Alternating-Offer Bargaining with the Global Games Information Structure

Anton Tsoy*

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Abstract

In this study, I examine the alternating-offer bilateral bargaining model with private correlated values. The correlation of values is modeled via the global games information structure. I focus on the double limits of perfect Bayesian equilibria as offers become frequent and the correlation approaches perfect. I characterize the Pareto frontier of the double limits and show that it is efficient, but the surplus split generally differs from the Nash Bargaining split. I then construct a double limit that approximates the Nash Bargaining split in the ex-post surplus, but with a delay. Further, I prove the Folk theorem when the range of the buyer's values coincides with the range of the seller's costs: any feasible and individually rational ex-ante payoff profile can be approximated by a double limit.

Keywords: *bargaining delay, alternating offers, incomplete information, private correlated values, Coase conjecture, global games.*

JEL Codes: *C78, D82.*

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1 Introduction

In many markets, parties negotiate prices bilaterally and their privately known valuations are correlated. Examples of such markets include over-the-counter markets for financial assets, real estate, private equity, and durable goods. However, the bargaining literature has focused exclusively on one-sided or two-sided independent private information but not on two-sided correlated private information.

To fill this gap, I study frequent-offer limits of perfect Bayesian equilibria (PBEs) in an alternating-offer bilateral bargaining model with private correlated values. The main conclusion is that even when values become almost perfectly correlated, the bargaining outcome can differ in both the surplus split and the delay from the complete information outcome. In this limit, while the players' private information about values is precise, a variety of bargaining outcomes is possible because of the lack of common knowledge about values. In this sense, the results stress the role of public rather than private information in predicting the bargaining outcome.

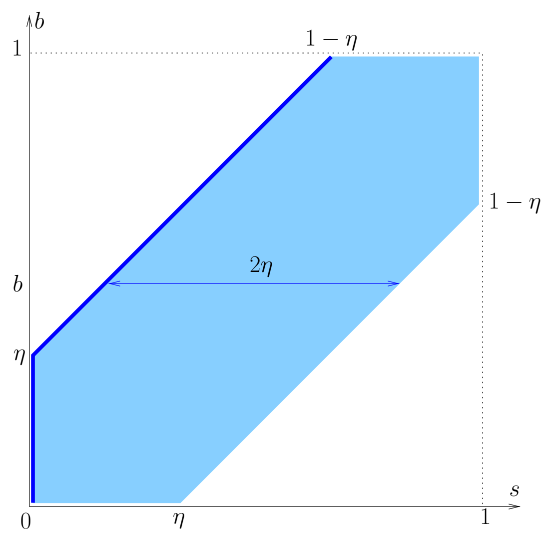

Figure 1: ***Distribution of types***
Types (s, b) are uniformly distributed on the diagonal stripe of width 2η inside the unit square. The bold line depicts the support of buyer's optimistic conjectures.

I use the global games information structure to capture the correlation of values, which is commonly used in the global games literature (Carlsson and Van Damme [1993], Morris and Shin [1998, 2000]). Specifically, players' types are uniformly distributed on a "diagonal stripe" of width 2η inside the unit square. (See Figure 1). Players' values are strictly increasing functions of their own types. Thus, the buyer with a higher value assigns positive probability to an interval of seller types with higher costs. The global games information structure is quite rich and incorporates a variety of correlations ranging from almost perfect (narrow stripe) to independent values (wide stripe). I assume that the gains from trade are positive for any realization of types.

My focus is on PBE limits as first offers become frequent, and then the correlation approaches perfect. I refer to such limits as *double limits.* This order of limits is particularly interesting, because the reverse order of limits boils down to the well-studied complete information game (Rubinstein [1982], Binmore et al. [1986]). Given the strong notion of the correlation between values (types assign positive mass only to an interval of the opponent's types) and positive gains from trade for any realization of types, one might expect that as the correlation between the values becomes almost perfect the bargaining outcome converges to the unique complete information outcome in which equally patient players immediately split the surplus equally. I refer to such a split as the *Nash split,* because it coincides with the Nash Bargaining Solution (Nash [1953]).

I find that even for an almost perfect correlation, a variety of bargaining outcomes can be sustained as frequent-offer PBE limits, and I obtain the type of multiplicity that is common in bargaining models with two-sided independent private information about values. Recall that with independent values, multiple outcomes can be sustained as frequent-offer PBE limits with *optimistic conjectures* where, for example, each buyer type puts probability one on the lowest seller type after the seller's deviation. Under optimistic conjectures, there is a continuation equilibrium with the following Coasian property: As offers become frequent, the deviator gets the lowest share of the surplus over all equilibrium continuation plays. However, when values are highly correlated, such optimistic conjectures result only in a marginal updating of beliefs. Nevertheless, I show that for any level of correlation, I can construct a continuation equilibrium with optimistic conjectures with the following *contagious Coasian property*: The frequent-offer limit gives just as harsh a punishment to the deviator as in the case of independent values.

The key to my construction of the continuation equilibrium is the combination of the Coasian argument and the contagion arguments, which is similar to the global games literature. Specifically, I first consider the game between seller type 0 and buyer types below η that under optimistic conjectures assign probability one to the seller type 0. (See Figure 1). The Coasian property in this game with one-sided private information starts the contagion. By the Coasian argument, the buyer types below η have a low maximal willingness to pay that implies low screening prices for the seller types that are "slightly" above seller type 0. This fact, in turn, leads to a larger set of buyer types at the bottom with a low willingness to pay, which in turn, implies low screening prices for a larger set of seller types at the bottom. This process continues until we cover all buyer types and show that their willingness to pay is relatively low, which guarantees that the seller's utility is low in such a continuation equilibrium.

The contagious Coasian property sustains a variety of on-path dynamics and reduces the problem of characterizing various equilibrium sets to the construction of appropriate on-path strategies. There are three main results of the paper. The first result is the characterization of the ex-ante Pareto frontier of double limits. On the Pareto frontier, the bargaining outcomes are efficient, as in the complete information case. However, in general, the surplus split differs from the Nash split. To characterize the Pareto frontier, I construct PBEs with the following *segmentation dynamics*. There are endogenously defined segments, and the seller signals a particular segment of types with her first offer. This offer is accepted by a sufficiently large set of buyer types, which ensures that the trade will occur immediately for a sufficiently large set of players' types. The rest of the buyer types reject this offer. And after the rejection, the war-of-attrition type of dynamics emerge in which both sides gradually concede to less favorable price offers. This war-of-attrition dynamics ensure that none of the seller types finds it optimal to mimic another segment, and the buyer types who accept the seller's first offer find it optimal to do so. By appropriately choosing the price offer in each segment and increasing the number of segments as the correlation approaches perfect, I can construct PBEs that approximate any point on the Pareto frontier or approximate ex-post the complete information outcome, and do so without delay in the limit.

The second result is the construction of frequent-offer PBE limits that approximate the ex-post Nash split as correlation approaches perfect, but with a delay. Such outcomes exhibit realistic *two-sided screening dynamics* that can be described as follows: Both sides start from extreme price offers and gradually their offers converge. All types on each side pool on price offers, but gradually separate by the time they give in and accept the opponent's offer. By constructing accepted price offers that are close to the prices that attain the Nash split for accepting types, I ensure that such a bargaining outcome approximates the ex-post Nash split. However, the gradual acceptance leads to inefficient bargaining delays.

The third result proves the Folk theorem when the range of values coincides with the range of costs. In this case, I use the frequent-offer PBE limits with the two-sided screening dynamics in which the accepted prices are close to the values of the accepting types. These limits ensure that the acceptances occur very slowly, and there is no trade at the limit. Given the characterization of the Pareto frontier, I obtain the Folk theorem when players have access to the public correlation device in the beginning of the game: any feasible, individually rational bargaining outcome is sustainable as a double limit.

The results stress the role of higher-order uncertainty in bargaining models with private information about values. Specifically, suppose that the buyer and the seller types equal $\frac{1}{2}$. Then for any integer m and any positive ε, the correlation of types can be sufficiently high (the diagonal stripe in Figure 1 sufficiently narrow) so that there is a m'th level of mutual knowledge between the buyer and the seller that their types are within ε of $\frac{1}{2}$. However, no matter how high the correlation of types is, it is only common knowledge between players that their types are in $[0, 1]$. The analyst who wishes to predict the bargaining outcome in this environment might be tempted to ignore the higher-order uncertainty and choose to approximate this environment with the complete information game between buyer and seller types equal to $\frac{1}{2}$. In particular, the analyst might feel confident in using the Nash Bargaining Solution as a reduced form for the complete-information bargaining outcome, which is standard in the applied work. I demonstrate that this assumption is far from innocuous. For example, the Folk theorem shows that under certain conditions, no predictions about the expected payoffs can be made apart from those implied by feasibility and individual rationality constraints.

Related Literature This paper is related to several strands of literature. First, the literature on bargaining with private information about values focuses on one-sided private information about independent or interdependent values and two-sided private information about independent values. The main result in the literature on one-sided private information about independent values is the Coase conjecture showing that the complete information bargaining outcome is sensitive to even a small amount of private information (see Fudenberg et al. [1985], Gul and Sonnenschein [1988], Grossman and Perry [1986], Ausubel and Deneckere [1992a], Gul et al. [1986]). Deneckere and Liang [2006], Fuchs and Skrzypacz [2013], Gerardi et al. [2014] explore the model with one-sided private information about interdependent values where only one party knows the quality of the object, which determines the values of both parties.[1] When the uninformed party makes offers, there is a unique equilibrium as in the case of independent values. In models with two-sided independent private information optimistic conjectures generally sustain a variety of equilibrium dynamics, and the literature studies particular classes of equilibria and

[1]Vincent [1989] provide an earlier analysis of this model.

restricts its attention to one-sided offers.[2] Ausubel and Deneckere [1992b] shows that in a rich subclass of sequential equilibria for the no-gap case, essentially no trade happens as the offers become frequent. Cramton [1984] constructs an equilibrium where first the seller gradually reveals her type and then screens the buyer types. Cho [1990] considers a class of equilibria in which the seller's price offers perfectly separate types in every round. Both equilibria in Cramton [1984] and Cho [1990] converge to an immediate trade at a price equal to the lowest valuation of the buyer as offers become frequent. Cramton [1992] considers the model with two-sided offers where parties strategically choose the amount of delay to signal their values.[3] This paper complements these papers by constructing a rich class of equilibria with segmentation and two-sided screening dynamics in a new environment with correlated values and two-sided offers. Further, it shows that a variety of bargaining outcomes is possible even when the support of players' beliefs (but not higher-order beliefs) is small.

The papers in the bargaining literature closest to this one are Feinberg and Skrzypacz [2005] and Weinstein and Yildiz [2013]. Feinberg and Skrzypacz [2005] show that the Coase conjecture is not robust to second-order uncertainty. In the current paper, any finite-order uncertainty becomes small as the correlation between values increases, and my model stresses the role of higher-order uncertainty in bargaining. Weinstein and Yildiz [2013] show that the complete-information game is not robust to the perturbations in higher-order beliefs. However, their result involves complex and somewhat artificial types, while my type space has a natural structure of private information about values commonly used in the bargaining literature. Put differently, I impose stronger common knowledge restrictions compared to Weinstein and Yildiz [2013], in particular, it is common knowledge between players how types are mapped into values.

Although I use the information structure from the global games literature (Carlsson and Van Damme [1993], Morris and Shin [1998, 2000]), the multiplicity of double limits is quite different from the selection results in that literature. In this sense, my results are closer to Weinstein and Yildiz [2007, 2013], who apply the contagion argument to show that there is no robust refinement of rationalizability.[4] Morris and Shin [2012] show

[2] The exception is Ausubel and Deneckere [1993a] that allows offers by both sides and justifies the restriction to one-sided offers with the welfare perspective.

[3] See also Fudenberg and Tirole [1983] for an analysis of the model with two bargaining rounds, and Chatterjee and Samuelson [1987] for a characterization of the bargaining dynamics under the additional restriction that the type and action space consist of only two types and two offers. Watson [1998] analyze the uncertainty about discount factors.

[4] See also Angeletos et al. [2007], Chassang [2010] who show the multiplicity of equilibria in dynamic environments less related to mine.

how the contagious adverse selection can lead to a market break down in a static trading game. However, I show that even when players have a great flexibility in exchanging offers, the inefficient trade delay can arise as an equilibrium outcome. Similar to Morris and Shin [2012], I stress that the public information ensures efficiency through building common knowledge among players, as opposed to the reduction in the adverse selection as proposed in Daley and Green [2012], Asriyan et al. [2017], Duffie et al. [forthcoming].

The segmentation dynamics of PBEs that approximate the Pareto frontier of double limits are similar to the war-of-attrition dynamics in reputational bargaining (Abreu and Gul [2000], Kambe [1999], Compte and Jehiel [2002], Wolitzky [2012], Fanning [2016]). In those models, commitment types require a fixed share of the surplus, and the war-of-attrition dynamics emerge, in which rational types mimic certain commitment types. In my model these dynamics arise despite the fact that all types are rational. My paper also gives the connection between the trade dynamics and primitives such as the values of players, while the reputational bargaining literature is silent on where the commitment types come from.

The paper is organized as follows. Section 2 describes the game. Section 3 presents the main results. Section 4 derives the optimal punishment for off-path deviations. Section 5 concludes by reviewing the results and directions for future research. Omitted proofs are in Appendices A and B. Additional results and auxiliary technical lemmas are provided in the Online Appendix.

2 The Model

Types and Values The buyer (he) and the seller (she) negotiate the price of an indivisible object. The seller's type s and the buyer's type b are jointly uniformly distributed on the diagonal stripe of width 2η inside the unit square, $\Omega_\eta = \{(s,b) \in [0,1]^2 : s-\eta \leq b \leq s+\eta\}$. (See Figure 1). The Lebesgue σ-algebra on Ω_η is denoted by $\mathcal{F}_\eta$, the uniform distribution on Ω_η is denoted by $\mathbb{P}_\eta$, and the expectation with respect to $\mathbb{P}_\eta$ is denoted by $\mathbb{E}_\eta$. I denote the diagonal inside the unit square by $\Omega_0 = \{(s,b) \in [0,1] : s = b\}$, and the uniform distribution on the diagonal by $\mathbb{P}_0$.

The uncertainty parameter $\eta \in (0,1]$ controls the degree of the correlation between types. By varying η, the model spans a variety of environments from independent ($\eta = 1$) to almost perfectly correlated types ($\eta \approx 0$). If $\overline{\pi}(x) = \min\{1, x+\eta\}$ and $\underline{\pi}(x) = \max\{0, x-\eta\}$, then given their types, players' prior beliefs about the opponent's type are uniform on $B_s = [\underline{\pi}(s), \overline{\pi}(s)]$ for seller type s and on $S_b = [\underline{\pi}(b), \overline{\pi}(b)]$ for buyer type b. Such an information structure is similar to the global games information structure: Types s and b serve as noisy private signals about the underlying quality, where the noise has bounded support as in Morris and Shin [1998].

The value of an object for a type b buyer is $v(b)$, and the cost of selling an object for a type s seller is $c(s)$, where v and c are strictly increasing, continuously differentiable functions with derivatives bounded from below and above by positive $\underline{\ell}$ and ℓ, resp.[5] I additionally impose the following technical condition on v and c: there is $D > 0$ such that $\left|{}^{d^k v(x)}/{}_{dx^k}\right|/k! < D$ and $\left|{}^{d^k c(x)}/{}_{dx^k}\right|/k! < D$ for all $x \in [0,1]$ and $k = 1, 2, \ldots$. (All polynomial functions satisfy this condition). The monotonicity of v and c implies that the values are positively correlated. The uncertainty about the type of opponent translates into the uncertainty about the opponent's value.

The gains from trade are denoted by $\Pi(s,b) \equiv v(b) - c(s)$, and the upper bound on the gains from trade is denoted by $\Sigma \equiv v(1) - c(0)$. I use the short-hand notation $\Pi(x)$ for $\Pi(x,x)$. I assume that the minimal gains from trade $\xi \equiv \min\limits_{(s,b)\in\Omega_\eta} \{\Pi(s,b)\}$ are positive. This assumption does not preclude the possibility that $c(1) > v(0)$, and hence, in general there is not a single price that gives non-negative utility to all types.

The Game Bargaining occurs in rounds $n = 1, 2, \ldots$. The length of the time interval between bargaining rounds is $\Delta > 0$. Players discount the future at a common discount rate $r > 0$. And the common discount factor is denoted by $\delta \equiv e^{-r\Delta}$.

The seller is active in odd rounds, and the buyer is active in even rounds. An active player can either accept the last offer of the opponent or make a counter-offer. Once a price offer is accepted, the game ends and the payoffs are determined. If the trade happens in round N at price p, the buyer's utility is $\delta^{N-1}(v(b)-p)$, and the seller's utility is $\delta^{N-1}(p - c(s))$. If a trade does not occur in finite time, then both players get a payoff of zero.

A history H_n is a sequence of rejected price offers up to round $n-1$, if by the beginning of round n no trade has occurred. A pure strategy of the buyer σ_b^n is a measurable function that maps any buyer type b and history H_n to the acceptance decision or the counter-

[5] When types are independent ($\eta = 1$), there is no loss of generality by assuming that types are uniformly distributed. For any distribution of values, there is a transformation of functions v and c that preserves the distribution of values and changes the distribution of types to uniform on the unit interval. With correlated types, this is no longer true, because no such transformation is guaranteed to preserve the correlation. I consider a general class of functions v and c but restrict the distribution of types to be uniform.

offer. The posterior belief of buyer μ_b^n is a measurable function that maps any buyer type b and any history H_n to a probability distribution of seller types. The strategy σ_s^n and the posterior beliefs μ_s^n are defined analogously for the seller. The pure strategies are extended appropriately to behavioral strategies.

A *perfect Bayesian equilibrium (PBE)* consists of a pair of strategy profiles (σ_b^n, σ_s^n) and beliefs (μ_b^n, μ_s^n) that satisfy the sequential rationality and the following conditions on the beliefs: (a) Bayes' rule is applied to update beliefs whenever possible; (b) μ_b^n and μ_s^n do not change in even and odd rounds, respectively; and (c) for any history H_n, $\mu_b^n \in \Delta(S_b)$ and $\mu_s^n \in \Delta(B_s)$. This is a natural adaptation of the PBE (Fudenberg and Tirole [1991]) to my environment with correlated values and the bounded support of beliefs. The last requirement states that both on and off the equilibrium path, players assign positive probability only to the types of opponent that they initially considered possible, that is, in B_s or S_b.

An *outcome* in the bargaining game is the mapping from types (s, b) to the time of trade $\tau(s, b)$ and the price of a trade $\rho(s, b)$. Hence, τ and ρ are random variables on the probability space $(\Omega_\eta, \mathcal{F}_\eta, \mathbb{P}_\eta)$. I focus on PBEs, in which players' on-path strategies are pure. Given this restriction, I define a *PBE outcome,* $(N(s, b)\Delta, p(s, b))$, as the outcome induced by equilibrium strategies where $N(s, b)$ and $p(s, b)$ are the round and the price of a trade between types s and b in the PBE.

Double Limits I focus on double limits of equilibria as first offers become frequent, $\Delta \to 0$ or equivalently $\delta \to 1$, and then the correlation becomes perfect, $\eta \to 0$. I call (τ, ρ) the *frequent-offer PBE limit* of a sequence of PBEs indexed by $\delta \to 1$ if equilibrium outcomes $(N\Delta, p)$ converge in probability to the outcome (τ, ρ) as $\delta \to 1$, that is, for any $\varepsilon > 0$,

$$\limsup_{\delta \to 1} \mathbb{P}_\eta \left(|N\Delta - \tau| > \varepsilon \text{ or } |p - \rho| > \varepsilon \right) = 0.$$

The double limits are defined as follows:

Definition 1. *The sequence of outcomes of frequent-offer PBE limits, (τ_η, ρ_η), indexed by $\eta \to 0$ converges to the limit (τ, ρ) as $\eta \to 0$, which is denoted by $(\tau_\eta, \rho_\eta) \underset{\eta \to 0}{\to} (\tau, \rho)$, if for any $\varepsilon > 0$,*

$$\limsup_{\eta \to 0} \mathbb{P}_\eta \left(|\tau_\eta - \tau| > \varepsilon \text{ or } |\rho_\eta - \rho| > \varepsilon \right) = 0.$$

I call (τ, ρ) the double limit, *if there is a sequence of frequent-offer PBE limits $(\tau_\eta, \rho_\eta) \underset{\eta \to 0}{\to}$ (τ, ρ).*

Definition 1 states that as $\eta \to 0$, deviations larger than ε in the trade delay and the price in the frequent-offer PBE limit (τ_η, ρ_η) from that in (τ, ρ) become unlikely. I denote by DL the set of all double limits.

Since my focus is on double limits, the complete information model is a natural benchmark. Let

$$\overline{y}(s,b) \equiv \frac{\delta c(s) + v(b)}{1+\delta} \text{ and } \underline{y}(s,b) \equiv \frac{c(s) + \delta v(b)}{1+\delta}$$

be equilibrium offers of the seller and the buyer, respectively, when values $v(b)$ and $c(s)$ are common knowledge (Rubinstein [1982]). In the complete information game, the first offer is accepted, and as $\delta \to 1$, $\overline{y}(s,b)$ and $\underline{y}(s,b)$ converge to an equal surplus split, which I denote by $y^*(s,b) \equiv \frac{1}{2}(c(s)+v(b))$. When $s = b$, I write $y^*(s)$ instead of $y^*(s,b)$, and the same notation is adopted for $\overline{y}$ and $\underline{y}$. The complete-information outcome provides noncooperative foundations to the axiomatic Nash Bargaining Solution (Nash [1953]). Hence, I refer to the equal surplus split as the *Nash Bargaining split*, or simply, the *Nash split*.

2.1 The Set of Individually Rational Payoffs

In this subsection, I introduce the set of individually rational expected payoffs and its Pareto frontier. The following lemma gives weak bounds on equilibrium prices and payoffs:

Lemma 1. *In any PBE and after any history,*

1. *The seller accepts with probability one any offer above* $\underline{y}(1)$*, and the buyer rejects with probability one any offer above* $\overline{y}(1)$*.*

2. *The buyer accepts with probability one any offer below* $\overline{y}(0)$*, and the seller rejects with probability one any offer below* $\underline{y}(0)$*.*

3. *The seller's continuation utility is at least* $\max\{\underline{y}(0) - c(s), 0\}$*, and the buyer's continuation utility is at least* $\max\{v(b) - \overline{y}(1), 0\}$*.*

Proof. See Appendix A.1. □

When the variation in values across types is large, then $\underline{y}(0)$ and $\overline{y}(1)$ are far apart and Lemma 1 puts only weak bounds on the price of a trade even when η is very small. These bounds are standard in the bargaining literature (e.g., Grossman and Perry [1986], Watson [1998]). By Lemma 1, the seller cannot do better than if she convinces the buyer

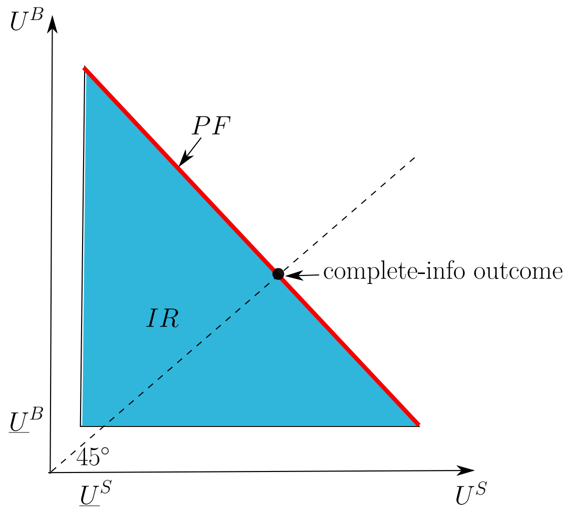

Figure 2: ***The set of feasible, ex-ante individually rational payoffs and its Pareto frontier***

The complete-information outcome in the figure corresponds to the immediate equal surplus split, $(\tau, \rho) = (0, y^*(s, b))$.

that her cost is the highest possible and the buyer admits that his value is the highest possible. Moreover, the buyer always has the option to trade immediately at price $\underline{y}(1)$ by admitting that he has the highest valuation and by recognizing that the seller has the highest cost. This option combined with the opportunity to reject all seller's offers gives the lower bound on the buyer's utility in Lemma 1.

If $U_\eta^S(\tau, \rho) = \mathbb{E}_\eta \left[e^{-r\tau(s,b)}(\rho(s, b) - c(s))\right]$ and $U_\eta^B(\tau, \rho) = \mathbb{E}_\eta \left[e^{-r\tau(s,b)}(v(b) - \rho(s, b))\right]$ are the players' expected ex-ante payoffs from the outcome (τ, ρ), then the set of ex-ante payoffs supported by double limits is:

$$E = \left\{ \lim_{\eta \to 0} \left(U_\eta^S(\tau_\eta, \rho_\eta), U_\eta^B(\tau_\eta, \rho_\eta) \right) : (\tau_\eta, \rho_\eta) \underset{\eta \to 0}{\to} (\tau, \rho) \in DL \right\}.$$

The convex hull of the closure of E is denoted by $\mathcal{E}$. Furthermore, when players have access to the public randomization device in the beginning of the game, then $\mathcal{E}$ is the set of ex-ante expected payoff profiles that can be approximated by the double limits in my model.

Let us put some preliminary restrictions on $\mathcal{E}$. Fix η and any outcome (τ_η, ρ_η) of the frequent-offer PBE limit. Clearly, the *feasibility constraint* holds:

$$U_\eta^S(\tau_\eta, \rho_\eta) + U_\eta^B(\tau_\eta, \rho_\eta) \leq \mathbb{E}_\eta[\Pi(s, b)].$$

Moreover, Lemma 1 implies that the seller's utility is at least $\underline{U}_{\eta}^{S} \equiv \mathbb{E}_{\eta}[\max\{y^{*}(0) - c(s), 0\}]$; and symmetrically, the buyer's utility is at least $\underline{U}_{\eta}^{B} \equiv \mathbb{E}_{\eta}[\max\{v(b) - y^{*}(1), 0\}]$. Hence, the *ex-ante individual rationality constraints* holds:

$$U_{\eta}^{S}(\tau_{\eta}, \rho_{\eta}) \geq \underline{U}_{\eta}^{S} \text{ and } U_{\eta}^{B}(\tau_{\eta}, \rho_{\eta}) \geq \underline{U}_{\eta}^{B}.$$

Denoting the limits of $\mathbb{E}_{\eta}[\Pi(s,b)], \underline{U}_{\eta}^{S}$, and $\underline{U}_{\eta}^{B}$ as $\eta \to 0$ by $\Pi, \underline{U}^{S}$, and $\underline{U}^{B}$, respectively, I get that $\mathcal{E} \subseteq IR$, where

$$IR = \{(U^{S}, U^{B}) : U^{S} + U^{B} \leq \Pi, U^{S} \geq \underline{U}^{S}, \text{ and } U^{B} \geq \underline{U}^{B}\}, \tag{1}$$

is the set of feasible, ex-ante individually rational payoffs. The ex-ante Pareto frontier is:

$$PF = \{(U^{S}, U^{B}) : U^{S} + U^{B} = \Pi, U^{S} \geq \underline{U}^{S}, \text{ and } U^{B} \geq \underline{U}^{B}\}.$$

(See Figure 2 for the illustration). For simplicity of exposition, I focus on characterizing the ex-ante payoffs in the double limits. Interim versions of the main results can be found in the Online Appendix.

3 Main Results

This section shows that a variety of bargaining outcomes arises as double limits. Subsecion 3.1 derives the optimal punishment for detectable deviations from the equilibrium path. Subsection 3.2 characterizes the Pareto frontier of double limits. Subsection 3.3 shows that requiring the double limit to approximate the Nash split does not necessarily mean efficiency. Subsection 3.4 proves the Folk theorem.

3.1 Optimal Punishment

The literature on bargaining shows that two-sided offers and two-sided independent private information can create a variety of bargaining dynamics by punishing deviations from the equilibrium path with optimistic conjectures: the opponent of the deviator assigns probability one to the weakest type of the deviator, and this way the deviator gets a low continuation utility after the deviation. Similarly, I construct continuation equilibria with optimistic conjectures to deter deviations. In my model, it is a priori not clear whether optimistic conjectures are as efficient in deterring deviations. Indeed, when η is small, players' beliefs can be updated only marginally within the support B_s or S_b. Nevertheless, in the next lemma, I show that when offers are frequent, the deviator's utility in the punishing continuation equilibrium is independent of η and approximates the lower bound in Lemma 1. I call a deviation by the seller *detectable*, if *all* buyer types assign it probability zero on the equilibrium path.

Lemma 2 (**Contagious Coasian Property**). *Fix a PBE. Consider a history H_n that contains no buyer's deviations and ends with the seller's detectable deviation. If after history H_n, the seller's posterior beliefs are the truncation of the prior from above at $\overline{b}$ and from below at $\underline{b}$, then for any $\varepsilon > 0$, there is $\delta(\varepsilon) \in (0,1)$ such that for all $\delta \in (\delta(\varepsilon), 1)$, there is a continuation equilibrium in which the continuation utility of any remaining seller type $s \in [\underline{\pi}(\underline{b}), \overline{\pi}(\overline{b})]$ is at most* $\max\{y^*(\underline{b}) - c(s), 0\} + \varepsilon$.

Remark 1. *I can switch the buyer and the seller in Lemma 2 to derive the punishing continuation equilibrium for the buyer's detectable deviations. In this case, the continuation utility of any buyer type b is at most* $\max\{v(b) - y^*(\overline{s}), 0\} + \varepsilon$, *where $\overline{s}$ is defined analogously to $\overline{b}$ in Lemma 2.*

I refer to the continuation equilibrium in Lemma 2 as the *seller punishing (continuation) equilibrium with $\underline{b}$*. It has several useful properties. First, the punishment is the harshest possible. It approximates the lower bound on the seller's utility in Lemma 1 that does not depend on η. Second, this equilibrium punishes all seller types simultaneously, and hence, the buyer does not need to know which seller type deviated. Further, the convergence is uniform, that is, $\delta(\varepsilon)$ does not depend on s or $\underline{b}$.

Remark 2. *However, $\delta(\varepsilon)$ in Lemma 2 does depend on η, and it converges to one as $\eta \to 0$.*[6] *Thus, the order of limits is important for the results. Intuitively, when η decreases, each buyer type's optimistic conjecture assigns probability one to a higher seller type, who in turn believes that the buyer type comes from a higher interval in which every buyer type assigns probability one to higher seller types, and so on. This fact results in the higher maximal willingness to pay of buyer types, which in turn, increases the seller's reservation price.*

The proof of Lemma 2 builds on the existing results to construct the continuation equilibrium strategies for types $b \in [\underline{b}, \underline{b}+\eta]$ and $s = \underline{b}$ that exhibit Coasian dynamics, and

[6] To see this, note that the right-hand side of the inequality (36) in Lemma 11 in Appendix A.1 converges to $\underline{y}(s)$ as $\eta \to 0$. Hence, for sufficiently small η, the seller never accepts offers below $\underline{y}(s) - \varepsilon$ and so, is guaranteed utility of at least $\underline{y}(s) - c(s) - \varepsilon$ after any history.

then uses the contagion argument to extend the construction and the Coasian dynamics to higher types. The proof is deferred till Section 4.

3.2 Pareto Frontier

Lemma 2 in the previous subsection shows that the high correlation of values does not restrict the ability to punish players for detectable deviations as long as offers are frequent. This lemma gives the bounds in terms of payoffs on the outcomes potentially attainable by double limits. The next theorem characterizes the Pareto frontier of bargaining outcomes that are sustainable as double limits.

Theorem 1 (Pareto Frontier). *It obtains*

1. *if* $(U^S, U^B) \in PF$, *then* $(U^S, U^B) \in \mathcal{E}$;

2. $(0, y^*(s, b)) \in DL$.

The first part of Theorem 1 states that any point of the ex-ante Pareto frontier can be approximated by some double limit. The $\mathcal{E} \subseteq IR$ implies that the ex-ante Pareto frontier of payoff profiles that are approximated by double limits is efficient. However, the surplus split in general differs from the Nash split. The second part states that it is still possible to approximate the complete-information outcome $(0, y^*(s, b))$ with double limits, that is, for $\eta \approx 0$ I can construct PBEs in which the ex-post surplus split is close to equal and the trade delay is short with high probability.

The key to Theorem 1 is the construction of PBEs with segmentation dynamics in which players establish common knowledge of a narrow range of values early in the game, and this way, avoid lengthy bargaining delays. I next construct these PBEs.

Proof of Theorem 1. The proof proceeds in five steps.

Step 1: Bargaining outcomes spanning the Pareto frontier I first construct bargaining outcomes that approximate the expected payoff profiles in Theorem 1 when types are uniformly distributed on the diagonal Ω_0. For $x \in [0, 1]$ and $\gamma \in (0, 1)$, I define the trade price as

$$q(x, \gamma) \equiv (1 - \gamma) y^*(x) + \gamma \max\{c(x), y^*(0)\}, \tag{2}$$

which is the convex combination of the complete-information price for types $s = x$ and $b = x$ and the minimal price that type $s = x$ can accept in any frequent-offer PBE limit.

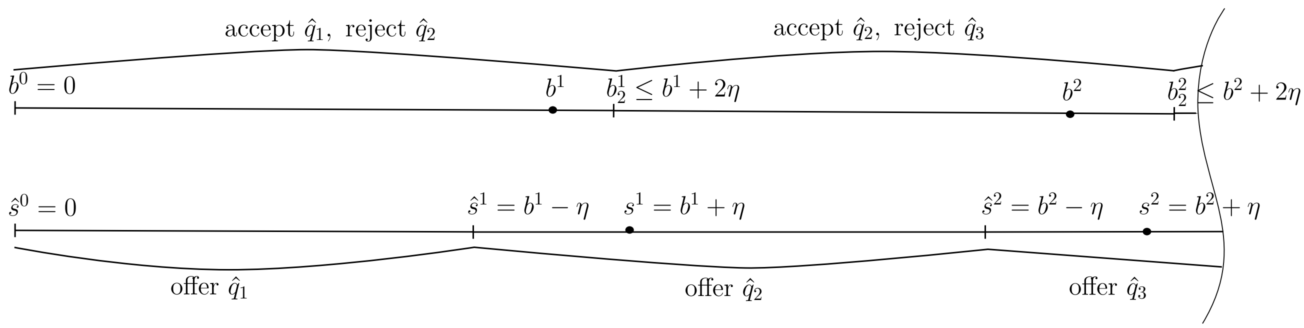

Figure 3: ***Illustration of On-Path Strategies in Step 2 of the Proof***
In round 1, types $s \in [0, \hat{s}^1)$ offer $\hat{q}_1$, types $s \in [\hat{s}^1, \hat{s}^2)$ offer $\hat{q}_2$, types $s \in [\hat{s}^2, \hat{s}^3)$ offer $\hat{q}_3$, and so on. If the first round offer is $\hat{q}_1$, then all buyer types accept it; if the first round offer is $\hat{q}_2$, then types $b < b_2^1$ reject it and types $b \geq b_2^1$ accept it; if the first round-offer is $\hat{q}_3$, then types $b < b_2^2$ reject it and types $b \geq b_2^2$ accept it; and so on.

In the bargaining outcome $(0, q(s, \gamma))$, trade is immediate at price $q(s, \gamma)$. Step 5 below shows that for any $\gamma \in (0, 1)$, the outcome $(0, q(s, \gamma))$ is a double limit. Therefore, while varying γ from 0 to 1, I span all ex-ante expected payoff profiles on the segment of Pareto frontier that connects $(\frac{1}{2}\Pi, \frac{1}{2}\Pi)$ and $(\underline{U}^S, \Pi - \underline{U}^S)$. To span the segment of Pareto frontier that connects $(\frac{1}{2}\Pi, \frac{1}{2}\Pi)$ and $(\Pi - \underline{U}^B, \underline{U}^B)$, I can use the same argument with the roles of the seller and the buyer reversed and instead of $q(x, \gamma)$ use trade prices

$$p(x, \gamma) \equiv (1 - \gamma)y^*(x) + \gamma \min\{v(x), y^*(1)\}. \tag{3}$$

This way I can span the whole ex-ante Pareto frontier with double limits. This proves the first part of Theorem 1. Further, as $\gamma \to 0$, the outcome $(0, q(s, \gamma))$ for all types converges to the outcome $(0, y^*(s, s))$, which in turn approximates the complete-information outcome, $(0, y^*(s, b))$, as $\eta \to 0$. This proves the second part of Theorem 1.

Step 2: Construction of segments Fix $\gamma \in (0, 1)$. To approximate $(0, q(s, \gamma))$, I partition the set of types into segments of length $\sqrt{\eta}$ and construct PBEs in which the seller reveals in the first round her segment with a price offer "close" to $q(s, \gamma)$. This is accepted by "most" types of buyer except for a "small" interval of types near the boundaries of the segments that engage in the war-of-attrition type of bargaining in the continuation.

In this step, I describe the on-path strategies in round 1, when the seller reveals her segment. (See Figure 3 for the illustration). The number of segments is $Z \equiv \lfloor 1/\sqrt{\eta} \rfloor$. Let

$b^0 = s^0 = 0, b^Z = s^Z = 1$, and define for $z = 1, \dots, Z-1$,

$$b^z = b^{z-1} + \sqrt{\eta}, \tag{4}$$

$$s^z = b^z + \eta. \tag{5}$$

I refer to types b^z and s^z for $z = 1, \dots, Z-1$ as *boundary types*. Let $\hat{s}^0 = 0$,$\hat{s}^Z = 1$, and define for $z = 1, \dots, Z-1$,

$$\hat{s}^z = b^z - \eta, \tag{6}$$

$$q_z = q(b^z, \gamma), \tag{7}$$

$$\hat{q}_z = (1-\delta^2)c(\hat{s}^z) + \delta^2 q_z, \tag{8}$$

and $\hat{q}_Z = q(b^Z, \gamma)$. I specify that in the PBE, in round 1, for $z = 0, \dots, Z-1$ seller types $s \in [\hat{s}^z, \hat{s}^{z+1})$ offer $\hat{q}_{z+1}$.[7] The interpretation is that seller types signal one of Z segments in round 1. Offers q_z and $\hat{q}_z$ increase with z, i.e., higher segments are associated with higher offers. First round offer $\hat{q}_z$ is constructed so that type $\hat{s}^z$ is indifferent between trading at price $\hat{q}_z$ now and trading at q_z in two rounds. I suppose that η is sufficiently small so that $\sqrt{\eta} > 4\eta$, and[8]

$$c(x + 3\sqrt{\eta}) < q(x, \gamma) < v(x - 3\sqrt{\eta}), \text{ for all } x \in [0, 1], \tag{9}$$

$$q(\min\{x + \eta, 1\}, \gamma) < y^*(x), \text{ for all } x \in [\sqrt{\eta} - \eta, 1]. \tag{10}$$

Step 3: War-of-attrition dynamics near boundaries In this step, I construct equilibrium strategies on- and off-path after the on-path play in round 1. Suppose in round 1, the seller types $s \in [\hat{s}^z, \hat{s}^{z+1})$ make offer $\hat{q}_{z+1}$ for some $z = 0, \dots, Z-1$. Observe that since $\hat{s}^{z+1} - \hat{s}^z \geq \sqrt{\eta} - \eta > 2\eta$ for all $z = 0, \dots, Z-1$, each buyer type expects to receive one of at most two offers on the equilibrium path in round 1.

On-path strategies after round 1 For $z = 0$, only buyer types $b \in [0, \overline{\pi}(\hat{s}^1))$ expect offer $\hat{q}_1$, and they accept it. I now consider $z = 1, \dots, Z-1$. Let $b_1^z = \overline{\pi}(\hat{s}^{z+1})$ and $s_2^z = \hat{s}^z$. Consider $(b_n^z)_{n=2}^\infty$ and $(s_n^z)_{n=3}^\infty$ such that b_n^z is strictly decreasing for even n and

[7] Here and further, all the intervals of types are closed for $z = Z-1$, e.g., types $s \in [\hat{s}^{Z-1}, \hat{s}^Z]$ offer $\hat{q}_Z$.

[8] Since $c(x) < q(x, \gamma) < v(x)$ for all $x \in [0, 1]$, and c, v, and $q(\cdot, \gamma)$ are continuous functions, the inequality (9) holds for sufficiently small η. The inequality (10) holds, as $y^*(x) > q(x)$ for $x \in (0, 1]$, and $y^*(\cdot)$ and $q(\cdot, \gamma)$ are continuous functions.

is constant for odd n, s_n^z is strictly decreasing for even n and is constant for odd n, and that satisfy

$$v(b_n^z) - \hat{q}_{z+1} = \delta^2 \left(\alpha_n^S(v(b_n^z) - q_z) + (1-\alpha_n^S)(v(b_n^z) - \hat{q}_{z+1})\right) \text{ for } n \text{ even}, \tag{11}$$

$$q_z - c(s_n^z) = \delta^2 \left(\alpha_n^B(\hat{q}_{z+1} - c(s_n^z)) + (1-\alpha_n^B)(q_z - c(s_n^z))\right) \text{ for } n \text{ odd}, \tag{12}$$

where

$$\alpha_n^S = \frac{s_{n+1}^z - s_{n-1}^z}{b_n^z + \eta - s_{n-1}^z}, \tag{13}$$

$$\alpha_n^B = \frac{b_{n-1}^z - b_{n+1}^z}{b_{n-1}^z - s_n^z + \eta}. \tag{14}$$

Lemma 3. *Fix $z = 1, \ldots, Z-1$. There are sequences $(b_n^z)_{n=2}^\infty$ and $(s_n^z)_{n=3}^\infty$ as specified above such that $b_n^z \downarrow b^z$ and $s_n^z \uparrow s^z$, and $b_2^z - b^z \in (0, 2\eta]$ and $s^z - s_3^z \in (0, 2\eta]$.*

The proof of this and the other lemmas in this subsection are in Appendix $A.2$. On the equilibrium path, only types $s \in [\hat{s}^z, \hat{s}^{z+1})$ make offer $\hat{q}_{z+1}$, and only buyer types $b \in [\underline{\pi}(\hat{s}^z), \overline{\pi}(\hat{s}^{z+1}))$ expect to receive it. The on-path strategies are as follows:

- For even n, if offer $\hat{q}_{z+1}$ is made in round $n-1$, then buyer types $b \in [b_n^z, b_{n-1}^z)$ accept it, types $b \in [\underline{\pi}(\hat{s}^z), b_n^z)$ reject it and make a counter-offer $\underline{y}(0)$. If offer q_z is made in round $n-1$, then all (remaining) buyer types $b \in [\underline{\pi}(\hat{s}^z), b_{n-1}^z)$ accept it. (See Figure 3 for the illustration of thresholds b_n^z for $n = 2$).

- For odd n, types $s \in [s_{n-1}^z, s_n^z)$ make offer q_z, and types $s \in [s_{n-1}^z, \hat{s}^{z+1})$ make offer $\hat{q}_{z+1}$.

Remark 3. *Threshold types b_n^z (similarly, s_n^z) are chosen so that they are just indifferent between accepting $\hat{q}_{z+1}$ in round n or rejecting it and accepting it in round $n+2$. Indeed, acceptance of $\hat{q}_{z+1}$ brings utility $v(b_n^z) - \hat{q}_{z+1}$ (the left-hand side of* (11)*). On the other hand, after the rejection of $\hat{q}_{z+1}$, the seller lowers her offer to q_z with probability α_n^S and with complementary probability offers $\hat{q}_{z+1}$ in round $n+1$ (the right-hand side of* (11)*). Note that α_n^S is the probability that type b_n^z assigns to the seller offering q_z. Type b_n^z believes that seller types $s \in [s_{n-1}^z, b_n^z + \eta]$ remain in the game in round n, and types $s \in [s_{n-1}^z, s_{n+1}^z)$ will offer q_z in the next round. This logic gives equation* (13)*. (Equations* (12) *and* (14) *for the seller threshold types are interpreted analogously). Observe that since types are correlated, the probability* (13) *that the seller lowers her offer to q_z depends on the threshold type b_n^z, which would not be the case if the types were independent.*

Off-path strategies after round 1

- If the buyer makes an offer different from $\underline{y}(0)$, then the play switches to the buyer punishing equilibrium in Remark 1 with $\overline{s} = \hat{s}^{z+1}$ if the seller rejects it. The seller rejects such an offer if and only if it brings her a lower utility compared to the continuation utility in the buyer punishing equilibrium with $\overline{s} = \hat{s}^{z+1}$.

- If the buyer type $b \in [b^z_{n-1}, \overline{\pi}(\hat{s}^{z+1}))$ remains in the game in even round n, because of the deviation from the acceptance strategy in some previous round, and in round $n-1$ the seller offers $\hat{q}_{z+1}$ or q_z (and the seller did not deviate prior to $n-1$), then type b accepts the seller's offer in round n.

- If the buyer type $b \in [\underline{\pi}(\hat{s}^z), \overline{\pi}(\hat{s}^{z+1}))$ counter-offers $\underline{y}(0)$ in round n to the seller's offer q_z made by seller types in $[s^z_{n-2}, s^z_{n-1})$ in round $n-1$, then in round $n+1$, seller types in $[s^z_{n-2}, s^z_{n-1})$ make offer q_z again, which is accepted by the buyer type $b \in [\underline{\pi}(\hat{s}^z), \overline{\pi}(\hat{s}^{z+1}))$ in round n.

- After any history without any previous seller's detectable deviations, if the seller makes an offer different from $\hat{q}_{z+1}$ or q_z, then if the buyer rejects it, the play switches to the seller punishing equilibrium in Lemma 2 with $\underline{b} = 0$. The buyer rejects such an offer if and only if it brings him a lower utility compared to the continuation utility in the seller punishing equilibrium with $\underline{b} = 0$.

- If the seller makes an offer $\hat{q}_{z+1}$ when the equilibrium strategies prescribe her to offer q_z, then it is rejected by the buyer types who detect this deviation. In the next round, the seller makes offer q_z that is accepted by all remaining types of buyer.

- If the seller makes an offer q_z when the equilibrium strategies prescribe her to offer $\hat{q}_{z+1}$, then it is accepted by all remaining types of buyer.

The next lemma verifies that these strategies constitute the equilibrium in the continuation after the on-path play in round 1.

Lemma 4. *Fix a PBE and $z = 0, \dots, Z-1$. Suppose on-path in round 1, seller types in $[\hat{s}^z, \hat{s}^{z+1})$ make offer $\hat{q}_{z+1}$. There is $\hat{\delta} \in (0,1)$ such that for all $\delta \in (\hat{\delta}, 1)$ the strategies specified above constitute equilibrium in the continuation game.*

Since Z is finite, it follows from Lemma 4 that the constructed strategies constitute continuation equilibria for all $z = 0, \dots, Z-1$ for sufficiently large δ.

Remark 4. *The constructed continuation equilibrium has dynamics similar to the war-of-attrition game in which all buyer types pool at the lower offer q_z and gradually accept the seller's offer starting from the top, and seller types pool at the higher offer $\hat{q}_{z+1}$ and gradually accept the buyer's offer starting from the bottom. The difference is that in my construction, the buyer types pool at the lowest offer $\underline{y}(0)$ and gradually accept the higher offer $\hat{q}_{z+1}$, and the seller types at the bottom gradually reveal themselves with an offer q_z that is accepted by all buyer types. This construction ensures that the buyer after round 1 does not reveal that his type is above $\underline{\pi}(\hat{s}^z)$, which would happen if the buyer types did not pool at $\underline{y}(0)$, but instead made partially revealing offers. Thus, the seller's utility in the punishing equilibrium is close to* $\max\{0, y^*(0) - c(s)\}$ *(instead of* $\max\{0, y^*(\underline{\pi}(\hat{s}^z)) - c(s)\}$*), which in turn makes trades at prices q_z and $\hat{q}_{z+1}$ (which might be below $y^*(\underline{\pi}(\hat{s}^z))$) possible. Note that the difference between these two dynamics vanishes as $\delta \to 1$.*

Step 4: Verification of equilibrium conditions In this step, I construct the rest of the off-path strategies and verify that these strategies indeed constitute the PBE for sufficiently large δ. The following off-path histories have not been covered yet in the previous step:

- Suppose that in round 1 the seller type $s \in [\hat{s}^z, \hat{s}^{z+1})$ deviates to an offer $q \in Q^S \equiv \{\hat{q}_1, \ldots, \hat{q}_Z\}$. If $q < \hat{q}_{z+1}$, then it is accepted by the buyer. If $q > \hat{q}_{z+1}$, then it is rejected by the buyer; and in round 3, type s offers $\hat{q}_{z+1}$ and the game proceeds as on the equilibrium path.

- After any deviation of the seller in round 1 to offers not in Q^S, players switch to the seller punishing equilibrium in Lemma 2 with $\underline{b} = 0$, if such an offer gets rejected. The buyer rejects such an offer if and only if it brings him a lower utility compared to the continuation utility in the seller punishing equilibrium with $\underline{b} = 0$.

Lemma 5. *Strategies constructed in Steps 2-4 constitute the PBE for sufficiently large δ.*

Step 5: Proof of convergence On the equilibrium path, for any $z = 1, \ldots, Z-1$, buyer types in $[b_2^z, \underline{\pi}(\hat{s}^{z+1})] \subseteq [b^z + 2\eta, b^{z+1} - 2\eta]$ receive offer $\hat{q}_z$ and accept it in round 2. Since $b^{z+1} - b^z \geq \sqrt{\eta}$, the fraction of buyer types that trade in round 2 is of the order $1 - {}^{4\eta}/_{\sqrt{\eta}} \underset{\eta\to 0}{\to} 1$. Moreover, such types trade at prices close to $q(b, \gamma)$, since $|q_z - \hat{q}_z| \underset{\delta\to 1}{\to} 0$ and $\max_{x \in [b^{z-1}+2\eta, b^z - 2\eta]} |q_z - q(x, \gamma)| \underset{\eta\to 0}{\to} 0$. This argument implies the convergence at the double limit of the bargaining outcome arising from the constructed PBE strategies to $(0, q(s, \gamma))$. $\square$

Remark 5. *In the proof of Theorem* 1*, the number of segments is* Z*, which goes to infinity at rate* $1/\sqrt{\eta}$ *as* $\eta \to 0$*. This construction implies the probability of delay longer than two rounds of order* $Z\eta = \sqrt{\eta} \underset{\eta \to 0}{\to} 0$*. However, if I fix the number of segments at* $Z_0 = 1/\sqrt{\eta_0}$ *for some* η_0 *sufficiently small so that the construction above is valid, then the probability of delay longer than two rounds is of order* η*, which has an intuitive interpretation: the inefficiency is of the same order as the players' uncertainty about each others' values.*

3.3 Nash Split with Delay

In this subsection, I construct a double limit that approximates the Nash split but with a bargaining delay.

Auxiliary War-of-Attrition Game A preliminary step is to consider the auxiliary continuous-time war-of-attrition game in which players only choose the acceptance time but not offers. Types and payoffs are as in the original model. Suppose paths of offers are exogenously fixed and given by the continuous, strictly decreasing function q_t^S for the seller and the continuous, strictly increasing function q_t^B for the buyer. I suppose that for some $T < \infty$, $q_t^S = q_T^S$ and $q_t^B = q_T^S$ for $t \geq T$, i.e., offers are constant starting from time T. Before the bargaining starts, each player chooses his or her acceptance time conditional on his or her type, and privately commits to it, i.e., he or she cannot revise it over the course of the game. Thus, the game is essentially static. When one of the parties accepts, the trade occurs at the accepted price. If both parties accept simultaneously at time t, then the price is $\frac{q_t^B + q_t^S}{2}$. I denote such a game by $\mathcal{G}(q_t^S, q_t^B)$.

I consider Bayesian Nash equilibria (BNE) of $\mathcal{G}(q_t^S, q_t^B)$ in threshold strategies. The threshold acceptance strategies are described by the strictly decreasing $b_t : [0, \infty) \to [0, 1]$ and the strictly increasing $s_t : [0, \infty) \to [0, 1]$ such that for any t, all buyer types above b_t and all seller types below s_t accept the opponent's offer at t. I further restrict b_t and s_t to be right-continuous with discontinuities possible only at 0 and T. The game $\mathcal{G}(q_t^S, q_t^B)$ has the following BNE in threshold strategies.

Lemma 6. *Suppose there are types* $b_\infty \in (2\eta, 1-3\eta)$, $b_0 \in (b_\infty+2\eta, 1]$, $s_0 \in [0, b_\infty-\eta)$*, time* $T < \infty$*, and thresholds strategies* b_t *and* s_t *that for* $s_\infty \equiv b_\infty+\eta$, $\overline{s}_T \equiv s_\infty-2\eta$, $\overline{b}_T \equiv b_\infty+2\eta$ *satisfy*

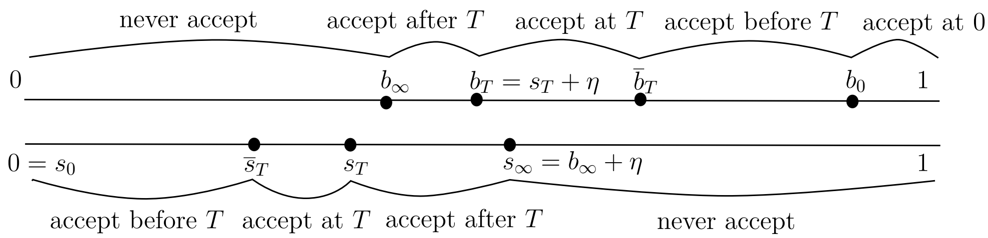

Figure 4: ***Illustration of strategies in Lemma*** 6
Buyer types in $[0, b_\infty]$ *never accept; the acceptance strategy of buyer types* $b \in (b_\infty, b_T)$ *is given by* (15)*; buyer types in* $[b_T, \bar{b}_T]$ *accept at* T*; the acceptance strategy of buyer types* $b \in (\bar{b}_T, b_0)$ *is given by* (18)*; buyer types in* $[b_0, 1]$ *accept at time zero. The seller's strategy is analogous.*

1. *for* $t \in [T, \infty)$*, it holds* $v(b_\infty) - q_T^S > 0$ *and* $q_T^B - c(s_\infty) > 0$*, and* b_t *and* s_t *solve*

$$r(v(b_t) - q_T^S) = \frac{\dot{s}_t}{b_t - s_t + \eta}(q_T^S - q_T^B), \tag{15}$$

$$r(q_T^B - c(s_t)) = -\frac{\dot{b}_t}{b_t - s_t + \eta}(q_T^S - q_T^B), \tag{16}$$

$$b_T = s_T + \eta, \lim_{t\to\infty} b_t = b_\infty, \lim_{t\to\infty} s_t = s_\infty; \tag{17}$$

2. *for* $t \in [0, T)$*, it holds* $v(b_t) - q_t^S > 0$ *and* $q_t^B - c(s_t) > 0$*, and* b_t *and* s_t *solve*

$$r(v(b_t) - q_t^S) = -\dot{q}_t^S, \tag{18}$$

$$r(q_t^B - c(s_t)) = \dot{q}_t^B \tag{19}$$

$$s_t|_{t=0} = s_0, b_t|_{t=0} = b_0, \lim_{t\uparrow T} b_t = \bar{b}_T, \lim_{t\uparrow T} s_t = \bar{s}_T \tag{20}$$

Then b_t *and* s_t *constitute a BNE in the war-of-attrition game* $\mathcal{G}(q_t^S, q_t^B)$*.*

Figure 4 illustrates strategies in Lemma 6. After time T, the dynamics are similar to the war-of-attrition dynamics constructed in Step 3 of the proof for Theorem 1: offers are constant and both sides gradually accept the opponent's offer so that cutoff types b_t and s_t are just indifferent between accepting at t and marginally delaying the acceptance. In fact, conditions (15) and (16) are continuous-time counter-parts of (11) and (12), respectively. Incentives to wait after time T come from the positive probability that the counter-offer is accepted by the opponent.

Note why threshold strategies are optimal for types that accept after time T. If types were independent, then the optimality would follow from the single crossing property of payoffs. For example, the buyer with a higher value is more impatient and accepts earlier, given the likelihood of the seller's concession. With correlated types, this logic might not apply. If the seller with higher costs accepts faster than the seller with lower costs, this creates a countervailing force that makes higher types of the buyer more patient. However, this situation does not occur when both sides follow threshold strategies and types are positively correlated, as the buyer with higher value also assigns positive probability to higher types of the seller, who accept later in the game. Since both forces point in the same direction, the optimality of the threshold strategies holds.

Before time T, by $\bar{b}_T > \bar{s}_T + \eta$, each side assigns probability zero to his or her offer being accepted and the only incentive for waiting comes from opponent's more favorable offers in the future. Thus, the two-sided screening dynamics emerge. For the types above $\bar{b}_T$, the buyer is screened by a path of the seller's decreasing offers q_t^S; while for the types below $\bar{s}_T$, the seller is screened by a path of the buyer's increasing offers q_t^B. The optimality of threshold strategies here follows from the standard skimming property: higher types of the buyer and lower types of the seller are more impatient, and thus, accept earlier. Because of the two-sided screening dynamics, there is necessarily a delay, as long as $q_0^B < q_T^B$ and $q_0^S > q_T^S$.

Inefficient Limits In the original model, I am interested in approximating with frequent-offer PBE limits the bargaining dynamics in a particular class of BNEs in Lemma 6 that allow the approximation of the Nash split with delay and that are also used in the proof of the Folk theorem. Specifically, given $\beta \in (0,1)$, b_0 and s_0 are:

$$\beta y^*(b_0) + (1-\beta)v(b_0) = y^*(1), \tag{21}$$

$$\beta y^*(s_0) + (1-\beta)c(s_0) = y^*(0). \tag{22}$$

For $t \in [0, T)$,

$$q_t^S = \beta y^*(b_t) + (1-\beta)v(b_t), \tag{23}$$

$$q_t^B = \beta y^*(s_t) + (1-\beta)c(s_t). \tag{24}$$

Buyer types above b_0 and seller types below s_0 immediately accept $y^*(1)$ and $y^*(0)$, respectively, and until time T, the threshold types b_t and s_t accept prices that are convex combinations of the Nash prices and their values. I can get b_t and s_t for $t \in [0, T)$ by solving the system $(18) - (20)$ with s_0, b_0, q_t^S, and q_t^B, and find $\bar{b}_T$ and $\bar{s}_T$ such that $\bar{b}_T = \bar{s}_T + 3\eta$. Set $q_T^S \equiv \lim_{t\uparrow T} q_t^S$,$q_T^B \equiv \lim_{t\uparrow T} q_t^B$, $b_\infty \equiv \bar{b}_T - 2\eta$,$s_\infty \equiv \bar{s}_T + 2\eta$. I can solve $(15)-(17)$ to get b_t and s_t for $t \geq T$, and this way determine the BNE in game $\mathcal{G}(q_t^S, q_t^B)$. The bargaining outcome of this BNE is denoted by $(\tau_\eta^\beta, \rho_\eta^\beta)$. The next lemma approximates $(\tau_\eta^\beta, \rho_\eta^\beta)$ with the frequent-offer PBE limit.

Lemma 7 (**Inefficient Limit**). *For any $\beta \in (0,1)$, there is $\eta(\beta) > 0$ such that for all $\eta \in (0, \eta(\beta))$ there is a frequent-offer PBE limit (τ, ρ) such that $\tau(s,b) = \tau_\eta^\beta(s,b)$ and $\rho(s,b) = \rho_\eta^\beta(s,b)$ whenever $b > \bar{b}_T + \eta$ or $s < \bar{s}_T - \eta$.*

Proof Outline of Lemma 7. The complete the proof is in Appendix A.3, and here, I outline the main steps. Let $N \equiv \lfloor T/\Delta \rfloor$, and suppose without loss of generality that it is even. I construct decreasing thresholds b_n, increasing thresholds s_n, and discrete offer paths q_n^S and q_n^B such that $b_{N-2} = \bar{b}_T$ and $s_{N-1} = \bar{s}_T$; $q_n^S = q_T^S$ for even $n \geq N$, and $q_n^B = q_T^B$ for odd $n \geq N+1$; and paths q_n^S and q_n^B approximate q_t^S and q_t^B for $n < N$. I then construct PBEs with the following on-path strategies:[9]

- In an odd round n, if offer q_{n-1}^S is made in the previous round, then it is accepted by all remaining seller types. If offer $\underline{y}(0)$ is made in round $n-1$, seller types in $[s_n, 1]$ offer $\overline{y}(1)$, and seller types in $[s_{n-1}, s_n)$ offer q_n^B.
- In an even round n, if offer q_{n-1}^B is made in the previous round, then it is accepted by all remaining buyer types. If offer $\overline{y}(1)$ is made in round $n-1$, then buyer types in $[0, b_n]$ offer $\underline{y}(0)$ and buyer types in $(b_n, b_{n-1}]$ offer q_n^S.

The construction of (b_n, s_n, q_n^B, q_n^S) is different before and after round N. The difference comes from the fact that types that reveal themselves in rounds before N assign probability zero to their opponent revealing him or herself (due to the bounded support of beliefs). While after round N, there is a positive probability that the opponent reveals him or herself.

For $n \geq N$, I replicate Step 3 in the proof of Theorem 1 to construct the discrete-time war-of-attrition with constant revealing offers q_T^S and q_T^B. For $n < N$, only one side actively reveals itself and so, unlike in the case $n \geq N$, there is no need to carefully choose b_n and s_n to guarantee that both sides have incentives to reveal themselves at the appropriate rate. Hence, the tuple (b_n, s_n, q_n^B, q_n^S) is a discretization of continuous-time paths (b_t, s_t, q_t^B, q_t^S). As before, I punish detectable deviations with punishing equilibria in Lemma 2 and Remark 1.

[9]Note the difference between these strategies and the strategies in $\mathcal{G}(q_t^S, q_t^B)$: here, the seller reveals that her type is in $[s_{n-1}, s_n)$ with offer q_n^B, while in $\mathcal{G}(q_t^S, q_t^B)$, offer q_t^B is made by the buyer and accepted by seller type s_t. This construction guarantees the optimality of such strategies when players deviate and do not reveal themselves in the prescribed round.

I can use PBEs with on-path strategies described by (b_n, s_n, q_n^B, q_n^S) to complete the proof. Whenever $b > \bar{b}_T + \eta$ or $s < \bar{s}_T - \eta$, the bargaining outcome is determined by on-path strategies before round N, and because $\mathbb{P}_\eta(\{(s,b) : b > \bar{b}_T + \eta \text{ or } s < \bar{s}_T - \eta\}) \underset{\eta \to 0}{\to} 1$, strategies of these types determine the bargaining outcome at the double limit. Thus, the convergence to $(\tau_\eta^\beta, \rho_\eta^\beta)$ as $\delta \to 1$ follows from the convergence of (b_n, s_n, q_n^B, q_n^S) to (b_t, s_t, q_t^B, q_t^S) as $\delta \to 1$. □

Next, I state the main result of this subsection. Consider sequence $\beta \to 1$ and let $\hat{\eta}(\beta) \equiv \min\{1-\beta, \eta(\beta)\}$, where $\eta(\cdot)$ is as in Lemma 7. Note that $\hat{\eta}(\beta) \underset{\beta \to 1}{\to} 0$. Denoted by $(\tau^\dagger, \rho^\dagger)$ the limit of frequent-offer PBE limits $(\tau_{\hat{\eta}(\beta)}^\beta, \rho_{\hat{\eta}(\beta)}^\beta)$ as $\beta \to 1$. The next theorem shows that $(\tau^\dagger, \rho^\dagger)$ is the desired double limit that attains the Nash split, but is inefficient.

Theorem 2 (**Nash Split with Delay**). *For any* $\varepsilon > 0$, $\lim_{\beta \to 1} \mathbb{P}_{\hat{\eta}(\beta)} \left(|\rho_{\hat{\eta}(\beta)}^\beta - y^*(s,b)| > \varepsilon \right) = 0$. *Further, there is* $\underline{\tau} > 0$ *such that* $\lim_{\beta \to 1} \mathbb{E}_{\hat{\eta}(\beta)}[\tau_{\hat{\eta}(\beta)}^\beta] > \underline{\tau}$.

Proof of Theorem 2. As $\beta \to 1$, q_t^S and q_t^B in (23) and (24) converge to $y^*(b_t)$ and $y^*(s_t)$, respectively. For types above $\bar{b}_T + \hat{\eta}(\beta)$ and below $\bar{s}_T - \hat{\eta}(\beta)$, the bargaining outcome is determined by the acceptance strategy of only one side, and as $\beta \to 1$, the accepted price converges to $y^*(s,b)$ for such types. This implies the convergence of $\rho_{\hat{\eta}(\beta)}^\beta$ to $y^*(s,b)$ as $\beta \to 1$. The positive expected delay $\mathbb{E}_{\hat{\eta}(\beta)}[\tau_{\hat{\eta}(\beta)}^\beta]$ follows from the fact that in this case, the left-hand sides of (18) and (19) are bounded from above by $r\Sigma$ and so, the acceptance is gradual. □

3.4 Folk Theorem

In this subsection, I prove the Folk theorem when the ranges of v and c coincide.

Let $v_0(\cdot)$ and $c(\cdot)$ be strictly increasing functions such that $v_0(0) = c(0)$, $v_0(1) = c(1)$, and $v_0(x) > c(x)$ for all $x \in (0,1)$. Suppose that v_0 and c satisfy the differentiability assumptions on v and c in Section 2. Note that both v_0 and c span values in $[c(0), c(1)]$, and the surplus is zero for $x = 0$ and $x = 1$, but is positive for $x \in (0,1)$.[10] Given $\xi > 0$ and if the seller's cost is given by $c(s)$ and the buyer's value is given by $v(b) = v_0(b) + \xi$ for $b \in [0,1]$, then I obtain the specification of v and c as in the baseline model.

I am interested in characterizing the limit of set $\mathcal{E}$ as $\xi \to 0$, which I denote by $\mathcal{E}_0$ where the convergence is in the Hausdorff metric. I denote by IR_0 the limit of IR at

[10] An example of such functions is any strictly concave v_0 and strictly convex c satisfying the differentiability conditions with $c(0) = v_0(0)$ and $c(1) = v_0(1)$ (e.g., $c(x) = e^x - 1$ and $v_0(x) = e - e^{1-x}$).

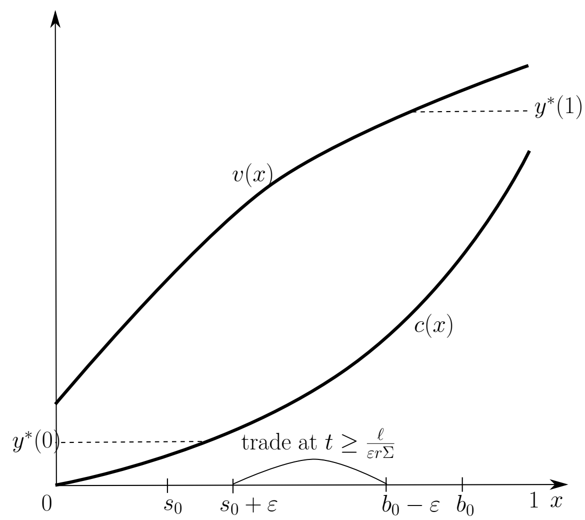

Figure 5: ***Illustration for the proof of Theorem*** 3

$\xi \to 0$. If $\Pi_0 \equiv \lim_{\eta \to 0} \mathbb{E}_\eta[v_0(b) - c(s)]$, then

$$IR_0 = \left\{ (U^S, U^B) : U^S + U^B \leq \Pi_0, U^S \geq 0, \text{ and } U^B \geq 0 \right\}.$$

The difference between IR_0 and IR is that the reservation utility of both sides is zero in the former, and positive in the latter. As $\xi \to 0$, $y^*(0)$ converges to $c(0)$ ($y^*(1)$ converges to $v(1)$). Thus, fewer seller (respectively, buyer) types get positive utility from trading at $y^*(0)$ (respectively, $y^*(1)$). In the limit $\xi \to 0$, $\underline{U}^S(x)$ and $\underline{U}^B(x)$ are equal to zero for any $x \in [0, 1]$. The next theorem states the Folk theorem at the limit $\xi \to 0$.

Theorem 3 (**Folk Theorem**)**.** *It obtains* $\mathcal{E}_0 = IR_0$*.*

The following is the reasoning for the proof of Theorem 3. Consider the configuration depicted in Figure 5. I have already characterized in Theorem 1 the Pareto frontier of $\mathcal{E}$ and so, it remains to show that the point $(\underline{U}^S, \underline{U}^B)$ can be approximated as a double limit when $\xi \to 0$. As $\xi \to 0$, the fraction of buyer (seller) types that get positive utility from trading at price $y^*(1)$ (respectively, $y^*(0)$) goes to zero, while for the rest of types $\underline{U}^B(b) = 0$ (respectively, $\underline{U}^B(s) = 0$). I use Lemma 7 to construct PBEs with the standoff dynamics. Initially, offers $y^*(1)$ and $y^*(0)$ are accepted by buyer types above some b_0 and seller types below some s_0. The remaining types follow the two-sided screening dynamics as in the previous subsection with prices of trade that are close to the values or costs of the threshold types ($\beta \approx 0$). The latter guarantees that the trade happens slowly over time and ensures that the expected utility is close to zero for these types. As $\xi \to 0$, the set of buyer (seller) types that get positive utility from a trade at price $y^*(1)$ (respectively, $y^*(0)$) disappears and so, b_0 goes to 1 (respectively, s_0 goes to 0). Thus, I construct PBEs, in which the ex-ante expected utilities of both sides are close to zero.

Proof of Theorem 3. Theorem 1 has already proven that the Pareto frontiers of $\mathcal{E}$ coincides with PF, the Pareto frontier of IR. As $\xi \to 0$, PF converges to PF_0, the Pareto frontier of IR_0. Hence, the Pareto frontier of $\mathcal{E}_0$ coincides with PF_0. Note that the worst payoff profile $(\underline{U}^S, \underline{U}^B)$ converges to $(0,0)$ as $\xi \to 0$. Thus, to prove the Folk theorem, it is sufficient to construct a sequence of double limits indexed by $\xi \to 0$ that have ex-ante expected payoffs converging to $(0,0)$.

Fix $\varepsilon > 0$. Consider PBEs constructed in Lemma 7 that approximate $(\tau_\eta^{\varepsilon^2}, \rho_\eta^{\varepsilon^2})$ as $\delta \to 1$. Plugging (24) into (19) with $\beta = \varepsilon^2$, I get

$$\dot{q}_t^B = r\varepsilon^2(y^*(s_t) - c(s_t)) \leq r\Sigma\varepsilon^2,$$

hence, $q_t^B - q_0^B \leq r\Sigma\varepsilon^2 t$. On the other hand, since $c'(x) \leq \ell$ and $v'(x) \leq \ell$ for all $x \in [0,1]$, it follows from (24) that $q_t^B - q_0^B \geq \ell(s_t - s_0)$. Therefore,

$$s_t - s_0 \leq \frac{r\Sigma}{\ell}\varepsilon^2 t.$$

Thus, in $(\tau_\eta^{\varepsilon^2}, \rho_\eta^{\varepsilon^2})$, types $s \geq s_0 + \varepsilon$ trade after a delay of at least ${}^{\ell}\!/\!_{\varepsilon r\Sigma}$. By equation (22), $\beta y^*(s_0) + (1-\beta)c(s_0) = y^*(0)$, and so, $s_0 \leq c^{-1}(y^*(0)) \to 0$ as $\xi \to 0$. (Since $y^*(0) - c(0) \underset{\xi \to 0}{\to} 0$). I choose a ξ small enough so that $s_0 < \varepsilon$ and so, $[2\varepsilon, 1] \subseteq [s_0 + \varepsilon, 1]$. Analogously, in $(\tau_\eta^{\varepsilon^2}, \rho_\eta^{\varepsilon^2})$, types $b \in [0, b_0 - \varepsilon]$ trade after a delay of at least ${}^{\ell}\!/\!_{\varepsilon r\Sigma}$, and $[0, 1-2\varepsilon] \subseteq [0, b_0 - \varepsilon]$. Therefore, only a fraction of types of order ε trade with a delay that does not exceed ${}^{\ell}\!/\!_{\varepsilon r\Sigma}$, while the expected payoff of the rest of the types is at most $\Sigma \exp(-{}^{\ell}\!/\!_{\varepsilon\Sigma})$. This holds for all η sufficiently small, and so, holds in the limit outcome $(\tau^{\varepsilon^2}, \rho^{\varepsilon^2})$ of $(\tau_\eta^{\varepsilon^2}, \rho_\eta^{\varepsilon^2})$ as $\eta \to 0$. By taking $\varepsilon \to 0$, I get the desired sequence of double limits. $\square$

4 Proof of the Contagious Coasian Property

Subsection 3.1 introduced the Contagious Coasian property of punishing equilibria. The key insight formalized in Lemma 2 is that the utility of the deviator in the punishing equilibrium is independent of η. In this section, I outline the construction of such punishing equilibria and provide intuition for how the contagion argument can obtain Lemma 2. (The full analysis is in Appendix B).

The proof proceeds in five steps. I start with the analysis of the *auxiliary game*, in which the buyer is restricted to either accepting the last seller's offer or making a counter-offer $\underline{y}(0)$; and the buyer holds optimistic conjecture, while the seller holds her original beliefs. Specifically, the buyer puts probability one on the lowest type in the support S_b, i.e.,[11]

$$\mu_b^n[\underline{\pi}(b)] = 1, \tag{25}$$

for all $b \in [0,1]$. In Steps 1-4, I construct the PBE in this auxiliary game with Coasian dynamics, namely, as offers become frequent, the buyer's highest willingness to pay uniformly approaches $\max\{y^*(0), c(\underline{\pi}(b))\}$. In Step 5, I show that it is also the PBE, even when the buyer can make counter-offers different from $\underline{y}(0)$. This immediately implies the result in Lemma 2 when $\underline{b} = 0$ and $\bar{b} = 1$. I generalize the argument to arbitrary $0 \leq \underline{b} < \bar{b} \leq 1$ in Appendix B.5, which completes the proof of Lemma 2.

Step 1: Standard Coasian dynamics for types $b \in [0, \eta]$ and $s = 0$ It is useful to start with the argument for independent values, $\eta = 1$. In this case, the buyer's optimistic conjectures put probability one on the lowest seller type, $s = 0$, for all buyer types. Thus, the auxiliary game with the buyer types pooling at $\underline{y}(0)$ is essentially the standard game with one-sided offers, one-sided private information between the buyer types $b \in [0,1]$ and the seller type $s = 0$ analyzed in Fudenberg et al. [1985], Gul et al. [1986], Ausubel and Deneckere [1989]. I can use an argument similar to Fudenberg et al. [1985] to construct the continuation equilibrium in weak-Markov strategies where

- the buyer accepts any seller's offer less than or equal to his willingness to pay $P(b)$, where $P(\cdot)$ is weakly increasing and right-continuous;

- the seller optimally screens the buyer given the willingness to pay function $P(\cdot)$.

[11] Such beliefs can be justified in the original model by the following trembles in the model with a finite number of types and a finite grid of price offers. Seller's and buyer's types come from $\{k/K\}_{k=1}^K$ for some integer K, and price offers come from a discrete set $\mathcal{P}$. Seller type s trembles with probability $(1-s)^m/2$ for some integer m, and conditional on trembling, she chooses a price offer uniformly from $\mathcal{P}$. As $m \to \infty$, the probability of tremble converges to zero. Yet, conditional on the buyer type b, the probability that the tremble comes from seller type $\underline{\pi}(b)$ is

$$\frac{(1-\underline{\pi}(b))^n}{(1-\underline{\pi}(b))^m + \sum_{s \in S_b \setminus \{\underline{\pi}(b)\}} (1-s)^m} \underset{m \to \infty}{\to} 1,$$

as $1 - s < 1 - \underline{\pi}(b)$ for $s \in S_b \setminus \{\underline{\pi}(b)\}$.

By the same argument as in Gul et al. [1986] or Ausubel and Deneckere [1989], such an equilibrium has the Coasian property: as $\delta \to 1$, all screening offers of type $s = 0$ are close to $y^*(0)$. This implies that the willingness to pay function $P(\cdot)$ converges to the constant function $P(b) = y^*(0)$.

In contrast to $\eta = 1$, when $\eta < 1$, only the buyer types below η place probability one on the seller type $s = 0$. For such types, I use the above argument to construct the PBE strategies in the auxiliary game that have Coasian dynamics. (See Appendix B.1). However, this argument implies that the willingness to pay $P(b)$ is close to $y^*(0)$ only for buyer types below η.

Step 2: Strategies for types $b > \eta$ and $s > 0$ The key insight of Lemma 2 is that the contagion argument extends the Coasian property to the rest of the types. Since the buyer holds optimistic conjectures, players can disagree on the equilibrium path play. Hence, I refer to the path that the seller expects as the equilibrium path in the auxiliary game. In this step, I describe the PBE on-path strategies in the auxiliary game for types $b > \eta$ and $s > 0$, and in the next step, construct them. The PBE strategies off-path are constructed in Appendix B.2.

Like in Step 1, I construct the equilibrium *on-path* strategies for types $b > \eta$ and $s > 0$, in which for any history without buyer's deviation

- the buyer accepts according to the right-continuous, weakly increasing *willingness to pay* function $P(\cdot)$;
- each seller type $s > 0$ optimally screens the buyer given the willingness to pay $P(b)$ of types $b \in B_s$.

If the lowest seller's offer in the past is p and the buyer follows strategy $P(\cdot)$, then $\beta \equiv \inf\{b : P(b) \geq p\}$ is the highest remaining buyer type. For a given function $P(\cdot)$, the screening problem of seller type $s > 0$ can be formulated recursively. The lowest type of the buyer in B_s with willingness to pay no greater than $c(s)$ is denoted by $\overline{\beta}(s) \equiv \max\{\sup\{b : P(b) \leq c(s)\}, \underline{\pi}(s)\}$.[12] I look for an optimal screening strategy for the seller given $P(\cdot)$ such that type s sells only to buyer types above $\overline{\beta}(s)$. Then for s and $\beta \geq \overline{\beta}(s)$, the expected profit $R(\beta, s)$ normalized by $(\beta - \overline{\beta}(s))$ satisfies the Bellman equation

$$R(\beta, s) = \max_{b \in B_s \cap [\overline{\beta}(s), \beta]} \left\{ (\beta - b)(P(b) - c(s)) + \delta^2 R(b, s) \right\}. \tag{26}$$

[12] By convention, $\overline{\beta}(s) = \underline{\pi}(s)$ whenever $P(b) > c(s)$ for all b.

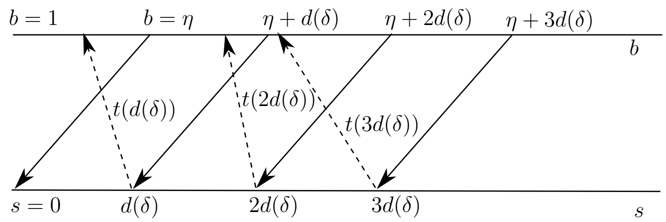

Figure 6: ***Illustration of the construction of*** $P(\cdot)$ ***and*** $t(\cdot,\cdot)$
Solid arrows indicate the beliefs of buyer types under optimistic conjectures: Type b assigns probability one to seller type $\underline{\pi}(b)=b-\eta$ for $b>\eta$. Dashed arrows indicate the cut-off type $t(s)$ in the first round of screening by seller type s. The key observation in Step 3 is that the dashed arrows originating in types $(kd(\delta),(k+1)d(\delta)]$ always point to buyer types below $\eta+kd(\delta)$, which allows the iterative construction of $P(\cdot)$ and $t(\cdot,\cdot)$.

The set of maximizers in (26) is denote by $T(\beta,s)$, and $t(\beta,s)\equiv\sup T(\beta,s)$. That is, it is optimal for the seller of type s to sell to all types above $t(\beta,s)$ given that the highest remaining buyer type is β. I can also recover via $\sigma(\beta,s)=P(t(\beta,s))$ the corresponding price offer of seller type s. A special role in the analysis is played by the first screening cut-off and price offer, and I denote them as $t(s)\equiv t(\overline{\pi}(s),s)$ and $\sigma(s)\equiv\sigma(\overline{\pi}(s),s)$, respectively.

Note that if buyer type $b>\eta$ rejects $P(b)$, then he is still the highest type in the support of beliefs of type $\underline{\pi}(b)$, as only types above b accept $P(b)$. Hence, after the rejection of $P(b)$, seller type $\underline{\pi}(b)$ will restart her screening and again offer $\sigma(\underline{\pi}(b))$. Thus, the willingness to pay $P(b)$ is the price at which type b is indifferent between accepting it and accepting the price $\sigma(\underline{\pi}(b))$ in the next screening round, i.e., $v(b)-P(b)=\delta^2\left(v(b)-\sigma(\underline{\pi}(b))\right)$, or after rearranging the terms,

$$P(b)=(1-\delta^2)v(b)+\delta^2\sigma(\underline{\pi}(b)),\ \text{for}\ b\in(\eta,1]. \tag{27}$$

Step 3: Construction of $P(\cdot)$ **and** $t(\cdot,\cdot)$ In this step, I construct functions $P(\cdot)$ and $t(\cdot,\cdot)$ such that $P(\cdot)$ satisfies (27), and $t(\cdot,\cdot)$ is the supremum of the set of solutions to (26). (See Figure 6 for the illustration).

I have already constructed in Step 1, $t(\beta,0)$ and $P(\cdot)$ for $b\in[0,\eta]$. In Lemma 22 in Appendix B.3, I prove that there is $d(\delta)>0$ such that each seller type s sells to at least types $[\overline{\pi}(s)-d(\delta),\overline{\pi}(s)]$ in the first round of screening, i.e., $t(s)\leq\overline{\pi}(s)-d(\delta)$. This implies that any seller type $s\in(0,d(\delta)]$ does not screen buyer types above η, for whom I have already constructed the willingness to pay $P(b)$. Thus, I can derive the optimal screening policy $t(\cdot,s)$ of seller types $s\in(0,d(\delta)]$ such that $t(s)\leq\eta$. Now, buyer types

$b \in (\eta, \eta + d(\delta)]$ assign probability one to the seller types $\underline{\pi}(b)$, for whom I have just derived $t(\underline{\pi}(b))$ and shown that $t(\underline{\pi}(b)) \leq \eta$. Thus, the formula (27) determines $P(b)$ for $b \in (\eta, \eta + d(\delta)]$. Lemma 17 in Appendix $B.3$ shows that $\sigma(s)$ is right-continuous and weakly increasing in s. This fact combined with the formula (27) and the continuity and strict monotonicity of $v(\cdot)$ implies that $P(\cdot)$ on $[0, \eta+d(\delta)]$ is right-continuous and strictly increasing.

I next proceed to seller types $s \in (d(\delta), 2d(\delta)]$ and buyer types $b \in (\eta+d(\delta), \eta+2d(\delta)]$, and by the same argument construct functions $P(\cdot)$ and $t(\cdot, \cdot)$ for them. Then, to types $s \in (2d(\delta), 3d(\delta)]$ and $b \in (\eta + 2d(\delta), \eta + 3d(\delta)]$, and so on until I cover all types. Since $d(\delta) > 0$, I will do so in a finite number of steps.

Step 4: Contagious Coasian Property of $P(\cdot)$ This step proves that as offers become frequent, the buyer's willingness to pay approaches the minimal price that the seller can accept in any equilibrium.

Lemma 8. *For any* $\varepsilon > 0$*, there is a* $\delta(\varepsilon) \in (0, 1)$ *such that for all* $\delta \in (\delta(\varepsilon), 1)$*, it holds* $\max_{b\in[0,1]} |P(b) - P^*(b)| < \varepsilon$*, where* $P^*(b) = \max\{y^*(0), c(\underline{\pi}(b))\}$*.*

Before proceeding with the proof of Lemma 8, I make the following preliminary observation. By iteratively applying (27) and the fact that $\sigma(\underline{\pi}(b)) = P(t(\underline{\pi}(b))) \leq P(b - d(\delta))$ I have that for any $b \in (\eta, 1]$ there is an integer J such that

$$P(b) \leq (1 - \delta^2) \sum_{j=0}^{J-1} \delta^{2j} v(b - jd(\delta)) + \delta^{2J} P(\eta). \tag{28}$$

Thus, it follows from (28) that the willingness to pay of any buyer type is connected to the willingness to pay of type η. This connection illustrates the importance of higher-order beliefs in the analysis. Under optimistic conjectures, type b assigns probability one to seller type $\underline{\pi}(b)$. Seller type $\underline{\pi}(b)$ targets buyer type $t(\underline{\pi}(b)) \leq b - d(\delta)$, when she makes her first screening offer. These are second-order beliefs of type b. Buyer type $t(\underline{\pi}(b))$ in turn assigns probability one to seller type $\underline{\pi}\left(t(\underline{\pi}(b))\right)$, who in turn targets buyer type $t\left(\underline{\pi}\left(t(\underline{\pi}(b))\right)\right)$, when she makes her first screening offer. These are third- and forth-order beliefs of type b. This process can go on until it reaches buyer types below η. Here, buyer type b knows that the seller assigns probability zero to buyer types below η. Moreover, if η is small, this fact might be a mutual knowledge up to some finite order. However, the willingness to pay of buyer types below η is important in determining $P(b)$, because on a

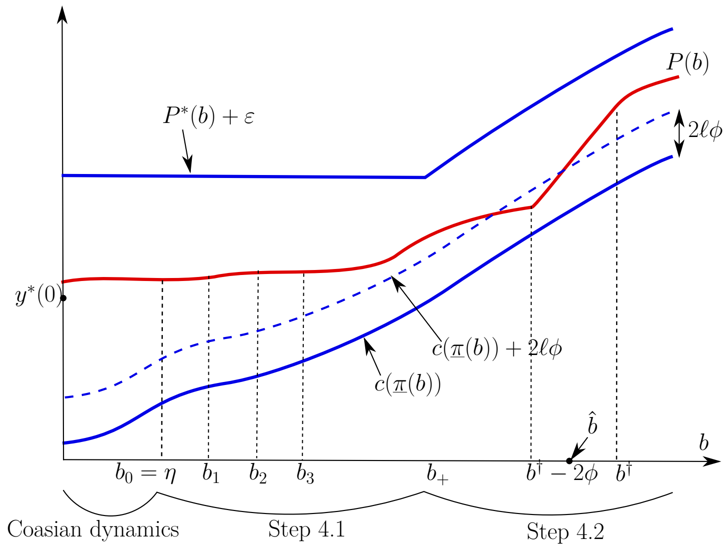

Figure 7: ***Illustration for the proof of the Contagious Coasian Property***

sufficiently high order of beliefs, the seller does assign positive probability to buyer types below η.

If J in (28) does not change with δ, then Lemma 8 would immediately follow from (28), because the second term in (28) would converge to $y^*(0)$, while the first would converge to zero as $\delta \to 1$. However, this is not the case, because $d(\delta) \underset{\delta\to 1}{\to} 0$ and therefore $J \underset{\delta\to 1}{\to} \infty$. Instead, the argument proceeds as follows. I denote $s_+ \equiv c^{-1}(y^*(0))$ as the seller type whose cost is equal to $y^*(0)$. The proof of Lemma 8 separately considers buyer types that assign probability one to seller types below s_+ and above s_+. The difference between those cases is similar to that between the "gap" and "no-gap" cases in the bargaining literature: types $s < s_+$ get positive utility from trading at the lowest price $y^*(0)$ (so there is a "gap" between $c(s)$ and $y^*(0)$), while types $s > s_+$ get negative utility from trading at $y^*(0)$ (so there is "no gap" between $c(s)$ and $y^*(0)$). Let b_+ be such that $\underline{\pi}(b_+) = s_+$.

Step 4.1: Buyer types $b \in (\eta, b_+)$ The types $b \leq \eta$ assign probability one to seller type 0. Thus, I can use the standard Coasian property to show that the willingness to pay of buyer types below η approaches $y^*(0)$ as offers become frequent. Therefore, I can then build the contagion argument to determine the willingness to pay of the rest of the buyer types. I use the following lemma to determine the willingness to pay of buyer types $b < b_+$.

Lemma 9. *There is a function $f(\phi, \delta)$ that satisfies the following conditions:*

1. *for any fixed* $\delta \in (0,1)$ *and* $\phi \in (0, \frac{1}{2}\eta)$*, if for some* $\hat{b} \in (\eta - \phi, 1]$*, it holds that*

$$c(\underline{\pi}(\hat{b})) + 2\ell\phi < P(\hat{b} - \phi), \tag{29}$$

then

$$P(\min\{\hat{b} + \phi, 1\}) < P(\hat{b}) + f(\phi, \delta); \tag{30}$$

2. *for any fixed* $\phi > 0$*,* $\lim_{\delta \to 1} f(\phi, \delta) = 0$*.*

I now illustrate how Lemma 9 can be used to build the contagion argument. Given $\phi \in (0, \frac{1}{2}\eta)$, I denote $b_k \equiv \eta + \phi k$ and $s_k \equiv \underline{\pi}(b_k)$ for integer $k = 0, \ldots, K$, where K is the largest k such that $c(\underline{\pi}(b_k)) + 2\ell\phi < y^*(0)$. (See Figure 7 for the illustration). The choice of K implies

$$c(\underline{\pi}(b_k)) + 2\ell\phi < y^*(0) < \overline{y}(0) \leq P(b_k - \phi), \tag{31}$$

which in turn gives inequality (29) with $\hat{b} = b_k$ for each $k = 0, \ldots, K$.

Further, I show that for any $\varepsilon > 0$, the willingness to pay of buyer type b_K is below $y^*(0) + \varepsilon$ for δ sufficiently close to one.[13] I start by considering $k = 0$. By Lemma 9, the willingness to pay of type $b_1 = b_0 + \phi$ cannot differ by more than $f(\phi, \delta)$ from the willingness to pay of type $b_0 = \eta$. The intuition is similar to that for the Coasian property in the "gap" case. The willingness to pay of buyer type b_1 is determined by the screening policy of seller type $s_1 = \underline{\pi}(b_1)$. If the difference $P(b_1) - P(\eta)$ is large, then type s_1 spends significant time screening buyer types above $b_0 = \eta$. However, if (29) is satisfied for $\hat{b} = b_1$, then the profit of type s_1 from selling to types below $b_0 = \eta$ is positive and creates a temptation for type s_1 to speed up the screening of types above $b_0 = \eta$. As a result, the difference in $P(b_1) - P(\eta)$ is bounded from above by $f(\phi, \delta)$ that converges to zero as offers become frequent.

I now consider type $b_2 = \phi + 2\eta$. Again, the profit from types below b_1 creates a temptation for type s_2 not to screen for a significant time those types above b_1. Hence, by Lemma 9, the willingness to pay of type b_2 is at most $f(\phi, \delta)$ away from the willingness to pay of type b_1, and at most $2f(\phi, \delta)$ away from $P(\eta)$. Similarly, I get that for any $k \leq K$,

$$P(b_k + \phi) - P(\eta) \leq (k+1) f(\phi, \delta).$$

[13] Note that for any $b < b_+$, $c(\underline{\pi}(b)) < y^*(0)$. Thus, I can find ϕ sufficiently small such that $b_K > b$ and then the argument covers all such types.

Given δ be such that $f(\phi,\delta)/\phi < \frac{1}{2}\varepsilon$, then

$$P(b_K) = P(b_{K-1}+\phi) \le Kf(\phi,\delta) + P(\eta) \le \tfrac{1}{\phi}f(\phi,\delta) + P(\eta) < \frac{\varepsilon}{2} + P(\eta).$$

Since by the standard Coasian property, $P(\eta) \le y^*(0) + \frac{1}{2}\varepsilon$ for sufficiently large δ, I have that $P(b_K) \le y^*(0) + \varepsilon$.

Remark 6. *This contagion argument resembles the one in the global games literature (see, e.g., Morris and Shin [1998] or Weinstein and Yildiz [2013]). In the latter, there are regions of types for whom certain actions are dominant. Then for types that are close to the boundaries of those regions, actions become dominant, because such types assign sufficiently high probability to types in dominance regions, and this argument can be repeated to expand the dominance regions. Similarly, my model shows that only a small set of buyer types below η has a willingness to pay close to $y^*(0)$ by the standard Coasian dynamics. However, seller types below s_1 are tempted by the possibility to sell to buyer types below η, and hence, quickly screen types above η. This screening leads to the willingness to pay close to $y^*(0)$ of a larger set of buyer types below b_1, which in turn, ensures that seller types below s_2 quickly lower their price offers to $y^*(0)$, and so on. In the end, the willingness to pay is close to $y^*(0)$ for buyer types that are much higher than η.*

Step 4.2: Buyer types $b \in [b_+, 1]$ The above argument is not valid for types above b_+. Indeed, such types know that the seller will never charge a price below her costs and so, their willingness to pay can not converge to $y^*(0)$. Showing that it converges to $c(\underline{\pi}(b))$ is a bit subtle, but the idea is similar to Lemma 9. For any such type $b^\dagger$, I show that it is possible to find $\hat{b} < b^\dagger$ such that for the seller type $s^\dagger \equiv \underline{\pi}(b^\dagger)$, the temptation to sell to types below $\hat{b}$ is sufficiently strong so that seller type $s^\dagger$ does not spend significant time screening buyer types above $\hat{b}$. This lack of screening implies that the willingness to pay of type $b^\dagger$ is "close" to $c(\underline{\pi}(b^\dagger))$. The difficulty with this argument is that the profit from types both above and below $\hat{b}$ for type $s^\dagger$ converges to zero as $\delta \to 1$. In this sense, this step is like the "no-gap" case in the classical bargaining literature. The following lemma extends the contagion argument to buyer types above b_+.

Lemma 10. *For any $\varepsilon > 0$, there are a $\tilde{\phi}(\varepsilon) > 0$ and a $\tilde{\delta}(\varepsilon) \in (0,1)$ such that for any $\phi \in (0, \tilde{\phi}(\varepsilon))$ and $\delta \in (\tilde{\delta}(\varepsilon), 1)$, if for some $b^\dagger \in [\eta, 1]$ it holds that*

$$c(\underline{\pi}(b^\dagger)) + 2\ell\phi \ge P(b^\dagger - 2\phi), \tag{32}$$

then

$$P(b^\dagger) < c(\underline{\pi}(b^\dagger)) + \varepsilon. \tag{33}$$

Step 5: Verification of equilibrium conditions So far, I have considered the auxiliary game, in which the buyer is restricted to counter-offer $\underline{y}(0)$ if he rejects the seller's offer. In the last step, I verify that even when the buyer can make offers different from $\underline{y}(0)$, I can specify continuation strategies so that he does not have incentives to do so. I specify that if the buyer deviates from the pooling offer $\underline{y}(0)$, then the seller switches to optimistic beliefs, i.e.,

$$\mu_s^n[\overline{\pi}(s)] = 1. \tag{34}$$

These optimistic conjectures are never updated in the continuation. In this case, buyer types $b \in [\eta, 1]$ and seller types $s \in [0, 1-\eta]$ believe that they play a complete information game against type $\underline{\pi}(b)$ and $\overline{\pi}(s)$, respectively, and play corresponding equilibrium strategies in the continuation. Seller types $s \in (1-\eta, 1]$ best respond to buyer type 1's strategy, and buyer types $b \in [0, \eta)$ best respond to seller type 0's strategy. Thus, for any $\varepsilon > 0$ and sufficiently large δ, if the buyer deviates, he expects to trade at a price above $y^*\left(\underline{\pi}(b), \overline{\pi}(\underline{\pi}(b))\right) - \varepsilon$. On the other hand, for a sufficiently large δ, on the equilibrium path, the buyer expects to trade at a price below $\max\left\{y^*(0), c(\underline{\pi}(b))\right\} + \varepsilon$ with no delay. These strategies make the buyer's deviations from $\underline{y}(0)$ unprofitable for small ε, because $\max\left\{y^*(0), c(\underline{\pi}(b))\right\} < y^*\left(\underline{\pi}(b), \overline{\pi}(\underline{\pi}(b))\right)$ for all b.

5 Discussion

In this paper, I study the alternating-offer bargaining model with a global games information structure. Despite the strong notion of correlation – each type assigns a positive probability only to a set of opponent types – a variety of equilibrium dynamics can be sustained even when the correlation between values is close to perfect. To conclude, I discuss the implications of the analysis, alternative bargaining protocols, robustness of the results to the model of correlation, and directions for future research.

The Role of Higher-Order Uncertainty and Public Information The results stress the role of higher-order uncertainty in bargaining. For example, suppose that the analyst does not know the primitives of the model, in particular functions v and c and parameters δ and η, but can observe m'th-order beliefs of players for arbitrary but finite

m. That is, she observes that, say, the seller's cost and all higher-order beliefs about the buyer's value and the seller's cost up to order m. The rules of the bargaining game imply the trivial prediction that the price can range between the seller's cost and the buyer's value, and the expected delay can be any non-negative number. Can one refine this prediction with a given degree of accuracy ε_0 by hiring such an analyst?

The answer is "no". For any buyer or seller type $x \in (0,1)$, any $\varepsilon \in (x, 1-x)$, and any integer m; there is $\overline{\eta}$ such that for all $\eta < \overline{\eta}$, type x has m'th level of mutual knowledge that $s \in [x-\varepsilon, x+\varepsilon]$ and $b \in [x-\varepsilon, x+\varepsilon]$. Hence, the m'th order beliefs provide information about functions v and c only in the ε neighborhood of x. I can specify functions v and c outside this neighborhood so that $v(0) < c(x-\varepsilon)$ and $c(1) > v(x+\varepsilon)$, and use double limits constructed in the proofs of Theorems 1 and 3 to approximate up to ε_0 any surplus split and any expected delay. Thus, the analyst who observes only finite-order beliefs cannot refine the trivial prediction. On the other hand, if the analyst observes only players' values and the range of values and costs (which is common knowledge among players), then by Lemma 1 such an analyst can predict that the price is between $y^*(0)$ and $y^*(1)$, and this way, potentially refine the trivial prediction.

The analysis delineates the effect of private and public information on bargaining outcomes, and stresses the role of the latter in obtaining predictions in bargaining models. In the limit of $\eta \to 0$, the model approaches the complete-information model in the sense that the infinite hierarchies of beliefs of types s and b, or Harsanyi's types (Harsanyi [1967]), approach types with common knowledge of values $c(s)$ and $v(b)$ in the product topology. Specifically, for $\eta \to 0$, the uncertainty about values vanishes, and in this sense, the information asymmetry disappears. At the same time, no matter how small η is, it is only common knowledge among players that types are in Ω_η, and in this sense, the public information remains coarse for any η. The results demonstrate that as long as the public information is coarse, the complete-information outcome can be a poor approximation of the bargaining outcome both in the surplus split and efficiency. This finding suggests that the application of the Nash bargaining Solution as a reduced form for the bargaining outcome with small information asymmetry could be less compelling in environments with scarce public information, such as over-the-counter financial markets.

Alternative Bargaining Protocols Suppose that instead of alternating offers, after the seller's offer in an odd round, time $\Delta_s > 0$ elapses, and after the buyer's offer in an even round, time $\Delta_b > 0$ elapses. Given $\alpha \equiv {}^{\Delta_b}/_{\Delta_s + \Delta_b} \in (0,1)$, the complete information outcome in this case is the immediate trade at a price that approaches $y^*_\alpha(s,b) = (1 -$

$\alpha)v(b) + \alpha c(s)$ as $\delta \to 1$. All of the results immediately extend to this more general model with $y^*_\alpha(s,b)$ replacing $y^*(s,b)$ in all the expressions.

Further, suppose that the buyer is restricted to only offer some unacceptable y^B below $c(0)$. Since an offer below $c(0)$ can never be accepted by the seller, such a game is essentially the game with one-sided offers by the seller. Note that in the proof of the Contagion Coasian property and the characterization of the Pareto frontier, I have constructed equilibria in which all buyer types pool at the lowest acceptable price $\underline{y}(0)$. I can extend these proofs to the case where buyer only makes the unacceptable counter-offer y^B. In particular, after replacing $y(s,b)$ with $v(b)$ and $y^*(x)$ with $v(x)$, and appropriately adjusting the definition of the set PF, I get Theorem 1 for the case of one-sided offers.

Robustness to the Model of Correlation The assumption that the support of beliefs is bounded makes it harder to sustain equilibrium dynamics by ruling out the standard punishment with optimistic conjectures. However, it simplifies the construction of the on-path dynamics. For example, in the PBEs constructed in the proof of Theorem 1, each buyer type expects to receive one of at most two offers in the first round, and hence, the seller can only mimic types in the adjacent segment without being detected. To explore the role of the bounded support, my companion paper (Tsoy [2016]) considers an alternative model of correlation, in which types are distributed according to the affiliated distribution with full support on the unit square. Tsoy [2016], additionally, uses a version of the standard refinement in the bargaining literature (see Bikhchandani [1992], Grossman and Perry [1986], Rubinstein [1985]) that the support of beliefs does not expand. The main result is that the delay is necessary to attain the Nash split with double limits. In particular, this result means that the segmentation dynamics in Theorem 1, or more generally, any efficient bargaining dynamics cannot approximate the Nash split. Tsoy [2016] also shows that the two-sided screening dynamics in Theorem 2 can still be sustained in the model with the full support of beliefs. In this sense, the double limit with two-sided screening dynamics is robust to the assumptions about the details of the correlation between values, while the efficient dynamics are not.

Frequent-offer PBE Limits for Fixed η My main results describe double limits. However, Lemma 2 is true for any η, and the main result is that the level of punishment does not depend on η. This result can be useful in the analysis of the bargaining outcomes for fixed η, as it immediately provides the optimal punishment for detectable deviations. Also, the proofs of Theorems 1 and 7 explicitly construct PBEs that have drastically different efficiency properties for sufficiently large δ and small η. These results suggest that the equilibrium set is large for η small, but not necessarily converging to zero.

Directions for Future Research There are several interesting directions for future research. First, I prove the contagious Coasian property for any η, but focus on the analysis of the equilibrium set in the limit $\eta \to 0$. The analysis of the equilibrium set for intermediary levels of correlation is an interesting avenue for future research. Second, Remark 5 constructs PBEs with the fixed number of segments in which the inefficiency is of order η. It remains an open question whether one could improve this bound. Third, my results are of the Folk-theorem type. Tsoy [2016] shows that some of the bargaining outcomes in this paper are sensitive to the details of the correlation between types. Further refinements of the model predictions are left for future research. Forth, the order of limits is important to the results: there is a multiplicity of outcomes when I first take $\delta \to 1$, then $\eta \to 0$, but there is a convergence to the complete information outcome under the reverse order of limits. Exploring how the relative speed of convergence of δ and η affects the equilibrium set would be interesting. Finally, the prediction that the amount of public information can affect bargaining delays when the precision of private information is fixed can be tested in the lab.

A Appendix

A.1 Proofs for Subsection 3.1

I first derive general bounds on acceptable offers that imply Lemma 1. Let P_b be the supremum of offers accepted by type b of the buyer with positive probability in equilibrium (both on- and off-path), and analogously, let P_s be the supremum of offers rejected by type s of the seller with positive probability in equilibrium, p_b be the infimum of prices rejected by type b of the buyer with positive probability in equilibrium, p_s be the infimum of prices accepted by type s of the seller with positive probability in equilibrium.

Lemma 11. *For all b, s,*

$$P_b \leq (1-\delta)\sum_{k=0}^{\infty}\delta^{2k}\left(v(\overline{\pi}^{(2k)}(b)) + \delta c(\overline{\pi}^{(2k+1)}(b))\right), \tag{35}$$

$$p_s \geq (1-\delta)\sum_{k=0}^{\infty}\delta^{2k}\left(c(\underline{\pi}^{(2k)}(s)) + \delta v(\underline{\pi}^{(2k+1)}(s))\right). \tag{36}$$

Proof. First, by the definition of P_s, type b of buyer can guarantee himself the continuation utility arbitrarily close to $\delta(v(b) - \max_{s\in S_b} P_s)$ by making an offer arbitrarily close to $\max_{s\in S_b} P_s$ whenever he is active. Hence, $\delta(v(b) - \max_{s\in S_b} P_s) \leq v(b) - P_b$. Second, let U_s be the supremum of the continuation utilities of type s on- and off-path if the trade does not occur in the current round. If type s of the seller rejects an offer, she cannot guarantee more than $\max\{\delta(\max_{b\in B_s} P_b - c(s)), \delta^2 U_s\}$, which implies $U_s \leq \delta(\max_{b\in B_s} P_b - c(s))$. Hence, $P_s - c(s) \leq \delta(\max_{b\in B_s} P_b - c(s))$. Therefore,

$$\begin{aligned} P_b &\leq (1-\delta)v(b) + \delta\max_{s\in S_b} P_s \\ &\leq (1-\delta)v(b) + \delta\max_{s\in S_b}\left((1-\delta)c(s) + \delta\max_{b'\in B_s} P_{b'}\right) \\ &= (1-\delta)\left(v(b) + \delta c(\overline{\pi}(b))\right) + \delta^2\max_{s\in S_b}\max_{b'\in B_s} P_{b'}. \end{aligned}$$

By iterating this inequality, I obtain the first inequality in the statement of the lemma. The argument for the second inequality is symmetric. □

***Proof of Lemma* 1.** Parts 1 and 2 follow directly from Lemma 11 and the monotonicity of functions v and c. The last part follows from parts 1 and 2, and the fact that players can make unacceptable offers and guarantee payoff 0. □

A.2 Proofs for Subsection 3.2

The following lemma is the key mathematical fact in the proof of Lemma 4.

Lemma 12. *Consider $b_\infty \in (2\eta, 1-3\eta)$, $s_\infty = b_\infty + \eta$, q^B, q^S that satisfy*

$$c(s_\infty) < q^B < q^S < v(b_\infty). \tag{37}$$

There exists $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$ there exist positive sequences $(x_k, y_k)_{k=1}^{\infty}$ that converge to $(0,0)$ and satisfy the recursive system

$$\begin{cases} x_{k+1} = (1-\alpha^B(y_{k+1}))x_k - \alpha^B(y_{k+1})y_{k+1}, \\ y_{k+1} = (1-\alpha^S(x_k))y_k - \alpha^S(x_k)x_k, \\ x_1 + y_1 = 2\eta, \end{cases} \tag{38}$$

where

$$\alpha^B(y) = \frac{(1-\delta^2)(q^B - c(s_\infty - y))}{\delta^2(q^S - q^B)}, \tag{39}$$

$$\alpha^S(x) = \frac{(1-\delta^2)(v(b_\infty + x) - q^S)}{\delta^2(q^S - q^B)}. \tag{40}$$

Moreover,

$$x_k + y_k < 2\eta, \text{ for } k = 2, 3, \ldots, \tag{41}$$

$$x_k + y_{k+1} < 2\eta, \text{ for } k = 1, 2, \ldots. \tag{42}$$

Proof. See the Online Appendix. □

Proof of Lemma 3. Fix $z = 1, \ldots, Z-1$. By (9),

$$\begin{aligned} v(b^z) > v(b^z - 2\eta) \geq v(b^z - \sqrt{\eta}) \geq v(b^{z+1} - 3\sqrt{\eta}) > q(b^{z+1}, \gamma) = q_{z+1} > \\ > q_z = q(b^z, \gamma) > c(b^z + 3\sqrt{\eta}) = c(\hat{s}^z + \eta + 3\sqrt{\eta}) > c(\hat{s}^{z+1}) > c(s^z). \end{aligned} \tag{43}$$

Moreover, since $|q_z - \hat{q}_z| \underset{\delta \to 1}{\to} 0$ for all $z = 1, \ldots, Z-1$ and q_z is increasing in z, for sufficiently large δ it holds $q_z < \hat{q}_{z+1}$. Hence, $c(s^z) < q_z < \hat{q}_{z+1} < v(b^z)$. Since $b^z = b^{z-1} + \sqrt{\eta}$ and $\sqrt{\eta} > 4\eta$, $b^z \in (2\eta, 1-3\eta)$. Therefore, $q^S = \hat{q}_{z+1}, q^B = q_z, s_\infty = s^z, b_\infty = b^z$ satisfy conditions of Lemma 12, and so, for sufficiently large δ, there exist positive sequences $(x_k, y_k)_{k=1}^{\infty}$ that satisfy (38). I construct sequences of threshold types b_n^z,s_n^z,α_n^S, and α_n^B as follows. Let $b_1^z = \overline{\pi}(\hat{s}^{z+1})$, and for for $k = 1, 2, \ldots$, define $b_{2k}^z = b^z + x_k$, $s_{2k-1}^z = s^z - y_k$, $\alpha_{2k}^S = \alpha^S(x_k)$, $\alpha_{2k-1}^B = \alpha^B(y_k)$. This implies that b_n^z and s_n^z so constructed satisfy (11), (12), (13), (14). Since x_k and y_k are positive and $\alpha^B(y) > 0$ whenever $y > 0$, it follows from (38) that $x_{k+1} - x_k = -\alpha^B(y_{k+1})(x_k + y_{k+1}) < 0$ for all k, and analogously, $y_{k+1} - y_k < 0$. Hence, b_n^z and s_n^z are strictly decreasing in even rounds and strictly increasing in odd rounds, resp. Since (x_k, y_k) converges to $(0, 0)$, the limits of b_n^z and s_n^z are b^z and s^z, resp. By (41), $b^z + x_k \leq s^z - y_k + \eta$ or $b_{2k}^z \leq s_{2k-1}^z + \eta$, and so,

$$s^z - \eta = b^z \leq b_{2k}^z \leq s_{2k-1}^z + \eta \leq s^z + \eta = b^z + 2\eta.$$

Hence, $\max\{b_2^z - b^z, s^z - s_3^z\} \leq 2\eta$. This completes the proof. □

Proof of Lemma 4. After round 1, only types $s \in [\hat{s}^z, \hat{s}^{z+1})$ and types $b \in [\underline{\pi}(\hat{s}^z), \overline{\pi}(\hat{s}^{z+1}))$ remain. By (11) and (13), type b_n^z is indifferent between accepting in even round n and accepting in round $n + 2$. (Recall Remark 3). Note that (13) gives the probability that threshold buyer type b_n^z assigns in round n to the seller's type being in $[s_{n-1}^z, s_{n+1}^z)$, and hence, to seller lowering the price in round $n + 1$. To see this, observe that this probability is equal to $(s_{n+1}^z - \max\{s_{n-1}^z, \underline{\pi}(b_n^z)\})/(\overline{\pi}(b_n^z) - \max\{s_{n-1}^z, \underline{\pi}(b_n^z)\})$ for even $n > 2$, and to $(s_3^z - \underline{\pi}(b_2^z))/(\overline{\pi}(b_2^z) - \underline{\pi}(b_2^z))$ for $n = 2$. For $n > 2$, using (41) I get that $s_{n-1}^z = s^z - y_{n/2} = b^z + \eta - y_{n/2} > b^z + x_{n/2} - \eta = b_n^z - \eta = \underline{\pi}(b_n^z)$, and so,

$$\alpha_n^S = \frac{s_{n+1}^z - s_{n-1}^z}{b_n^z + \eta - s_{n-1}^z} = \frac{s_{n+1}^z - \max\{s_{n-1}^z, \underline{\pi}(b_n^z)\}}{\overline{\pi}(b_n^z) - \max\{s_{n-1}^z, \underline{\pi}(b_n^z)\}}.$$

For $n = 2$, using $x_1 + y_1 = 2\eta$ I get that $s_1^z = s^z - y_1 = b^z + \eta - y_1 = b^z + x_1 - \eta = b_2^z - \eta = \underline{\pi}(b_2^z)$, and so,

$$\alpha_2^S = \frac{s_3^z - s_1^z}{b_2^z + \eta - s_1^z} = \frac{s_3^z - \underline{\pi}(b_2^z)}{\overline{\pi}(b_2^z) - \underline{\pi}(b_2^z)}.$$

The likelihood that type b assigns in round n to the seller offering q^z in the next round is given by $(s_{n+1}^z - s_{n-1}^z)/(b + \eta - s_{n-1}^z)$, which is decreasing in b. Thus, if $b > b_n^z$, then type b has both higher value and assigns smaller probability to the seller lowering offer to q^z in round $n+1$ and so, all types above b_n^z strictly prefer to accept in round n. Similarly, all types below b_n^z strictly prefer to accept in round $n + 2$. Hence, the acceptance strategy given by b_n^z is optimal for the buyer. Similarly, (12) and (14) imply that the seller prefers to make offer q_z as prescribed by strategy s_n^z, but not earlier or later.

For $z = 0, \ldots, Z - 1$, by (10) and (43),

$$\begin{aligned}
q_{z+1} &= q(b^{z+1}, \gamma) = q(s^{z+1} - \eta, \gamma) = q(\hat{s}^{z+1} + \eta, \gamma) < y^*(\hat{s}^{z+1}), \\
q_z &= q(b^z, \gamma) > y^*(0), \\
q_{z+1} &< v(b^z - 2\eta), \\
q_z &> c(\hat{s}^{z+1}).
\end{aligned}$$

Moreover, $|q_{z+1} - \hat{q}_{z+1}| \underset{\delta \to 1}{\to} 0$. By following the equilibrium strategy, any seller type $s \in [\hat{s}^z, \hat{s}^{z+1})$ gets at least $q_z - c(s)$, and any buyer type $b \in [\underline{\pi}(\hat{s}^z), \overline{\pi}(\hat{s}^{z+1}))$ gets at least $v(b) - \hat{q}_{z+1}$. On the other hand, when δ is sufficiently large, by Lemma 2, if the seller deviates from offers q_z and $\hat{q}_{z+1}$, then she gets utility uniformly close to $\max\{0, y^*(0) - c(s)\} < q_z - c(s)$; and by Remark 1, if the buyer deviates from counter-offer $\underline{y}(0)$, then he gets utility uniformly close to $\max\{0, v(b) - y^*(\hat{s}^{z+1})\} < v(b) - \hat{q}_{z+1}$. Hence, for sufficiently large δ, such deviations are not profitable. This completes the proof. □

Proof of Lemma *5.* By Lemma 4, after on-path play in round 1 the continuation strategies constitute the PBE for sufficiently large δ. By the construction of $\hat{q}_z > y^*(0)$ and Lemma 2, the seller in round 1 does not have incentives to deviate to offers outside Q^S.

I next verify that the seller has no incentives to deviate in the first round. Note that only buyer types $b \in [b^z - 2\eta, b^z)$ are uncertain about the offer they receive in round 1. (All the rest of the types b know which offer the seller makes in round 1 on-path, as $S_b \subseteq [\hat{s}^z, \hat{s}^{z+1})$ for some z). Only seller types $s \in [\hat{s}^z - 2\eta, \hat{s}^z + 2\eta]$ assign positive probability to such buyer types. Thus, I consider separately the following two cases:

Case 1: Type $s \in [\hat{s}^z, \hat{s}^z + 2\eta]$ deviates to $\hat{q}_z$ or type $s \in [\hat{s}^z - 2\eta, \hat{s}^z)$ deviates to $\hat{q}_{z+1}$ Such deviations to offers in $\{\hat{q}_z\}_{z=1,\ldots,Z}$ may not be detected by the buyer. Since $b_2^z > b^z = \hat{s}^z + \eta = \overline{\pi}(\hat{s}^z)$ and $b_2^{z-1} < b^{z-1} + 2\eta = b^z + 2\eta - \sqrt{\eta} < \hat{s}^z - \eta = \underline{\pi}(\hat{s}^z)$, all buyer types in $B_{\hat{s}^z}$ accept offer $\hat{q}_z$, but always reject $\hat{q}_{z+1}$. By (8), $\hat{q}_z - c(\hat{s}^z) = \delta^2(q_z - c(\hat{s}^z))$ and so, seller type $\hat{s}^z$ is indifferent between offering $\hat{q}_z$ that is accepted for sure in round 2, and offering $\hat{q}_{z+1}$ that is rejected for sure and then offering q_z that is accepted for sure in round 4. By the single-crossing property of payoffs and the fact that types $s \in [\hat{s}^z, \hat{s}^z + 2\eta]$ assign a positive probability to $\hat{q}_{z+1}$ being accepted in round 2, seller types $s \in [\hat{s}^z, \hat{s}^z + 2\eta]$ strictly prefer to follow their equilibrium strategy. Analogously, seller types $\hat{s}^z \in [\hat{s}^z - 2\eta, \hat{s}^z)$ strictly prefer to trade at $\hat{q}_z$ in round 2, rather than at q_z in round 4. Since $b^z \geq b^{z-1} + \sqrt{\eta}$, all seller types in $[\hat{s}^z - 2\eta, \hat{s}^z)$ expect the buyer to always reject $\hat{q}_{z+1}$. Thus, such types prefer offering $\hat{q}_z$ rather than deviating to $\hat{q}_{z+1}$ and then making offer q_z in round 4.

Case 2: Types deviating to offers in $\{\hat{q}_z\}_{z=1,\ldots,Z}$ that are always detected by the buyer Deviation of the seller in round 1 to a higher offer in Q^S is rejected for sure for sufficiently large δ, as the buyer expects that the seller will return to the equilibrium path and decrease her price offer in the next round. Hence, such a deviation is not profitable (as it causes additional delay with any gain in the price). Deviation from $\hat{q}_{z+1}$ to a lower offer in Q^S is accepted for sure, but for sufficiently large δ, is dominated by offering $\hat{q}_{z+1}$ and counter-offering q_z if it gets rejected. □

A.3 Proofs for Subsection 3.3

***Proof of Lemma* 6.** I prove that in $\mathcal{G}(q_t^S, q_t^B)$, the threshold strategy b_t is a best response for the buyer to the seller's threshold strategy s_t. (The argument is symmetric for the seller). Observe that since $v(b_t) - q_t^S > 0$ for all $t \in [0, \infty)$, all buyer types $b \in (b_\infty, 1]$ prefer to accept at some t to never trading.

Step 1: buyer type $b \in [\bar{b}_T, 1]$ Since $s_\infty = \underline{\pi}(\bar{b}_T) \leq \underline{\pi}(b_t)$ for $t \leq T$, buyer type b assigns probability zero to his offer being accepted. Thus, he chooses the acceptance time t to maximize $e^{-rt}(v(b) - q_t^S)$ for which the first-order condition is given by the equation (18). The sufficiency of the first-order condition follows from the single-crossing property of payoffs. Since it is optimal for type b_0 to accept q_0^S at $t = 0$, by the single-crossing property it is also a best-response for any type $b > b_0$.

Step 2: buyer type $b \in (b_\infty, b_T)$ Note that $\mathbb{P}_\eta$ is affiliated, and denote its c.d.f. by F. Then $F(s|b) = (\max\{\min\{s, \overline{\pi}(b)\} - \underline{\pi}(b), 0\}) / (\overline{\pi}(b) - \underline{\pi}(b))$ is the c.d.f. of the buyer type b's beliefs, and let $f(s|b)$ be the corresponding p.d.f. For any $b \in (b_\infty, b_T)$, there exists time t such that $b = b_t$. I show that if b_t satisfies the equation (15) and the seller follows the threshold strategy s_t, then it is optimal for type b_t to accept at time t.

Let $t_S(s)$ be the inverse of s_t, i.e., $t_S(s) = \inf\{t : s \leq s_t\}$ for $s \in [0, s_\infty)$ and $t_S(s) = \infty$ for $s \in [s_\infty, 1]$. $t_S(s)$ gives the time when seller type s accepts if she follows threshold strategy s_t Analogously, let $t_B(b)$ be the inverse of b_t. Fix buyer type $b \in (b_\infty, b_T)$. Note that such a type assigns positive probability to his (lower) offer being accepted by the buyer and has a lower value than type $\bar{b}_T$, who assigns zero probability to his offer being accepted and for whom it is optimal to accept at T. Thus, it is without loss to assume that type b chooses from acceptance times in $t \in [T, \infty)$, as any time $t < T$ is dominated by T. Buyer type $b \in (b_\infty, b_T)$ chooses the acceptance time $t \geq T$ to maximize

$$u(b,t) = \int_0^{s_t} e^{-rt_S(s)}(v(b) - q_T^B)dF(s|b) + (1 - F(s_t|b))e^{-rt}(v(b) - q_T^S).$$

The first-order condition for this problem is

$$(q_T^S - q_T^B)f(s_t|b)\dot{s}_t = r(v(b) - q_T^S)(1 - F(s_t|b)). \tag{44}$$

(Note that types $b \in (b_\infty, b_T)$ assign positive probability only to types $s \in (s_T, 1]$ that accept at times $t > T$, and that for $t > T$, s_t is differentiable, as it is part of the solution to (15) and

(16)). From the first-order condition (44), for $b \in (b_\infty, b_T)$,

$$\begin{aligned} u(b_T, t_B(b_T)) - u(b, t_B(b)) &= \int_b^{b_T} \left(\tfrac{\partial}{\partial b} u(\tilde{b}, t_B(\tilde{b})) + \tfrac{\partial}{\partial t} u(\tilde{b}, t_B(\tilde{b})) t'_B(\tilde{b}) \right) d\tilde{b} \\ &= \int_b^{b_T} \tfrac{\partial}{\partial b} u(\tilde{b}, t_B(\tilde{b})) d\tilde{b}, \end{aligned} \tag{45}$$

In Claim 1 below, I show that $u(b,t)$ satisfies the smooth single crossing difference (SSCD) condition in $(b, -t)$. Together with the envelope formula (45), this verifies the conditions of Theorem 4.2 in Milgrom [2004] and proves that type b_t's best response to the threshold strategy s_t is to accept at time t, which completes the proof of Step 2.

Claim 1. *$u(b,t)$ satisfies the SSCD condition in $(b, -t)$ for $b \in (b_\infty, b_T)$ and $t \in [T, \infty)$.*

Proof: I will show the following conditions are satisfied, which imply the SSCD.

1. $u(b,t)$ satisfies the strict single crossing difference condition in $(b, -t)$, i.e., for all $\tilde{t} > t$ and $\tilde{b} > b$,

$$u(b,t) - u(b,\tilde{t}) \geq 0 \quad \Longrightarrow \quad u(\tilde{b},t) - u(\tilde{b},\tilde{t}) > 0.$$

2. for all t, if $\partial u(b,t)/\partial t = 0$, then for all $\delta > 0$, $\partial u(b, t-\delta)/\partial t \geq 0$ and $\partial u(b, t+\delta)/\partial t \leq 0$.

Let us start with the strict single crossing difference condition. Consider $b < \tilde{b}$ and $t < \tilde{t}$ and suppose that

$$u(b,t) \geq u(b,\tilde{t}). \tag{46}$$

I will show that $u(\tilde{b},t) > u(\tilde{b},\tilde{t})$. Define function

$$g(u|b,t) = e^{-ru}(v(b) - q_T^B) 1\{u \leq t)\} + e^{-rt}(v(b) - q_T^S) 1\{u > t\}.$$

Note that $g(\cdot|b,t)$ is decreasing in u, and

$$u(b,t) = \int_0^1 g(t_S(s)|b,t) dF(s|b). \tag{47}$$

Then

$$\begin{aligned} \int_0^1 g(t_S(s)|b,t) dF(s|b) &= u(b,t) \\ &\geq u(b,\tilde{t}) \\ &= \int_0^1 g(t_S(s)|b,\tilde{t}) dF(s|b) \\ &\geq \int_0^1 g(t_S(s)|b,\tilde{t}) dF(s|\tilde{b}), \end{aligned}$$

where the first inequality follows from (46), and the second inequality follows from the fact that $g(t_S(\cdot)|b,\tilde{t})$ is decreasing in s and $F(\cdot|\tilde{b})$ first-order stochastically dominates $F(\cdot|b)$ (as F is affiliated). This implies that

$$u(b,t) = \int_0^{s_t} e^{-rt_S(s)}(v(b)-q_T^B)dF(s|b) + (1-F(s_t|b))e^{-rt}(v(b)-q_T^S)$$
$$\geq \int_0^{s_{\tilde{t}}} e^{-ru}(v(b)-q_T^B)dF(s|\tilde{b}) + (1-F(s_{\tilde{t}}|\tilde{b}))e^{-r\tilde{t}}(v(b)-q_T^S),$$

or equivalently,

$$v(b)\left(\int_0^{s_t} e^{-ru}dF(s|b) + (1-F(s_t|b))e^{-rt} - \int_0^{s_{\tilde{t}}} e^{-ru}dF(s|\tilde{b}) - (1-F(s_{\tilde{t}}|\tilde{b}))e^{-r\tilde{t}}\right)$$
$$\geq q_T^S - \int_0^{s_{\tilde{t}}} e^{-ru}q_T^B dF(s|\tilde{b}) - (1-F(s_{\tilde{t}}|\tilde{b}))e^{-r\tilde{t}}q_T^S. \quad (48)$$

I will show that the left-hand side of (48) is positive and so, the left-hand side would increase if I substitute $v(\tilde{b})$ instead of $v(b)$. This in turn implies that $u(\tilde{b},t) > u(\tilde{b},\tilde{t})$ and completes the proof of the strict single crossing difference. Let $h(u|t) = e^{-ru}1\{u<t)\} + e^{-rt}1\{u \geq t\}$, which is decreasing in u. Then the left-hand side of (48) is equal to

$$\begin{aligned} & v(b)\left(\int_0^1 h(t_S(s)|t)dF(s|b) - \int_0^1 h(t_S(s)|\tilde{t})dF(s|\tilde{b})\right) \\ \geq & v(b)\left(\int_0^1 h(t_S(s)|t)dF(s|\tilde{b}) - \int_0^1 h(t_S(s)|\tilde{t})dF(s|\tilde{b})\right) \\ = & v(b)\int_0^1 (h(t_S(s)|t) - h(t_S(s)|\tilde{t}))dF(s|\tilde{b}) > 0, \end{aligned}$$

where the first inequality follows from $F(\cdot|\tilde{b})$ first-order stochastically dominates $F(\cdot|b)$ and $h(\cdot|t)$ decreasing, and the last term is strictly positive by $t < \tilde{t}$.

Now, let us show the second requirement of the SSCD condition. Suppose $\partial u(b,t)/\partial t = 0$. By taking the partial derivative

$$\tfrac{\partial}{\partial t}u(b,t) = e^{-rt}\left[(q_T^S - q_T^B)f(s_t|b)\dot{s}_t - r(v(b)-q_T^B)(1-F(s_t|b))\right],$$

I get that

$$\tfrac{\partial}{\partial t}u(b-\delta,t) = e^{-rt}(1-F(s_t|b-\delta))\left[(q_T^S-q_T^B)\frac{f(s_t|b-\delta)}{1-F(s_t|b-\delta)}\dot{s}_t - (r(v(b-\delta)-q_T^S)\right].$$

Since $v(b-\delta) \leq v(b)$ and $f(s_t|b-\delta)/1-F(s_t|b-\delta) \geq f(s_t|b)/1-F(s_t|b)$ (by the affiliation of f), it follows that $\partial u(b-\delta,t)/\partial t \geq 0$. Showing that $\partial u(b+\delta,t)/\partial t \leq 0$ is analogous. *q.e.d.*

Step 3: buyer type $b \in [0, b_\infty] \cap [b_T, \bar{b}_T)$ Note that type $b \in [0, b_\infty]$ has a lower value and assigns a higher probability to his (lower) offer being accepted than any type in (b_∞, b_T). Hence, it is optimal for such types to never accept. Analogously, types $b \in [b_T, \bar{b}_T)$ prefer to accept at T. $\square$

Before proceeding with the proof of Theorem 7, I derive the lower bound on the players' expected utilities in the BNE of $\mathcal{G}(q_t^S, q_t^B)$ in Lemma 6. Denote such utilities by $U_{\mathcal{G}}^S(s)$ for seller type s and $U_{\mathcal{G}}^B(b)$ for buyer type b.

Lemma 13. *There exists $\tilde{\varepsilon} > 0$ such that for sufficiently small η, $U_{\mathcal{G}}^S(s) > \tilde{\varepsilon}$ for all $s \in [0, 1]$ and $U_{\mathcal{G}}^B(b) > \tilde{\varepsilon}$ for all $b \in [0, 1]$.*

Proof. In the game $\mathcal{G}(q_t^S, q_t^B)$, type s can accept q_T^B at time T and so, her expected utility is at least $e^{-rT}(q_T^B - c(s))$, which is positive for $s \in [0, s_\infty + 2\eta]$ and sufficiently small η. Moreover, type $s > s_\infty + 2\eta$ knows for sure that her offer will be accepted by the buyer by time T, and this offer is above $\min\{\beta y^*(s - \eta) + (1 - \beta) v(s - \eta), y^*(1)\} \geq \min\{y^*(s - \eta), y^*(1)\}$. Hence, I have

$$U_{\mathcal{G}}(s) \geq \begin{cases} e^{-rT}\left(q_T^B - c(s)\right) & \text{for } s \in [0, s_\infty + 2\eta], \\ e^{-rT}\left(\min\{y^*(s - \eta), y^*(1)\} - c(s)\right) & \text{for } s \in (s_\infty + 2\eta, 1], \end{cases}$$

and the right-hand side is bounded from zero in both cases. The argument for the buyer is symmetric. $\square$

***Proof of Lemma** 7.* I construct a frequent-offer PBE limit that coincides with $(\tau_\eta^\beta, \rho_\eta^\beta)$ whenever $b > \bar{b}_T + \eta$ or $s < \bar{s}_T - \eta$. Threshold strategies b_n and s_n and paths of offers q_n^S and q_n^B on the equilibrium path are described in the text, and I proceed to the construction of $(b_n, s_n, q_n^S, q_n^B)_{n=1}^\infty$ in Steps 1 and 2. In Step 3, I specify off-path strategies and verify equilibrium conditions.

Recall that $N \equiv \lfloor T/\Delta \rfloor$, and I set $b_{N-2} = b_{N-1} = \bar{b}_T$ and $s_{N-1} = s_N = \bar{s}_T$. Without loss of generality suppose N is even.

Step 1: Construction of on-path strategies after round N This step essentially replicates Step 3 in the proof of Theorem 1. I set that for $n \geq N$, $q_n^S = q_T^S$ and $q_n^B = q_T^B$, i.e., offers are constant. By following the same line of the argument as in Lemma 12, I can show that Lemma 12 in Appendix A.2 holds with functions $\alpha^B(y)$ and $\alpha^S(x)$ in (39) and (40) replaced by

$$\alpha^B(y) = \frac{(1 - \delta^2)(q^B - c(s_\infty - y))}{\delta(q^S - c(s_\infty - y)) - \delta^2(q^B - c(s_\infty - y))}, \tag{49}$$

$$\alpha^S(x) = \frac{(1 - \delta^2)(v(b_\infty + x) - q^S)}{\delta(v(b_\infty + x) - q^B) - \delta^2(v(b_\infty + x) - q^S)}. \tag{50}$$

I then use this result and replicate the construction in Step 3 of proof of Theorem 1 to obtain threshold types $b_n, n \geq N$, and $s_n, n > N$, such that $b_n \downarrow b_\infty, s_n \uparrow s_\infty, b_N \leq s_{N+1} + \eta$, and that satisfy

$$v(b_n) - q_T^S = \delta\alpha_n^S(v(b_n) - q_T^B) + \delta^2(1 - \alpha_n^S)(v(b_n) - q_T^S) \text{ for } n \text{ even}, \tag{51}$$

$$q_T^B - c(s_n) = \delta\alpha_n^B(q_T^S - c(s_n)) + \delta^2(1 - \alpha_n^B)(q_z - c(s_n)) \text{ for } n \text{ odd}, \tag{52}$$

where α_n^S and α_n^B are defined by

$$\alpha_n^S = \frac{s_{n+1} - s_{n-1}}{b_n + \eta - s_{n-1}}, \tag{53}$$

$$\alpha_n^B = \frac{b_{n-1} - b_{n+1}}{b_{n-1} - s_n + \eta}. \tag{54}$$

Equations (51) and (52) are counter-parts of (11) and (12) with the difference coming from the fact that here both sides make unacceptable offers until they make a revealing offer, which is accepted by all opponent's types. Equation (51) implies that type b_n is indifferent between revealing himself to be in $[b_n, b_{n-1})$ in round n with price offer q_T^S and revealing that $b \in [b_{n+2}, b_n)$ in round $n+2$. Any type $b > b_n$ has a higher value and assigns a lower probability to the seller making offer q_T^B in round $n+1$ (as $(s_{n+1} - s_n)/(b + \eta - s_{n-1})$ is decreasing in b), hence, he strictly prefers to offer q_T^B in round n to offering q_T^B in round $n+2$. Similarly, types $b < b_n$ strictly prefer to make offer q_T^B in round $n+2$. Hence, equation (51) guarantees that after round N, if the seller follows on-path strategy, the on-path strategy is optimal for buyer types $b \leq b_{N-1}$. The argument is symmetric for the seller types $s \geq s_N$.

Step 2: Construction of on-path strategies before round N Let $\tilde{\varepsilon} > 0$ be as in Lemma 13, and fix

$$\varepsilon_0 \equiv \frac{1}{2} \min \left\{ \tilde{\varepsilon}, \min_{t \in [0,T]} \{q_t^S - y^*(b_t), y^*(s_t) - q_t^B\} \right\}.$$

Let

$$s_n = s_{n+1} - \Delta \frac{ds_t}{dt}|_{t=(n+1)\Delta} \text{ for odd } n \leq N - 3,$$

$$b_n = b_{n+1} - \Delta \frac{db_t}{dt}|_{t=(n+1)\Delta} \text{for even } n \leq N - 4.$$

For odd $n \le N-3$, set $b_n = b_{n-1}$, and for even $n \le N-2$, set $s_n = s_{n-1}$. I construct paths q_n^S (for n odd) and q_n^B (for n even) backwards in time starting from $N-1$ that satisfy

$$v(b_n) - q_n^S = \delta^2(v(b_n) - q_{n+2}^S), \text{ for even } n \le N-2, \tag{55}$$

$$q_n^B - c(s_n) = \delta^2(q_{n+2}^B - c(s_n)), \text{ for odd } n \le N-1. \tag{56}$$

Let n_0 be the largest n such that either $q_n^S > y^*(1) - \varepsilon_0$ or $q_n^B < y^*(0) + \varepsilon_0$. Suppose that the former is the case in round n_0 and n_0 is even. (The other case is analogous). Then I redefine the rounds to be $n - n_0$ instead of n, i.e., s_{n_0+1} and $q_{n_0+1}^B$ become s_1 and q_1^B, resp., b_{n_0+2} and $q_{n_0+2}^S$ become b_2 and q_2^S, resp., and so on. Note that this way $q_1^B \ge y^*(0) + \varepsilon_0$ and $q_2^S \le y^*(1) - \varepsilon_0$.

I extrapolate (b_n, s_n, q_n^B, q_n^S) to continuous time by the linear extrapolation. The following claim follows from the construction of (b_n, s_n, q_n^B, q_n^S).

Claim 2. *The linear extrapolation of* (b_n, s_n, q_n^B, q_n^S) *to continuous time converges uniformly to* (b_t, s_t, q_t^B, q_t^S) *on* $[0, T]$ *as* $\delta \to 1$*, i.e., for any* $\varepsilon > 0$*, there is* $\delta^\dagger(\varepsilon) \in (0,1)$ *such that for all* $\delta \in (\delta^\dagger(\varepsilon), 1)$ *it holds*

$$\sup_{t\in[0,T]} \left| b_n|_{n=\lfloor \frac{t}{\Delta} \rfloor} - b_t \right| < \varepsilon, \quad \textit{and} \quad \sup_{t\in[0,T]} \left| s_n|_{n=\lfloor \frac{t}{\Delta} \rfloor} - s_t \right| < \varepsilon,$$
$$\sup_{t\in[0,T]} \left| q_n^B|_{n=\lfloor \frac{t}{\Delta} \rfloor} - q_t^B \right| < \varepsilon, \quad \textit{and} \quad \sup_{t\in[0,T]} \left| q_n^S|_{n=\lfloor \frac{t}{\Delta} \rfloor} - q_t^S \right| < \varepsilon,$$

where $\Delta = -\frac{1}{r}\ln\delta$*.*

Claim 2 implies the convergence of outcomes of the PBEs that I am constructing to $(\tau_\eta^\beta, \rho_\eta^\beta)$ for types $b > \bar{b}_T + \eta$ and $s < \bar{s}_T - \eta$, as such types assign probability one to the trade happening when they reveal themselves at some time before T.

Step 3: Construction of off-path strategies and verification of equilibrium I describe how the seller's deviations are punished and strategies are symmetric after the buyer's deviations.

Since revealing price offers become more favorable over time, the lowest on-path utility is attained in the beginning of the game. Thus, Lemma 13 and Claim 2 imply that there is $\delta^*(\varepsilon_0) \in (0,1)$ such that for $\delta \in (\delta^*(\varepsilon_0), 1)$, the seller's on-path utility in any round is greater than ε_0, and in addition,

$$q_n^S > y^*(b_n) + \tfrac{1}{2}\varepsilon_0, \tag{57}$$

for any $n \le N$. Moreover, I will specify below that seller's offer q_1^B in round 1 is accepted both on- and off-path. Thus, the seller's on-path continuation utility in any round is at least

$$\max\{q_1^B - c(s), \varepsilon_0\} > \max\{y^*(0) - c(s), 0\} + \tfrac{1}{2}\varepsilon_0. \tag{58}$$

Let us choose $\delta(\varepsilon)$ as in Lemma 2 such that for any $\underline{b}$, in the seller punishing equilibrium with $\underline{b}$, the seller's continuation utility is less than $\max\{y^*(\underline{b}) - c(s), 0\} + \frac{1}{2}\varepsilon_0$. I suppose that δ is greater than $\max\{\delta(\varepsilon), \delta^*(\varepsilon)\}$.

The following seller's deviations are possible:

1. If the seller makes in round n an offer different from $\overline{y}(1)$ or q_n^B, the play switches to the seller punishing equilibrium with $\underline{b} = 0$ described in Lemma 2. Such a deviation is not profitable, as the utility from punishment is at most $\max\{y^*(0) - c(s), 0\} + \frac{1}{2}\varepsilon_0$, which by (58) is the lower bound on the on-path utility in any round.

2. Suppose type $s \in [s_{n-1}, s_n)$ of the seller mimics a lower type and offers $q_{n'}^B < q_n^B$ in odd round n. I specify that in equilibrium, such an offer is accepted by the buyer irrespective of whether the buyer detects such an offer as a deviation or not, and if it gets rejected, then the subsequent play returns to the equilibrium path. By (56), such a deviation is not profitable for type s. Since the play returns to the main path in case the deviating offer is rejected, the buyer indeed prefers to accept such an offer, as it is lower than subsequent on-path offers.

3. Suppose type $s \in [s_{n-1}, s_n)$ offers $\overline{y}(1)$ instead of q_n^B in odd round n. Such a deviation is detected by the buyer only when $b < \underline{\pi}(s_n)$. In equilibrium, buyer types who detect this deviation switch to optimistic conjectures as in Lemma 2 with $\underline{b} = 0$. Note that these types assign probability one to the seller type $\underline{\pi}(b)$.

   3.1 If the seller type is in $[s_{n-1}, s_n) \cap [0, \underline{\pi}(\underline{\pi}(s_n)))$, then the seller assigns probability one to the buyer detecting the deviation. In this case, the play proceeds as in the punishing equilibrium in Lemma 2 with $\underline{b} = 0$.

   3.2 If the seller type is in $[s_{n-1}, s_n) \cap [\underline{\pi}(\underline{\pi}(s_n)), 1]$, then the seller does one of two things depending on which one brings higher expected utility in the continuation:

   - offer q_{n+2}^B in round $n + 2$, which is accepted if and only if $b \geq \underline{\pi}(s_n)$. If such an offer is rejected, then they follow their strategies in the punishing equilibrium;
   - they follow their strategies in the punishing equilibrium.

   By the single-crossing property of payoffs and construction of offers in (55), even if the buyer does not detect the deviation, type s prefers to reveal herself in round n rather than in round $n + 2$. (See the argument in the end of Step 1 of this proof). On top of that, delaying the revelation increases the probability that the buyer detects the deviation, which triggers the punishing equilibrium making in turn the deviation even less appealing for the seller (by the inequality (58)).

4. Suppose type $s \in [s_{n-1}, s_n)$ offers $\overline{y}(1)$ instead of q_k^B in rounds $k = n, n+2, \dots, n'$. The strategies in the continuation are analogous to the previous case: buyer types who detect such a deviation switch to the seller punishing equilibrium, and the seller types either switch to strategies in the punishing equilibrium or do so after making offer $q_{n'+2}^B$ in round $n'+2$, whichever brings higher continuation payoff.

5. Suppose in round n, the buyer makes offer q_n^S and the seller deviates and rejects such an offer in round $n+1$. Since the buyer follows the on-path strategy up to round n, the seller's type is below $\overline{\pi}(b_{n-1})$. I specify that after such a deviation, the play switches to the seller punishing equilibrium with $\underline{b} = b_n$ described in Lemma 2. By (57), the seller's continuation on-path utility in round $n+1$ is at least $y^*(b_n) + \varepsilon_0 - c(s)$. At the same time, the seller punishing equilibrium with $\underline{b} = b_n$ brings utility less than $y^*(b_n) - c(s) + \frac{1}{2}\varepsilon_0$, which makes such a deviation unprofitable.

$\square$

B Appendix

This appendix constructs and analyzes the *punishing equilibrium*, the continuation equilibrium with optimistic conjectures of the buyer and original beliefs of the seller. The structure of this section is as follows. I first consider in Appendices $B.1$-$B.4$ the auxiliary game, in which the buyer holds optimistic conjectures and is restricted to either accept the last seller's offer or make a counter-offer $\underline{y}(0)$. Appendix $B.1$ constructs punishing equilibrium strategies for buyer types below η and seller type $s = 0$, and proves the Coasian property for such types. Appendix $B.2$ describes PBE strategies for the rest of the types. Appendix $B.3$ contains the preliminary analysis of such strategies. Appendix $B.4$ shows that the willingness to pay of the buyer is uniformly (in type) close to $\max\{y^*(0), c(\underline{\pi}(b))\}$. The argument in these sections is provided for the case $\underline{b} = 0$ and $\overline{b} = 1$. Finally, I show in Appendix $B.5$ how the argument generalizes to $0 \leq \underline{b} < \overline{b} \leq 1$, and collect all the steps to complete the proof of Lemma 2.

B.1 Appendix for Step 1: Standard Coasian dynamics for types $b \in [0, \eta]$ and $s = 0$

Observe that when the buyer has optimistic conjectures, buyer types $b \in [0, \eta]$ assign probability one to the seller type $s = 0$, and hence, if I restrict attention only to those types, my game is essentially the game with one-sided private information and one-sided offers.[14] Hence, for those types, I can use existing results to construct the punishing equilibrium strategies that

[14] Recall that I consider the auxiliary game with the buyer pooling on $\underline{y}(0)$.

exhibit Coasian dynamics. my construction and results in this subsection follow the argument in Fudenberg et al. [1985] and Ausubel and Deneckere [1989], and so I simply sketch the argument.

Note that seller type $s = 0$ is indifferent between accepting buyer's offer $\underline{y}(0)$ in the current round and making a counter-offer $\overline{y}(0)$ that is guaranteed to be accepted (by Lemma 1) in the next round. I focus on PBEs of the auxiliary game, in which the seller never accepts the buyer's price offer. I follow Fudenberg et al. [1985] to construct PBE in weak-Markov strategies in this auxiliary game. Let right-continuous and weakly increasing function $P(b)$ be the maximal willingness to pay of type b. In equilibrium, after any history, type b accepts any offer less than or equal to $P(b)$, and rejects any offer above $P(b)$. By the standard argument,

Lemma 14. *In the PBE of the auxiliary game, after any history, the posterior of seller type $s = 0$ is a truncation of prior from above at some β.*

Let $\sigma(\beta, H_n)$ be the probability distribution over seller's offers in odd round n given the history H_n of rejected offers up to round n and the highest remaining buyer type β in the beginning of round n. By the same argument as in Lemma 3 in Fudenberg et al. [1985], I can show that there exists $\bar{\beta} \in (0, \eta]$ such that $P(b) = (1 - \delta^2)v(b) + \delta^2\overline{y}(0)$ for $b \in [0, \bar{\beta}]$ and $\sigma(\beta, H_n) = \overline{y}(0)$ for $\beta \in [0, \bar{\beta}]$ constitute a continuation equilibrium in any subgame with the highest buyer type below $\bar{\beta}$. That is, $\bar{\beta}$ is sufficiently small so that given $P(\cdot)$, the seller optimally chooses not to screen buyer's types and offer $\overline{y}(0)$: and the willingness to pay $P(b)$ of types $b \in [0, \bar{\beta}]$ is such that they are just indifferent between accepting $P(b)$ and rejecting it and accepting the seller's offer $\overline{y}(0)$ in the following round. Again, as in Lemma 3 in Fudenberg et al. [1985], $\bar{\beta} > 0$ implies that the game ends in at most N^* rounds. I can then follow the steps in the proof of Proposition 1 in Fudenberg et al. [1985] to construct weak-Markov equilibrium strategies $P(b)$ and $\sigma(\beta, H_{n-1})$. To summarize,

Lemma 15. *There exists a PBE in weak-Markov strategies (P, σ) in the auxiliary game such that for some $\bar{\beta}$, $P(b) = (1 - \delta^2)v(b) + \delta^2\overline{y}(0)$ for $b \in [0, \bar{\beta}]$.*

By the same argument as in the Uniform Coase Conjecture[15] in Ausubel and Deneckere [1989], I can show that the constructed weak-Markov equilibrium exhibits Coasian dynamics:

Lemma 16. *For any $\varepsilon > 0$, there exists $\delta_1(\varepsilon) < 1$ such that for all $\delta \in (\delta_1(\varepsilon), 1)$, in the weak-Markov equilibrium in Lemma 15, $\sigma(\eta, H_n) < y^*(0) + \varepsilon$ after any history H_n.*

Remark 7. The Uniform Coase conjecture in Ausubel and Deneckere [1989] holds for a general class of demand functions, which are their counter-parts of my function v. Hence, for any $\underline{b} \in [0, 1]$, if in the argument above I replace the lowest type of the seller $s = 0$ with $s = \underline{b}$ and the buyer types $b \in [0, \eta]$ with $b \in [\underline{b}, \underline{b} + \eta]$, $\delta_1(\varepsilon)$ in Lemma 16 will not change.[16]

[15]See Theorem 5.4 in Ausubel and Deneckere [1989] and the discussion after the statement of Theorem 5.4 for the "gap" version of the Uniform Coase Conjecture, which is the relevant result in my analysis.

[16]Of course, $y^*(0)$ in Lemma 16 should be replaced by $y^*(\underline{b})$.

B.2 Appendix for Step 2: Strategies for types $b > \eta$ and $s > 0$

On-path strategies for buyer types $b \in (\eta, 1]$ and seller types $s \in (0, 1]$ are described in the main text. Specifically, I specified that as long as seller's previous offers were above $P(b)$, buyer type b accepts the current offer if and only if it is less than or equal to $P(b)$; and that if the highest remaining buyer type β belongs to $[\overline{\beta}(s), \overline{\pi}(s)]$, seller type s makes offer $\sigma(\beta, s) = P(t(\beta, s))$.

Note that if $\underline{\pi}(s) < \overline{\beta}(s)$ and $\beta \in [\underline{\pi}(s), \overline{\beta}(s))$, then $P(b) \leq c(s)$ for all remaining types b in B_s below β, and so it is optimal for the seller type s to make unacceptable offers for the rest of the game and get utility 0. I specify that in this case, $\sigma(\beta, s) = \overline{y}(s, \overline{\pi}(s))/\delta^2 + v(\overline{\pi}(s))(1-\delta^2)/\delta^2$, which is guaranteed to be rejected by the buyer (as $P(b) < c(s) < \overline{y}(s, \overline{\pi}(s))$ for $b < \beta$) and so, is optimal for seller type s who believes that $b \in [\underline{\pi}(s), \beta)$.

Remark 8. *Type $\underline{\pi}(s)$ in this case is willing to accept any offer less than or equal to $\overline{y}(s, \overline{\pi}(s))$. Observe that in my original game with two-sided offers, I can specify instead that type s in this case assigns probability one to buyer type $b = \overline{\pi}(s)$, in which case $\overline{y}(s, \overline{\pi}(s))$ would be indeed her equilibrium offer in the continuation (as she believes that the buyer type $b = \overline{\pi}(s)$ in turn assigns probability 1 to the seller type s).*

Next, I specify off-path strategies. Suppose that in the beginning of an odd round n, I have

- the buyer's threshold acceptance strategy $P^\dagger(b)$ in round $n+1$, i.e., offer p in round $n+1$ is accepted if and only if $p \leq P^\dagger(b)$;
- $\overline{\beta}^\dagger(s)$ that satisfies $\overline{\beta}^\dagger(s) = \max\left\{\sup\{b : P^\dagger(b) \leq c(s)\}, \underline{\pi}(s)\right\}$, i.e., if the buyer follows $P^\dagger$, then seller type s gets positive payoff only from allocating to types above $\overline{\beta}^\dagger(s)$;
- the seller's screening policy $\sigma^\dagger(\beta, s)$ for any $\beta \in B_s$.

Suppose that in round n, the seller makes an offer p, and the buyer rejects it. Let $\beta^\ddagger = \inf\{b : P^\dagger(b) \geq p\}$ be the highest buyer type that rejects p under strategy $P^\dagger$. Then I specify the continuation strategies as follows:

- For $b < \beta^\ddagger$, the buyer did not deviate from the strategy $P^\dagger$ in round $n+1$, and I specify that $P^\ddagger(b) = P^\dagger(b)$ for $b < \beta^\ddagger$.
- For $s < \beta^\ddagger + \eta$, even if the buyer deviated from $P^\dagger$ in round $n+1$, the seller does not detect such a deviation in round $n+2$. In the continuation, the seller makes offers $\sigma^\dagger(\beta, s)$ for $\beta \in [\underline{\pi}(s), \overline{\pi}(s)]$. Thus, for such seller types, I specify $\sigma^\ddagger(\beta, s) = \sigma^\dagger(\beta, s)$.
- For $b \in [\beta^\ddagger, \beta^\ddagger + 2\eta)$, the buyer deviated from $P^\dagger$ in round $n+1$, but believes that the deviation is not detected, as he assigns probability one to seller type $s < \beta^\ddagger + \eta$. He expects such seller types to follow $\sigma^\ddagger\left(\beta^\ddagger, s\right)$ defined in the previous step, and I specify

$$P^\ddagger(b) = (1-\delta^2)v(b) + \delta^2\sigma^\ddagger\left(\beta^\ddagger, \underline{\pi}(b)\right), \text{ for } b \in [\beta^\ddagger, \beta^\ddagger + 2\eta). \tag{59}$$

Note that since $\sigma^\ddagger\left(\beta^\ddagger, s\right)$ is defined in the previous step for $s < \beta^\ddagger + \eta$ and $\underline{\pi}(b) < \beta^\ddagger + \eta$ for $b \in [\beta^\ddagger, \beta^\ddagger + 2\eta)$, (59) is well-defined.

- For $s \in [\beta^\ddagger + \eta, \beta^\ddagger + 3\eta)$, the seller detects the buyer's deviation. I specify that in round $n+2$, such a type believes that the buyer type is uniformly distributed on $[\underline{\pi}(s), \beta^\ddagger + 2\eta]$. I define

$$\overline{\beta}^\ddagger(s) = \max\left\{\sup\{b \leq \beta^\ddagger + 2\eta : P^\ddagger(b) \leq c(s)\}, \underline{\pi}(s)\right\}, \text{ for } s \in [\beta^\ddagger + \eta, \beta^\ddagger + 3\eta).$$

  Let $\sigma^\ddagger(\beta, s)$ for $\beta \in [\overline{\beta}^\ddagger(s), \beta^\ddagger + 2\eta)$ be the optimal screening policy of seller type $s \in [\beta^\ddagger + \eta, \beta^\ddagger + 3\eta)$, given that the buyer's willingness to pay function is $P^\ddagger(b)$. Note that only buyer types below $\beta^\ddagger + 2\eta$ are relevant for screening by the seller type s, and $P^\ddagger(b)$ was defined in the previous step for such types. For $\beta \in [\underline{\pi}(s), \overline{\beta}^\ddagger(s))$, I specify that $\sigma^\ddagger(\beta, s) = \overline{y}(s, \overline{\pi}(s))/\delta^2 + v(\overline{\pi}(s))(1-\delta^2)/\delta^2$.

- For $b \in [\beta^\ddagger + 2\eta, \beta^\ddagger + 4\eta)$, the buyer knows that his deviation in round $n+1$ is detected (as he assigns probability one to seller type $\underline{\pi}(b) \in [\beta^\ddagger + \eta, \beta^\ddagger + 3\eta)$). I specify the new willingness to pay function for such types as follows:

$$P^\ddagger(b) = (1-\delta^2)v(b) + \delta^2 \sigma^\ddagger\left(\beta^\ddagger + 2\eta, \underline{\pi}(b)\right), \text{ for } b \in [\beta^\ddagger + 2\eta, \beta^\ddagger + 4\eta). \tag{60}$$

  Note that $\underline{\pi}(b) < \beta^\ddagger + 3\eta$ for $b \in [\beta^\ddagger + 2\eta, \beta^\ddagger + 4\eta)$, and $\sigma^\ddagger(\beta, s)$ was specified for $s < \beta^\ddagger + 3\eta$ in the previous step. Thus, (60) is well-defined.

For general $k \geq 2$,

- For $s \in \left[\beta^\ddagger + k\eta, \beta^\ddagger + (k+2)\eta\right)$, the seller detects the buyer's deviation. I specify that such types believe that the buyer type is uniformly distributed on $[\underline{\pi}(s), \beta^\ddagger + (k+1)\eta]$. I define

$$\overline{\beta}^\ddagger(s) = \max\left\{\sup\{b \leq \beta^\ddagger + (k+1)\eta : P^\ddagger(b) \leq c(s)\}, \underline{\pi}(s)\right\}, \quad \text{for } s \in \left[\beta^\ddagger + k\eta, \beta^\ddagger + (k+1)\eta\right).$$

  Let $\sigma^\ddagger(\beta, s)$ for $\beta \in [\overline{\beta}^\ddagger(s), \beta^\ddagger + (k+1)\eta)$ be the optimal screening policy of seller type $s \in [\beta^\ddagger + \eta, \beta^\ddagger + (k+2)\eta)$, given that the buyer's willingness to pay function is $P^\ddagger(b)$. Note that only buyer types below $\beta^\ddagger + (k+1)\eta$ are relevant for screening by the seller type s, and $P^\ddagger(b)$ was defined in the previous step for such types. For $\beta \in [\underline{\pi}(s), \overline{\beta}^\ddagger(s))$, I specify that $\sigma^\ddagger(\beta, s) = \overline{y}(s, \overline{\pi}(s))/\delta^2 + v(\overline{\pi}(s))(1-\delta^2)/\delta^2$.

- For $b \in [\beta^\ddagger + (k+1)\eta, \beta^\ddagger + (k+3)\eta)$, the buyer knows that his deviation in round $n+1$ is detected (as he assigns probability one to seller type $\underline{\pi}(b) \in [\beta^\ddagger + k\eta, \beta^\ddagger + (k+2)\eta)$). I specify the new willingness to pay function for such types as follows:

$$P^{\ddagger}(b) = (1-\delta^2)v(b)+\delta^2\sigma^{\ddagger}\left(\beta^{\ddagger}+(k+1)\eta, \underline{\pi}(b)\right), \text{ for } b \in [\beta^{\ddagger}+(k+1)\eta, \beta^{\ddagger}+(k+3)\eta). \quad (61)$$

Note that $\underline{\pi}(b) < \beta^{\ddagger} + (k+2)\eta$ for $b \in [\beta^{\ddagger} + (k+1)\eta, \beta^{\ddagger} + (k+3)\eta)$, and $\sigma^{\ddagger}(\beta, s)$ was specified for $s < \beta^{\ddagger} + (k+2)\eta$ in the previous step. Thus, (61) is well-defined.

Since $\eta > 0$, this way I construct in a finite number of steps functions $P^{\ddagger}$ and $\sigma^{\ddagger}$.

Now, the equilibrium strategies are constructed as follows. In the beginning of the game, set $\tilde{P} = P$ and $\tilde{\sigma} = \sigma$, where P and σ are as described in the main text. The following deviations are possible:

- If the seller deviates to p from $\tilde{\sigma}(\beta, s)$, then such an offer is guaranteed to be accepted if $p \le \tilde{P}(\underline{\pi}(s))$, and in this case the game ends. If $p > \tilde{P}(\underline{\pi}(s))$ and the buyer rejects it, then the seller makes an offer $\tilde{\sigma}(\beta', s)$ in the next round, where $\beta' = \min\left\{\beta, \inf\{b : \tilde{P}(b) \ge p\}\right\}$, and players continue following $\tilde{P}$ and $\tilde{\sigma}$ in the continuation.
- If the buyer deviates and the offer $p < \tilde{P}(b)$ is rejected, then I set $P^{\dagger} = \tilde{P}$ and $\sigma^{\dagger} = \tilde{\sigma}$ in the argument above, and compute corresponding $P^{\ddagger}$ and $\sigma^{\ddagger}$. I set new $\tilde{P}$ and $\tilde{\sigma}$ to $P^{\ddagger}$ and $\sigma^{\ddagger}$, resp., and the game proceeds according to these new strategies, with the play after deviations being as I just specified for the original $\tilde{P}$ and $\tilde{\sigma}$.

B.3 Appendix for Step 3: Construction of $P(\cdot)$ and $t(\cdot, \cdot)$

In this appendix, I derive several properties of equilibrium functions $P(\cdot)$ and $t(\cdot, \cdot)$. I first derive properties of the function $\sigma(\cdot, \cdot)$. The central result is the bound $d(\delta)$ on $\overline{\pi}(s) - t(s)$ in Lemma 19 that I use in the main text to construct functions $P(\cdot)$ and $t(\cdot, \cdot)$. In the end, I show that the constructed strategies indeed constitute the PBE in the auxiliary game.

The next lemma shows properties of the seller's screening offers σ.

Lemma 17. *The following hold:*

1. *$\sigma(\beta, s)$ is weakly increasing in β on $[\overline{\beta}(s), \overline{\pi}(s)]$, and $\sigma(\beta, s)$ is weakly increasing in s on $[0, \underline{\pi}^{-1}(\beta))$.*
2. *$\sigma(s)$ is right-continuous and weakly increasing in s.*

Proof. Part 1: On $[\overline{\beta}(s), \overline{\pi}(s)]$, increasing β strictly increases the gradient of the objective in (26), and so the correspondence $T(\beta, s)$ is weakly increasing in β. This implies that $t(\beta, s)$ is weakly increasing in β, and so is $\sigma(\beta, s) = P(t(\beta, s))$.

To prove the monotonicity of $\sigma(\beta, s)$ in s, suppose to contradiction that for some $\tilde{\beta}$ and some $s_1 < s_2 < \underline{\pi}^{-1}(\tilde{\beta})$, $\sigma(\tilde{\beta}, s_2) < \sigma(\tilde{\beta}, s_1)$. Note that if $\overline{\beta}(s_2) \geq \min\{\overline{\pi}(s_1), \tilde{\beta}\}$, then $\sigma(\tilde{\beta}, s_2) = \overline{y}(s, \overline{\pi}(s))/\delta^2 + v(\overline{\pi}(s))(1-\delta^2)/\delta^2 > c(s) \geq P(b)$ for all $b \leq \tilde{\beta}$, which implies that $\sigma(\tilde{\beta}, s_2) > \sigma(\tilde{\beta}, s_1)$. Thus, suppose that $\min\{\overline{\pi}(s_1), \tilde{\beta}\} > \overline{\beta}(s_2)$. I consider the case $\tilde{\beta} \leq \overline{\pi}(s_1)$, and the argument is analogous for $\tilde{\beta} > \overline{\pi}(s_1)$. Let $\mathcal{B} \equiv [\underline{\pi}(s_1), \tilde{\beta}]$. Let

$$\pi(\beta, b, s) \equiv \begin{cases} (\beta - b)(P(b) - c(s)) & \text{for } b \in [\overline{\beta}(s), \tilde{\beta}], \\ (\overline{\beta}(s) - \beta)(b - \overline{\beta}(s_2)) & \text{for } b \in [\underline{\pi}(s_1), \overline{\beta}(s)). \end{cases}$$

Then for $s = s_1, s_2$, the value function $R(\beta, s)$ solves the Bellman equation:

$$R(\beta, s) = \max_{b \in B_s \cap [\underline{\pi}(s_1), \beta]} \left\{ \pi(\beta, b, s) + \delta^2 R(b, s) \right\}. \tag{62}$$

Indeed, equation (62) allows the seller to choose $b \in [\underline{\pi}(s), \beta]$ as opposed to the constraint $b \in [\overline{\beta}(s), \beta]$ in (26). However, by construction of $\pi(\beta, b, s)$, it is never optimal for type s to choose $b < \overline{\beta}(s)$, as $\pi(\beta, b, s) < 0$ for $b < \overline{\beta}(s)$ and so, it is dominated by $b = \overline{\beta}(s)$. Thus, the solutions to (62) and (26) coincide for $\beta \geq \overline{\beta}(s)$. Observe that for $\beta \in [\underline{\pi}(s_1), \overline{\beta}(s))$, $R(\beta, s) = -(\overline{\beta}(s) - \beta)^2/(1-\delta^2) < 0$.

I will show that the maximized function in (62) satisfies the single crossing property in $(b; s)$ on $\mathcal{B} \times \{s_1, s_2\}$. This implies, by Theorem 4' in Milgrom and Shannon [1994], that $t(\tilde{\beta}, s_1) \leq t(\tilde{\beta}, s_2)$ and so, $\sigma(\tilde{\beta}, s_1) \leq \sigma(\tilde{\beta}, s_2)$, which leads to a contradiction and completes the proof of monotonicity of $\sigma(\beta, s)$.

Since $\partial^2 \pi(\beta, b, s)/\partial s \partial b = c'(s) > 0$, $\pi(\beta, b, s)$ has strict single crossing property in $(b; s)$, and so it is sufficient to show that $R(b, s)$ has single crossing property in $(b; s)$. I show that for $\beta_2, \beta_1 \in \mathcal{B}$ such that $\beta_2 > \beta_1$, it holds

$$R(\beta_2, s_2) > R(\beta_1, s_2), \tag{63}$$

which is a stronger property than the strict single crossing property. There are three cases possible.

Case 1: $\beta_1 < \beta_2 \leq \overline{\beta}(s_2)$. Then

$$\begin{aligned} R(\beta_2, s_2) - R(\beta_1, s_2) &= -\frac{1}{1-\delta^2}(\overline{\beta}(s_2) - \beta_2)^2 + \frac{1}{1-\delta^2}(\overline{\beta}(s_2) - \beta_1)^2 \\ &= \frac{1}{1-\delta^2}(\beta_2 - \beta_1)(2\overline{\beta}(s_2) - \beta_2 - \beta_1) > 0 \end{aligned}$$

Case 2: $\beta_1 \leq \overline{\beta}(s_2) < \beta_2$. Then $R(\beta_1, s_2) \leq 0 < R(\beta_2, s_2)$.

Case 3: $\overline{\beta}(s_2) < \beta_1 < \beta_2$. Then by the optimality of $t(\beta_2, s_2)$ when $\beta = \beta_2$,

$$\pi(\beta_2, t(\beta_2, s_2), s_2) + \delta^2 R(t(\beta_2, s_2), s_2) \geq \pi(\beta_2, t(\beta_1, s_2), s_2) + \delta^2 R(t(\beta_1, s_2), s_2), \tag{64}$$

and so, for $R(\beta_2, s_2) > R(\beta_1, s_2)$ it is sufficient that

$$\pi(\beta_2, t(\beta_1, s_2), s_2) + \delta^2 R(t(\beta_1, s_2), s_2) > \pi(\beta_1, t(\beta_1, s_2), s_2) + \delta^2 R(t(\beta_1, s_2), s_2). \tag{65}$$

Inequality (65) is equivalent to

$$(\beta_2 - \beta_1)(P(t(\beta_1, s_2)) - c(s_2)) > 0,$$

which holds, as $P(t(\beta_1, s_2)) > c(s_2)$ whenever $\beta_1 > \overline{\beta}(s_2)$. I conclude that indeed for $\beta_2 > \beta_1$, $R(\beta_2, s_2) > R(\beta_1, s_2)$ and so, function $\pi(\beta, b, s) + \delta^2 R(b, s)$ has the single crossing property in $(b; s)$ on $\mathcal{B} \times \{s_1, s_2\}$.

Part 2: To show that $\sigma(s)$ is right-continuous in s, suppose to contradictions that there is a sequence $s_k \downarrow s^*$ such that $\lim_{k\to\infty} \sigma(s_k) \equiv \bar{\sigma} > \sigma(s^*)$, which is equivalent to $\lim_{k\to\infty} \sigma(\overline{\pi}(s_0), s_k) \equiv \bar{\sigma} > \sigma(\overline{\pi}(s_0), s^*)$. By the right-continuity of $P(\cdot)$, this implies that $\lim_{k\to\infty} t(\overline{\pi}(s_0), s_k) \equiv \bar{t} > t(\overline{\pi}(s_0), s^*)$. By the generalization of Theorem of the Maximum in Ausubel and Deneckere [1993b], $\bar{t} \in T(\overline{\pi}(s_0), s^*)$, which however, contradicts the fact that $t(\overline{\pi}(s_0), s^*) = \sup T(\overline{\pi}(s_0), s^*)$. Monotonicity of $\sigma(s)$ follows from the monotonicity of $\sigma(\beta, s)$ in both arguments proven in the first part. □

The next lemma is the preliminary lower bound on the expected profit of the seller.

Lemma 18. *For any $b \in [0, 1]$, it holds*

$$P(b) \geq c(\underline{\pi}(b)) + (1 - \delta^2)\xi, \tag{66}$$

Moreover, there exists $\delta_0 \in (0, 1)$ such that for any $\delta \in (\delta_0, 1)$ and any $s \in [0, 1 - \eta]$, it holds

$$R(\overline{\pi}(s), s) \geq \tfrac{\xi^2}{4\ell}(1 - \delta)^2. \tag{67}$$

Proof. For any type b of the buyer,

$$\begin{aligned} P(b) &= (1 - \delta^2)v(b) + \delta^2 P(t(\underline{\pi}(b))) \\ &\geq (1 - \delta^2)v(b) + \delta^2 c(\underline{\pi}(b)) \\ &\geq c(\underline{\pi}(b)) + (1 - \delta^2)\xi, \end{aligned}$$

where the first inequality follows from the fact that the seller does not make offers below her costs, the second inequality is by $v(b) - c(\underline{\pi}(b)) \geq \xi$.

To prove inequality (67), consider type $s \in [0, 1-\eta]$ of the seller and suppose she makes offer $p \equiv c(s) + (1-\delta^2)\frac{\xi}{2}$ in the first round of screening. Let

$$\tilde{b} \equiv \inf\{b \in [0,1] : c(\underline{\pi}(b)) > c(s) - (1-\delta^2)\tfrac{\xi}{2}\},$$

and $b^* \equiv \max\{\tilde{b}, \underline{\pi}(s)\}$. By (66), for types $b \in [b^*, \overline{\pi}(s)]$,

$$P(b) \geq c(\underline{\pi}(b)) + (1-\delta^2)\xi \geq c(s) + (1-\delta^2)\tfrac{\xi}{2}$$

and so, such types accept the offer p. I next compute the mass of these types. I consider separately two cases.

Case 1: $\tilde{b} = 0$. Then $b^* = \underline{\pi}(s) = 0$ and so, the mass of types in $[b^*, \overline{\pi}(s)]$ is greater than η. This implies that $R(\overline{\pi}(s), s) \geq \eta(p - c(s)) = \frac{1}{2}(1-\delta^2)\xi\eta$.

Case 2: $\tilde{b} > 0$ and so, by continuity of c it holds $c(\underline{\pi}(\tilde{b})) = c(s) - \frac{1}{2}(1-\delta^2)\xi$. Since $\max_{s\in[0,1]} c'(s) \leq \ell$, $c(s) \leq c(\underline{\pi}(\tilde{b})) + \ell(s - \underline{\pi}(\tilde{b}))$ and so,

$$s - \underline{\pi}(\tilde{b}) \geq \tfrac{1}{\ell}(c(s) - c(\underline{\pi}(\tilde{b}))) = (1-\delta^2)\tfrac{\xi}{2\ell}.$$

Hence,

$$\begin{aligned}
\overline{\pi}(s) - b^* &\geq \min\{\overline{\pi}(s) - \tilde{b}, \overline{\pi}(s) - \underline{\pi}(s)\} \\
&\geq \min\{s - \underline{\pi}(\tilde{b}), \eta\} \\
&\geq \min\{(1-\delta^2)\tfrac{\xi}{2\ell}, \eta\}.
\end{aligned}$$

For $\delta \in (\sqrt{\max\{0, 1-2\eta\ell/\xi\}}, 1)$, the lower bound equals $(1-\delta^2)\xi/2\ell$. Therefore, type s is guaranteed to get at least $(1-\delta^2)\xi/2\ell \times \frac{1}{2}(1-\delta^2)\xi \geq \frac{\xi^2}{4\ell}(1-\delta)^2$. This bound is smaller than the bound obtained in Case 1 whenever $\delta > \max\{0, 1-2\eta\ell/\xi\}$ and so, for $\delta_0 \equiv \sqrt{\max\{0, 1-2\eta\ell/\xi\}}$ the inequality (67) holds for $\delta > \delta_0$. $\square$

I can now derive $d(\delta)$ used in the main text to construct on-path strategies.

Lemma 19. *There exists $\delta_0 \in (0,1)$ such that for any $\delta \in (\delta_0, 1)$ and any $s \in [0, 1-\eta]$, it holds*

$$\overline{\pi}(s) - t(s) \geq d(\delta), \tag{68}$$

where

$$d(\delta) \equiv \tfrac{\xi^2}{4\ell\Sigma}(1-\delta)^3. \tag{69}$$

Proof. Fix $s \in [0, 1-\eta]$. Let $x = \overline{\pi}(s) - t(s)$. Then

$$\begin{aligned} R(\overline{\pi}(s), s) &= x\left(P(\overline{\pi}(s) - x) - c(s)\right) + \delta^2 R(\overline{\pi}(s) - x, s) \\ &\leq x\left(P(\overline{\pi}(s) - x) - c(s)\right) + \delta^2 R(\overline{\pi}(s), s), \end{aligned}$$

where the inequality follows from $R(\overline{\pi}(s), s) \geq x\left(P(t(\overline{\pi}(s) - x, s)) - c(s)\right) + R(\overline{\pi}(s) - x, s)$. Thus, for any $\delta \in (\delta_0, 1)$

$$x \geq \frac{R(\overline{\pi}(s), s)(1-\delta^2)}{P(\overline{\pi}(s) - x) - c(s)} \geq \tfrac{\xi^2}{4\ell}(1-\delta)^2 \frac{(1-\delta^2)}{\Sigma} \geq \tfrac{\xi^2}{4\ell\Sigma}(1-\delta)^3.$$

where δ_0 is the bound on δ in Lemma 18, and the second inequality is by (67) and the fact that $P(\overline{\pi}(s) - x) - c(s) \leq v(\overline{\pi}(s) - x) - c(s) \leq \Sigma$. □

To conclude this appendix, I prove that strategies that I constructed in Step 3 in the main text and in Appendix $B.2$ constitute the PBE in the auxiliary game.

Lemma 20. *Strategies constructed in Step 3 in the main text and in Appendix $B.2$ constitute the PBE in the auxiliary game.*

Proof. By Lemma 15, the constructed strategies are part of equilibrium for types $b \in [0, \eta]$ and $s = 0$. By construction in Step 3 in the main text and Appendix $B.2$, screening policies are optimal for the seller both on- and off-path. I am left to show that the buyer has no incentives to deviate from strategy P on-path (off-path the acceptance strategy is given by some $P^\dagger$ described in Appendix $B.2$, and the argument is analogous).

If the highest remaining type of the buyer exceeds b, then type b interprets the previous seller's offers as the seller's deviations and expects the seller to restart the screening. From equation (27), it follows that any offer above $P(b)$ would be rejected by buyer type b. I next show that prices below $P(b)$ are accepted by buyer b. Suppose to contradiction that the seller makes price offer $p < P(b)$ and buyer type b strictly prefers to reject p. If $P(\beta) = P(b)$, then in the continuation, the seller type $\underline{\pi}(b)$ restarts her screening policy and so by equation (27), it is optimal for type b to accept p. Suppose $p \leq P(\beta) < P(b)$. First, suppose that $\underline{\pi}(b) \leq \beta$. Then the next price offer of seller type $\underline{\pi}(b)$ is $\sigma(\beta, \underline{\pi}(b)) \geq \sigma(\beta, \underline{\pi}(\beta))$ by Lemma 17. It follows from equation (27) that

$$v(\beta) - p \geq \delta^2(v(\beta) - \sigma(\beta, \underline{\pi}(\beta))),$$

from which it follows

$$\begin{aligned}(1-\delta^2)v(b) &> (1-\delta^2)v(\beta)\\ &\geq p - \delta^2\sigma(\beta, \underline{\pi}(\beta))\\ &\geq p - \delta^2\sigma(\beta, \underline{\pi}(b)).\end{aligned}$$

Hence,

$$v(b) - p > \delta^2(v(b) - \sigma(\beta, \underline{\pi}(b))). \tag{70}$$

Buyer type b believes that the seller type is $\underline{\pi}(b)$. Thus, by the construction of equilibrium strategies off-path in Appendix $B.2$, after the deviation type b expects to accept $\sigma(\beta, \underline{\pi}(b))$ in the next round. But then by (70), type b strictly prefers to accept p, which gives a contradiction.

Now, suppose that $\underline{\pi}(b) \in (\beta, \beta+2\eta]$. Then by the specification of strategies in Appendix $B.2$, seller type $s = \underline{\pi}(b)$ detects the buyer's deviation and her beliefs are uniform on $[\underline{\pi}(s), \beta + 2\eta]$. By Lemma 17, $\sigma(\beta + 2\eta, \underline{\pi}(b)) \geq \sigma(\beta + 2\eta, \underline{\pi}(\beta)) \geq \sigma(\beta, \underline{\pi}(\beta))$, and by the same argument as above for $\underline{\pi}(b) \leq \beta$, I get the contradiction. The cases $\underline{\pi}(b) \in (\beta + k\eta, \beta + (k+1)\eta]$ for some $k = 1, \ldots$ are considered analogously. This completes the proof of optimality of strategy P. $\square$

B.4 Appendix for Step 4: Contagious Coasian property of $P(\cdot)$

This appendix proves Lemma 8, which characterizes the limit of $P(\cdot)$ as $\delta \to 1$. The argument proceeds in three steps. In subsection $B.4.1$, I first prove Lemma 9 that covers types below b_+. In subsection $B.4.2$, I then prove Lemma 10 that covers types above b_+. Finally, in subsection $B.4.3$, I combine the two lemmas combined to yield the proof of Lemma 8.

B.4.1 Buyer types $b \in (\eta, b_+)$

Proof of Lemma 9. Fix $\delta \in (0, 1)$ and $\phi \in (0, \frac{1}{2}\eta)$. Suppose that for some $\hat{b} \in (\eta - \phi, 1]$, (29) holds. I construct $f(\phi, \delta)$ so that the upper bound in (30) on the willingness to pay of type $b^\dagger \equiv \min\{\hat{b} + \phi, 1\}$ holds, and $\lim_{\delta\to 1} f(\phi, \delta) = 0$ for any $\phi > 0$. (See Figure 8 for the illustration of quantities in the proof of lemma).

Denote by $s^\dagger \equiv \underline{\pi}(b^\dagger) = b^\dagger - \eta$ and $\hat{s} \equiv \underline{\pi}(\hat{b}) = \max\{\hat{b} - \eta, 0\}$ types of the seller, to which types of the buyer $b^\dagger$ and $\hat{b}$, resp., assign probability one. Since $\phi \in (0, \frac{1}{2}\eta)$ and $b^\dagger - \hat{b} \leq \phi$,

$$\max\{s^\dagger - \eta, 0\} = \max\{b^\dagger - 2\eta, 0\} < b^\dagger - 2\phi \leq \hat{b} - \phi < b^\dagger = s^\dagger + \eta$$

and so, $[\hat{b} - \phi, b^\dagger] \subset B_{s^\dagger}$. I have

$$c(s^\dagger) + \ell\phi < c(\hat{s}) + 2\ell\phi < P(\hat{b} - \phi), \tag{71}$$

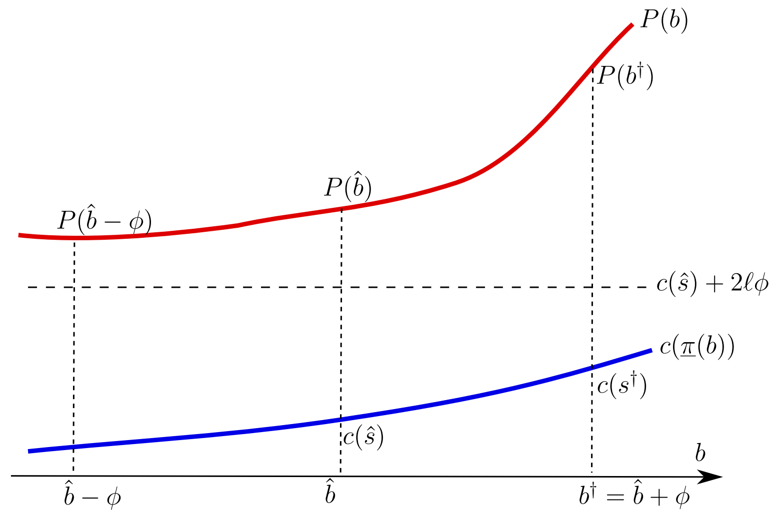

Figure 8: ***Illustration for Lemma*** 9

where the first inequality is by $\max_{s\in[0,1]} c'(s) \le \ell$ and $s^\dagger - \hat{s} \le \phi$, and the second is by (29) and $\hat{s} = \underline{\pi}(\hat{b})$.

Let $K \le \infty$ be the first round of screening when seller type $s^\dagger$ makes an offer below $P(\hat{b})$. Then

$$R(b^\dagger, s^\dagger) \le \int_{\hat{b}}^{b^\dagger} (P(b) - c(\hat{s}))db + \delta^{2K} R(\hat{b}, s^\dagger). \tag{72}$$

Let $M(\delta) = \lfloor (1-\delta)^{-1/2} \rfloor$ and consider an alternative screening policy, in which type $s^\dagger$ makes a sequence of offers $(a_m)_{m=1}^{M(\delta)}$ such that $a_m = v(b^\dagger) + {}^{m}/_{M(\delta)}(c(s^\dagger) - v(b^\dagger))$ and sells with probability one in $M(\delta)$ rounds. Since in round m, the seller sells to types with $P(b) \in [a_m, a_{m-1})$ at price a_m, the loss in profit from such types compared to the maximal profit that could be extracted from such types is at most $a_{m-1} - a_m = (v(b^\dagger) - c(s^\dagger))/M(\delta) \le \Sigma/M(\delta)$. Recall that $\overline{\beta}(s^\dagger) = \max\left\{\sup\{b : P(b) \le c(s^\dagger)\}, \underline{\pi}(s^\dagger)\right\}$. By (71), $\overline{\beta}(s^\dagger) \le \hat{b} - \phi$. Then

$$\begin{aligned} R(b^\dagger, s^\dagger) &\ge \delta^{2M(\delta)} \left(\int_{\overline{\beta}(s^\dagger)}^{b^\dagger} (P(b) - c(s^\dagger))db - \tfrac{\Sigma}{M(\delta)}(b^\dagger - \overline{\beta}(s^\dagger)) \right) \\ &\ge \delta^{2M(\delta)} \left(\int_{\overline{\beta}(s^\dagger)}^{b^\dagger} (P(b) - c(s^\dagger))db - \tfrac{\Sigma}{M(\delta)} \right), \end{aligned} \tag{73}$$

where the first inequality is by the optimality of the seller's screening policy, and the second inequality is by $b^\dagger - \overline{\beta}(s^\dagger) \le 1$. Combining (72) and (73),

$$\int_{\hat{b}}^{b^\dagger} (P(b) - c(\hat{s}))db + \delta^{2K} R(\hat{b}, s^\dagger) \ge \delta^{2M(\delta)} \left(\int_{\overline{\beta}(s^\dagger)}^{b^\dagger} (P(b) - c(s^\dagger))db - \tfrac{\Sigma}{M(\delta)} \right). \tag{74}$$

Then I have

$$\begin{aligned}\left(1-\delta^{2M(\delta)}\right)\Sigma\phi+\delta^{2K}R(\hat b,s^\dagger) &\geq \left(1-\delta^{2M(\delta)}\right)\int_{\hat b}^{b^\dagger}(P(b)-c(s^\dagger))db+\delta^{2K}R(\hat b,s^\dagger)\\ &\geq \delta^{2M(\delta)}\left(\int_{\overline{\beta}(s^\dagger)}^{\hat b}(P(b)-c(s^\dagger))db-\tfrac{\Sigma}{M(\delta)}\right)\\ &\geq \delta^{2M}\left(R(\hat b,s^\dagger)-\tfrac{\Sigma}{M(\delta)}\right),\end{aligned}\tag{75}$$

where the first inequality is by $P(b)-c(s^\dagger)\leq v(b)-c(s^\dagger)\leq\Sigma$, the second is by (74), the third is by $R(\hat b,s^\dagger)\leq\int\limits_{\overline{\beta}(s^\dagger)}^{\hat b}(P(b)-c(s^\dagger))db$. When the highest remaining type is $\hat b$, type $s^\dagger$ of the seller can make offer $P(\hat b-\phi)$, which is accepted at least by types in $(\hat b-\phi,\hat b)$ and so, using (71),

$$R(\hat b,s^\dagger)\geq\phi(P(\hat b-\phi)-c(s^\dagger))>\phi^2\ell>0.\tag{76}$$

Finally, dividing (75) by $R(\hat b,s^\dagger)$ and using (76), I get

$$\delta^{2K}\geq\delta^{2M(\delta)}-\tfrac{\Sigma}{R(\hat b,s^\dagger)}\left(\tfrac{\delta^{2M(\delta)}}{M(\delta)}+\phi\left(1-\delta^{2M(\delta)}\right)\right)>\delta^{2M(\delta)}-\tfrac{\Sigma}{\phi^2\ell}\left(\tfrac{\delta^{2M(\delta)}}{M(\delta)}+\phi\left(1-\delta^{2M(\delta)}\right)\right).\tag{77}$$

Buyer type $b^\dagger$ prefers to purchase at his willingness to pay $P(b^\dagger)$ rather than wait until the screening round K when price drops below $P(\hat b)$ and so, $v(b^\dagger)-P(b^\dagger)\geq\delta^{2K}(v(b^\dagger)-P(\hat b))$, which implies that

$$\begin{aligned}P(b^\dagger)-P(\hat b)&\leq(1-\delta^{2K})(v(b^\dagger)-P(\hat b))\\ &\leq(1-\delta^{2K})(v(b^\dagger)-c(\hat s))\\ &\leq(1-\delta^{2K})\Sigma\\ &<\left(1-\delta^{2M(\delta)}+\tfrac{\Sigma}{\phi^2\ell}\left(\tfrac{\delta^{2M(\delta)}}{M(\delta)}+\phi\left(1-\delta^{2M(\delta)}\right)\right)\right)\Sigma,\end{aligned}\tag{78}$$

where I used (77) in the last line. Denoting the last expression by $f(\phi,\delta)$ gives the desired bound. (Indeed, $\lim_{\delta\to1}\delta^{M(\delta)}=\lim_{\delta\to1}e^{\ln\delta/\sqrt{1-\delta}}=1$ and $\lim_{\delta\to1}M(\delta)=\infty$, and so, $\lim_{\delta\to1}1-\delta^{2M(\delta)}+{}^{\Sigma}/_{\phi^2\ell}\left(\delta^{2M(\delta)}/M(\delta)+\phi\left(1-\delta^{2M(\delta)}\right)\right)=0$). □

B.4.2 Buyer types $b\in[b_+,1]$

Before proceeding to the proof of Lemma 10, I prove two auxiliary results. The first auxiliary lemma shows that there cannot be large discontinuities in function $P(\cdot)$.

Lemma 21. *For any $\varepsilon>0$, there exists $\underline{\delta}(\varepsilon)\in(0,1)$ such that for any $\delta\in(\underline{\delta}(\varepsilon),1)$ the following hold*

1. *for any interval* $(\underline{p}, \overline{p}) \subset [P(0), P(1)]$ *of length at least* ε*, there exists* $b \in [0,1]$ *such that* $P(b) \in (\underline{p}, \overline{p})$*;*

2. *for any type* s *of the seller and any type* $b \in B_s$ *of the buyer,*

$$P(b) - \sigma(b, s) \leq \varepsilon. \tag{79}$$

Proof. Part 1: Suppose to contradiction that there exist $\varepsilon > 0, \underline{p}, \overline{p} > \underline{p} + \varepsilon$ such that for any δ arbitrarily close to 1, either $P(b) \geq \overline{p}$ or $P(b) \leq \underline{p}$ for all b. By equation (27), for any b and $\delta \in \left(\sqrt{1 - \varepsilon/2\Sigma}, 1\right)$,

$$P(b) - \sigma(\underline{\pi}(b)) = (1 - \delta^2)\,(v(b) - \sigma(\underline{\pi}(b))) \leq (1 - \delta^2)\Sigma < \tfrac{\varepsilon}{2}. \tag{80}$$

Let $\tilde{b} \equiv \sup\{b : P(b) < \underline{p}\}$. Consider type $\hat{b} = \tilde{b} + \frac{1}{4}d(\delta)$ and $\check{b} = \tilde{b} - \frac{1}{4}d(\delta)$, where $d(\delta)$ is as in Lemma 19. Let $\underline{\delta}(\varepsilon) = \max\{\delta_0, \sqrt{1 - \varepsilon/2\Sigma}\}$, where δ_0 is as in Lemma 19. Suppose $\delta \in (\underline{\delta}(\varepsilon), 1)$. Applying Lemma 19 to the seller type $\underline{\pi}(\hat{b})$, I get that $\check{b} > t(\underline{\pi}(\hat{b}))$. Hence, $P(\hat{b}) - \sigma(\underline{\pi}(\hat{b})) = P(\hat{b}) - P(t(\underline{\pi}(\hat{b}))) \geq P(\hat{b}) - P(\check{b}) \geq \varepsilon$, which contradicts (80).

Part 2: Consider $s \in [0,1]$ and $b \in B_s$. By Lemma 17, $\sigma(b, \cdot)$ is weakly increasing and so, since $s \geq \underline{\pi}(b)$, $\sigma(b, s) \geq \sigma(b, \underline{\pi}(b))$. Hence,

$$P(b) - \sigma(b, s) \leq P(b) - \sigma(b, \underline{\pi}(b)) = P(b) - \sigma(\underline{\pi}(b)), \tag{81}$$

which by (80) is less than ε for $\delta \in (\underline{\delta}(\varepsilon), 1)$. □

The second auxiliary lemma gives the bound on the distance between two types in terms of δ and the difference in their willingness to pay.

Lemma 22. *For any* $\varepsilon > 0$*, there exists* $\overline{\delta}(\varepsilon) \in (0,1)$ *such that for any* $\delta \in (\overline{\delta}(\varepsilon), 1)$ *and any* $b'', b' \in [0,1]$,

$$P(b'') - P(b') \geq \varepsilon \text{ implies } b'' - b' \geq \mathcal{C}(\varepsilon)(1 - \delta)^2,$$

where $\mathcal{C}(\varepsilon)$ *is a strictly increasing function of* ε *with* $\mathcal{C}(0) = 0$*.*

Proof. Set $\delta_1(\varepsilon) \in (0,1)$ such that for any $\delta \in (\delta_1(\varepsilon), 1)$, $\ln\left(1 - \frac{\varepsilon}{\Sigma}\right)/(4 \ln \delta) \geq 1$ and $1-\delta/(-\ln \delta) > \frac{1}{2}$, and let δ_0 be the bound on δ from Lemma 19. Set $\underline{\delta}(\varepsilon) = \max\{\delta_0, \delta_1(\varepsilon)\}$.

Consider ε,b', and b'' such that $P(b'') - P(b') \geq \varepsilon > 0$. Define $\hat{l}$ and $t_l, l = 0, \ldots, \hat{l} + 1$ recursively as follows. Recall that $t(s)$ gives the lowest type to whom seller type s allocates in the first round of screening. Let $t_0 = b''$ and $t_l = t\left(\underline{\pi}(t_{l-1})\right)$ for $l = 1, \ldots, \hat{l} + 1$, where $\hat{l}$ is the largest integer such that $t_{\hat{l}} \geq b'$. By Lemma 19,

$$t_{l-1} - t_l \geq d(\delta), \tag{82}$$

where $d(\delta) = \xi^2(1-\delta)^3/4\ell\Sigma$ is defined in (69) in Lemma 19. Since $d(\delta) > 0$, $\hat{l}$ is finite and sequence $(t_l)_{l=0}^{\hat{l}+1}$ is strictly decreasing. By equation (27) and $\sigma(s) = P(t(s))$,

$$\begin{aligned}
P(b'') &= (1-\delta^2)v(b'') + \delta^2\sigma(\underline{\pi}(b'')) \\
&= (1-\delta^2)v(b'') + \delta^2 P(t(\underline{\pi}(b''))) \\
&= (1-\delta^2)v(b'') + \delta^2 P(t_1) \\
&= (1-\delta^2)v(t_0) + (1-\delta^2)\delta^2 v(t_1) + (1-\delta^2)\delta^4\sigma(\underline{\pi}(t_1)) \\
&\cdots \\
&= (1-\delta^2)\sum_{l=0}^{\hat{l}}\delta^{2l}v(t_l) + \delta^{2(\hat{l}+1)}P(t_{\hat{l}+1}).
\end{aligned}$$

Then

$$\begin{aligned}
P(b'') - P(b') &= (1-\delta^2)\sum_{l=0}^{\hat{l}}\delta^{2l}v(t_l) + \delta^{2(\hat{l}+1)}P(t_{\hat{l}+1}) - P(b') \\
&\leq (1-\delta^2)\sum_{l=0}^{\hat{l}}\delta^{2l}v(t_l) - (1-\delta^{2(\hat{l}+1)})P(b') \\
&< (1-\delta^{2(\hat{l}+1)})(v(b'') - P(b')),
\end{aligned}$$

where the first inequality is by $P(\cdot)$ weakly increasing and $b' \geq t_{\hat{l}+1}$, and the second inequality is by $v(\cdot)$ strictly increasing and $t_l < t_0 = b''$. Since $v(b'') - P(b') \geq P(b'') - P(b') \geq \varepsilon > 0$,

$$\delta^{2(\hat{l}+1)} \leq 1 - \frac{P(b'') - P(b')}{v(b'') - P(b')} \leq 1 - \frac{\varepsilon}{\Sigma},$$

which implies that

$$\hat{l} \geq \frac{\ln\left(1 - \frac{\varepsilon}{\Sigma}\right)}{2\ln\delta} - 1. \tag{83}$$

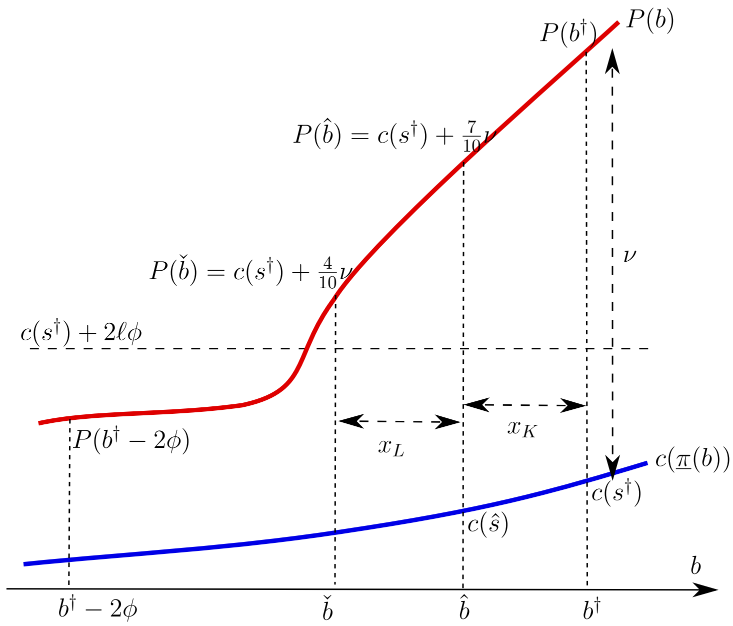

Figure 9: ***Illustration for Lemma*** 10

Then I have

$$
\begin{aligned}
b'' - b' &= \sum_{l=1}^{\hat{l}} (t_{l-1} - t_l) + t_{\hat{l}} - b' \\
&\geq d(\delta)\hat{l} \\
&\geq d(\delta)\left(\frac{\ln\left(1 - \frac{\varepsilon}{\Sigma}\right)}{2\ln\delta} - 1\right) \\
&= \tfrac{\xi^2}{4\ell\Sigma}(1-\delta)^3\left(\frac{\ln\left(1 - \frac{\varepsilon}{\Sigma}\right)}{2\ln\delta} - 1\right) \\
&\geq \tfrac{\xi^2}{16\ell\Sigma}\ln\left(1 - \frac{\varepsilon}{\Sigma}\right)\frac{(1-\delta)^3}{\ln\delta} \\
&> -\tfrac{\xi^2}{32\ell\Sigma}\ln\left(1 - \frac{\varepsilon}{\Sigma}\right)(1-\delta)^2,
\end{aligned}
$$

where I used the definition of t_l in the first line; (82) in the second line; (83) in the third line; definition of $d(\delta)$ in the forth line; $\delta > \delta_1(\varepsilon)$ in the last two lines. Defining $\mathcal{C}(\varepsilon) \equiv -\xi^2 \ln\left(1 - \frac{\varepsilon}{\Sigma}\right)/(32\ell\Sigma)$, I get the desired result. $\square$

I can now prove Lemma 10.

Proof of Lemma 10. Suppose to contradiction that there exists $\varepsilon > 0$ such that there exists a sequence of $\phi > 0$ and $\delta \in (0,1)$ converging to 0 and 1, resp., such that for any ϕ and δ, for some $b^\dagger \in [0,1]$, it holds $c(\underline{\pi}(b^\dagger)) + 2\ell\phi \geq P(b^\dagger - 2\phi)$ (inequality (32)), but

$$P(b^\dagger) \geq c(\underline{\pi}(b^\dagger)) + \varepsilon. \tag{84}$$

I will derive a contradiction.

Let $s^\dagger \equiv \underline{\pi}(b^\dagger)$ and $\nu \equiv P(b^\dagger) - c(s^\dagger) \geq \varepsilon > 0$. (See Figure 9 for the illustration of different variables in the proof). By Lemma 21, there exists $\underline{\delta}(\varepsilon) \in (0,1)$ such that for all $\delta \in (\underline{\delta}(\varepsilon), 1)$, in any interval $(\underline{p}, \overline{p}) \subset [P(0), P(1)]$ of length greater than $\frac{1}{10}\varepsilon$ (and so, greater than $\frac{1}{10}\nu$), there exists b such that $P(b) \in (\underline{p}, \overline{p})$; and $P(b) - P(t(b,s)) \leq \frac{1}{10}\varepsilon \leq \frac{1}{10}\nu$ for any $s \in [0,1]$ and $b \in B_s$. I further consider $\delta > \underline{\delta}(\varepsilon)$.

Lower bound on x_K: Let $K \leq \infty$ be the first round of screening, in which seller type $s^\dagger$ makes an offer less than or equal to $c(s^\dagger) + \frac{7}{10}\nu$. Let $\hat{b}$ be the lowest buyer type that buys in round K. By Part 1 of Lemma 21, $P(\hat{b}) > c(s^\dagger) + \frac{6}{10}\nu$. In the first K rounds of screening, type $s^\dagger$ allocates to the mass $x_K \equiv b^\dagger - \hat{b}$ of buyer types.

Consider the following alternative screening strategy, in which type $s^\dagger$ screens buyer types above $\hat{b}$ in $M_K = \lfloor \min\{\frac{K}{2}, (1-\delta)^{-1}\} \rfloor < K$ rounds instead of K. Let a_k be the price offer that type $s^\dagger$ makes in round k in the on-path equilibrium screening strategy. Define $q_k \equiv P(b^\dagger) + \left(a_{K-1} - P(b^\dagger)\right) k/M_K$, $k = 1, 2, .., M_K$. In the alternative strategy, type $\hat{s}$ makes offer $p_k = \min\{q_k, a_k\}$ in rounds $k \leq M_K$, makes offer a_K in round $M_K + 1$, and continues following the equilibrium strategy from then on (i.e., offers $a_{K+1}, a_{K+2}, \dots$).

The total loss from using this alternative strategy is at most $\frac{4}{10}\nu x_K / M_K$. Indeed, in each round the loss of type $s^\dagger$ compared to the maximum surplus that can be extracted is at most $(P(b^\dagger) - P(\hat{b}))/M_K \leq \frac{4}{10}\nu/M_K$, and she allocates to a mass x_K of buyer types. Moreover, there is no loss due to discounting, as by construction, the allocation to all buyer types happens sooner under the alternative strategy than under the equilibrium strategy. At the same time, by speeding up the screening, type $s^\dagger$ gains at least $\left(\delta^{2M_K} - \delta^{2K}\right) V_K$, where V_K is the continuation utility of type $s^\dagger$ after she makes price offer a_K and follows the equilibrium strategy further. By the optimality of the screening strategy of type $s^\dagger$,

$$\tfrac{4}{10}\nu \tfrac{x_K}{M_K} \geq \left(\delta^{2M_K} - \delta^{2K}\right) V_K. \tag{85}$$

Lower bound on x_L: Consider seller type $\hat{s} \equiv \underline{\pi}(\hat{b})$, and let L be the first round of screening, in which type $\hat{s}$ makes an offer below $c(s^\dagger) + \frac{4}{10}\nu$ on the equilibrium path. Suppose in round L, type $\hat{s}$ allocates to all buyer types above some $\check{b}$. By Part 1 of Lemma 21, $P(\check{b}) > c(s^\dagger) + \frac{3}{10}\nu$. Denote by $x_L \equiv \hat{b} - \check{b}$ the mass of buyer types to whom type $\hat{s}$ allocates in first L rounds, by V_L the continuation utility of the seller type $\hat{s}$ after round L of equilibrium screening, and let $M_L = \lfloor \min\{\frac{L}{2}, (1-\delta)^{-1}\} \rfloor < L$. By the analogous argument as with the lower bound on x_K, for the optimality of the screening strategy of type $\hat{s}$, it is necessary that

$$\tfrac{4}{10}\nu\tfrac{x_L}{M_L} \geq \left(\delta^{2M_L} - \delta^{2L}\right) V_L. \tag{86}$$

Lower bound on V_K *:* Type $s^\dagger$ can offer $c(s^\dagger) + \frac{3}{10}\nu$ in round $K+1$ of the screening instead of following the equilibrium screening policy. The mass of buyer types that accept such a price is at least x_L and so,

$$V_K \geq \tfrac{3}{10}\nu x_L. \tag{87}$$

Lower bound on V_L *:* In round $L+1$ of screening by type $\hat{s}$, only buyer types below $\check{b}$ remain and $P(\check{b}) > c(s^\dagger) + \frac{3}{10}\nu$. Type $\hat{s}$ can offer $c(s^\dagger) + \frac{1}{10}\nu$ instead of following the equilibrium screening strategy. Denote by b'' and b' the highest and lowest buyer types, resp., that accept such an offer. By Part 1 of Lemma 21, $P(b'') - P(b') > \frac{1}{10}\nu \geq \frac{1}{10}\varepsilon$. By Lemma 22, there is $\overline{\delta}(\varepsilon)$ such that whenever $\delta > \overline{\delta}(\varepsilon)$, it holds $b'' - b' \geq \mathcal{C}(\frac{1}{10}\varepsilon)(1-\delta^2)$ and so,

$$V_L \geq \tfrac{1}{10}\nu\mathcal{C}(\tfrac{\varepsilon}{10})(1-\delta)^2, \tag{88}$$

where $\mathcal{C}(\varepsilon)$ is the function defined in Lemma 22.

Deriving the contradiction: Multiplying inequalities (85), (86), (87), (88), I get

$$\tfrac{16}{3}x_K \geq M_K(1-\delta)M_L(1-\delta)\left(\delta^{2M_K} - \delta^{2K}\right)\left(\delta^{2M_L} - \delta^{2L}\right)\mathcal{C}(\tfrac{\varepsilon}{10}). \tag{89}$$

I first show that the right-hand side of (89) converges to a positive number along the subsequence as $\delta \to 1$. Let us first find the lower bound on the limit of $M_K(1-\delta)(\delta^{2M_K} - \delta^{2K})$. Consider a subsequence of δ such that δ^K and δ^L converge. Since type $b^\dagger$ prefers to buy at price $P(b^\dagger)$ rather than wait until price drops below $P(\hat{b})$ in round K,

$$\delta^{2K} \leq \frac{v(b^\dagger) - P(b^\dagger)}{v(b^\dagger) - P(\hat{b})} \leq \frac{v(b^\dagger) - c(s^\dagger) - \nu}{v(b^\dagger) - c(s^\dagger) - \frac{7}{10}\nu} \leq \frac{\Sigma - \nu}{\Sigma - \frac{7}{10}\nu} \leq \frac{\Sigma - \varepsilon}{\Sigma - \frac{7}{10}\varepsilon} < 1. \tag{90}$$

It follows from (90) that

$$\lim_{\delta\to1}(1-\delta)K = \lim_{\delta\to1}(-\ln\delta)K > -\tfrac{1}{2}\ln\left[\frac{\Sigma - \varepsilon}{\Sigma - \frac{7}{10}\varepsilon}\right] \equiv \mathcal{A}(\varepsilon) > 0,$$

and so, $\lim_{\delta\to1}\delta^K \leq e^{-\mathcal{A}(\varepsilon)}$. Since $M_K = \lfloor \min\{\frac{1}{2}K, (1-\delta)^{-1}\}\rfloor$,

$$\begin{aligned}\lim_{\delta\to1} M_K(1-\delta) &= \min\{\lim_{\delta\to1}\tfrac{1}{2}K(1-\delta), 1\} \\ &\geq \min\{\tfrac{1}{2}\mathcal{A}(\varepsilon), 1\} > 0.\end{aligned}$$

If $\lim_{\delta\to 1}\delta^K \le e^{-3}$, or equivalently, $\lim_{\delta\to 1}(-\ln\delta)K \ge 3$, then

$$\lim_{\delta\to 1}\delta^{2M_K} = e^{-2\min\{\lim_{\delta\to 1}\frac{K}{2}(-\ln\delta),1\}} = e^{-2},$$

and so,

$$\lim_{\delta\to 1}(\delta^{2M_K} - \delta^{2K}) \ge e^{-2} - e^{-6} > 0.$$

If $\lim_{\delta\to 1}\delta^K > e^{-3}$, or equivalently, $\lim_{\delta\to 1}(-\ln\delta)K < 3$, then

$$\begin{aligned}\lim_{\delta\to 1}(\delta^{2M_K} - \delta^{2K}) &\ge \lim_{\delta\to 1}(\delta^{K} - \delta^{2K})\\ &> \min\{e^{-3} - e^{-6}, e^{-\mathcal{A}(\varepsilon)} - e^{-2\mathcal{A}(\varepsilon)}\} > 0,\end{aligned}$$

where the first inequality is by $M_K \le \frac{1}{2}K$, and the second inequality is by the fact that the function $x - x^2$ attains its minimum on $[e^{-3}, e^{-\mathcal{A}(\varepsilon)}]$ either at e^{-3} or $e^{-\mathcal{A}(\varepsilon)}$. Summarizing, I get

$$\lim_{\delta\to 1} M_K(1-\delta)(\delta^{2M_K} - \delta^{2K}) \ge \min\{\tfrac{1}{2}\mathcal{A}(\varepsilon), 1\} \times \min\{e^{-3} - e^{-6}, e^{-\mathcal{A}(\varepsilon)} - e^{-2\mathcal{A}(\varepsilon)}\} \equiv \mathcal{B}(\varepsilon) > 0$$

Analogously, since type $\hat{b}$ prefers to buy at price $P(\hat{b})$ rather than wait until price drops to $P(\check{b})$,

$$\delta^{2L} \le \frac{v(\hat{b}) - P(\hat{b})}{v(\hat{b}) - P(\check{b})} \le \frac{v(\hat{b}) - c(s^\dagger) - \frac{6}{10}\nu}{v(\hat{b}) - c(s^\dagger) - \frac{4}{10}\nu} \le \frac{\Sigma - \frac{6}{10}\nu}{\Sigma - \frac{4}{10}\nu} \le \frac{\Sigma - \frac{6}{10}\varepsilon}{\Sigma - \frac{4}{10}\varepsilon} < 1, \tag{91}$$

and I can proceed as above to show that for some constant $\mathcal{B}_1(\varepsilon) > 0$, it holds

$$\lim_{\delta\to 1} M_L(1-\delta)(\delta^{2M_L} - \delta^{2L}) \ge \mathcal{B}_1(\varepsilon).$$

Therefore, the limit of the right-hand side of (89) as $\delta \to 1$ is bounded from below by $\mathcal{B}(\varepsilon)\mathcal{B}_1(\varepsilon)\mathcal{C}(\frac{\varepsilon}{10}) > 0$.

At the same time, for $\phi < \phi_1 \equiv \varepsilon/(20\ell)$, inequality (32) implies $P(b^\dagger - 2\phi) \le c(s^\dagger) + 2\ell\phi \le c(s^\dagger) + \frac{1}{10}\varepsilon \le c(s^\dagger) + \frac{1}{10}\nu$ and so, $\{b \in B_{s^\dagger} : P(b) > c(s^\dagger) + \frac{1}{10}\nu\} \subseteq [b^\dagger - 2\phi, b^\dagger]$. Therefore, $x_K \le 2\phi$ and the left-hand side of (89) converges to zero as $\phi \to 0$, which gives the desired contradiction and proves the lemma. □

B.4.3 Proof of Lemma 8

Fix $\varepsilon > 0$. Since by Lemma 1, $P(b) \ge \max\{\overline{y}(0), c(\underline{\pi}(b))\} \ge \max\{y^*(0), c(\underline{\pi}(b))\}$, it is sufficient to show that there exists $\delta(\varepsilon) \in (0,1)$ such that for any $\delta \in (\delta(\varepsilon), 1)$ and $b \in [0,1]$, either $P(b) \le c(\underline{\pi}(b)) + \varepsilon$ or $P(b) \le y^*(0) + \varepsilon$.

Let us collect all results that I have proven so far:

1. By Lemma 16, there exists $\delta_1(\varepsilon) \in (0,1)$ such that for any $\delta \in (\delta_1(\varepsilon), 1)$, $P(b) \le y^*(0) + \frac{1}{2}\varepsilon$

for any $b \in [0, \eta]$.

2. By Lemma 10, there exists $\tilde{\phi}(\varepsilon) > 0$ and $\tilde{\delta}(\varepsilon) \in (0,1)$ such that for any $\phi \in (0, \tilde{\phi}(\varepsilon))$ and $\delta \in (\tilde{\delta}(\varepsilon), 1)$, for any $b^\dagger \in [\eta, 1]$,

$$c(\underline{\pi}(b^\dagger)) + 2\ell\phi \geq P(b^\dagger - 2\phi) \quad \text{implies} \quad P(b^\dagger) < c(\underline{\pi}(b^\dagger)) + \tfrac{\varepsilon}{2}. \tag{92}$$

   Let

$$\phi_\varepsilon \equiv \tfrac{1}{2} \min \left\{ \tilde{\phi}(\varepsilon), \tfrac{1}{2}\eta, \tfrac{1}{2\ell}(y^*(0) - c(0) - \tfrac{\varepsilon}{2}) \right\}. \tag{93}$$

3. Let function $f(\cdot, \cdot)$ be as in Lemma 9. By Lemma 9, there exists $\hat{\delta}(\varepsilon) \in (0,1)$ such that for any $\delta \in (\hat{\delta}(\varepsilon), 1)$, $f(\phi_\varepsilon, \delta) < \frac{1}{2}\varepsilon\phi_\varepsilon$; and for any $\hat{b} \in (\eta - \phi_\varepsilon, 1]$,

$$c(\underline{\pi}(\hat{b})) + 2\ell\phi_\varepsilon < P(\hat{b} - \phi_\varepsilon) \quad \text{implies} \quad P(\min\{\hat{b} + \phi_\varepsilon, 1\}) < P(\hat{b}) + f(\phi_\varepsilon, \delta). \tag{94}$$

I let $\delta(\varepsilon) \equiv \max\{\delta_1(\varepsilon), \tilde{\delta}(\varepsilon), \hat{\delta}(\varepsilon)\}$, and fix any $\delta \in (\delta(\varepsilon), 1)$. Suppose there exists b such that

$$P(b) > c(\underline{\pi}(b)) + \varepsilon. \tag{95}$$

I will show that this implies that

$$P(b) \leq y^*(0) + \varepsilon \tag{96}$$

Since $\delta > \delta_1(\varepsilon)$, by Lemma 16, the inequality (96) holds whenever $b \leq \eta$. Hence, I consider the case $b > \eta$.

Define K as follows. If there is a non-negative integer $k < \lfloor (b-\eta)/\phi_\varepsilon \rfloor + 1$ such that $c(\underline{\pi}(b - k\phi_\varepsilon)) + 2\ell\phi_\varepsilon \geq P(b - k\phi_\varepsilon - \phi_\varepsilon)$, then let K be the smallest such integer. If no such k exists, let $K = \lfloor (b-\eta)/\phi_\varepsilon \rfloor + 1$. Observe that $K \geq 1$. Indeed, if $K = 0$, then since $P(\cdot)$ is weakly increasing, $c(\underline{\pi}(b)) + 2\ell\phi_\varepsilon \geq P(b - \phi_\varepsilon) \geq P(b - 2\phi_\varepsilon)$. Then (92) implies that $P(b) < c(\underline{\pi}(b)) + \frac{1}{2}\varepsilon$, which contradicts inequality (95).

By the construction of K, for any $k = 1, \ldots, K-1$, it holds $c(\underline{\pi}(b - k\phi_\varepsilon)) + 2\ell\phi_\varepsilon < P(b - k\phi_\varepsilon - \phi_\varepsilon)$. Then (94) holds for $\hat{b} = b - k\phi_\varepsilon, k = 1, \ldots, K-1$, and so,

$$P(b - k\phi_\varepsilon + \phi_\varepsilon) < P(b - k\phi_\varepsilon) + f(\phi_\varepsilon, \delta). \tag{97}$$

Summing inequalities (97) over all $k = 1, \ldots, K-1$, I get

$$P(b) < P(b - (K-1)\phi_\varepsilon) + (K-1)f(\phi_\varepsilon, \delta). \tag{98}$$

To complete the argument, I consider separately two cases.

Case 1: Suppose $K = \lfloor (b-\eta)/\phi_\varepsilon \rfloor + 1$. This implies that $b - K\phi_\varepsilon \leq \eta$. By (93), $b - K\phi_\varepsilon >$

$\eta - \phi_\varepsilon$ and

$$c(\underline{\pi}(b - K\phi_\varepsilon)) + 2\ell\phi_\varepsilon = c(0) + 2\ell\phi_\varepsilon < y^*(0) - \frac{\varepsilon}{2} < \overline{y}(0) \le P(b - K\phi_\varepsilon - \phi_\varepsilon),$$

where the last inequality is by Lemma 1. Hence, by (94) applied to $\hat{b} = b - K\phi_\varepsilon$, the inequality (97) holds for $k = K$ as well and so, combined with the inequality (98),

$$\begin{aligned} P(b) &< P(b - K\phi_\varepsilon) + Kf(\phi_\varepsilon, \delta) \\ &\le P(b - K\phi_\varepsilon) + \tfrac{1}{\phi} f(\phi_\varepsilon, \delta) \\ &\le P(b - K\phi_\varepsilon) + \tfrac{\varepsilon}{2}, \\ &\le P(\eta) + \tfrac{\varepsilon}{2}, \\ &\le y^*(0) + \varepsilon, \end{aligned}$$

where I used $K \le (b - \eta + \phi_\varepsilon)/\phi_\varepsilon \le b/\phi_\varepsilon \le 1/\phi_\varepsilon$ in the second inequality; $f(\phi_\varepsilon, \delta)/\phi_\varepsilon \le \frac{1}{2}\varepsilon$ for $\delta > \hat{\delta}(\varepsilon)$ in the third inequality; $P(\cdot)$ weakly increasing in the forth inequality; $P(\eta) \le y^*(0) + \frac{1}{2}\varepsilon$ for $\delta > \delta_1(\varepsilon)$ in the last inequality. Therefore, inequality (96) holds in this case.

Case 2: Suppose that $K < \lfloor (b - \eta)/\phi_\varepsilon \rfloor + 1$. Then by the definition of K and monotonicity of $c(\cdot)$,

$$c(\underline{\pi}(b - (K-1)\phi_\varepsilon)) + 2\ell\phi_\varepsilon \ge c(\underline{\pi}(b - K\phi_\varepsilon)) + 2\ell\phi_\varepsilon \ge P(b - K\phi_\varepsilon - \phi_\varepsilon).$$

Applying (92) with $b^\dagger = b - (K-1)\phi_\varepsilon$, I get

$$P(b - (K-1)\phi_\varepsilon) < c(\underline{\pi}(b - (K-1)\phi_\varepsilon)) + \tfrac{\varepsilon}{2}. \tag{99}$$

On the other hand, by the definition of K, for $k = 1, \dots, K-1$, $c(\underline{\pi}(b - k\phi_\varepsilon)) + 2\ell\phi_\varepsilon < P(b - k\phi_\varepsilon - \phi_\varepsilon)$. Hence, by the inequality (98), inequality (95), and the argument as in the previous case, I get

$$\begin{aligned} P(b - (K-1)\phi_\varepsilon) &> P(b) - (K-1)f(\phi_\varepsilon, \delta) \\ &\ge P(b) - \tfrac{1}{\phi_\varepsilon} f(\phi_\varepsilon, \delta) \\ &\ge P(b) - \tfrac{\varepsilon}{2} \\ &> c(\underline{\pi}(b)) + \tfrac{\varepsilon}{2} \\ &\ge c(\underline{\pi}(b - (K-1)\phi_\varepsilon)) + \tfrac{\varepsilon}{2}, \end{aligned}$$

which is a contradiction to (99). Thus, this case is not possible.

B.5 Appendix: Proof of Lemma 2

So far, I considered the case $\underline{b} = 0$ and $\bar{b} = 1$. Now, consider the general case $0 \leq \underline{b} < \bar{b} \leq 1$. I first restrict attention to buyer types $b \in [\underline{b}, \bar{b}]$ and seller types $s \in [\underline{b}, \bar{b}]$, and suppose that the optimistic conjectures of the buyer in this case put probability one on the lowest seller type above $\underline{b}$, i.e., $\mu_b^n[\max\{\underline{\pi}(b), \underline{b}\}] = 1$. I can proceed as in Appendices $B.1 - B.4$ to construct the PBE of the auxiliary game with the buyer's strategy given by the willingness to pay function $P(b|\underline{b})$, and show that it exhibits the Contagious Coasian property. Importantly, the bounds on δ that I derived in these steps do not depend on $\underline{b} = 0$, and hence, I can generalize Lemma 8 as follows:

Lemma 23. *For any $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0, 1)$ such that for all $\delta \in (\delta(\varepsilon), 1)$ and all $0 \leq \underline{b} < \underline{b} \leq 1$, it holds*

$$\max_{b \in [\underline{b}, \bar{b}]} |P(b|\underline{b}) - \max\{y^*(\underline{b}), c(\underline{\pi}(b))\}| < \varepsilon.$$

I now collect all steps to complete the proof of Lemma 2. Step 5 in the main text verifies that the PBEs of the auxiliary game are also PBEs in the original game for sufficiently high δ. The continuation utility of any seller type $s \in [\underline{\pi}(\underline{b}), \overline{\pi}(\bar{b})]$ in the PBEs of the auxiliary game that I constructed is bounded above by $P\left(\min\{\overline{\pi}(s), \bar{b}\}\right) - c(s)$. By Lemma 23, for $\delta \in (\delta(\varepsilon), 1)$ and any $s \in [\underline{\pi}(\underline{b}), \overline{\pi}(\bar{b})]$,

$$P\left(\min\{\overline{\pi}(s), \bar{b}\}\right) - c(s) \leq \max\{y^*(\underline{b}) - c(s), 0\} + \varepsilon,$$

which gives the desired conclusion of Lemma 2.

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Online Appendix A (Not for Publication)

B.6 Interim Versions of Theorems 1 and 3

I first introduce interim counter-parts of sets $E, \mathcal{E}$, and IR. Define $U_\eta^S(s|\tau,\rho) = \mathbb{E}_\eta[e^{-r\tau(s,b)}(\rho(s,b) - c(s))|s]$ and $U_\eta^B(b|\tau,\rho) = \mathbb{E}_\eta[e^{-r\tau(s,b)}(v(b) - \rho(s,b))|b]$ players' expected payoffs from the bargaining outcome (τ,ρ) at the interim stage, i.e., after types s and b are realized. For any $x \in [0,1]$, let

$$E(x) = \left\{ \lim_{\eta\to 0} \left(U_\eta^S(x|\tau_\eta,\rho_\eta), U_\eta^B(x|\tau_\eta,\rho_\eta) \right) : (\tau_\eta,\rho_\eta) \underset{\eta\to 0}{\to} (\tau,\rho) \in DL \right\}$$

be the set of all interim expected payoff profiles of types on the diagonal $s = x$ and $b = x$ generated by double limits. Let $\mathcal{E}(x)$ be the convex hull of the closure of $E(x)$. Note that when players have access to the public randomization device in the beginning of the game, $\mathcal{E}(x)$ is the set of interim expected payoff profiles of types $s = x$ and $b = x$ that can be approximated by double limits of my model.

I put some preliminary restrictions on $\mathcal{E}(x)$. Clearly, the *feasibility constraint* holds:

$$U_\eta^S(x|\tau_\eta,\rho_\eta) + U_\eta^B(x|\tau_\eta,\rho_\eta) \le \Pi(x).$$

Moreover, Lemma 1 implies that the seller's utility is at least $\underline{U}^S(s) \equiv \max\{y^*(0) - c(s), 0\}$ in any frequent-offer PBE limit, and symmetrically, the buyer's utility is at least $\underline{U}^B(b) \equiv \max\{v(b) - y^*(1), 0\}$. Hence, for any η, any outcome of the frequent-offer PBE limit (τ_η,ρ_η) satisfies the following *interim individual rationality constraints*:

$$U_\eta^S(s|\tau_\eta,\rho_\eta) \ge \underline{U}^S(s) \text{ and } U_\eta^B(b|\tau_\eta,\rho_\eta) \ge \underline{U}^B(b).$$

Denote by

$$IR(x) = \{(U^S, U^B) : U^S + U^B \le \Pi(x), U^S \ge \underline{U}^S(x), \text{ and } U^B \ge \underline{U}^B(x)\} \tag{100}$$

the set of feasible, interim individually rational payoffs of types $s = x$ and $b = x$, and by

$$PF(x) = \{(U^S, U^B) : U^S + U^B = \Pi(x), U^S \ge \underline{U}^S(x), \text{ and } U^B \ge \underline{U}^B(x)\}$$

its Pareto frontier. Then I have $\mathcal{E}(x) \subseteq IR(x)$.

The following is the interim counter-part of Theorem 1. The proof follows directly from the argument in the proof of Theorem 1.

Theorem 4. *If $(U^S, U^B) \in PF(x)$ for some $x \in [0,1]$, then $(U^S, U^B) \in \mathcal{E}(x)$.*

I now turn to the interim version of the Folk Theorem. Suppose that the buyer's value is given by $v_0(b)+\xi$ and the seller's cost is $c(s)$, where v_0, c, and ξ are introduced in Section 3.4. I am interested in the limit of the set $\mathcal{E}(x)$ as $\xi \to 0$, which I denote by $\mathcal{E}_0(x)$. Denote by $IR_0(x), x \in [0,1]$ the limit of $IR(x)$ as $\xi \to 0$. It is easy to see that

$$IR_0(x) = \{(U^S, U^B) : U^S + U^B \leq \Pi_0(x), U^S \geq 0, \text{ and } U^B \geq 0\} \text{ for } x \in [0,1],$$

where $\Pi_0(x) \equiv v_0(x) - c(x)$. The following theorem is the interim counter-part of Theorem 3.

Theorem 5. $\mathcal{E}_0(x) = IR_0(x)$ *for all* $x \in [0,1]$.

Proof. The argument for the Pareto frontier of $\mathcal{E}_0(x)$ is analogous to that for $\mathcal{E}$ in the proof of Theorem 3. Note that for $x=0$ or $x=1$, $IR_0(x) = \{(0,0)\}$. Then in the double limit that I constructed in Theorem 1, types $s=0$ and $b=0$ trade at price close to $y^*(0)$. Since $v(0)-c(0) \underset{\xi\to 0}{\to} 0$, the expected utilities of those types converge to 0 as $\xi \to 0$. Similarly, the expected utilities of types $s=1$ and $b=1$ converge to 0 as $\xi \to 0$, which proves that $\mathcal{E}_0(x) = IR_0(x), x \in \{0,1\}$. Now, consider $x \in (0,1)$. There exists ξ small enough such that $y^*(0) < c(x) < v(x) < y^*(1)$. In the double limit constructed in the proof of Theorem 3, the utility of types $s=x$ and $b=x$ converges to 0 as $\xi \to 0$, which proves that $\mathcal{E}_0(x) = IR_0(x)$. $\square$

B.7 Proof of Lemma 12

Let $V^B \equiv v(b_\infty) - q^S$, $V^S \equiv q^B - c(s_\infty)$, and $\Delta P \equiv q^S - q^B$. Denote $\phi \equiv \frac{1-\delta^2}{\delta^2 \Delta P}$, $\alpha_B \equiv \alpha^B(0) = \phi V^S$, and $\alpha_S \equiv \alpha^S(0) = \phi V^B$. By (37), $\alpha_B > 0$ and $\alpha_S > 0$. Note that $\phi \underset{\delta\to 1}{\to} 0$. I choose δ sufficiently large so that α_S and α_B are less that 1.

System (38) has steady states $(z,-z), z \in \mathbb{R}$. I am interested in the positive trajectory that approaches the steady state $(0,0)$. Around this steady state the linearized system can be written in the matrix form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1-\alpha_B+\alpha_S\alpha_B & -\alpha_B(1-\alpha_S) \\ -\alpha_S & 1-\alpha_S \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}.$$

The matrix has eigenvalues 1 and $\lambda \equiv (1-\alpha_B)(1-\alpha_S) \in (0,1)$. Since one of eigenvalues is equal to 1, the steady state is unstable, and I cannot conclude that in the neighborhood of the steady state, the non-linear system will converge to the steady state, neither that the trajectory will stay positive. Therefore, I construct a particular trajectory that satisfies the desired properties. The proof proceeds in three steps.

Step 1: Conjectured solution In the first step, I conjecture the form of solution and use the method of indeterminate coefficients to derive it. The following preliminary claim gives the Taylor expansion of $\alpha^B(y)$ and $\alpha^S(x)$.

Claim 3. *For any $x \in (0,1)$ and $y \in (0,1)$,*

$$\alpha^B(y) \equiv \alpha_B + \phi \sum_{l=1}^{\infty} \gamma_l^B y^l, \tag{101}$$

$$\alpha^S(x) \equiv \alpha_S + \phi \sum_{l=1}^{\infty} \gamma_l^B x^l, \tag{102}$$

where $\gamma_l^B \equiv \frac{d^l c(s_\infty)/dx^l}{l!}$ and $\gamma_l^S \equiv \frac{d^l v(b_\infty)/dx^l}{l!}$.

Note that by the regularity conditions on v and c, $\gamma_l^B < D$ and $\gamma_l^S < D$ for all l.

I conjecture that there exists $(\mu_i^x, \mu_i^y)_{i=1}^{\infty}$ such that the solution (38) takes form[17]

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \sum_{i=1}^{\infty} \lambda^{ik} \begin{pmatrix} \lambda^{i/2}\mu_i^x \\ \mu_i^y \end{pmatrix}, \text{ for } k = 1, 2, \dots, \tag{103}$$

and in addition, satisfies for all $i = 1, 2, \dots,$

$$|\mu_i^x| \leq u_\delta M^i \text{ and } |\mu_i^y| \leq u_\delta M^i \tag{104}$$

for some positive M and u_δ such that

$$M < 1 < \frac{1}{\lambda(1+u_\delta)}. \tag{105}$$

Given this conjecture, I next derive expressions for coefficients μ_i^x and μ_i^y, and in the next step, I will verify that for δ sufficiently close to 1, upper bounds (104) indeed hold.

Series (103) defining (x_k, y_k) are absolutely convergent, as they are dominated by the absolutely convergent series $u_\delta \sum_{i=1}^{\infty} \lambda^{ik} M^i$. Plugging the conjectured solution (103) into system (38), I get

$$\begin{cases} \sum_{i=1}^{\infty} \lambda^{i(k+\frac{1}{2})}(\mu_i^x - \mu_i^x \lambda^i - \alpha_B(\mu_i^x + \mu_i^y \lambda^{\frac{i}{2}})) = \phi \left(\sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=1}^{\infty} \mu_i^y \lambda^{i(k+1)} \right)^l \right) \left(\sum_{i=1}^{\infty} \lambda^{i(k+\frac{1}{2})}(\mu_i^x + \mu_i^y \lambda^{\frac{i}{2}}) \right), \\ \sum_{i=1}^{\infty} \lambda^{ik}(\mu_i^y - \mu_i^y \lambda^i - \alpha_S(\mu_i^x \lambda^{\frac{i}{2}} + \mu_i^y)) = \phi \left(\sum_{l=1}^{\infty} \gamma_l^S \left(\sum_{i=1}^{\infty} \mu_i^x \lambda^{ik} \right)^l \right) \left(\sum_{i=1}^{\infty} \lambda^{ik}(\mu_i^x \lambda^{\frac{i}{2}} + \mu_i^y) \right). \end{cases} \tag{106}$$

Consider the first equation in system (106). Since $\lambda \in (0,1)$ and $|\mu_i^y| \leq u_\delta M^i$, by Mertens' Theorem,

$$\sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=1}^{\infty} \mu_i^y \lambda^{i(k+1)} \right)^l = \sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=l}^{\infty} \sum_{i_1+\dots+i_l=i} \mu_{i_1}^y \cdot \dots \cdot \mu_{i_l}^y \lambda^{i(k+1)} \right). \tag{107}$$

[17] This is a natural guess given the eigenvalues of the linearized system.

The series in (107) is absolutely convergent by

$$\begin{aligned}
& \sum_{l=1}^{\infty}\sum_{i=l}^{\infty}\left|\lambda^{i(k+1)}\gamma_l^B \sum_{i_1+\cdots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y\right| \\
\leq \; & D\sum_{l=1}^{\infty}\sum_{i=l}^{\infty}\lambda^{i(k+1)} \sum_{i_1+\cdots+i_l=i} \left|\mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y\right| \\
\leq \; & D\sum_{l=1}^{\infty}\sum_{i=l}^{\infty}\lambda^{i(k+1)} \sum_{i_1+\cdots+i_l=i} u_\delta^l M^i \\
= \; & D\sum_{l=1}^{\infty} u_\delta^l \sum_{i=l}^{\infty}\lambda^{i(k+1)} M^i \binom{i-1}{l-1} \\
= \; & D\sum_{l=1}^{\infty} u_\delta^l \left(\frac{\lambda^{k+1}M}{1-\lambda^{k+1}M}\right)^l \\
\leq \; & D\sum_{l=1}^{\infty} u_\delta^l \left(\frac{\lambda M}{1-\lambda M}\right)^l,
\end{aligned}$$

where the first inequality arises via the triangle inequality and $|\gamma_l^B| < D$, the second inequality follows from (104), the first equality arises from the fact that the number of compositions of i into exactly l parts is $\binom{i-1}{l-1}$, the second equality is by summing over i, the third inequality is by $\lambda^{k+1} < \lambda < 1$. The resulting series is convergent whenever $u_\delta \frac{\lambda M}{1-\lambda M} < 1$, which holds by (105). Therefore, by Fubini's Theorem, exchanging the order of summation in (107) results in

$$\begin{aligned}
\sum_{l=1}^{\infty}\gamma_l^B\left(\sum_{i=1}^{\infty}\mu_i^y\lambda^{i(k+1)}\right)^l &= \sum_{l=1}^{\infty}\gamma_l^B\left(\sum_{i=l}^{\infty}\sum_{i_1+\cdots+i_l=i}\mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y\lambda^{i(k+1)}\right) \\
&= \sum_{i=1}^{\infty}\lambda^{i(k+1)}\sum_{l=1}^{i}\sum_{i_1+\cdots+i_l=i}\gamma_l^B\mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y.
\end{aligned}$$

By the absolute convergence of both series on the right-hand side of (106), the product on the right-hand side is equal to the Cauchy product, and so I can rewrite system (106) as follows

$$\begin{cases}
\sum_{i=1}^{\infty}\lambda^{i(k+\frac{1}{2})}\left[\mu_i^x - \mu_i^x\lambda^i - \alpha_B(\mu_i^x + \mu_i^y\lambda^{\frac{i}{2}}) - \phi\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{\frac{j}{2}} + \mu_{i-j}^y\lambda^{\frac{i}{2}})\sum\limits_{l=1}^{j}\gamma_l^B\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y\right)\right] = 0, \\
\sum_{i=1}^{\infty}\lambda^{ik}\left[\mu_i^y - \mu_i^y\lambda^i - \alpha_S(\mu_i^x\lambda^{\frac{i}{2}} + \mu_i^y) - \phi\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{\frac{i-j}{2}} + \mu_{i-j}^y)\sum\limits_{l=1}^{j}\gamma_l^S\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^x \cdot \ldots \cdot \mu_{j_l}^x\right)\right] = 0.
\end{cases}$$

Setting all coefficient at $\lambda^{i(k+1/2)}$ and λ^{ik} equal to zero results in the system

$$\begin{cases} \mu_i^x - \mu_i^x \lambda^i - \alpha_B(\mu_i^x + \mu_i^y \lambda^{\frac{i}{2}}) = \phi \sum\limits_{j=1}^{i-1} \left((\mu_{i-j}^x \lambda^{\frac{j}{2}} + \mu_{i-j}^y \lambda^{\frac{i}{2}}) \sum\limits_{l=1}^{j} \gamma_l^B \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^y \cdot \dots \cdot \mu_{j_l}^y \right), \\ \mu_i^y - \mu_i^y \lambda^i - \alpha_S(\mu_i^x \lambda^{\frac{i}{2}} + \mu_i^y) = \phi \sum\limits_{j=1}^{i-1} \left((\mu_{i-j}^x \lambda^{\frac{i-j}{2}} + \mu_{i-j}^y) \sum\limits_{l=1}^{j} \gamma_l^S \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^x \cdot \dots \cdot \mu_{j_l}^x \right). \end{cases} \tag{108}$$

Using notation $A_i \equiv \begin{pmatrix} 1-\lambda^i - \alpha_B & -\alpha_B \lambda^{i/2} \\ -\alpha_S \lambda^{i/2} & 1-\lambda^i-\alpha_S \end{pmatrix}$, $\mu_i \equiv \begin{pmatrix} \mu_i^x \\ \mu_i^y \end{pmatrix}$, and

$$\varphi_i = \begin{pmatrix} \varphi_i^x \\ \varphi_i^y \end{pmatrix} \equiv \begin{pmatrix} \phi \sum\limits_{j=1}^{i-1} \left((\mu_{i-j}^x \lambda^{j/2} + \mu_{i-j}^y \lambda^{i/2}) \sum\limits_{l=1}^{j} \gamma_l^B \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^y \cdot \dots \cdot \mu_{j_l}^y \right) \\ \phi \sum\limits_{j=1}^{i-1} \left((\mu_{i-j}^x \lambda^{(i-j)/2} + \mu_{i-j}^y) \sum\limits_{l=1}^{j} \gamma_l^S \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^x \cdot \dots \cdot \mu_{j_l}^x \right) \end{pmatrix}, \tag{109}$$

I can write the system in matrix form as

$$A_i \mu_i = \varphi_i.$$

Since $\det(A_i) = (1-\lambda^i)(\lambda - \lambda^i) > 0$, for $i \geq 2$, matrix A_i is invertible, and I can solve for all μ_i (with the exception of $i = 1$)

$$\mu_i = A_i^{-1} \varphi_i. \tag{110}$$

Equation (110) expresses μ_i through $\mu_1, \dots, \mu_{i-1}$. For $i = 1$, the equations in (108) are linearly dependent (as $\det(A_i) = 0$) and the relation between μ_1^x and μ_1^y is given by

$$\mu_1^x = \lambda^{-\frac{1}{2}} \frac{\alpha_B}{\alpha_S} (1-\alpha_S) \mu_1^y. \tag{111}$$

Equations (110) and (111) give the desired expressions for μ_i^x and μ_i^y through the parameters of the model.

Step 2: Verify bounds In this step, I verify that for μ_i given by (111) and (110), bounds (104) and (105) indeed hold and so, my derivation in Step 1 is justified. Fix any $M < 1$. Let

$$u_\delta = \lambda^{-1/4} - 1 \tag{112}$$

Then $\frac{1}{\lambda(1+u_\delta)} = \lambda^{-3/4} > 1$ and so, (105) holds. If $\frac{\alpha_B}{\alpha_S}(1-\alpha_S) \leq \lambda^{\frac{1}{2}}$, then I set

$$\begin{aligned} \mu_1^y &= u_\delta M, \\ \mu_1^x &= \lambda^{-\frac{1}{2}} \frac{\alpha_B}{\alpha_S} (1-\alpha_S) \mu_1^y, \end{aligned} \tag{113}$$

and otherwise, set

$$\begin{aligned} \mu_1^x &= u_\delta M, \\ \mu_1^y &= \lambda^{\frac{1}{2}} \frac{\alpha_S}{\alpha_B(1-\alpha_S)} \mu_1^x. \end{aligned} \tag{114}$$

The next claim verifies (104).

Claim 4. There exists $\hat{\delta} \in (0,1)$ *such that for any* $\delta \in (\hat{\delta}, 1)$ *such that for* μ_1^x *and* μ_1^y *defined above in* (113), (114) *and* μ_i^x *and* μ_i^y *defined in* (110), *bounds* (104) *hold.*

Proof. The proof is by induction on i. By (113) and (114), $|\mu_1^x| \leq u_\delta M$ and $|\mu_i^y| \leq u_\delta M$, which proves the base of induction. Now, I prove the inductive step. Suppose that the statement is true for all $j < i$. I show that $|\mu_i^x| < u_\delta M^i$ and $|\mu_i^y| < u_\delta M^i$. I can find the closed-form solution to system (110),

$$|\mu_i^x| = \frac{|(1-\lambda^i-\alpha_S)\varphi_i^x + \alpha_B \lambda^{i/2} \varphi_i^y|}{(1-\lambda^i)(\lambda-\lambda^i)} \leq \frac{4\max\{1-\lambda^i, \alpha_S, \alpha_B\} \cdot \max\{|\varphi_i^x|, |\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)}$$

and the same upper bound holds for $|\mu_i^y|$. Thus, it is sufficient to show that

$$\frac{4\max\{(1-\lambda^i), \alpha_S, \alpha_B\} \cdot \max\{|\varphi_i^x|, |\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)} < u_\delta M^i.$$

Notice that $\frac{\alpha_S}{1-\lambda^i} < \frac{\alpha_S}{1-\lambda}$ for $i \geq 2$, and by l'Hospital rule $\lim_{\delta\to 1} \frac{\alpha_S}{1-\lambda} = \lim_{\delta \to 1} \frac{\alpha_S}{\alpha_S+\alpha_B-\alpha_S\alpha_B} = \frac{V^S}{V^S+V^B} < 1$. Hence, for sufficiently large δ and all $i \geq 2$, I have $\frac{\alpha_S}{1-\lambda^i} < 1$, and by an analogous argument, $\frac{\alpha_B}{1-\lambda^i} < 1$. Therefore, $\frac{\max\{1-\lambda^i, \alpha_S, \alpha_B\}}{1-\lambda^i} \leq 1$ for sufficiently large δ and it remains to show that

$$\frac{4\max\{|\varphi_i^x|, |\varphi_i^y|\}}{\lambda-\lambda^i} < u_\delta M^i$$

for sufficiently large δ. I will show that $\frac{|\varphi_i^x|}{(\lambda-\lambda^i)u_\delta M^i} < \frac{1}{4}$, and by symmetric argument, $\frac{|\varphi_i^y|}{(\lambda-\lambda^i)u_\delta M^i} <$

$\frac{1}{4}$. Recall from (109) that

$$
\begin{aligned}
\varphi_i^x =&\phi \sum_{j=1}^{i-1}\left((\mu_{i-j}^x \lambda^{\frac{j}{2}} + \mu_{i-j}^y \lambda^{\frac{i}{2}}) \sum_{l=1}^{j} \gamma_l^B \sum_{j_1+\cdots+j_l=j} \mu_{j_1}^y \cdot \dots \cdot \mu_{j_l}^y \right) \\
\leq&\phi \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} \left(\sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} |\mu_{i-j}^x \mu_{j_1}^y \cdot \dots \cdot \mu_{j_l}^y| \right) + \phi\lambda^{\frac{i}{2}} \sum_{j=1}^{i-1}\sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} |\mu_{i-j}^y \mu_{j_1}^y \cdot \dots \cdot \mu_{j_l}^y| \\
\leq&\phi \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} u_\delta^{l+1} M^i + \phi\lambda^{\frac{i}{2}} \sum_{j=1}^{i-1}\sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} u_\delta^{l+1} M^i \\
=&\phi \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} \sum_{l=1}^{j} |\gamma_l^B| u_\delta^{l+1} M^i \binom{j-1}{l-1} + \phi\lambda^{\frac{i}{2}} \sum_{j=1}^{i-1}\sum_{l=1}^{j} |\gamma_l^B| u_\delta^{l+1} M^i \binom{j-1}{l-1} \\
\leq&2\phi u_\delta M^i D \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} \sum_{l=1}^{j} u_\delta^l \binom{j-1}{l-1} \\
\leq&2\phi u_\delta M^i D \sum_{j=1}^{i-1} \lambda^{\frac{j}{2}} u_\delta \left(1+u_\delta\right)^{j-1} \\
=&2\phi u_\delta M^i D \frac{u_\delta \lambda^{\frac{1}{2}} (1-\lambda^{\frac{i-1}{2}}(1+u_\delta)^{i-1})}{1-\lambda^{\frac{1}{2}}(1+u_\delta))} \\
=&2\phi u_\delta M^i D \lambda^{\frac{1}{2}} (1-\lambda^{\frac{i-1}{2}}),
\end{aligned}
$$

where the first inequality is due to the triangle inequality, the second inequality arises via the inductive hypothesis, the first equality makes use of the fact that the number of compositions of j into exactly l parts is $\binom{j-1}{l-1}$, the forth inequality is by $\lambda^j > \lambda^i$ for $j < i$ and $|\gamma_l^B| < D$, the fifth inequality is by summing over l, the second equality is the summation over j, the last equality is plugging in $u_\delta = \lambda^{-1/4} - 1$. Thus, I need to show that

$$
2\phi D \frac{\lambda^{\frac{1}{4}}(1-\lambda^{\frac{i-1}{4}})}{\lambda - \lambda^i} < \frac{1}{4}. \tag{115}
$$

This inequality holds for sufficiently large δ, as $\phi \to 0$ as $\delta \to 1$ and

$$
\lim_{\delta\to 1} \frac{\lambda^{\frac{1}{4}}(1-\lambda^{\frac{i-1}{4}})}{\lambda - \lambda^i} = \lim_{\delta\to 1} \frac{1-\lambda^{\frac{i-1}{4}}}{1-\lambda^{i-1}} = \lim_{\delta\to 1} \frac{-\frac{i-1}{4}\lambda^{\frac{i-5}{4}}}{-i\lambda^{i-2}} = \frac{i-1}{4i}.
$$

This completes the proof of the inductive step and the claim. $\square$

Step 3: Check solution In this step, I verify that the candidate trajectories (x_k, y_k) given by (103) that I have constructed indeed satisfy all conditions of Lemma 12. The convergence to $(0,0)$ follows immediately by taking the limit $k \to \infty$ of (103) and noting that $\lambda < 1$. Note that

I still have one free parameter left M that pins down μ_1^x and μ_1^y in (113) or (114). I choose M so that

$$x_1 + y_1 = \lambda^{3/2}\mu_i^x + \lambda\mu_i^y = 2\eta,$$

and so, the initial condition in (38) is satisfied. In follows from (103) that x_k and y_k are decreasing in k and so,

$$x_k + y_{k+1} \leq x_k + y_k \leq x_1 + y_1 = 2\eta,$$

verifying inequalities (41) and (42).

Finally, I show that x_k and y_k are positive. Observe that

$$\begin{aligned} x_k &= \sum_{i=1}^{\infty} \lambda^{i(k+1/2)}\mu_i^x \\ &= \lambda^{k+1/2}\left(\mu_1^x + \sum_{i=2}^{\infty}\lambda^{i(k+1/2)}\mu_i^x\right) \\ &\geq \lambda^{k+1/2}\left(\mu_1^x - \sum_{i=2}^{\infty}\lambda^{i(k+1/2)}u_\delta M^i\right) \\ &\geq \lambda^{k+1/2}\left(\mu_1^x - u_\delta\frac{\lambda^{2k+1}M^2}{1-\lambda^{(k+1/2)}M}\right). \end{aligned}$$

Hence, for sufficiently large k, x_k is positive whenever μ_1^x is. Since from (111) the sign of μ_1^y and μ_1^x is the same, y_k is positive for sufficiently large k. Thus, I have shown that x_k and y_k are positive starting from some k_0. By rearranging terms in the first equation of (38), $x_k = \frac{x_{k+1}+\alpha^B(y_{k+1})y_{k+1}}{1-\alpha^B(y_{k+1})}$. Observe that for δ sufficiently close to 1, $\alpha^B(y) \in (0,1)$ for all $y > 0$. Hence, x_k is positive whenever x_{k+1} and y_{k+1} are positive. Analogously, it can be shown from the second equation of (38) that for sufficiently large δ, y_k is positive whenever x_{k+1} and y_{k+1} are positive. This proves that x_k, y_k are positive for all $k = 1, 2, \dots$, when δ is sufficiently close to 1.