

School Choice under Partial Fairness*

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Abstract

We generalize the school choice problem by defining a notion of *allowable priority violations*. In this setting, a weak axiom of stability (*partial stability*) allows only certain priority violations. We introduce a class of algorithms called the *Student Exchange under Partial Fairness (SEPF)*. Each member of this class gives a *partially stable* matching that is not *Pareto dominated* by another *partially stable* matching (i.e. *constrained efficient* in the class of *partially stable* matchings). Moreover, any *constrained efficient* matching that *Pareto improves* upon a *partially stable* matching can be obtained via an algorithm within the SEPF class. We characterize the unique algorithm in the SEPF class satisfying a desirable incentive property. The extension of the model to an environment with weak priorities enables us to provide a characterization result which proves the counterpart of the main result in [Erdil and Ergin \(2008\)](#).

Keywords : School choice, stability, efficiency

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1 Introduction

School choice has become an important policy tool for school districts to provide students with the opportunity to choose their school. In a school choice problem, students submit their preferences over schools to a central placement authority and the authority decides on an assignment (or a *matching*) based on schools' capacities, schools' priorities over students, and submitted preferences. The main criteria are improving students' welfare (efficiency) and respecting schools' priorities (fairness). Unfortunately, efficiency and fairness are incompatible in this setting (Balinski and Sönmez, 1999). This incompatibility has led school districts and scholars to search for a compromise between efficiency and fairness. The current work contributes to this research program.

Our approach is to improve efficiency by allowing priority violations. We generalize the standard school choice problem by including a set of *allowable priority violations*. The standard fairness notion, *stability*, is projected onto this framework in a straightforward way, *partial stability* (Section 2.2). We propose a class of algorithms, *Student Exchange under Partial Fairness (SEPF)*, such that each algorithm in this class gives a *partially stable* matching that is not *Pareto dominated* by another *partially stable* matching (in other words, *constrained efficient* in the class of *partially stable* matchings). One can begin with any *partially stable* matching and each algorithm in the SEPF class selects a *constrained efficient* matching that *weakly Pareto dominates* the initial matching. Moreover, each such matching can be obtained by an algorithm in the SEPF class (Theorem 1).¹

The set of allowable priority violations is taken as a premise of the model, and the concept has several interpretations. One interpretation involves the following scenario (Section 4): the school district can ask students' consent for priority violations and then selects a matching that takes the students' consent into account (Kesten, 2010). The idea is to design school choice rules² such that students have incentives to consent for their priorities to be violated: a consenting student is never hurt by her decision to consent. This is an indispensable property which assures that the idea of consent is operational. When priorities are consented to be violated, stability constraints are relaxed and each student's welfare can be weakly improved. A rule motivated by this interpretation is the *Efficiency Adjusted Deferred Acceptance Mechanism (EADAM)* (Kesten, 2010). The EADAM gives students incentives to

¹In an independent study, Combe et al. (2018) study a similar class of rules. Motivated by the teacher reassignments in France, they propose a class of algorithms. Each algorithm in this class starts with an initial matching and in each iteration the set of blocking pairs shrinks in order to improve the welfare of the teachers. They prove non-existence of a strategy-proof rule in this class.

²A *school choice rule* is a systematic way of matching students with schools for each school choice problem.

consent and finds a *constrained efficient* matching. Our characterization result implies that the EADAM is in the SEPF class (Proposition 1). We argue that a particular rule, the *Top Priority (TP) Rule*, stands out in the SEPF class: The TP is the unique *constrained efficient* rule under which students are not hurt by consenting (Theorem 2). An immediate corollary is the outcome equivalence of the EADAM and the TP rule: they select the same outcome for the same preference and consent profiles. Unfortunately, no rule within the SEPF class is immune to violations through misrepresentation of preferences (Proposition 3). This incompatibility is indeed more general: a *constrained efficient* rule can never be *strategy-proof* (Theorem 3).

Another interpretation (Section 5) pertains to the case where certain *types* of priorities can be violated, while other priorities must always be respected. For instance, the Boston Public School System (BPSS) has recently removed proximity from the priority structure and has started prioritizing students based on sibling status only (Dur et al., 2018). In another example, the Recovery School District in New Orleans replaced an efficient but non-stable matching rule with a stable one after including the private scholarship schools, whose priorities cannot be violated by law (Pathak, 2017). Similarly, some school districts include both exam and regular schools. Exam schools’ priorities are determined based on centralized test scores and cannot be violated by law. Regular schools’ priorities are exogenously determined and respecting them is plausible but not necessary (Abdulkadiroğlu, 2011; Sönmez and Ünver, 2011). These examples show that school districts may consider allowing some priorities to be violated, which would render further efficiency gains possible. An intuitive approach to elicit these efficiency gains is to remove these priority classes. Nevertheless, we show (in Section 5.1) that suppressing a priority class may yield a perverse consequence: it may make each student worse off compared to the outcome of the student-proposing Deferred Acceptance (DA) rule where all existing priority classes are kept. On the other hand, the SEPF class produces matchings that weakly improve on the outcome of the student-proposing DA rule and implicitly “determines” which priorities (among those which are allowed to be violated) to violate in order to obtain these efficiency gains.

Our basic model has strict priorities, but its extension to an environment with weak priorities is straightforward (Section 7). We show that a characterization result analogous to Theorem 1 holds for this domain, which completes the main result of Erdil and Ergin (2008) by proving its converse. This result reveals an important connection between the SEPF and Stable Improvement Cycles algorithm proposed by Erdil and Ergin (2008).

Related literature

The school choice problem is introduced by [Abdulkadiroğlu and Sönmez \(2003\)](#). A major concern in a school choice problem is *stability*: for each school s , there should not be a student who prefers s to her assigned school and another student assigned to s with a lower priority at s , and there should not also be an unfilled seat at s if a student prefers s to her assigned school. There are rules which always select *stable* matchings. The student-proposing DA rule is such an example ([Gale and Shapley, 1962](#)). The student-proposing DA gives the student-optimal stable matching (SOSM), which is the best matching in terms of the students’ welfare among all the stable matchings ([Gale and Shapley, 1962](#); [Balinski and Sönmez, 1999](#)). Furthermore, the student-proposing DA is *strategy-proof*: revealing preferences truthfully is a weakly dominant strategy for each student ([Dubins and Freedman, 1981](#); [Roth, 1982](#)). But there is a serious drawback of the SOSM: it is not *Pareto efficient* ([Balinski and Sönmez, 1999](#); [Abdulkadiroğlu and Sönmez, 2003](#)). Moreover, this inefficiency can be quite severe ([Kesten, 2010](#)) and there is empirical support for this insight: in NYC high school match, possible welfare gains over the SOSM is significant ([Abdulkadiroğlu et al., 2009](#)). The inefficiency of the SOSM is actually due to a deeper issue: stability and efficiency are incompatible ([Gale and Shapley, 1962](#); [Roth, 1982](#); [Balinski and Sönmez, 1999](#)).

A possible remedy for the SOSM’s inefficiency is to relax stability. One alternative in this direction is to focus on efficiency via the Top Trading Cycles (TTC) rule, which always gives an efficient matching ([Abdulkadiroğlu and Sönmez, 2003](#); [Hakimov and Kesten, 2018](#); [Morrill, 2013, 2015](#)). Another alternative is to weaken the stability notion. Such a weakening is *reasonable stability*: a matching is *reasonably stable* if whenever a student i ’s priority is violated at school s then there does not exist a stable matching in which i is assigned to s .³ Another weakening is α -*equitability*: a matching with a priority violation is not deemed as unfair if a student’s objection to that priority violation is counter-objectioned by another student ([Alcade and Romero-Medina, 2017](#)). [Alva and Manjunath \(2019\)](#) define another weakening of stability: a rule is *stable-dominating* if it selects an allocation that Pareto-improves *some* stable allocation at every preference profile. They show that the SOSM is the only *stable-dominating* and *strategy-proof* rule. [Troyan et al. \(2016\)](#) and [Morrill \(2016\)](#) focus on the alternative ways to improve efficiency by relaxing the fairness constraint. In particular, under [Morrill \(2016\)](#)’s definition of fairness, a student i ’s priority at school s needs to be respected only if there exists a “legitimate” matching in which i is assigned to s . [Morrill \(2016\)](#) introduces an iterative procedure to find the set of “legitimate” matchings. He shows that there is a unique legitimate set of matchings and a unique *Pareto efficient*

³This notion is first discussed in the working paper version of [Kesten \(2010\)](#).

and legitimate matching which corresponds to EADAM’s outcome when all students consent. According to [Troyan et al. \(2016\)](#), a matching is *essentially stable* if any objection of student i to her priority violation at school s initiates a rejection chain that results in her rejection from s . They show that the EADAM produces an *essentially stable* matching.

When students consent for the violation of their priorities, at the SOSM, students assigned to certain schools cannot be made better off, and the EADAM can be redefined by taking these schools into account ([Tang and Yu, 2014](#)). Moreover, the outcome of the EADAM is supported as the *strong Nash Equilibrium* of the preference revelation game under the DA ([Bando, 2014](#)). In the affirmative action context, a variant of the EADAM has recently been proposed as a *minimally responsive* rule (that is, a rule such that changing the affirmative action parameter in favor of the minorities never results in a matching that makes each minority student weakly worse off) ([Doğan, 2016](#)). Regarding the interpretation where certain priorities can be violated, one example is the NYC school match: motivated by the observation that the efficiency loss under the SOSM is significant, school districts have been considering allowing such violations anywhere but exam schools ([Abdulkadiroğlu, 2011](#)). Another example of this approach can also be seen in [Afacan et al. \(2017\)](#).⁴

The idea that possible welfare gains can be captured by improvement cycles is first proposed by [Erdil and Ergin \(2008\)](#) in the context of coarse priorities of schools. This idea inspired the rules proposed in some other works ([Ehlers et al., 2014](#); [Abdulkadiroğlu, 2011](#)) and in the current work as well.

2 The Model

We first present the standard school choice problem, and then introduce the extended model with allowable priority violations.

2.1 School Choice Problem

A **school choice problem** consists of the following elements:

⁴They allow certain priority violations based on the information that the parents will not appeal these violations. The authors introduce the *Efficiency-Corrected Deferred Acceptance Mechanism (ECDA)* algorithm, which finds a *constrained efficient sticky stable* matching. Besides informational issues, our paper differs from theirs in two major aspects. First, we introduce a class of rules selecting *all constrained efficient* matchings and thus provide a full characterization result. Second, we show that there exists a *unique* rule in this class that satisfies desired incentive properties.

- a finite set of students $I = \{i_1, i_2, \dots, i_n\}$;
- a finite set of schools $S = \{s_1, s_2, \dots, s_m\}$;
- a strict priority profile of schools $\succ = (\succ_s)_{s \in S}$ where \succ_s is the complete priority order of school s over I ;⁵
- a capacity vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats at school s ;
- a strict preference profile of students $P = (P_i)_{i \in I}$ such that P_i is student i 's strict preferences over $S \cup \{\emptyset\}$, where \emptyset stands for the option of being unassigned with $q_\emptyset = |I|$.

Let R_i denote the *at-least-as-good-as* preference relation associated with P_i , that is: $s R_i s'$ if and only if $s P_i s'$ or $s = s'$. Let $R = (R_i)_{i \in I}$ denote the weak preference profile of all students.

A **matching** $\mu : I \rightarrow S \cup \{\emptyset\}$ is a function such that for each $s \in S$, $|\mu^{-1}(s)| \leq q_s$. A matching μ **violates the priority** of student $i \in I$ at school $s \in S$ if there exists another $j \in I$ such that $\mu(j) = s$, $s P_i \mu(i)$, and $i \succ_s j$. A matching μ is **fair** if for each $i \in I$ and $s \in S$, it does not violate the priority of student i at school s . A matching μ is **individually rational** if for each $i \in I$, $\mu(i) R_i \emptyset$. A matching μ is **non-wasteful** if there do not exist a student $i \in I$ and a school $s \in S$ such that $s P_i \mu(i)$ and $|\mu^{-1}(s)| < q_s$. A matching μ is **stable** if it is fair, individually rational, and non-wasteful.

A matching μ **weakly Pareto dominates** matching μ' if for each $i \in I$, $\mu(i) R_i \mu'(i)$. A matching μ **Pareto dominates** μ' if μ weakly Pareto dominates μ' , and, $\mu(j) P_j \mu'(j)$ for some $j \in I$. A matching μ is **Pareto efficient** if it is not Pareto dominated by another matching μ' . A stable matching μ is **student optimal stable matching** (SOSM) if it is not Pareto dominated by another stable matching.

2.2 School Choice Problem with Allowable Priority Violations

A **school choice problem with allowable priority violations** (or simply a **problem**) is a school choice problem where some priority violations are allowed. These violations are given by a correspondence $C : S \rightrightarrows I \times I$, where $(i, j) \in C(s)$ means that the priority of i at s is allowed to be violated by a student j . We require that $(i, j) \in C(s)$ only if $i \succ_s j$.⁶ We impose the following restriction on C throughout the paper.

⁵In Section 7, we allow priorities to be weak.

⁶This requirement follows directly from the definition of *priorities* (see Section 2.1).

Assumption 1. *If $(i, j) \in C(s)$, then for each $j' \in I$ such that $i \succ_s j' \succ_s j$, $(i, j') \in C(s)$.*

Assumption 1 implies that if the violation of priority of $i \in I$ at $s \in S$ by some students is allowable, then there is a *cutoff student* j with $i \succ_s j$ such that i 's priority at s can only be violated by the students whose priorities are weakly higher than j (and by definition, lower than i).⁷

Throughout the paper, we fix I, S, \succ and q . Thus, a problem is defined by (R, C) .⁸ A **rule** ψ is a systematic procedure that selects a matching for each problem. For problem (R, C) , we denote the matching selected by rule ψ with $\psi_{(R,C)}$ and the assignment of student i in $\psi_{(R,C)}$ with $\psi_{(R,C)}(i)$.

There are several interpretations of the correspondence C , such as those that consider them to be the set of priorities that the central authority can sacrifice, or those that come from the consenting decisions of students, such as in Kesten (2010). We discuss these interpretations in detail in Sections 4 and 5; yet, for now, we remain agnostic and give the most general treatment.

The addition of correspondence C relaxes the fairness constraint in the following manner: priorities for which violations are allowable (according to C) are not taken into account when considering fairness. By ignoring these priorities, one can define weaker notions of fairness and stability. A matching μ is **partially fair** if for each $i, j \in I$ and $s \in S$, $\mu(j) = s$, $s P_i \mu(i)$, and $i \succ_s j$ imply $(i, j) \in C(s)$. A matching μ is **partially stable** if it is *partially fair*, *individually rational*, and *non-wasteful*.

A matching μ is **constrained efficient** if it is *partially stable* and is not Pareto dominated by any other *partially stable* matching. A rule ψ is **constrained efficient** if it selects a *constrained efficient* matching for any problem.

The notion of *partial stability* stands as a compromise between efficiency and stability. Indeed, one can easily see how the interpolation between the two ends works by investigating the extremes. In one extreme, when no priority violation is allowable, the notion of *partial stability* collapses to that of stability and the only *constrained efficient* matching is the SOSM. In particular, the SOSM *Pareto dominates* any other *partially stable* matching and the set of *partially stable* matchings constitutes a lattice. In the other extreme, when any priority violation is allowable, each matching vacuously satisfies *partial fairness* and *constrained efficiency* is equivalent to *Pareto efficiency*. For some problems, there is no

⁷This assumption is necessary for our main results: Theorem 1 would not hold without it. In Appendix B, we give an example where Assumption 1 does not hold and Theorem 1 fails.

⁸Since the preference relation is strict, we can define the problem by (P, C) as well.

partially stable matching that *Pareto dominates* or *Pareto dominated* by any other *partially stable* matching. Thus, we do not observe lattice structure when the set of *partially stable* matchings are considered.⁹

3 The Student Exchange under Partial Fairness

We now present a class of algorithms to characterize the set of *constrained efficient* matchings that improve the students' welfare upon a *partially stable* matching.

3.1 The Algorithm

Given a *partially stable* matching μ , for each school $s \in S$, let

- $D_\mu(s) = \{i \in I : s P_i \mu(i)\}$ (the set of students who prefer s to their assignments under μ)
- $X_\mu(s) = \{i \in D_\mu(s) : \forall j \in [D_\mu(s) \setminus (\{k \in I : (k, i) \in C(s)\} \cup \{i\})], i \succ_s j\}$ (the set of students who are eligible for a *partially fair* exchange involving school s).¹⁰

Let $G = (V, E)$ be a directed graph with the set of vertices V , and the set of directed edges E , which is a set of ordered pairs of V .

For each matching μ , $G(\mu) = (I, E(\mu))$ is the **(directed) application graph** associated with μ where the set of directed edges $E(\mu) \subseteq I \times I$ is as follows: $ij \in E(\mu)$ (that is, i *points to* j) if and only if $s = \mu(j)$ and $i \in X_\mu(s)$.

A set of edges $\{i_1i_2, i_2i_3, \dots, i_ni_{n+1}\}$ is a **path** if the vertices i_1, i_2, \dots, i_{n+1} are distinct, and a **cycle** if the vertices i_1, i_2, \dots, i_n are distinct and $i_1 = i_{n+1}$. We say that a **cycle** $\phi = \{i_1i_2, i_2i_3, \dots, i_ni_1\}$ is **solved** when for each $ij \in \phi$, student i is assigned to $\mu(j)$ to obtain a new matching. Formally, we denote the *solution of a cycle* $\phi = \{i_1i_2, i_2i_3, \dots, i_ni_1\}$ by the operation \circ ; that is, $\eta = \phi \circ \mu$ if and only if for each $ij \in \phi$, $\eta(i) = \mu(j)$, and for each $i' \notin \{i_1, i_2, \dots, i_n\}$, $\eta(i') = \mu(i')$. The following algorithm is built on a *partially stable* matching and is defined by *solving* cycles iteratively:

⁹One can easily see this point by considering an example composed of one school with one seat and two students.

¹⁰Note that this set is always well-defined. In particular, whenever $D_\mu(s)$ is nonempty, the student with the highest priority for s among the students in $D_\mu(s)$ is always in $X_\mu(s)$. (See Lemma 3 in Appendix C for a formal treatment.) Moreover, for any $i, j \in D_\mu(s)$ with $i \succ_s j$, if $j \in X_\mu(s)$, then i is also in $X_\mu(s)$ by the virtue of Assumption 1.

The Student Exchange under Partial Fairness (SEPF) Algorithm:¹¹

Step 0 Let μ_0 be a *partially stable* matching.

Step $k \geq 1$ Given a matching μ_{k-1} ,

(k.1) if there is no cycle in $G(\mu_{k-1})$, then the algorithm terminates and μ_{k-1} is the matching obtained;

(k.2) otherwise, *solve* one of the cycles in $G(\mu_{k-1})$, say ϕ_k , and let $\mu_k = \phi_k \circ \mu_{k-1}$.

The algorithm terminates in a finite number of steps due to finite number of students and schools and strict preferences. Moreover, the maximum number of possible steps is $0.5|I|(|S| - 1)$.¹² Since the SEPF involves two selections which are not predetermined, it is a class of algorithms: (1) We only require μ_0 to be *partially stable*, and impose no other restrictions on it, and (2) at each step of the algorithm, we require “one of the cycles” to be *solved* without specifying which one.

One obtains (possibly) different outcomes by choosing different initial *partially stable* matchings and different cycles at each step of the algorithm.¹³ Two questions follow immediately: which initial *partially stable* matching and which cycle selection procedure should be used? We provide a detailed answer to the second question in detail in Section 4. For the initial *partially stable* matching, there is a strong argument for choosing the SOSM: Many school districts use the student-proposing DA. Thus, the SOSM is the status quo outcome and school districts might be constrained by giving each student at least what she gets under the status quo. For this reason, we focus on algorithms within the SEPF class starting with the SOSM and present the results by using the SOSM as the initial *partially stable* matching.

3.2 A Characterization Result

Our first result establishes the relationship between *constrained efficiency* and the SEPF class: For any problem, if one starts with the SOSM, each matching obtained by an algorithm within the SEPF class is *constrained efficient* and *weakly Pareto dominates* the SOSM.

¹¹Example 2 in Appendix J demonstrates how the algorithm works.

¹²This is because there can be at most $|I|(|S| - 1)$ improvements in a problem, and at each step a cycle is *solved*, implying that there are at least two improvements obtained in each step.

¹³The outcome of the SEPF is *a set of* matchings. In Example 2 in Appendix J, there are two matchings obtained by algorithms within the SEPF class.

Moreover, each *constrained efficient* matching that *weakly Pareto dominates* the SOSM is attainable through some algorithm within the SEPF which starts with the SOSM.

Let $\Psi(R, C)$ denote the set of all *constrained efficient* matchings that *weakly Pareto dominate* the SOSM with respect to the problem (R, C) . Let $\Pi(R, C)$ denote the set of all matchings that can be obtained via an algorithm in the SEPF class starting with the SOSM under (R, C) .

Theorem 1. *For each problem (R, C) , a matching is constrained efficient and weakly Pareto dominates the SOSM if and only if it is obtained by an algorithm within the SEPF class starting with the SOSM. That is, $\Psi(R, C) = \Pi(R, C)$.*

Proof. See Appendix C.¹⁴ □

This result states that a group of reasonable normative properties characterize the set of the SEPF outcomes. An important remark is that these properties include a restriction: the characterization concerns only the set of *constrained efficient* matchings that *weakly Pareto dominate* the SOSM. In general, there may exist *constrained efficient* matchings that do not *weakly Pareto dominate* the SOSM, and the SEPF starting with the SOSM does not find such matchings. Thus, a relevant question is whether this restriction is at the expense of an important property. To answer it, we remove this restriction in Section 6 and seek *constrained efficient* rules, whereas additionally imposing incentive compatibility. But, it turns out that *constrained efficiency* and incentive compatibility are incompatible (Theorem 3).

Theorem 1 is a general result without any reference on the nature of allowable priority violations. While there might be different interpretations of allowable priority violations, our characterization is a starting point to obtain strong results for the model where priority violations are only possible with students' consent. We analyze this model next.¹⁵

4 Priority Violation via Students' Consent

An example where the notion of *partial fairness* is pertinent is the model where priorities are violated as a result of consenting decisions of students, as in Kesten (2010). Before making observations about this interpretation, we emphasize the relationship between the SEPF

¹⁴In Appendix C, we provide the proof for the general case in which μ_0 can be any *partially stable* matching.

¹⁵In Section 5, we discuss another interpretation of our model inspired by a recent change in a school choice policy: we show that our approach can be adopted for the case where a school district is willing to violate certain types of priorities.

and **Efficiency Adjusted Deferred Acceptance Mechanism (EADAM)** introduced by [Kesten \(2010\)](#).

4.1 The SEPF and EADAM

The motivation behind the EADAM is to explore the source of inefficiency of the SOSM due to fairness constraints and improve it on the efficiency dimension. An important observation made by [Kesten \(2010\)](#) is that the priority of student i at school s might not help her to get a better school under the student-proposing DA at all. If this is the case, giving i the lowest priority at s instead of her current priority would not change her assignment and the DA would possibly select a matching that *Pareto dominates* the SOSM with the original priorities. Motivated by this observation, [Kesten \(2010\)](#) introduces the EADAM in a setting that allows students to *consent* for the violation of their own priorities. This would correspond to student i declaring the violation of her priority at school s by some other student as allowable. That is, there is a clear connection between our setup and the one considered in [Kesten \(2010\)](#). Yet, the setup considered in [Kesten \(2010\)](#) is more restrictive than ours: It assumes that a student can either consent for priority violation by each student at each school, or not consent for any violation by any student at any school. To capture this difference, we define the following notion.

Definition 1. *Correspondence C satisfies **all-or-nothing property** if, for all $i \in I$, only one of the following is true:*

- $(i, j) \in C(s)$ for each $s \in S$ and $j \in I$ such that $i \succ_s j$, or,
- $(i, j) \notin C(s)$ for each $s \in S$ and $j \in I$.

The all-or-nothing property is used only in Section 4.1 in order to establish the connection between the SEPF class and the EADAM. The property is crucial for the EADAM to be well-defined under our setup. Since all-or-nothing property merely puts a restriction on the correspondence C , the algorithms in the SEPF class would still be well-defined. The following result is immediate from Theorem 1 and from Theorem 1 of [Tang and Yu \(2014\)](#).

Proposition 1. *For each problem (R, C) where C satisfies all-or-nothing property, the matching obtained by the EADAM can also be obtained by an algorithm in the SEPF class starting with the SOSM.*

Each particular algorithm in the SEPF class (starting with the SOSM) defines a *constrained efficient* rule that *Pareto dominates* the SOSM (we call these rules as the **class of SEPF rules**), and the EADAM is a rule in this class (Proposition 1). It turns out that the EADAM

is a special member of the class of SEPF rules: it defines a *unique* rule that satisfies a desirable incentive compatibility property related to the consenting interpretation. Before we state this result, we provide some insights.

4.2 The Concept of Underdemanded Schools

At each step of an algorithm in the SEPF class, only the welfare of students in the *solved* cycle improves. Clearly, if a student’s current school is not demanded by other students, then she is not part of any cycle and a welfare improvement is not possible for her. To formalize this idea, we introduce the concept of *underdemanded* schools.¹⁶

A school s has **no demand** at μ if there does not exist a student i who prefers s to $\mu(i)$. A school s is **underdemanded** at μ if it has *no demand* at μ or if no student on a path to a student in $\mu^{-1}(s)$ is part of a cycle in the graph $G(\mu)$.

Let (R, C) be a problem and ψ be a rule in the SEPF class. Assume that the underlying SEPF algorithm terminates at Step K and the outcome is the matching μ_{K-1} ; that is, $\psi_{(R,C)} = \mu_{K-1}$. Take a step $k \leq K - 1$. If student i is not pointed to by another student in the graph $G(\mu_k)$, then school $\mu_k(i)$ has *no demand* at $\mu_{k'}$ for any $k' \geq k$.¹⁷ Consequently, a student assigned to an *underdemanded* school in the matching selected at Step k is not part of any cycle at any step $k' \geq k$. Consequently, $\psi_{(R,C)}(i) = \mu_k(i)$.¹⁸ Consistent with this observation, we say that a student is **permanently matched at μ** if she is assigned to an *underdemanded* school at μ , and a student is **temporarily matched at μ** if she is not *permanently matched*.

4.3 Incentives to Consent

A student’s decision to consent (or not consent) to priority violations is a strategic one. Consequently, the main issue for a rule based on the idea of consent is whether students have incentives for *consenting*. The mechanism designer would like to give incentives to the students to consent, because each additional consent relaxes the *partial fairness* constraint and provides the mechanism designer with the opportunity of choosing a matching closer to

¹⁶See also Kesten and Kurino (2013) and Tang and Yu (2014) for a discussion of the same concept. The terminology in Tang and Yu (2014) is different: the authors refer to a school with *no demand* as a **tier-0 underdemanded** school and an *underdemanded* school as a **tier-k underdemanded** school for $k > 0$.

¹⁷This easily follows from Remark 3 in Appendix C.

¹⁸See Lemma 8 in Appendix E for a formal statement and proof.

the efficiency frontier. Consequently, one should look for rules that guarantee a student will not be made worse off if she consents.

Definition 2. A rule ψ **provides incentives to consent** if for any problem, for each $i \in I$ and $s \in S$, student i who consents for s does not get a better assignment by not consenting for s . That is, take any (R, C) and define C' as $C'(s) = C(s) \setminus \cup_{j \in I} \{(i, j)\}$ and $C'(s') = C(s')$ for each $s' \in S \setminus \{s\}$. A rule ψ provides incentives to consent if $\psi_{(R, C)}(i) R_i \psi_{(R, C')}(i)$ for each $i \in I$ and $s \in S$.¹⁹

Our goal is to search for rules that *provide incentives to consent* and always yield *constrained efficient* outcomes that *weakly Pareto dominate* the SOSM. Our characterization (Theorem 1) reduces this search to the class of SEPF rules. But, a rule within the SEPF class may not *provide incentives to consent*.²⁰ Next, we introduce a rule, which *provides incentives to consent*, and it turns out that it is the unique such rule in the SEPF class. We refer to this rule as the Top Priority rule.

4.4 The Top Priority Rule

The Top Priority (TP) rule is a member of the class of SEPF rules, thus its underlying algorithm corresponds to a specific cycle selection procedure.²¹ In order to construct this cycle selection procedure, we introduce some definitions.

For each matching μ , let $G^T(\mu) = (I, E^T(\mu))$ be the **Top Priority graph** associated with μ where the set of directed edges $E^T(\mu)$ is defined as follows: $ij \in E^T(\mu)$ if and only if student i has the highest priority for $\mu(j)$ among the students who are *temporarily matched* at μ and point to j in $G(\mu)$.

The **Top Priority (TP) Algorithm**:²²

Step 0 Let μ_0 be the SOSM.²³

Step $k \geq 1$ Given a matching μ_{k-1} ,

¹⁹One can easily extend the definition by constructing $C'(s)$ by only removing a student pair (i, j) from $C(s)$. All our results are robust to this extension.

²⁰In Example 2 in Appendix J, student i_2 is better off by not consenting for s_3 under any rule selecting matching μ' .

²¹Note that the class of SEPF rules is defined such that the SOSM is the initial *partially stable* matching (Section 4.1).

²²Example 3 in Appendix J demonstrates how the TP algorithm works.

²³The TP algorithm can be defined with any *stable* matching as the initial matching. All our results are robust to this extension.

(k.1) if there is no cycle in $G(\mu_{k-1})$, then the algorithm terminates and μ_{k-1} is the matching obtained;

(k.2) otherwise, *solve* one of the cycles in $G^T(\mu_{k-1})$, say ϕ_k , and let $\mu_k = \phi_k \circ \mu_{k-1}$.

By definition, if i points to j in $G^T(\mu)$, then i points to j in $G(\mu)$ as well. This implies the following remark.

Remark 1. *Any cycle that appears in $G^T(\mu)$ also appears in $G(\mu)$.*

The following lemma states that if there is a cycle in $G(\mu)$, there is a cycle in $G^T(\mu)$ as well.

Lemma 1. *Let μ be a partially stable matching. There is a cycle in the graph $G(\mu)$ if and only if there is a cycle in the graph $G^T(\mu)$.*

Proof. See Appendix D. □

Remark 1 and Lemma 1 imply that the TP algorithm is in the SEPF class. Consequently, the TP algorithm is nothing more than a cycle selection procedure within the class of SEPF rules. The only restriction we impose on the cycle selection procedure is that it selects a cycle in $G^T(\mu_k)$. This raises a question: Does it matter which cycle in $G^T(\mu_k)$ is chosen? We demonstrate that the outcome of the TP algorithm does not depend on the order of cycles *solved*. Thus, the TP algorithm defines a *rule*.

Proposition 2. *The TP algorithm defines a rule in the class of SEPF rules.*

Proof. See Appendix E. □

Actually, the TP rule stands out: it is the *unique* SEPF rule that provides *incentives to consent*.

Theorem 2. *A rule is constrained efficient, provides incentives to consent, and improves the SOSM if and only if it is the TP rule.*

Proof. See Appendix F. □

Recall that the EADAM satisfies the following property: for any student, *consenting* for all schools does not hurt any student (this follows from Proposition 3 by Kesten (2010)). This implies that when C satisfies all-or-nothing property, the EADAM is outcome-equivalent to the TP rule: they select the same outcome for the same preference and consent profile. The following important implication of this result is immediate: in Kesten (2010)'s setup, one cannot do better than the EADAM without sacrificing *incentives to consent*.

Alternatively, one can interpret *incentives to consent* as a protection from objections by the students due to priority violation. In particular, if a student objects to a priority violation under a rule that *provides incentives to consent*, then she cannot be assigned to a better school when her priorities are respected under the same rule. Thus, even if the students are not asked to consent (the case we consider in the next section), *incentives to consent* can be considered as a desirable feature.

5 Violating Certain Types of Priorities

We now discuss another interpretation of C . In particular, we show that our approach can be adopted for the case where a school district allows certain types of priority violations, and compare our approach to an alternative where certain priorities are removed by school districts.

The school districts are using priority classes when they rank the students. For instance, the Boston Public Schools System (BPSS) until recently used four priority classes: Sibling+Walk-Zone, Sibling, Walk-Zone, and Others. Each school was giving the highest priority to the students in the Sibling+Walk-Zone priority class, the second highest priority to the students in the Sibling priority class, and third highest priority to the students in the Walk-Zone priority class. The ranks of the students within each priority class were determined according to a random draw. Recently, the BPSS decided to suppress walk-zone priorities and now uses only two priority classes: Sibling and Others (Dur et al., 2018). This demonstrates that although walk-zone priority is important for the BPSS, there is enough justification for the removal of this priority class. The school choice problem with allowable priority violations captures this phenomenon: the acquiescence to violate walk-zone priorities can be modelled as (i, j) being in $C(s)$ if and only if $i \succ_s j$ and either i has walk-zone priority at school s or i has sibling+walk-zone priority and j has sibling priority at s .²⁴ Thus, instead of removing a priority class, the mechanism designer may consider modelling these priority violations as allowable and using a rule within the SEPF class.

An important clarification needs to be made at this point. Our motivation for proposing the SEPF class is to improve the SOSM in terms of efficiency. On the other hand, a school district’s motivation in suppressing certain priority classes is **not** necessarily to improve efficiency, and our proposal does not capture other concerns behind suppressing a priority class. Yet, our purpose here is to point out a side effect of this policy: *suppressing priorities may*

²⁴ $C(s)$ obtained in this manner satisfies Assumption 1.

have *perverse efficiency implications*. Below, we compare the SEPF to the approach of suppressing some priority classes on the efficiency dimension alone to point out the unintended consequences of the latter approach.

5.1 Suppressing a Priority Class vs Allowing Violation of Priorities

One might think that when a priority class is suppressed, it reduces the number of rejection chains (the source of inefficiency of the SOSM), and leads to efficiency gains. Example 1 indicates this is not always true.

Example 1. *There are two schools, $S = \{a, b\}$, and three students $I = \{i, j, k\}$. Each school has one available seat. The preferences of students are: $b P_i a P_i \emptyset$, $a P_j b P_j \emptyset$, $a P_k b P_k \emptyset$. Each school uses four priority classes: Sibling+Walk-Zone, Sibling, Walk-Zone, and Other. Student i has sibling priority at school a and walk-zone priority at school b . Student j and k belong to the “other” priority class for both schools. Suppose the random draw favors j most and i least. In this problem, the SOSM assigns i to b and j to a . But, when the walk-zone priority is suppressed, the SOSM assigns i to a and j to b without changing the assignment of k .*

Example 1 shows that suppressing walk-zone priorities might result in a *stable* (with respect to relaxed priorities) matching which is *Pareto inferior* to the SOSM (under the original priorities). Moreover, this approach might weaken the fairness aspect of the outcome (with respect to the suppressed priorities) without improving it in terms of efficiency and yield an undesirable allocation (with respect to the balance between efficiency improvements, fairness, and other possible concerns). On the other hand, this balance is already embedded in the class of SEPF rules, since each rule in this class always yields *stable* and *constrained efficient* matchings (with respect to relaxed priorities) which are *Pareto superior* to the SOSM, and walk-zone priorities are violated only if they lead to inefficiencies.

6 Manipulation Through Preference Revelation

We say that a rule is **strategy-proof** if, given a profile of allowable priority violations C , truthful revelation of preferences is a weakly dominant strategy for each $i \in I$. Unfortunately, there is no *strategy-proof* rule whose outcome *Pareto dominates* the SOSM (Kesten and

Kurino, 2013; Abdulkadiroğlu et al., 2009).²⁵ Since each rule within the SEPF class *Pareto improves* over the SOSM, the following is obtained as a corollary of this result.

Proposition 3. *There is no rule within the class of SEPF rules that satisfies strategy-proofness.*

This result, combined with Theorem 1, demonstrates that *constrained efficiency*, *weakly Pareto dominating* the SOSM, and *strategy-proofness* are incompatible. Given this incompatibility result, since *constrained efficiency* is an indispensable property in our model, the only possible way to gain *strategy-proofness* is to consider all the *constrained efficient* matchings instead of only the ones which *weakly Pareto dominate* the SOSM. The impossibility extends.

Theorem 3. *There is no strategy-proof and constrained efficient rule.*

Proof. See Appendix G. □

The notion of *strategy-proofness* used here requires truthful revelation to be a dominant strategy for every C . In the settings where C is exogenously given, one may wonder whether there is a characterization of C 's for which *strategy-proofness* and *constrained efficiency* are compatible. In Appendix H, we provide sufficient conditions that guarantee existence of *strategy-proof* and *constrained efficient* rules and propose such a rule.

7 The SEPF for Weak Priorities

School districts usually rank students using some predetermined criteria such as proximity and sibling status. Thus, many students end up being grouped under the same priority classes. Since the student-proposing DA is defined only under strict priority orders, school districts use exogenously determined tie-breaking rules to order students within priority classes. Unfortunately, exogenous tie-breaking causes efficiency loss (Erdil and Ergin, 2008; Abdulkadiroğlu et al., 2009; Kesten, 2010). A solution for this efficiency loss problem is the Stable Improvement Cycles (SIC) algorithm (Erdil and Ergin, 2008). The SIC algorithm takes the SOSM for a given tie-breaking rule and then improves the assignment by utilizing

²⁵Both papers consider the environments where schools have indifferences among students and some tie-breaking rule is needed. Nevertheless, our setup can easily be considered as a special case where each indifference class consists of one student only. Kesten and Kurino (2013)'s result is more general than that of Abdulkadiroğlu et al. (2009), as it does not require the existence of a null school. Since we allow for the null school, both theorems are applicable in our setup. In a recent paper Alva and Manjunath (2019) provides a stronger result such that student-proposing DA is the unique strategy-proof rule that (weakly) Pareto improves any stable matching.

trade cycles between students. The SIC algorithm defines a class of rules, since at each step there may exist more than one cycle and each cycle selection procedure possibly gives a different outcome. An alternative solution is a modified version of the EADAM (defined in Section V.D. of [Kesten \(2010\)](#)), which we denote as *the EADAM for weak priorities*. It turns out that there is a close connection between the SIC and the EADAM for weak priorities, which becomes apparent through their relations to the SEPF.

We extend our model by allowing each $s \in S$ to have a weak priority order denoted by \succsim_s . We denote the strict priority order of school s by \succ_s and the associated indifference relation by \sim_s . For this section, we assume that the set of allowable priority violations is not part of the problem (indeed, we construct this object). Thus, a problem is given by a preference profile R . All the other definitions remain the same as they are defined in Section 2.1. Given a problem R and a priority profile \succsim , let $f^{\succsim}(R)$ denote the set of student optimal stable matchings in the problem. If $i \sim_s j$, then by definition, the assignment of j to s while i prefers s to her assignment does not violate *fairness*. Consequently, $f^{\succsim}(R)$ may no longer be a singleton.²⁶ Moreover, the common method of using a single tie-breaker, generating a strict priority profile \succ' and finding the SOSM according to \succ' may not yield a matching in $f^{\succsim}(R)$. That is, letting $\mathcal{T}(\succsim)$ denote the set of strict priority profiles obtained by breaking the ties in \succsim ,²⁷ we have $f^{\succsim}(R) \subseteq \bigcup_{\succ' \in \mathcal{T}(\succsim)} f^{\succ'}(R)$ where, as discussed in [Erdil and Ergin \(2008\)](#), the inclusion may be proper. This observation calls for an improvement of the SOSM with single tie-breaking, such that the end matching is in $f^{\succsim}(R)$.

Given a weak priority profile \succsim , let a strict priority profile \succ' be obtained by breaking the ties in \succsim . For each $i \in I$ and $s \in S$, construct the set of allowable priority violations under \succ' as follows:²⁸

$$(i, j) \in C(s) \text{ if and only if } j \in (\{j' : i \succ'_s j'\} \cap \{j' : i \sim_s j'\}) \quad \text{for each } j \in I. \quad (1)$$

Given the constructed set of allowable priority violations C , we can now start with the SOSM under \succ' and then select a rule within the SEPF class according to C . We refer to this procedure as the **SEPF for weak priorities**. Let $\Omega(R, \succ')$ denote the set of all matchings that can be obtained via the SEPF for weak priorities with a given \succ' according to this procedure.

The SEPF for weak priorities inherits the *constrained efficiency* of the SEPF class. Con-

²⁶For a given strict priority profile \succ , we define $f^{\succ}(R)$ similarly. It is worth mentioning that $f^{\succ}(R)$ is a singleton and it contains the outcome of the student-proposing DA rule under R and \succ .

²⁷Formally, $\mathcal{T}(\succsim)$ is the set of priority profiles such that $i \succ_s j$ implies $i \succ'_s j$ for all $i, j \in I$ and $s \in S$.

²⁸The set of allowable priority violations constructed in this manner satisfies Assumption 1.

sequently, an adaptation of the characterization result in Theorem 1 applies: for a given tie-breaking rule, the set of outcomes obtained by the SEPF for weak priorities is characterized by the set of student optimal stable matchings that *Pareto dominate* the SOSM obtained under the given tie-breaking rule.

Proposition 4. *Given R and \succsim , let $\succ' \in \mathcal{T}(\succsim)$ and $\mu_0 \in f^{\succ'}(R)$. A matching is student optimal stable and weakly Pareto dominates μ_0 if and only if it is obtained by the SEPF for weak priorities. That is, $\mu \in f^{\succ}(R)$ and Pareto dominates μ_0 if and only if $\mu \in \Omega(R, \succ')$.*

Proof. See Appendix I. □

It is easy to see that the SEPF for weak priorities and the SIC algorithm of Erdil and Ergin (2008) are outcome equivalent, i.e. for the same problem, the same tie-breaking rule, and the same cycle selection procedure they select the same matching. As a result, Proposition 4 provides a counterpart of Erdil and Ergin (2008)'s Theorem 1: not only does the SIC procedure always find a student optimal stable matching, but it also finds *all* the student optimal stable matchings for any given tie-breaking rule.

Another interesting side result of Proposition 4 is the connection between the SIC and EADAM for weak priorities: the EADAM for weak priorities is a member of the SIC algorithms class.

8 Conclusion

This study introduces the *school choice problem with allowable priority violations*. The main result is a characterization of a class of algorithms, the SEPF, each of which always yields a *constrained efficient matching weakly Pareto dominating* the SOSM. A relevant interpretation is the setup where school district officials ask for students' consent for the violation of their priorities (Kesten, 2010). Clearly, students having incentives to consent is indispensable in this setting. Our proposal, the TP rule, is the unique rule that satisfies this property within the SEPF class. Indeed, the mechanism designer can also attempt to provide more incentives. One such rule may be the one that favors consenting students in tie-breaking or gives the consenting students higher priorities when they apply to the school system for the higher grades. The characterization of such rules is left for future research.

Our framework also applies to settings where some priority violations are deemed feasible. We demonstrate that the SEPF class under such violations (marked as allowable) has superior welfare properties compared to suppressing these priorities. Another example for

allowable priority violations is a setting with centralized placement of students to exam and regular schools. Whereas the priorities to exam schools are legal constraints which cannot be violated, the regular schools are more flexible in terms of their priorities. One can then simply adopt the framework offered in Section 2.2 and specify that priority violations in private schools are allowed for each pair of students. Each cycle selection rule within the SEPF, including the “uniform cycle selection rule” which *solves* each cycle at each step with equal probabilities, is guaranteed to produce a *constrained efficient* matching in this case.

Appendices

A The EADAM

Before providing the definition of the EADAM, we first present a notion used in the definition. If student i is tentatively accepted by school s at some step t and is rejected by s in a later step t' of the student-proposing DA, and there exists another student j who is rejected by s in step $t'' \in \{t, t + 1, \dots, t' - 1\}$, then i is called an **interrupter** for s and (i, s) is called an **interrupting pair** of step t' .

Under EADAM, each student reports her preferences over the schools and her decision to consent or not. EADAM selects its outcome through the following algorithm:

Round 0: Run the student-proposing DA algorithm.

Round $k > 0$: Find the last step of the student-proposing DA run in Round $k - 1$ in which a consenting interrupter is rejected from the school for which she is an interrupter. Identify all the interrupting pairs of that step with consenting interrupters. For each identified interrupting pair (i, s) , remove s from the preferences of i without changing the relative order of the other schools. Rerun student-proposing DA algorithm with the updated preference profile. If there are no more consenting interrupters, then stop.

B Necessity of Assumption 1

We demonstrate a problem (R, C) where Assumption 1 does not hold and there are two *partially stable* matchings $\mu, \tilde{\mu}$ such that $\tilde{\mu}$ *Pareto dominates* μ , but there are no cycles in $G(\mu)$. Consequently, Theorem 1 fails: any algorithm in the SEPF class that starts with μ yields μ , despite μ not being *constrained efficient*.

Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, and $q_s = 1$ for each $s \in S$. Take any C such that $(i_1, i_3) \notin C(s_2)$, $(i_1, i_4) \in C(s_2)$, and $(i_3, i_4) \notin C(s_2)$. Let the students' preferences and schools' priorities be as follows:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	\succ_{s_1}	\succ_{s_2}	\succ_{s_3}	\succ_{s_4}
s_2	s_3	s_4	s_2	i_1	i_2	i_3	i_4
s_1	s_2	s_2	s_4	\vdots	i_1	i_2	i_3
\vdots	\vdots	s_3	\vdots	\vdots	i_3	\vdots	\vdots
\vdots	\vdots	\vdots	\vdots	\vdots	i_4	\vdots	\vdots

The matchings $\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4)\}$ and $\tilde{\mu} = \{(i_1, s_1), (i_2, s_3), (i_3, s_4), (i_4, s_2)\}$ are *partially stable*.²⁹ Also, $\tilde{\mu}$ *Pareto dominates* μ . But, since $X_\mu(s_1) = \emptyset$, $X_\mu(s_2) = \{i_1\}$, $X_\mu(s_3) = \{i_2\}$, and $X_\mu(s_4) = \{i_3\}$, there are no cycles in $G(\mu)$, and the algorithm terminates.

Since $i_3 \in D_\mu(s_2)$, $i_3 \succ_{s_2} i_4$ and $(i_3, i_4) \notin C(s_2)$ (that is, i_3 prevents i_4 from pointing to i_2 in $G(\mu)$), $i_4 \notin X_\mu(s_2)$. Similarly, i_3 is prevented from pointing to i_2 by i_1 . This would not occur if Assumption 1 was retained. Since $(i_1, i_3) \notin C(s_2)$ and $i_3 \succ_{s_2} i_4$, Assumption 1 requires that $(i_1, i_4) \notin C(s_2)$, and $\tilde{\mu}$ would not be *partially stable*.

C Proof of Theorem 1

The statement in the theorem is for the case where the initial matching μ_0 is the SOSM. Instead of this special case, we provide the proof of a more general result such that μ_0 can be any *partially stable* matching. That is, we show that for each problem (R, C) and for each *partially stable* matching μ_0 , a matching is *constrained efficient* and *weakly Pareto dominates* μ_0 if and only if it is obtained by an algorithm in the SEPF class starting with μ_0 .

Remark 2. *In the graph $G(\mu)$, if i points to j , then i points to each student in $\mu^{-1}(\mu(j))$.*

Remark 2 follows from a simple observation: i points to j if and only if $i \in X_\mu(\mu(j))$ and thus i also points to student i' if $\mu(i') = \mu(j)$.

For a given problem (R, C) and a *partially stable* initial matching μ_0 , consider an algorithm in the SEPF class. Let K be the last step of the algorithm and μ_k be the matching selected by the algorithm at Step $k \in \{1, \dots, K-1\}$. A cycle is *solved* at each step of the algorithm, which implies that the students in the cycle are better off and no student is worse off at the new matching obtained by solving the cycle. Thus, the matching at each step *Pareto dominates* the matching in the previous step. This implies that for a student i , if a school s is better than $\mu_k(i)$, then it is also better than $\mu_{k-1}(i)$.

Remark 3. *For each $k \geq 1$ and each $s \in S$, $D_{\mu_k}(s) \subseteq D_{\mu_{k-1}}(s)$.*

²⁹The matching μ is the SOSM and *partial stability* of μ follows immediately. For $\tilde{\mu}$, the only student who *envies* another student is i_1 . She prefers $s_2 = \tilde{\mu}(i_4)$ to $\tilde{\mu}(i_1)$, and $i_1 \succ_{s_2} i_4$. Nevertheless, $(i_1, i_4) \in C(s_2)$.

A consequence of Remark 3 is that, if i points to j in the graph $G(\mu_{k-1})$ and i is not better off at Step k , then in the graph $G(\mu_k)$, i points to the students who are assigned to school $\mu_{k-1}(j)$ at μ_k . In particular, if i points to j and both are not better off at a step of the algorithm, then i points to j in the next step as well. To see this, let cycle $\phi_k = \{i_1 i_2, i_2 i_3 \dots, i_n i_1\}$ be *solved* in the graph $G(\mu_{k-1})$ such that $\mu_k = \phi_k \circ \mu_{k-1}$. Suppose i points to j in the graph $G(\mu_{k-1})$ and $i, j \notin \{i_1, i_2, \dots, i_n\}$. By definition of the graph $G(\mu_{k-1})$, $i \in X_{\mu_{k-1}}(s)$ where $s = \mu_{k-1}(j)$. Since $\mu_k(i) = \mu_{k-1}(i)$, $i \in D_{\mu_k}(s)$. Let $i' \in D_{\mu_k}(s)$ be such that $i' \succ_s i$. By Remark 3, $i' \in D_{\mu_{k-1}}(s)$. Thus, since $i \in X_{\mu_{k-1}}(s)$ and $i' \succ_s i$, we have $(i', i) \in C(s)$. We conclude that for each student $\tilde{j} \in D_{\mu_k}(s)$ with a higher priority than student i at school s , $(\tilde{j}, i) \in C(s)$. Consequently, $i \in X_{\mu_k}(s)$. Since $s = \mu_k(j) = \mu_{k-1}(j)$, i points to j in the graph $G(\mu_k)$.

Remark 4. *If i points to j in $G(\mu_{k-1})$ and both students' assignments do not change at Step k , then i points to j in $G(\mu_k)$.*

C.1 Proof of the “if” part

Lemma 2. *Each matching obtained by an algorithm within the SEPF class is partially stable.*

Proof. (i) *Partial fairness.* Let μ_k be the matching at step $k \in \{0, 1, \dots, K-1\}$ of an algorithm in the SEPF class. We prove this statement by induction on k . The initial matching, μ_0 , is *partially fair*.

As an inductive hypothesis, suppose μ_{k-1} is *partially fair*. Take any student-school pair (i, s) such that $s P_i \mu_k(i)$. At each step of the algorithm, each student is either better off (she is in the *solved* cycle) or she is assigned to the same school as in the previous step. Thus, for each $\ell \in I$, $\mu_k(\ell) R_\ell \mu_{k-1}(\ell)$. Since $s P_i \mu_k(i)$, this implies that $s P_i \mu_{k-1}(i)$ and $i \in D_{\mu_{k-1}}(s)$. Take any $j \in \mu_k^{-1}(s)$ with $(i, j) \notin C(s)$. If $j \in \mu_{k-1}^{-1}(s)$, then by *partial fairness* of μ_{k-1} , $j \succ_s i$. Alternatively, suppose $j \notin \mu_{k-1}^{-1}(s)$. Since $j \in \mu_k^{-1}(s)$, j is in the cycle *solved* at Step k . Thus, $j \in X_{\mu_{k-1}}(s)$. By assumption, $(i, j) \notin C(s)$, thus, $i \in D_{\mu_{k-1}}(s) \setminus \{i' \in I : (i', j) \in C(s)\}$. Since $j \in X_{\mu_{k-1}}(s)$, we have $j \succ_s i$. Thus, μ_k does not violate the priority of i at s and it is *partially fair*. The induction follows.

(ii) *Individual rationality.* Since μ_0 is *individually rational* and each student is weakly better off at each step of the SEPF algorithm, its outcome is *individually rational*.

(iii) *Non-wastefulness.* The initial matching μ_0 is *non-wasteful* and at each step students are assigned to (weakly) better schools. By the definition of the SEPF algorithm, for each school s , the number of students assigned to s at each step remains the same as it is under

μ_0 . Thus, if $|\mu_0^{-1}(s)| = q_s$, then each matching obtained by the SEPF assigns q_s students to s . Suppose $|\mu_0^{-1}(s)| < q_s$. Since μ_0 is *non-wasteful*, the set $D_{\mu_0}(s)$ is empty. By Remark 3, at each step k , $D_{\mu_k}(s)$ is empty. Thus, each matching obtained by the SEPF satisfies *non-wastefulness*. \square

Lemma 3. *For each partially stable matching μ and $s \in S$, $X_\mu(s) = \emptyset$ if and only if $D_\mu(s) = \emptyset$.*

Proof. (Only if) Let $X_\mu(s) = \emptyset$. Suppose $D_\mu(s) \neq \emptyset$. By strict priorities, there exists i who has higher priority for s than any other student in $D_\mu(s)$. By definition, $i \in X_\mu(s)$; a contradiction. *(If)* It follows directly from the definition of the set $X_\mu(s)$. \square

Lemma 4. *Let μ and η be partially stable matchings such that η Pareto dominates μ . For each $s \in S$, $|\eta^{-1}(s)| = |\mu^{-1}(s)|$.*

Proof. Take a school $s \in S$. First, we show that $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$. Suppose $|\eta^{-1}(s)| > |\mu^{-1}(s)|$ and let $i \in \eta^{-1}(s) \setminus \mu^{-1}(s)$. Since η Pareto dominates μ and preferences are strict, this implies that $s P_i \mu(i)$. But since $|\mu^{-1}(s)| < q_s$, this violates *non-wastefulness* of μ . This implies that for each $s \in S$, $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$.

Second, we show that $|\eta^{-1}(s)| \geq |\mu^{-1}(s)|$. Suppose $|\eta^{-1}(s)| < |\mu^{-1}(s)|$. Adding together all schools and using the previous finding, we have: $\sum_{s \in S} |\eta^{-1}(s)| < \sum_{s \in S} |\mu^{-1}(s)|$. But, since η Pareto dominates μ and both matchings are *partially stable*, if a student is assigned to a school under μ , then she is also assigned to a school under η . This implies that $\sum_{s \in S} |\eta^{-1}(s)| \geq \sum_{s \in S} |\mu^{-1}(s)|$, a contradiction. \square

Lemma 5. *For a given problem (R, C) , a matching obtained by an algorithm within the SEPF class is constrained efficient.*

We use the following concept in the proof: Given $G = (V, E)$, a path $\{i_1 i_2, i_2 i_3, \dots, i_n i_{n+1}\}$ is a **chain** if for each $j \in V$, $j i_1 \notin E$ and $i_{n+1} j \notin E$, and vertex i_1 is called the **tail** of this chain.

Proof. Let μ be a matching obtained by an algorithm within the SEPF class. By Lemma 2, μ is *partially stable*. We show that there does not exist a *partially stable* matching that Pareto dominates μ . On the contrary, suppose a *partially stable* matching $\tilde{\mu}$ Pareto dominates μ . Note that, by definition of the SEPF class, there is no cycle in the graph $G(\mu)$. There are two possible cases.

Case 1: *There are no chains in $G(\mu)$.* Then, for each $s \in S$, $X_\mu(s) = \emptyset$. By Lemma 3, this implies that $D_\mu(s) = \emptyset$. Thus, at μ , each student is assigned to her best school and μ is *Pareto efficient*.

Case 2: *There is a chain in $G(\mu)$.* Let I_1 be the set of students who are not pointed to by any other student.³⁰ Let $i_1 \in I_1$ with $\mu(i_1) = s_1$. Since i_1 is not pointed to by any student, $X_\mu(s_1) = \emptyset$. Then, by Lemma 3, $D_\mu(s_1) = \emptyset$. Since $\tilde{\mu}$ *Pareto dominates* μ , there does not exist $i \in I$ such that $\mu(i) \neq s_1$ but $\tilde{\mu}(i) = s_1$. Thus, $\tilde{\mu}^{-1}(s_1) \subseteq \mu^{-1}(s_1)$. By Lemma 4, $|\tilde{\mu}^{-1}(s_1)| = |\mu^{-1}(s_1)|$. Thus, $\mu^{-1}(s_1) = \tilde{\mu}^{-1}(s_1)$. Since i_1 is chosen arbitrarily, this holds for each $s \in S$ such that $\mu^{-1}(s) \subseteq I_1$. Let S_1 denote the set of these schools. That is, for each $s \in S_1$, $\mu^{-1}(s) = \tilde{\mu}^{-1}(s)$.

There exists at least one student in $I \setminus I_1$ such that she is pointed to only by students in I_1 . Otherwise, there is a cycle in $G(\mu)$ or there are no chains in $G(\mu)$, a contradiction. Let I_2 be the set of such students and take some $i_2 \in I_2$, $s_2 = \mu(i_2)$. We first show that there does not exist $j \in I$ such that $\mu(j) \neq s_2$ but $\tilde{\mu}(j) = s_2$. To see why, assume the contrary. Since $\tilde{\mu}$ *Pareto dominates* μ , this implies that $s_2 P_j \mu(j)$ and thus $j \in D_\mu(s_2)$. Nevertheless, we must have $j \notin X_\mu(s_2)$. This is because otherwise $j \in I_1$ (recall that i_2 is pointed to only by students in I_1). Thus, by the previous paragraph, we must have $\mu(j) = \tilde{\mu}(j)$, a contradiction. We conclude that $j \notin X_\mu(s_2)$ and it implies that for some $j' \in D_\mu(s_2)$, $(j', j) \notin C(s_2)$ and $j' \succ_{s_2} j$. Let i be the student with highest priority for s_2 among such j' . Note that $i \in X_\mu(s_2)$,³¹ and since i_2 is pointed to only by students in I_1 , $i \in I_1$. Then, since i is assigned to the same school both under μ and $\tilde{\mu}$, the assignment of j to s_2 under matching $\tilde{\mu}$ violates the priority of i at s_2 , which contradicts with *partial fairness* of $\tilde{\mu}$. Thus, there does not exist a student j such that $\mu(j) \neq s_2$ but $\tilde{\mu}(j) = s_2$. Once again, by Lemma 4, $\mu^{-1}(s_2) = \tilde{\mu}^{-1}(s_2)$. Let S_2 denote the set of the schools such that for each $s \in S_2$, $\mu^{-1}(s) \subseteq I_2$.

Now we can continue in the same manner. If there is a student in $I \setminus (I_1 \cup I_2)$, who is pointed to by a student in $G(\mu)$, then at least one of them, say i_3 , is pointed to only by students in $I_1 \cup I_2$. By the same argument, the same students are assigned to $\mu(i_3)$ under both μ and $\tilde{\mu}$.

Repeating the argument once more, we conclude that all students in a chain in $G(\mu)$ have the same assignment under μ and $\tilde{\mu}$. The students who are not in a chain in $G(\mu)$, i.e. the ones who are not pointed to by a student and who are not pointing to a student, are contained in

³⁰Note that, by definition, I_1 contains all the students who are at the tail of some chain in $G(\mu)$.

³¹Otherwise, there must be another student $i' \in D_\mu(s_2)$, $(i', i) \notin C(s_2)$ and $i' \succ_{s_2} i$. But by Assumption 1, $(i', i) \notin C(s_2)$ and $i \succ_{s_2} j$ implies that $(i', j) \notin C(s_2)$. This would contradict with i being the student with highest priority for s_2 among $j' \in D_\mu(s_2)$ with $(j', j) \notin C(s_2)$ and $j' \succ_{s_2} j$.

I_1 ; and, as argued above, they have the same assignment under μ and $\tilde{\mu}$ as well. Thus, the matchings μ and $\tilde{\mu}$ coincide for all students. We conclude that $\mu = \tilde{\mu}$ and $\tilde{\mu}$ cannot *Pareto dominate* μ . \square

Since the matching obtained at each step of an algorithm within the SEPF class *improves* the matching in the previous step, it *improves* over μ_0 . This completes the proof of the “if” part of the theorem.

C.2 Proof of the “only if” part

Let μ_0 be a *partially stable* matching. We prove that each *constrained efficient* matching that *Pareto dominates* μ_0 can be obtained by an algorithm within the SEPF class. We first introduce an auxiliary notion called **improvement cycle**.

Definition 3. An **improvement cycle** ϕ over a matching μ is a cycle such that for each $ij \in \phi$, $\mu(j) P_i \mu(i)$.

The following lemma is the crucial first step in the construction of a cycle which appears in the graph used by the algorithms in the SEPF class.

Lemma 6. Let μ and η be partially stable matchings such that η Pareto dominates μ . Then, there exists a set of disjoint improvement cycles $\Phi = \{\phi_1, \dots, \phi_m\}$ such that $\eta = \phi_m \circ \dots \circ \phi_1 \circ \mu$.

Proof. Let $N \subseteq I$ be the set of students who are strictly better off under η . Let $\tilde{G}(\mu, \eta) = (N, E)$ be a directed graph where the edges $E \subseteq N \times N$ are such that each $i \in N$ points to a unique student in $N \cap \mu^{-1}(\eta(i))$, and each student in N is pointed to by a unique student.

We claim that such a graph $\tilde{G}(\mu, \eta)$ can be constructed. By Lemma 4, for each s , $|\mu^{-1}(s)| = |\eta^{-1}(s)|$. Thus, $\mu^{-1}(s) \neq \eta^{-1}(s)$ implies $|\mu^{-1}(s) \setminus \eta^{-1}(s)| = |\eta^{-1}(s) \setminus \mu^{-1}(s)|$. Moreover, each $i \in \mu^{-1}(s) \setminus \eta^{-1}(s)$ is pointed to by one of the students in $\eta^{-1}(s) \setminus \mu^{-1}(s)$. Thus, one can construct the graph $\tilde{G}(\mu, \eta)$.³²

Since each student in N is pointed to by a unique student and points to a unique student in N , each student is in a cycle and no two cycles intersect. By construction, each of these disjoint cycles is an *improvement cycle* over μ , and the matching η is obtained by *solving* these cycles in any order. \square

³²Clearly, there are multiple graphs that one can construct following this method, but any of these graphs are sufficient to demonstrate the existence of a set of improvement cycles.

An improvement cycle does not necessarily appear in the graph constructed by the SEPF algorithm. To complete our proof, we prove a result similar to Lemma 6 for the cycles that appear in the graph constructed by the SEPF at each step.

Lemma 7. *Let μ and η be partially stable matchings such that η Pareto dominates μ . Then, there exists a sequence of cycles $(\gamma_1, \dots, \gamma_n)$ such that:*

- γ_1 appear in $G(\mu)$;
- for each $k \in \{2, \dots, n\}$, γ_k appears in $G(\gamma_{k-1} \circ \dots \circ \gamma_1 \circ \mu)$;
- $\gamma_n \circ \dots \circ \gamma_1 \circ \mu = \eta$.

Proof. By Lemma 6, one can construct a set of improvement cycles $\Phi = \{\phi_1, \dots, \phi_m\}$. The result is trivial for the case where all the cycles in Φ appear in $G(\mu)$: it follows that there are disjoint cycles in $G(\mu)$, and solving them in any order leads to η . To prove the alternative case, we assume that none of the cycles in Φ appears in $G(\mu)$. This assumption is without loss of generality because of the following observation: If a cycle $\phi \in \Phi$ appears in $G(\mu)$, then this cycle is *solved* first and $\mu' = \phi \circ \mu$ is obtained. If another cycle $\phi' \in \Phi$ also appears in $G(\mu)$, by the fact that all cycles in Φ are disjoint and by Remark 4, it also appears in $G(\mu')$. Thus, we can *solve* ϕ' and obtain $\mu'' = \phi' \circ \mu'$. Following this logic, whenever a subset of cycles in Φ appear in $G(\mu)$, these cycles are *solved* first. Consequently, we focus on the case where none of the *improvement cycles* appear in $G(\mu)$.

To show the existence of a cycle in $G(\mu)$, first we prove that for any $\phi \in \Phi$ and any $ij \in \phi$, there exists some $k \in I$ such that $kj \in G(\mu)$ and $lk \in \phi'$ for some $\ell \in I, \phi' \in \Phi$. Take any $\phi \in \Phi$ and $ij \in \phi$.

- If $i \in X_\mu(\mu(j))$, then this edge also appears on $G(\mu)$ by construction. Moreover, i is a part of ϕ , which implies that there exists some $\ell \in I$ with $li \in \phi$.
- If $i \notin X_\mu(\mu(j))$, there exists a student i' such that $i' \in D_\mu(\mu(j))$, $(i', i) \notin C(\mu(j))$ and $i' \succ_{\mu(j)} i$. Let k be the student with the highest priority for the school $\mu(j)$ among such students. Note that $k \in X_\mu(\eta(i))$,³³ and thus $kj \in G(\mu)$. Moreover, we claim k is in an *improvement cycle* in Φ ; that is, there exists $\phi' \in \Phi$ such that $lk \in \phi'$ for some $\ell \in I$. If $\eta(k) = \mu(k)$, then, since $\mu(j) P_k \eta(k)$, $(k, i) \notin C(\mu(j))$ and $k \succ_{\mu(j)} i$, there is a priority violation at $\eta(i)$, which is not allowed, contradicting *partial stability*

³³Note the subtle role played by Assumption 1 here. If $k \notin X_\mu(\eta(i))$, then there must be another student $\ell \in I$ such that $\ell \in D_\mu(\mu(j))$, $(\ell, k) \notin C(\mu(j))$ and $\ell \succ_{\mu(j)} k$. But by Assumption 1, $(\ell, k) \notin C(\mu(j))$ and $k \succ_{\mu(j)} i$ imply that $(\ell, i) \notin C(\mu(j))$. This contradicts with i being the student with the highest priority for $\mu(j)$ among $i' \in D_\mu(\mu(j))$ with $(i', i) \notin C(\mu(j))$ and $i' \succ_{\mu(j)} i$.

of η . Thus, $\eta(k) P_k \mu(k)$, which implies that k is in an *improvement cycle* in Φ . Consequently, $\ell k \in \phi'$ for some $\ell \in I, \phi' \in \Phi$.

Thus, for any student j who is in an improvement cycle $\phi \in \Phi$, there exists another student k such that $kj \in G(\mu)$ and k is in an improvement cycle $\phi' \in \Phi$. But since the set of students in improvement cycles is finite, and each student is pointed to by another student in this set, there exists a cycle denoted by γ_1 , in $G(\mu)$.

We next show that the matching $\gamma_1 \circ \mu$ *Pareto dominates* μ and is (*weakly*) *Pareto dominated* by η . For any $kj \in \gamma_1$, note that $(\gamma_1 \circ \mu)(k) = \mu(j)$.

- If $kj \in \phi$ for some $\phi \in \Phi$, then $\mu(j) = \eta(k)$.
- If $kj \notin \phi$ for any $\phi \in \Phi$, we claim that $\eta(k) R_k \mu(j)$. Suppose $\mu(j) P_k \eta(k)$, that is, $k \in D_\eta(\mu(j))$. Consider the student $i \in I$ such that $ij \in \phi$ for some $\phi \in \Phi$. Since $ij \notin G(\mu)$, we must have $(k, i) \notin C(\eta(i))$ and $k \succ_{\eta(i)} i$. Thus, matching η *violates the priority of student k at school $\eta(i)$* , a contradiction.

Thus, under the matching $\gamma_1 \circ \mu$, each student in γ_1 is better off than the matching μ and weakly worse off than the matching η . Each remaining student is assigned to the same school to which she is assigned under μ , which implies that the matching $\gamma_1 \circ \mu$ *Pareto dominates* μ and is *weakly Pareto dominated* by η . Moreover, by the same argument in Lemma 2, $\gamma_1 \circ \mu$ is *partially stable*. If the matching $\gamma_1 \circ \mu$ is equivalent to η , the proof is complete. If not, we use the same argument inductively: By Lemma 6, there is a set of distinct *improvement cycles*, such that the matching η is obtained by *solving* these cycles over $\gamma_1 \circ \mu$ and one can construct a cycle which appears in the graph used by SEPF. \square

D Proof of Lemma 1

Let I_μ^t denote the set of *temporarily matched* students at μ . (Only if) Since there is a cycle in $G(\mu)$, the set of *temporarily matched* students, I_μ^t , is nonempty. By definition of the graph $G^T(\mu)$, each student in I_μ^t is pointed to by a unique student in I_μ^t . Thus, by finiteness of I_μ^t , there exists a cycle in $G^T(\mu)$. In particular, each cycle in $G^T(\mu)$ is formed by the students in I_μ^t . (If) It follows from Remark 1.

E Proof of Proposition 2

As argued in Section 4.4, Lemma 1 and Remark 1 imply that the TP algorithm is a member of the SEPF class.

Lemma 8. *Let φ be a rule within the SEPF class. For a given problem (R, C) and partially stable initial matching μ_0 , suppose student i is assigned to an underdemanded school s at Step k of the underlying algorithm of φ . Then, at each further step, she is not part of any cycle; thus, she is assigned to s at any further step.*

Proof. Let K be the last step of the algorithm underlying the rule φ and let μ_k be the matching selected at Step $k \in \{1, \dots, K-1\}$. Remember that if a school s is *underdemanded* at μ_k , either the school has *no demand* at μ_k or no student on a path to a student in $\mu_k^{-1}(s)$ is part of a cycle in $G(\mu_k)$. Let $S_k^u \subseteq S$ denote the set of *underdemanded* schools at μ_k . We first argue that S_k^u has a specific structure. Let S_k^1 be the set of schools that has *no demand* at μ_k . By definition, $S_k^1 \subseteq S_k^u$. Let S_k^2 be the set of schools such that, for a school $s_2 \in S_k^2$, the students in $\mu_k^{-1}(s_2)$ are pointed to only by students in $\bigcup_{s_1 \in S_k^1} \mu_k^{-1}(s_1)$. Let S_k^3 be the set of schools such that, for $s_3 \in S_k^3$, the students in $\mu_k^{-1}(s_3)$ are pointed to only by students in $\bigcup_{s \in S_k^1 \cup S_k^2} \mu_k^{-1}(s)$. Continuing in this manner, we can decompose S_k^u into groups such that $S_k^u = S_k^1 \cup S_k^2 \cup \dots \cup S_k^m$ for some $m \geq 1$. In this decomposition, given a $\tilde{m} \in \{2, \dots, m\}$, for any $s_{\tilde{m}} \in S_k^{\tilde{m}}$, the students in $\mu_k^{-1}(s_{\tilde{m}})$ are pointed to only by students in $\bigcup_{s \in S_k^1 \cup \dots \cup S_k^{\tilde{m}-1}} \mu_k^{-1}(s)$.

Take any $s_1 \in S_k^1$. Since s_1 has *no demand* at μ_k , $X_{\mu_k}(s_1) = \emptyset$. By Lemma 3, $D_{\mu_k}(s_1) = \emptyset$. Moreover, by Remark 3, $D_{\mu_k}(s_1) = \emptyset$ implies that, for each $k' \geq k$, $D_{\mu_{k'}}(s_1) = \emptyset$. Since by definition, $X_{\mu}(s_1) \subseteq D_{\mu}(s_1)$, for each $k' \geq k$, $X_{\mu_{k'}}(s_1) = \emptyset$. Consequently, any $i \in \mu_k^{-1}(s_1)$ is not part of any cycle in steps $k' \geq k$.

If $S_k^u = S_k^1$, the argument is complete. If not, there exists at least one school in $s_2 \in S_k^2$. Otherwise, some students in $(S_k^u \setminus S_k^1)$ must be a part of a cycle, a contradiction. Let $s_2 \in S_k^2$. Suppose, for some student $i \in I$, $i \in (D_{\mu_k}(s_2) \setminus X_{\mu_k}(s_2))$. Then, there exists $j \in (D_{\mu_k}(s_2) \setminus \{j' \in I : (j', i) \in C(s_2)\})$ such that $j \succ_{s_2} i$ and j points to the students in $\mu_k^{-1}(s_2)$. But we know that $j \in \mu_k^{-1}(s_1)$ for some $s_1 \in S_k^1$ (recall that students in $\mu_k^{-1}(s_2)$ are pointed to only by students in $\bigcup_{s_1 \in S_k^1} \mu_k^{-1}(s_1)$), and, by the argument above, j is not a part of a cycle in steps $k' \geq k$. Therefore, at each step k' , $i \notin X_{\mu_{k'}}(s_2)$. Thus, at each Step $k' \geq k$, the students assigned to s_2 are pointed to only by the students who are assigned to schools with *no demand*, and thus they are not part of any cycle.

Now we can continue in the same manner. For any $\tilde{m} \in \{2, \dots, m\}$, if $S_k^u \setminus (S_k^1 \cup \dots \cup$

$S_k^{\tilde{m}-1}$) is nonempty, then there is at least one school in $S_k^{\tilde{m}}$. Otherwise, some students in $S_k^u \setminus (S_k^1 \cup \dots \cup S_k^{\tilde{m}-1})$ must be a part of a cycle, a contradiction. Let $s_{\tilde{m}} \in S_k^{\tilde{m}}$. By the same argument above, at each step $k' \geq k$, the students assigned to $s_{\tilde{m}}$ are pointed to only by the students who are assigned to *underdemanded* schools, and thus they are not part of any cycle. Repeating the same argument once more, we conclude that all students who are assigned to a school in S_k^u are not part of any cycle at any step $k' \geq k$; thus, they are not part of any cycle. The result follows. \square

Lemma 8 justifies the language we use for the students assigned to *underdemanded* schools at some matching μ : by definition, a *permanently matched* student i at μ is assigned to an *underdemanded* school at μ and she is not part of any cycle through an algorithm within the SEPF class which selects μ at some step. Thus, at each *constrained efficient* matching which *weakly Pareto dominates* μ , i is assigned to $\mu(i)$.

Lemma 9. *For a given problem (R, C) and partially stable initial matching μ_0 , let μ_{k-1} be the matching selected at Step $k-1$ of the TP algorithm. In the graph $G^T(\mu_{k-1})$, let cycle $\phi_k = \{i_1i_2, i_2i_3, \dots, i_ni_1\}$ be solved by the TP algorithm such that $\mu_k = \phi_k \circ \mu_{k-1}$. Then, for each $i \notin \{i_1, i_2, \dots, i_n\}$, if i points to j in $G^T(\mu_{k-1})$, then i points to j in $G^T(\mu_k)$.*

Proof. A student can be pointed to by at most one student in $G^T(\mu_{k-1})$. Since i points to j in $G^T(\mu_{k-1})$, $i \notin \{i_1, i_2, \dots, i_n\}$ implies $j \notin \{i_1, i_2, \dots, i_n\}$. Since $\mu_{k-1}(i) = \mu_k(i)$ and $\mu_{k-1}(j) = \mu_k(j)$, by Remark 4 and Lemma 1, i points to j in $G(\mu_k)$. Suppose student $i' \neq i$ points to j in $G^T(\mu_k)$. Thus, $i' \in D_{\mu_k}(\mu_k(j))$ and i' is *temporarily matched* at μ_k . Since $\mu_k(j) = \mu_{k-1}(j)$, by Remark 3, $i' \in D_{\mu_{k-1}}(\mu_{k-1}(j))$. Moreover, since i is *temporarily matched* at μ_k , i' has a higher priority at the school $\mu_{k-1}(j)$ than i . Thus, i (but not i') points to j in the graph $G^T(\mu_{k-1})$ implies that i' is *permanently matched* at μ_{k-1} . This contradicts Lemma 8, which implies that for each $k' \geq k-1$, i' must be *permanently matched* at $\mu_{k'}$. \square

Lemma 10. *For a given problem (R, C) and partially stable initial matching μ_0 , let μ_{k-1} be the matching selected at Step $k-1$ of the TP algorithm. If a cycle ϕ in the graph $G^T(\mu_{k-1})$ is not solved at Step k , then ϕ appears in the graph $G^T(\mu_k)$.*

Proof. Let $ij \in \phi$. By Remark 4, i points to j in $G(\mu_k)$. Since this holds for each edge in ϕ , ϕ is a cycle in $G(\mu_k)$. Thus, i is *temporarily matched* at μ_k . By Lemma 9, this implies that i points to j in $G^T(\mu_k)$. Since this holds for each $ij \in \phi$, the graph $G^T(\mu_k)$ contains cycle ϕ . \square

Lemma 11. *Let μ_0 be a partially stable matching for problem (R, C) . Consider a cycle selection order denoted by $\Phi = (\phi_1, \phi_2, \dots, \phi_K)$ such that ϕ_k appears in $G^T(\mu_{k-1})$ for all*

$k \in \{1, 2, \dots, K\}$ where $\mu_k = \phi_k \circ \mu_{k-1}$. Let μ be the outcome of the TP algorithm under Φ . If there exists $\tilde{k} \in \{1, \dots, K-1\}$ such that $\phi_{\tilde{k}+1}$ appears in $G^T(\mu_{\tilde{k}-1})$, then the TP algorithm selects μ for the cycle selection order $\hat{\Phi} = (\phi_1, \dots, \phi_{\tilde{k}-1}, \phi_{\tilde{k}+1}, \phi_{\tilde{k}}, \phi_{\tilde{k}+2}, \dots, \phi_K)$.

Proof. Let ν_k be the matching selected at Step k of the TP algorithm under $\hat{\Phi}$. Since in the first $\tilde{k}-1$ steps the same cycles are *solved* under both $\hat{\Phi}$ and Φ , we have $\mu_k = \nu_k$ for all $k \leq \tilde{k}-1$. Thus, $G^T(\mu_{\tilde{k}-1}) = G^T(\nu_{\tilde{k}-1})$. That is, $\phi_{\tilde{k}+1}$ and $\phi_{\tilde{k}}$ exist in $G^T(\nu_{\tilde{k}-1})$. Moreover, $\phi_{\tilde{k}+1}$ and $\phi_{\tilde{k}}$ are disjoint. When $\phi_{\tilde{k}+1}$ is *solved* at Step k , by Lemma 10, $\phi_{\tilde{k}}$ exists in $G^T(\nu_k)$. We have $\mu_{\tilde{k}+1} = \nu_{\tilde{k}+1}$ and $G^T(\mu_{\tilde{k}+1}) = G^T(\nu_{\tilde{k}+1})$ since the cycles are disjoint and only the students in $\phi_{\tilde{k}+1}$ and $\phi_{\tilde{k}}$ improved to the same schools. Then, ϕ_k appears in $G^T(\nu_k)$ and $\mu_k = \nu_k$ for all $k \geq \tilde{k}+1$. \square

Lemma 12. *For a given problem (R, C) and partially stable initial matching μ_0 , the outcome of the TP algorithm is independent of the order of cycles solved.*

Proof. We prove the result by constructing a cycle selection order, Φ , which generates the same outcome as any other cycle selection order, $\hat{\Phi}$, under the TP algorithm. The construction of Φ first requires a tie-breaker vector. Let $\pi = (\pi_i)_{i \in I}$ be such a vector, where π_i is the number assigned to student $i \in I$ and $\pi_j \neq \pi_k$ for all $j \neq k$. Given the π , the order Φ is determined as follows:

“At Step $k \geq 0$, given matching μ_k :

1. Let A_k be the set of cycles in $G^T(\mu_k)$.
2. Consider the cycles in $\cup_{\tilde{k} \leq k} A_{\tilde{k}}$ that are not *solved* yet. (By Lemma 10, this equals to A_k .) Among those, select the cycle to *solve* according to the following (lexicographic) cycle selection order:
 - (a) for all m and m' such that $m < m' \leq k$, all cycles in A_m are *solved* before the cycles in $A_{m'} \setminus A_m$;
 - (b) for all $m \leq k$, the cycles in A_m are *solved* according to the highest tie-breaker number of the student in the cycle.
3. *Solve* the cycle according to this order and obtain the new matching μ_{k+1} .”

Suppose that under Φ , the TP algorithm terminates at Step K and yields μ_{K-1} . Take any other cycle selection order $\hat{\Phi}$ one can have when the TP algorithm is applied to the problem (R, C) . First note that all cycles in A_0 necessarily appear under any cycle selection order. By Lemma 10, they are *solved* under $\hat{\Phi}$. By Lemma 11, we can rearrange the order of

cycles such that first $|A_0|$ steps are the same as those of Φ , and the final outcome of $\hat{\Phi}$ is unchanged. This produces, say, $\tilde{\Phi}$, whose final outcome is the same as $\hat{\Phi}$ and whose first $|A_0|$ steps are the same as Φ . Since first $|A_0|$ steps are the same, the cycles in $\cup_{\tilde{k} \leq |A_0|} A_{\tilde{k}}$ all appear under $\tilde{\Phi}$. By Lemma 10, these cycles are *solved* when the TP algorithm is run under $\tilde{\Phi}$. One can then reapply Lemma 11 and obtain another cycle selection order that yields the same outcome as when the TP algorithm is run under $\hat{\Phi}$, and yields the same matchings in the first $|A_0| + |A_{|A_0|}|$ steps. One can then continue until the cycle selection order whose final outcome is the same as $\hat{\Phi}$ and whose first K steps are the same as Φ . Thus, the TP algorithm produces the same outcome under $\hat{\Phi}$ and Φ . \square

By Lemma 1 and Remark 1, the TP algorithm is in the SEPF class. By Lemma 12, any cycle selection order gives the same matching under the TP algorithm. Thus, the TP algorithm produces a unique matching and it defines a rule in the SEPF class.

F Proof of Theorem 2

We provide the proof for the TP rule starting with an arbitrary *stable* matching μ_0 .

F.1 Proof of the “If” Part

The TP rule yields a *constrained efficient* matching that improves over the initial *stable* matching μ_0 .³⁴ Thus, we only need to show that the TP rule *provides incentives to consent*. We use the following lemma in the proof.

Lemma 13. *Let i be a permanently matched student at μ for (R, C) . Then, i is permanently matched at μ for each problem (R, C') where C and C' coincide except for i 's consents.*

Proof. If i does not point to j in the graph $G(\mu)$, then i 's consent for $\mu(j)$ does not affect who points to j in $G(\mu)$.³⁵ Since i is *permanently matched* at μ , by the definition of an *underdemanded* school, either $\mu(i)$ has *no demand* at μ or each path to i ends in a student assigned to a school with *no demand* at μ . For the former case, a school having *no demand*

³⁴This is a direct consequence of Theorem 1 and Proposition 2.

³⁵The following argument clarifies this. If $\mu(i) R_i \mu(j)$, then $i \notin D_\mu(\mu(j))$. Thus, i 's consent for $\mu(j)$ is never used in the construction of $G(\mu)$. Therefore, it does not determine who points to j . If $\mu(j) P_i \mu(i)$, then, there is another student i' pointing to j such that i' has a higher priority than i at $\mu(j)$ and does not consent for priority violation by i and students with lower priority than i to be assigned to $\mu(j)$. But then the consent of i does not determine who points to j , because there is a higher priority and non-consenting student.

depends only on the students' preferences. Thus, a student cannot change it through her consent decision. Now assume the latter case. First, realize that the only way to change the *underdemanded* status of the school is through changing the directed edges in the paths leading to i . Nevertheless, if i changes her consent for a school that no student in a path leading to i is assigned to, she cannot affect the edges in the paths leading to i . This means that we can restrict attention to changes in consents to schools to which some students in the paths leading to i are assigned. Let j be a student on a path to i . Clearly, i does not point to j (otherwise, there would be a cycle involving i and i would not be *permanently matched* at μ). Thus, each path to i remains the same regardless of the consents of i for the schools which the students on these paths are assigned to. This means that i remains *permanently matched* at μ regardless of her consenting behavior for the schools of students in the paths leading to i . Thus, $\mu(i)$ remains *underdemanded* at μ for each problem (R, C') where C and C' coincide except i 's consents. We conclude that i is *permanently matched* at μ for such a problem. \square

Next, we show that, under the TP rule, the placement of a student does not depend on her consent decision. Let (R, C) be a problem and μ_0 be an initial *partially stable* matching. Let K be the last step of the TP algorithm and μ_k be the matching selected at Step $k \in \{1, \dots, K-1\}$. A *temporarily matched* student can potentially affect the graph $G(\mu_{k-1})$ by her consenting decision, but not $G^T(\mu_{k-1})$. This is because the only way in which a *temporarily matched* student i affects $G^T(\mu_{k-1})$ by not consenting for s is by being the top priority student among those who are *temporarily matched* and who point to $\mu_{k-1}^{-1}(s)$ in $G(\mu_{k-1})$. But in this case, i points to $\mu_{k-1}^{-1}(s)$ under $G^T(\mu_{k-1})$ anyway. Thus, her consenting decision is irrelevant. Moreover, the consent decision of a *temporarily matched* student does not affect the set of *underdemanded* schools. Thus, at each step k , the consent of only the *permanently matched* students at μ_{k-1} is relevant for the graph $G^T(\mu_{k-1})$. Therefore, the only students who can be better off by not consenting under the TP rule are the *permanently matched* students. Nevertheless, by Lemma 8, each *permanently matched* student at μ_{k-1} is assigned to the same school under the matching given by the TP rule. Thus, not consenting would not make her better off. Moreover, by Lemma 13, a *permanently matched* student at μ_{k-1} is *permanently matched* at μ_{k-1} regardless of her consenting decisions. Thus, she can not change her status through consenting decisions either. That is, whenever a student's *consent* matters at some step k of the TP rule, then that student is already assigned to her school under the matching given by the TP rule at an earlier step $k' < k$, and her consenting decision cannot affect her matching under the TP rule.

F.2 Proof of the “Only If” Part

Lemma 14. *Let (R, C) be a problem and μ_0 be an initial stable matching. Let K be the last step of the TP algorithm and μ_k be the matching selected at Step $k \in \{1, \dots, K - 1\}$. Let ψ be a rule that provides incentives to consent and gives a constrained efficient matching that weakly Pareto dominates μ_0 . For each $k \in \{1, \dots, K - 1\}$, $\psi_{(R,C)}$ weakly Pareto dominates μ_k .*

Proof. We prove this result by contradiction. We construct a consent profile C^* for which $\psi_{(R,C^*)}$ is not a *constrained efficient* matching.

Let $A_0 = \emptyset$ and ϕ_k be the cycle solved in $G^T(\mu_{k-1})$ for each $k \in \{1, \dots, K - 1\}$ under the TP rule. Suppose that there is a step $\tilde{k} \in \{1, \dots, K\}$ where $\psi_{(R,C)}$ does not *weakly Pareto dominate* $\mu_{\tilde{k}}$. Let k be the first such step. That is, for each $k' < k$ and $i \in I$, $\psi_{(R,C)}(i) R_i \mu_{k'}(i)$. Let $\phi_k = \{i_1 i_2, i_2 i_3, \dots, i_n i_1\}$. Since we chose k to be the *first* step at which the matching chosen is not *weakly Pareto dominated*, there exists a student in $\{i_1, i_2, \dots, i_n\}$ who prefers her assignment under μ_k to $\psi_{(R,C)}$. Without loss of generality, suppose $\mu_k(i_1) P_{i_1} \psi_{(R,C)}(i_1)$. Note that $\mu_k(i_1) = \mu_{k-1}(i_2)$, and denote $\mu_{k-1}(i_2)$ with s_1 .

Let $A_1 := A_0 \cup \{(i_1, s_1)\}$. Construct the following consent profile C^1 : $(i_1, j) \notin C^1(s_1)$ for all $j \in I$ and the *consent* profile for the remaining schools/students is the same as C , i.e., $C^1(s) = C(s)$ for all $s \neq s_1$ and $C^1(s_1) = C(s_1) \setminus \cup_{j \in I} (i_1, j)$. Now, we consider two possible cases:

Case 1: Suppose $(i_1, j) \notin C(s_1)$ for all $j \in I$. In this case, $C = C^1$. Thus, we do not change the original consent profile. Since $s_1 P_{i_1} \psi_{(R,C^1)}(i_1)$ and i_1 has the highest priority at s_1 among the *temporarily matched* students who prefer s_1 to their assignment at μ_{k-1} , assigning a *temporarily matched* student, who is not in $\mu_{k-1}^{-1}(s_1)$ and prefers s_1 to her matched school under μ_{k-1} , to s_1 would violate the *partial fairness* of $\psi_{(R,C^1)}$. Consequently, the *temporarily matched* students at μ_{k-1} can not be assigned to s_1 under $\psi_{(R,C^1)}$. Moreover, the *permanently matched* students at μ_{k-1} must be assigned to their schools in μ_{k-1} at any *constrained efficient* matching which *weakly Pareto dominates* μ_{k-1} .³⁶ Thus, they can not be assigned to s_1 under $\psi_{(R,C^1)}$ either. At each *partially stable* matching *weakly Pareto dominating* μ_{k-1} , the number of students assigned to s_1 is $|\mu_0^{-1}(s_1)| = q_{s_1}$ (Lemma 4). Thus, the students in $\mu_{k-1}^{-1}(s_1)$ can not be assigned to another school than s_1 under $\psi_{(R,C^1)}$. Consequently, $\mu_{k-1}^{-1}(s_1) = \psi_{(R,C^1)}^{-1}(s_1)$. In

³⁶Since $\psi_{(R,C^1)}$ is *constrained efficient* and *weakly Pareto dominates* μ_{k-1} , the matching $\psi_{(R,C^1)}$ is obtained by solving a sequence of cycles which all appear in the graph used by the rules in the SEPF class (Theorem 1). But, by Lemma 8, a *permanently matched* student at μ_{k-1} is not a part of any cycle. Thus, she must be assigned to her school under μ_{k-1} .

particular, since $\mu_{k-1}(i_2) = s_1$, she is assigned to s_1 under $\psi_{(R,C^1)}$. That is, if $(i_1, j) \notin C(s_1)$ for all $j \in I$, then i_2 is assigned a school worse than $\mu_k(i_2) = \mu_{k-1}(i_3)$ in $\psi_{(R,C^1)}$.

Case 2: Suppose $(i_1, \bar{j}) \in C(s_1)$ for some $\bar{j} \in I$. Since ψ provides incentives to consent, $\psi_{(R,C)}(i_1) R_{i_1} \psi_{(R,C^1)}(i_1)$. Since $s_1 P_{i_1} \psi_{(R,C)}(i_1)$, we have $s_1 P_{i_1} \psi_{(R,C^1)}(i_1)$. There are two possibilities:

Case 2.1: $\psi_{(R,C^1)}^{-1}(s_1) = \mu_{k-1}^{-1}(s_1)$. Student i_2 is assigned to a school worse than $\mu_k(i_2) = \mu_{k-1}(i_3)$ in $\psi_{(R,C^1)}$ as explained in Case 1.

Case 2.2: $\psi_{(R,C^1)}^{-1}(s_1) \neq \mu_{k-1}^{-1}(s_1)$. Since both $\psi_{(R,C^1)}$ and μ_{k-1} are *partially stable* and *Pareto dominate* μ_0 , by Lemma 4, $|\psi_{(R,C^1)}^{-1}(s_1)| = |\mu_{k-1}^{-1}(s_1)|$. Then, there exists a $j \in (\psi_{(R,C^1)}^{-1}(s_1) \setminus \mu_{k-1}^{-1}(s_1))$. We have two subcases:

Case 2.2.a: $\mu_{k-1}(j) P_j s_1$. Then, j is assigned to a school worse than $\mu_{k-1}(j)$ in $\psi_{(R,C^1)}$.

Case 2.2.b: $s_1 P_j \mu_{k-1}(j)$. Since, by Lemma 4, $|\psi_{(R,C^1)}^{-1}(\mu_{k-1}(j))| = |\mu_{k-1}^{-1}(\mu_{k-1}(j))|$, there exists a $j' \in (\psi_{(R,C^1)}^{-1}(\mu_{k-1}(j)) \setminus \mu_{k-1}^{-1}(\mu_{k-1}(j)))$. If $\mu_{k-1}(j') P_{j'} \mu_{k-1}(j)$, then j' is assigned to a school worse than $\mu_{k-1}(j')$ in $\psi_{(R,C^1)}$. Alternatively, consider the case where $\mu_{k-1}(j) P_{j'} \mu_{k-1}(j')$. We claim that we can find a student \tilde{j} where \tilde{j} is assigned to a school worse than $\mu_{k-1}(\tilde{j})$ under $\psi_{(R,C^1)}$.

In order to see why, first note that $\mu_{k-1}(j) P_{j'} \mu_{k-1}(j')$. Thus, $j' \in D_{\mu_{k-1}}(\mu_{k-1}(j))$. But recall that $(i_1, j) \notin C^1(s_1)$ and $\psi_{(R,C^1)}(j) = s_1$. Thus, by *partial fairness* of ψ , $j \succ_{s_1} i_1$. Nevertheless, i_1 points to i_2 in $G^T(\mu_{k-1})$, which implies that j is *permanently matched* under μ_{k-1} . For this, either (i) j' is *permanently matched* under μ_{k-1} as well, or (ii) there exists a student j'' *temporarily matched* under μ_{k-1} , such that $j'' \succ_{\mu_{k-1}(j)} j'$ and $(j'', j') \notin C^1(\mu_{k-1}(j))$. In case (i), take a student in $(\psi_{(R,C^1)}^{-1}(\mu_{k-1}(j')) \setminus \mu_{k-1}^{-1}(\mu_{k-1}(j')))$. In case (ii), take a student in $(\psi_{(R,C^1)}^{-1}(\mu_{k-1}(j'')) \setminus \mu_{k-1}^{-1}(\mu_{k-1}(j'')))$. Continue until finding a student \tilde{j} such that $\mu_{k-1}(\tilde{j}) P_{\tilde{j}} \psi_{(R,C^1)}(\tilde{j})$. Because we are always following students assigned to *underdemanded* schools, the process cannot cycle and eventually ends up with such a student.

Let us summarize everything we have done so far. We began with Step k where ψ does not weakly *Pareto dominate* μ_k . Next, we found a student-school pair (i_1, s_1) , with the property that $s_1 P_{i_1} \psi_{(R,C)}(i_1)$. Then, we found a consent profile C^1 where $(i_1, i') \notin C^1(s_1)$ for all $i' \in I$, and a step $k^1 \leq k$ with the following property: for some $\ell \in I$, $\mu_{k^1}(\ell) P_\ell \psi_{(R,C^1)}(\ell)$ and for all $\bar{k} < k^1$ and all \bar{i} , $\psi_{(R,C^1)}(\bar{i}) P_{\bar{i}} \mu_{\bar{k}}(\bar{i})$.

Now, we repeat the whole argument. Let Step k^1 be as defined in the previous paragraph, and $(i_2, s_2) := (\ell, \mu_{k^1}(\ell))$ be the student-school pair. By construction, this pair satisfies the

property that $s_2 P_{i_2} \psi_{(R,C^1)}(i_2)$. Let $A_2 := A_1 \cup \{(i_2, s_2)\}$. Consider C^2 where $(i_2, i') \notin C^2(s_2)$ for all $i' \in I$, and the consent profile for the remaining schools/students is the same as C^1 . Following the exact same argument, one can find a step $k^2 \leq k^1$ with the following property: for some $\ell \in I$, $\mu_{k^2}(\ell) P_\ell \psi_{(R,C^2)}(\ell)$.

In general, at each step m , given A_{m-1} and k^{m-1} , take this pair, and let $(i_m, s_m) := (\ell, \mu_{k^{m-1}}(\ell))$. Define $A_m = A_{m-1} \cup \{(i_m, s_m)\}$, find a consent profile C^m where $(i_m, i') \notin C^m(s_m)$ for all $i' \in I$, and a step $k^m \leq k^{m-1}$ with the following property: for some $\ell \in I$, $\mu_{k^m}(\ell) P_\ell \psi_{(R,C^m)}(\ell)$.

Realize that k^m is a weakly decreasing sequence and A_m is expanding at each step. These two facts, combined with the finiteness of student and school sets, imply that eventually the next pair (i_{m+1}, s_{m+1}) will be a pair that is already in A_m . That is, the process will run into a cycle. Fix the consent profile C^m and Step k^m at this moment, and denote them by C^* and k^* , respectively. Now we have a consent profile C^* , a step k^* , and a cycle of students $\phi = \{i_1 i_2, i_2 i_3, \dots, i_m i_1\}$ which appears in $G^T(\mu_{k^*})$, with the following property: “for each $n \in \{1, \dots, m\}$, $\mu_{k^*}(i_n) P_{i_n} \psi_{(R,C^*)}(i_n)$.” Since the *solution* of this cycle ϕ does not violate *partial fairness* of $\psi_{(R,C^*)}$ and does not make any student worse off, $\psi_{(R,C^*)}$ cannot be *constrained efficient*. \square

Let (R, C) be problem and μ_0 be an initial *stable* matching. Let K be the last step of the TP algorithm and μ_k be the matching selected at Step $k \in \{1, \dots, K-1\}$. Let ψ be a rule which *provides incentives to consent* and gives a *constrained efficient* matching that *weakly Pareto dominates* μ_0 . Note that $TP_{(R,C)} = \mu_{K-1}$. By Lemma 14, $\psi_{(R,C)}$ *weakly Pareto dominates* μ_k for each $k \in \{1, \dots, K-1\}$, which implies that $\psi_{(R,C)}$ *weakly Pareto dominates* $TP_{(R,C)}$. Since both $\psi_{(R,C)}$ and $TP_{(R,C)}$ are *constrained efficient*, this implies that $\psi_{(R,C)} = TP_{(R,C)}$. This completes the proof.

G Proof of Theorem 3

Consider the following problem: $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$ and $q_s = 1$ for all $s \in S$. The preferences and the priorities are: $s_1 P_{i_1} s_2 P_{i_1} s_3$, $s_1 P_{i_2} s_2 P_{i_2} s_3$, $s_3 P_{i_3} s_1 P_{i_3} s_2$, $i_3 \succ_{s_1} i_1 \succ_{s_1} i_2$, $i_1 \succ_{s_2} i_2 \succ_{s_2} i_3$, and $i_1 \succ_{s_3} i_2 \succ_{s_3} i_3$.

Assume that $C(s_1) = \{(i_1, i_2)\}$, $C(s_2) = C(s_3) = \emptyset$. This problem has three *partially stable* matchings:³⁷ $\mu := \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$, $\mu' := \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$, and $\mu'' :=$

³⁷Any matching where i_1 is assigned to s_3 violates the priority of i_1 for s_2 , and any matching where i_3 is

$\{(i_1, s_2), (i_2, s_3), (i_3, s_1)\}$. Note that μ'' is *Pareto dominated* by μ and the only *constrained efficient* matchings are μ and μ' .

Let ψ be a *constrained efficient* and *strategy-proof* rule. Suppose μ' is the outcome of ψ . If i_1 reveals $P'_{i_1} : s_1 P'_{i_1} s_3 P'_{i_1} s_2$, in the new problem, the only *constrained efficient* matching is μ .³⁸ Then, ψ must select μ in the new problem. Thus, i_1 can gain from misreporting if ψ selects μ' in the original problem.

Alternatively, suppose ψ selects μ in the original problem. If i_2 reveals $P'_{i_2} : s_1 P'_{i_2} s_3 P'_{i_2} s_2$, in the new problem, the only *constrained efficient* matching is μ' .³⁹ Then ψ must select μ' in this problem. Thus, i_2 is better off by misreporting if ψ selects μ in the original problem.⁴⁰

H Strategy-proof and Constrained Efficient Rules

This section considers C 's which, when fixed, present no tension between *strategy-proofness* and *constrained efficiency*. The two extremes mentioned at the end of Section 2.2 are two clear examples of such C 's: when all priorities are allowed to be violated, the Top Trading Cycles (TTC) rule is *strategy-proof* and *constrained efficient*. When no priorities are allowed to be violated, the student-proposing DA rule is *constrained efficient* and *strategy-proof*. The following proposition provides other sufficient conditions that guarantee the existence of such a rule.

Given a correspondence C , we say that a student $i \in I$ is **fully consenting** if $(i, j) \in C(s)$ for each $s \in S$ and $j \in I$ such that $i \succ_s j$. A student is **non-consenting** if $(i, j) \notin C(s)$ for each $s \in S$ and $j \in I$. A student is **partially consenting** if she is neither *fully consenting* nor *non-consenting*. Note that, by definition, C satisfies all-or-nothing property if and only if the number of *partially consenting* students is zero.

Proposition 5. *If*

- *the correspondence C satisfies all-or-nothing property, or,*

assigned to s_2 violates the priority of i_3 for s_1 .

³⁸Any matching where i_1 is assigned to s_2 violates the priority of i_1 for s_3 , and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1 . The only *partially stable* matchings are μ and $\{(i_1, s_3), (i_2, s_2), (i_3, s_1)\}$, but the former *Pareto dominates* the latter.

³⁹Any matching where i_1 is assigned to s_3 violates the priority of i_1 for s_2 , and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1 . Also, under μ , i_2 's priority for s_3 is violated. The only *partially stable* matchings are μ' and μ'' , but the former *Pareto dominates* the latter.

⁴⁰To prove Theorem 3, one can also modify the problem used in the proof of Proposition 1 in Ehlers and Westkamp (2011) in order to construct a counterexample. Their proof gives an example which subsumes the case where some schools are consented by all students, and the remaining schools are not consented by any students.

- the total number of partially consenting and non-consenting students is less than or equal to two,

then the following rule is strategy-proof and constrained efficient: Run the student-proposing DA rule by considering all the capacity and the partially consenting and non-consenting students. Then, run the TTC rule for the remaining capacity and the fully consenting students.

Proof. Strategy-proofness: Both the student-proposing DA rule and the TTC rule are *strategy-proof*. Moreover, all seats are available when the student proposing DA rule is applied and the students considered under the TTC rule cannot affect the assignment selected by the DA rule via preference manipulation. Thus, no student can gain from misreporting her preferences.

Partial stability: Since priorities of all students considered under the student-proposing DA rule are respected and only the *fully consenting* students are considered under the TTC rule, the outcome is *partially fair*. Moreover, *non-wastefulness* and *individual rationality* of both rules imply the *non-wastefulness* and *individual rationality* of the outcome. Thus, the outcome is *partially stable*.

Constrained efficiency: First note that, when there are at most two students, by [Ergin \(2002\)](#), the student-proposing DA rule selects a *Pareto efficient* allocation. Thus, the outcome under the latter condition is *Pareto efficient*. Recall that all seats are available when the student-proposing DA rule is applied. Thus, we cannot form a welfare improvement cycle including students considered under both student-proposing DA and TTC. That is, any welfare improvement under the former condition require priorities of some *non-consenting* students to be violated. \square

I Proof of Proposition 4

Let C be constructed according to Equation 1. We claim that a matching μ is *partially stable* with respect to \succ' and C if and only if it is *stable* with respect to \succsim . First, note that the definitions of *individual rationality* and *non-wastefulness* are the same across two concepts, so that the equivalence of *partial fairness* with respect to \succ' and C and *fairness* with respect to \succsim implies the desired result. We claim that if μ is *stable* with respect to \succsim , then it is *partially stable* with respect to \succ' and C . Suppose it is not. If μ is not *partially fair*, then there exist $i, j \in I$ and $s \in S$ such that (i) $\mu(j) = s$, (ii) $s P_i \mu(i)$, (iii) $i \succ'_s j$, and (iv) $(i, j) \notin C(s)$. But by Equation 1, $(i, j) \notin C(s)$ implies that either $i \succ_s j$ or $j \succ_s i$. Since

$\succ' \in \mathcal{T}(\succ)$, by (iii), we conclude that $i \succ_s j$. But then, we have (i) $\mu(j) = s$, (ii) $s P_i \mu(i)$ and (iii) $i \succ_s j$. Thus, μ is not *fair* with respect to \succ , a contradiction. Now, assume μ is *partially stable* with respect to \succ' and C . If μ is not *fair* with respect to \succ , then there exist $i, j \in I$ and $s \in S$ such that (i) $\mu(j) = s$, (ii) $s P_i \mu(i)$ and (iii) $i \succ_s j$. But (iii) implies that $i \succ'_s j$ and $(i, j) \notin C(s)$. Thus, μ cannot be *partially fair*, a contradiction. Due to this equivalence, we conclude that *constrained efficiency* with respect to \succ' and C is equivalent to student optimal stability with respect to \succ and *Pareto dominating* μ_0 . Thus, Theorem 1 implies the desired result.

J Examples

Example 2. (This example is based on Example 3 in Kesten (2010), pp. 1310) Let $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, $q_{s_x} = 1$ for $x = 1, \dots, 4$ and $q_{s_5} = 2$. Assume that $(i, j) \in C(s)$ for each $s \in S$ and $i, j \in I$ with $i \succ_s j$. The students' preferences and schools' priorities are as follows:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	\succ_{s_1}	\succ_{s_2}	\succ_{s_3}	\succ_{s_4}	\succ_{s_5}
s_2	s_3	s_3	s_1	s_1	s_4	i_2	i_3	i_1	i_4	\vdots
s_1	s_1	s_4	s_2	s_5	s_1	i_1	i_6	i_6	i_3	\vdots
s_3	s_5	s_2	s_4	\vdots	s_3	i_5	i_4	i_2	i_6	\vdots
\vdots	\vdots	\vdots	\vdots	\vdots	s_2	i_6	i_1	i_3	\vdots	\vdots
\vdots	\vdots	\vdots	\vdots	\vdots	s_5	i_4	\vdots	\vdots	\vdots	\vdots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	i_3	\vdots	\vdots	\vdots	\vdots

The SOSM for this problem is $\mu_0 = \{(i_1, s_3), (i_2, s_1), (i_3, s_2), (i_4, s_4), (i_5, s_5), (i_6, s_5)\}$ (marked with boxes above). The sets X_{μ_0} are as follows:

$$\begin{aligned}
X_{\mu_0}(s_1) &= \{i_1, i_4, i_5, i_6\}, & X_{\mu_0}(s_2) &= \{i_1, i_4, i_6\}, & X_{\mu_0}(s_3) &= \{i_2, i_3, i_6\} \\
X_{\mu_0}(s_4) &= \{i_3, i_6\}, & X_{\mu_0}(s_5) &= \emptyset
\end{aligned}$$

The graph $G(\mu_0)$ is given in Figure 1.⁴¹

There are four cycles in this graph: $\phi_1 = \{i_3 i_4, i_4 i_3\}$, $\phi_2 = \{i_1 i_3, i_3 i_1\}$, $\phi_3 = \{i_1 i_2, i_2 i_1\}$, and

⁴¹As defined in Section 3.1, each graph in the algorithm is on the set of students. For convenience and tractability, we include the school which is assigned to the student in the current matching as well.

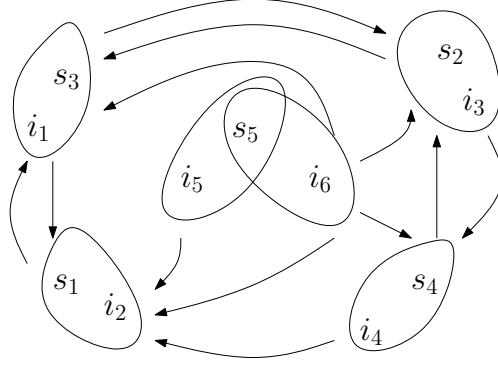


Figure 1: Graph $G(\mu_0)$

$\phi_4 = \{i_1i_3, i_3i_4, i_4i_2, i_2i_1\}$. First, as an illustration, we demonstrate how the algorithm proceeds when one follows a specific cycle selection rule. In $G(\mu_0)$, assume that the cycle selection rule requires ϕ_1 to be *solved*. Once ϕ_1 is *solved*, $\mu_1 = \{(i_1, s_3), (i_2, s_1), (i_3, s_4), (i_4, s_2), (i_5, s_5), (i_6, s_5)\}$ is obtained. The sets X_{μ_1} are as follows:

$$\begin{aligned} X_{\mu_1}(s_1) &= \{i_1, i_4, i_5, i_6\}, & X_{\mu_1}(s_2) &= \{i_1, i_6\}, & X_{\mu_1}(s_3) &= \{i_2, i_3, i_6\} \\ X_{\mu_1}(s_4) &= \{i_6\}, & X_{\mu_1}(s_5) &= \emptyset, \end{aligned}$$

The graph $G(\mu_1)$ is given in Figure 2.

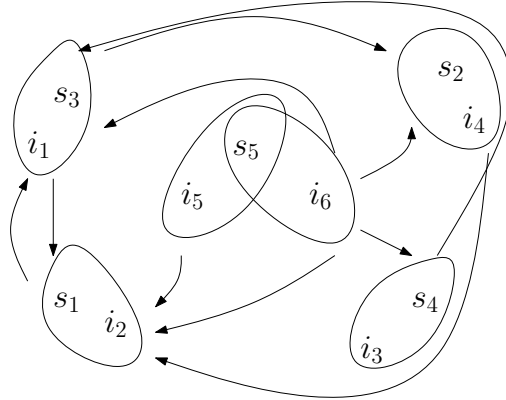


Figure 2: Graph $G(\mu_1)$

In the graph $G(\mu_1)$, there are two cycles: $\phi'_1 = \{i_1i_4, i_4i_2, i_2i_1\}$ and $\phi''_1 = \{i_1i_2, i_2i_1\}$. Assume that the cycle selection rule requires ϕ'_1 to be *solved*. Then, $\mu_2 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$ is obtained. The graph $G(\mu_2)$ is given in Figure 3.

is conducted first, such that among the students pointing to a particular student, say i , in $G(\mu_0)$, the top priority student is selected, and only this student points to i . This gives the graph in Figure 4.

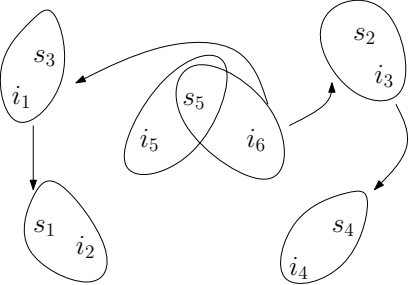


Figure 4: The highest priority students are chosen before *permanently matched* students are removed from the application graph.

This graph has no cycles, but the application graph $G(\mu_0)$ does. Thus, when step (i) is skipped, in general, we end up with a matching which is not *constrained efficient*. The TP algorithm, on the other hand, ignores the *permanently matched* students when selecting the student with the highest priority for a given school: students i_5 and i_6 are *permanently matched* at μ_0 (note that $\mu_0(i_5) = \mu_0(i_6) = s_5$ has *no demand* at μ_0) and the edges that originate from these students are removed in step (i) resulting in the subgraph of $G(\mu_0)$ in Figure 5.

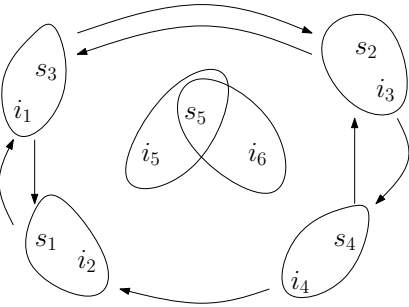


Figure 5: The subgraph of $G(\mu_0)$ after *permanently matched* students' demands are ignored.

Among the students pointing to a student i in this graph, the student with the highest priority at school $\mu_0(i)$ is selected and in the graph $G^T(\mu_0)$, only that student points to i (Figure 6).

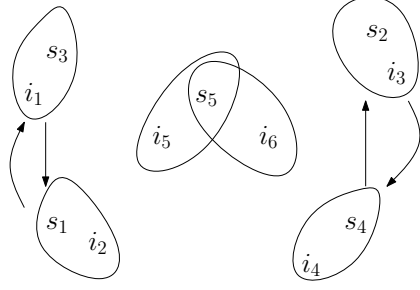


Figure 6: Graph $G^T(\mu_0)$

There are two cycles in this graph: $\phi_1 = (i_3 i_4, i_4 i_3)$ and $\phi_3 = (i_1 i_2, i_2 i_1)$. The TP algorithm proceeds by *solving* both of these cycles simultaneously and the matching

$$\mu_1 = \{(i_1, s_1), (i_2, s_3), (i_3, s_4), (i_4, s_2), (i_5, s_5), (i_6, s_5)\}$$

is obtained.⁴² The graph $G(\mu_1)$ is given in Figure 7.

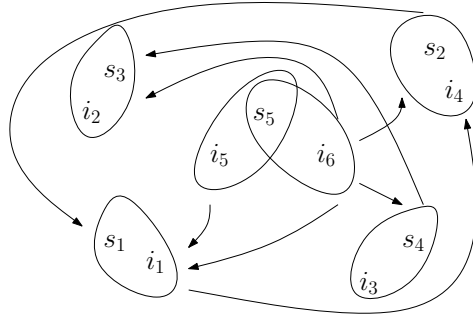


Figure 7: Graph $G(\mu_1)$

Once again, the TP algorithm proceeds by first ignoring the demands of *permanently matched* students, who are $i_2, i_3, i_5,$ and i_6 (Figure 8).

Since no student is pointed to by more than one student, the graph $G^T(\mu_1)$ is the same as

⁴²Actually, in the TP algorithm, only one cycle is *solved* at each step. But, as we argue in the proof of Proposition 2 (see Appendix E), the order of cycles *solved* is not consequential. Equivalently, each cycle in the graph used by the TP algorithm can be *solved* simultaneously.

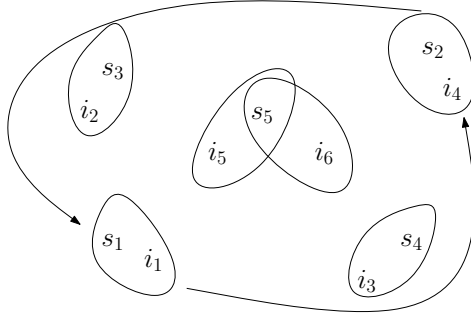


Figure 8: The subgraph of $G(\mu_1)$ after *permanently matched* students' demands are ignored.

in Figure 8. By *solving* the only cycle (i_1i_4, i_4i_1) in the graph $G^T(\mu_1)$, the matching

$$\mu_2 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$$

is obtained. In the graph $G(\mu_2)$, there is no cycle (see Figure 9). Thus, the TP-algorithm stops and the matching obtained by the TP algorithm is μ_2 . This matching is also the one obtained by the (generalized) EADAM.

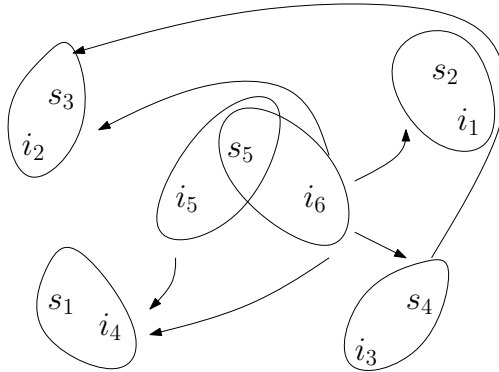


Figure 9: Graph $G(\mu_2)$

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