Who's afraid of aggregating money metrics?*

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Abstract. We provide an axiomatic justification to aggregate money metrics. The key axiom requires the approval of richer-to-poorer transfers that preserve the overall efficiency of the distribution. This transfer principle, together with the basic axioms anonymity, continuity, monotonicity, and a version of welfarism, characterizes a standard social welfare function defined over money metric utilities.

Keywords. Money metric utility · Transfer principle · Efficiency

JEL classification. D61 · D63 · D71 · I31

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1 Introduction

The money metric utility of an individual is the minimum income, computed at reference prices, that she needs to reach a bundle that is at least as good as her actual bundle (McKenzie, 1957; Samuelson, 1974). Money metric utility forms the basis of applied welfare analysis. It is for example standard practice to evaluate policy reforms by the change in money metric utility using as reference prices the pre-reform prices (the Hicksian equivalent variation) or post-reform prices (the Hicksian compensating variation).[1]

However, several theoretical objections have been raised against the aggregation of money metrics.[2] The most powerful critique came from Blackorby and Donaldson (1988). They show that the money metric utility function is in general not concave. This implies that a standard (quasiconcave) social welfare function defined over money metrics may fail to approve transfers from richer to poorer individuals.

Figure 1 illustrates the problem. Individuals 1 and 2 have identical preferences over the goods a and b. A bundle δ is transferred from the richer individual 2 to the poorer individual 1.[3] The distances between the straight lines represent the changes in money metric utility (for some reference price vector). Clearly, the transfer is leaky: the gain in money metric utility of the poorer individual is smaller than the loss of the richer individual. Therefore, only a social welfare function exhibiting a sufficiently high degree of inequality aversion would approve the depicted transfer. Moreover, by changing the shape of the indifference curves, the leak can be made arbitrarily large. This means that no social welfare function approves all richer-to-poorer transfers, with the exception of Rawlsian social welfare functions—such as maximin or leximin—that assign absolute priority to the poorer of the two individuals.

This observation has given rise to two far-reaching and opposing responses. Blackorby and Donaldson (1988, p. 129) conclude negatively, stating that "social welfare analysis based on money metrics is flawed." Fleurbaey and Maniquet (2011, p. 21), by contrast, conclude that "this observation ..., instead of undermining the approach, can serve to justify the maximin or the leximin as aggregation criteria." Although the two responses are diametrically opposed, they share the premise that the approval of richer-to-poorer transfers is an essential requirement for all social welfare rankings.

We question this premise. We argue that not every richer-to-poorer transfer is an unequivocal improvement. Such a transfer, while improving equity, may have the side effect of worsening the overall efficiency of the distribution. To see this, note that the transfer in Figure 1 transforms an efficient distribution—with equal marginal rates of

[1] See Slesnick (1998) for an overview.

[2] See Fleurbaey (2009, pp. 1052-1055) for a discussion.

[3] The transfer depicted in Figure 1 yields post-transfer bundles that are convex combinations of the pre-transfer bundles. Blackorby and Donaldson (1988) impose this as a restriction on richer-to-poorer transfers. This restriction is not essential, however, and we will not impose it in our analysis.

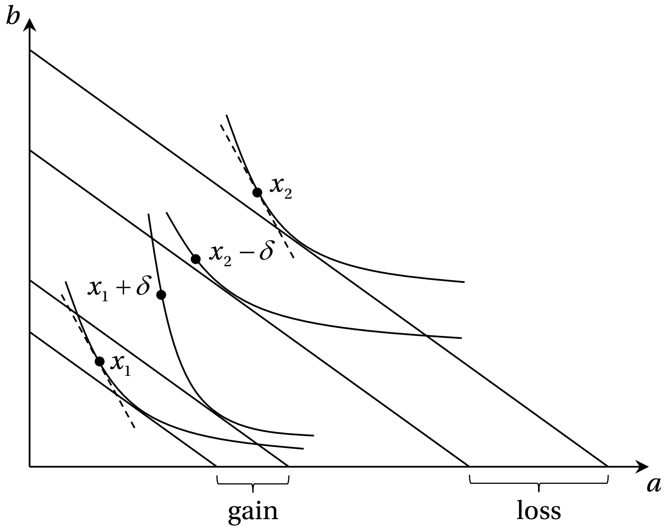

Figure 1. A transfer in goods that is leaky in money metrics

substitution—into an inefficient distribution. Hence, the judgment of whether a particular transfer improves social welfare depends on the position one takes with respect to the equity-efficiency trade-off. By insisting that all transfers must be approved, regardless of the associated efficiency losses, Blackorby and Donaldson (1988) and Fleurbaey and Maniquet (2011) implicitly take the extreme stance that gives absolute priority to equity over efficiency. In this light, it is not surprising that they arrive at such strong conclusions.

We introduce a transfer principle that requires to approve only those transfers that preserve efficiency. This obviously requires a way to measure efficiency. Rather than choosing among the many efficiency measures that have been proposed in the literature—see Diewert (1985) for an overview—we focus on what they have in common. All these measures quantify efficiency by what could be disposed of without lowering any individual's utility. Formally, they measure the distance between the actual societal bundle (listing the total amounts of all goods) and the Scitovksy boundary (collecting the minimum societal bundles that can deliver to each individual the same utility level as her actual utility level). We define an efficiency-preserving transfer as a transfer that changes neither the societal bundle, nor the Scitovksy boundary. All efficiency measures unanimously agree that such a transfer preserves efficiency. Our transfer principle demands that only efficiency-preserving transfers have to be approved.[4]

[4]Chambers and Hayashi (2012) also make use of the Scitovsky boundary to study social welfare rankings, but for a very different purpose. Their interest lies in informational parsimony. They require the social ranking of two distributions to depend solely on the aggregate data contained in the corresponding Scitovsky boundaries. This excludes any concern for equity, which is precisely our focus.

We combine the efficiency-preserving transfer principle with the basic axioms anonymity, continuity, monotonicity, and a version of welfarism. Our main result has two implications. First, a continuous, strictly increasing, and Schur-concave social welfare function defined over money metric utilities satisfies all five axioms.[5] Contrary to the conclusion of Blackorby and Donaldson (1988), the use of money metrics in social welfare analysis can be justified. In particular, since any standard social welfare function over money metrics is admissible, it is not necessary—contrary to the conclusion of Fleurbaey and Maniquet (2011)—to adopt a Rawlsian social welfare function. Second, and more strikingly, the opposite is also true: only if the social ranking can be represented in this particular form, then it satisfies all axioms. In sum, we show not only that one can, but also that one must aggregate money metrics.

We proceed as follows. Section 2 introduces notation and the five axioms. Section 3 presents and discusses the main result. Section 4 concludes.

2 Axioms

2.1 Preliminaries

The set of individuals in society is $N = \{1, 2, \dots, n\}$ with $n \geq 2$. Each individual i has a bundle x_i in $\mathcal{X} = \mathbb{R}^m_+$ with $m \geq 2$. For two bundles x and y in $\mathcal{X}$, we write $x \geq y$ if $x_k \geq y_k$ for each $k = 1, 2, \dots, m$, we write $x > y$ if $x \geq y$ and $x \neq y$, and we write $x \gg y$ if $x_k > y_k$ for each $k = 1, 2, \dots, m$. We denote the boundary of the set $A \subseteq \mathcal{X}$ by ∂A.[6] The sum of two subsets A and B of $\mathcal{X}$ is defined to be the set of all sums of an element of A and an element of B. That is, $A + B = \{z \in \mathcal{X} \mid z = x + y \text{ with } x \text{ in } A \text{ and } y \text{ in } B\}$.

Each individual i has a preference relation R_i over bundles in $\mathcal{X}$. As usual, $x \, R_i \, y$ means that bundle x is at least as good as bundle y according to individual i, whereas P_i and I_i denote the corresponding strict preference and indifference relations. We write $x \, R_i \, A$ to denote that bundle x is at least as good as all bundles in set A according to individual i. For a bundle-preference pair (x_i, R_i), the better-than set is $B(x_i, R_i) = \{y \in \mathcal{X} \mid y \, R_i \, x_i\}$. We sometimes use B_i as shorthand for $B(x_i, R_i)$. Individual preferences belong to $\mathcal{R}$, the set of complete, transitive, continuous, monotone, and convex preference relations.[7]

A distribution $X = (x_1, x_2, \dots, x_n)$ in $\mathcal{X}^n$ contains a bundle for each individual in N.

[5] Schur-concavity is a weak version of concavity that is standard in the literature on inequality measurement. See, e.g., Dasgupta, Sen, and Starrett (1973).

[6] Let $\|x - y\|$ be the Euclidean distance between bundles x and y. The boundary of the set A (relative to $\mathcal{X}$) is defined as $\partial A = \{x \in A \mid \text{for each } \varepsilon > 0, \text{ there is a bundle } y \text{ in } \mathcal{X} \setminus A \text{ such that } \|x - y\| < \varepsilon\}$.

[7] A preference relation R_i is complete if $x \, R_i \, y$ or $y \, R_i \, x$ for all x and y in $\mathcal{X}$. It is transitive if $x \, R_i \, y$ and $y \, R_i \, z$ imply $x \, R_i \, z$ for all x, y, and z in $\mathcal{X}$. It is continuous if each better-than set and each worse-than set is closed. It is monotone if $x \gg y$ implies $x \, P_i \, y$ for all x and y in $\mathcal{X}$. It is convex if each better-than set is convex.

We refer to the sum of all bundles $x_1 + x_2 + \cdots + x_n$ as the societal bundle. A preference profile $R = (R_1, R_2, \dots, R_n)$ in $\mathcal{R}^n$ contains a preference relation for each individual in N.

A social ranking specifies for each preference profile a social preference relation over all distributions.[8] Formally, a social ranking $\succsim$ maps each preference profile R in $\mathcal{R}^n$ into a complete and transitive social preference relation $\succsim_R$ on $\mathcal{X}^n$. We use $X \succsim_R Y$ to denote that distribution X is at least as good as distribution Y in terms of social welfare. The relations $\succ_R$ and $\sim_R$ denote the corresponding strict social preference and social indifference relations.

2.2 *Three basic axioms*

We define three basic axioms. Anonymity requires that switching the bundles of two individuals with the same preferences does not change social welfare.

ANONYMITY: For each preference profile R in $\mathcal{R}^n$, for each distribution X in $\mathcal{X}^n$, and for all individuals i and j in N such that $R_i = R_j$, we have $(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \sim_R (x_1, \dots, x_j, \dots, x_i, \dots, x_n)$.

Continuity ensures that small changes in distributions do not lead to large changes in their social ranking.

CONTINUITY: For each preference profile R in $\mathcal{R}^n$, for all distributions X and Y in $\mathcal{X}^n$, and for each sequence of distributions $\{X^k\}_{k \in \mathbb{N}}$ that converges to X, if $X^k \succsim_R Y$ for each k in $\mathbb{N}$, then $X \succsim_R Y$, and if $Y \succsim_R X^k$ for each k in $\mathbb{N}$, then $Y \succsim_R X$.

Although continuity excludes leximin, the axiom is compatible with social preference relations arbitrarily close to leximin.

Monotonicity imposes that increasing all amounts in some individual's bundle improves social welfare.

MONOTONICITY: For each preference profile R in $\mathcal{R}^n$ and for all distributions X and Y in $\mathcal{X}^n$, if $x_i \geq y_i$ for each individual i in N and $x_i \gg y_i$ for some individual i in N, then $X \succ_R Y$.

Individual preferences are monotone. Therefore, monotonicity of the social ranking is implied by the Pareto principle (obtained by replacing $\geq$ in the monotonicity axiom by R_i and $\gg$ by P_i). Note that, conversely, the combination of all our axioms implies the Pareto principle (see Lemma 1 below).

[8]Thus, we focus on social comparisons for a fixed population with a given preference profile. It is straightforward to extend the analysis to comparisons across societies with different population sizes and preference profiles.

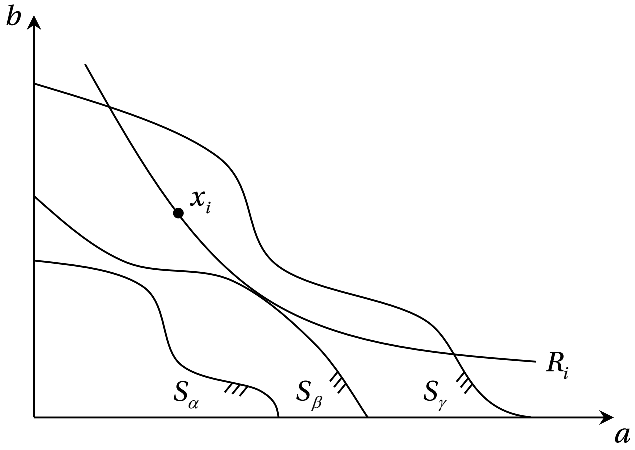

Figure 2. General example

2.3 Reference set welfarism

We impose reference set welfarism as our fourth axiom.[9] To the best of our knowledge, this version of welfarism underlies all existing approaches that base social welfare rankings on ordinal and non-comparable individual preferences (see Fleurbaey and Blanchet, 2013, Chapter 4, for an overview). For elementary axiomatic underpinnings of reference set welfarism, see Cato (2016) and Piacquadio (2017).

Reference set welfarism prescribes two steps to rank distributions. The first step uses a list of reference sets to cardinalize each individual situation—a bundle-preference pair—into a utility value. The second step uses the resulting vectors of individual utility values to rank distributions. We now turn to a discussion of reference sets and their use in reference set welfarism.

Reference sets are sets of bundles. Consider a list of nested reference sets. Each reference set is labelled using a real number, with larger sets receiving larger numbers. These real numbers are used to cardinalize bundle-preference pairs. Each bundle-preference pair is assigned a utility value equal to the number of the reference set that is just tangent to the indifference curve through the bundle. Figure 2 shows three nested reference sets, labelled by the nonnegative real numbers α, β, and γ, with $\alpha < \beta < \gamma$. For the depicted bundle-preference pair (x_i, R_i), the assigned utility value is β.

We stress that the obtained utility values are treated as interpersonally comparable. Two individuals whose indifference curves are tangent to the same reference set are assigned the same utility value, and hence are regarded as equally well-off. The choice of a list of reference sets determines how interpersonal comparisons are made and must therefore be based on value judgments. Our axioms, and especially the transfer princi-

[9]Fleurbaey (2009) refers to reference set welfarism as the equivalence approach.

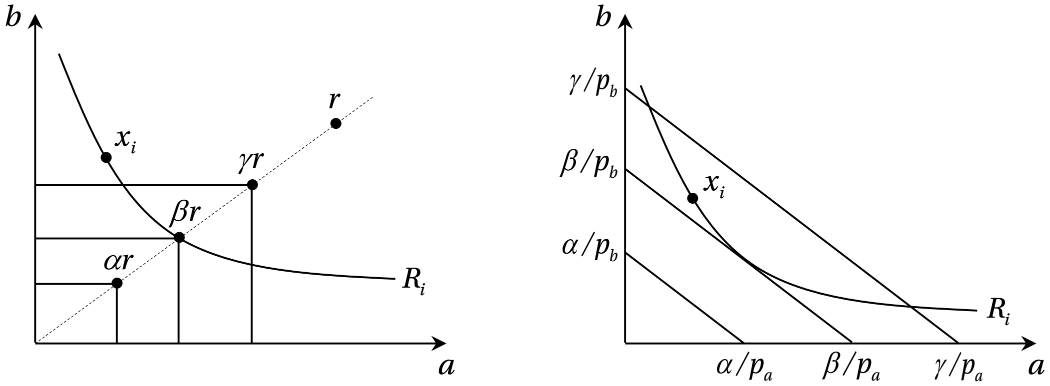

Figure 3. Quantity metric and money metric utility

ple, make these value judgments explicit and, as we will demonstrate, put considerable structure on the shape of the reference sets.

We now formalize the properties of a list of reference sets $S = (S_\lambda)_{\lambda\in\mathbb{R}_+}$. A list contains a compact reference set $S_\lambda \subseteq \mathcal{X}$ for each λ in $\mathbb{R}_+$, starts from the origin ($S_0 = \{0\}$), expands in a strictly nested way ($\lambda < \mu$ implies $S_\lambda \subset S_\mu$ and $\partial S_\lambda \cap \partial S_\mu = \emptyset$), and has no gaps (the union of all boundaries $\cup_{\lambda\in\mathbb{R}_+}\partial S_\lambda$ is equal to the set of bundles $\mathcal{X}$). Because individual preferences are monotone, we can, without loss of generality, require additionally that each list satisfies free disposal (if x belongs to S_λ, then each bundle $y \leq x$ belongs to S_λ). Let $\mathcal{S}$ be the set of all lists of reference sets that satisfy these properties.

For a given list of reference sets S in $\mathcal{S}$, the utility value assigned to a bundle-preference pair (x_i, R_i) is the greatest number λ for which $x_i\, R_i\, S_\lambda$. Accordingly, the reference set utility function u_S is defined as

$$u_S(x_i, R_i) = \max\{\lambda \mid x_i\, R_i\, S_\lambda\} \quad \text{for each } x_i \text{ in } \mathcal{X} \text{ and each } R_i \text{ in } \mathcal{R}. \tag{1}$$

The properties of the preferences in $\mathcal{R}$ and of the lists of reference sets in $\mathcal{S}$ ensure that the reference set utility function u_S in equation (1) is well defined, unique, and continuous (in bundles). This utility function represents the preference relation, i.e., for all bundles x and y in $\mathcal{X}$, we have $u_S(x, R_i) \geq u_S(y, R_i)$ if and only if $x\, R_i\, y$.

Before we state the axiom reference set welfarism, we define two prominent reference set utility functions.[10] The quantity metric utility function, illustrated in the left-hand panel of Figure 3, is defined by equation (1) with $S_\lambda = \{x \in \mathcal{X} \mid x \leq \lambda r\}$ for a fixed reference bundle $r \gg 0$.[11] Quantity metric utilities were introduced by Samuelson (1977) and Pazner and Schmeidler (1978) in welfare economics. The money metric utility function,

[10]See the discussion in Deaton and Muellbauer (1980, pp. 179-182).

[11]We use 0 to denote a vector of zeroes of appropriate length.

which we denote by u_p, is illustrated in the right-hand panel of Figure 3. The function u_p is defined by equation (1) with $S_\lambda = \{x \in \mathcal{X} \mid \sum_{k=1}^{m} p_k x_k \leq \lambda\}$ for a fixed reference price vector $p \gg 0$. Money metric utilities were introduced by McKenzie (1957) and Samuelson (1974), and applied in welfare economics by Deaton (1980), King (1983), Ravallion and van de Walle (1991), Creedy and Hérault (2012), and Chiappori and Meghir (2014), among others.

Reference set welfarism requires that welfare comparisons are based on reference set utility values only. For a list of reference sets S, a distribution X, and a preference profile R, we abbreviate the vector of reference set utilities $(u_S(x_1, R_1), u_S(x_2, R_2), \ldots, u_S(x_n, R_n))$ by $u_S(X, R)$.

Reference set welfarism: There exists a list of reference sets S in $\mathcal{S}$ and a binary relation $\succsim^*$ defined over reference set utility vectors in $\mathbb{R}^n_+$ such that, for each preference profile R in $\mathcal{R}^n$ and for all distributions X and Y in $\mathcal{X}^n$, we have

$$X \succsim_R Y \quad \text{if and only if} \quad u_S(X, R) \succsim^* u_S(Y, R),$$

where u_S is the reference set utility function defined in equation (1).

We conclude this section by combining reference set welfarism with anonymity, continuity, and monotonicity. A social ranking satisfies these four axioms if and only if the social ranking can be represented by a continuous, strictly increasing, and symmetric social welfare function defined over reference set utilities. We state this straightforward result without proof.

Lemma 1: *A social ranking $\succsim$ satisfies anonymity, continuity, monotonicity, and reference set welfarism if and only if there exists a list of reference sets S in $\mathcal{S}$ and a continuous, strictly increasing, and symmetric social welfare function $W : \mathbb{R}^n_+ \to \mathbb{R}$ such that, for each preference profile R in $\mathcal{R}^n$ and for all distributions X and Y in $\mathcal{X}^n$, we have*

$$X \succsim_R Y \quad \textit{if and only if} \quad W(u_S(X, R)) \geq W(u_S(Y, R)),$$

where u_S is the reference set utility function defined in equation (1).

The four axioms in Lemma 1 leave open the question of which list of reference sets to use.[12] Our final axiom, the efficiency-preserving transfer principle, will determine the shape of the reference sets.

[12]The social ranking in Lemma 1 satisfies the Pareto principle. Indeed, the social welfare function W is strictly increasing and the utility function u_S is a representation of individual preferences.

2.4 An efficiency-preserving transfer principle

Underlying the conclusions of Blackorby and Donaldson (1988) and Fleurbaey and Maniquet (2011) that we discussed in the introduction, is their acceptance of a strong transfer principle that we define as follows.[13]

Societal-bundle-preserving transfer principle: For each preference profile R in $\mathcal{R}^n$, for all distributions X and Y in $\mathcal{X}^n$, and for all individuals i and j in N such that $R_i = R_j$, if X is obtained from Y by a richer-to-poorer transfer from j to i ($y_i \ll x_i \ll x_j \ll y_j$ and $x_k = y_k$ for $k \neq i, j$) that preserves the societal bundle ($x_i + x_j = y_i + y_j$), then $X \succsim_R Y$.

The transfer in this principle preserves the societal bundle, but may considerably worsen the efficiency of how this societal bundle is distributed. The transfer depicted in Figure 1 illustrates this point: it preserves the societal bundle, but takes us from an efficient distribution (where the marginal rates of substitution are equal) to an inefficient distribution (where they are unequal). We introduce a weaker transfer principle that requires only the approval of those transfers that preserve efficiency.

We develop a concept of efficiency-preservation based on the two building blocks of the efficiency measurement literature: the actual societal bundle and the Scitovsky boundary. The Scitovksy boundary collects the minimum societal bundles able to deliver to each individual the utility level she obtains in the actual distribution (Scitovsky, 1942). Formally, for a distribution X and a preference profile R, the Scitovsky set is defined as the sum of the better-than sets $B_1 + B_2 + \cdots + B_n$ and the Scitovsky boundary is $\partial(B_1 + B_2 + \cdots + B_n)$. A distribution is efficient only if its societal bundle lies on the Scitovksy boundary: for the given societal resources, no individual can be made better off without making any other individual worse off. Common to all efficiency measures in the literature is that they quantify inefficiency as the distance between the societal bundle and the Scitovsky boundary. What distinguishes these efficiency measures is how they define distance. Diewert (1985) provides a general overview.

Figure 4 gives an example for the case of two individuals and two goods. Both panels of the figure show distribution $X = (x_1, x_2)$ and the corresponding better-than sets B_1 and B_2. The societal bundle is $x_1 + x_2$ and the Scitovsky boundary is the boundary of the Scitovsky set $B_1 + B_2$. Note that distribution X is not efficient: the societal bundle does not lie on the Scitovsky boundary, but rather in the interior of the Scitovsky set. The two panels illustrate the two dominant approaches in the efficiency measurement literature, referred to by Diewert (1985) as the quantity-oriented and price-oriented approaches. The left-hand panel illustrates the quantity-oriented measures of

[13] This transfer principle only considers transfers among individuals with the same preferences. Fleurbaey and Trannoy (2003) show that without this restriction the transfer principle directly clashes with the Pareto principle. See Weymark (2017) for an overview of impossibility results in this vein.

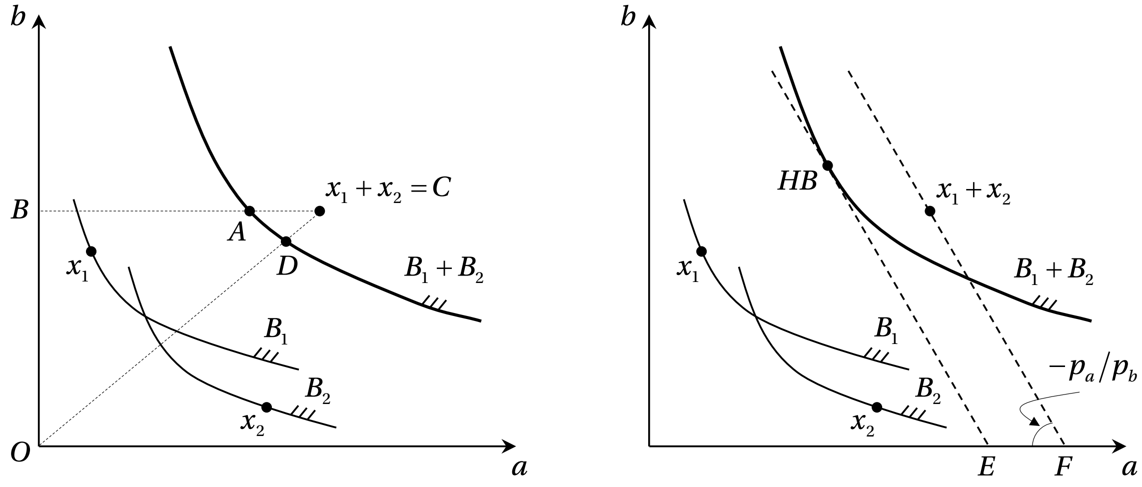

Figure 4. Quantity-oriented and price-oriented efficiency measures

Allais (1943) and Debreu (1951). Allais (1943) measures inefficiency as AC/BC, the relative distance between the societal bundle and the efficient bundle A that contains less only of a numéraire good (here good a). Debreu (1951) measures inefficiency as DC/OC, the relative distance between the societal bundle and the efficient bundle D that is proportional to the societal bundle. The right-hand panel illustrates the price-oriented approach, proposed by Hicks (1942) and Boiteux (1951). For the given reference price vector $p = (p_a, p_b)$, inefficiency equals $p_a \times EF$, or the distance, expressed in expenditure terms, between the societal bundle and the cheapest bundle HB on the Scitovsky boundary.

Our efficiency-preserving transfer principle requires the approval of each richer-to-poorer transfer that keeps the societal bundle and the Scitovsky boundary fixed.[14] These restrictions on the transfer guarantee, as shown above, that all efficiency measures unanimously agree that the transfer preserves efficiency.

Efficiency-preserving transfer principle: For each preference profile R in $\mathcal{R}^n$, for all distributions X and Y in $\mathcal{X}^n$, and for all individuals i and j in N such that $R_i = R_j$, if X is obtained from Y by a richer-to-poorer transfer from j to i ($y_i \ll x_i \ll x_j \ll y_j$ and $x_k = y_k$ for $k \neq i, j$) that preserves efficiency ($x_i + x_j = y_i + y_j$ and $B(x_i, R_i) + B(x_j, R_j) = B(y_i, R_i) + B(y_j, R_j)$), then $X \succsim_R Y$.

[14]This axiom does not cover all cases where the distributions before and after the transfer are both efficient (equal marginal rates of substitution). Indeed, in some such cases the Scitovsky boundaries do not coincide, but rather intersect at the societal bundle. As is easy to show, a stronger axiom that would also cover these cases clashes with the other axioms in Theorem 1.

Both the efficiency-preserving and the societal-bundle-preserving transfer principles generalize the unidimensional Pigou-Dalton transfer principle. The Pigou-Dalton transfer principle requires approval of richer-to-poorer transfers in income (the single good) that preserve total income (the societal bundle). Because preserving the efficiency of the distribution reduces to preserving the societal bundle in the unidimensional setting, the Pigou-Dalton transfer principle is also efficiency-preserving. We claim that the efficiency-preserving transfer principle captures an aspect of the unidimensional Pigou-Dalton transfer principle that the societal-bundle-preserving transfer principle does not. The former two transfer principles are silent on the equity-efficiency trade-off because the considered transfers improve equity, without changing efficiency. The societal-bundle-preserving transfer principle, in contrast, does take a stance regarding the equity-efficiency trade-off and, as we have argued, an extreme stance. Transfers that only preserve the societal bundle improve equity, but may cause arbitrarily large efficiency losses. Our efficiency-preserving transfer principle does not exclude such an extreme stance, but is moreover compatible with more moderate ethical positions.

3 Result

Recall that Lemma 1 leaves open the choice of the reference set utility function. Theorem 1 singles out the money metric utility function by adding our efficiency-preserving transfer principle to the four axioms in Lemma 1. A natural additional consequence is that the social welfare function must be Schur-concave.[15] The proof of Theorem 1 is in the appendix.

Theorem 1. *A social ranking $\succsim$ satisfies anonymity, continuity, monotonicity, reference set welfarism, and the efficiency-preserving transfer principle if and only if there exists a vector p in $\mathbb{R}^m_{++}$ and a continuous, strictly increasing, and Schur-concave social welfare function $W : \mathbb{R}^n_+ \to \mathbb{R}$ such that, for each preference profile R in $\mathcal{R}^n$ and for all distributions X and Y in $\mathcal{X}^n$, we have*

$$X \succsim_R Y \quad \textit{if and only if} \quad W\left(u_p(X,R)\right) \geq W\left(u_p(Y,R)\right),$$

where u_p is the money metric utility function using p as the reference price vector.

Theorem 1 gives necessary and sufficient conditions for a social ranking to satisfy the five axioms. The sufficiency part states that any standard social welfare function defined over money metric utilities satisfies the axioms. More strikingly, the necessity part states

[15] A function $W : \mathbb{R}^n_+ \to \mathbb{R}$ is Schur-concave if $W(Qx) \geq W(x)$ for each vector x in $\mathbb{R}^n_+$ and for each bistochastic matrix Q. A matrix Q in $\mathbb{R}^{n\times n}_+$ is bistochastic if each row sum and each column sum is equal to 1.

that the axioms are satisfied by this particular social ranking exclusively. We discuss in turn the sufficiency and necessity parts of the theorem.

To understand the sufficiency part, note that if a transfer preserves efficiency, then it also preserves the sum of money metric utilities. We show this in the first part of the proof. It follows immediately that any Schur-concave welfare function defined over money metric utilities will approve efficiency-preserving transfers.

The sufficiency part stands in sharp contrast to Blackorby and Donaldson (1988). They show that a standard social welfare function defined over money metric utilities fails to satisfy the stronger—and in their view essential—societal-bundle-preserving transfer principle.[16] Theorem 1 demonstrates that aggregating money metrics is perfectly justified if one adds the sensible requirement of preserving efficiency while transferring goods. The sufficiency part of the theorem furthermore contrasts with Fleurbaey and Maniquet (2011). They too view the societal-bundle-preserving transfer principle as essential, but conclude that a Rawlsian social welfare function must be used. Indeed, if one drops continuity and imposes the societal-bundle-preserving transfer principle together with the three remaining basic axioms anonymity, monotonicity, and reference set welfarism, then the only option is to use leximin as the aggregation criterion.[17] Our result shows that the efficiency-preserving transfer principle admits the whole spectrum of inequality aversion, ranging from the inequality neutral sum-utilitarian case to the extremely inequality averse Rawlsian case.

The necessity part of Theorem 1 shows that we must use money metric utilities. No other reference set utilities can be used. We illustrate the intuition by showing that aggregating quantity metric utilities—a prominent alternative to aggregating money metric utilities—is not permissible. The second part of our proof generalizes this intuition. Figure 5 depicts an efficiency-preserving transfer of a bundle δ from individual 2 to individual 1. Because the indifference curves are piece-wise linear and parallel, and because the sum of the kink points remains the same before and after the transfer ($k_1 + k_2 = k'_1 + k'_2$), the Scitovsky boundary also remains the same. The bundle r is the reference bundle. The gain and loss in quantity metric utilities can be read from the horizontal axis. If we rotate the lower segments of the four indifference curves upwards, then we still have an efficiency-preserving transfer that must be approved. Yet, the gain in quantity metric utility of the recipient becomes smaller, whereas the loss of the donor remains the same. It is clear that the gain can be made arbitrarily small without changing the loss. By consequence, no continuous social welfare function defined over quantity metric utilities

[16] It is easy to show that, more generally, no social ranking satisfies the societal-bundle-preserving transfer principle together with anonymity, continuity, monotonicity, and reference set welfarism.

[17] Leximin, combined with any reference set utility function, satisfies the societal-bundle-transfer principle and the three other axioms. Keeping continuity, but weakening monotonicity (by requiring $X \succ_R Y$ only if $x_i \gg y_i$ for each individual i in N), yields maximin instead of leximin.

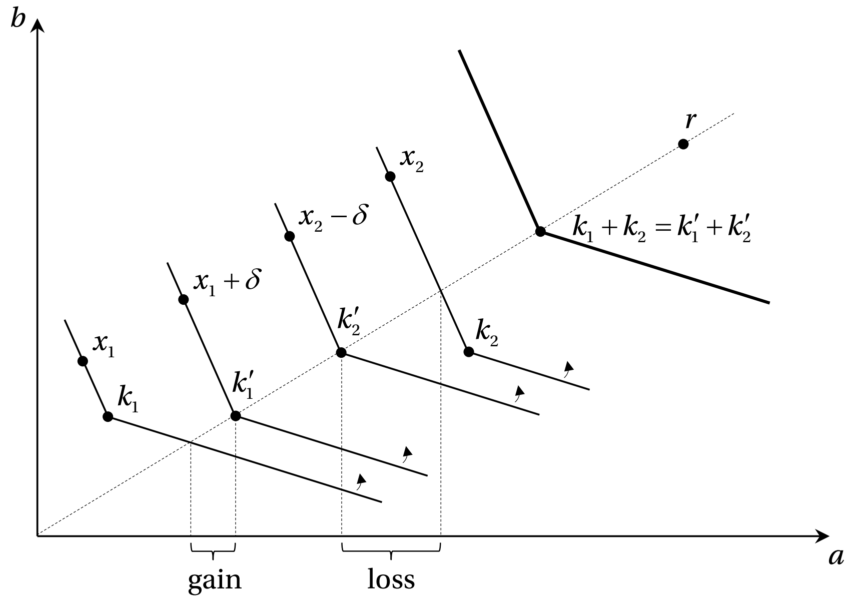

Figure 5. Aggregating quantity metric utilities is not permissible

can approve all transfers.

4 Conclusion

We have provided an ethical justification to aggregate money metrics. Our core axiom is a transfer principle based on transfers that preserve the overall efficiency of the distribution. This efficiency-preserving transfer principle—in combination with four basic axioms—characterizes a continuous, strictly increasing, and Schur-concave social welfare function defined over money metric utilities.

We conclude with two questions for further research. First, our result justifies the use of a standard unidimensional social welfare function defined over money metrics. This raises the question of whether the use of a unidimensional poverty or inequality measure over money metrics is also justified. For poverty, this requires the identification of the poor—a challenge in a setting with heterogeneous preferences—and to incorporate a focus on the poor into the axioms.[18] For inequality, the so-called normative approach can be used. In this approach, inequality is defined as the social welfare gain that could be obtained by optimally redistributing the societal bundle.[19] In a setting with homogeneous preferences, we have argued elsewhere that it is important to decompose this social welfare gain into an equity gain and an efficiency gain, with only the equity

[18]See Decancq, Fleurbaey, and Maniquet (2014) for a proposal.

[19]An alternative is to define inequality directly, rather than deriving it from a social welfare function. The key issue in this direct approach is to replace the monotonicity axiom by an invariance axiom. For example, a ratio-scale invariance axiom could require that, for a profile of homothetic preferences, if all bundles are multiplied by the same factor, then inequality remains the same.

gain capturing true inequality (Bosmans, Decancq, and Ooghe, 2015). We leave the extension to heterogeneous preferences for future work.

Second, the main theorem does not tell us which reference price vector should be used to compute money metric utilities. A pragmatic solution is to fix a particular set of reference price vectors and to only focus on welfare comparisons that are robust to the choice of price vectors within this set. A more fundamental, but more challenging approach, is to think of appealing axioms—presumably depending on the particular context—that would reduce the set of admissible reference price vectors.

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Appendix: Proof of Theorem 1

Sufficiency. Each Schur-concave function is symmetric. Hence, by Lemma 1 a social ranking $\succsim$ that can be represented by a continuous, strictly increasing, and Schur-concave social welfare function W defined over money metrics utilities satisfies anonymity, continuity, monotonicity, and reference set welfarism. We now show that efficiency-preserving transfers do not decrease social welfare.
The sum of money metric utilities is

$$\sum_{i \in N} \min\{p' y_i \mid y_i \, R_i \, x_i\},$$

which equals

$$\min\left\{p' \sum_{i \in N} y_i \mid y_i \, R_i \, x_i \text{ for all } i \in N\right\}. \tag{2}$$

If the Scitovsky set $\{\sum_{i \in N} y_i \mid y_i \, R_i \, x_i \text{ for all } i \in N\}$ remains unchanged by a transfer of a bundle of goods, then the sum of money metrics given by equation (2) remains unchanged as well, irrespective of the choice of the reference price vector p. Each efficiency-preserving transfer therefore corresponds to a mean-preserving progressive transfer in the space of money metric utilities. Consequently, a Schur-concave social welfare function defined over money metric utilities does not decrease welfare after an efficiency-preserving transfer.

Necessity. Let $\succsim$ be a social ranking that satisfies anonymity, continuity, monotonicity, reference set welfarism, and the efficiency-preserving transfer principle. Lemma 1 applies. Lemma 2 below will show that, in addition, these axioms imply that the boundaries of all reference sets are linear and parallel.

Together, Lemmas 1 and 2 establish that there exists a reference price vector p in $\mathbb{R}^m_{++}$ such that the utilities obtained using equation (1) are money metric utilities up to a strictly increasing transformation. If we choose this transformation function to be the identity function, then the reference set utilities are equal to money metric utilities. Because efficiency-preserving transfers correspond to mean-preserving transfers in the space of money metric utilities, the continuous, strictly increasing, and symmetric social welfare function W singled out in Lemma 1 must be Schur-concave in order to satisfy the efficiency-preserving transfer principle.

We now prove Lemma 2.

LEMMA 2: *A social ranking $\succsim$ satisfies anonymity, continuity, monotonicity, reference set welfarism, and the efficiency-preserving transfer principle only if the boundaries of all reference sets in the associated list $S = (S_\lambda)_{\lambda \in \mathbb{R}_+}$ are linear and parallel.*

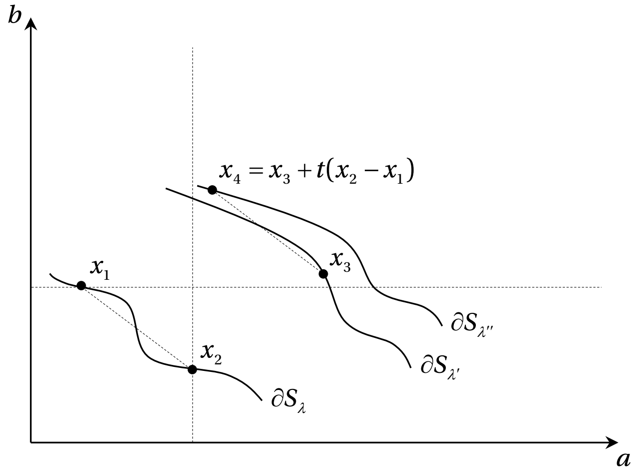

Figure A1.

We proceed by contradiction. Suppose that the boundaries associated with a list of reference sets $S = (S_\lambda)_{\lambda \in \mathbb{R}_+}$ are not everywhere linear and parallel. Then it follows that

- there exist distinct bundles x_1 and x_2 in $\mathcal{X}$ such that x_1 and x_2 belong to ∂S_λ for some $\lambda > 0$,
- there exists a bundle x_3 such that $x_3 \gg x_1$, $x_3 \gg x_2$, and x_3 belongs to $\partial S_{\lambda'}$ for some $\lambda' > \lambda$,
- there exists a bundle x_4 such that $x_4 \gg x_1$, $x_4 \gg x_2$, and $x_4 = x_3 + t(x_2 - x_1)$ for some t in $\mathbb{R}$,

but x_4 does not belong to $\partial S_{\lambda'}$.

Figure A1 illustrates such a case for $m = 2$. Note that bundle x_4 belongs to $\partial S_{\lambda''}$, with $\lambda'' > \lambda'$ (the case $\lambda'' < \lambda'$ is analogous).

We consider the lowest possible reference set boundary that is tangent to the closed line segment $[x_1, x_2]$ and denote it by ∂S_α.[20] The boundary ∂S_α is uniquely defined and the tangent bundles $\partial S_\alpha \cap [x_1, x_2]$ form a closed set. From that set, we select the tangent bundle that is closest to bundle x_1 and call it y. Bundle y is unique and distinct from bundle x_2.[21] Figure A2 illustrates the reference set boundary ∂S_α and bundle y for $m = 2$. Analogously, we consider the lowest possible reference set boundary that is tangent to the closed line segment $[x_3, x_4]$ and denote it by ∂S_β. We select the tangency

[20] By construction, we have that $\alpha \leq \lambda$.

[21] Suppose $y = x_2$. Bundle x_2 must then be a tangent bundle by definition. This is possible only if $\alpha = \lambda$ (otherwise, the boundaries ∂S_α and ∂S_λ are different, but tangent in x_2, which is not admitted). But if $\alpha = \lambda$, then also x_1 is a tangent bundle that is obviously closer to x_1 than x_2 is. Hence, we must have $y = x_1 \neq x_2$, a contradiction.

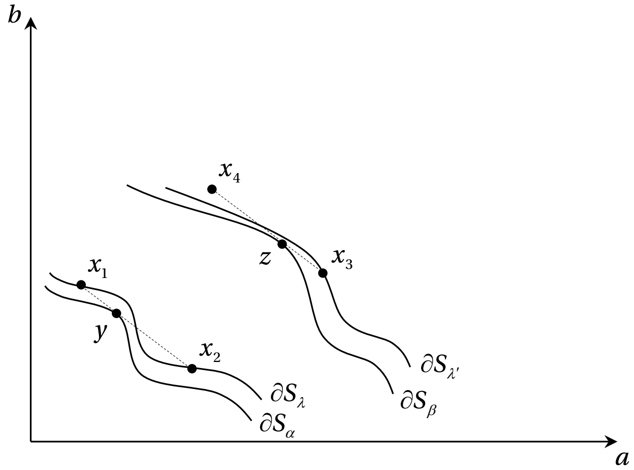

Figure A2.

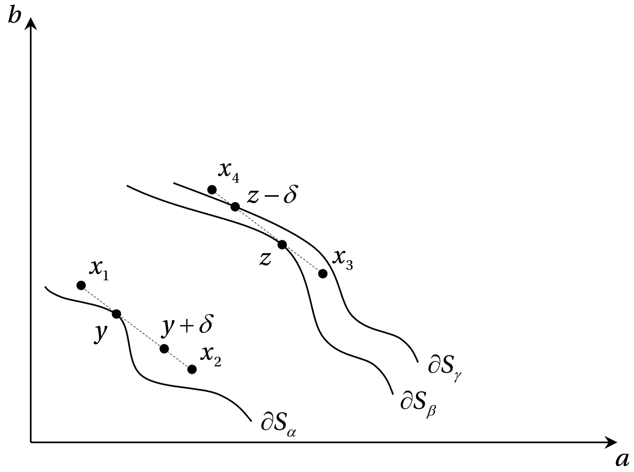

Figure A3.

bundle z in $\partial S_\beta \cap [x_3, x_4]$ that is closest to bundle x_4. Bundle z is unique and distinct from x_4.[22] Figure A2 illustrates the reference set boundary ∂S_β and bundle z for $m = 2$.

By construction, we have that $z \gg y \gg 0$. In addition, no bundle on the closed line segment $[y, x_2]$ belongs to the interior of S_α and no bundle on the half-open line segment $(z, x_4]$ belongs to the interior or boundary of S_β. Because $y \neq x_2$ and $z \neq x_4$, we can choose a bundle δ such that no bundle in the closed line segment $[y, y+\delta] \subseteq [y, x_2]$ belongs to the interior of S_α and no bundle in the half-open line segment $(z, z-\delta] \subseteq (z, x_4]$ belongs to the interior or boundary of S_β. Figure A3 illustrates the construction of $y + \delta$ and $z - \delta$ for $m = 2$. The original boundaries ∂S_λ and $\partial S_{\lambda'}$ have been removed from the figure for clarity, whereas the boundary ∂S_γ through the bundle $z - \delta$ is added. Note that $\gamma > \beta$.

[22] If $z = x_4$, then $\beta = \lambda'' > \lambda'$. This is not possible because we must have $\beta \leq \lambda'$ since the boundary ∂S_β is the lowest possible boundary.

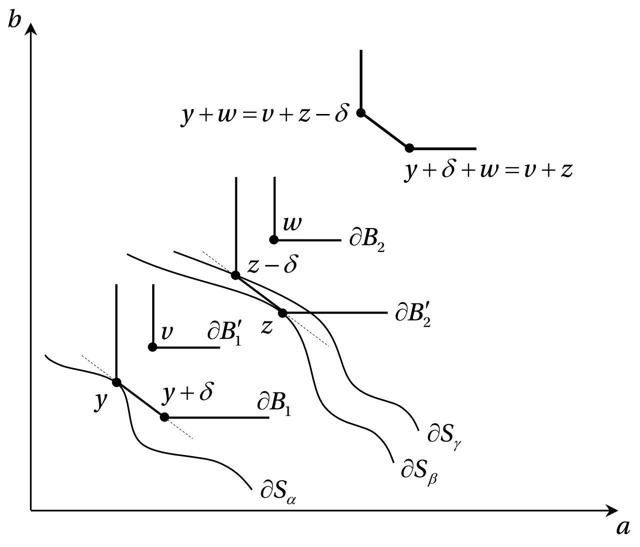

Figure A4.

Based on these line segments, we now construct four kinked indifference surfaces. The first indifference surface is based on the closed line segment $[y, y+\delta]$. Construct the set B_1 that contains the bundles that vector-dominate the bundles in $[y, y+\delta]$, i.e., $B_1 = \{x \in \mathcal{X} \,|\, x \geq x' \text{ for some } x' \text{ in } [y, y+\delta]\}$. By construction, the set B_1 is closed, monotone (if x belongs to B_1, then also $x' \geq x$ belongs to B_1), and convex. Hence, the boundary ∂B_1 is an indifference surface of a continuous, monotone, and convex preference relation. Analogously, we construct another indifference surface, denoted by $\partial B_2'$, based on the set of bundles B_2' that vector-dominate the bundles in $[z, z-\delta]$. Figure A4 illustrates the boundaries ∂B_1 and $\partial B_2'$ for $m = 2$.

To construct the final two indifference surfaces, we consider an arbitrary bundle v that satisfies $v \gg y$, $z \gg v$, and $z - \delta \gg v$, and we define bundle w such that $w = (z-\delta) + (v-y) \gg z - \delta$. We construct two other indifference surfaces, denoted by $\partial B_1'$ and ∂B_2, based on the sets of bundles B_1' and B_2 that vector-dominate the bundles v and w, respectively. By construction, the four indifference surfaces have no bundles in common and can therefore be assumed to belong to the same preference relation in $\mathcal{R}$.

We consider an efficiency-preserving transfer between two individuals with an identical preference ordering that contains these four indifference surfaces. We assign the lower two indifference surfaces ∂B_1 and $\partial B_1'$ to individual 1 and the upper two indifference surfaces ∂B_2 and $\partial B_2'$ to individual 2. Let y and w be the bundles of individuals 1 and 2 before the transfer and let v and $z - \delta$ be their bundles after the transfer. This richer-to-poorer transfer is efficiency-preserving. It preserves the societal bundle: the societal bundle before the transfer is $y + w$, the societal bundle after the transfer is $v + (z - \delta)$, and we have that $y + w = v + (z - \delta)$. The transfer also preserves the Scitovsky set: the Scitovsky set before the transfer is equal to the set of bundles that vector-dominate at least one bundle in $[y + w, (y + \delta) + w]$, the Scitovsky set after the transfer is equal to the set of bundles that vector-dominate at least one bundle in $[v + (z - \delta), v + z]$, and we have that $[y + w, (y + \delta) + w] = [v + (z - \delta), v + z]$.

Finally, note that bundle v can be chosen arbitrarily close to y with $v \gg y$. Choosing a closer bundle such as v implies by definition that bundle w shifts closer to bundle $z - \delta$ and that the indifference surfaces through v and w shift accordingly. The resulting transfer remains efficiency-preserving. It is thus possible to construct a sequence of efficiency-preserving transfers with v approaching y arbitrarily close, and as a result, w approaching $z - \delta$ arbitrarily close.

Since W is Schur-concave, each transfer in this sequence does not decrease social welfare. Because the social welfare function W is continuous in the space of reference set utilities and the reference set utility function is continuous in bundles, in the limit we get $W(\alpha, \beta, \ldots) \geq W(\alpha, \gamma, \ldots)$. But, because $\gamma > \beta$ and W is strictly increasing, we obtain a contradiction.