Rational expectation of mistakes and a measure of error-proneness

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Abstract

We characterize axiomatically a stochastic choice model, the *Consistent-Mistakes Model* (CMM), that describes an error-prone decision maker's choices. In contrast to random utility models, CMMs generate closed-form choice probability. Under the axioms, we uniquely identify from the choices an expected utility function that represents the decision maker's true preference and a propensity function that describes how likely an alternative is to be chosen. We introduce a measure of error-proneness and show that the logit model of mistakes is a CMM with a constant measure of error-proneness, characterized by a strong version of the independence axiom from expected utility theory. We analyze the properties of models of mistakes.

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1 Introduction

People often make mistakes. To study mistakes, we use models that permit choice randomness. One of the most prominent is the random utility model (Thurstone (1927)). When it is used to model mistakes, the model says that each alternative i has true utility u_i, but the decision maker observes a noisy signal $u_i + \varepsilon_i$. The decision maker chooses the alternative that has the highest signal. Since ε_i's are random, the decision maker chooses each alternative with some probability.

Although the random utility model of mistakes (RUMM) has been widely used in economics, several issues remain unaddressed. First, how do we distinguish a model of mistakes from a model of taste shocks? In a random utility model, we can interpret ε_i's either as noise or as taste shocks. If we interpret ε_i's as taste shocks, the true utility of alternative i is $u_i + \varepsilon_i$. If we interpret ε_i's as noise, the true utility of alternative i is u_i. However, the two interpretations seem to induce identical choice behavior. Hence, when we observe a decision maker's stochastic choices, it seems unclear whether we should attribute choice randomness to mistakes or random taste shocks.

Second, imagine that we want to understand, in a textbook macroeconomic model, what will happen if we allow decision makers to make mistakes. Suppose we introduce mistakes into the textbook model via the RUMM. The original textbook model may be simple and tractable, but the new model becomes intractable, because the choice probability generated by the RUMM is not in closed form in general. Some RUMMs are tractable; for instance, the widely used *logit* RUMM generates closed-form choice probability. However, if we can only examine the logit case, it becomes unclear how much the results are driven by the logit assumption.

This paper characterizes a simple model of mistakes that addresses these issues. We study a two-stage model of mistakes. At the first stage, the decision maker chooses stochastically from a set of *menus*. Each menu is a set of *risky* alternatives (lotteries). At the second stage, the decision maker chooses a lottery stochastically from the menu.

First, the dynamic and risky environment helps us distinguish a model of mistakes from a model of taste shocks. Take the random utility model as an example. In a static setting, the decision maker's choice behavior is the same whether we interpret ε_i's as noise or as taste shocks. In a dynamic setting, however, when it is a model of taste shocks, the *expected utility* of a menu consisting of n alternatives is often defined as the expected maximum of $u_i + \varepsilon_i$'s, $\mathbb{E} \max_{i \in \{1,\dots,n\}} \{u_i + \varepsilon_i\}$. When it is a model of mistakes, the expected utility of the menu is often defined as the *rational expectation of true utility*, $\sum_{i=1}^{n} \rho_i u_i$, in which ρ_i is the choice probability of alternative i.[1] Thus, the dynamic setting enables us to focus on a model of mistakes. Since we need to analyze the decision maker's expected utility (not just utility) of menus, we study choices among risky alternatives to elicit the decision maker's expected utility function.[2]

Second, our model of mistakes always generates closed-form choice probability, and nests the logit RUMM as a special case. Clearly, this model will differ from the RUMM, but will not conflict with the substantial amount of research based on the logit RUMM following McKelvey and Palfrey (1995).

The primitive of our model is a *stochastic choice function* (SCF) ρ_t that describes how the decision maker chooses at stage t. For example, $\rho_1(\{a,b\},\{a,b,c\})$ describes the probability of choosing menu a or b if the decision maker confronts menus a, b, c at the first stage, and $\rho_2(\{p\}, a)$ describes the choice probability of lottery p from menu a. We impose axioms on the SCF. Among other axioms, one main axiom follows from the assumption that the decision maker forms a rational expectation of her own future mistakes, and another main axiom requires that the decision maker's true preference (induced by the SCF) satisfies the independence axiom from expected utility theory.

[1] Under the interpretation of taste shocks, the formula $\mathbb{E} \max_{i \in \{1,\dots,n\}} \{u_i + \varepsilon_i\}$ can be found in Train (2009) and Fudenberg and Strzalecki (2015). Under the interpretation of mistakes, the formula $\sum_{i=1}^{n} \rho_i u_i$ can be found in McKelvey and Palfrey (1998). In the former case, the decision maker enjoys the option value of larger menus. In the latter case, the decision maker suffers from mistakenly choosing alternatives with low true utility from larger menus.

[2] For these ideas to be implemented, it is sufficient to examine a two-stage model we describe above. We generalize the model to a setting that is similar to Kreps and Porteus (1978) in the Appendix.

The main theorem establishes that the SCF satisfies our axioms if and only if it has the following representation: There exists an expected utility function U and a *propensity function* ϕ such that for each lottery p, $U(p) = \sum p(x)U(x)$ as usual; for each menu $a = \{p_1, \ldots, p_n\}$ and each set of menus A,

$$U(a) = \sum_{p_i \in a} \rho_2(\{p_i\}, a)U(p_i), \tag{1}$$

$$\rho_1(\{a\}, A) = \frac{\phi(U(a))}{\sum_{b \in A} \phi(U(b))} \quad \text{and} \quad \rho_2(\{p_i\}, a) = \frac{\phi(U(p_i))}{\sum_{p_j \in a} \phi(U(p_j))}.$$

Equation (1) reflects the decision maker's rational expectation of future mistakes. It computes the expected true utility the decision maker gains by choosing from a. The propensity function is a strictly increasing function that converts an alternative's utility u into a measure of propensity for choosing that alternative, $\phi(u) > 0$. We call this representation of an SCF a *Consistent-Mistakes Model* (CMM).[3]

We introduce a simple measure of *error-proneness*—that is, the propensity for making mistakes—for comparative static analyses. Consider two decision makers, labeled 1 and 2. Decision maker 2 is said to be *more error-prone* than decision maker 1 if decision maker 2 always chooses inferior lotteries with higher probability. In a CMM, this happens if and only if

$$\frac{\phi_2'}{\phi_2} \leq \frac{\phi_1'}{\phi_1};$$

that is, the normalized increase rate of decision maker 1's propensity function is greater than that of decision maker 2's. We take ϕ/ϕ' as a measure of error-proneness.

Using the measure of error-proneness, we study the logit RUMM. We note that the logit RUMM is the CMM that exhibits a *constant measure of error-proneness*, and show that among CMMs, the logit RUMM is characterized by a strong stochastic version of the independence axiom from expected utility theory. The logit RUMM may be violated, for

[3]A similar representation has been discussed in non-axiomatic work by Chen et al. (1997) and Hofbauer and Sandholm (2002). A detailed discussion follows in Section 5.

example, if the decision maker becomes more error-prone as the utilities of all available alternatives increase. We show how the CMM can accommodate the violation without sacrificing tractability.

We analyze the properties of models of mistakes and show that, in a CMM or RUMM, a simple monotonicity property may be violated. In particular, the expected utility of a menu may not be increasing in the utilities of the lotteries in the menu. This observation is specific to neither the CMM nor the RUMM. First, whenever a model of mistakes nests the logit RUMM as a special case, monotonicity fails. Second, we show that for a wide range of models of mistakes, including the CMM and the RUMM, there are three simple properties that cannot hold simultaneously.

The paper proceeds as follows. Section 2 introduces the axioms, the representation theorem, and the measure of error-proneness. In Section 3, we characterize the logit RUMM. Section 4 studies some properties of models of mistakes. Section 5 discusses the related literature.

2 A Two-Stage Model of Mistakes

There are two choice stages. At the first stage, the decision maker confronts a *decision problem*, which is a set of *menus*. The decision maker chooses a menu from the decision problem at the first stage. A menu is a set of *lotteries*. At the second stage, the decision maker chooses a lottery from the menu to consume. Choices may be stochastic at both stages. We sometimes call menus and lotteries *alternatives*.

2.1 The Choice Domain and the Primitive

Let $\mathcal{Z}$ be an arbitrary set of outcomes, and $\Delta(\mathcal{Z})$ the set of simple lotteries on $\mathcal{Z}$. For any set X, let $\mathcal{K}(X)$ denote the collection of nonempty finite subsets of X. Therefore, the set of menus is $\mathcal{M} := \mathcal{K}(\Delta(\mathcal{Z}))$, and the set of decision problems is $\mathcal{D} := \mathcal{K}(\mathcal{M})$. Typical decision problems are denoted by A, B, C, D; typical menus by a, b, c, d; and typical lotteries by p, q, r, s. As usual, we do not distinguish between an outcome x and a degenerate lottery δ_x that assigns probability 1 to x. For any set of lotteries $p_1, \ldots, p_n$, $\sum_{i=1}^n \alpha_i p_i$ represents the lottery whose probability of outcome x is equal to $\sum_{i=1}^n \alpha_i p_i(x)$, in which $\alpha_i \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$.

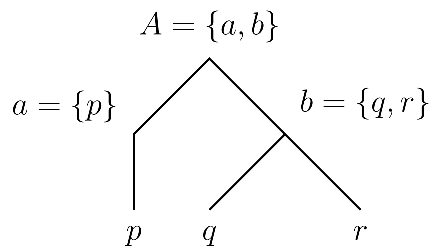

Figure 1: The decision problem is A, which consists of two menus, a and b. Menu a consists of only one lottery p. Menu b consists of two lotteries, q and r.

The decision maker makes mistakes stochastically at both stages. The following pair of functions is the primitive of our model that describes the decision maker's choices.

Definition 1 *A pair of functions* $\rho_1 : \mathcal{D} \times \mathcal{D} \to [0, 1]$ *and* $\rho_2 : \mathcal{M} \times \mathcal{M} \to [0, 1]$ *is called a stochastic choice function (SCF) if for any decision problems* A, B *and menus* a, b, $\rho_1(A, A) = 1$, $\rho_2(a, a) = 1$, $\rho_1(A, B) = \sum_{c \in A} \rho_1(\{c\}, B)$, *and* $\rho_2(a, b) = \sum_{p \in a} \rho_2(\{p\}, b)$.

When the menu is b, $\rho_2(a, b)$ represents the probability that any lottery contained in a is chosen. Similarly, when the decision problem is B, $\rho_1(A, B)$ represents the probability that any menu contained in A is chosen.

When we say that a decision maker makes mistakes, implicitly we mean that she has a stable true preference. Since she is error-prone, she will not reveal the true preference deterministically. However, she may reveal the true preference statistically. Based on an SCF, we define the decision maker's true preference over lotteries as follows.

Definition 2 *For any* $p, q \in \Delta(\mathcal{Z})$, *we say that* p *is preferred to* q ($p \succsim q$) *if* $\rho_2(\{p\}, \{p\} \cup a) \geq \rho_2(\{q\}, \{q\} \cup a)$ *for any menu* $a \in \mathcal{M}$ *such that* $p, q \notin a$.

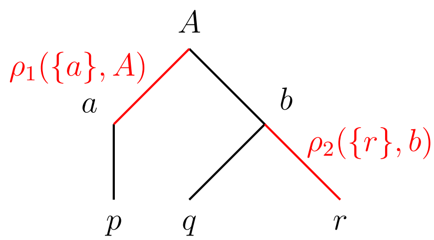

Figure 2: At the first stage, the decision maker chooses menu a from decision problem A with probability $\rho_1(\{a\}, A)$. Suppose she chooses b at the first stage. At the second stage, she chooses lottery r from menu b with probability $\rho_2(\{r\}, b)$.

The decision maker reveals that she prefers p to q if p is always chosen over a with higher probability than q over a, for any a that does not contain p, q. We could have similarly defined the decision maker's true preference over menus, but it is unnecessary now; the true preference over menus will become useful in Section 4. Below, we impose axioms on the SCF (ρ_1, ρ_2) and sometimes on the induced true preference $\succsim$.

2.2 Axioms and the Representation

We first consider three axioms that have appeared in the literature.

Axiom 1 *(Positivity) For any $p \in a \in \mathcal{M}$ and $b \in A \in \mathcal{D}$, $\rho_1(\{b\}, A) > 0$ and $\rho_2(\{p\}, a) > 0$.*

Axiom 2 *(Luce Independence) For any $a, b, c, d \in \mathcal{M}$ and $A, B, C, D \in \mathcal{D}$ such that $(a \cup b) \cap (c \cup d) = \emptyset$ and $(A \cup B) \cap (C \cup D) = \emptyset$,*

1. *$\rho_1(A, A \cup C) \geq \rho_1(B, B \cup C)$ implies $\rho_1(A, A \cup D) \geq \rho_1(B, B \cup D)$;*
2. *$\rho_2(a, a \cup c) \geq \rho_2(b, b \cup c)$ implies $\rho_2(a, a \cup d) \geq \rho_2(b, b \cup d)$.*

Axiom 3 *(vNM Independence) For any $p, q, r \in \Delta(\mathcal{Z})$ and $\alpha \in (0, 1)$, $p \succ q$ implies $\alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r$.*

Positivity is from McFadden (1973). In our setting, it states that every alternative has some chance to be (mistakenly) chosen. *Luce Independence* is from Gul et al. (2014).

Together with other axioms, it ensures that our representation will be a Luce rule (Luce (1959)). One well-known violation of the Luce rule is the duplication problem (Debreu (1960)), which violates *Luce Independence*. Since the main idea of our paper is unrelated to this violation, we maintain *Luce Independence*. More importantly, *Luce Independence* ensures that the decision maker will reveal her true preference consistently; that is, if she chooses p over a set of lotteries a more often than q over a, then she always chooses p with higher probability than q over any other set of lotteries that does not contain p, q.

The third axiom, *vNM Independence*, is needed to identify the decision maker's expected utility function. Note that the primitive of our model is the SCF. Therefore, although *vNM Independence* appears to be identical to the independence axiom from expected utility theory, they have different behavioral content. Also note that *vNM Independence* only concerns second-stage choices. Following *vNM Independence*, we impose two technical assumptions on the second-stage choices.

Axiom 4 *(Continuity) For any $p, q \in \Delta(\mathcal{Z})$ and $a \in \mathcal{M}$, $\rho_2(\{\alpha p + (1-\alpha)q\}, \{\alpha p + (1-\alpha)q\} \cup a)$ is continuous in α.*

Axiom 5 *(Unboundedness) For any $a \in \mathcal{M}$ and $\alpha \in (0,1)$, there exist lotteries $p, q \notin a$ such that $\rho_2(\{p\}, \{p\} \cup a) < \alpha$ and $\rho_2(\{q\}, \{q\} \cup a) > \alpha$.*

Since we have imposed *vNM Independence*, it is natural to consider the continuity axiom from expected utility theory. However, imposing the vNM continuity axiom on the decision maker's true preference can still allow $\rho_2(\{\alpha p + (1-\alpha)q\}, \{\alpha p + (1-\alpha)q\} \cup a)$ to change discontinuously as α changes. Our *Continuity* axiom is a natural extension of the vNM continuity axiom to the current setting. *Unboundedness* is a stochastic version of the standard unboundedness assumption expressed using choice probability.

Our last axiom connects the decision maker's belief about menus to lotteries. To state it, we first define *comparable lotteries*. For any menu $a = \{p_1, \ldots, p_n\} \in \mathcal{M}$, we can find a lottery denoted by p_a such that for any outcome $x \in \mathcal{Z}$,

$$p_a(x) = \sum_{i=1}^{n} \rho_2(\{p_i\}, a) \times p_i(x);$$

that is, the probability that p_a assigns to each outcome x is equal to the probability that the decision maker gets x after (i) she chooses some p_i from a and (ii) p_i's risk resolves. We call p_a the comparable lottery of menu a.

Axiom 6 *(Rational Expectation of Mistakes) For $a_1, \ldots, a_n \in \mathcal{M}$, $\rho_1(\{a_i\}, \{a_1, \ldots, a_n\}) = \rho_2(\{p_{a_i}\}, \{p_{a_1}, \ldots, p_{a_n}\})$.*

If the decision maker understands the probability with which she chooses each alternative in the menu a_i, she will notice that a_i and p_{a_i} induce the same probability distribution over outcomes. The only difference is that with the alternative a_i, it is the decision maker's choice that induces the distribution over outcomes, while with p_{a_i}, the decision maker does not need to make any choice—p_{a_i} itself is the distribution over outcomes. If the decision maker only cares about what distribution over outcomes she obtains, and does not care whether the distribution arises from her choices, then choosing between a_i's is equivalent to choosing between p_{a_i}'s. Therefore, we require that the decision maker's error-prone choice behavior be identical in those two situations.

In Theorem 1, we show that these axioms are equivalent to the following representation.

Definition 3 *An SCF (ρ_1, ρ_2) is a Consistent-Mistakes Model (CMM) if there exists a function $U : \Delta(\mathcal{Z}) \cup \mathcal{M} \to \mathbb{R}$ and a surjective strictly increasing continuous function $\phi : U(\Delta(\mathcal{Z}) \cup \mathcal{M}) \to \mathbb{R}_{++}$ such that for any $p \in \Delta(\mathcal{Z})$, $a = \{p_1, \ldots, p_n\} \in \mathcal{M}$, and $A \in \mathcal{D}$,*

$$U(p) = \sum p(x)U(x), \tag{2}$$

$$U(a) = \sum_{i=1}^{n} \rho_2(\{p_i\}, a)U(p_i), \tag{3}$$

and

$$\rho_1(\{a\}, A) = \frac{\phi(U(a))}{\sum_{b\in A}\phi(U(b))} \quad \textit{and} \quad \rho_2(\{p_i\}, a) = \frac{\phi(U(p_i))}{\sum_{p_j\in a}\phi(U(p_j))}. \tag{4}$$

When U, ϕ satisfy the equations above, we say that (U, ϕ) represents ρ. A lottery's utility is given by equation (2), the standard expected utility equation. For a decision problem a, equation (3) says that the decision maker forms a rational expectation about the expected utility that she will get if she actually chooses from a. Although the decision maker seems to understand her future choice probability, this does not imply that she can avoid mistakes. For example, an investor may be well aware that she does not always invest optimally, but she may understand, on average, how often she makes mistakes and how bad those mistakes typically are. Following the tradition in economics, we assume that the decision maker's expectation is rational (unbiased).

The decision maker makes mistakes when choosing. The function ϕ, called the *propensity function*, in equation (4) translates an alternative's utility into a measure of the propensity for choosing that alternative. Since the propensity function is the same for both stages, the way the decision maker makes mistakes at both stages is the same. One way to interpret equation (4) is that, for example, when confronting two alternatives, the decision maker may know that one alternative (such as an investment opportunity) gives her expected utility 1 and the other gives 0, but she can only identify the better alternative with probability $\frac{\phi(1)}{\phi(1)+\phi(0)}$. Since ϕ is increasing, alternatives with higher utility will be chosen more often.

The propensity function describes the decision maker's error-proneness. For example, let $\phi(u) = u^k$ ($u \in \mathbb{R}_{++}$). Higher k implies fewer mistakes. In the limiting case in which k is arbitrarily large, the best alternatives will be chosen with certainty. At the other extreme, when ϕ becomes a constant function in the limit, the decision maker chooses uniformly randomly. A formal comparative static analysis will be presented after we introduce the main theorem.

Our first result is the representation theorem that establishes the equivalence between the axioms and the CMM.

Theorem 1 *An SCF ρ is a CMM if and only if it satisfies Axioms 1–6. Moreover, suppose (U, ϕ) represents ρ. Then, $\left(\tilde{U}, \tilde{\phi}\right)$ also represents ρ if and only if there exist $\alpha_1, \alpha_2 > 0$ and β such that $\tilde{U} = \alpha_1 U + \beta$ and $\phi(u) = \alpha_2 \tilde{\phi}(\alpha_1 u + \beta)$.*

The necessity is routine. The sufficiency proof consists of three steps. First, we show that the true preference $\succsim$ over lotteries induced by the second-stage SCF ρ_2 satisfies the three vNM axioms. Recall that a lottery p is preferred to q if p is always chosen over a with higher probability than q over a, for any menu a that does not contain p, q. *Luce Independence* ensures that $\succsim$ is complete. *Positivity*, *Luce Independence*, and *Unboundedness* are needed to show that the preference is transitive. Together with *vNM Independence* and *Continuity*, we know that $\succsim$ has an expected utility representation; that is, we can identify an expected utility function $\hat{U}$ defined for all lotteries. This gives us equation (2). As usual, $\hat{U}$ is unique up to a positive affine transformation.

Next, we show that the second-stage SCF ρ_2 satisfies a richness assumption used by Gul et al. (2014) due to *Positivity*, *Continuity*, and *Unboundedness*. Therefore, thanks to their Theorem 1, *Luce Independence* and richness imply the existence of a positive function V that evaluates each lottery such that $\rho_2(\{p_i\}, \{p_1, \ldots, p_n\}) = \frac{V(p_i)}{\sum_{j=1}^{n} V(p_j)}$. The axioms impose several restrictions on V. The most important restriction stems from the observation that V represents $\succsim$. Specifically, $\hat{U}$ is an expected utility representation of $\succsim$, while V only needs to be a utility representation of $\succsim$. Hence, V is a monotone transformation of $\hat{U}$ (but not necessarily a positive affine transformation of $\hat{U}$); that is, given V and $\hat{U}$, the restriction on V is that there is a unique strictly increasing function ϕ such that $V(p) = \phi\left(\hat{U}(p)\right)$ for each lottery p.

Lastly, the first-stage choices provide a revealed-preference foundation for interpreting the decision maker's second-stage choice randomness as mistakes. Define a function U such that $U(p) = \hat{U}(p)$ for any lottery p, and $U(a) = \hat{U}(p_a)$ for any menu a. According to the definition of comparable lotteries, equation (3) holds because $\hat{U}$ is linear. *Rational Expectation of Mistakes* pins down equations for the first-stage choices: For any decision problem $A = \{a_1, \ldots, a_m\}$,

$$\rho_1(\{a_i\}, A) = \rho_2(\{p_{a_i}\}, \{p_{a_1}, \ldots, p_{a_m}\}) = \frac{\phi(U(p_{a_i}))}{\sum_{j=1}^m \phi(U(p_{a_j}))} = \frac{\phi(U(a_i))}{\sum_{j=1}^m \phi(U(a_j))}.$$

Therefore, equation (4) holds.

To see how ϕ is uniquely identified from choices and how it describes error-proneness more concretely, fix an expected utility function U that represents $\succsim$. Focus on the second stage. Consider four lotteries, p, q, r, s, such that $U(p) = 0$, $U(q) = 1$, $U(r) = 100$, and $U(s) = 101$. Suppose we observe that the decision maker chooses q over p with probability 90% ($\rho_2(\{q\}, \{p, q\}) = 90\%$), and that she chooses s over r with probability 60% ($\rho_2(\{s\}, \{r, s\}) = 60\%$). Intuitively, this reveals that the decision maker makes fewer mistakes when the expected utilities of all available alternatives are lower. In other words, if the decision maker will get about 100 utils (confronting $\{r, s\}$), she is less likely to choose the *marginally* better lottery, compared to the case in which she confronts $\{p, q\}$. This allows us to pin down the rate of increase of ϕ for each level of expected utility.

This example also suggests that the rate of increase of ϕ should be related to measuring the decision maker's error-proneness. In the next subsection, we formalize this observation.

2.3 A Comparative Measure of Error-Proneness

Consider two decision makers, labeled I and II. We say that decision maker II is more error-prone than decision maker I if decision maker II always chooses the inferior alternative with higher probability. Let ρ_t^{I} and ρ_t^{II} describe decision maker I's and II's stochastic choice behavior at stage t, respectively.

Definition 4 *Decision maker II is more error-prone than decision maker I if for any* $p, q \in \Delta(\mathcal{Z})$, $\rho_2^{\mathrm{II}}(\{p\}, \{p, q\}) \geq \rho_2^{\mathrm{II}}(\{q\}, \{p, q\})$ *implies* $\rho_2^{\mathrm{I}}(\{p\}, \{p, q\}) \geq \rho_2^{\mathrm{II}}(\{p\}, \{p, q\})$.

Although the decision maker also makes mistakes at the first stage, her first-stage choices reflect how she thinks about her second-stage choices. To focus exclusively on error-proneness, Definition 4 only involves second-stage choices.[4] The definition says that if decision maker II is more error-prone than decision maker I, then whenever decision maker II reveals that she prefers p over q, decision maker I not only prefers p over q as well, but also chooses the preferred lottery p with higher probability.

Proposition 1 *Suppose decision maker I's and II's SCFs are CMMs. Then, decision maker II is more error-prone than decision maker I if and only if there exist $(U^{\mathrm{I}}, \phi^{\mathrm{I}})$ and $(U^{\mathrm{II}}, \phi^{\mathrm{II}})$ representing decision maker I's and II's SCFs, respectively, such that $U^{\mathrm{I}}(p) = U^{\mathrm{II}}(p)$ for any $p \in \Delta(\mathcal{Z})$, and $\frac{\phi^{\mathrm{II}}(u)}{\phi^{\mathrm{I}}(u)}$ is decreasing in u.*

Proof. Suppose $U^{\mathrm{I}}(p) = U^{\mathrm{II}}(p)$ for any lottery p, and $\frac{\phi^{\mathrm{II}}(u)}{\phi^{\mathrm{I}}(u)}$ is decreasing in u. For any lotteries p, q such that $\rho_2^{\mathrm{II}}(\{p\}, \{p, q\}) \geq \rho_2^{\mathrm{II}}(\{q\}, \{p, q\})$, we have $\frac{\phi^{\mathrm{II}}(U^{\mathrm{II}}(p))}{\phi^{\mathrm{II}}(U^{\mathrm{II}}(p))+\phi^{\mathrm{II}}(U^{\mathrm{II}}(q))} \geq \frac{\phi^{\mathrm{II}}(U^{\mathrm{II}}(q))}{\phi^{\mathrm{II}}(U^{\mathrm{II}}(p))+\phi^{\mathrm{II}}(U^{\mathrm{II}}(q))}$. Therefore, $U^{\mathrm{II}}(p) \geq U^{\mathrm{II}}(q)$. Define $u_h := U^{\mathrm{I}}(p) = U^{\mathrm{II}}(p)$ and $u_l := U^{\mathrm{I}}(q) = U^{\mathrm{II}}(q)$. Since $\frac{\phi^{\mathrm{II}}(u_h)}{\phi^{\mathrm{I}}(u_h)} \leq \frac{\phi^{\mathrm{II}}(u_l)}{\phi^{\mathrm{I}}(u_l)}$, we have

$$\frac{\phi^{\mathrm{I}}(u_h)}{\phi^{\mathrm{I}}(u_h) + \phi^{\mathrm{I}}(u_l)} \geq \frac{\phi^{\mathrm{II}}(u_h)}{\phi^{\mathrm{II}}(u_h) + \phi^{\mathrm{II}}(u_l)}$$

as desired.

Now, suppose we know that decision maker II is more error-prone than I. In a CMM, $\rho_2^{\mathrm{I}}(\{p\}, \{p, q\}) \geq \rho_2^{\mathrm{I}}(\{q\}, \{p, q\}) \iff U^{\mathrm{I}}(p) \geq U^{\mathrm{I}}(q)$ and $\rho_2^{\mathrm{II}}(\{p\}, \{p, q\}) \geq \rho_2^{\mathrm{II}}(\{q\}, \{p, q\}) \iff U^{\mathrm{II}}(p) \geq U^{\mathrm{II}}(q)$. Therefore, the hypothesis $\rho_2^{\mathrm{II}}(\{p\}, \{p, q\}) \geq \rho_2^{\mathrm{II}}(\{q\}, \{p, q\}) \Rightarrow \rho_2^{\mathrm{II}}(\{p\}, \{p, q\}) \leq \rho_2^{\mathrm{I}}(\{p\}, \{p, q\})$ implies that

$$U^{\mathrm{II}}(p) \geq U^{\mathrm{II}}(q) \Rightarrow U^{\mathrm{I}}(p) \geq U^{\mathrm{I}}(q). \tag{5}$$

Equation (5) seems to allow the case in which $U^{\mathrm{II}}(p) > U^{\mathrm{II}}(q)$, but $U^{\mathrm{I}}(p) = U^{\mathrm{I}}(q)$. However, as in Ghirardato et al. (2004), when U^{I} and U^{II} are affine functionals on a linear space, this

[4] The definition is the same as Definition 11 of pairwise-selectiveness in Fudenberg et al. (2015). Their paper uses this condition to characterize properties of the cost function in their representation. We uses this condition to study the propensity function. Both their cost function and our propensity function determine how the decision maker makes mistakes.

case can be ruled out. According to Corollary B.3 of Ghirardato et al. (2004), for some $\lambda > 0$ and δ, $U^{\mathrm{I}}(p) = \lambda U^{\mathrm{II}}(p) + \delta$ for any lottery p. Because of the uniqueness of the CMM, we can without lost of generality pick the U^{II} such that $\lambda = 1$ and $\delta = 0$. In that case, $U^{\mathrm{I}}(\Delta(\mathcal{Z}) \cup \mathcal{M}) = U^{\mathrm{II}}(\Delta(\mathcal{Z}) \cup \mathcal{M})$; that is, ϕ^{I} and ϕ^{II} share the same domain. Now, for any lottery p, q such that $U^{\mathrm{II}}(p) \geq U^{\mathrm{II}}(q)$, we define u_h and u_l similarly. We must have $\frac{\phi^{\mathrm{I}}(u_h)}{\phi^{\mathrm{I}}(u_h)+\phi^{\mathrm{I}}(u_l)} \geq \frac{\phi^{\mathrm{II}}(u_h)}{\phi^{\mathrm{II}}(u_h)+\phi^{\mathrm{II}}(u_l)}$, which implies that $\frac{\phi^{\mathrm{II}}(u_h)}{\phi^{\mathrm{I}}(u_h)} \leq \frac{\phi^{\mathrm{II}}(u_l)}{\phi^{\mathrm{I}}(u_l)}$. Therefore, we know that $\frac{\phi^{\mathrm{II}}(u)}{\phi^{\mathrm{I}}(u)}$ is decreasing on the common domain of ϕ^{I} and ϕ^{II}. ■

To better understand this result, let us apply it to the case in which the propensity functions are differentiable. We omit the proof of the following corollary.

Corollary 1 *Suppose that $(U^{\mathrm{I}}, \phi^{\mathrm{I}})$ and $(U^{\mathrm{II}}, \phi^{\mathrm{II}})$ represent decision maker I's and II's SCFs, respectively, such that $U^{\mathrm{I}}(p) = U^{\mathrm{II}}(p)$ for any $p \in \Delta(\mathcal{Z})$ and $\phi^{\mathrm{I}}, \phi^{\mathrm{II}}$ are differentiable. Then, decision maker II is more error-prone than decision maker I if and only if $\frac{D\phi^{\mathrm{I}}(u)}{\phi^{\mathrm{I}}(u)} \geq \frac{D\phi^{\mathrm{II}}(u)}{\phi^{\mathrm{II}}(u)}$ for all u.*[5]

The result above says that if ρ^{II} is more error-prone than ρ^{I}, the normalized rate of increase of the propensity function ϕ^{II}, $\frac{D\phi^{\mathrm{II}}}{\phi^{\mathrm{II}}}$, should be lower than that of ϕ^{I}. Based on this result, it is natural to let $\frac{\phi(u)}{D\phi(u)} > 0$ be the *measure of error-proneness*. A decision maker with higher $\frac{\phi(u)}{D\phi(u)}$ is more error-prone.

3 The Logit Model and Stochastic vNM Invariance

Below, we establish a relation between the widely used logit RUMM and the independence axiom from expected utility theory. The result suggests a new reason why, in addition to the well-known duplication problem, the logit RUMM may be undesirable. To begin with, note that previous discrete choice literature often does not distinguish a Luce rule from a logit model. A Luce rule says that each alternative $i \in \{1, \ldots, n\}$ has a Luce value $v_i > 0$

[5] To simplify notations, we use $D\phi^{\gamma}$ to denote the derivative of ϕ^{γ}, $\gamma =$ I,II. We use either $D\phi$ or ϕ' to denote the derivative of ϕ when ϕ has no superscript.

and is chosen with probability $\frac{v_i}{\sum_{j=1}^n v_j}$. In a logit RUMM, each alternative i has true utility u_i, but the decision maker observes a noisy signal $u_i + \varepsilon_i$ of the true utility. The noise terms ε_i's follow some i.i.d. extreme value type I distribution. Since in an RUMM, the probability of choosing alternative $i \in \{1, \ldots, n\}$ is $\Pr[u_i + \varepsilon_i \geq u_j + \varepsilon_j, j \in \{1, \ldots, n\}]$, it can be shown that for some $\lambda > 0$, alternative i is chosen with probability $\frac{\exp\{u_i/\lambda\}}{\sum_{j=1}^n \exp\{u_j/\lambda\}}$.[6] Usually, economists will identify v_i's from data and let $u_i = \lambda \log v_i$.

The CMM is always a Luce rule, but it is a logit RUMM if and only if

$$\phi(u) = e^{u/\lambda} \tag{6}$$

(up to a positive scalar multiplication) for some $\lambda > 0$. The CMM can separate the logit RUMM from the Luce rule because in a CMM, u_i is the expected utility of alternative i, and $\lambda \log v_i$ does not necessarily give us alternative i's expected utility.

We want to understand what conditions characterize the logit special case of the CMM. First, note that when a CMM exhibits a *constant measure of error-proneness* ($\phi/D\phi = \lambda$), simple calculations show that equation (6) holds; that is, the logit RUMM is the CMM with a constant measure of error-proneness. The axiom below characterizes the CMM with a constant measure of error-proneness.

Axiom 7 *(Stochastic vNM Invariance) For any $p, q, r, s \in \Delta(\mathcal{Z})$ and $\alpha \in (0, 1)$, $\rho_2(\{\alpha p + (1-\alpha)r\}, \{\alpha p + (1-\alpha)r, \alpha p + (1-\alpha)s\}) = \rho_2(\{\alpha q + (1-\alpha)r\}, \{\alpha q + (1-\alpha)r, \alpha q + (1-\alpha)s\})$.*

Stochastic vNM Invariance says that when the common component p in $\alpha p + (1-\alpha)r$ and $\alpha p + (1-\alpha)s$ is replaced with q, the propability of choosing either lottery should not change. This axiom is related to the independence axiom in expected utility theory. Consider a classic example from the Allais paradox. Suppose there is a continuum of ex ante identical students. Each is asked to choose between two lotteries, s_1 and s_2. Lottery s_1 gives one million dollars with probability 11% and zero otherwise, and lottery s_2 gives

[6] See Luce (1959), McFadden (1973), and Train (2009).

five million dollars with probability 10% and zero otherwise. Say that τ percent of students prefer s_2 over s_1. Next, each student is asked to choose between another two lotteries, r_1 and r_2. Lottery r_1 gives one million dollars with certainty, and lottery r_2 gives zero dollars with probability 1%, one million dollars with probability 89%, and five million dollars with probability 10%. Say that τ' percent of students prefer r_2 over r_1. Clearly, τ may be different from τ'.

However, *Stochastic vNM Invariance* requires that $\tau = \tau'$. To see this, first note that because the students are ex ante identical, if (ρ_1, ρ_2) is an arbitrary student's SCF, then $\rho_2(\{s_2\}, \{s_1, s_2\}) = \tau$ and $\rho_2(\{r_2\}, \{r_1, r_2\}) = \tau'$. Next, from the relation between s_1, s_2, r_1, r_2 illustrated in the table below, we can see that *Stochastic vNM Invariance* implies $\tau = \tau'$.

$r_1 = 11\% \times p + 89\% \times r$	$r_2 = 11\% \times q + 89\% \times r$
\$1M	89% \$1M 1% \$0 10% \$5M
$s_1 = 11\% \times p + 89\% \times s$	$s_2 = 11\% \times q + 89\% \times s$
89% \$0 11% \$1M	90% \$0 10% \$5M

In the table, p and r both give one million dollars with certainty, s gives zero dollars with certainty, and q gives five million dollars with probability 10/11 and zero dollars with probability 1/11.

From the table we can also see that if every student's preference over lotteries satisfies the independence axiom in expected utility theory, then a student chooses s_2 over s_1 if and only if she chooses r_2 over r_1. This implies $\tau = \tau'$. Thus, the violation of *Stochastic vNM Invariance* is related to the violation of the independence axiom in expected utility theory.

A CMM does not have to satisfy *Stochastic vNM Invariance*, but it must satisfy *vNM Independence*, which is also related to the independence axiom in expected utility theory.

Does the violation of the independence axiom in expected utility theory above suggest that *vNM Independence* is often violated as well? Not necessarily. In the example above, it can be verified that as long as both τ and τ' are above 1/2, or both are below 1/2, *vNM Independence* is not violated. Indeed, in 2015 and 2016, we asked students from a PhD-level microeconomics course at the University of Michigan to choose from $\{s_1, s_2\}$ and then from $\{r_1, r_2\}$. In both years, the total number of students was around 50, with τ around 45 and τ' around 35. We do not find evidence that *vNM Independence* is violated.

The following result establishes the relation between the logit RUMM and *Stochastic vNM Invariance.*

Proposition 2 *Suppose an SCF is a CMM such that $U(\Delta(\mathcal{Z})) = \mathbb{R}$. The SCF is a CMM with a constant measure of error-proneness if and only if it satisfies Stochastic vNM Invariance.*

Suppose the decision maker's SCF is a CMM. The fact that *Stochastic vNM Invariance* is almost always violated means that the decision maker usually does not have a constant measure of error-proneness. One plausible alternative assumption is that $\frac{D\phi(u)}{\phi(u)}$ may be higher at lower u, but lower at higher u; that is, the decision maker is more error-prone when the utilities of all available alternatives are higher. The RUMM may be able to accommodate this, but it is likely intractable. The CMM can easily accommodate this without sacrificing tractability. For instance, suppose $\phi(u) = u^\lambda$ $(u \geq 0)$. Then,

$$\frac{\phi(u)}{D\phi(u)} = \frac{u}{\lambda};$$

that is, the measure of error-proneness increases with u.

Let us point out another way to characterize the logit RUMM. In the previous literature, the CMM briefly appears in Hofbauer and Sandholm (2002).[7] In a somewhat different setting, they show that the logit model is the only intersection between the random utility model and

[7] We thank an anonymous referee for referring us to this result.

the CMM. Below, we adapt their finding to our setting to provide another characterization of the logit RUMM.

Proposition 3 *Suppose an SCF (ρ_1, ρ_2) is a CMM such that $U(\Delta(\mathcal{Z})) = \mathbb{R}$. The following statements are equivalent:*

1. *For some strictly positive density function $f : \mathbb{R} \to \mathbb{R}_{++}$, $\rho_2(\{p_i\}, \{p_1, \ldots, p_n\}) = \Pr[U(p_i) + \varepsilon_i \geq U(p_j) + \varepsilon_j, \forall j]$ for any $p_1, \ldots, p_n \in \Delta(\mathcal{Z})$, in which ε_j's are i.i.d. random variables whose density function is f;*

2. *The SCF is a CMM with a constant measure of error-proneness.*

Proof. The second statement implies the first, because we can let ε_j's be i.i.d. according to some extreme value type I distribution (McFadden (1973)), whose density function is strictly positive. To see why part 1 implies part 2, first note that since $U(\Delta(\mathcal{Z})) = \mathbb{R}$, Lemma 4 in the Appendix shows that for each $u \in \mathbb{R}$, there are infinitely many lotteries whose expected utility is equal to u. Thus, for each integer n and each vector $(u_1, \ldots, u_n) \in \mathbb{R}^n$, we can find a set of distinct lotteries $p_1, \ldots, p_n$ such that $U(p_j) = u_j$ for $j = 1, \ldots, n$. Next, since ε_j's are i.i.d. and the first statement holds, for each integer n, we apply Hofbauer and Sandholm's (2002) Proposition 2.3 to show that the only way for ρ_2 to satisfy $\rho_2(\{p_i\}, \{p_1, \ldots, p_n\}) = \frac{\phi(U(p_i))}{\sum_{p_j \in a} \phi(U(p_j))}$ for any n lotteries $p_1, \ldots, p_n$ is to have $\phi(u) = e^{u/\lambda}$ for some positive λ (up to a positive scalar multiplication). Since this conclusion does not depend on n, we know that the CMM has to have a constant measure of error-proneness. ■

4 Risk from Mistakes vs. Standard Objective Risk

Two types of risks appear in our framework. The risk associated with lotteries is the standard objective risk. The other, risk from mistakes, is due to the decision maker's stochastic error-prone choices. Decision makers' choice mistakes have caused big losses for banks and other financial institutions in the past. Economists have classified this type of risk as an important case of *operational risk* in one of the most influential banking regulations, Basel II. In this section, we analyze risk from mistakes and discuss how it differs from standard risk.

We start with a counterintuitive observation: The expected utility that a decision maker gets from a menu may not be increasing in the utilities of its lotteries. In Definition 2, we only define the decision maker's true preference over lotteries. Now, let us similarly define the decision maker's true preference over menus: For any $a, b \in \mathcal{M}$, we say that a is preferred to b ($a \succsim b$) if

$$\rho_1(\{a\}, \{a\} \cup A) \geq \rho_1(\{b\}, \{b\} \cup A)$$

for any decision problem $A \in \mathcal{D}$ such that $a, b \notin A$.

Let us introduce a simple property of the SCF.

Definition 5 *An SCF is weakly monotone if* $p_1 \succsim q_1$ *and* $p_2 \succsim q_2$ *imply* $\{p_1, p_2\} \succsim \{q_1, q_2\}$.

Note that an adapted version of weak monotonicity holds for standard risk under expected utility theory: If $p_1 \succsim q_1$ and $p_2 \succsim q_2$, then $\alpha p_1 + (1-\alpha)p_2 \succsim \alpha q_1 + (1-\alpha)q_2$.

Surprisingly, the logit RUMM violates weak monotonicity, which means that weak monotonicity will be violated for any class of models of mistakes that nests the widely used logit RUMM as a special case. To see why weak monotonicity is violated, consider a menu consisting of two lotteries, $\{p, q_n\}$. Say the utility of lottery p is 2, and the utility of q_n is $-n$ ($n \geq 0$). If we replace p with r, whose utility is equal to 2.1, and replace q_n with s, whose utility is 1, we seem to have obtained a better menu, $\{r, s\}$. However, think of the logit RUMM with $\phi(u) = e^u$. As n gets arbitrarily large,

$$\lim_{n \to -\infty} U(\{p, q_n\}) = 2,$$

because $U(\{p, q_n\}) = \rho_2(\{p\}, \{p, q_n\}) \times 2 + \rho_2(\{q_n\}, \{p, q_n\}) \times (-n)$, and it can be verified that $\lim\limits_{n\to-\infty} -n \times \rho_2(\{q_n\}, \{p, q_n\}) = 0$. In contrast,

$$U(\{r,s\}) = \frac{\exp\{1\} + 2.1\exp\{2.1\}}{\exp\{1\} + \exp\{2.1\}} \approx 1.825 < 2.$$

Therefore, for n large enough, $\{p, q_n\} \succ \{r, s\}$, even though $r \succ p$ and $s \succ q_n$.

The intuition is as follows. In a model of mistakes, having strictly better lotteries in a menu does not necessarily improve the menu, because the choice probability distribution depends on the lotteries. By having the obviously worse lottery q_n instead of s, it is easier for the decision maker to avoid the worse lottery.

Such a violation of weak monotonicity is not limited to this particular logit RUMM with $\phi(u) = e^u$. It is easy to prove that all logit RUMMs violate weak monotonicity, because when $\phi(u) = e^{u/\lambda}$, $\lim\limits_{u\to-\infty} u\phi(u) = 0$. Many other CMMs and RUMMs also violate weak monotonicity. The following result says that in the CMM, some limiting behavior of ϕ can tell us whether weak monotonicity is violated.

Proposition 4 *Suppose an SCF is a CMM such that $U(\Delta(\mathcal{Z})) = \mathbb{R}$. If $\lim\limits_{u\to-\infty} u\phi(u) = 0$, then ρ is not weakly monotone.*

This is not a coincidence. Some properties are natural in the context of standard risk, but conflict in the context of risk from mistakes. To illustrate this, we present below a general impossibility result. The proposition above will become a corollary of it.

4.1 An Impossibility Theorem

To show this result, we need only to work with the decision maker's true utility function $U : \Delta(\mathcal{Z}) \cup \mathcal{M} \to \mathbb{R}$. We do not require U to be an expected utility function, nor do we need to specify how the decision maker's SCF depends on U. Therefore, the result below applies to a wide range of models, including the CMM and RUMM. We say that U is *monotone* if

$$U(p_1) > U(q_1), U(p_2) \geq U(q_2) \Rightarrow U(\{p_1, p_2\}) > U(\{q_1, q_2\}).$$

It should be clear by now why monotonicity may seem reasonable and why it may not. We say that U satisfies *betweenness* if

$$U(p) \geq U(q) \Rightarrow U(p) \geq U(\{p, q\}) \geq U(q).$$

The idea is that even though the decision maker makes mistakes, she understands that the true utility she gets by choosing from a menu will be between the best lottery's utility and the worst lottery's utility. Clearly, both the CMM and RUMM satisfy betweenness. We say that U satisfies *reducibility* if

$$\lim_{n\to\infty} U(q_n) = -\infty \Rightarrow \lim_{n\to\infty} U(\{p, q_n\}) = U(p).$$

Reducibility means that as q_n gets arbitrarily bad, it become more and more obvious for the decision maker to not choose q_n. Since q_n will not be chosen in the limit, it can be ignored in the limit from the menu $\{p, q_n\}$; that is, p is as good as $\{p, q_n\}$ in the limit. Although this property may seem less appealing, it is nonetheless satisfied by the logit RUMM.

We can write down a version of monotonicity, betweenness, and reducibility for standard risk and expected utility theory, which can hold simultaneously.[8] In contrast, the result below shows that there is some tension between these three simple properties.

Theorem 2 *Suppose $U(\Delta(\mathcal{Z})) = \mathbb{R}$, and for each $u \in \mathbb{R}$, there exist two distinct lotteries p, q such that $U(p) = U(q) = u$. Monotonicity, betweenness, and reducibility cannot hold simultaneously.*

The proof of this result is simple. Suppose there is a U such that all three conditions hold. Take two different lotteries p and q_0 such that $U(p) = U(q_0) = u$. By betweenness, we know that $U(\{p, q_0\}) = u$. Find a sequence $\{q_n\}$ such that $U(q_{n+1}) < U(q_n)$ and $\lim_{n\to\infty} U(q_n) =$

[8] A lottery version of monotonicity is as follows: If $p_1 \succsim q_1$ and $p_2 \succsim q_2$, then $\alpha p_1 + (1-\alpha)p_2 \succsim \alpha q_1 + (1-\alpha)q_2$. A lottery version of betweenness is as follows: If $p \succsim q$, then $p \succsim \alpha p + (1-\alpha)q \succsim q$. A lottery version of reducibility is as follows: $\lim_{\alpha\to 1} \alpha p + (1-\alpha)q \sim p$.

$-\infty$. Then, by monotonicity and reducibility, we know that

$$u > U(\{p, q_n\}) > \lim_{n\to\infty} U(\{p, q_n\}) = u.$$

Therefore, we reach a contradiction.

5 Related Literature

The most popular stochastic choice model is the random utility model (Thurstone (1927) and Block and Marschak (1960)). Motivated by the fact that the random utility model is usually intractable, we provide a decision theoretic foundation for an alternative model, the CMM, which is tractable. Similar to what Hofbauer and Sandholm (2002) have observed, we confirm that under certain assumptions, the only intersection between the random utility model and the CMM is the logit model.

Chen et al. (1997) first study the CMM. They examine an equilibrium notion in which players' behavior follows equations that are similar to the CMM. We focus on other aspects of the CMM. We characterize the CMM axiomatically, which shows how it can be uniquely identified or falsified using individual choice data. We analyze its comparative measure of error-proneness, the logit special case, and other behavioral properties.

A CMM is a Luce rule (Luce (1959)). In the previous literature, we often do not distinguish between the Luce rule and the logit model (either of mistakes or of taste shocks). However, in our setting, the logit RUMM is a special case of the CMM. This is because by analyzing the decision maker's choices over lotteries, we identify the expected utility function separately. A similar argument about why the Luce rule and the logit model are different also appears in Chen et al. (1997) and Hofbauer and Sandholm (2002), but in both papers, utilities are given. We explain how to identify the true expected utility function from error-prone choices. We show that a strong stochastic version of the independence axiom from expected utility theory characterizes the logit model, and that the logit model has a constant measure of error-proneness.

In a recent critique, Apesteguia and Ballester (2016) show that the random utility model violates some monotonicity property: Confronting two lotteries, the choice probability of the riskier lottery may increase as the decision maker becomes more risk-averse. Their monotonicity property is different from ours in Section 4. The CMM also violates their monotonicity property.

Gul and Pesendorfer (2006) also study stochastic choices in a setting with risky alternatives. They introduce a linearity condition that requires

$$\rho_2(\{p_i\}, \{p_1, \dots, p_n\}) = \rho_2(\{\alpha p_i + (1-\alpha)q\}, \{\alpha p_1 + (1-\alpha)q, \dots, \alpha p_n + (1-\alpha)q\}), \quad (7)$$

for any lotteries $p_1, \dots, p_n, q$ and $\alpha \in (0, 1)$. This condition is stronger than *Stochastic vNM Invariance* in two ways. First, *Stochastic vNM Invariance* only considers binary choices. Second, for binary choices, what *Stochastic vNM Invariance* requires is implied by equation (7).

Our paper is related to several papers on dynamic stochastic choices, such as Gul et al. (2014), Fudenberg and Strzalecki (2015), and Fudenberg et al. (2015). Gul et al. introduce a model to address the duplication problem (Debreu (1960)). In a dynamic setting, their decision maker can detect and delete duplicates in dynamic problems. Our *Luce Independence* axiom is adapted from Gul et al. They use a richness assumption to establish the equivalence between their version of *Luce Independence* and the Luce rule. In our paper, with lotteries, the richness assumption is replaced with *Continuity* and *Unboundedness*. We use their Theorem 1 to show that under our axioms, the SCF is a Luce rule.

Fudenberg and Strzalecki (2015) are the first to use an axiomatic approach to extend the logit model to a dynamic setting. The dynamic setting they consider has finitely many stages and flow payoffs.[9] They establish the relation between aversion to bigger menus and preference for postponing making choices. They offer two equivalent representations of the

[9] We extend the CMM to a similar dynamic setting in the Appendix.

SCF. In the first, choice randomness comes from taste shocks. The utility of a menu is equal to $\mathbb{E}\max\{u_i + \varepsilon_i\}$ minus some cost of making choices, which captures the aversion to bigger menus. In the second representation, the stochastic choice results from maximizing a menu's expected true utility (similar to our equation (3)) minus an adjusted entropy cost function. Similar to the second representation but in a static setting, Fudenberg et al. (2015) propose a model whose cost function is not necessarily adjusted entropy, and the SCF is not necessarily logistic. Cost functions are ruled out in our paper.

Other papers have studied dynamic deterministic choices. Krishna and Sadowski (2014) study the decision maker's preference over state-contingent infinite-horizon decision problems; the states represent taste shocks and follow a subjective Markov process. Cooke (2017) and Piermont et al. (2016) study how from dynamic choices, one can reveal the taste-related information that the decision maker has learned through past consumption. In both papers, learning is endogenous.

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A Appendix

Proof of Theorem 1: We first prove the necessity of the axioms. A CMM is a Luce rule according to (4). Hence, according to Gul et al. (2014), ρ_1 satisfies the first part of *Luce Independence*, and ρ_2 satisfies the second part. In a Luce rule, $p \succsim q$ if and only if $\phi(U(p)) \geq \phi(U(q))$. Since ϕ is strictly increasing, we know that $p \succsim q$ if and only if $U(p) \geq U(q)$. When restricted to $\Delta(\mathcal{Z})$, U is an expected utility function. Therefore, *vNM Independence* holds. According to equations (3) and (4), *Rational Expectation of Mistakes* holds.

Since $\phi(U(p)), \phi(U(a)) > 0$ for any lottery p and menu a, *Positivity* holds. Note that $U(\Delta(\mathcal{Z})) = U(\Delta(\mathcal{Z}) \cup \mathcal{M})$, because for any $a = \{p_1, \ldots, p_n\} \in \mathcal{M}$, $U(a)$ is some weighted average of $U(p_i)$'s. Since ϕ is surjective, $\phi(U(\mathcal{Z})) = \mathbb{R}_{++}$. Therefore, *Unboundedness* holds. Lastly, because for any $p, q \in \Delta(\mathcal{Z})$, $U(\alpha p + (1-\alpha)q)$ and $U(\{\alpha p + (1-\alpha)q\})$ are equal and continuous in α, and ϕ is continuous, one can verify that *Continuity* holds.

Next, we prove the sufficiency of the axioms.

Lemma 1 *The preference $\succsim$ is complete and transitive.*

Proof. From *Luce Independence*, we know that $\succsim$ is complete because for any $\rho_2(\{p\}, \{p\} \cup a)$ and $\rho_2(\{q\}, \{q\} \cup a)$ such that $p, q \notin a$, the former is either greater than or less than the latter. Say it is greater. *Luce Independence* implies that $\rho_2(\{p\}, \{p\} \cup b) \geq \rho_2(\{q\}, \{q\} \cup b)$ for any $b \in \mathcal{M}$ such that $p, q \notin b$. Therefore, $p \succsim q$.

To prove transitivity, suppose $\rho_2(\{p\}, \{p\} \cup a) \geq \rho_2(\{q\}, \{q\} \cup a)$ and $\rho_2(\{q\}, \{q\} \cup b) \geq \rho_2(\{r\}, \{r\} \cup b)$, in which $p, q, r \in \Delta(\mathcal{Z})$, $p, q \notin a$, and $q, r \notin b$. If any two of p, q, r are identical, clearly we have $p \succsim r$. Otherwise, we can apply *Luce Independence* to know that $\rho_2(\{p\}, \{p\} \cup a) \geq \rho_2(\{q\}, \{q\} \cup a)$ implies

$$\rho_2(\{p\}, \{p, r\}) \geq \rho_2(\{q\}, \{q, r\}), \tag{8}$$

and $\rho_2(\{q\},\{q\}\cup b) \geq \rho_2(\{r\},\{r\}\cup b)$ implies

$$\rho_2(\{q\},\{p,q\}) \geq \rho_2(\{r\},\{p,r\}). \tag{9}$$

Due to *Positivity*, $\rho_2(\{r\},\{p,r\}) > 0$. By *Unboundedness*, we can find a new lottery $s \in \Delta(\mathcal{Z})$ such that

$$\rho_2(\{s\},\{p,s\}) < \rho_2(\{r\},\{p,r\}). \tag{10}$$

Equations (8), (9), and (10) show that s is distinct from p, q, r. By *Luce Independence*, we have $\rho_2(\{p\},\{p,s\}) \geq \rho_2(\{q\},\{q,s\}) \geq \rho_2(\{r\},\{r,s\})$. Therefore, $p \succsim r$. ■

Lemma 2 *For any $p, q, r \in \Delta(\mathcal{Z})$, $p \succ q \succ r$ implies that there exist $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$.*

Proof. Following similar arguments from the previous lemma, we can find a lottery s such that $\rho_2(\{p\},\{p,s\}) > \rho_2(\{q\},\{q,s\}) > \rho_2(\{r\},\{r,s\})$, in which $s \neq p, q, r$. Note that $p = 1 \cdot p + 0 \cdot r$. By *Continuity*, we can find an α near 1 such that $\rho_2(\{\alpha p + (1-\alpha)r\},\{\alpha p + (1-\alpha)r, s\}) > \rho_2(\{q\},\{q,s\})$, if $\alpha p + (1-\alpha)r$ is distinct from s. If the α we pick satisfies $\alpha p + (1-\alpha)r = s$, we can find another $\alpha' \neq \alpha$ near 1 such that $\rho_2(\{\alpha' p + (1-\alpha')r\},\{\alpha' p + (1-\alpha')r, s\}) > \rho_2(\{q\},\{q,s\})$, in which case $\alpha' p + (1-\alpha')r$ has to be different from s. To find β, similar arguments apply. ■

The lemma above shows that the familiar vNM continuity axiom is satisfied by the preference $\succsim$. Knowing that $\succsim$ is complete and transitive, and satisfies *vNM Independence* and the vNM continuity axiom, we know that there exists a function $\hat{U} : \Delta(\mathcal{Z}) \to \mathbb{R}$ that represents $\succsim$ such that

$$\hat{U}(p) = \sum_{x \in \mathcal{Z}} p(x)\hat{U}(x). \tag{11}$$

By writing $\hat{U}(x)$ in equation (11), we mean $\hat{U}(\delta_x)$, as we do not distinguish between δ_x and

x. Define a function $U : \Delta(\mathcal{Z}) \cup \mathcal{M} \to \mathbb{R}$ such that for any $a = \{p_1, \ldots, p_m\} \in \mathcal{M}$,

$$U(a) := \hat{U}(p_a),$$

and for any $p \in \Delta(\mathcal{Z})$, $U(p) := \hat{U}(p)$. We immediately have equation (2). By construction,

$$\begin{aligned} U(a) &= \hat{U}(p_a) = \sum_{x \in \mathcal{Z}} \left(\sum_{i=1}^{m} \rho_2(\{p_i\}, a) p_i(x) \hat{U}(x) \right) & (12) \\ &= \sum_{i=1}^{m} \rho_2(\{p_i\}, a) \hat{U}(p_i). & (13) \end{aligned}$$

Hence, equation (3) holds.

Finally, we show that equation (4) holds and ϕ's properties, as stated in Definition 3, are satisfied.

Lemma 3 *For any $a \in \mathcal{M}$ and $\alpha \in (0, 1)$, we can find $p, q, r \in \Delta(\mathcal{Z})$ such that $p, q, r \notin a$, $\rho_2(\{p\}, \{p\} \cup a) = \alpha$, $\rho_2(\{q\}, \{q\} \cup a) > \alpha$, $\rho_2(\{r\}, \{r\} \cup a) < \alpha$, and $p = \beta q + (1 - \beta) r$ for some $\beta \in (0, 1)$.*

Proof. By *Unboundedness*, there exists $q^{(0)} \notin a$ and $r^{(0)} \notin a$ such that $\rho_2(\{q^{(0)}\}, \{q^{(0)}\} \cup a) > \alpha$ and $\rho_2(\{r^{(0)}\}, \{r^{(0)}\} \cup a) < \alpha$. By *Continuity* and the Intermediate Value Theorem, we can find $\beta^{(0)}$ such that

$$\rho_2(\{\beta^{(0)} q^{(0)} + (1 - \beta^{(0)}) r^{(0)}\}, \{\beta^{(0)} q^{(0)} + (1 - \beta^{(0)}) r^{(0)}\} \cup a) = \alpha,$$

if $\beta^{(0)} q^{(0)} + (1 - \beta^{(0)}) r^{(0)} \notin a$.

Suppose $\beta^{(0)} q^{(0)} + (1 - \beta^{(0)}) r^{(0)} \in a$. We can apply *Unboundedness* to find $q^{(1)} \notin a$ and $r^{(1)} \notin a$ such that $1 > \rho_2\left(\left\{q^{(1)}\right\}, \left\{q^{(1)}\right\} \cup a\right) > \rho_2(\{q\}, \{q\} \cup a)$ and $0 < \rho_2\left(\left\{r^{(1)}\right\}, \left\{r^{(1)}\right\} \cup a\right) < \rho_2(\{r_1\}, \{r_1\} \cup a)$. We know that $1 > \rho_2\left(\left\{q^{(1)}\right\}, \left\{q^{(1)}\right\} \cup a\right)$ and $0 < \rho_2\left(\left\{r^{(1)}\right\}, \left\{r^{(1)}\right\} \cup a\right)$

because of *Positivity*. Again, we can find $\beta^{(1)}$ such that

$$\rho_2\left(\left\{\beta^{(1)}q^{(1)}+\left(1-\beta^{(1)}\right)r^{(1)}\right\},\left\{\beta^{(1)}q^{(1)}+\left(1-\beta^{(1)}\right)r^{(1)}\right\}\cup a\right)=\alpha,$$

if $\beta^{(1)}q^{(1)}+\left(1-\beta^{(1)}\right)r^{(1)}\notin a$.

Suppose $\beta^{(1)}q^{(1)}+\left(1-\beta^{(1)}\right)r^{(1)}\in a$ again. We can repeat the above procedure. Since a is finite, after at most $(|a|+1)$ rounds, we will be able to find lotteries $q^{(i)}, r^{(i)}, \beta^{(i)}q^{(i)}+\left(1-\beta^{(i)}\right)r^{(i)}\notin a$ such that $\rho_2(\{q^{(i)}\},\ \{q^{(i)}\}\cup a)>\alpha$, $\rho_2(\{r^{(i)}\},\{r^{(i)}\}\cup a)<\alpha$, and $\rho_2\left(\left\{\beta^{(i)}q^{(i)}+\left(1-\beta^{(i)}\right)r^{(i)}\right\},\left\{\beta^{(i)}q^{(i)}+\left(1-\beta^{(i)}\right)r^{(i)}\right\}\cup a\right)=\alpha$ for some integer $i\leq|a|$. Let $q:=q^{(i)}$, $r:=r^{(i)}$, and $p:=\beta^{(i)}q^{(i)}+\left(1-\beta^{(i)}\right)r^{(i)}$. ■

Lemma 4 *For any $a\in\mathcal{M}$, $\alpha\in(0,1)$, and $u\in U(\Delta(\mathcal{Z})\cup\mathcal{M})$, there exists infinitely many $p\in\Delta(\mathcal{Z})$ such that $\rho_2(\{p\},\{p\}\cup a)=\alpha$, and infinitely many $p'\in\Delta(\mathcal{Z})$ such that $U(p')=u$.*

Proof. By *Unboundedness*, according to Lemma 3, there exist $q_1, r_1, p_1\notin a$ such that $p_1=\beta_1q_1+(1-\beta_1)r_1$, $\rho_2(\{q_1\},\{q_1\}\cup a)>\alpha$, $\rho_2(\{r_1\},\{r_1\}\cup a)<\alpha$, and $\rho_2(\{p_1\},\{p_1\}\cup a)=\alpha$. Consider supp($q_1$) and supp($r_1$). Since they are finite, we can find $x\in$supp(q_1)$\cup$supp(r_1) such that $x\succsim z$ for all $z\in$supp(q_1)$\cup$supp(r_1), and $y\in$supp(q_1)$\cup$supp(r_1) such that $z\succsim y$ for all $z\in$supp(q_1)$\cup$supp(r_1). Applying *Unboundedness* again, we can find a lottery $q_2\notin a$ such that $1>\rho_2(\{q_2\},\{q_2\}\cup a)>\rho_2(\{x\},\{x\}\cup a)$. We must have $1>\rho_2(\{q_2\},\{q_2\}\cup a)$ because of *Positivity*. We do not need to worry about $x\in a$, because if that is the case, we can always apply Lemma 3 to find some other lottery s such that $s\sim x$ and $s\notin a$. The same arguments go through if we replace x with s. Without loss of generality, let us assume that $x\notin a$ for simplicity.

We claim that supp(q_2) must contain some outcome that is strictly better than x, and hence supp(q_2) $\neq$supp(q_1),supp(r_1). If this claim is not true, then $x\succsim q_2$, because $\succsim$ has a standard expected utility representation over $\Delta(\mathcal{Z})$. However, $x\succsim q_2$ implies that $\rho_2(\{x\},\{x\}\cup a)\geq\rho_2(\{q_2\},\{q_2\}\cup a)$ for any a such that $x, q_2\notin a$, which is a contradiction. Similarly, we can find $r_2\notin a$ such that $0<\rho_2(\{r_2\},\{r_2\}\cup a)<\rho_2(\{y\},\{y\}\cup a)$, and hence supp$(r_2)$ $\neq$supp(q_1),supp(r_1). Again, we have assumed without loss of generality that $y \notin a$. By *Continuity*, we can find $\beta_2 \in (0,1)$ such that $\rho_2(\{p_2\},\{p_2\} \cup a) = \alpha$, in which $p_2 := \beta_2 q_2 + (1-\beta_2)r_2 \notin a$. Clearly, p_2 is distinct from p_1, q_1, r_1 since both q_2 and r_2 have different supports from q_1 and r_1. Again, we do not have to worry that $\beta_2 q_2 + (1-\beta_2)r_2 \in a$; otherwise, we could use the same procedure in Lemma 3 to find some other $q_2^{(i)}, r_2^{(i)}, p_2^{(i)}$ such that they are all different from q_1, r_1, p_1 and satisfy $\rho_2\left(\left\{q_2^{(i)}\right\}, \left\{q_2^{(i)}\right\} \cup a\right) > \alpha$, $\rho_2\left(\left\{r_2^{(i)}\right\}, \left\{r_2^{(i)}\right\} \cup a\right) < \alpha$, and $\rho_2\left(\left\{\beta_2^{(i)} q_2^{(i)} + \left(1-\beta_2^{(i)}\right) r_2^{(i)}\right\}, \left\{\beta_2^{(i)} q_2^{(i)} + \left(1-\beta_2^{(i)}\right) r_2^{(i)}\right\} \cup a\right) = \alpha$ for some integer $i \leq |a|$ and $\beta_2^{(i)} \in (0,1)$.

We can repeat the above procedure to find a sequence of p_j, q_j, r_j, $j = 1, 2, \ldots$. Each q_j and r_j will invite new elements to the support, and hence generate countably infinitely many distinct $p_j \notin a$ such that $\rho_2(\{p_j\},\{p_j\} \cup a) = \alpha$.

Lastly, for each $u \in U(\Delta(\mathcal{Z}) \cup \mathcal{M})$, we can also find infinitely many $p' \in \Delta(\mathcal{Z})$ such that $U(p') = u$. First, since $U(a)$ is a weighted average of $U(p_i)$'s for any menu $a = \{p_1, \ldots, p_n\}$, we know that $U(\Delta(\mathcal{Z}) \cup \mathcal{M}) = U(\Delta(\mathcal{Z}))$. For any $u \in U(\Delta(\mathcal{Z}) \cup \mathcal{M})$, there is some $p' \in \Delta(\mathcal{Z})$ such that $U(p') = u$. From the proof above, we know that there are infinitely many lotteries. Take a set of lotteries b such that $p' \notin b$. Then, we already know that we can find a sequence of lotteries, $p'_1, p'_2, \ldots$, such that for any $j \in \mathbb{N}$, $p'_j \notin b$ and

$$\rho_2(\{p'_j\},\{p'_j\} \cup b) = \rho_2(\{p'\},\{p'\} \cup b). \tag{14}$$

Equation (14) and *Luce Independence* imply that $p'_j \sim p'$ for each $j \in \mathbb{N}$. Because when restricted to $\Delta(\mathcal{Z})$, U represents $\succsim$, we know that $U(p'_j) = u$. ■

Due to Lemma 4 and *Luce Independence*, we can apply Theorem 1 of Gul et al. (2014) to ρ_2. Then, we know that there exists a surjective function $V : \Delta(\mathcal{Z}) \to \mathbb{R}_{++}$ such that $p \succsim q$ if and only if $V(p) \geq V(q)$, and for any $a = \{p_1, \ldots, p_n\}$,

$$\rho_2(\{p_i\}, a) = \frac{V(p_i)}{\sum_{j=1}^n V(p_j)}.$$

To see why we can do this, first note that applying the independence condition from Gul et al. (2014) to ρ_2 is identical to assuming the second part of *Luce Independence*. Second, applying their richness condition to ρ_2 requires that for any $a, c \in \mathcal{M}$ and $\alpha \in (0, 1)$, there exists a menu $b \in \mathcal{M}$ such that $b \cap c = \emptyset$ and $\rho_2(b, a \cup b) = \alpha$. Lemma 4 shows for any $a, c \in \mathcal{M}$ and $\alpha \in (0, 1)$, we can find a lottery p such that $\rho_2(\{p\}, \{p\} \cup a) = \alpha$. We only have to show that $\{p\} \cap c = \emptyset$ before we let $b = \{p\}$. Since $c \in \mathcal{M}$ is a finite set of lotteries, and there are infinitely many lotteries p that satisfy $\rho_2(\{p\}, \{p\} \cup a) = \alpha$, we can assume without loss of generality that $\{p\} \cap c = \emptyset$. Therefore, we know that ρ_2 satisfies both conditions in Gul et al.'s (2014) Theorem 1.

Since both U and V represent $\succsim$ on $\Delta(\mathcal{Z})$, we know that there exists a strictly increasing function $\phi : U(\mathcal{L}) \to \mathbb{R}_{++}$ such that $V(p) = \phi(U(p))$ for any $p \in \Delta(\mathcal{Z})$. Since V is surjective, ϕ must also be surjective. By *Continuity*, ϕ must be continuous.

Lastly, we want to prove that for each decision problem $A = \{a_1, \ldots, a_m\} \in \mathcal{D}$,

$$\rho_1(\{a_i\}, A) = \frac{\phi(U(a_i))}{\sum_{j=1}^{m} \phi(U(a_j))}.$$

By equation (12), $U(a_j) = U(p_{a_j})$ for $j = 1, \ldots, m$. By *Rational Expectation of Mistakes*,

$$\begin{aligned}
\rho_1(\{a_i\}, A) &= \rho_2(\{p_{a_i}\}, \{p_{a_1}, \ldots, p_{a_m}\}) \\
&= \frac{\phi(U(p_{a_i}))}{\sum_{j=1}^{m} \phi(U(p_{a_j}))} \\
&= \frac{\phi(U(a_i))}{\sum_{j=1}^{m} \phi(U(a_j))}.
\end{aligned}$$

For uniqueness, we only prove the necessity. Suppose (U, ϕ) represents ρ. If $\left(\tilde{U}, \tilde{\phi}\right)$ also represents ρ, then U and $\tilde{U}$ both represent $\succsim$. Since U and $\tilde{U}$ are both expected utility functions on $\Delta(\mathcal{Z})$, we know that there exists $\alpha_1 > 0, \beta \in \mathbb{R}$ such that $\tilde{U}(p) = \alpha_1 U(p) + \beta$ for any $p \in \Delta(\mathcal{Z})$. For any $a \in \mathcal{M}$, $U(a) = U(p_a)$ and $\tilde{U}(a) = \tilde{U}(p_a) = \alpha_1 U(p_a) + \beta = \alpha_1 U(a) + \beta$. Therefore, $\tilde{U} = \alpha_1 U + \beta$ holds for $\Delta(\mathcal{Z}) \cup \mathcal{M}$. Since Luce values are unique up to a positive scalar multiplication, $\phi(U(p)) = \alpha_2\tilde{\phi}(\tilde{U}(p))$ for some $\alpha_2 > 0$. Therefore, $\phi(u) = \alpha_2\tilde{\phi}(\alpha_1 u+\beta)$.

■

Proof of Proposition 2: We first prove the necessity. Suppose (ρ_1, ρ_2) is a CMM such that $U(\Delta(\mathcal{Z})) = \mathbb{R}$ and $\phi(u) = \beta e^{u/\lambda}$ $(\lambda > 0, \beta > 0)$. Then,

$$\begin{aligned}
& \rho_2(\{\alpha p + (1-\alpha)r\}, \{\alpha p + (1-\alpha)r, \alpha p + (1-\alpha)s\}) \\
= & \frac{\exp\{\alpha/\lambda \cdot U(p) + (1-\alpha)/\lambda \cdot U(r)\}}{\exp\{\alpha/\lambda \cdot U(p) + (1-\alpha)/\lambda \cdot U(r)\} + \exp\{\alpha/\lambda \cdot U(p) + (1-\alpha)/\lambda \cdot U(s)\}} \\
= & \frac{\exp\{(1-\alpha)/\lambda \cdot U(r)\}}{\exp\{(1-\alpha)/\lambda \cdot U(r)\} + \exp\{(1-\alpha)/\lambda \cdot U(s)\}} \\
= & \frac{\exp\{\alpha/\lambda \cdot U(q) + (1-\alpha)/\lambda \cdot U(r)\}}{\exp\{\alpha/\lambda \cdot U(q) + (1-\alpha)/\lambda \cdot U(r)\} + \exp\{\alpha/\lambda \cdot U(q) + (1-\alpha)/\lambda \cdot U(s)\}} \\
= & \rho_2(\{\alpha q + (1-\alpha)r\}, \{\alpha q + (1-\alpha)r, \alpha q + (1-\alpha)s\}).
\end{aligned}$$

Therefore, *Stochastic vNM Invariance* holds.

Conversely, if *Stochastic vNM Invariance* holds, we know that $\rho_2(\{\frac{1}{2}p+\frac{1}{2}r\}, \{\frac{1}{2}p+\frac{1}{2}r, \frac{1}{2}p+\frac{1}{2}s\}) = \rho_2(\{\frac{1}{2}q+\frac{1}{2}r\}, \{\frac{1}{2}q+\frac{1}{2}r, \frac{1}{2}q+\frac{1}{2}s\})$, which means

$$\begin{aligned}
& \frac{\phi(1/2 \cdot U(p) + 1/2 \cdot U(r))}{\phi(1/2 \cdot U(p) + 1/2 \cdot U(r)) + \phi(1/2 \cdot U(p) + 1/2 \cdot U(s))} \\
= & \frac{\phi(1/2 \cdot U(q) + 1/2 \cdot U(r))}{\phi(1/2 \cdot U(q) + 1/2 \cdot U(r)) + \phi(1/2 \cdot U(q) + 1/2 \cdot U(s))}.
\end{aligned}$$

Since we are focusing on ρ_2, in which case ϕ's domain is $\mathbb{R}$, for any $u_1, u_2, \Delta u \in \mathbb{R}$, we can find $r, s \in \Delta(\mathcal{Z})$ such that $U(r) = 2u_1, U(s) = 2u_2$, and $p, q \in \Delta(\mathcal{Z})$ such that $U(p) = 0, U(q) = 2\Delta u$. Then, for all $u_1, u_2, \Delta u \in \mathbb{R}$,

$$\frac{\phi(u_1)}{\phi(u_2)} = \frac{\phi(u_1 + \Delta u)}{\phi(u_2 + \Delta u)}. \tag{15}$$

Fixing u_1, equation (15) implies that $\phi(u_2 + \Delta u) = \varphi_{u_1}(u_2)\eta_{u_1}(\Delta u)$, in which $\varphi_{u_1}(u_2) =$

$\phi(u_2)/\phi(u_1)$, $\eta_{u_1}(\Delta u) = \phi(u_1 + \Delta u)$. Since ϕ is strictly increasing, positive and continuous, by Theorem 2 in Chapter 3 in Aczél (1966), $\phi(u) = \beta \exp\{\alpha u\}$ for some $\alpha, \beta > 0$. Let $\lambda = 1/\alpha$.

■

A.1 The CMM in a Dynamic Setting

Now, we move to a more general dynamic setting similar to that of Kreps and Porteus (1978). We characterize the CMM in the new setting; notations adopted here will be different from other parts of the paper. There is a finite integer T, and for each time $t \in \{0, \ldots, T\}$, there is a set Z_t of possible payoffs. Generic elements of Z_t are denoted by x_t, y_t, z_t. Let L_T be the set of simple lotteries on Z_T. Recursively, let $\mathcal{M}_t$ be the collection of nonempty finite subsets of L_t, and let L_{t-1} be the set of simple lotteries on $Z_{t-1} \times \mathcal{M}_t$. Generic elements of L_t are denoted by p_t, q_t, r_t, and generic elements of $\mathcal{M}_t$ by A_t, B_t, C_t, D_t. We do not distinguish between a pair of consumption and a next-stage decision problem (x_t, A_{t+1}) and a degenerate lottery $\delta_{(x_t, A_{t+1})}$ that assigns probability 1 to the pair (x_t, A_{t+1}). Mixtures of lotteries are defined as usual. For notational convenience, we sometimes write $p_{t+1}, L_{t+1}, A_{t+1}, \mathcal{M}_{t+1}$ even when $t = T$. One could treat L_{T+1} and $\mathcal{M}_{T+1}$ as empty sets.

A *decision problem* at time t is an element of $\mathcal{M}_t$. An *alternative* at time t is an element of L_t. We define $H_1 := Z_0$, and for each $t \in \{2, \ldots, T\}$, $H_t := H_{t-1} \times Z_{t-1}$. A generic element of H_t denoted by $h_t = (x_0, \ldots, x_{t-1})$ is called a *payoff histories* h_t at time t. For simplicity, we write $h_{t+1} = (x_0, \ldots, x_t)$ as (h_t, x_t) if $h_t = (x_0, \ldots, x_{t-1})$.

At each time t, the decision maker confronting a decision problem A_t chooses an alternative $p_t \in A_t$. The alternative p_t is a probability distribution over pairs of current-stage consumption and a next-stage decision problem. Suppose the pair $(x_t, A_{t+1}) \in$supp(p_t) is realized. The decision maker consumes x_t and then makes another choice from A_{t+1} at the next stage. Specifically, the following function describes how, after each payoff history h_t, the decision maker chooses at time t.

Definition 6 *A function $\rho_{h_t} : \mathcal{M}_t \times \mathcal{M}_t \to [0,1]$ is the decision maker's SCF at time t after the payoff history h_t if $\rho_{h_t}(A_t, A_t) = 1$ and $\rho_{h_t}(A_t, B_t) = \sum_{p_t \in A_t} \rho_{h_t}(\{p_t\}, B_t)$.*

We call the set of functions

$$\varrho := \left\{ \rho_{h_t} : \begin{array}{c} \rho_{h_t} \text{ is the decision maker's SCF at time } t \text{ after the payoff} \\ \text{history } h_t \text{ for some } t \in \{0, \ldots, T\} \text{ and } h_t \in H_t \end{array} \right\}$$

the decision maker's SCF. From the SCF, we define the decision maker's true preference over alternatives at time t after the payoff history h_t.

Definition 7 *For any $p_t, q_t \in L_t$, we say that p_t is preferred to q_t ($p_t \succsim_{h_t} q_t$) at time t after the payoff history h_t if $\rho_{h_t}(\{p_t\}, \{p_t\} \cup A_t) \geq \rho_{h_t}(\{q_t\}, \{q_t\} \cup A_t)$ for any $A_t \in \mathcal{M}_t$ such that $p_t, q_t \notin A_t$.*

The axioms below are adapted from Section 2.2.

Axiom 8 *(Positivity*) At any time t after any h_t, for any $p_t \in A_t \in \mathcal{M}_t$, $\rho_{h_t}(\{p_t\}, A_t) > 0$.*

Axiom 9 *(Luce Independence*) At any time t after any h_t, for any $A_t, B_t, C_t, D_t \in \mathcal{M}_t$ such that $(A_t \cup B_t) \cap (C_t \cup D_t) = \emptyset$, $\rho_{h_t}(A_t, A_t \cup C_t) \geq \rho_{h_t}(B_t, B_t \cup C_t)$ implies $\rho_{h_t}(A_t, A_t \cup D_t) \geq \rho_{h_t}(B_t, B_t \cup D_t)$.*

Axiom 10 *(vNM Independence*) At any time t after any h_t, for any $p_t, q_t, r_t \in L_t$ and $\alpha \in (0,1)$, $p_t \succ_{h_t} q_t$ implies $\alpha p_t + (1-\alpha) r_t \succ_{h_t} \alpha q_t + (1-\alpha) r_t$.*

Axiom 11 *(Continuity*) At any time t after any h_t, for any $p_t, q_t \in L_t$ and $A_t \in \mathcal{M}_t$, $\rho_{h_t}(\{\alpha p_t + (1-\alpha) q_t\}, \{\alpha p_t + (1-\alpha) q_t\} \cup A_t)$ is continuous in α.*

Axiom 12 *(Unboundedness*) At any time t after any h_t, for any $A_t \in \mathcal{M}_t$ and $\alpha \in (0,1)$, there exist $p_t, q_t \notin A_t$ such that $\rho_{h_t}(\{p_t\}, \{p_t\} \cup A_t) < \alpha$ and $\rho_{h_t}(\{q_t\}, \{q_t\} \cup A_t) > \alpha$.*

For any decision problem $A_t = \left\{p_t^{(1)}, \ldots, p_t^{(n)}\right\}$ at time t after the payoff history h_t, we define its comparable lottery $\pi_{h_t}(A_t) \in L_t$ as follows:

$$\pi_{h_t}(A_t)(x_t, A_{t+1}) = \sum_{i=1}^{n} \rho_{h_t}\left(\left\{p_t^{(i)}\right\}, A_t\right) \times p_t^{(i)}(x_t, A_{t+1}).$$

The idea of the axiom below is similar to *Rational Expectation of Mistakes*: The decision maker has a correct belief about the probability that she will end up with any (x_t, A_{t+1}) after choosing from A_t. Since $\pi_{h_t}(A_t)$ generates the same distribution over (x_t, A_{t+1})'s, the decision maker should "identify" A_t with the degenerate decision problem $\{\pi_{h_t}(A_t)\}$.

Figure 3: Solid lines represent available alternatives in a decision problem. Dotted lines represent objective probabilities from lotteries. The probability of choosing (x_t, A_{t+1}) from A_t is equal to the probability that the alternative $\pi_{h_t}(A_t)$ assigns to (x_t, A_{t+1}).

Axiom 13 *(Rational Expectation of Mistakes*) At any time $t < T$ after any h_t, for any $x_t \in Z_t$ and $A_{t+1} \in \mathcal{M}_{t+1}$, $(x_t, A_{t+1}) \sim_{h_t} (x_t, \{\pi_{(h_t, x_t)}(A_{t+1})\})$.*

The last axiom is a simple temporal consistency assumption adapted from Kreps and Porteus (1978).

Axiom 14 *(Temporal Consistency) At any time $t < T$ after any h_t, $(x_t, \{p_{t+1}\}) \succsim_{h_t} (x_t, \{q_{t+1}\})$ if and only if $p_{t+1} \succsim_{(h_t, x_t)} q_{t+1}$.*

These axioms characterize the following extension of the CMM, whose functional form is similar to the representation in Lemma 4 in Kreps and Porteus (1978).

Definition 8 *An SCF ϱ is a Generalized Consistent-Mistakes Model (GCMM) if there exists, for each history h_t, a function $U_{h_t} : L_t \to \mathbb{R}$, a function $W_{h_t} : \{(x_t, u_{t+1}) : u_{t+1} = U_{(h_t,x_t)}(p_{t+1})$ for some $p_{t+1} \in L_{t+1}\} \to \mathbb{R}$ that is strictly increasing and continuous in its second argument, and a function $\phi_{h_t} : U_{h_t}(L_t) \to \mathbb{R}_{++}$ that is surjective, strictly increasing, and continuous such that for any time t, history h_t, $p_t \in L_t$, $x_t \in Z_t$, and $A_{t+1} \in \mathcal{M}_{t+1}$,*

$$U_{h_t}(p_t) = \sum_{(y_t,B_{t+1})\in supp(p_t)} p_t(y_t, B_{t+1}) U_{h_t}(y_t, B_{t+1}), \tag{16}$$

$$U_{h_t}(x_t, A_{t+1}) = W_{h_t}\left(x_t, \sum_{q_{t+1}\in A_{t+1}} \rho_{(h_t,x_t)}\left(\{q_{t+1}\}, A_{t+1}\right) U_{(h_t,x_t)}\left(q_{t+1}\right)\right), \tag{17}$$

and

$$\rho_{(h_t,x_t)}(\{q_{t+1}\}, A_{t+1}) = \frac{\phi_{(h_t,x_t)}(U_{(h_t,x_t)}(q_{t+1}))}{\sum_{r_{t+1}\in A_{t+1}} \phi_{(h_t,x_t)}(U_{(h_t,x_t)}(r_{t+1}))}. \tag{18}$$

Equation (16) is the standard expected utility equation. Equation (17) shows that after each history, the decision maker has a correct belief about how she will choose from the next-stage decision problem. Specifically, the decision maker uses her evaluation of the comparable lottery $\pi_{(h_t,x_t)}(A_{t+1})$ of A_{t+1} to evaluate A_{t+1}. Then, to evaluate (x_t, A_{t+1}), the decision maker uses an aggregator W_{h_t} to aggregate the current-stage consumption x_t and the utility of $\pi_{(h_t,x_t)}(A_{t+1})$. Note that equation (17) also implies

$$U_{h_t}(x_t, \{q_{t+1}\}) = W_{h_t}(x_t, U_{(h_t,x_t)}(q_{t+1})). \tag{19}$$

Lastly, after each history h_t, the decision maker uses the propensity function ϕ_{h_t} to convert an alternative's utility into a measure of propensity for choosing that alternative.

This representation should be applied backward to a decision problem. Starting from the last stage, after each history h_T, equation (16) evaluates each alternative in L_T. Then, equation (18) tells us what the choice probability distribution is over any A_T. Equation (17) shows how to evaluate each pair of (x_{T-1}, A_T), after which we are ready to evaluate each alternative in L_{T-1}, and so on.

Theorem 3 *An SCF ϱ is a GCMM if and only if it satisfies Axioms 9–15. Moreover, if an SCF ϱ is a GCMM, then each U_{h_t} is unique up to a positive affine transformation; fixing U_{h_t}'s, W_{h_t}'s are unique and ϕ_{h_t}'s are unique up to a positive scalar multiplication.*

Many steps in establishing the equivalence between the axioms and the representation are similar to the proof of Theorem 1. We only briefly describe how we can prove the sufficiency of the axioms. First, similar to Lemmas 1 and 2, for each history h_t, we can show that $\succsim_{h_t}$ satisfies the three axioms from expected utility theory. This helps us pin down U_{h_t}. Similar to Lemmas 3 and 4, we can show that the richness assumption in Gul et al. (2014) holds. Therefore, we obtain a Luce value $V_{h_t}(p_t)$ for each alternative $p_t \in L_t$. Since U_{h_t} and V_{h_t} represent the same preference, we can pin down ϕ_{h_t}. Lastly, *Rational Expectation of Mistakes** and *Temporal Consistency* imply that there must be a strictly increasing function $f_{(h_t,x_t)}$ such that

$$U_{h_t}(x_t, A_{t+1}) = f_{(h_t,x_t)}(U_{(h_t,x_t)}(\pi_{(h_t,x_t)}(A_{t+1}))).$$

This function becomes the W_{h_t} function in equation (18).