^i(\theta_0^i, \theta_1^{-i}; \theta_0^{-i}) \middle| x_0, \theta_0 \right]
\end{aligned}
$$

it follows that for all $\theta_0 \in \Theta_0$,

$$
U^i(\theta_0) = V_0^i(\theta_0) + \Psi_0^i(\theta_0) = 0. \tag{10}
$$

Furthermore, this mechanism, $(\chi_t, \psi_t)_{t=0}^{T+1}$, satisfies wp-EPIC. For $t = 1, \cdots, T+1$, the fact that $(\chi_t, \psi_t)_{t=0}^{T+1}$ satisfies wp-EPIC at $\theta_{0:t}$ immediately follows from the fact that $(\chi_t, g_t)_{t=0}^{T+1}$ satisfies wp-EPIC at $(\bar{\theta}_0, \theta_{1:t})$. In addition, for $t = 0$, we substitute (10) and

$$
\begin{aligned}
&u^i(\chi_0(\hat{\theta}_0^i; \theta_0^{-i}), \theta_0^i; \theta_0^{-i}) \\
=&v_0^i(\chi_0(\hat{\theta}_0^i; \theta_0^{-i}), \theta_0) + \delta \mathbb{E}\left[V_1^i(\theta_1) + G_1^i(\bar{\theta}_0, \theta_1) \middle| \chi_0(\hat{\theta}_0^i; \theta_0^{-i}), \theta_0 \right] \\
=&v_0^i(\chi_0(\hat{\theta}_0^i; \theta_0^{-i}), \theta_0) + \delta \mathbb{E}\left[V_1^i(\theta_1) + \Psi_1^i(\hat{\theta}_0^i, \theta_0^{-i}, \theta_1) - p^i(\hat{\theta}_0^i, \theta_1^{-i}; \theta_0^{-i}) \middle| \chi_0(\hat{\theta}_0^i; \theta_0^{-i}), \theta_0 \right]
\end{aligned}
$$

for (4) to verify (1). Accordingly, we also have wp-EPIC for i in period 0.

Remark 2. Assuming private values, Athey and Segal (2013) establish an efficient mechanism that satisfies wp-EPIC, irrespective of the transition probability functions, $(\mu_t)_{t=0}^{T+1}$. This is an example of efficient mechanisms $(\chi_t, g_t)_{t=0}^{T+1}$, whose surplus is extracted by strong detectability in the initial period.

Next, we consider a condition for a targeted allocation rule to be implementable. As we discussed in Section 4, weak detectability is crucial.

Proposition 2. *Given an allocation rule* $(\chi_t)_{t=0}^{T+1}$*, suppose that for all* $i \in \mathcal{I}$*,* $t \in \{0, \cdots, T\}$*, and* $\theta_t^{-i} \in \Theta_t^{-i}$*,* Θ_t^i *is weakly detectable with* $\bar{\Gamma}_t^i(\theta_t^{-i})$*. Then, there exists a mechanism* $(\chi_t, \psi_t)_{t=0}^{T+1}$ *that satisfies wp-EPIC.*

We can construct $(\chi_t, \psi_t)_{t=0}^{T+1}$ by applying Lemma 2 backward. Since we apply Lemma 2 multiple times, we add subscripts to (u^i, p^i, U^i) to denote periods. For $t = T$, and for each $\theta_T^{-i} \in \Theta_T^{-i}$, we apply Lemma 2 with

$$u_T^i(x_T, \theta_T^i; \theta_T^{-i}) \equiv v_T^i(x_T, \theta_T)$$

to obtain p_{T+1}^i and U_T^i that satisfy (3) and (4). We set $\psi_{T+1}^i(\theta_T, \theta_{T+1}^{-i}) \equiv p_{T+1}^i(\theta_T^i, \theta_{T+1}^{-i}; \theta_T^{-i})$. Here, ψ_{T+1}^i does not depend on the reports until $T-1$.

After constructing $(\psi_s^i)_{s=t+2}^{T+1}$ such that each ψ_s^i is independent of the reports until $s-2$, we construct ψ_{t+1}^i in the following manner. For each $\theta_t^{-i} \in \Theta_t^{-i}$, we apply Lemma 2 with

$$u_t^i(x_t, \theta_t^i; \theta_t^{-i}) \equiv v_t^i(x_t, \theta_t) + \delta \mathbb{E}\left[V_{t+1}^i(\theta_{t+1}) \middle| x_t, \theta_t \right] + \delta^2 \mathbb{E}\left[\Psi_{t+2}^i(\theta_{t+1}, \theta_{t+2}) \middle| x_t, \theta_t \right]$$

to obtain p_{t+1}^i and U_t^i that satisfy (3) and (4). Note that Ψ_{t+2}^i is independent of the report of θ_t^i because $(\psi_s^i)_{s=t+2}^{T+1}$ does not depend on the reports until t. We set $\psi_{t+1}^i(\theta_t^i, \theta_t^{-i}, \theta_{t+1}^{-i}) \equiv p_{t+1}^i(\theta_t^i, \theta_{t+1}^{-i}; \theta_t^{-i})$. Here, ψ_{t+1}^i does not depend on the reports until $t-1$ either.

Iterating this process and setting $\psi_0^i \equiv 0$, we obtain a wp-EPIC mechanism $(\chi_t, \psi_t)_{t=0}^{T+1}$.

Remark 3. Liu (2017) also studies implementability of allocation rules in dynamic environments. His Theorem 3.1 claims that when one assumes his Assumption 2 (convex independence)—which is essentially equivalent to *strong* detectability with $\bar{\Gamma}_t^i(\theta_t^{-i})$ for all $(i, t, \theta_t^{-i}, \chi_t)$—we can implement arbitrary allocation rules. Recall that (i) strong detectability implies weak detectability, and (ii) Proposition 2 relies only on *weak* detectability. Hence, Proposition 2 uses a weaker assumption than Theorem 3.1 in Liu (2017).

Liu (2017) also proves that his Assumption 2 is a sufficient condition for full surplus extraction. Propositions 1 and 2 also provide a sufficient condition but a weaker one. We need strong detectability only in the initial period (for $t = 0$) to make the participation constraint binding, and we implement an efficient allocation rule with *weak* detectability in later periods (for $t = 1, 2, \cdots, T$).

5.2 Backup by Strong Detectability in Later Periods

We can further weaken the assumptions of Propositions 1 and 2. In Subsection 5.1, we have used only the correlation between θ_t^i and θ_{t+1}^{-i} to induce truthtelling of θ_t^i. From now on, we also use agent i's own type in the next period, θ_{t+1}^i, as an ex post signal for the reporting problem of θ_t^i to obtain weaker conditions. Example 3 illustrates the idea.

Example 3. Consider a three-stage problem, in which $\left|\Theta_0^{-i}\right| = \left|\Theta_1^{-i}\right| = |\Theta_2^i| = 1$; $\Theta_0^i = \{L_0, R_0\}$; $\Theta_1^i = \{A_1, B_1, C_1, D_1\}$; $\Theta_2^{-i} = \{E_2, F_2, G_2\}$; and $|X_t| = 1$ for $t = 0, 1, 2$. The state transition functions μ_1 and μ_2 are described in Table 1.

	A_1	B_1	C_1	D_1
$\mu_1(\cdot; L_0)$	0.1	0.4	0.4	0.1
$\mu_1(\cdot; R_0)$	0.4	0.1	0.1	0.4

	E_2	F_2	G_2
$\mu_2(\cdot; A_1)$	0.6	0.2	0.2
$\mu_2(\cdot; B_1)$	0.4	0.5	0.1
$\mu_2(\cdot; C_1)$	0.4	0.1	0.5
$\mu_2(\cdot; D_1)$	0.2	0.4	0.4

Table 1: The state transition of Example 3.

Since the allocation spaces are singleton, the targeted allocation rule is trivially implementable. We consider whether full surplus extraction is guaranteed for this problem. If we consider only the correlation between θ_t^i and θ_{t+1}^{-i}, there are no ex post signals in period 0 (because $|\Theta_1^{-i}| = 1$). Hence, agent i's type is not strongly detectable with $\bar{\Gamma}_0^i$.

However, there exists a mechanism that leaves no information rent. The following two observations are crucial.

1. Θ_0^i is strongly detectable with $(X_0, \Theta_0^i, \chi_0, \Theta_1, \mu_1)$ (although Θ_0^i is not strongly detectable with $\bar{\Gamma}_0^i = (X_0, \Theta_0^i, \chi_0, \Theta_1^{-i}, \mu_1^{-i})$). In words, if we regard agent i's own type in period 1, θ_1^i, as an ex post signal of θ_0^i, strong detectability is satisfied in period 0.

2. Θ_1^i is strongly detectable with $\bar{\Gamma}_1^i = (X_1, \Theta_1^i, \chi_1, \Theta_2^{-i}, \mu_2^{-i})$, indicating that we can achieve arbitrary EPV in period 1.

Using strong detectability with $(X_0, \Theta_0^i, \chi_0, \Theta_1, \mu_1)$ (rather than $(X_0, \Theta_0^i, \chi_0, \Theta_1^{-i}, \mu_1^{-i})$), we apply Lemma 1 to the reporting problem of θ_0^i, with $U_0^i(\theta_0^i) \equiv 0$ and $u_0^i(x_0, \theta_0^i) = v_0^i(x_0, \theta_0^i)$.

Then, we obtain $p_1^i : \Theta_0^i \times \Theta_1^i \to \mathbb{R}$ such that

$$\begin{aligned} 0 =& v_0^i(\chi_0(\theta_0^i), \theta_0^i) + \delta \mathbb{E}\left[p_1^i(\theta_0^i, \theta_1^i) \middle| \chi_0(\theta_0^i), \theta_0^i \right]; \\ 0 \geq& v_0^i(\chi_0(\hat{\theta}_0^i), \theta_0^i) + \delta \mathbb{E}\left[p_1^i(\hat{\theta}_0^i, \theta_1^i) \middle| \chi_0(\hat{\theta}_0^i), \theta_0^i \right] \quad \text{for all } \hat{\theta}_0^i \in \Theta_0^i. \end{aligned}$$

Importantly, unlike the analysis in the previous section, p_1^i depends on agent i's own type in the next period. The equation and inequality above imply that *if* we can set each agent's EPV in period t to the one specified by p_1^i (i.e., if we can set $V_1^i(\theta_1^i) + \Psi_1(\theta_0^i, \theta_1^i) = p_1^i(\theta_0^i, \theta_1^i)$), *then* wp-EPIC in period 0 and the no-information-rent property are satisfied.

In this case, it is indeed possible because Θ_1^i is strongly detectable in period 1 (with $\bar{\Gamma}_1^i = (X_1, \Theta_1^i, \chi_1, \Theta_2^{-i}, \mu_2^{-i})$). For each $\theta_0^i \in \Theta_0^i$, we apply Lemma 1 to the reporting problem of θ_1^i with $U_1^i(\theta_1^i; \theta_0^i) \equiv p_1^i(\theta_0^i, \theta_1^i)$ and $u_1^i(x_1, \theta_1^i; \theta_0^i) = v_1^i(x_1, \theta_1^i)$. Using $p_2^i(\cdot, \cdot; \theta_0^i)$ obtained from Lemma 1 as the payment rule in period 2 (i.e., defining $\psi_2^i(\theta_0^i, \theta_1^i, \theta_2^{-i}) \equiv p_2^i(\theta_1^i, \theta_2^{-i}; \theta_0^i)$) we can satisfy wp-EPIC in period 1, achieving the EPV specified by p_1^i. The constructed $(\chi_t, \psi_t^i)_{t=0}^2$ satisfies wp-EPIC for i and leaves no information rent for i.

As illustrated in Example 3, if strong detectability is satisfied in period $t+1$, we can use the EPV from $t+1$ itself as an "incentive payment" for the period-t report because we can provide an arbitrary EPV in period $t+1$ (without collapsing wp-EPIC in period $t+1$). In this case, we can use not only θ_{t+1}^{-i} but also θ_{t+1}^i as the ex post signal of θ_t^i. As we can see in Proposition 2, strong detectability in later periods is not a necessary condition for implementing a targeted allocation rule. However, if it is satisfied in later periods, we can generate finer signal spaces, with which strong and weak detectability are more likely to be satisfied in earlier periods.

In general, strong detectability with $(X_{t+1}, \Theta_{t+1}^i, \chi_{t+1}(\cdot; \theta_{t+1}^{-i}), \Theta_{t+2}^{-i}, \mu_{t+2}^{-i}(\cdot, \cdot; \theta_{t+1}^{-i}))$ might be satisfied only if θ_{t+1}^{-i} belongs to a particular subset, say, $B_{t+1}^{-i} \subset \Theta_{t+1}^{-i}$. In such a case, we can use a partial approach. If $\theta_{t+1}^{-i} \in B_{t+1}^{-i}$ is realized, then we also use θ_{t+1}^i as an ex post signal of θ_t^i. Otherwise, we only use the event "θ_{t+1}^{-i} is realized" as the ex post signal of θ_t^i, and we do not distinguish between the realization of $(\theta_{t+1}^i, \theta_{t+1}^{-i})$ and $(\hat{\theta}_{t+1}^i, \theta_{t+1}^{-i})$ for $\theta_{t+1}^i \neq \hat{\theta}_{t+1}^i$. To make the above argument formally, we introduce the following notations:

Definition 6. Given $B_{t+1}^{-i} \subset \Theta_{t+1}^{-i}$, we define $\Theta_{t+1}[B_{t+1}^{-i}]$ as a partition of Θ_{t+1} such that

$$\begin{aligned} \left\{(\theta_{t+1}^{i}, \theta_{t+1}^{-i})\right\} &\in \Theta_{t+1}[B_{t+1}^{-i}] \quad \text{for all } \theta_{t+1}^{-i} \in B_{t+1}^{-i} \text{ and } \theta_{t+1}^{i} \in \Theta_{t+1}^{i}; \text{ and} \\ \left\{(\hat{\theta}_{t+1}^{i}, \theta_{t+1}^{-i})\right\}_{\hat{\theta}_{t+1}^{i} \in \Theta_{t+1}^{i}} &\in \Theta_{t+1}[B_{t+1}^{-i}] \quad \text{for all } \theta_{t+1}^{-i} \notin B_{t+1}^{-i}. \end{aligned}$$

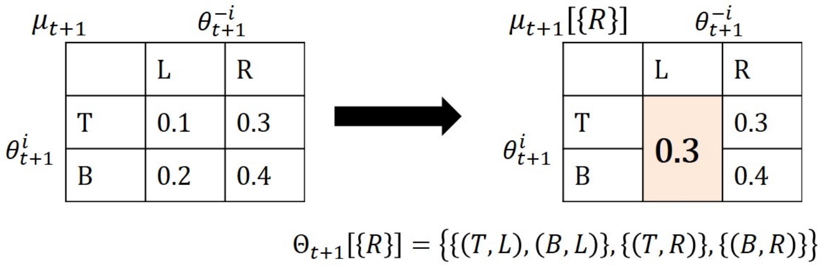
We define $\mu_{t+1}[B_{t+1}^{-i}]$: $X_t \times \Theta_t \to \Delta(\Theta_{t+1}[B_{t+1}^{-i}])$ as the conditional probability function such that $\mu_{t+1}[B_{t+1}^{-i}](s; x_t, \theta_t)$ represents the probability that the event that $s \in \Theta_{t+1}[B_{t+1}^{-i}]$ occurs after (x_t, θ_t). Formally, we define

$$\mu_{t+1}[B_{t+1}^{-i}](s; x_t, \theta_t) \equiv \sum_{\theta_{t+1} \in s} \mu_{t+1}(\theta_{t+1}; x_t, \theta_t).$$

We call B_{t+1}^{-i} a *backup set*. We also define the signal structure generated by $(\theta_t^{-i}, B_{t+1}^{-i})$ as follows:

$$\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i}) \equiv (X_t, \Theta_t^i, \chi_t(\cdot; \theta_t^{-i}), \Theta_{t+1}[B_{t+1}^{-i}], \mu_{t+1}[B_{t+1}^{-i}](\cdot, \cdot; \theta_t^{-i})).$$

Note that $(\Theta_{t+1}[\Theta_{t+1}^{-i}], \mu_{t+1}[\Theta_{t+1}^{-i}])$ is equivalent to $(\Theta_{t+1}, \mu_{t+1})$, and $(\Theta_{t+1}[\emptyset], \mu_{t+1}[\emptyset])$ is equivalent to $(\Theta_{t+1}^{-i}, \mu_{t+1}^{-i})$. Accordingly, $\Gamma_t^i(\theta_t^{-i}, \emptyset) = \bar{\Gamma}_t^i(\theta_t^{-i})$.



	L	R
T	0.1	0.3
B	0.2	0.4

	L	R
T	0.3	0.3
B		0.4

$\Theta_{t+1}[\{R\}] = \{\{(T, L), (B, L)\}, \{(T, R)\}, \{(B, R)\}\}$

Figure 1: An example of $\Theta_{t+1}[B_{t+1}^{-i}]$ and $\mu_{t+1}[B_{t+1}^{-i}]$. $\Theta_{t+1}^{i} = \{T, B\}$, $\Theta_{t+1}^{-i} = \{L, R\}$, and $B_{t+1}^{-i} = \{R\}$.

Example 4. Figure 1 illustrates $\Theta_{t+1}[B_{t+1}^{-i}]$ and $\mu_{t+1}[B_{t+1}^{-i}]$ given some $(x_t, \theta_t) \in X_t \times \Theta_t$. In this example, $\Theta_{t+1}^{-i} = \{L, R\}$, and $B_{t+1}^{-i} = \{R\}$. Since $L \notin B_{t+1}^{-i}$, we do not distinguish between (T, L) and (B, L). On the other hand, since $R \in B_{t+1}^{-i}$, the realizations of (T, R)

and (B, R) can be used as distinct realizations of ex post signals. It follows that:

$$\Theta_{t+1}^{-i}[\{R\}] = \{\{(T, L), (B, L)\}, \{(T, R)\}, \{(B, R)\}\}$$

and

$$\begin{aligned} \mu_{t+1}[\{R\}](\{(T, L), (B, L)\}; x_t, \theta_t) =& 0.1 + 0.2 = 0.3; \\ \mu_{t+1}[\{R\}](\{(T, R)\}; x_t, \theta_t) =& 0.3; \\ \mu_{t+1}[\{R\}](\{(B, R)\}; x_t, \theta_t) =& 0.4. \end{aligned}$$

Hence, $\mu_{t+1}[\{R\}](\cdot; x_t, \theta_t) = (0.3, 0.3, 0.4)$, and we can check the detectability conditions with such three-dimensional vectors.

When we apply Lemmas 1 and 2 for $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$, the payment rule p_{t+1}^i which is thereby generated satisfies

$$p_{t+1}^i(\theta_t^i, \theta_{t+1}^i, \theta_{t+1}^{-i}; \theta_t^{-i}) = p_{t+1}^i(\theta_t^i, \hat{\theta}_{t+1}^i, \theta_{t+1}^{-i}; \theta_t^{-i}) \quad \text{for } \theta_{t+1}^{-i} \notin B_{t+1}^{-i} \text{ and for all } \theta_{t+1}^i, \hat{\theta}_{t+1}^i \in \Theta_{t+1}^i$$

because $p_{t+1}^i : \Theta_t^i \times \Theta_{t+1}[B_{t+1}^{-i}] \to \mathbb{R}$. As a result, p_{t+1}^i can be expressed in the following way:

$$\begin{aligned} & \psi_{t+1}^i(\theta_t, \theta_{t+1}^{-i}) + \mathbf{1}_{\{\theta_{t+1}^{-i} \in B_{t+1}^{-i}\}} \cdot \left(V_{t+1}^i(\theta_{t+1}) + \Psi_{t+1}^i(\theta_t, \theta_{t+1})\right) \\ =& p_{t+1}^i(\theta_t^i, \theta_{t+1}; \theta_t^{-i}). \end{aligned}$$

Hence, if the planner can set an arbitrary on-path EPV $V_{t+1}^i(\theta_{t+1}) + \Psi_{t+1}^i(\theta_t, \theta_{t+1})$ to every $\theta_{t+1}^{-i} \in B_{t+1}^{-i}$ (i.e., if strong detectability is satisfied at every $\theta_{t+1}^{-i} \in B_{t+1}^{-i}$), then (i) strong detectability with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$ guarantee that the planner can also choose an arbitrary on-path EPV at θ_t^{-i}; and (ii) weak detectability with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$ guarantee that the planner can implement the targeted allocation rule at θ_t^{-i}.

Checking strong detectability sequentially, we can construct a sequence of backup sets.

Definition 7. $(B_t^{-i})_{t=1}^{T+1}$, where $B_t^{-i} \subset \Theta_t^{-i}$ for each t, is a *sequence of backup sets for agent i along $(\chi_t)_{t=1}^{T+1}$* if both the following hold:

1. $B_{T+1}^{-i} = \emptyset$.

2. For $t = 1, 2, \cdots, T$, $\theta_t^{-i} \in B_t^{-i}$ only if Θ_t^i is strongly detectable with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$.

When B_{t+1}^{-i} becomes larger, the generated partition, $\Theta_{t+1}[B_{t+1}^{-i}]$, becomes finer. Accordingly, for all $\hat{B}_{t+1}^{-i} \supset B_{t+1}^{-i}$, if Θ_t^i is strongly (weakly) detectable with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$, Θ_t^i is also strongly (weakly) detectable with $\Gamma_t^i(\theta_t^{-i}, \hat{B}_{t+1}^{-i})$. Hence, we can obtain the sequence of the *largest* backup sets by replacing Condition 2 of Definition 7 with this revised condition:

2'. For $t = 1, 2, \cdots, T$, $\theta_t^{-i} \in B_t^{-i}$ *if and* only if Θ_t^i is strongly detectable with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$.

The sequence of the largest backup sets forms an upper envelope of all sequences of backup sets. That is, if $(\hat{B}_t^{-i})_{t=1}^{T+1}$ is the sequence of the largest backup set and $(B_t^{-i})_{t=1}^{T+1}$ is a sequence of backup sets, then $\hat{B}_t^{-i} \supset B_t^{-i}$ holds for $t = 1, 2, \cdots, T+1$. If we want to maximize the chance to satisfy strong or weak detectability, we can concentrate on the sequence of the largest backup sets.

If Θ_0^i is strongly detectable with $\Gamma_0^i(\theta_0^{-i}, B_1^{-i})$ for every $\theta_0^{-i} \in \Theta_0^{-i}$ where $(B_t^{-i})_{t=1}^{T+1}$ is a sequence of backup sets, then we can choose agent i's EPV in period 0, $\{V_0^i(\theta_0) + \Psi_0^i(\theta_0)\}_{\theta_0 \in \Theta_0}$, arbitrarily. In particular, we can select $V_0^i(\theta_0) + \Psi_0^i(\theta_0) = 0$ for all $\theta_0 \in \Theta_0$. We now generalize Proposition 1.

Theorem 1 (Extraction)**.** *Let $(B_t^{-i})_{t=1}^{T+1}$ be a sequence of backup sets for agent i along $(\chi_t)_{t=0}^{T+1}$. Suppose that (i) there exists a payment rule $(g_t^i)_{t=0}^{T+1}$ that makes $(\chi_t, g_t^i)_{t=0}^{T+1}$ satisfy wp-EPIC for i, and (ii) for all $\theta_0^{-i} \in \Theta_0^{-i}$, Θ_0^i is strongly detectable with $\Gamma_0^i(\theta_0^{-i}, B_1^{-i})$. Then there exists $(\psi_t^i)_{t=0}^{T+1}$ such that $(\chi_t, \psi_t^i)_{t=0}^{T+1}$ satisfies wp-EPIC and leaves no information rent for i.*

Similarly, for a sequence of backup sets $(B_t^{-i})_{t=1}^{T+1}$, if weak detectability is satisfied for every period and at every $\theta_t^{-i} \in \Theta_t^{-i} \setminus B_t^{-i}$, then the implementability of the targeted allocation rule $(\chi_t)_{t=0}^{T+1}$ is guaranteed. This generalizes Proposition 2.

Theorem 2 (Implementation)**.** *Let $(B_t^{-i})_{t=1}^{T+1}$ be a sequence of backup sets for agent i along $(\chi_t)_{t=0}^{T+1}$. Suppose that for all $t \in \{0, 1, \cdots, T\}$, and for all $\theta_t^{-i} \in \Theta_t^{-i} \setminus B_t^{-i}$, Θ_t^i*

is weakly detectable with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$. *Then there exists* $(g_t^i)_{t=0}^{T+1}$*, such that* $(\chi_t, g_t^i)_{t=0}^{T+1}$ *satisfies wp-EPIC for* i.[15]

Theorems 1 and 2 include Propositions 1 and 2 as special cases: Propositions 1 and 2 fix $B_t^{-i} = \emptyset$ for $t = 1, \cdots, T$.

Combining Theorems 1 and 2, we obtain a condition for full surplus extraction which is guaranteed solely by the properties of the information structure $(\Theta_t, \mu_t)_{t=0}^{T+1}$.

6 Discussion

6.1 Tightness

Theorems 1 and 2 are "tight" in the following sense. Detectability conditions are the most likely to be satisfied if the signal space is the finest; i.e., the backup set is *full*: $B_{t+1}^{-i} = \Theta_{t+1}^{-i}$ if $t < T$ and $B_{t+1}^{-i} = \emptyset$. Accordingly, if strong or weak detectability is not satisfied with the finest signal space generated by the full backup set, some information rent must be left or implementation is impossible with some valuations.

Formally, first, if strong detectability in the initial period is violated even with the full backup set, then we can always find a sequence of flow valuation functions $(v_t^i)_{t=0}^T$ with which we must leave some information rent to agents to implement the targeted allocation rule:

Theorem 3. *Given* $(\chi_t)_{t=0}^{T+1}$*, suppose either (i) that there exists* $\theta_0^{-i} \in \Theta_0^{-i}$ *such that* Θ_0^i *is not strongly detectable with* $\Gamma_0^i(\theta_0^{-i}, \Theta_1^{-i})$*, or (ii) that* $T = 0$ *and there exists* $\theta_0^{-i} \in \Theta_0^{-i}$ *such that* Θ_0^i *is not strongly detectable with* $\Gamma_0^i(\theta_0^{-i}, \emptyset)$. *Under either hypothesis, there exists* $(v_t^i)_{t=0}^T$ *with which all wp-EPIC mechanisms leave at least some information rent for* i.

Second, if weak detectability is violated in some periods even with the full backup set, then we can always find $(v_t^i)_{t=0}^T$ with which we cannot satisfy wp-EPIC along with the targeted allocation rule:

---

[15] Since strong detectability (satisfied at B_t^{-i}) implies weak detectability, the condition of Theorem 2 ensures that weak detectability is satisfied for all $t \leq T$ and $\theta_t^{-i} \in \Theta_t^{-i}$.

Theorem 4. *Given* $(\chi_t)_{t=0}^{T+1}$*, suppose either (i) that there exists* $t \in \{0, 1, \cdots, T-1\}$ *and* $\theta_t^{-i} \in \Theta_t^{-i}$ *such that* Θ_t^i *is not weakly detectable with* $\Gamma_t^i(\theta_t^{-i}, \Theta_{t+1}^{-i})$*, or (ii) that there exists* $\theta_T^{-i} \in \Theta_T^{-i}$ *such that* Θ_T^i *is not weakly detectable with* $\Gamma_T^i(\theta_T^{-i}, \emptyset)$*. Under either hypothesis, there exists* $(v_t^i)_{t=0}^T$ *with which no mechanism is wp-EPIC for* i*.*

The proof follows immediately from Lemmas 1 and 2, so it is omitted here.

The necessary conditions provided in Theorems 3 and 4 coincide with the sufficient conditions provided in Theorems 1 and 2, respectively, when $T = 0$ (the case of static allocation problems with ex post signals, as considered in Mezzetti (2004, 2007)). Accordingly, the proposed hypotheses are the tight necessary and sufficient conditions for the conclusions when $T = 0$.

When $T > 0$, the sufficient conditions of Theorems 1 and 2 are strictly stronger than the necessary conditions of Theorems 3 and 4, which indicates that they are not tight necessary and sufficient conditions. This is because in Theorems 1 and 2, we do not attempt to control the EPV in $t+1$, when $\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i} \setminus B_{t+1}^{-i}$ is realized. For such θ_{t+1}^{-i}, although the set of EPV $V_{t+1}^i(\cdot, \theta_{t+1}^{-i}) + \Psi_{t+1}^i(\theta_t, \cdot, \theta_{t+1}^{-i})$ that is sustainable with wp-EPIC in $s \geq t+1$ is difficult to characterize, there still remain some degrees of freedom. Here, while we have only limited control of EPV in $t+1$, a specific vector of EPV in $t+1$, which we want to use for inducing truthtelling in t, may be available.[16] Example 5 illustrates this difficulty.

Example 5. Consider a three-stage problem, in which $\left|\Theta_0^{-i}\right| = \left|\Theta_1^{-i}\right| = \left|\Theta_2^i\right| = 1$; $\Theta_0^i = \{A_0, B_0\}$; $\Theta_1^i = \{C_1, D_1, E_1\}$; $\Theta_2^{-i} = \{F_2, G_2\}$; and $|X_t| = 1$ for $t = 0, 1, 2$. The state transition μ_1, μ_2 is summarized in Table 2. Since the allocation space is singleton, the implementability of the targeted allocation rule is trivial. We will consider whether or not we can detect θ_0^i without leaving information rent.

In this example, the assumption of Theorem 1 is not met. $\left|\Theta_1^{-i}\right| = 1$ implies that the only ex post signal that we can use for detecting agent i's type in period 0 is agent i's own type realization in period 1, θ_1^i. However, since Θ_1^i is not strongly detectable in period 1 ($\mu_2(D_1) = \left[\mu_2(C_1) + \mu_2(E_1)\right]/2$), the backup set in period 1 is empty. Therefore, the

---

[16] If $\theta_{t+1}^{-i} \in B_{t+1}^{-i}$, we can use an arbitrary EPV, $\left(V_{t+1}^i(\cdot, \theta_{t+1}^{-i}) + \Psi_{t+1}^i(\theta_t, \cdot, \theta_{t+1}^{-i})\right) \in \mathbb{R}^{|\Theta^i_{t+1}|}$; thus, we do not have this problem.

	C_1	D_1	E_1
$\mu_1(\cdot; A_0)$	0.5	0.3	0.2
$\mu_1(\cdot; B_0)$	0.2	0.3	0.5

	F_2	G_2
$\mu_2(\cdot; C_1)$	0.7	0.3
$\mu_2(\cdot; D_1)$	0.5	0.5
$\mu_2(\cdot; E_1)$	0.3	0.5

Table 2: State transition of Example 5.

assumption of Theorem 1 is not satisfied; thus, it cannot guarantee full surplus extraction in this example.

However, full surplus extraction is achievable for all valuation functions in Example 5. This is because (i) we do not have any restriction on the ratio of the continuation payoff from C_1 to that from E_1 because $\mu_2(C_1) \notin \mathrm{co}\left(\{\mu_2(D_1), \mu_2(E_1)\}\right)$ and $\mu_2(E_1) \notin \mathrm{co}\left(\{\mu_2(C_1), \mu_2(D_1)\}\right)$, and (ii) it follows from $\mu_1(D_1; A_0) = \mu_1(D_1; B_0) = 0.3$ that agent i's incentive for reporting in period 0 is independent of the EPV at D_1. Here, while it is impossible to achieve an arbitrary EPV vector $(U^i_1(C_1), U^i_1(D_1), U^i_1(E_1))$ depending on the report in period 0, an arbitrary $(U^i_1(C_1), U^i_1(E_1))$ is available (for some $U^i_1(D_1)$) and this is sufficient for detecting θ^i_0 without leaving information rent.

6.2 The Direct Use of Distant Intertemporal Correlations

The incentive for truthtelling of θ^i_t is ultimately provided by the correlation between θ^i_t and θ^{-i}_{t+s} for $s \geq 1$. Hence, we can obtain a sufficient condition by considering the conditional probability that $(\theta^{-i}_{t+1}, \theta^{-i}_{t+2}, \cdots, \theta^{-i}_{T+1})$ given θ^i_t. To study the above idea, Liu (2017) introduced the marginal state distributions of the distant future periods:

$$\begin{aligned}&\mu^{-i}_{t,t+s}(\theta^{-i}_{t+s}; x_t, x_{t+1}, \cdots, x_{t+s-1}, \theta_t)\\ \equiv&\sum_{\tilde{\theta}^i_{t+s}} \sum_{\theta_{t+1:t+s-1}} \mu_{t+1}(\theta_{t+1}; x_t, \theta_t) \cdots \mu_{t+s-1}(\theta_{t+s-1}; x_{t+s-2}, \theta_{t+s-2}) \mu_{t+s}(\tilde{\theta}^i_{t+s}, \theta^{-i}_{t+s}; x_{t+s-1}, \theta_{t+s-1}).\end{aligned}$$

Liu (2017) proposed a sufficient condition for detecting agent i's types, by checking strong detectability with this $\mu^{-i}_{t,t+s}$. This approach complements ours. In Example 5, $\mu^{-i}_{0,2}(A_0) = (0.56, 0.44) \neq \mu^{-i}_{0,2}(B_0) = (0.44, 0.56)$. Since each θ^i_0 generates a convex-independent belief of θ^{-i}_2, Liu's approach is applicable to Example 5.

Conversely, in Example 3, our backup-set approach is applicable while Liu's is not. As we have seen, Theorem 1 is applicable. However, each type in the initial period generate the same belief about all the future types of the other agents: $\mu_{0,2}^{-i}(L_0) = \mu_{0,2}^{-i}(R_0) = (0.4, 0.3, 0.3)$. Accordingly, Liu's approach is inapplicable.[17]

6.3 Infinite Horizon

So far, for simplicity, we have focused on a finite horizon. Nevertheless, our results can be extended to environments with an infinite horizon. First, we can straightforwardly extend our results for satisfying the no-information-rent property without substantial changes. Second, to implement an allocation rule in a problem with an infinite horizon, we need some additional conditions. It is well known that one should scale up Crémer-McLean lotteries as the correlation between agents' types becomes weaker. Accordingly, when the intertemporal correlation between agents' types vanishes as $t \to \infty$, the incentive payment may be unbounded; therefore, we cannot use either the one-shot deviation principle or the deposit scheme for keeping the participation constraint. In the online appendix, we propose a sufficient condition for implementing allocation rules in an infinite horizon, imposing the uniform lower bound on correlation intensity.

7 Concluding Remarks

We have proposed mechanisms that implement a targeted allocation rule and achieve full surplus extraction, from conditions on the intertemporal correlation of agents' types. In our mechanism, unlike that of Crémer and McLean (1988), no one wants to deviate even if he observed all the information available at each time point.

We believe that we can apply the techniques developed in this paper to some real-world problems. Nevertheless, if we accept that the generic possibility of full surplus extraction "cast[s] doubt on the value of the current mechanism design paradigm as a model of institutional design" (McAfee and Reny (1992), p. 400), our results suggest that this critique of Crémer-McLean might be more severe in dynamic environments.

---

[17] We cannot obtain a tight necessary and sufficient condition by combining our approach with Liu's. See the online appendix for the detail.

Appendix

A Proofs

A.1 Proof of Lemma 1

Sufficiency: By strong detectability with $(X, \Theta^i, \chi, S, \pi)$ and the separating hyperplane theorem, there exists $\lambda : \Theta^i \times S \to \mathbb{R}$ such that

$$\mathbb{E}\left[\lambda(\theta^i, s) \middle| \chi(\theta^i), \theta^i\right] = 0; \tag{11}$$

$$\mathbb{E}\left[\lambda(\theta^i, s) \middle| \chi(\theta^i), \hat{\theta}^i\right] < 0 \quad \text{for all } \hat{\theta}^i \in \Theta^i \setminus \{\theta^i\}. \tag{12}$$

Let

$$p^i(\theta^i, s) \equiv \delta^{-1}\left[U^i(\theta^i) - u^i(\chi(\theta^i), \theta^i)\right] + \alpha \cdot \lambda(\theta^i, s)$$

where $\alpha \in \mathbb{R}_{++}$ is a sufficiently large scalar.

By (11), for all α, (3) is satisfied for all $\theta^i \in \Theta^i$. Furthermore, by (12), letting α sufficiently large, (4) is also satisfied for all $(\theta^i, \hat{\theta}^i) \in \Theta^i \times \Theta^i$, as desired. ■

Necessity: Assume that there exists $\bar{\theta}^i \in \Theta^i$ such that

$$\pi(\chi(\bar{\theta}^i), \bar{\theta}^i) \in \text{co}\left(\left\{\pi(\chi(\bar{\theta}^i), \hat{\theta}^i)\right\}_{\hat{\theta}^i \in \Theta^i \setminus \{\bar{\theta}^i\}}\right). \tag{13}$$

Pick $\delta = 1$ and u^i such that for all $x \in X$, $u^i(x, \bar{\theta}^i) = 0$ and $u^i(x, \theta^i) = 1$ for all $\theta^i \neq \bar{\theta}^i$. Take $p^i : \Theta^i \times S \to \mathbb{R}$ arbitrarily. By (13), there exists $\alpha \in \Delta(\Theta^i \setminus \{\bar{\theta}^i\})$ such that

$$\sum_{\theta^i \neq \bar{\theta}^i} \alpha(\theta^i) \mathbb{E}\left[p^i(\bar{\theta}^i, s) \middle| \chi(\bar{\theta}^i), \theta^i\right] = \mathbb{E}\left[p^i(\bar{\theta}^i, s) \middle| \chi(\bar{\theta}^i), \bar{\theta}^i\right] = U^i(\bar{\theta}^i).$$

Hence, there exists $\eta(p^i) \in \Theta^i \setminus \{\bar{\theta}^i\}$ such that

$$U^i(\bar{\theta}^i) \leq \mathbb{E}\left[p^i(\bar{\theta}^i, s) \middle| \chi(\bar{\theta}^i), \eta(p^i)\right]. \tag{14}$$

On the other hand, to satisfy (3) and (4) for $(\bar{\theta}^i, \eta(p^i))$,

$$U^i(\eta(p^i)) = 1 + \mathbb{E}\left[p^i(\eta(p^i), s)\middle|\, \chi(\eta(p^i)), \eta(p^i)\right] \geq 1 + \mathbb{E}\left[p^i(\bar{\theta}^i, s)\middle|\, \chi(\bar{\theta}^i), \eta(p^i)\right] \tag{15}$$

is necessary. Combining (14) and (15), we obtain that $U^i(\eta(p^i)) \geq U^i(\bar{\theta}^i) + 1$ is necessary. Therefore, there is no p^i that satisfies (3), (4) and

$$U^i(\theta^i) < \; U^i(\bar{\theta}^i) + 1 \tag{16}$$

for all $\theta^i \in \Theta^i \setminus \{\bar{\theta}^i\}$. ■

Remark 4. In particular, $U^i(\theta^i) = 0$ for all $\theta^i \in \Theta^i$ is not achievable, as it satisfies (16).

A.2 Proof of Lemma 2

Sufficiency: First, we construct an ordered partition $\{H(k)\}_{k=1}^K$ of Θ^i, and corresponding lotteries $\lambda : \{1, \cdots, K\} \times S \to \mathbb{R}$. By assumption, we can find $\bar{\theta}^i \in \Theta^i$ that satisfies (5) for $\bar{\Theta}^i = \Theta^i$. By the separating hyperplane theorem, there exists $\lambda(1, \cdot)$ that satisfies

$$\begin{aligned} &\mathbb{E}\left[\lambda(1, s)\middle|\, \chi(\bar{\theta}^i), \bar{\theta}^i\right] = 0; \\ &\mathbb{E}\left[\lambda(1, s)\middle|\, \chi(\bar{\theta}^i), \theta^i\right] < 0 \text{ for } \theta^i \in \Theta^i \text{ s.t. } \chi(\theta^i) \neq \chi(\bar{\theta}^i). \end{aligned}$$

Let

$$H(1) \equiv \left\{\theta^i \in \Theta^i : \mathbb{E}\left[\lambda(1, s)\middle|\, \chi(\bar{\theta}^i), \theta^i\right] \geq 0\right\}.$$

Note that by construction, $\chi(\theta^i) = \chi(\bar{\theta}^i)$ must hold for all $\theta^i \in H(1)$.

Given $H(1), H(2), \cdots, H(k-1)$, we will construct $H(k)$ as follows. Again, by assumption, we can find $\bar{\theta}^i \in \Theta^i \setminus (\cup_{l=1}^{k-1} H(l))$ that satisfies (5) for $\bar{\Theta}^i = \Theta^i \setminus (\cup_{l=1}^{k-1} H(l))$. By the separating hyperplane theorem, there exists $\lambda(k, \cdot)$ that satisfies

$$\begin{aligned} &\mathbb{E}\left[\lambda(k, s)\middle|\, \chi(\bar{\theta}^i), \bar{\theta}^i\right] = 0; \\ &\mathbb{E}\left[\lambda(k, s)\middle|\, \chi(\bar{\theta}^i), \theta^i\right] < 0 \text{ for } \theta^i \in \Theta^i \setminus (\cup_{l=1}^{k-1} H(l)) \text{ s.t. } \chi(\theta^i) \neq \chi(\bar{\theta}^i). \end{aligned} \tag{17}$$

Let

$$H(k) \equiv \left\{\theta^i \in \Theta^i \setminus (\cup_{l=1}^{k-1} H(l)) : \mathbb{E}\left[\lambda(k,s) \middle| \chi(\bar{\theta}^i), \theta^i\right] \geq 0\right\}$$

We can proceed with this until $\cup_{k=1}^{K} H(k) = \Theta^i$.

Using this λ, we specify p^i given arbitrary u^i. U^i is always defined by (3). First, let $p^i(\theta^i, s) = 0$ for all $\theta^i \in H(K)$. Since the allocation, payments and continuation payoffs are fixed for the reports within $H(K)$, (4) is satisfied for $(\theta^i, \hat{\theta}^i) \in H(K) \times H(K)$.

Suppose that $p^i(\theta^i, s)$ is defined for $\theta^i \in \cup_{l=k+1}^{K} H(l)$, and (4) is satisfied for $(\theta^i, \hat{\theta}^i) \in (\cup_{l=k+1}^{K} H(l)) \times (\cup_{l=k+1}^{K} H(l))$. For $\theta^i \in H(k)$, let

$$\begin{aligned} & p^i(\theta^i, s) \\ \equiv & \max_{\check{\theta}^i \in H(k), \hat{\theta}^i \in \cup_{l=k+1}^{K} H(l)} \left\{ \delta^{-1} \left[u^i(\chi(\hat{\theta}^i), \check{\theta}^i) - u^i(\chi(\check{\theta}^i), \check{\theta}^i)\right] + \mathbb{E}\left[p^i(\hat{\theta}^i, s) \middle| \chi(\hat{\theta}^i), \check{\theta}^i\right] \right\} \\ & + \alpha \cdot \lambda(k, s) \end{aligned}$$

where $\alpha \in \mathbb{R}_{++}$ is a sufficiently large scalar.

Since $\chi(\theta^i) = \chi(\hat{\theta}^i)$ and $p^i(\theta^i, s) = p^i(\hat{\theta}^i, s)$ holds for all $\theta^i, \hat{\theta}^i \in H(k)$, (4) is satisfied for $(\theta^i, \hat{\theta}^i) \in H(k) \times H(k)$. Furthermore, by construction of the terms in the max operator and λ^i, (4) is satisfied for $(\theta^i, \hat{\theta}^i) \in H(k) \times (\cup_{l=k+1}^{K} H(l))$ as well. Finally, by (17), changing the value of α, we can provide type $\theta^i \in \cup_{l=k+1}^{K} H(l)$ arbitrarily strong punishment when he misreports $\hat{\theta}^i \in H(k)$. Therefore, (4) is satisfied for $(\theta^i, \hat{\theta}^i) \in (\cup_{l=k+1}^{K} H(l)) \times H(k)$ with a large, but fixed α. Hence, (4) is satisfied for $(\theta^i, \hat{\theta}^i) \in (\cup_{l=k}^{K} H(l)) \times (\cup_{l=k}^{K} H(l))$.

Since $\{H(k)\}_{k=1}^{K}$ is a partition of Θ^i, at the end, we can construct p^i (and U^i) that satisfies (3) for all $\theta^i \in \Theta^i$, and (4) for all $(\theta^i, \hat{\theta}^i) \in \Theta^i \times \Theta^i$, as desired. ■

Necessity: Assume that there exists $\bar{\Theta}^i \in \Theta^i$ such that for all $\theta^i \in \bar{\Theta}^i$,

$$\pi(\chi(\theta^i), \theta^i) \in \text{co}\left(\left\{\pi(\chi(\theta^i), \hat{\theta}^i)\right\}_{\hat{\theta}^i \in \bar{\Theta}^i \text{ s.t. } \chi(\hat{\theta}^i) \neq \chi(\theta^i)}\right). \tag{18}$$

Let $\delta = 1$ and

$$u^i(x, \theta^i) = \begin{cases} 0 & \text{if } x = \chi(\theta^i); \\ 1 & \text{otherwise.} \end{cases}$$

It follows from (18) that for each $\theta^i \in \bar{\Theta}^i$, there exists $\eta(\theta^i; p^i) \in \bar{\Theta}^i$ such that $\chi(\theta^i) \neq \chi(\eta(\theta^i; p^i))$ and

$$\mathbb{E}\left[p^i(\theta^i, s)\middle|\, \chi(\theta^i), \theta^i\right] \leq \mathbb{E}\left[p^i(\theta^i, s)\middle|\, \chi(\theta^i), \eta(\theta^i; p^i)\right]. \tag{19}$$

On the other hand, to satisfy (3) and (4),

$$U^i(\theta^i) = 0 + \mathbb{E}\left[p^i(\theta^i, s)\middle|\, \chi(\theta^i), \theta^i\right] \geq 1 + \mathbb{E}\left[p^i(\hat{\theta}^i, s)\middle|\, \chi(\hat{\theta}^i), \theta^i\right] \tag{20}$$

is necessary for each $\theta^i, \hat{\theta}^i \in \Theta^i$ such that $\chi(\theta^i) \neq \chi(\hat{\theta}^i)$.

From (19) and (20), we have that

$$U^i(\eta(\theta^i; p^i)) > U^i(\theta^i) \tag{21}$$

for all $\theta^i \in \Theta^i$ is necessary. Recall that we can find such a $\eta(\theta^i; p^i)$ for all $\theta^i \in \bar{\Theta}^i$.

Claim. There exists a cycle of i's type, $h(1), h(2), \cdots, h(N) \in \bar{\Theta}^i$ such that $N > 1$, $h(n+1) = \eta(h(n); p^i)$ for $n = 1, 2, \cdots, N-1$, and $h(1) = \eta(h(N); p^i)$.

Proof. Start from an arbitrary element of $\bar{\Theta}^i$, and name it $h(1)$. Let $h(2) \equiv \eta(h(1); p^i)$. By definition of η, $h(2) \neq h(1)$. For $k > 1$, after constructing $h(1), \cdots, h(k)$, if $\eta(h(k); p^i) = h(l)$ for some $l \in \{1, 2, \cdots, k-1\}$, $h(l), h(l+1), \cdots, h(k)$ constitutes a cycle. Otherwise, let $h(k+1) \equiv \eta(h(k); p^i)$. By definition of η, $h(k+1) \neq h(k)$. Finally, when $k = |\bar{\Theta}^i|$, it follows from $\{h(1), \cdots, h(|\bar{\Theta}^i| - 1)\} = \bar{\Theta}^i \setminus \{h(|\bar{\Theta}^i|)\}$ and $\eta(h(|\bar{\Theta}^i|); p^i) \in \bar{\Theta}^i \setminus \{h(|\bar{\Theta}^i|)\}$ that there exists $l \in \{1, \cdots, |\bar{\Theta}^i| - 1\}$ such that $\eta(h(|\bar{\Theta}^i|); p^i) = h(l)$. Accordingly, we can always find a cycle. ■

Proof of the necessity part of Lemma 2, continued. Suppose towards a contradiction that there exists p^i such that (21) holds for all $(\theta^i, \hat{\theta}^i) \in \Theta^i \times \Theta^i$. By the claim, we can find $h(1), h(2), \cdots, h(N) \in \bar{\Theta}^i$ such that

$$U^i(h(1)) < U^i(h(2)) < \cdots < U^i(h(N)) < U^i(h(1)).$$

This is a contradiction. ■

A.3 Proof of Propositions 1 and 2

Propositions 1 and 2 are the special cases of Theorems 1 and 2.

A.4 Proof of Theorem 1

For $t \in \{1, \cdots, T+1\}$, fix $\bar{\theta}_{0:t-1} \in \Theta_{0:t-1}$ arbitrarily, and define $g^i_{t,s} : \Theta_{t:s} \to \mathbb{R}$ by

$$g^i_{t,s}(\theta_{t:s}) \equiv g^i_s(\bar{\theta}_{0:t-1}, \theta_{t:s}).$$

Since $(\chi_s, g_s)_{s=0}^{T+1}$ is wp-EPIC for i at $(\bar{\theta}_{0:t-1}, \theta_{t:s})$, $(\chi_s, g_{t,s})_{s=t}^{T+1}$ is wp-EPIC for i at all $\theta_{t:s} \in \Theta_{t:s}$.

For $t \in \{1, \cdots, T+1\}$, for all $\theta_{0:t-1} \in \Theta^i_{0:t-1} \times \Theta^{-i}_0 \times B^{-i}_{1:t-1}$ (i.e., $\theta^{-i}_s \in B^{-i}_s$ for all $s \leq t-1$), once $\theta^{-i}_t \in \Theta^{-i}_t \setminus B^{-i}_t$ realizes, we set

$$\psi^i_t(\theta_{0:t-1}, \theta^i_t, \theta^{-i}_t) = g^i_{t,t}(\theta^i_t, \theta^{-i}_t) + \phi^i_t(\theta_{0:t-1}, \theta^{-i}_t)$$

for some $\phi^i_t(\theta_{0:t-1}, \theta^{-i}_t)$ (the value is specified later, but it does not depend on θ^i_t), and

$$\psi^i_s(\theta_{0:t-1}, \theta^i_t, \theta^{-i}_t, \theta_{t+1:s}) = g^i_{t,s}(\theta^i_t, \theta^{-i}_t, \theta_{t+1:s})$$

for all $s \geq t+1$, $\theta_{t+1:s} \in \Theta_{t+1:s}$. Then, wp-EPIC of $(\chi_s, g_{t,s})_{s=t}^{T+1}$ ensures wp-EPIC of $(\chi_s, \psi_s)_{s=t}^{T+1}$ for agent i at $(\theta_{0:t-1}, \theta^i_t, \theta^{-i}_t) \in \Theta^i_{0:t} \times \Theta^{-i}_0 \times B^{-i}_{1:t-1} \times (\Theta^{-i}_t \setminus B^{-i}_t)$, and $(\theta_{0:t-1}, \theta^i_t, \theta^{-i}_t, \theta_{t+1:s}) \in \Theta^i_{0:s} \times \Theta^{-i}_0 \times B^{-i}_{1:t-1} \times (\Theta^{-i}_t \setminus B^{-i}_t) \times \Theta^{-i}_{t+1:s}$, for $s \in \{t+1, \cdots, T+1\}$. This construction guarantees wp-EPIC of i at $\theta_{0:t} \in \Theta_{0:t} \setminus \left(\Theta^i_{0:t} \times \Theta^{-i}_0 \times B^{-i}_{1:t}\right)$ for $t \in \{1, \cdots, T+1\}$.

To satisfy wp-EPIC at $\theta_0 \in \Theta_0$ and $\theta_{0:t} \in \Theta^i_{0:t} \times \Theta^{-i}_0 \times B^{-i}_{1:t}$ for $t \geq 1$, we will construct $\phi^i_t : \Theta^i_{0:t-1} \times \Theta^{-i}_0 \times B^{-i}_{1:t-1} \times \Theta^{-i}_t \to \mathbb{R}$ for $t = 0, 1, \cdots, T$ by the following procedure, and then set $\psi^i_t(\theta_{0:t}) = \phi^i_t(\theta_{0:t-1}, \theta^{-i}_t)$ for $t = \{0, 1, \cdots, T\}$, $\theta_{0:t} \in \Theta^i_{0:t} \times \Theta^{-i}_0 \times B^{-i}_{1:t}$.

Step 0**:** Let $\phi_0^i(\theta_0^{-i}) = 0$ for all $\theta_0^{-i} \in \Theta_0^{-i}$. By assumption, for all $\theta_0^{-i} \in \Theta_0^{-i}$, Θ_0^i is strongly detectable with $\Gamma_0^i(\theta_0^{-i}, B_1^{-i})$. Hence, applying Lemma 1 with $U_0^i(\theta_0^i; \theta_0^{-i}) = 0$ for all $\theta_0^i \in \Theta_0^i$ and

$$u_0^i(x_0, \theta_0^i; \theta_0^{-i}) = v_0^i(x_0, \theta_0) + \delta \mathbb{E}\left[\mathbf{1}_{\{\theta_1^{-i} \notin B_1^{-i}\}}\left(V_1^i(\theta_1) + G_{1,1}^i(\theta_1)\right)\Big| \chi_0(\theta_0), \theta_0\right],$$

we obtain $p_1^i(\cdot, \cdot; \theta_0^{-i}) : \Theta_0^i \times \Theta_1 \to \mathbb{R}$ that satisfy

$$p_1^i(\theta_0^i, \theta_1^i, \theta_1^{-i}; \theta_0^{-i}) = p_1^i(\theta_0^i, \hat{\theta}_1^i, \theta_1^{-i}; \theta_0^{-i}) \text{ for all } \theta_1^i, \hat{\theta}_1^i \in \Theta_1^i \text{ and } \theta_1^{-i} \notin B_1^{-i}, \tag{22}$$

$$\begin{aligned} 0 = {} & v_0^i(\chi_0(\theta_0), \theta_0) \\ & + \delta \mathbb{E}\left[\mathbf{1}_{\{\theta_1^{-i} \notin B_1^{-i}\}}\left(V_1^i(\theta_1) + G_{1,1}^i(\theta_1)\right) + p_1^i(\theta_0^i, \theta_1; \theta_0^{-i})\Big| \chi_0(\theta_0), \theta_0\right] \end{aligned} \tag{23}$$

for all $\theta_0^i \in \Theta_0^i$, and

$$\begin{aligned} 0 \geq {} & v_0^i(\chi_0(\hat{\theta}_0^i, \theta_0^{-i}), \theta_0) \\ & + \delta \mathbb{E}\left[\mathbf{1}_{\{\theta_1^{-i} \notin B_1^{-i}\}}\left(V_1^i(\theta_1) + G_{1,1}^i(\theta_1)\right) + p_1^i(\hat{\theta}_0^i, \theta_1; \theta_0^{-i})\Big| \chi_0(\hat{\theta}_0^i, \theta_0^{-i}), \theta_0\right] \end{aligned} \tag{24}$$

for all $(\theta_0^i, \hat{\theta}_0^i) \in \Theta_0^i \times \Theta_0^i$. Let $\phi_1^i(\theta_0, \theta_1^{-i}) \equiv 0$ for $\theta_1^{-i} \in B_1^{-i}$ and

$$\phi_1^i(\theta_0, \theta_1^{-i}) \equiv p_1^i(\theta_0^i, \theta_1^i, \theta_1^{-i}; \theta_0^{-i}) \quad \text{for } \theta_1^{-i} \notin B_1^{-i},$$

Note that (22) ensures that ϕ_1^i is independent of θ_1^i. Moreover, since

$$V_1^i(\theta_1) + \Psi_1^i(\theta_0, \theta_1) = V_1^i(\theta_1) + G_{1,1}^i(\theta_1) + \phi_1^i(\theta_0, \theta_1^{-i}) \quad \text{for } \theta_1^{-i} \notin B_1^{-i},$$

if $V_1^i(\theta_1) + \Psi_1^i(\theta_0, \theta_1) = p_1^i(\theta_0^i, \theta_1; \theta_0^{-i})$ holds for $\theta_1^{-i} \in B_1^{-i}$, (23) implies the no-information-rent property, and (23) and (24) imply wp-EPIC for i at $(\theta_0^i, \theta_0^{-i})$ for all $\theta_0^i \in \Theta_0^i$.

Step t **(for** $0 < t < T$**):** Fix each $(\theta_{0:t-1}, \theta_t^{-i}) \in \Theta_{0:t-1}^i \times \Theta_0^{-i} \times B_{1:t}^{-i}$. Since $\theta_t^{-i} \in B_t^{-i}$, Θ_t^i is strongly detectable with $\Gamma_t^i(\theta_t^{-i}, B_{t+1}^{-i})$. Hence, applying Lemma 1 with $U_t^i(\theta_t^i; \theta_{0:t-1}, \theta_t^{-i}) =$

$p_t^i(\theta_{t-1}^i, \theta_t^i, \theta_t^{-i}; \theta_{0:t-2}, \theta_{t-1}^{-i})$ (whose value is specified in Step $t-1$) for all $\theta_t^i \in \Theta_t^i$ and

$$u_t^i(x_t, \theta_t^i; \theta_{0:t-1}, \theta_t^{-i}) = v_t^i(x_t, \theta_t) + \delta \mathbb{E}\left[\mathbf{1}_{\{\theta_{t+1}^{-i} \notin B_{t+1}^{-i}\}} \left(V_{t+1}^i(\theta_{t+1}) + G_{t+1,t+1}^i(\theta_{t+1}) \right) \middle| \chi_t(\theta_t), \theta_t \right],$$

we obtain $p_{t+1}^i(\cdot, \cdot; \theta_{0:t-1}, \theta_t^{-i}) : \Theta_t^i \times \Theta_{t+1} \to \mathbb{R}$ that satisfy

$$\begin{aligned} p_{t+1}^i(\theta_t^i, \theta_{t+1}^i, \theta_{t+1}^{-i}; \theta_{0:t-1}, \theta_t^{-i}) &= p_{t+1}^i(\theta_t^i, \hat{\theta}_{t+1}^i, \theta_{t+1}^{-i}; \theta_{0:t-1}, \theta_t^{-i}) \\ &\quad \text{for all } \theta_{t+1}^i, \hat{\theta}_{t+1}^i \in \Theta_{t+1}^i \text{ and } \theta_{t+1}^{-i} \notin B_{t+1}^{-i}, \end{aligned} \tag{25}$$

$$\begin{aligned} & p_t^i(\theta_{t-1}^i, \theta_t; \theta_{0:t-2}, \theta_{t-1}^{-i}) \\ &= v_t^i(\chi_t(\theta_t), \theta_t) \\ &\quad + \delta \mathbb{E}\left[\mathbf{1}_{\{\theta_{t+1}^{-i} \notin B_{t+1}^{-i}\}} \left(V_{t+1}^i(\theta_{t+1}) + G_{t+1,t+1}^i(\theta_{t+1}) \right) + p_{t+1}^i(\theta_t^i, \theta_{t+1}; \theta_{0:t-1}, \theta_t^{-i}) \middle| \chi_t(\theta_t), \theta_t \right] \end{aligned} \tag{26}$$

for all $\theta_t^i \in \Theta_t^i$, and

$$\begin{aligned} & p_t^i(\theta_{t-1}^i, \theta_t; \theta_{0:t-2}, \theta_{t-1}^{-i}) \\ &\geq v_t^i(\chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t) \\ &\quad + \delta \mathbb{E}\left[\mathbf{1}_{\{\theta_{t+1}^{-i} \notin B_{t+1}^{-i}\}} \left(V_{t+1}^i(\theta_{t+1}) + G_{t+1,t+1}^i(\theta_{t+1}) \right) + p_{t+1}^i(\hat{\theta}_t^i, \theta_{t+1}; \theta_{0:t-1}, \theta_t^{-i}) \middle| \chi_t(\hat{\theta}_t^i, \theta_t^{-i}), \theta_t \right] \end{aligned} \tag{27}$$

for all $(\theta_t^i, \hat{\theta}_t^i) \in \Theta_t^i \times \Theta_t^i$. Let $\phi_{t+1}^i(\theta_{0:t-1}, \theta_t, \theta_{t+1}^{-i}) \equiv 0$ for $\theta_{t+1}^{-i} \in B_{t+1}^{-i}$ and

$$\phi_{t+1}^i(\theta_{0:t-1}, \theta_t, \theta_{t+1}^{-i}) \equiv p_{t+1}^i(\theta_t^i, \theta_{t+1}^i, \theta_{t+1}^{-i}; \theta_{0:t-1}, \theta_t^{-i}) \quad \text{for } \theta_{t+1}^{-i} \notin B_{t+1}^{-i}.$$

Note that (25) ensures that ϕ_{t+1}^i is independent of θ_{t+1}^i. Moreover, since

$$V_{t+1}^i(\theta_{t+1}) + \Psi_{t+1}^i(\theta_{0:t+1}) = V_{t+1}^i(\theta_{t+1}) + G_{t+1,t+1}^i(\theta_{t+1}) + \phi_{t+1}^i(\theta_{0:t}, \theta_{t+1}^{-i}) \quad \text{for } \theta_{t+1}^{-i} \notin B_{t+1}^{-i},$$

if $V_{t+1}^i(\theta_{t+1}) + \Psi_{t+1}^i(\theta_{0:t+1}) = p_{t+1}^i(\theta_t^i, \theta_{t+1}; \theta_{0:t-1}, \theta_0^{-i})$ holds for $\theta_{t+1}^{-i} \in B_{t+1}^{-i}$, (26) implies that $V_t^i(\theta_t) + \Psi_t^i(\theta_{0:t}) = p_t^i(\theta_{t-1}^i, \theta_t; \theta_{0:t-2}, \theta_{t-1}^{-i})$, and (26) and (27) imply wp-EPIC for i at $(\theta_{0:t-1}, \theta_t^i, \theta_t^{-i})$ for all $\theta_t^i \in \Theta_t^i$.

Step T: Fix each $(\theta_{0:T-1}, \theta_T^{-i}) \in \Theta^i_{0:T-1} \times \Theta_0^{-i} \times B^{-i}_{1:T}$. Since $\theta_T^{-i} \in B_T^{-i}$, Θ_T^i is strongly detectable with $\Gamma_T^i(\theta_T^{-i}, \emptyset)$. Hence, applying Lemma 1 with $U_T^i(\theta_T^i; \theta_{0:T-1}, \theta_T^{-i}) = p_T^i(\theta^i_{T-1}, \theta_T; \theta_{0:T-2}, \theta^{-i}_{T-1})$ (whose value is specified in Step $T-1$) for all $\theta_t^i \in \Theta_t^i$ and

$$u_T^i(x_T, \theta_T^i; \theta_{0:T-1}, \theta_T^{-i}) = v_T^i(x_T, \theta_T),$$

we obtain $p^i_{T+1}(\cdot, \cdot; \theta_{0:T-2}, \theta^{-i}_{T-1}) : \Theta_T^i \times \Theta^{-i}_{T+1} \to \mathbb{R}$ (recall that it is independent of θ^i_{T+1}) that satisfy

$$p_T^i(\theta^i_{T-1}, \theta_T; \theta_{0:T-2}, \theta^{-i}_{T-1}) = v_T^i(x_T, \theta_T) + \delta \mathbb{E}\left[p^i_{T+1}(\theta_T^i, \theta^{-i}_{T+1}; \theta_{0:T-1}, \theta_T^{-i}) \middle| \, \chi_T(\theta_T), \theta_T \right] \quad (28)$$

for all $\theta_T^i \in \Theta_T^i$, and

$$p_T^i(\theta^i_{T-1}, \theta_T; \theta_{0:T-2}, \theta^{-i}_{T-1}) \geq v_T^i(x_T, \hat{\theta}_T^i, \theta_T^{-i}) + \delta \mathbb{E}\left[p^i_{T+1}(\hat{\theta}_T^i, \theta^{-i}_{T+1}; \theta_{0:T-1}, \theta_T^{-i}) \middle| \, \chi_T(\hat{\theta}_T^i, \theta_T^{-i}), \theta_T \right] \quad (29)$$

for all $(\theta_T^i, \hat{\theta}_T^i) \in \Theta_T^i \times \Theta_T^i$. Let $\phi^i_{T+1}(\theta_{0:T}, \theta^{-i}_{T+1}) \equiv p^i_{T+1}(\theta^i_T, \theta^{-i}_{T+1}; \theta_{0:T-1}, \theta_T^{-i})$. Then, (28) and (29) imply wp-EPIC for i at $(\theta_{0:T-1}, \theta_T^i, \theta_T^{-i})$ for all $\theta_T^i \in \Theta_T^i$. Furthermore, (28) implies that $V_T^i(\theta_T) + \Psi_T^i(\theta_{0:T}) = V_T^i(\theta_T) + \Phi_T^i(\theta_{0:T}) = p_T^i(\theta^i_{T-1}, \theta_T; \theta_{0:T-1}, \theta^{-i}_{T-1})$ holds, as demanded in Steps 0 to $T-1$. ■

A.5 Proof of Theorem 2

We will show that for $t = 0, \cdots, T+1$, there exists $(g_{t,k})_{k=t}^{T+1}$ that makes $(\chi_k, g^i_{s,k})_{k=s}^{T+1}$ wp-EPIC for i and is independent of the reports until $t-1$. By assumption on v^i_{T+1} and X_{T+1}, by letting $g^i_{T+1,T+1}(\theta_{T+1}) = 0$ for all $\theta_{T+1} \in \Theta_{T+1}$, $(\chi_{T+1}, g^i_{T+1,T+1})$ is trivially wp-EPIC.

Suppose that for $s = t+1, \cdots, T+1$, there exists a continuation payment rule $(g^i_{s,k})_{k=s}^{T+1}$ that makes $(\chi_k, g^i_{s,k})_{k=s}^{T+1}$ wp-EPIC for i. We will construct $(g^i_{t,k})_{k=t}^{T+1}$ that makes $(\chi_k, g^i_{t,k})_{k=t}^{T+1}$ wp-EPIC for i. Let $g^i_{t,t}(\theta_t) = 0$ for all θ_t. When $\theta^{-i}_{t+1} \in \Theta^{-i}_{t+1} \setminus B^{-i}_{t+1}$ realizes, we set $g^i_{t,t+1}(\theta_t, \theta^i_{t+1}, \theta^{-i}_{t+1}) = g^i_{t+1,t+1}(\theta^i_{t+1}, \theta^{-i}_{t+1}) + \phi^i_{t+1}(\theta_t, \theta^{-i}_{t+1})$ where the value of $\phi^i_{t+1}(\theta_t, \theta^{-i}_{t+1})$ is specified later, and $g^i_{t,k}(\theta_t, \theta_{t+1}, \theta_{t+2:k}) = g^i_{t+1,k}(\theta_{t+1}, \theta_{t+2:k})$ for all $k \in \{t+2, \cdots, T+1\}$, $\theta_{t+2:k} \in \Theta_{t+2:k}$. By the induction hypothesis, wp-EPIC for i at $(\theta_t, \theta_{t+1}, \theta_{t+2:k}) \in \Theta_t \times$

$\Theta^i_{t+1} \times (\Theta^{-i}_{t+1} \setminus B^{-i}_{t+1}) \times \Theta_{t+2:k}$ is satisfied for $k = \{t+1, \cdots, T+1\}$.

For each $\theta^{-i}_t \in \Theta^{-i}_t$ by assumption, Θ^i_t is weakly detectable with $\Gamma^i_t(\theta^{-i}_t, B^{-i}_{t+1})$. Hence, applying Lemma 2 with

$$u^i_t(x_t, \theta^i_t; \theta^{-i}_t) = v^i_t(x_t, \theta_t) + \delta \mathbb{E}\left[\mathbf{1}_{\{\theta^{-i}_{t+1} \notin B^{-i}_{t+1}\}} \left(V^i_{t+1}(\theta_{t+1}) + G^i_{t+1,t+1}(\theta_{t+1}) \right) \middle| \chi_t(\theta_t), \theta_t \right]$$

we can obtain $U^i_t(\cdot; \theta^{-i}_t) : \Theta^i_t \to \mathbb{R}$ and $p^i_{t+1}(\cdot, \cdot; \theta^{-i}_t) : \Theta^i_t \times \Theta_{t+1} \to \mathbb{R}$ that satisfy

$$p^i_{t+1}(\theta^i_t, \theta_{t+1}; \theta^{-i}_t) = p^i_{t+1}(\theta^i_t, \hat{\theta}^i_{t+1}, \theta^{-i}_{t+1}; \theta^{-i}_t) \text{ for all } \theta^i_{t+1}, \hat{\theta}^i_{t+1} \in \Theta^i_{t+1} \text{ and } \theta^{-i}_{t+1} \notin B^{-i}_{t+1},$$

$$\begin{aligned} & U^i_t(\theta^i_t; \theta^{-i}_t) \\ & = v^i_t(\chi_t(\theta_t), \theta_t) \\ & \quad + \delta \mathbb{E}\left[\mathbf{1}_{\{\theta^{-i}_{t+1} \notin B^{-i}_{t+1}\}} \left(V^i_{t+1}(\theta_{t+1}) + G^i_{t+1,t+1}(\theta_{t+1}) \right) + p^i_{t+1}(\theta^i_t, \theta_{t+1}; \theta^{-i}_t) \middle| \chi_t(\theta_t), \theta_t \right] \end{aligned} \tag{30}$$

for all $\theta^i_t \in \Theta^i_t$, and

$$\begin{aligned} & U^i_t(\theta^i_t; \theta^{-i}_t) \\ & \geq v^i_t(\chi_t(\hat{\theta}^i_t, \theta^{-i}_t), \theta_t) \\ & \quad + \delta \mathbb{E}\left[\mathbf{1}_{\{\theta^{-i}_{t+1} \notin B^{-i}_{t+1}\}} \left(V^i_{t+1}(\theta_{t+1}) + G^i_{t+1,t+1}(\theta_{t+1}) \right) + p^i_{t+1}(\hat{\theta}^i_t, \theta_{t+1}; \theta^{-i}_t) \middle| \chi_t(\hat{\theta}^i_t, \theta^{-i}_t), \theta_t \right] \end{aligned} \tag{31}$$

for all $\theta^i_t, \hat{\theta}^i_t \in \Theta^i_t$.

For $\theta^{-i}_{t+1} \notin B^{-i}_{t+1}$, we set

$$\phi^i_{t+1}(\theta^i_t, \theta^{-i}_t, \theta^{-i}_{t+1}) \equiv p^i_{t+1}(\theta^i_t, \theta^i_{t+1}, \theta^{-i}_{t+1}; \theta^{-i}_t) \quad \text{for } \theta^{-i}_{t+1} \notin B^{-i}_{t+1}.$$

Note that (25) ensures that ϕ^i_{t+1} is independent of θ^i_{t+1}. Moreover, since

$$V^i_{t+1}(\theta_{t+1}) + G^i_{t,t+1}(\theta_{t:t+1}) = V^i_{t+1}(\theta_{t+1}) + G^i_{t+1,t+1}(\theta_{t+1}) + \phi^i_{t+1}(\theta_{0:t}, \theta^{-i}_{t+1}) \quad \text{for } \theta^{-i}_{t+1} \notin B^{-i}_{t+1},$$

if $V^i_{t+1}(\theta_{t+1}) + G^i_{t,t+1}(\theta_{t:t+1}) = p^i_{t+1}(\theta^i_t, \theta^i_{t+1}, \theta^{-i}_{t+1}; \theta^{-i}_t)$ holds for $\theta^i_{t+1} \in B^{-i}_{t+1}$, (30) and (31) imply that $(\chi_k, g^i_{t,k})^{T+1}_{k=t}$ is wp-EPIC for i at $(\theta^i_t, \theta^{-i}_t)$ for all $\theta^i_t \in \Theta^i_t$.

Such EPV can actually be given for $\theta_{t+1}^{-i} \in B_{t+1}^{-i}$ keeping wp-EPIC. Applying the same argument as Theorem 1, fixing $(\theta_t, \theta_{t+1}^{-i}) \in \Theta_t \times B_{t+1}^{-i}$, we can construct $\left\{c_s^i(\cdot, \cdot; \theta_t, \theta_{t+1}^{-i})\right\}_{s=t+1}^{T+1}$, where $c_s^i(\cdot, \cdot; \theta_t, \theta_{t+1}^{-i}) : \Theta_{t+1}^i \times \Theta_{t+2} \to \mathbb{R}$, such that is wp-EPIC at $(\theta_{t+1}^i, \theta_{t+2}^{-i}, \theta_{t+2:s})$ for all $(\theta_{t+1}^i, \theta_{t+2:s}) \in \Theta_{t+1}^i \times \Theta_{t+2:s}$ and $p_{t+1}^i(\theta_t^i, \theta_{t+1}^i, \theta_{t+1}^{-i}; \theta_t^{-i}) = V_{t+1}^i(\theta_{t+1}^i, \theta_{t+1}^{-i}) + C_{t+1}^i(\theta_{t+1}^i; \theta_t, \theta_{t+1}^{-i})$ for all $\theta_{t+1}^{-i} \in B_{t+1}^{-i}$. Define

$$g_{t,s}(\theta_{t:s}) \equiv c_s^i(\theta_{t+1}^i, \theta_{t+2:s}; \theta_t, \theta_{t+1}^{-i}) \text{ for } s = t+1, \cdots, T+1,\ \theta_{t:s} \in \Theta_t \times \Theta_{t+1}^i \times B_{t+1}^{-i} \times \Theta_{t+2:s}.$$

The constructed $(\chi_k, g_{t,k}^i)_{k=t}^{T+1}$ is wp-EPIC for i at every $\theta_{t:s} \in \Theta_{t:s}$ for $s \geq t$.

Iterating this process, finally we can obtain $(g_{0,t}^i)_{t=0}^{T+1}$ that makes $(\chi_t, g_{0,t}^i)_{t=0}^{T+1}$ wp-EPIC for i. Defining $g_t^i \equiv g_{0,t}^i$, we obtain wp-EPIC $(\chi_t, g_t^i)_{t=0}^{T+1}$. ■

References

Aoyagi, M. (1998): "Correlated Types and Bayesian Incentive Compatible Mechanisms with Budget Balance," *Journal of Economic Theory*, 151, 142–151.

Athey, S. and I. Segal (2013): "An Efficient Dynamic Mechanism," *Econometrica*, 81, 2463–2485.

Bergemann, D. and J. Välimäki (2010): "The Dynamic Pivot Mechanism," *Econometrica*, 78, 771–789.

Cavallo, R., D. C. Parkes, and S. Singh (2010): "Efficient Mechanisms with Dynamic Populations and Dynamic Types," *Working Paper*.

Crémer, J. and R. P. McLean (1985): "Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist when Demands are Interdependent," *Econometrica*, 53, 345–361.

——— (1988): "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56, 1247–1257.

Fudenberg, D. and D. Levine (1994): "Efficiency and Observability with Long-Run and Short-Run Players," *Journal of Economic Theory*, 62, 103–135.

Fudenberg, D., D. Levine, and E. Maskin (1994): "The Folk Theorem with Imperfect Public Information," *Econometrica*, 62, 997–1039.

He, W. and J. Li (2016): "Efficient Dynamic Mechanisms with Interdependent Valuations," *Games and Economic Behavior*, 97, 166–173.

Hörner, J., S. Takahashi, and N. Vieille (2015): "Truthful Equilibria in Dynamic Bayesian Games," *Econometrica*, 83, 1–53.

Jehiel, P. and B. Moldovanu (2001): "Efficient Design with Interdependent Valuations," *Econometrica*, 69, 1237–1259.

Liu, H. (2017): "Efficient Dynamic Mechanisms in Environments with Interdependent Valuations: the Role of Contingent Transfers," *Theoretical Economics*, forthcoming.

McAfee, R. P. and P. J. Reny (1992): "Correlated Information and Mechanism Design," *Econometrica*, 60, 395–421.

Mezzetti, C. (2004): "Mechanism Design with Interdependent Valuations: Efficiency," *Econometrica*, 72, 1617–1626.

——— (2007): "Mechanism Design with Interdependent Valuations: Surplus Extraction," *Economic Theory*, 31, 473–488.

Nath, S., O. Zoeter, Y. Narahari, and C. R. Dance (2015): "Dynamic Mechanism Design with Interdependent Valuations," *Review of Economic Design*, 19, 211–228.

Obara, I. (2008): "The Full Surplus Extraction Theorem with Hidden Actions," *The BE Journal of Theoretical Economics*, 8.

Rochet, J.-C. (1987): "A Necessary and Sufficient Condition for Rationalizability in a Quasi-Linear Context," *Journal of Mathematical Economics*, 16, 191–200.