Pareto Efficiency and Identity*

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Abstract

This paper examines the set of Pareto efficient allocations in a finite period Mirrlees (1971, 1976) economy; each period represents a lifetime for an agent who cares about the utility of his descendants. In making Pareto comparisons, we use an interim concept of efficiency, and consider an individual as indexed not only by his date of birth but also by the history of events up to his birth, including his own type. That is, we assume the child of a high skilled parent is a *different person* than the child of a low skilled parent, even if both children have the same skill level. Our contributions are characterization of these efficient allocations and their implementation.

We completely characterize the set of efficient allocations under *full information*. We show that for efficient allocations, implicit inheritance taxes from the perspective of the *parent's* type, can be either progressive or regressive. Further, imposing no taxes of any kind, coupled with each agent owning his own production, results in a Pareto efficient allocation.

Under *private information*, we completely characterize the set of Pareto efficient allocations for the two-period economy where skill types take on two values, and again show that implicit inheritance taxes can be either progressive or regressive, again relative to the parent's type. For more general multi-period economies with private information, we show that the reciprocal Euler condition of Rogerson (1985) and Golosov, Kocherlakota, and Tsyvinski (2003) holds as a necessary condition, but as an inequality and that the expected value of implicit inheritance tax rates conditional on a parent's history are weakly negative. Finally, we derive conditions such that given private information, no taxes of any kind, coupled with each agent owning his own production, results in a Pareto efficient allocation.

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1 Introduction

Pareto efficiency can be an enormously useful concept when considering government policy. It allows the economist to make distinct those interventions necessary to avoid outcomes which are undesirable by any plausible measure versus those interventions which are desirable only if some individuals are deemed more deserving than others. In dynamic or intergenerational economies, however, Pareto efficiency is considered problematic precisely because it can be unclear which preferences orderings "count" when making Pareto comparisons.

In this paper, we examine the implications on the set of Pareto efficient allocations of requiring that individuals born after the beginning of time not only count, but that they must also be indexed by the history that preceded their births. That is, an individual born after one history is a *different person* (for the purposes of Pareto comparisons across allocations) than an individual born at the same date but after a different history. Specifically, we consider a repeated $T+1$ period Mirrlees (1971, 1976) economy similar to that of Farhi-Werning (2007), where each period represents a lifetime for an agent, but such an agent cares about the utility of his descendants. When individuals are indexed not only by their dates of birth and their own skill levels, but also by the skill levels of all their ancestors, Pareto efficiency no longer requires the consideration of Harsanyi-Rawls type insurance motives against the realization of one's own type or the types of one's ancestors. However, altruism towards descendants does imply that Pareto efficiency consider the insurance motives of parents toward the type realizations of their children and other descendants. Technically, we consider a form of *interim efficiency* (Holmström and Myerson(1983)) in that agents know their own type realization and that of their ancestors but not the types of their descendants.[1]

[1]The distinction between ex-post, ex-ante and interim efficiency was introduced in the overlapping-generations literature in the debate following the Lucas (1972), where only ex-post efficiency was considered. In Lucas' model, where agents live for two periods, if one ignores calendar date, an individual faces two uncertainties, one regarding the market segment where he is born as young, the other the segment where he is allocated when old. For this economy Muench (1975) suggested an ex-ante efficiency criterion, and Peled (1982, 1984) an interim concept, where social planner has the same information as the agent have when they make their decisions. In other OLG models (such as Peled (1982) or Rangel and Zeckhauser (2001)) uncertainty is on the aggregate endowment and its distribution between young and old. Our definition of efficiency is an extension of the interim criterion to repeated Mirrlees economies.

The basic idea is that it is certainly *feasible* for allocations to depend on an individual's skill and his ancestors' history of skills. Moving from an allocation in which, say, a highly skilled individual from a long line of highly skilled individuals is treated better than those born at the same time but with different histories to an allocation which treats individuals more equally would generally be objected to by the highly skilled individual from a long line of similarly highly skilled individuals. Allowing this individual to "count" is precisely what our interim definition accomplishes. That is, an allocation A is Pareto efficient if and only if there is no other feasible allocation B such that B would be weakly preferred by all individuals and strictly preferred by one. But this definition requires one to specify exactly the relevant reference set of individuals who compare the two allocations, and here we specify that set as individuals who know their own type as well as the type of each of their ancestors. Whether such "votes" correspond to actual political outcomes is an interesting but distinct issue that we do not discuss here.

Given this definition of Pareto efficiency, we completely characterize the set of Pareto efficient allocations under full information, and show the necessary and sufficient conditions for efficiency are *identical* to the efficiency conditions when periods are interpreted as dates in a $T+1$ lived individual's life as opposed to generations, with the single exception that the appropriate inter-temporal condition comparing a parent's marginal utility with each possible type of child holds as a weak inequality instead of an equality. In fact, this inter-temporal condition holds as a strict inequality if and only if the child receives a positive Pareto weight in the social planner's problem which produces the Pareto efficient allocation. We further show that for Pareto efficient allocations, implicit inheritance taxes can be either progressive or regressive and that imposing no taxes of any kind, coupled with each agent owning his own production and being able to make actuarially fair non-negative voluntary conditional bequests (which we label laissez-faire), results in a Pareto efficient allocation.

It is important to note that our broadening the set of Pareto efficient allocations by taking into account the preferences of those who know their type realization and those of their ancestors in no way makes the set of Pareto efficient allocations trivially large. These conditions still take into account that parents have desires to trade off their own consumption and those of their descendants (who they care about). Thus, for instance, our result on the efficiency of the laissez-faire allocation depends crucially on the existence of insurance markets to make actuarially fair conditional bequests. A different market structure, such as only allowing parents to make voluntary non contingent bequests, would generally lead to an inefficient allocation, even under our broad definition of efficiency. (An allocation which *forces* parents to make large unconditional transfers could be efficient).

For the case of private information, we completely characterize the set of Pareto efficient allocations for the two-period economy of Farhi-Werning (2010) subject to skill types taking on two values, and again show that implicit inheritance taxes can be either progressive or regressive relative to the parent's type. For more general $T+1$ period economies with private information, we show that the reciprocal Euler condition of Rogerson (1985) and Golosov, Kocherlakota, and Tsyvinski (2003) holds as a necessary condition, but again as an inequality. We further show, following the logic of Kocherlakota (2005), that the expected value of implicit inheritance tax rates conditional on a parent's history are weakly negative in any constrained efficient allocation. Finally, we derive conditions such that given private information, no taxes of any kind, coupled with each agent owning his own production and being able to make actuarially fair non-negative conditional bequests, results in a Pareto efficient allocation.

2 The Model

Consider a repeated Mirrlees economy with $T+1$ dates (generations), $t \in \{0, 1, \ldots, T\}$ where at each date t, there exists a unit continuum of agents with *family names* $i \in [0, 1]$. At each date t, each agent i has a *type* $\theta_{i,t} \in \Theta$, where Θ is a finite set of positive real numbers, $\Theta = \{\theta^0, \ldots, \theta^N\}$. For a type θ agent to produce $y \geq 0$ units of the single consumption good, he must exert $\frac{y}{\theta}$ units of labor effort. Agents live for one period, but an agent from family i at each date $t < T$ (a parent) is associated with a single agent from family i at date $t+1$ (his child). Thus, associated with each agent from family i at date t is the history of shocks of his dynasty, $\theta_i^t = \{\theta_{i,0}, \ldots, \theta_{i,t}\}$. Let $\pi(\theta|\theta^-)$ denote the probability that a type θ^- parent has a type θ child. The transition matrix π and the initial distribution on types f are such that for all $\theta \in \Theta$, $f(\theta) = \sum_{\theta^-} \pi(\theta|\theta^-) f(\theta^-)$, so the fraction of agents of each type is constant over time, and such that for all pairs (θ, θ^-), $\pi(\theta|\theta^-) > 0$, which ensures that all future θ paths are possible given past realizations.

An allocation is a collection $\{\{c_{i,t}(\theta^t), y_{i,t}(\theta^t)\}_{t=0}^T\}_{i\in[0,1]}$. A *symmetric* allocation is an allocation such that for all (i, j), t and $\theta_i^t = \theta_j^t$, $c_{i,t}(\theta_i^t) = c_{j,t}(\theta_j^t)$ and $y_{i,t}(\theta_i^t) = y_{j,t}(\theta_j^t)$, and thus is denoted $\{c_t(\theta^t), y_t(\theta_t)\}_{t=0}^T$. Let $\theta^{s,t} = \{\theta_s, \ldots, \theta_t\}$ denote a family's realized types between dates s and t. Symmetric allocations are feasible if

$$\sum_{t=0}^{T} \sum_{\theta^t} \frac{1}{R^t} f(\theta_0)\pi(\theta_1|\theta_0) \ldots \pi(\theta_t|\theta_{t-1})(c_t(\theta^t) - y_t(\theta^t)) \leq 0, \tag{1}$$

where $R > 1$. From here on, we consider only symmetric allocations.

Agents have identical preferences over the consumption-labor pairs of themselves and their descendants. We assume an agent born at date t with family history θ^t ranks allocations according to

$$\begin{aligned} U_t(\theta^t) \quad \equiv \quad & u(c_t(\theta^t)) - h(\frac{y_t(\theta^t)}{\theta_t}) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1}) \\ & [u(c_s(\theta^t, \theta^{t+1,s})) - h(\frac{y_s(\theta^t, \theta^{t+1,s})}{\theta_s})], \end{aligned} \tag{2}$$

where u is assumed differentiable, strictly increasing, strictly concave, and with $\lim_{c\to 0} u'(c) = \infty$. We also assume that:

$$h(\ell) = \ell^\psi, \text{ for some } \psi > 1 \tag{3}$$

This functional form of h ensures that the utility possibilities set is convex when θ realizations are private information. The ranking (2) also assumes restrictions regarding altruism toward descendants. In particular, a parent cares about the expectation of the discounted dynastic utility of his child, with $\beta \in (0, 1)$. Exponential discounting ensures time consistency regarding the preferences of a parent toward his grandchild and the preferences of his child toward that grandchild.

The type realization θ_t of an agent born at date t to family i is alternatively assumed to be publicly observed (the *full information* case) or privately observed by that agent (the *private information* case). In both cases, all other objects, such as an agent's consumption, c, and production, y, are assumed publicly observed with one exception: in the case of private information, an agent's labor effort, $\frac{y}{\theta}$, is also assumed to be privately observed by that agent. If not, one could infer an agent's θ type from his labor effort and output.

In the full information case, a symmetric allocation $\{c_t(\theta^t), y_t(\theta^t)\}_{t=0}^{T}$ specifies consumption and required output levels as functions of the date and a family's *realized* type outcomes. In the private information case, an allocation is understood to specify consumption and required output levels as functions of the date, and the family's *announcements* of type outcomes, where the announcement of θ_t is made by the family member living at date t.

In the private information case, an allocation is considered *incentive compatible* if for all t, θ^t, and $\hat{\theta} \neq \theta_t$,

$$\begin{aligned} U_t(\theta^t) \quad &\geq \quad u(c_t(\theta^{t-1}, \hat{\theta})) - h(\frac{y_t(\theta^{t-1}, \hat{\theta})}{\theta_t}) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1}) \\ &\qquad [u(c_s(\theta^{t-1}, \hat{\theta}, \theta^{t+1,s})) - h(\frac{y_s(\theta^{t-1}, \hat{\theta}, \theta^{t+1,s})}{\theta_s})]. \end{aligned} \tag{4}$$

In words, this condition requires that an agent born at date t to a family with history θ^t weakly prefers to announce his true type given that his descendants will truthfully announce their true types.

Let the full information *symmetric utility possibilities set* be the set of all $\{\{U_t(\theta^t)\}_{\theta^t \in \Theta^t}\}_{t=0}^{T}$

vectors that can be generated by an allocation satisfying the resource condition (1). Let the private information utility possibilities set be such collections that also satisfy the incentive condition (4). A feasible symmetric allocation is considered Pareto efficient if no other feasible symmetric allocation generates a profile $\{\{U_t(\theta^t)\}_{\theta^t\in\Theta^t}\}_{t=0}^T$ that weakly dominates it and strictly dominates it for at least one t and θ^t. Likewise, a feasible, incentive compatible symmetric allocation is considered constrained Pareto efficient if no other feasible, incentive compatible symmetric allocation generates a dominating utility profile.

Lemma 1 *The full information and private information utility possibilities sets are each convex.*

Proof. Define $\overline{u}_t(\theta^t) \equiv u(c_t(\theta^t))$ and $\overline{h}_t(\theta^t) \equiv (\frac{y_t(\theta^t)}{\theta_t})^\psi$. Then a symmetric allocation can be considered a specification $\{\overline{u}_t(\theta^t), \overline{h}_t(\theta^t)\}_{t=0}^T$. The utility of each type $U_t(\theta^t)$ is then a linear function of the allocation. The incentive condition becomes for all t, θ^t, and $\hat{\theta} \neq \theta_t$,

$$\begin{aligned} U_t(\theta^t) \geq\ & \overline{u}_t(\theta^{t-1}, \hat{\theta}) - (\frac{\hat{\theta}}{\theta_t})^\psi \overline{h}_t(\theta^{t-1}, \hat{\theta}) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \dots \pi(\theta_s|\theta_{s-1}) \\ & [\overline{u}_s(\theta^{t-1}, \hat{\theta}, \theta^{t+1,s}) - \overline{h}_s(\theta^{t-1}, \hat{\theta}, \theta^{t+1,s})], \end{aligned} \tag{5}$$

which is a linear function of the allocation. Finally, the resource constraint becomes

$$\sum_{t=0}^{T} \sum_{\theta^t} \frac{1}{R^t} f(\theta_0)\pi(\theta_1|\theta_0)\dots\pi(\theta_t|\theta_{t-1})(u^{-1}(\overline{u}_t(\theta^t)) - \theta_t \overline{h}_t(\theta^t)^{\frac{1}{\psi}}) \leq 0, \tag{6}$$

a condition that a convex function of the allocation be weakly less than zero. ■

Define the planner's problem as that of maximizing a weighted sum of lifetime utilities, where $\gamma_t(\theta^t) \geq 0$ is the weight the planner gives to an agent born at date t into a family with history θ^t:

$$PP: \quad \max_{c_t(\theta^t), y_t(\theta^t)} \sum_{t=0}^{T} \sum_{\theta^t} f(\theta_0)\pi(\theta_1|\theta_0)\dots\pi(\theta_t|\theta_{t-1})\gamma_t(\theta^t)U_t(\theta^t) \tag{7}$$

subject to $c_t(\theta^t)$ and $y_t(\theta^t)$ non-negative for all t and θ^t, resource feasibility (1), and, for the case of private information, incentive compatibility (4). The principal difference between our approach and that in Phelan (2006) or Farhi-Werning (2007, 2010) is that here $\gamma_t(\theta^t)$ is allowed to depend on θ^t. Farhi-Werning (2007) essentially restricts $\gamma_t(\theta^t)$ to be equal across all θ^t, whereas Phelan (2006) essentially restricts γ_t to be equal across dates t as well.

Note that if we let

$$\Gamma_t(\theta^t) \equiv \gamma_t(\theta^t) + \sum_{s=1}^{t} \beta^s \gamma_{t-s}(\theta^{t-s}) = \gamma_t(\theta^t) + \beta\Gamma_{t-1}(\theta^{t-1}),$$

(or $\Gamma_t(\theta^t)$ is the weight the planner puts on the *instantaneous* utility of a type (t, θ^t) agent), then the planner's problem becomes

$$PP: \max_{c_t(\theta^t), y_t(\theta^t)} \sum_{t=0}^{T} \sum_{\theta^t} f(\theta_0)\pi(\theta_1|\theta_0)\ldots\pi(\theta_t|\theta_{t-1})\Gamma_t(\theta^t)(u(c_t(\theta^t)) - h(\frac{y_t(\theta^t)}{\theta_t})) \tag{8}$$

again subject to $c_t(\theta^t)$ and $y_t(\theta^t)$ non-negative for all t and θ^t, resource feasibility (1), and, for the case of private information, incentive compatibility (4). Note that given weights $\Gamma_t(\theta^t)$, one can also back out the implied weights $\gamma_t(\theta^t) = \Gamma_t(\theta_t) - \beta\Gamma_{t-1}(\theta^{t-1})$. In particular, note that although for arbitrary weights the condition $\gamma_t(\theta^t) \geq 0$ implies that the weights $\Gamma_t(\theta^t) \geq 0$, that $\Gamma_t(\theta^t) \geq 0$ for all t, θ^t does *not* imply $\gamma_t(\theta^t) \geq 0$ for all t, θ^t.

3 Full Information

In this section, we consider the implications of Pareto efficiency when θ realizations are public information. We first establish the correspondence between efficient allocations and solutions to the planner's problem.

Lemma 2 *Every Pareto efficient symmetric allocation solves the planner's problem for some weights $\Gamma_t(\theta^t) > 0$ for all $t \geq 0$ and $\theta^t \in \Theta^t$. Likewise, if a symmetric allocation solves the planner's problem given weights $\Gamma_t(\theta^t) > 0$ for all $t \geq 0$ and $\theta^t \in \Theta^t$, then it is Pareto efficient* *if and only if* $\Gamma_t(\theta^t) - \beta\Gamma_{t-1}(\theta^{t-1}) \geq 0$ *for all* $t \geq 0$ *and* $\theta^t \in \Theta^t$ *(or, alternatively, if* $\gamma_t(\theta^t) \geq 0$ *for all* t *and* θ^t*).*

Proof. If a symmetric allocation $\{c_t(\theta^t), y_t(\theta^t)\}_{t=0}^T$ is Pareto efficient, then it must lie on the frontier of the utility possibilities set. The convexity of the utility possibilities set then ensures that it solves PP for some specification $\gamma_t(\theta^t) \geq 0$ for all t and θ^t. Next, the assumptions on u and h ensure any Pareto efficient allocation has $c_t(\theta^t)$ and $y_t(\theta^t)$ each strictly positive for all t and θ^t. (Otherwise, one could generate a Pareto improvement by marginally increasing both $y_t(\theta^t)$ and $c_t(\theta^t)$.) The planner's first order condition with respect to $c_0(\theta^0)$ implies that

$$\gamma_0(\theta_0)u'(c_0(\theta^0)) = \lambda > 0, \tag{9}$$

where λ is the Lagrange multiplier on the resource constraint. Thus, $\gamma_0(\theta_0) = \Gamma_0(\theta_0) > 0$ for all θ_0. From the definition of $\Gamma_t(\theta^t)$, this ensures $\Gamma_t(\theta^t) > 0$ for all t and θ^t.

Next assume that $\{c_t(\theta^t), y_t(\theta^t)\}_{t=0}^T$ solves PP for weights $\Gamma_t(\theta^t) > 0$ for all t and θ^t such that $\Gamma_t(\theta^t) - \beta\Gamma_{t-1}(\theta^{t-1}) \geq 0$. This then implies that $\gamma_t(\theta^t) \geq 0$ for all t and θ^t. That the allocation is Pareto efficient is then immediate given that u is strictly increasing (non-satiation). ■

We now turn to the main characterization result for the full information economy: necessary and sufficient conditions for an allocation to be Pareto efficient.

Proposition 3 *Assume for all dates* t*, a dynasty's history of shocks* θ^t *is public information. Then allocation* $\{c_t(\theta^t), y_t(\theta^t)\}_{t=0}^T$ *is Pareto efficient if and only if it satisfies the resource condition (1) with equality and for all* t*, and* (θ^t, θ_{t+1})*,*

$$u'(c_t(\theta^t))\theta_t = h'(\frac{y_t(\theta^t)}{\theta_t}), \tag{10}$$

and

$$u'(c_t(\theta^t)) \geq \beta R u'(c_{t+1}(\theta^t, \theta_{t+1})). \tag{11}$$

Proof. From Lemma 2, if $\{c_t(\theta^t), y_t(\theta^t)\}_{t=0}^T$ is Pareto efficient, then it solves PP for some $\Gamma_t(\theta^t) > 0$ for all t, θ^t. That the resource condition must hold with equality is then immediate. That (10) must hold follows from comparing the planner's first order conditions with respect to $c_t(\theta^t)$ and $y_t(\theta^t)$. Next, the planner's first order condition with respect to $c_0(\theta^0)$ implies that

$$\gamma_0(\theta_0)u'(c_0(\theta^0)) = \lambda, \tag{12}$$

where λ is the Lagrange multiplier on the resource constraint. Thus $\gamma_0(\theta_0) > 0$ for all θ_0. Next, the planner's first order condition with respect to $c_t(\theta^t)$ implies for all $t \geq 0$ that

$$\gamma_{t+1}(\theta^{t+1}) = \frac{\Gamma_t(\theta^t)(u'(c_t(\theta^t)) - \beta R u'(c_{t+1}(\theta^{t+1})))}{R u'(c_{t+1}(\theta^{t+1}))}. \tag{13}$$

Applying this sequentially from $t = 0$ on then delivers that $u'(c_t(\theta^t)) \geq \beta R u'(c_{t+1}(\theta^{t+1}))$ is both necessary and sufficient for $\gamma_{t+1}(\theta^{t+1})$ to be non-negative.

Next, assume that $\{c_t(\theta^t), y_t(\theta^t)\}_{t=0}^T$ satisfies (1) at equality, condition (10) for all $t \leq T$ and θ^t, and (11) for all $t < T$ and θ^{t+1}. Since PP is a concave programming problem, these are sufficient for $\{c_t(\theta^t), y_t(\theta^t)\}_{t=0}^T$ to solve PP with weights $\gamma_t(\theta^t) \geq 0$ defined by (13). Lemma 2 then implies Pareto efficiency. ■

The intuition behind this result is that an allocation is Pareto efficient if it does not waste resources, that no individual can be made better off by varying his labor effort and letting him consume the resulting variation in output, and finally, that no type θ^t individual can be made better off by reducing his consumption by ϵ and increasing the consumption of his type θ_{t+1} child by $\epsilon \frac{R}{\pi(\theta_{t+1}|\theta_t)}$. Note that these conditions are *exactly* the same conditions as when periods are interpreted as dates in a $T+1$ lived individual's life as opposed to generations, with the single exception that condition (11) is a weak inequality instead of an equality. It is clear that if (11) is violated, then an allocation is inefficient: one could take from the parent and give to the child in manner which makes both the parent and the child better off. (That is, the condition is necessary). It is sufficient because if satisfied, the parent either wishes to make a zero or negative conditional gift to that type child depending on whether or not the condition holds as a weak or strict inequality. If the parent has a latent wish to make a zero gift (the condition holds as an equality), varying the consumption of the parent and child makes the parent worse off. If the parent has a latent wish to make a negative gift (the condition holds as a strict inequality), reducing the consumption of the parent makes the parent worse off and reducing the consumption of the child makes the child worse off.

We next show that progressive estate taxes are not necessary for Pareto efficiency.

3.1 Full Information Tax Implementation

From Proposition (3), zero marginal labor taxes are necessary for any decentralized implementation of a Pareto efficient allocation. This follows directly from the assumption of full information. Next, for any given allocation, define the implicit inheritance tax $\tau_{t+1}(\theta^{t+1})$ on a type θ^{t+1} child as solving:

$$u'(c_t(\theta^t)) = \beta R u'(c_{t+1}(\theta^{t+1}))(1 - \tau_{t+1}(\theta^{t+1})). \tag{14}$$

This is the inheritance tax necessary to induce a type θ^t parent to make exactly a zero transfer to his θ^{t+1} child if he takes the allocation as given (or as a property right) but then is allowed to make positive *or negative* type conditional transfers to his possible children at actuarially fair rates of return. (By an actuarially fair rate of return we mean that if a type θ^t parent gives up ϵ units of consumption himself, he can conditionally transfer $\epsilon\frac{R}{\pi(\theta_{t+1}|\theta_t)}$ units of consumption to his type θ_{t+1} child.)[2]

Here, and through the rest of this paper, we say that a tax $\tau_{t+1}(\theta^{t+1})$ is *progressive from the perspective of the parent*, or parent-progressive, if for all $(\theta^t, \hat{\theta}^t)$, $c_t(\theta^t) > c_t(\hat{\theta}^t)$ if and only

[2] Since such a tax implementation by design induces zero transfers, it raises no tax revenues (or if the implicit taxes are negative, costs nothing to the government).

if $\sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t)\tau_{t+1}(\theta^t, \theta_{t+1}) > \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t)\tau_{t+1}(\hat{\theta}^t, \theta_{t+1})$. This is the definition used by Farhi-Werning (2010). We say that a tax $\tau_{t+1}(\theta^{t+1})$ is *progressive from the perspective of the child*, or child-progressive if for all $(\theta^t, \theta_{t+1}, \hat{\theta}_{t+1})$, $c_{t+1}(\theta^t, \theta_{t+1}) > c_{t+1}(\theta^t, \hat{\theta}_{t+1})$, if and only if $\tau_{t+1}(\theta^t, \theta_{t+1}) > \tau_{t+1}(\theta^t, \hat{\theta}_{t+1})$. This definition is used in Kocherlakota (2005). (The definitions of parent-regressive and child-regressive, respectively, are the same as the definitions of parent-progressive and child-progressive with the inequalities for the taxes are reversed.)

The next result characterizes full information implied inheritance taxes in terms of the planner weights.

Proposition 4 *The full information inheritance tax is*

$$\tau_{t+1}(\theta^{t+1}) = \frac{-\gamma_{t+1}(\theta^{t+1})}{\beta\Gamma_t(\theta^t)}. \tag{15}$$

Proof. The proof is immediate from the first order conditions of the full information planner's problem. ■

Proposition (4) implies that in any Pareto efficient full information allocation, the implicit inheritance tax on a type θ^{t+1} child is weakly negative and strictly negative if the child receives positive weight $\gamma_{t+1}(\theta^{t+1})$ in the social planner's problem. Further, the higher the weight on the child of that particular type, and the lower the weight on his ancestors, the more inheritances to that type are subsidized.

If one sets $T = 1$ (so $t \in \{0, 1\}$), the resulting two period economy is very close to being a full-information version of the private information two-period model of Farhi-Werning (2010) (the only difference being that in Farhi-Werning (2010), second period agents have no θ type and cannot work) and thus sheds some light on their result regarding the optimality of parent-progressive inheritance taxes.

In particular, Farhi-Werning (2010) assumes a social welfare function that is equivalent to assuming that the Pareto weight on a child, γ_1, cannot depend on the type of the parent,

θ_0. Letting $T = 1$ and requiring $\gamma_1(\theta_0, \theta_1) = \gamma_1$ (a constant) delivers for all θ_1,

$$\tau_1(\theta_0, \theta_1) = \frac{-\gamma_1}{\beta\gamma_0(\theta_0)}. \tag{16}$$

Thus, the higher the consumption of the date $t = 0$ parent (which follows directly from a higher $\gamma_0(\theta_0)$), the higher (closer to zero) the common inheritance tax on his children. That is, when γ_1 is restricted to be a constant, inheritance taxes are parent-progressive.

But next consider allowing the Pareto weight of the child, $\gamma_1(\theta_0, \theta_1)$, to depend on the type of the parent, θ_0. (We can continue to restrict it to not depend on θ_1, the type of the child, for this example.) Then (15) becomes, again for all θ_1,

$$\tau_1(\theta_0, \theta_1) = \frac{-\gamma_1(\theta_0)}{\beta\gamma_0(\theta_0)}, \tag{17}$$

and implicit inheritance taxes in a full information Pareto efficient allocation can be parent-regressive through an appropriate choice of $\gamma_1(\theta_0)$. In particular, if we let $\theta_0 \in \{\underline{\theta}, \overline{\theta}\}$ with $\gamma_0(\overline{\theta_0}) > \gamma_0(\underline{\theta_0}) > 0$ and set $\gamma_1(\overline{\theta}) > 0$ and $\gamma_1(\underline{\theta}) = 0$, then inheritances taxes are negative for the "rich" ($\overline{\theta}$ types) and zero for the "poor" ($\underline{\theta}$ types) and thus parent-regressive. (It is straightforward to produce similar examples where taxes are child-progressive or child-regressive. Finally, it is possible with more than two types to produce examples where implicit taxes are neither progressive nor regressive from either perspective.)

3.2 Full Information Property Rights Implementation

In this section, we establish that although inheritance taxes may be necessary to achieve a *particular* Pareto efficient allocation, no taxes of any kind are necessary to achieve Pareto efficiency in general.

To this end, define the full information *laissez-faire allocation* for the static ($T = 0$)

economy, $\{c^{\ell,0}(\theta_0), y^{\ell,0}(\theta_0)\}$, as solving (for all θ_0)

$$V^{\ell,0}(\theta_0) \equiv \max_y \left(u(y) - h(\frac{y}{\theta_0}) \right), \tag{18}$$

with $c^{\ell,0}(\theta_0) = y^{\ell,0}(\theta_0)$. Here, $V^{\ell,0}(\theta_0)$ represents the value of the static autarkic allocation to an agent of type θ.

Next, for $T > 0$, recursively define

$$\begin{aligned} V^{\ell,T}(\theta_0) \quad \equiv \quad & \max_{c_0, y_0, \{c_t(\theta^{1,t}), y_t(\theta^{1,t})\}_{t=1}^T} u(c_0) - h(\frac{y_0}{\theta_0}) + \sum_{t=1}^T \beta^t \sum_{\theta^{1,t}} \pi(\theta_1|\theta_0) \ldots \pi(\theta_t|\theta_{t-1}) \\ & [u(c_t(\theta^{1,t})) - h(\frac{y_t(\theta^{1,t})}{\theta_t})], \end{aligned} \tag{19}$$

subject to a budget condition

$$c_0 - y_0 + \sum_{t=1}^T \frac{1}{R^t} \sum_{\theta^{1,t}} \pi(\theta_1|\theta_t) \ldots \pi(\theta_t|\theta_{t-1})(c_t(\theta^{1,t}) - y_t(\theta^{1,t})) \leq 0, \tag{20}$$

and a no-enslaving-your-descendants condition, for all $t \geq 1$ and $\theta^{1,t}$,

$$\begin{aligned} & u(c_t(\theta^{1,t})) - h(\frac{y_t(\theta^{1,t})}{\theta_t}) + \sum_{s=t+1}^T \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1}) \\ & [u(c_s(\theta^{1,t}, \theta^{t+1,s})) - h(\frac{y_s(\theta^{1,t}, \theta^{t+1,s})}{\theta_s})] \geq V^{\ell,T-t}(\theta_t). \end{aligned} \tag{21}$$

Here, $V^{\ell,T}(\theta_0)$ is the value to a date $t = 0$ agent of type θ_0 of having a property right to his own production, and further having the ability to control the production and consumption of his descendants, as long as the expected discounted consumption of himself and his descendants does not exceed their production (which implies the ability to make actuarially fair conditional transfers to descendants) and these descendants do no worse than what they could achieve on their own with a zero inheritance and these same rights. (Thus, condition (21) implies that conditional inheritances be non-negative.) The full information laissez-faire allocation for the $T+1$ date economy is then the solution to the problem defining $V^{\ell,T}(\theta_0)$ for all θ_0.

Proposition 5 *The full information laissez-faire allocation is Pareto efficient.*

Proof. It is straightforward to show that the laissez-faire allocation has (20) hold as an equality for each θ_0 type. (Otherwise, each θ_0 type could simply consume the extra resources himself.) That is, under the laissez-faire allocation, the expected discounted dynastic consumption of each θ_0 type equals its expected discounted dynastic output.

Suppose there exists an allocation that Pareto dominates the laissez-faire allocation. That $\beta > 0$ and $\pi(\theta|\theta^-) > 0$ for all (θ, θ^-) implies that at least one first generation θ_0 type agent is strictly better off in this Pareto-improving allocation. Further, since the allocation is a Pareto improvement, it satisfies (21) for all $t \geq 1$ and $(\theta^{1,t})$. Thus, it must be the case that for this first generation θ_0 type, (20) is violated, otherwise the first generation θ_0 type would have chosen it. That is, under the Pareto-improving allocation, the expected discounted dynastic consumption of at least one θ_0 type exceeds its expected discounted dynastic output. The society-wide resource constraint then implies that for at least one θ_0 type, say $\hat{\theta}_0$, its expected discounted dynastic output exceeds its expected discounted dynastic consumption. Further, since the new plan is a Pareto improvement, conditions (21) are satisfied for the $\hat{\theta}_0$ type's problem. Thus, the $\hat{\theta}_0$ type could have chosen his part of the Pareto-improving allocation with (20) holding as a strict inequality, which is a contradiction. ■

The Pareto efficient full information laissez-faire allocation can be implemented with no taxes whatsoever (no labor taxes, no inheritance taxes). It does generally require sufficient property rights and insurance markets, however. In particular, parents need to be able to make actuarially fair conditional transfers to their children, and children need to be protected from parents making negative transfers. If parents can make only non-negative *unconditional* bequests, a necessary condition for a type θ^t parent to be optimizing is

$$u'(c_t(\theta^t)) \geq \beta R \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t) u'(c_{t+1}(\theta^{t+1}). \tag{22}$$

But if (22) holds as an equality (which happens whenever $c_t(\theta^t) < y_t(\theta^t))$ and $c_{t+1}(\theta^{t+1})$ non-trivially depends on θ_{t+1} (which it will since high skill children have a more valuable endowment, the right to their own production, than low skill children), this implies (11) is violated for at least one θ_{t+1} realization (in particular, the lowest). In words, if parents can make only unconditional bequests, a planner can improve on the market allocation by providing insurance to parents against them having low skilled children.

Protection against negative bequests (a property right to one's own production) is in turn necessary to implement the particular Pareto efficient allocation we label as the laissez-faire allocation. In fact, these property rights for children not to receive a negative inheritance is where positive Pareto weights $\gamma_t(\theta^t)$ occur. Recall from (13) that if an allocation is efficient, the Pareto weight on a date $t+1$, type θ^{t+1} agent, $\gamma_{t+1}(\theta^{t+1})$, is given by

$$\gamma_{t+1}(\theta^{t+1}) \quad = \quad \frac{\Gamma_t(\theta^t)(u'(c_t(\theta^t)) - \beta R u'(c_{t+1}(\theta^{t+1})))}{R u'(c_{t+1}(\theta^{t+1}))}. \tag{23}$$

That is, a type θ^{t+1} agent is associated with a strictly positive Pareto weight when $u'(c_t(\theta^t)) > \beta R u'(c_{t+1}(\theta^{t+1}))$ or precisely when, under the laissez-faire allocation, his parent wishes he could make him a negative conditional transfer. Likewise, children who receive a positive inheritance, which from the parent's inter-temporal first order condition only occurs if (21) is slack and thus $u'(c_t(\theta^t)) = \beta R u'(c_{t+1}(\theta^{t+1}))$, have $\gamma_{t+1}(\theta^{t+1}) = 0$.

3.3 Full Information Bequest Implementation

One less appealing aspect of the *laissez-faire* allocation presented in the previous section is that it allows complete control by the date $t = 0$ agent on the consumption and output of all his descendants. Actual estate plans are, of course, far less controlling. However, the full information laissez-faire allocation can be achieved as the outcome of a much simpler bequest game, which is closer to real life allocations.

To this end, consider a bequest game where the $t = 0$, type θ_0 agent can choose his own consumption $c_0(\theta_0)$, his own output $y_0(\theta_0)$, and the amount he bequeathes to each type child, $b_1(\theta^1) = b_1(\theta_0, \theta_1)$, subject only to $b_1(\theta^1) \geq 0$ and $c_0(\theta_0) - y_0(\theta_0) + \frac{1}{R}\sum_{\theta_1} \pi(\theta_1|\theta_0)b_1(\theta^1) \leq 0$. The date $t = 1$, type θ^1 agent then likewise chooses his own consumption $c_1(\theta^1)$, his own output $y_1(\theta^1)$, and the amount he bequeathes to each type of his children, $b_2(\theta^2)$, subject only to $b_2(\theta^2) \geq 0$ and $c_1(\theta^1) - y_1(\theta^1) + \frac{1}{R}\sum_{\theta_2} \pi(\theta_2|\theta_1)b_1(\theta^2) \leq b_1(\theta^1)$. This continues until the date $t = T$, type θ^T agent chooses only his consumption and output, $c_T(\theta^T)$ and $y_T(\theta^T)$, subject to $c_T(\theta^T) - y_T(\theta^T) \leq b_T(\theta^T)$. Payoffs are unchanged. Let the outcome of this full information bequest game be denoted $\{c_t^b(\theta^t), y_t^b(\theta^t)\}_{t=0}^T$. We now prove that the Nash equilibrium outcome of this bequest game is the same as that of the laissez-faire property rights economy. From the proof it is clear that the equilibrium is a backward induction equilibrium.

Proposition 6 *The full information laissez-faire allocation* $\{c_t^\ell(\theta^t), y_t^\ell(\theta^t)\}_{t=0}^T$ *equals the full information bequest game allocation* $\{c_t^b(\theta^t), y_t^b(\theta^t)\}_{t=0}^T$.

Proof. Let $b_t^\ell(\theta^t)$ denote the implicit bequest associated with the laissez-faire allocation, defined, for $t = 0, 1, \ldots, T$, by

$$R\left(c_t^\ell(\theta^t) - y_t^\ell(\theta^t) + b_{t-1}^\ell(\theta^t)\right) = \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta^t)b_t^\ell(\theta^{t+1}) \tag{24}$$

where $b_{-1}^\ell(\theta^0) = b_T^\ell(\theta^{T+1}) = 0$. Solving backward from the T^{th} equation, we get:

$$b_t^\ell(\theta^t) \equiv c_t^\ell(\theta^t) - y_t^\ell(\theta^t) + \sum_{s=t+1}^{T} \frac{1}{R^{s-t}} \tag{25}$$

$$\sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t)\dots\pi(\theta_T|\theta_{T-1})(c_s^{\ell}(\theta^{t+1,s}) - y_s^{\ell}(\theta^{t+1,s})).$$

It is clear that $b_t^{\ell}(\theta^t) \geq 0$ for all (t, θ^t) (otherwise, 20 is violated). Next, consider a date t, type θ^T agent who receives bequest $b_T^{\ell}(\theta^T)$. We claim that he will choose $(c_T^{\ell}(\theta^T), y_T^{\ell}(\theta^T))$: If not, then the date $t = 0$, θ_0 agent could not have been optimizing. Next consider the date $t = T-1$, θ^{T-1} agent who receives bequest $b_{T-1}^{\ell}(\theta^{T-1})$. Since if he leaves bequests $b_T^{\ell}(\theta^T)$ and produces $y_{T-1}^{\ell}(\theta^{T-1})$, he eats $c_{T-1}^{\ell}(\theta^{T-1})$ and his descendants (of each type) consume and produce $(c_T^{\ell}(\theta^T), y^{\ell}(\theta^T))$, the date $t = T-1$, θ^{T-1} agent can implement the laissez-faire allocation from $T-1$ on. Further, he cannot afford, after receiving bequest $b_{T-1}^{\ell}(\theta^{T-1})$, any subsequent allocation he prefers. Otherwise, again, the date $t = 0$, θ_0 agent could not have been optimizing. Continuing to date $t = 0$ proves the result. ■

4 Private Information

With private information, characterization of the set of Pareto efficient allocations is less straightforward. In this section, we present our results in three subsections. In the first subsection, we consider a simplified two-period economy where only one generation, the first, has private skills. Here, we completely characterize the set of Pareto efficient allocations subject to the two-types restriction that $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and show that, like the economy with full information, in constrained efficient allocations, implicit inheritance taxes can be either parent-progressive or parent-regressive, depending on the particular constrained efficient allocation. In particular, we show the result of Farhi-Werning (2010) that optimal implicit inheritance taxes are parent-progressive depends crucially on their assumption that societal preferences toward unborn generations reflect, in their words, a "preference for equality" regarding the consumption of unborn generations. In our environment, in which a child born to a parent of one θ type is considered a different person (for the purpose of Pareto efficiency comparisons) than a child born to a parent of a different θ type, this progressivity result no longer holds.

In the second subsection, we consider the implications of private information more generally. In particular, we show that for $T+1$ period economies where all generations have private skills, the "reciprocal Euler condition" result of Golosov, Kocherlakota, and Tsyvinski (2003) (hereafter, GKT), itself a generalization of Rogerson (1985), holds as a necessary condition for any incentive constrained Pareto efficient allocation, but as an inequality as opposed to an equality as in GKT. This result establishes an interesting parallel to our related result in the full information environment. There, the normal (non-reciprocal) Euler condition $u'(c_t(\theta^t)) \geq \beta R u'(c_{t+1}(\theta^{t+1}))$ holds necessarily as an equality when only the first generation receives positive Pareto weights and holds as an inequality when future generations are given positive direct, history-dependent Pareto weights. Here, for the case of private information, we again show that the exact same necessary condition that holds as an *equality* when only the first generation receives positive Pareto weights — now the reciprocal Euler condition — holds as a weak *inequality* when future generations are given positive direct, history-dependent Pareto weights. This reciprocal Euler condition can then be used, following the logic of Kocherlakota (2005), to show that the expected value of implicit inheritance tax rates, conditional on a parent's history, are weakly negative. That is, while in the case of full information, implicit inheritance tax rates are always weakly negative, with private information they are instead weakly negative in expectation (conditional on a parent's history.)

Finally, we consider the efficiency of the laissez-faire allocation given private information. Here, we first show that if skill-levels θ are i.i.d. over time, then the laissez-faire allocation is Pareto efficient. We then consider the case in which θ is Markov.

4.1 Two-Period Models with Private Information

In this section, we completely characterize the set of constrained Pareto efficient allocations in the two-period version of our economy, with the additional restriction that parent types are restricted to be either low or high.

This is Farhi-Werning's (2010) (hereafter, FW) setup, with one basic exception: in FW, second period agents (children) cannot produce and do not have private productivity types. As a corollary of our characterization we show that implicit inheritance taxes are parent-progressive if the Pareto weights on children are restricted to be independent of their parent's type, as FW essentially assume in their social welfare function approach, but can be either parent-progressive or parent-regressive if the Pareto weights on children can depend on the type of their parent (or children of high types are assumed to be *different people*, for the purposes of Pareto comparisons, than the children of low types).

We begin with a proposition that completely characterizes the set of constrained efficient allocations. We let $MC(\theta) \equiv \frac{f(\theta)}{u'(c_0^*(\theta))}$ (that is, $MC(\theta)$ equals the marginal societal cost of providing utility from consumption to type θ), and we let $M\overline{U}(\theta) \equiv f(\theta)\left(\frac{1}{u'(c_0^*(\theta))} - \frac{\theta}{h'(y^*(\theta)/\theta)}\right)$, (that is, $M\overline{U}(\theta)$ equals the marginal societal cost of increasing both consumption and output for type θ such that the utility for type θ stays constant).

Proposition 7 *For the two-period economy $t \in \{0,1\}$ such that only $t = 0$ agents can produce, an incentive compatible, resource feasible allocation $(c_0^*(\theta), y^*(\theta), c_1^*(\theta))$, $\theta \in \{\underline{\theta}, \overline{\theta}\}$ is constrained Pareto efficient if and only if:*

1. *the resource constraint holds with equality,*

2. *the low type has a weak residual motive to work more, or*

$$u'(c_0^*(\underline{\theta}))\underline{\theta} \geq h'(\frac{y^*(\underline{\theta})}{\underline{\theta}}) \tag{26}$$

(and with equality if the incentive constraint that high types do not wish to falsely announce as low types, $u(c_0^*(\overline{\theta})) - h(\frac{y^*(\overline{\theta})}{\overline{\theta}}) + \beta u(c_1^*(\overline{\theta})) \geq u(c_0^*(\underline{\theta})) - h(\frac{y^*(\underline{\theta})}{\overline{\theta}}) + \beta u(c_1^*(\underline{\theta}))$, is slack),

3. *the high type has a weak residual motive to work less, or*

$$u'(c_0^*(\overline{\theta}))\overline{\theta} \leq h'(\frac{y^*(\overline{\theta})}{\overline{\theta}}) \tag{27}$$

(and with equality if the incentive constraint that low types do not wish to falsely announce as high types,, $u(c_0^*(\underline{\theta})) - h(\frac{y^*(\underline{\theta})}{\underline{\theta}}) + \beta u(c_1^*(\underline{\theta})) \geq u(c_0^*(\overline{\theta})) - h(\frac{y^*(\overline{\theta})}{\underline{\theta}}) + \beta u(c_1^*(\overline{\theta}))$, *is slack),*

4. *for each* $\theta \in \{\underline{\theta}, \overline{\theta}\}$ *the date* $t = 0$ *parent has a weak residual motive to transfer consumption from his child to himself, or*

$$u'(c_0^*(\theta)) \geq \beta R u'(c_1^*(\theta)). \tag{28}$$

5. *As part of the necessary and sufficient conditions for* $(c_0^*(\theta), y^*(\theta), c_1^*(\theta))$ *to be constrained efficient, one needs for* $\theta = \underline{\theta}$, $\theta^c = \overline{\theta}$ *as well as* $\theta = \overline{\theta}$, $\theta^c = \underline{\theta}$,

$$MC(\theta) \geq \frac{M\overline{U}(\theta^c)}{(\frac{\theta^c}{\theta})^{\psi} - 1} - \frac{M\overline{U}(\theta)}{(\frac{\theta}{\theta^c})^{\psi} - 1}. \tag{29}$$

Proof. That the resource constraint must hold as an equality is immediate. (Increasing first period consumption to raise the utility of each type of parent by an equal amount is incentive compatible and Pareto improving.) Thus, $(c_0^*(\theta), h^*(\theta), c_1^*(\theta))$ is Pareto efficient if and only if there exist $\gamma_0(\theta) \geq 0$ and $\gamma_1(\theta) \geq 0$ such that $(\overline{u}_0^*(\theta), \overline{h}^*(\theta), \overline{u}_1^*(\theta)) \equiv (u(c_0^*(\theta)), h(\frac{y^*(\theta)}{\theta}), u(c_1^*(\theta)))$ solve

$$\min_{\overline{u}_0(\theta), \overline{h}(\theta), \overline{u}_1(\theta)} \sum_{\theta} f(\theta)[u^{-1}(\overline{u}_0(\theta)) - \theta h^{-1}(\overline{h}(\theta)) + \frac{1}{R} u^{-1}(\overline{u}_1(\theta))] \tag{30}$$

subject to the incentive conditions that for $\theta = \underline{\theta}$, $\theta^c = \overline{\theta}$ as well as $\theta = \overline{\theta}$, $\theta^c = \underline{\theta}$,

$$\overline{u}_0(\theta) - \overline{h}(\theta) + \beta\overline{u}_1(\theta) \geq \overline{u}_0(\theta^c) - \overline{h}(\theta^c)(\frac{\theta^c}{\theta})^\psi + \beta\overline{u}_1(\theta^c), \tag{31}$$

and

$$\begin{aligned} & \sum_\theta f(\theta)[\gamma_0(\theta)(\overline{u}_0(\theta) - \overline{h}(\theta) + \beta\overline{u}_1(\theta)) + \gamma_1(\theta)\overline{u}_1(\theta)] \\ \geq & \sum_\theta f(\theta)[\gamma_0(\theta)(u_0^*(\theta) - h^*(\theta) + \beta u_1^*(\theta)) + \gamma_1(\theta)u_1^*(\theta)]. \end{aligned} \tag{32}$$

If each of the constraints is subtracted from the objective function to form a Lagrangian (with multipliers $f(\theta)\mu(\theta,\theta^c)$ on the incentive constraints and λ on equation (32)), then the derivatives of this Lagrangian with respect to $\overline{u}_0(\theta)$, $\overline{u}_1(\theta)$, and $\overline{h}(\theta)$ give rise to the following first order conditions:

$$f(\theta)u^{-1\prime}(\overline{u}_0(\theta)) = \frac{f(\theta)}{u'(c_0(\theta))} = f(\theta)\mu(\theta,\theta^c) - f(\theta^c)\mu(\theta^c,\theta) + \lambda f(\theta)\gamma_0(\theta), \tag{33}$$

$$\frac{f(\theta)}{R}u^{-1\prime}(\overline{u}_1(\theta)) = \frac{f(\theta)}{Ru'(c_1(\theta))} = f(\theta)\beta\mu(\theta,\theta^c) - f(\theta^c)\beta\mu(\theta^c,\theta) + \lambda f(\theta)(\beta\gamma_0(\theta) + \gamma_1(\theta)), \tag{34}$$

$$-f(\theta)h^{-1\prime}(\overline{h}(\theta))\theta = -\frac{f(\theta)\theta}{h'(\frac{y(\theta)}{\theta})} = -f(\theta)\mu(\theta,\theta^c) + f(\theta^c)(\frac{\theta}{\theta^c})^\psi\mu(\theta^c,\theta) - \lambda f(\theta). \tag{35}$$

If one adds (33) and (35), one can solve for

$$\mu(\theta,\theta^c) = \frac{f(\theta^c)}{f(\theta)}\frac{\frac{1}{u'(c_0(\theta^c))} - \frac{\theta^c}{h'(\frac{y(\theta^c)}{\theta^c})}}{(\frac{\theta^c}{\theta})^\psi - 1}. \tag{36}$$

If $\mu(\theta,\theta^c) = 0$, this then implies that $u'(c_0(\theta^c))\theta^c = h'(\frac{y(\theta^c)}{\theta^c})$. If $\mu(\theta,\theta^c) > 0$, it is necessary that the sign of the numerator, $\frac{1}{u'(c_0(\theta^c))} - \frac{\theta^c}{h'(\frac{y(\theta^c)}{\theta^c})}$, agrees with the sign of the denominator, $(\frac{\theta^c}{\theta})^\psi - 1$. If $\theta = \underline{\theta}$ and $\theta^c = \overline{\theta}$, then the denominator is positive and thus $\frac{1}{u'(c_0(\overline{\theta})} > \frac{\overline{\theta}}{h'(\frac{y(\overline{\theta})}{\overline{\theta}})}$, or $u'(c_0(\overline{\theta}))\overline{\theta} < h'(\frac{y(\overline{\theta})}{\overline{\theta}})$. If $\theta = \overline{\theta}$ and $\theta^c = \underline{\theta}$, then the denominator is negative and thus

$u'(c_0(\underline{\theta}))\underline{\theta} > h'(\frac{y(\theta)}{\overline{\theta}})$.

Next, if one solves (33) for $f(\theta)\mu(\theta,\theta^c) - f(\theta^c)\mu(\theta^c,\theta)$ and substitutes into (34), one has

$$\frac{1}{u'(c_0(\theta))} = \frac{1}{\beta R u'(c_1(\theta))} - \lambda\gamma_1(\theta). \tag{37}$$

Since $\gamma_1(\theta) \geq 0$, this implies that $u'(c_0(\theta)) \geq \beta R u'(c_1(\theta))$.

Finally, substituting (36) into (33) delivers

$$MC(\theta) - \frac{M\overline{U}(\theta^c)}{(\frac{\theta^c}{\theta})^{\psi} - 1} + \frac{M\overline{U}(\theta)}{(\frac{\theta}{\theta^c})^{\psi} - 1} = f(\theta)\lambda\gamma_0(\theta). \tag{38}$$

Since $f(\theta)\lambda\gamma_0(\theta) \geq 0$, this delivers (29) as a necessary condition.

For sufficiency, note that if one finds an allocation $(c_0(\theta), y(\theta), c_1(\theta))$ satisfying the derived necessary conditions, these solve the Lagrangian above (which in the transformed choice variables has a convex objective function and linear constraints and thus satisfies the Kuhn-Tucker conditions) with $\lambda\gamma_0(\theta) \geq 0$ defined by (38) and $\lambda\gamma_1(\theta) \geq 0$ defined by (37). ■

Taken together, that the resource constraint holding with equality and the more familiar looking conditions (26), (27), and (28) (without condition (29)) are necessary and sufficient for an allocation to be on the frontier of the utility possibilities set. The more unfamiliar looking condition (29) is necessary and sufficient to ensure that the frontier point associated with the allocation is on a downward sloping portion of the utilities possibilities set.

Next we show that implicit inheritance taxes are parent-progressive if the Pareto weight on the child of a high type parent is restricted to be equal to the Pareto weight on the child of a low type parent, or $\gamma_1(\overline{\theta}) = \gamma_1(\underline{\theta})$, but can be parent-progressive or parent-regressive if these Pareto weights are not so restricted.

Proposition 8 *Let $(c_0^*(\theta), y^*(\theta), c_1^*(\theta))$, $\theta \in \{\underline{\theta}, \overline{\theta}\}$ be a constrained efficient allocation for the two period economy $t \in \{0,1\}$ such that only $t = 0$ agents can produce, and let its associated implicit inheritance tax $\tau(\theta)$ solve $u'(c_0^*(\theta)) = \beta R u'(c_1^*(\theta))(1 - \tau(\theta))$. Then, if Pareto weights* *are restricted such that* $\gamma_1(\underline{\theta}) = \gamma_1(\overline{\theta}) > 0$, $\tau(\underline{\theta}) < \tau(\overline{\theta}) < 0$ *(or inheritance taxes are negative and parent-progressive). If instead* $\gamma_1(\underline{\theta}) \geq 0$ *and* $\gamma_1(\overline{\theta}) \geq 0$ *are unrestricted, then inheritance taxes remain (weakly) negative but can be either parent-progressive or parent-regressive.*

Proof. First, note that if one solves for $\tau(\theta)$ from its definition, one derives

$$\tau(\theta) = \frac{\beta R u'(c_1(\theta)) - u'(c_0(\theta))}{\beta R u'(c_1(\theta))}. \tag{39}$$

Next, consider the primal problem

$$\max_{c_0(\theta), y(\theta), c_1(\theta)} \sum_{\theta \in \{\underline{\theta}, \overline{\theta}\}} f(\theta)[\gamma_0(\theta)(u(c_0(\theta)) - h(\frac{y(\theta)}{\theta})) + (\beta\gamma_0(\theta) + \gamma_1(\theta))u(c_1(\theta))], \tag{40}$$

subject to (for $\theta = \underline{\theta}$ and $\theta^c = \overline{\theta}$ as well as for $\theta = \overline{\theta}$ and $\theta^c = \underline{\theta}$)

$$u(c_0(\theta)) - h(\frac{y(\theta)}{\theta}) + \beta u(c_1(\theta)) \geq u(c_0(\theta^c)) - h(\frac{y(\theta^c)}{\theta}) + \beta u(c_1(\theta^c)), \tag{41}$$

and

$$\sum_{\theta \in \{\underline{\theta}, \overline{\theta}\}} f(\theta)[c_0(\theta) - y(\theta) + \frac{1}{R} c_1(\theta)] \leq 0, \tag{42}$$

where $f(\theta)\mu(\theta, \theta^c)$ and λ are the respective Lagrange multipliers.

The necessary first order conditions with respect to $c_0(\theta)$ and $c_1(\theta)$ are

$$f(\theta)\gamma_0(\theta)u'(c_0(\theta)) + f(\theta)\mu(\theta, \theta^c)u'(c_0(\theta)) - f(\theta^c)\mu(\theta^c, \theta)u'(c_0(\theta)) - f(\theta)\lambda = 0 \tag{43}$$

and

$$f(\theta)(\beta\gamma_0(\theta) + \gamma_1(\theta))u'(c_1(\theta)) + \beta f(\theta)\mu(\theta, \theta^c)u'(c_1(\theta)) - \beta f(\theta^c)\mu(\theta^c, \theta)u'(c_1(\theta)) - \frac{1}{R} f(\theta)\lambda = 0 \tag{44}$$

Solving (43) and (44) for $f(\theta)\lambda$ and equating delivers

$$\begin{aligned} & \beta R u'(c_1(\theta))[f(\theta)\gamma_0(\theta) + f(\theta)\mu(\theta,\theta^c) - f(\theta^c)\mu(\theta^c,\theta)] + Rf(\theta)\gamma_1(\theta)u'(c_1(\theta)) \\ = \; & u'(c_0(\theta))[f(\theta)\gamma_0(\theta) + f(\theta)\mu(\theta,\theta^c) - f(\theta^c)\mu(\theta^c,\theta)], \end{aligned} \tag{45}$$

or

$$\begin{aligned} & (\beta R u'(c_1(\theta)) - u'(c_0(\theta)))[f(\theta)\gamma_0(\theta) + f(\theta)\mu(\theta,\theta^c) - f(\theta^c)\mu(\theta^c,\theta)] \\ = \; & -Rf(\theta)\gamma_1(\theta)u'(c_1(\theta)). \end{aligned} \tag{46}$$

Dividing each side by the expression in square brackets and $\beta R u'(c_1(\theta))$ delivers

$$\tau(\theta) \equiv \frac{\beta R u'(c_1(\theta)) - u'(c_0(\theta))}{\beta R u'(c_1(\theta))} = \frac{-f(\theta)\gamma_1(\theta)}{\beta(f(\theta)\gamma_0(\theta) + f(\theta)\mu(\theta,\theta^c) - f(\theta^c)\mu(\theta^c,\theta))}, \tag{47}$$

or

$$\tau(\theta) = \frac{-f(\theta)\gamma_1(\theta)}{\beta(f(\theta)\gamma_0(\theta) + f(\theta)\mu(\theta,\theta^c) - f(\theta^c)\mu(\theta^c,\theta))}. \tag{48}$$

Note from (43) that

$$f(\theta)\gamma_0(\theta) + f(\theta)\mu(\theta,\theta^c) - f(\theta^c)\mu(\theta^c,\theta)) = \frac{f(\theta)\lambda}{u'(c_0(\theta))}. \tag{49}$$

Thus,

$$\tau(\theta) = \frac{-1}{\beta\lambda}\gamma_1(\theta)u'(c_0(\theta)). \tag{50}$$

Next, assume that $\gamma_1(\theta) = \gamma_1$ (a constant) for all θ. This, (43), and (46) then imply that

$$\beta R\left(\frac{1}{u'(c_0(\underline{\theta}))} - \frac{1}{u'(c_0(\overline{\theta}))}\right) = \frac{1}{u'(c_1(\underline{\theta}))} - \frac{1}{u'(c_1(\overline{\theta}))}. \tag{51}$$

Thus, if $c_0(\underline{\theta}) \geq c_0(\overline{\theta})$, then $c_1(\underline{\theta}) \geq c_1(\overline{\theta})$.

Next, from $u'(c_0(\underline{\theta}))\underline{\theta} \geq h'(\frac{y(\underline{\theta})}{\underline{\theta}})$ and $u'(c_0(\overline{\theta}))\overline{\theta} \leq h'(\frac{y(\overline{\theta})}{\overline{\theta}})$, if $c_0(\underline{\theta}) \geq c_0(\overline{\theta})$ then $\frac{y(\underline{\theta})}{\underline{\theta}} < \frac{y(\overline{\theta})}{\overline{\theta}}$. Thus, if $c_0(\underline{\theta}) \geq c_0(\overline{\theta})$, then $c_1(\underline{\theta}) \geq c_1(\overline{\theta})$ and $\frac{y(\underline{\theta})}{\underline{\theta}} < \frac{y(\overline{\theta})}{\overline{\theta}}$, which violates incentive compatibility. Thus, $c_0(\underline{\theta}) < c_0(\overline{\theta})$, which (50) then implies $\tau(\underline{\theta}) < \tau(\overline{\theta})$, or inheritance taxes are progressive when $\gamma_1(\underline{\theta}) = \gamma_1(\overline{\theta}) > 0$.

Finally, if $\gamma_1(\underline{\theta}) = 0$ and $\gamma_1(\overline{\theta}) > 0$, then (50) implies $\tau(\underline{\theta}) = 0$ and $\tau(\overline{\theta}) < 0$, and thus $\tau(\underline{\theta}) > \tau(\overline{\theta})$ or inheritance taxes are regressive. ■

4.2 The Reciprocal Euler Condition and Expected Implicit Inheritance Taxes

In this section we consider the effect of private information for our general $T+1$ period economy where agents of all generations have private information about their type. First, we establish that in all constrained efficient allocations, the reciprocal Euler condition of Golosov, Kocherlakota and Tsvynski (2003) must hold as an *inequality.*[3]

Proposition 9 *For the $T+1$-period economy with $t \in \{0, \ldots, T\}$, an incentive compatible, resource feasible allocation $\{c_t^*(\theta^t), y_t^*(\theta^t)\}_{t=0}^{T}$ is constrained Pareto efficient only if for all $t < T$ and θ^t,*

$$\frac{1}{u'(c_t(\theta^t))} \leq \frac{1}{\beta R} \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t) \frac{1}{u'(c_{t+1}(\theta^{t+1}))}. \tag{52}$$

Proof. Since the resource constraint must hold with equality in any constrained efficient allocation, $\{c_t^*(\theta^t), y_t^*(\theta^t), \}_{t=0}^{T}$ is constraint efficient only if $\{\overline{u}_t^*(\theta^t), \overline{h}_t^*(\theta^t)\}_{t=0}^{T} \equiv \{u(c_t^*(\theta^t)), h(\frac{y_t^*(\theta^t)}{\theta_t})\}_{t=0}^{\infty}$ solves

$$\min_{u_t(\theta^t), h_t(\theta^t)} \sum_{t=0}^{T} \sum_{\theta^t} \frac{1}{R^t} f(\theta_0)\pi(\theta_1|\theta_0)\ldots\pi(\theta_t|\theta_{t-1})(u^{-1}(\overline{u}_t(\theta^t)) - \theta_t h^{-1}(\overline{h}_t(\theta^t))) \tag{53}$$

[3] The reciprocal Euler condition necessarily holding as an inequality, and the implications of this on expected inheritance taxes, is shown in Farhi-Werning (2007) as well. The results here generalize their finding to our definition of constrained Pareto efficiency (where individuals are indexed not just by date, but also by their dynasty's history of type realizations, including their own.)

subject to for all t, θ^t, and $\hat{\theta} \neq \theta_t$

$$\begin{aligned} & \overline{u}_t(\theta^t) - \overline{h}_t(\theta^t) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1})[\overline{u}_s(\theta^s) - \overline{h}_s(\theta^s)] \\ \geq \; & \overline{u}_t(\theta^{t-1}, \hat{\theta}) - (\frac{\hat{\theta}}{\theta_t})^{\psi} \overline{h}_t(\theta^{t-1}, \hat{\theta}) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1}) \\ & [\overline{u}_s(\theta^{t-1}, \hat{\theta}, \theta^{t+1,s}) - \overline{h}_s(\theta^{t-1}, \hat{\theta}, \theta^{t+1,s})], \end{aligned} \tag{54}$$

and for all t and θ^t

$$\begin{aligned} & \overline{u}_t(\theta^t) - \overline{h}_t(\theta^t) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1})[\overline{u}_s(\theta^s) - \overline{h}_s(\theta^s)] \\ \geq \; & \overline{u}^*_t(\theta^t) - \overline{h}^*_t(\theta^t) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1})[\overline{u}^*_s(\theta^s) - \overline{h}^*_s(\theta^s)]. \end{aligned} \tag{55}$$

Next note that if, for a particular $t < T$ and θ^t, one perturbs $\{\overline{u}^*_t(\theta^t), \overline{h}^*_t(\theta^t)\}_{t=0}^T$ by decreasing $\overline{u}_t(\theta^t)$ by $\Delta \geq 0$ and increasing $\overline{u}_{t+1}(\theta^{t+1})$ by Δ/β for all θ_{t+1}, and otherwise leaving the allocation unchanged, the perturbed policy remains in the constraint set. Further, it affects the objective function only in the terms for dates t and $t+1$ following history θ^t. Thus, if $\{c^*_t(\theta^t), y^*_t(\theta^t)\}_{t=0}^T$ is constrained Pareto efficient, a choice of $\Delta = 0$ must solve

$$\min_{\Delta} u^{-1}(\overline{u}^*_t(\theta^t) - \Delta) + \frac{1}{R} \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t) u^{-1}(\overline{u}^*_{t+1}(\theta^{t+1}) + \Delta/\beta) \tag{56}$$

subject to $\Delta \geq 0$. The necessary first order condition of this problem with respect to Δ (where μ is the multiplier on the constraint) is

$$-\frac{1}{u'(c_t(\theta^t))} + \frac{1}{\beta R} \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t) \frac{1}{u'(c_{t+1}(\theta^{t+1}))} = \mu. \tag{57}$$

That $\mu \geq 0$ then proves the result. ■

Note that unlike Rogerson (1985) and Golosov, Kocherlakota, and Tsyvinski (2003), where the reciprocal Euler condition holds as an equality as opposed to an inequality here, efficiency no longer implies a residual motive to save (or in our context, to make larger bequests) as in Rogerson (1985, page 70). That is, if (52) holds as an equality, one can use Jensen's inequality and the fact that $\frac{1}{x}$ is a convex function to show that $u'(c_t(\theta^t)) < \beta R \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t) u'(c_{t+1}(\theta^{t+1}))$. However, if (52) holds as a strict inequality, then it may be that $u'(c_t(\theta^t)) > \beta R \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t) u'(c_{t+1}(\theta^{t+1}))$, or that a date t parent of type θ^t has a residual motive to make a *negative* unconditional bequest. This is not surprising since our set of efficient allocations includes those which put high weight on these children.

The inequality form of the reciprocal Euler condition (52) can nevertheless be used to sign the conditional expected value of implicit inheritance tax rates. To consider implicit inheritance taxes when children have private types, however, we must first establish that our definition of implicit inheritance taxes continues to make sense when there is private information about the child's type. To do this, we consider the following implementation: first, endow each agent with a property right to the incentive constrained optimal plan, $\{c_t^*(\theta^t), y_t^*(\theta^t)\}_{t=0}^t$, but then allow each parent to make a positive or negative *unconditional* gift x to each of his possible children facing the following tax structure: If a date t parent of type θ^t gifts x, then his child of type θ^{t+1} receives transfer $xR(1-\tau_{t+1}(\theta^{t+1}))$. That is, the amount of inheritance Rx the child gets to keep depends on the child's announcement θ_{t+1}. Next, choose these taxes such that for all $t+1 \leq T$ and θ^{t+1},

$$u'(c_t^*(\theta^t)) = \beta R u'(c_{t+1}^*(\theta^{t+1}))(1-\tau_{t+1}(\theta^{t+1})), \tag{58}$$

which is the same implicit inheritance tax structure defined earlier. Such a tax system ensures that a choice of $x = 0$ is optimal for a type θ^t parent regardless of his beliefs regarding the reporting strategy of his children and thus implements the constrained efficient allocation. Given this, and following the logic of Kocherlakota (2005), the reciprocal Euler condition (52) and the definition of implicit taxes, (58), together imply

$$1 \leq \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t)(1 - \tau_{t+1}(\theta^{t+1})), \tag{59}$$

or

$$0 \geq \sum_{\theta_{t+1}} \pi(\theta_{t+1}|\theta_t)\tau_{t+1}(\theta^{t+1}). \tag{60}$$

That is, the expected value of $\tau_{t+1}(\theta^{t+1})$, conditional on θ^t, is weakly negative for all t and θ^t.

For the $T+1$ period economy, the counterintuitive results of Kocherlakota (2005) go through — having to provide incentives for generations born after $t = 0$ to truthfully reveal their type causes child-regressive inheritance taxes. To see this, note

$$\tau_{t+1}(\theta^{t+1}) = 1 - \frac{u'(c_t(\theta^t))}{\beta R u'(c_{t+1}(\theta^{t+1}))}. \tag{61}$$

Thus holding the consumption of a parent constant, the implicit inheritance tax rate is higher the lower the consumption of the child. Children having private information about their skill types makes the consumption of higher skilled children higher and thus causes child-regressive inheritance taxes. For reasons similar to what we show for two period economies, for the general $T+1$ period economy with private information, indexing agents by both dates and histories, as opposed to indexing them only by dates as in Farhi-Werning (2010), overturns their result that $\tau_{t+1}(\theta^{t+1})$ is necessarily parent-progressive.

4.3 Private Information Property Rights Implementation

In this section, we follow the results established under full information. In particular, we establish conditions under which no taxes of any kind are necessary to achieve Pareto efficiency.

To this end, first note that for the static ($T = 0$) economy, the private information laissez-faire allocation and the full information laissez-faire allocation are identical, since the static full information laissez-faire allocation is incentive compatible.

Next, as was the case with full information, for $T > 0$, recursively define

$$\begin{aligned} V^{\ell,T}(\theta_0) \;\; &\equiv \;\; \max_{c_0, y_0, \{c_t(\theta^{1,t}), y_t(\theta^{1,t})\}_{t=1}^{T}} u(c_0) - h(\frac{y_0}{\theta_0}) + \sum_{t=1}^{T} \beta^t \sum_{\theta^{1,t}} \pi(\theta_1|\theta_0) \ldots \pi(\theta_t|\theta_{t-1}) \\ & [u(c_t(\theta^{1,t})) - h(\frac{y_t(\theta^{1,t})}{\theta_t})], \end{aligned} \tag{62}$$

subject to a budget condition

$$c_0 - y_0 + \sum_{t=1}^{T} \frac{1}{R^t} \sum_{\theta^{1,t}} \pi(\theta_1|\theta_t) \ldots \pi(\theta_t|\theta_{t-1})(c_t(\theta^{1,t}) - y_t(\theta^{1,t})) \leq 0, \tag{63}$$

the no-enslaving-your-descendants condition, for all $t \geq 1$ and $\theta^{1,t}$,

$$\begin{aligned} & u(c_t(\theta^{1,t})) - h(\frac{y_t(\theta^{1,t})}{\theta_t}) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1}) \\ & [u(c_s(\theta^{1,t}, \theta^{t+1,s})) - h(\frac{y_s(\theta^{1,t}, \theta^{t+1,s})}{\theta_s})] \geq V^{\ell,T-t}(\theta_t). \end{aligned} \tag{64}$$

and, finally, an incentive condition such that a date t, type θ^t agent's descendants must be willing to truthfully reveal their types (assuming all of their descendants truthfully reveal), or for all $t \geq 1$, $\theta^{1,t}$, and $\hat{\theta}$,

$$\begin{aligned} & u(c_t(\theta^{1,t})) - h(\frac{y_t(\theta^{1,t})}{\theta_t}) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1}) \\ & [u(c_s(\theta^{1,t}, \theta^{t+1,s})) - h(\frac{y_n(\theta^{1,t}, \theta^{t+1,s})}{\theta_s})] \;\; \geq \\ & u(c_t(\theta^{1,t-1}, \hat{\theta})) - h(\frac{y_t(\theta^{1,t-1}, \hat{\theta})}{\theta_t}) + \sum_{s=t+1}^{T} \beta^{s-t} \sum_{\theta^{t+1,s}} \pi(\theta_{t+1}|\theta_t) \ldots \pi(\theta_s|\theta_{s-1}) \\ & [u(c_s(\theta^{1,t-1}, \hat{\theta}, \theta^{t+1,s}) - h(\frac{y_s(\theta^{1,t-1}, \hat{\theta}, \theta^{t+1,s})}{\theta_s})]. \end{aligned} \tag{65}$$

As with full information, once $V^{\ell,S}(\theta_0)$ is thus calculated for all $S \leq T$, the laissez-faire allocation with private information for the $T+1$ period economy corresponds to the solution of the problem defining $V^{\ell,T}(\theta_0)$. Thus the definition of the laissez-faire allocation with private information is the same as that with full information except that at each step of the derivation (from $T = 0$ backward), the date $t = 0$, type θ_0 agent faces a constraint that the plan he chooses for his descendants much induce them to truthfully reveal their types.

While this allocation is mathematically well defined, there are two complications associated with θ_t being private information: First, in interpreting the budget condition at each step in the derivation (which in turn defines the right hand side of no-slavery condition) we are implicitly assuming that a date t, type θ^t agent can make actuarily fair conditional transfers to his descendants conditional on his *true* type. Second, note that nowhere in this definition of the private information laissez-faire allocation is a condition that a date $t = 0$, θ_0 type agent be willing to truthfully reveal his type. If he isn't, then $V^{\ell,T-t}(\theta_t)$ on the right hand side of the no-slavery condition does not necessarily represent the utility a date t, type θ_t agent could achieve on his own when having a property right to his own production and choosing an incentive compatible, resource-feasible plan for himself and his descendants, when these descendants have these same rights. The incentive compatibility of the laissez-faire allocation for date $t = 0$ agents thus needs to be checked after-the-fact and presents the need for additional assumptions to ensure truth-telling at each step of the definition of the laissez-faire allocation by the date $t = 0$ agents.

To handle each of these complications, we consider two sets of alternative additional assumptions. The first case is where $\pi(\theta|\theta^-)$ is i.i.d. (or $\pi(\theta|\theta_0^-) = \pi(\theta|\theta_1^-) \equiv \pi(\theta)$ for all (θ_0^-, θ_1^-).) If $\pi(\theta|\theta^-)$ is i.i.d., the definition of an actuarily fair conditional transfer is independent of the parent's type. (His true type is irrelevant). Further, it directly implies that an allocation chosen by a date $t = 0$, θ_0 agent is in the constraint set of the problem of a date $t = 0$, $\hat{\theta}_0$ agent for all $(\theta_0, \hat{\theta}_0)$. These two facts then allow us to prove the following proposition.

Proposition 10 *If $\pi(\theta|\theta^-)$ is i.i.d., the private information laissez-faire allocation is incentive compatible and constrained Pareto efficient.*

Proof. The proof is essentially identical to the full information case. It is straightforward to show that the laissez-faire allocation with private information has (63) hold as an equality for each θ_0 type. (Otherwise, each θ_0 type could simply consume the extra resources himself without upsetting incentive compatibility.) So as with full information, under the private information laissez-faire allocation, the expected discounted dynastic consumption of each θ_0 type equals its expected discounted dynastic output.

Next suppose the private information laissez-faire allocation is incentive compatible (including incentive compatible for date $t = 0$ agents) and there exists another feasible incentive-compatible allocation that Pareto dominates the private information laissez-faire allocation. Again, that $\beta > 0$ and $\pi(\theta) > 0$ for all θ implies that at least one first generation θ_0 type agent is strictly better off in this Pareto-improving allocation. Further, since the allocation is a Pareto improvement, it satisfies (64) for all $t \geq 1$ and $(\theta^{1,t})$. Finally, since the allocation is incentive compatible, it satisfies (65) for all $t \geq 1$ and $(\theta^{1,t})$. Thus, it must be the case that for this first generation θ_0 type, (63) is violated, otherwise the first generation θ_0 type would have chosen it. That is, under the Pareto-improving allocation, the expected discounted dynastic consumption of at least one θ_0 type exceeds its expected discounted dynastic output. The society-wide resource constraint then implies that for at least one θ_0 type, say $\hat{\theta}_0$, its expected discounted dynastic output exceeds its expected discounted dynastic consumption. Further, since the new plan is an incentive-compatible Pareto improvement, conditions (64) and (65) are satisfied for the $\hat{\theta}_0$ type's problem. Thus, the $\hat{\theta}_0$ type could have chosen his part of the Pareto-improving allocation with (63) holding as a strict inequality.

Thus the only remaining possibility with no contradiction is that the private information laissez-faire allocation is not incentive compatible at date $t = 0$. At each step of the construction of the laissez-faire allocation, the incentive compatibility of *future* ($t \geq 1$) generations is already assured by (65) holding. The incentive compatibility of the date $t = 0$

generation is ensured by the fact that since $\pi(\theta)$ is i.i.d., the allocation chosen by any date $t = 0$ type $\hat{\theta}_0 \neq \theta_0$ agent is in the constraint set of the problem for the θ_0 type. Thus, if the date $t = 0$, θ_0 agent prefers the allocation chosen by the date $t = 0$, type $\hat{\theta}_0$ agent, he could have chosen it, establishing a contradiction. ■

Next, we consider an alternative set of assumptions that ensure that the private information laissez-faire allocation is constrained Pareto efficient. First, we assume that while a given θ type can claim to be a *lower* type than his true type, the reverse is not true. In essence, for the next proposition, we assume the existence of a skills test that agents can purposely do worse than they are capable of if they wish to pretend to be a lower type than their true type, but cannot do better than the outcome associated with their true type. In this case, we call an allocation incentive compatible if all types weakly prefer the allocation associated with the truth versus that of any lower type, with no corresponding condition that given types prefer not to claim to be higher types. Further, we assume if $\theta_1 > \theta_2$, $\pi(.|\theta_1)$ first-order dominates $\pi(.|\theta_2)$, or higher θ parents tend to have higher θ children.

Proposition 11 *For the $T+1$ period economy, consider the private information laissez-faire allocation $\{c_t^{\ell,T}(\theta^t), y_t^{\ell,T}(\theta^t)\}_{t=0}^T$ given agents can announce types only weakly lower than their true type and $\pi(.|\theta_2)$ first-order dominates $\pi(.|\theta_1)$ for all $\theta_2 > \theta_1$. Further, for all θ^1 let*

$$b^{\ell,T}(\theta_0, \theta_1) \equiv c_1^{\ell,T}(\theta^1) - y_1^{\ell,T}(\theta^1) + \sum_{t=2}^{T} \frac{1}{R^{t-1}} \sum_{\theta^{2,t}} \pi(\theta_2|\theta_1) \dots \pi(\theta_t|\theta_{t-1})(c_t^{\ell,T}(\theta^1, \theta^{2,t}) - y_t^{\ell,T}(\theta^1, \theta^{2,t})). \tag{66}$$

Suppose for all $S \leq T$ and θ_0, $b^{\ell,S}(\theta_0, \theta_1)$ is weakly decreasing in θ_1. Then for all $S \leq T$, the private information laissez-faire allocation $\{c_t^{\ell,S}(\theta^t), y_t^{\ell,S}(\theta^t)\}_{t=0}^S$ is incentive compatible and constrained Pareto Efficient.

Proof. First note that $b^{\ell,T}(\theta_0, \theta_1)$ is the implicit bequest a date $t = 1$, type θ^1 agent receives from his date $t = 0$, type θ_0 parent. Thus $b^{\ell,T}(\theta_0, \theta_1)$ being weakly decreasing in θ_1

is equivalent to each type of date $t = 0$ parent giving weakly larger implicit bequests to less skilled children.

That $b^{\ell,S}(\theta_0, \theta_1)$ is weakly decreasing in θ_1 for all $S \leq T$ and that $\pi(.|\theta_1)$ first-order dominates $\pi(.|\theta_2)$ if $\theta_1 > \theta_2$ directly implies that the allocation chosen by the $t = 0$, θ_0 type agent for himself and his descendants (for any S) is in the constraint set of every date $t = 0$, $\hat{\theta}_0 > \theta_0$ agent (satisfies conditions (63), (64) and (65)). This immediately implies then that $\{c_t^{\ell,S}(\theta^t), y_t^{\ell,S}(\theta^t)\}_{t=0}^S$ is incentive compatible, and the arguments from Proposition 10 then imply constrained Pareto efficiency. ■

Note Proposition 11 is conditional not just on assumptions about primitives (first order stochastic dominance and the ability to only announce types lower than one's true type) but also on an endogenous characteristic of the laissez-faire contract – that implicit bequests are monotonically decreasing in a child's type. However, this characteristic can be checked in any computed example. We have been unable to generate any computed examples where such monotonicity is violated.

4.4 Private Information Bequest Implementation

In the case where θ_t is privately observed, decentralization through bequests requires a more complicated bequest game than in the full information case. This is clear because if an agent with history θ^t can only state a vector of (θ^t, θ_{t+1})-dependent bequests, then the only incentive compatible bequests are bequests constant over θ_{t+1}: if any difference exists, all children will choose the bequest with largest value, regardless of their true types. Further, as shown in Golosov, Tsyvinski, Werning (2007), a robust characteristic of optimal Mirrleesian taxation schemes is their *time inconsistency*, implying that some restrictions must be put on descendants to limit full reoptimization. That is, a re-optimizing descendant would generally not choose to continue the allocation chosen by his ancestors since he will not consider the effect of changes on his own incentives to truthfully reveal his type or those of his ancestors.[4]

[4] Albanesi and Sleet (2006) show that assuming types are i.i.d. can allow tax systems (or bequest plans in our environment) to be *simple*, but they assume full commitment by the government to its tax plan.

In this section, we implement the laissez-faire allocation through a bequest game by allowing each agent to make conditional bequests, but also allow them to put additional restrictions on their child's own output, consumption and bequest behavior. The flavor of these restrictions is that if a child agrees to accept the bequest associated with his being a type θ, he agrees to choose his consumption, output and own bequests so that together, they are sufficiently unattractive to a different type $\hat{\theta}$ claiming to be a type θ.

To formulate these restrictions, consider the choice problem of a date T, type θ agent with inheritance b who faces a vector of utility restrictions parameterized by an $N-1$ length vector of utilities w, one for each $\hat{\theta} \neq \theta$. Let

$$V_T^T(\theta, b, w) = \max_{c,y} u(c) - h(\frac{y}{\theta}) \tag{67}$$

subject to a budget constraint,

$$c \leq y + b, \tag{68}$$

and a *threat keeping constraint* (similar to that in Fernandes and Phelan (2000)), for all $\hat{\theta} \neq \theta$

$$u(c) - h(\frac{y}{\hat{\theta}}) \leq w(\hat{\theta}). \tag{69}$$

Here, $V_T^T(\theta, b, w)$ is the value to a date T, θ type who receives an inheritance b, but must choose a (c, y) plan that gives each last period $\hat{\theta} \neq \theta$ type no more than $w(\hat{\theta})$ utils. Note that if $w = \vec{\infty}$ (where $\vec{\infty}$ represents an $N-1$ length vector of $+\infty$ values) and $b = 0$, then $V_T^T(\theta, 0, \vec{\infty}) = V^{\ell,0}(\theta)$, the value of the static laissez-faire allocation.

For $0 \leq t < T$, consider the choice problem of a date t, type θ agent with inheritance b, and an $N-1$ length vector of utility restrictions w, who chooses c and y as well as an N length vector of bequests (one for each possible type of child), B, and N, $N-1$ length vectors of utility restrictions, W (one $N-1$ length vector for each possible child type). Here,

$$V_t^T(\theta, b, w) = \max_{c,y,B,W} u(c) - h(\frac{y}{\theta}) + \beta \sum_{\theta'} \pi(\theta'|\theta) V_{t+1}^T(\theta', B(\theta'), W(\theta')) \tag{70}$$

subject to a budget constraint

$$c + \frac{1}{R} \sum_{\theta'} \pi(\theta'|\theta) B(\theta') \leq y + b, \tag{71}$$

an incentive constraint requiring for all $(\theta', \hat{\theta}')$,

$$V_{t+1}^T(\theta', B(\theta'), W(\theta')) \geq W(\hat{\theta}')(\theta'), \tag{72}$$

(where $W(\hat{\theta}')(\theta')$ is the utility restriction next period on an agent who announces $\hat{\theta}'$ regarding the utility received by a θ' agent), a no-slavery constraint requiring for all θ',

$$V_{t+1}^T(\theta', B(\theta'), W(\theta')) \geq V^{\ell, T-(t+1)}(\theta'). \tag{73}$$

and a threat keeping constraint requiring for all $\hat{\theta} \neq \theta$,

$$u(c) - h(\frac{y}{\hat{\theta}}) + \beta \sum_{\theta'} \pi(\theta'|\hat{\theta}) V_{t+1}^T(\theta', B(\theta'), W(\theta')) \leq w(\hat{\theta}). \tag{74}$$

Here, $V_t^T(\theta, b, w)$ is the value to a date t, type θ agent who receives a request b and must choose a plan which he can afford, in which each type child is willing to truthfully announce his type, in which each type child receives a lifetime payoff weakly greater than that of the laissez-faire allocation, and finally, which gives a payoff of no more than $w(\hat{\theta})$ to a date t, type $\hat{\theta}$ agent.

Our claim is that a date $t = 0$, type θ_0 agent (who has no inheritance, and thus $b = 0$, and no utility restrictions, and thus $w = \vec{\infty}$) will choose his own consumption, output, conditional bequests and conditional utility restrictions on his children, so that when they do likewise for their children (and so on) with possibly non-zero bequests and non-trivial utility restrictions, they will implement the laissez-faire allocation.

To see this, consider the two date ($T = 1$) environment. Suppose a date $t = 1$, type θ_1's father announced type θ_0 and specified a menu of bequests and utility restrictions (b, w) such that $b(\theta_1) = c_1^{\ell,1}(\theta_0, \theta_1) - y_1^{\ell,1}(\theta_0, \theta_1)$ (for all θ_1) and $w(\theta_1)(\hat{\theta}_1) = u(c_1^{\ell,1}(\theta_0, \hat{\theta}_1)) - h(\frac{y_1^{\ell,1}(\theta_0,\hat{\theta}_1)}{\hat{\theta}_1})$ (for all $(\theta_1, \hat{\theta}_1)$). Next suppose the date $t = 1$, type θ_1 announces his type truthfully and sets his consumption and output equal to $c_1^{\ell,1}(\theta_0, \theta_1)$ and $y_1^{\ell,1}(\theta_0, \theta_1)$, respectively. By construction, his budget condition (68) will be satisfied with equality. Further, his threat keeping constraint (69) is satisfied due to the incentive compatibility of the laissez-faire allocation. Next suppose the date $t = 1$, type θ_1 agent can choose a (c, y) combination that satisfies (68) and (69) that gives him a higher payoff. This contradicts the laissez-faire allocation solving the $t = 0$, θ_0 agent's optimization problem. Finally, the strict incentive compatibility of the laissez-faire allocation implies the date $t = 1$, θ_1 agent prefers to announce θ_1 truthfully. Thus the $t = 0$, θ_0 agent can implement the laissez-faire allocation by choosing $c = c_0^{\ell,1}(\theta_0)$, $y = y_0^{\ell,1}(\theta_0)$, $B(\theta_1) = c_1^{\ell,1}(\theta_0, \theta_1) - y_1^{\ell,1}(\theta_0, \theta_1)$ (for all θ_1) and $W(\theta_1)(\hat{\theta}_1) = u(c_1^{\ell,1}(\theta_0, \hat{\theta}_1)) - h(\frac{y_1^{\ell,1}(\theta_0,\hat{\theta}_1)}{\hat{\theta}_1})$ (for all $(\theta_1, \hat{\theta}_1)$). What remains is to show that the $t = 0$, θ_0 agent cannot choose a different (c, y, B, W) to achieve a higher payoff. Suppose he could. This would imply a consumption/output plan for himself and each child type $(c_0, y_0, c_1(\theta_1), y_1(\theta_1))$, where c_0 and y_0 are chosen directly by the $t = 0$ agent and $c_1(\theta_1)$ and $y_1(\theta_1)$ are chosen by his child of type θ_1. That $(c_0, y_0, c_1(\theta_1), y_1(\theta_1))$ satisfies (71) and (73) implies $(c_0, y_0, c_1(\theta_1), y_1(\theta_1))$ satisfies (63) and (64). Since if (c, y, B, W) increases the payoff to the date $t = 0$, θ_0 agent, it increases the objective function for the $T = 1$ laissez-faire allocation as well. Thus $(c_0, y_0, c_1(\theta_1), y_1(\theta_1))$ must violate (65) for some $(\theta_1, \hat{\theta}_1)$. But this directly implies the incentive constraint (72) is violated, a contradiction. The extension to economies with $T > 1$ is straightforward.

The basic intuition behind this result is that introducing additional constraints on ex-post reoptimizers can solve time consistency problems but still allow a step-by-step, or recursive, implementation. In our implementation, these additional constraints take the form of restrictions on the attractiveness of any given plan for off-path types. Regarding lessons for actual inheritance laws, we see this result as an argument for laws which give parents flexibility in designing enforceable arrangements regarding bequests to their children. Arrangements that may appear as if venal parents are trying to selfishly control their children may instead be efficient responses of altruistic parents who want to differentiate gifts to their children based on their innate characteristics, but need to worry about incentives.

5 Conclusion

In this paper, we have introduced a natural method of considering Pareto efficiency in dynamic economies with altruism toward children and uncertainty regarding their skill. We argued that under this method, efficiency requires taking into account the desire of individuals to insure against uncertainty, but only against the real uncertainty they face, for example, that of parents regarding the skill endowments of their children. Instead, this concept does not require individuals to insure against events that have already happened, and that characterize who they are. For example, there is no uncertainty on a child's part regarding the realization of his skill. If we look at the characterization of efficiency in terms of the Pareto weights on individuals used by the social planner, our definition is equivalent to one in which the weights over an individual may depend on the history of type realizations of his dynasty until his birth date. This is a natural application of the interim concept of efficiency (Muench (1975), Holmström and Myerson (1983)) in our environment.

By adopting this definition, we extend the set of Pareto efficient allocations. That this set of efficient allocations is strictly larger is easy to see in the case of complete information economies. The set of efficient allocations for these economies can be completely characterized. An allocation is efficient in this case if and only if every parent would not want to make, at that allocation, direct transfers to any of his child's type realizations.

Although this is a substantial extension of the efficient set, substantial restrictions are still imposed by the efficiency requirement, and they are naturally induced by the altruism of the parent. For the general case of private information we show that Pareto efficiency does not require any specific form of government intervention, such as progressive inheritance taxes, to correct market failures. In particular progressive inheritance taxation cannot be justified on efficiency grounds alone.

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