

## Rationing rules and stable coalition structures

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This paper introduces a model of coalition formation with claims. It assumes that agents have claims over the outputs that they could produce by forming coalitions. Outputs are insufficient to meet the claims and are rationed by a rule whose proposals of division induce each agent to rank the coalitions in which she can participate. As a result, a hedonic game of coalition formation emerges. Using resource monotonicity and consistency, we characterize the continuous rationing rules that induce hedonic games that admit core stability.

**KEYWORDS.** Coalition formation, hedonic games, core stability, rationing rules.

**JEL CLASSIFICATION.** C71, D63, D74.

### 1. INTRODUCTION

Agents such as individuals, firms, and institutions seek to form alliances with the aim of achieving profits to be divided according to their aspirations,<sup>1</sup> which are often too great to be satisfied. When there are many profitable coalitions and agents have conflicts over which coalitions to build, the rule used to distribute profits specifies what coalitions are likely to form. Examples of such situations arise in provision of local public goods and of club goods, formation of jurisdictions and of research teams, and others.

Consider, for instance, a community of households deciding whether to install a public facility that provides a certain benefit to each of them. Suppose that coalitions of households can equip themselves with that facility as long as it is efficient (joint benefits are greater than the provision cost) to do so and that there is a rule dividing coalitional costs according to individual benefits. Each household will seek to minimize its payment and will rank the coalitions of which it forms part accordingly. Consequently, which coalitions of households build the facility depends on the rule used. As a second

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<sup>1</sup>Aspirations are depicted by a one-dimensional quantifiable factor such as claim, demand, effort, etc., depending on the context.

scenario, consider a call for funding for research projects. Researchers form teams to apply for funding, knowing that the outcome depends on their competence. Typically, the money assigned to a funded project falls short of meeting the researchers' claims<sup>2</sup> and participation in a single team is a prerequisite of the call. Clearly, each researcher aims for the highest possible share of the funding, so the payoffs proposed by the rule, which takes the agents' claims as input, in the distribution of funding play a key role in how researchers rank the teams in which they may participate. Regardless of whether agents maximize payoffs or minimize payments, these two examples are formally identical. In this paper, we mainly use the claims interpretation.

We deal with problems consisting of the following ingredients: (i) a set of agents with their claims, which are commonly known, (ii) the set of coalitions formed by those agents, each one producing an output that is insufficient to meet the claims of its members, and (iii) a rule that dictates the payoffs for each member in each coalition, which, in turn, induces agents' preferences over coalitions. The structure of those preferences is decisive in the formation of the coalitions attained. The first two ingredients define the model of coalition formation with claims and the third one determines its solution. Whether the resulting preferences generate a stable partition of coalitions is an essential issue since its absence prevents agreements to form coalitions. Thus, analyzing which rules induce stable coalition structures is relevant not only in providing a better understanding of coalition formation, but also from a normative point of view when setting the rules.

The formal literature on rationing problems began with O'Neill (1982). In a rationing problem, the endowment is insufficient to meet all agents' claims, and a rule proposes a division such that every agent receives a nonnegative payoff that does not exceed her claim. There are characterizations of different rationing rules that satisfy appealing properties such as resource monotonicity and consistency. Resource monotonicity is a natural property that requires that when the endowment increases, each agent receives at least as much as she received initially. The idea behind consistency is the following: Consider a distribution given by a rule and assume that some agents take their payoffs and leave, and the situation is then reassessed. A consistent rule requires that the remaining agents in the reduced problem receive the same payoffs as they initially did.<sup>3</sup>

The literature on hedonic games, initiated by Drèze and Greenberg (1980), is based on the idea that each agent's preference relation over coalitions depends on the identities of the coalition members. Informally, a coalition structure (a partition of the set of agents into coalitions) is blocked by a coalition if each of its members strictly prefers that coalition to the one in which she is currently participating. A coalition structure is stable if there is no blocking coalition. The core is the set of all stable coalition structures.

<sup>2</sup>In this example, claims are linked to expertise, and although researchers may tend to overestimate their own expertise, objective measures such as curriculum vitae (CVs) bind them.

<sup>3</sup>Consistency is a property with a fertile history in the literature on social choice and cooperative game theory. This principle, adapted to diverse areas, differs in the precise definition of the reduced problems (see, for instance, Thomson and Lensberg 1989). For an extensive review of consistency, see Moulin (2003) and Thomson (2015).

It is known that hedonic games may have an empty core and the analysis of classes of hedonic games with nonempty core has been a central issue in this literature.<sup>4</sup>

Our approach to coalition formation problems with claims bridges these two branches of literature. Specifically, the question addressed in this paper is what rationing rules induce stable coalition structures in coalition formation problems with claims. In answering this question, we introduce a new class of hedonic games, called noncircular. This class includes the games that satisfy the common ranking property [Farrell and Scotchmer \(1988\)](#) and is contained in the class of stable hedonic games that satisfy the top coalition property ([Banerjee et al. 2001](#)) (our [Theorem 1](#)). Then we proceed to characterize rules that admit stable coalition structures. We find that only resource monotonic and consistent rationing rules (among continuous rules) guarantee the existence of stable coalition structures (our [Theorem 2](#)). Thus, nonconsistent rules commonly analyzed in rationing problems such as the Shapley value, [Shapley \(1953\)](#) fail to guarantee stability. However, a host of continuous rules satisfy resource monotonicity and consistency. The most important ones belong to the class of (symmetric and asymmetric) parametric rules<sup>5</sup> (see [Young 1987](#) and [Stovall 2014](#)). In [Proposition 1](#), we give a simple proof of the fact that parametric rules induce stable coalition structures. We find that they are not the only ones that generate stability. There are continuous nonparametric rules that also do so. This is sufficient grounds for justifying a characterization of rules beyond the class of parametric rules drawn up in [Theorem 2](#).

#### *Related literature*

Our work is inspired by [Pycia \(2012\)](#), who deals with a unified coalition formation model that includes many-to-one matching problems with externalities. He introduces the pairwise aligned property, which requires that any two agents who share two coalitions order them in the same way. Then he shows that fulfillment of that property guarantees stability in his coalition formation model. By contrast, we restrict ourselves to one-sided coalition formation problems—hedonic games—and we weaken the property of pairwise alignment by requiring only that any two agents who share two coalitions do not order them in opposite ways. Obviously, the assumption of this property requires other conditions to be met to achieve stability. We impose absence of rings (cyclicity among coalitions).<sup>6</sup> These two conditions define the class of noncircular hedonic games.

As an application, Pycia enriches the coalition formation model by assuming that coalitions produce outputs to be divided among their members according to their bargaining power. Then he characterizes the bargaining rules that induce stable coalition structures in coalition formation problems. Unlike Pycia, we do not specify a utility

<sup>4</sup>See, for instance, [Banerjee et al. \(2001\)](#), [Bogomolnaia and Jackson \(2002\)](#), and [Iehlé \(2007\)](#).

<sup>5</sup>Proportional, constrained equal awards, constrained equal losses, the Talmud rule ([Aumann and Maschler 1985](#)), the reverse Talmud rule ([Chun et al. 2001](#)), and the dictatorial rule with priority are parametric rules.

<sup>6</sup>There are different definitions of cyclicity among coalitions: For instance, [Chung \(2000\)](#) defines it for roommate problems with weak preferences, while [Inal \(2015\)](#) does it for hedonic games with strict preferences. To avoid confusion, as in computer science, we use the word “rings” to describe cyclicity among coalitions.

function for each agent, but rather a claim: we assume that each coalitional output is insufficient to reconcile all members' claims. Then we impose rationing rules that satisfy continuity, resource monotonicity, and consistency. Pycia has already established a link between consistency and stability (see footnote 5 in his paper). Exploiting this idea, we find that when consistency is coupled with resource monotonicity, it enables continuous rules to be characterized so as to guarantee stability. The intuition as to why this coupling is fruitful is the following: The exit (or entry) of some agents from (to) a coalition, regardless of whether it is accompanied by changes in output, does not affect the payoffs of the remaining agents in opposite directions. This generates weak pairwise alignment preferences without rings.

Two other articles bear some relationship with the current paper. [Alcalde and Revilla \(2004\)](#) explore the existence of stable research teams when each agent's preference depends on the identity of the members of the team with which she can collaborate. The assumption that agents' preferences satisfy the tops responsiveness condition<sup>7</sup> guarantees the existence of stable research teams. The formation of research teams facing a call for funding has been used to motivate our coalition formation problem with claims. However, unlike these authors, we do not impose restrictions on each agent's preference other than those induced by the rule. [Barberà et al. \(2015\)](#) consider coalitions in which individuals are endowed with productivity levels whose sum gives an output. Members of each coalition decide by a majority vote between meritocratic and egalitarian divisions of the output so that one coalition may choose meritocracy while another chooses egalitarianism. Like them, we endow each individual with a claim, but we assume that the output is insufficient to satisfy all claims and that its division among agents is dictated by a single rule. These two articles, like ours, analyze the core stability of hedonic games.

The rest of the paper is organized as follows: [Section 2](#) contains the preliminaries on rationing problems and on hedonic games used to define our coalition formation problem with claims. It also contains a result on hedonic games. The characterization and other results are presented in [Section 3](#). [Section 4](#) concludes.

## 2. COALITION FORMATION PROBLEM WITH CLAIMS

This section presents the preliminaries of two models that have been extensively analyzed in the literature: rationing problems and hedonic games. It also presents a result concerning hedonic games. By combining some ingredients of the two literatures, we define a coalition formation problem with claims.

First, we introduce some notation. There is an infinite set of potential agents, indexed by the natural numbers  $\mathbb{N}$ . Let  $\mathcal{N}$  denote the class of nonempty finite subsets of  $\mathbb{N}$ . Let  $\mathbb{R}_+$  denote the nonnegative real numbers. Given  $C \in \mathcal{N}$  for  $x, y \in \mathbb{R}^C$ , we use the vector inequalities  $x \geq y$  if  $x_i \geq y_i$  for all  $i \in C$ .

<sup>7</sup>This assumption is based on the idea of how each researcher thinks different colleagues can complement her abilities.

### 2.1 Rationing problems

A rationing problem involves a finite number of agents. Given  $C \in \mathcal{N}$  and  $i \in C$ , let  $d_i$  be agent  $i$ 's claim and let  $d = (d_i)_{i \in C}$  be the claims vector. Let  $E$  be the endowment to be divided among the agents in  $C$ . A *rationing problem* is a pair  $(d, E) \in \mathbb{R}_+^C \times \mathbb{R}_+$  such that  $\sum_{i \in C} d_i \geq E$ . Let  $\mathcal{B}^C$  denote the class of all rationing problems with the set of agents  $C$ . An *allocation* for  $(d, E) \in \mathcal{B}^C$  is a vector  $x_C = (x_i)_{i \in C}$  that satisfies the nonnegativity and claim boundedness conditions, i.e.,  $0 \leq x_C \leq d$ , and the efficiency condition  $\sum_{i \in C} x_i = E$ . A *rationing rule* (or *rule*) is a mapping  $F$  defined on  $\bigcup_{C \in \mathcal{N}} \mathcal{B}^C$  that associates to each rationing problem  $(d, E) \in \mathcal{B}^C$  an allocation  $x_C$ . The payoff of agent  $i$  is denoted by  $x_i = F_i(d, E)$ .

A rule  $F$  is continuous if small changes in the data of the problem do not lead to large changes in the solution.<sup>8</sup> Hereafter, we restrict ourselves to continuous rules.

We now introduce our axioms.

**AXIOM 1** (Resource monotonicity). *When the endowment increases, each agent receives at least as much as she did initially. Formally, for each  $C \in \mathcal{N}$ , each  $(d, E) \in \mathcal{B}^C$ , and each  $E' > E$ , if  $\sum_{i \in C} d_i \geq E'$ , then  $F(d, E') \geq F(d, E)$ .*

**AXIOM 2** (Consistency). *Consider a rationing problem and the allocation given by a rule for it. Assume that some agents leave with their payoffs and the situation of the remaining agents is then reassessed. Then the rule assigns the same payoffs to them as it did initially. Formally, for each  $C \in \mathcal{N}$ , each  $(d, E) \in \mathcal{B}^C$ , and each  $M \subset C$ , if  $x_C = F(d, E)$ , then  $x_M = F((d_i)_{i \in M}, \sum_{i \in M} x_i)$ . Bilateral consistency requires consistency only when  $|M| = 2$ .*

### 2.2 Hedonic games

Given  $N \in \mathcal{N}$ , let  $2^N \setminus \{\emptyset\}$  be the set of coalitions (subsets of  $N$ ). Each agent  $i \in N$  has a complete and transitive preference relation,  $\succsim_i$ , over the set of coalitions to which she belongs. Note that indifference is allowed. If  $i \in C \cap C'$ ,  $C \succsim_i C'$  means that agent  $i$  weakly prefers coalition  $C$  to  $C'$ . The preference profile of all agents  $\succsim_N = (\succsim_i)_{i \in N}$  defines a *hedonic game* denoted by  $(N, \succsim_N)$ . The class of all hedonic games is denoted by  $\mathcal{D}$ .

A *coalition structure* (or partition) of a finite set of agents  $N = \{1, \dots, n\}$  is a set  $\{C_1, \dots, C_k\}$  ( $k \leq n$  is a positive integer) such that (i) for each  $j = 1, \dots, k$ ,  $C_j \neq \emptyset$ , (ii)  $\bigcup_{j=1}^k C_j = N$ , and (iii) for each  $j, l \in \{1, \dots, k\}$  with  $j \neq l$ ,  $C_j \cap C_l = \emptyset$ .

A coalition structure  $\{C_1, \dots, C_k\}$  is *blocked* by a coalition  $C'$  if each member of  $C'$  is strictly better off in  $C'$  than in her component  $C_i$  to which she belongs. A coalition structure that admits no blocking coalitions is *stable*. The *core* is the set of all stable coalition structures. It is known that a hedonic game may have an empty core.

For coalition formation games, Pycia (2012) shows that under a rich domain of preferences and some coalition restrictions, there is a stable coalition structure for each preference profile if and only if agents' preferences satisfy pairwise alignment in each preference profile. Agents' preferences are pairwise aligned if any two agents rank coalitions

<sup>8</sup>A rule  $F$  is continuous if, for each  $(d, E) \in \mathcal{B}^C$  and for each sequence of problems  $\{(d^k, E^k)\}$  of elements of  $\mathcal{B}^C$ ,  $(d^k, E^k)$  converges to  $(d, E)$ , then the solution  $F(d^k, E^k)$  converges to  $F(d, E)$ .

that contain both of them in the same way.<sup>9</sup> Here we weaken the property of pairwise alignment and add another property to get stability for hedonic games.

DEFINITION 1. A preference profile  $\succsim_N$  satisfies *weak pairwise alignment* if for all  $C, C'$  and all  $i, j \in C \cap C'$ ,

$$\neg[C \succ_i C' \text{ and } C' \succ_j C].$$

That is, it cannot be the case that one agent ranks coalitions  $C$  and  $C'$  in one way while the other ranks them in the opposite way. This definition allows agent  $i$  to be indifferent between the two coalitions while agent  $j$  has a strict preference.

The class of all hedonic games that satisfy weak pairwise alignment is denoted by  $\mathcal{D}_{\text{WPA}} \subset \mathcal{D}$ .

Next, we introduce the second property. The lack of stability in hedonic games is caused by the existence of cycles among coalitions, which we refer to as rings. However, such structures do not always preclude stability (see Example 2). Indeed, it depends on the “position” of the ring in the preference profile under consideration (see Bloch and Diamantoudi 2011). There are several ways to define rings in preferences. We choose one that is coherent with the definition of weak pairwise alignment.

DEFINITION 2. A *ring* in a preference profile over coalitions  $\succsim_N$  is an ordered set of coalitions  $\{C_1, \dots, C_k\}$ ,  $k > 2$ , such that (subscripts modulo  $k$ )

$$C_{i+1} \succ_{S_i} C_i \text{ for all agents in } S_i = C_i \cap C_{i+1} \text{ and at least one agent with } \succ.$$

That is, there must be at least one agent at the intersection of any two consecutive coalitions,  $S_i$ , who strictly prefers  $C_{i+1}$  to  $C_i$ ; nonetheless, the other agents in  $S_i$  can be indifferent between the two coalitions. Thus, if cyclicity is interpreted as a transition from one coalition to the next, an agent is unable to implement such a transition unless her fellows at the intersection allow her to do so.<sup>10</sup> Hereafter a hedonic game that satisfies the weak pairwise aligned property and has no rings is called *noncircular*.

Next we show that noncircular hedonic games satisfy the “top coalition property” (Banerjee et al. 2001), which imposes a commonality of preferences among the players. This condition requires that for any nonempty subset  $S$  of players, we can find a coalition  $T \subseteq S$  such that all members of  $T$  prefer  $T$  to any other coalition that consists of some (or all) members of  $S$ . This is formally defined as follows.

DEFINITION 3. Given a nonempty set of agents  $V \subseteq N$ , a nonempty subset  $S \subseteq V$  is a *top coalition of  $V$*  if, for any  $i \in S$  and any  $T \subseteq V$  with  $i \in T$ , we have  $S \succsim_i T$ . A hedonic

<sup>9</sup>A preference profile over coalitions  $\succsim_N$  is pairwise aligned if, for all  $i, j \in C \cap C'$ ,  $C \succsim_i C' \iff C \succsim_j C'$ . This definition implies that if agent  $i$  is indifferent between two coalitions, so is agent  $j$ .

<sup>10</sup>Inal (2015) defines cyclicity by requiring that only one agent at the intersection of two consecutive coalitions strictly prefers the former over the latter. Pycia (2012) only requires a weak preference of a single agent at the intersection of any two consecutive coalitions with at least one strict preference. In both definitions, other members who belong to two consecutive coalitions can oppose the transition from one coalition to the next.

game satisfies the *top coalition property* if, for any nonempty set of agents  $V \subseteq N$ , there exists a top coalition of  $V$ .

This property is a generalization of the “common ranking property” (see Farrell and Scotchmer 1988), which requires the existence of a linear ordering over all the coalitions coincidental with any agent’s preferences. Hence, the following result may have some interest in itself because it defines a class of hedonic games that includes the class of games that satisfy the common ranking property (the proof is straightforward) and is contained in the class of games that satisfy the top coalition property, as shown in Theorem 1.<sup>11</sup>

**THEOREM 1.** *A noncircular hedonic game satisfies the top coalition property.*

**PROOF.** Let  $(N, \succsim_N)$  be a noncircular game. Let  $V \subseteq N$ , let  $|V| = v$ , and denote by  $\text{Ch}_i(V)$  the choice sets of agent  $i$  in coalition  $V$ . That is,  $\text{Ch}_i(V) = \{S \subseteq V : i \in S \text{ and } S \succsim_i T \text{ for all } T \subseteq V\}$ .<sup>12</sup> We show that, for a noncircular hedonic game, there exists at least one coalition  $S \subseteq V$  such that  $S \in \text{Ch}_i(V)$  for all  $i \in S$  and, therefore, it is a top coalition of  $V$ .

Assume to the contrary that there is no such  $S$ . Then  $S \neq \{i\}$  for each  $i \in V$  and, for each  $i \in V$  and for each  $S \in \text{Ch}_i(V)$ , there exist  $j \in V$ ,  $i \neq j$ , and  $S' \in \text{Ch}_j(V)$  such that  $S' \succ_j S$ . Using this fact, we define an iterative process with at most  $v - 1$  steps in which either the weak pairwise aligned property is not satisfied or there is a ring. At that point the process stops because a contradiction is reached.

Take any agent of  $V$  and denote her as agent 1.

**Step 1.** Let  $S_1 \in \text{Ch}_1(V)$  be a set chosen by agent 1. Then there exists an agent, say agent 2, with  $S_2 \in \text{Ch}_2(V)$ , such that  $S_2 \succ_2 S_1$ . If  $1 \in S_2$  and  $S_1 \succ_1 S_2$ , then the weak pairwise aligned property is violated. Otherwise, go to Step 2.

**Step**  $k \leq v - 1$ . Let  $S_k \in \text{Ch}_k(V)$  be a set chosen by agent  $k$ . Then there is an agent, say agent  $k + 1$ , with  $S_{k+1} \in \text{Ch}_{k+1}(V)$  such that  $S_{k+1} \succ_{k+1} S_k$ . Two cases are distinguished:

(i) We have  $S_{k+1} = S_i$ ,  $i \in \{1, \dots, k - 1\}$ .

- If  $S_{k+1} = S_{k-1}$  by Step  $k - 1$ , then  $S_k \succ_k S_{k-1}$  and the weak pairwise aligned property is violated.
- If  $S_{k+1} = S_i$ ,  $i \in \{1, \dots, k - 2\}$ , then  $S_i \succ_{k+1} S_k$ . By previous steps,  $S_{j+1} \succ_{j+1} S_j$  for all  $j = i, \dots, k$ . Then  $\{S_i, S_{i+1}, \dots, S_k\}$  is a ring.<sup>13</sup>

<sup>11</sup>The class of noncircular hedonic games does not have an inclusive relation with the class of hedonic games that satisfy the (weak) top-choice property (Karakaya 2011) (see the supplementary material of that paper) or with the class of hedonic games induced by separable preferences (Burani and Zwicker 2003). The notions of  $k$  acyclicity (Inal 2015) and ordinal balance (Bogomolnaia and Jackson 2002) are only defined for strict preferences.

<sup>12</sup>Note that if a hedonic game shows indifference between coalitions, any agent may have several choice sets.

<sup>13</sup>Note that any other agent at the intersection of two consecutive coalitions of the ring should order them in the same way or be indifferent; otherwise the weak pairwise aligned property is violated.

(ii) We have  $S_{k+1} \neq S_i$  for each  $i = 1, \dots, k - 1$ .

- If  $S_k \succ_k S_{k+1}$ , then the weak pairwise aligned property is violated.
- If  $S_i \succ_i S_{k+1}$ ,  $i \in \{1, \dots, k - 1\}$ , and given that, by the previous steps,  $S_{j+1} \succ_{j+1} S_j$  for all  $j = i + 1, \dots, k + 1$ , then  $\{S_{k+1}, S_i, \dots, S_k\}$  is a ring.

Otherwise, if  $k < v - 1$ , go to Step  $k + 1$ .

Note that if  $k = v - 1$ , given that  $S_v \neq \{v\}$  and there exists  $j$  such that  $S_j \succ_j S_v$ , then either  $S_v = S_i$ ,  $i \in \{1, \dots, v - 2\}$ , and a contradiction is reached in (i) or  $S_v \neq S_i$  for each  $i = 1, \dots, v - 2$  and a contradiction is reached in (ii).  $\square$

However, a hedonic game that satisfies the top coalition property may not satisfy the weak pairwise aligned property or may have rings, as the following examples illustrate.

EXAMPLE 1. <sup>14</sup> Let  $N = \{1, 2, 3\}$  and let  $\succ_N$  be

$$\begin{aligned} \{1\} \succ_1 \{12\} \succ_1 \{13\} \succ_1 \{123\}, \\ \{123\} \succ_2 \{12\} \succ_2 \{23\} \succ_2 \{2\}, \\ \{123\} \succ_3 \{23\} \succ_3 \{13\} \succ_3 \{3\}. \end{aligned}$$

This game satisfies the top coalition property but not the weak pairwise aligned property: Agent 1 strictly prefers  $\{12\}$  to  $\{123\}$ , while agent 2 orders these coalitions in the opposite way.  $\diamond$

EXAMPLE 2. Let  $N = \{1, 2, 3\}$  and let  $\succ_N$  be

$$\begin{aligned} \{123\} \succ_1 \{12\} \succ_1 \{13\} \succ_1 \{1\}, \\ \{123\} \succ_2 \{23\} \succ_2 \{12\} \succ_2 \{2\}, \\ \{123\} \succ_3 \{13\} \succ_3 \{23\} \succ_3 \{3\}. \end{aligned}$$

This game satisfies the top coalition property and has  $\{\{13\}, \{12\}, \{23\}\}$  as a ring.  $\diamond$

Banerjee et al. (2001) show that the top coalition property guarantees stability in hedonic games. Furthermore, if preferences are strict, a hedonic game has a unique stable coalition structure. Therefore, a hedonic game that satisfies weak pairwise alignment and has no rings admits at least one stable coalition structure.

### 2.3 The model and the solution

Let  $N \in \mathcal{N}$  be a set of agents and let  $2^N \setminus \{\emptyset\}$  be the set of coalitions (subsets of  $N$ ). For each  $i \in N$ , let  $d_i \in \mathbb{R}_+$  be the claim of agent  $i$  and let  $d_N = (d_i)_{i \in N} \in \mathbb{R}_+^N$  be the claims vector. For each  $C \subseteq N$ , let  $E(C)$  be the output of coalition  $C$ . A *coalition formation problem with claims* is a tuple  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$  satisfying  $\sum_{i \in C} d_i \geq E(C)$  for all  $C \in 2^N \setminus \{\emptyset\}$ .

<sup>14</sup>This is Example 3.5 in Bloch and Diamantoudi (2011).

For each  $C \in 2^N \setminus \{\emptyset\}$ , a rationing problem, derived from a coalition formation problem with claims, is defined as  $(d_C, E(C))$ , where  $d_C = (d_i)_{i \in C}$  is the claims vector<sup>15</sup> of the agents in coalition  $C$  and  $E(C)$  is the output of coalition  $C$ .

A rule is a mapping  $F$  that associates to each  $C \in 2^N \setminus \{\emptyset\}$  and each  $(d_C, E(C)) \in \mathcal{B}^C$  an allocation  $x_C = (x_i)_{i \in C}$  such that  $\sum_{i \in C} x_i = E(C)$ .

Let  $F(d_C, E(C)) = x_C$ . If  $C' \subset C$ , then  $F_{C'}(d_C, E(C)) = x_{C'}$ , where  $x_{C'} = (x_i)_{i \in C'}$  is the vector of payoffs of agents in coalition  $C'$ . In particular, the payoff of agent  $i$  in coalition  $C$  is denoted by  $x_i = F_i(d_C, E(C))$ .

Clearly, each agent prefers a higher payoff to a lower one and orders the coalitions to which she belongs accordingly. That is, if  $F_i(d_C, E(C)) \geq F_i(d_{C'}, E(C'))$ , then  $C \succsim_i C'$ .

Thus, a coalition formation problem with claims and a rule induces a hedonic game in which the *core stability* is the solution to be studied.

### 2.4 A numerical example

We conclude the section with a numerical example that illustrates how some rationing rules applied to a coalition formation problem with claims induce hedonic games.

Let  $N = \{1, 2, 3, 4\}$  be the set of agents with  $d = (10, 50, 50, 60)$ . Assume that coalitions  $\{13\}$ ,  $\{23\}$ ,  $\{123\}$ ,  $\{124\}$  have the outputs shown in Table 1, while the remaining outputs are 0.

Coalitions	{13}	{23}	{123}	{124}
Outputs	20	34	25	55

TABLE 1. Positive coalitional outputs.

Table 2 shows the individual payoffs obtained from different rules.

	{13}	{23}	{123}	{124}
Shapley value	(5, 15)	(17, 17)	(20/6, 65/6, 65/6)	(25/6, 145/6, 160/6)
CEA	(10, 10)	(17, 17)	(25/3, 25/3, 25/3)	(10, 45/2, 45/2)
CEL	(0, 20)	(17, 17)	(0, 25/2, 25/2)	(0, 45/2, 65/2)

TABLE 2. Divisions of coalitional outputs.

First, consider the *Shapley value* as the rationing rule used to distribute each coalitional output among its members. To compute it, line up the agents in all possible orderings. Beginning at the front of the line, pay each agent in full until her claim is satisfied. The Shapley value for each agent is the average payoff over all possible orderings. According to Table 2, the resulting preference profile given by the Shapley value is

$$\{13\} \succ_1 \{124\} \succ_1 \{123\} \succ_1 \{1\} \sim_1 \dots,$$

<sup>15</sup>Note that the components of  $d_C$  are the components of  $d_N$  restricted to the agents in  $C$ .

$$\begin{aligned} &\{124\} \succ_2 \{23\} \succ_2 \{123\} \succ_2 \{2\} \sim_2 \dots, \\ &\{23\} \succ_3 \{13\} \succ_3 \{123\} \succ_3 \{3\} \sim_3 \dots, \\ &\{124\} \succ_4 \{4\} \sim_4 \dots. \end{aligned}$$

This game satisfies pairwise alignment and weak pairwise alignment. Moreover,  $\{\{13\}, \{23\}, \{124\}\}$  is a ring, which is also a cycle in Pycia’s terms. It can be checked that the Shapley value applied to this example does not induce stable coalition structures.

Next, consider the *constrained equal awards* (CEA) rule, which divides each coalitional output as equally as possible under the constraint that no agent receives more than her claim. By contrast, the *constrained equal losses* (CEL) rule divides the total loss (the difference between the sum of claims and the coalitional output) as equally as possible under the constraint that no agent receives a negative amount. According to Table 2, the resulting preference profiles given by CEA and CEL ((a) and (b), respectively) are

$$\begin{array}{ll} \{13\} \sim_1 \{124\} \succ_1 \{123\} \succ_1 \{1\} \sim_1 \dots & \{13\} \sim_1 \{124\} \sim_1 \{123\} \succ_1 \{1\} \sim_1 \dots \\ \{124\} \succ_2 \{23\} \succ_2 \{123\} \succ_2 \{2\} \sim_2 \dots & \{124\} \succ_2 \{23\} \succ_2 \{123\} \succ_2 \{2\} \sim_2 \dots \\ \{23\} \succ_3 \{13\} \succ_3 \{123\} \succ_3 \{3\} \sim_3 \dots & \{13\} \succ_3 \{23\} \succ_3 \{123\} \succ_3 \{3\} \sim_3 \dots \\ \{124\} \succ_4 \{4\} \sim_4 \dots & \{124\} \succ_4 \{4\} \sim_4 \dots \end{array}$$

(a) (b)

The CEA rule induces the hedonic game (a), in which  $\{\{124\}, \{3\}\}$  is a stable coalition structure. The CEL rule induces the hedonic game (b), in which  $\{\{13\}, \{2\}, \{4\}\}$  and  $\{\{124\}, \{3\}\}$  are stable coalition structures. Both are noncircular hedonic games. Game (a) satisfies pairwise alignment. Moreover,  $\{\{23\}, \{124\}, \{13\}\}$  is a cycle according to Pycia’s definition because agent 2 strictly prefers  $\{124\}$  to  $\{23\}$ , agent 1 is indifferent between  $\{13\}$  to  $\{124\}$ , and agent 3 strictly prefers  $\{23\}$  to  $\{13\}$ , but it is not a ring. Game (b) does not satisfy pairwise alignment because agent 3 strictly prefers coalition  $\{13\}$  to  $\{123\}$ , whereas agent 1 is indifferent between them. Moreover,  $\{\{23\}, \{124\}, \{123\}\}$  is a cycle according to Pycia’s definition because agent 2 strictly prefers  $\{124\}$  to  $\{23\}$ , agent 3 strictly prefers  $\{23\}$  to  $\{123\}$ , and agent 1 is indifferent between  $\{123\}$  and  $\{124\}$ , but it is not a ring.

### 3. RATIONING RULES INDUCING STABILITY

In this section, we characterize the rules that induce stable coalition structures in hedonic games. We show that only rules that satisfy Axioms 1 and 2 induce hedonic games that are weak pairwise aligned (Lemma 2) and have no rings (Lemma 3). These two lemmas and Theorem 1 prove the characterization result of the paper (Theorem 2).

First, we introduce a lemma that says that the payoffs given by a consistent rule in a rationing problem can be obtained in an extended rationing problem in which one or more agents are added.

LEMMA 1. *Let  $(d_C, E(C))$  be a rationing problem and let  $F$  be a rule that satisfies consistency such that  $F(d_C, E(C)) = x_C$ . Then for each  $C'$ ,  $C \subset C'$ , there is another rationing problem  $(d_{C'}, \bar{E}(C'))$  such that  $F_i(d_{C'}, \bar{E}(C')) = x_i$  for all  $i \in C$ .*

PROOF. Let  $F(d_C, E(C)) = x_C$ , where  $F$  is consistent. Let  $C'$  be a coalition such that  $C \subset C'$ . Consider rationing problem  $(d_{C'}, E(C'))$ . Define  $\alpha(E(C')) = \sum_{i \in C} F_i(d_{C'}, E(C'))$ . Observe that, since  $F$  is continuous,  $\alpha$  is a continuous function of  $E(C')$  in the interval  $[0, \sum_{i \in C'} d_i]$  with  $\alpha(0) = 0$  and  $\alpha(\sum_{i \in C'} d_i) = \sum_{i \in C} d_i$ . By continuity of  $\alpha$ , there exists  $\bar{E}(C')$ ,  $0 \leq \bar{E}(C') \leq \sum_{i \in C'} d_i$ , such that  $\alpha(\bar{E}(C')) = \sum_{i \in C} x_i$ . Let  $F(d_{C'}, \bar{E}(C')) = y_{C'}$ . By construction,  $\sum_{i \in C} y_i = \sum_{i \in C} x_i$ , and by consistency,  $F_i(d_{C'}, \bar{E}(C')) = x_i$  for all  $i \in C$ .<sup>16</sup>  $\square$

LEMMA 2. *Only rules that satisfy Axioms 1 and 2 induce hedonic games that satisfy the weak pairwise aligned property.*

PROOF. On the one hand, we prove that if rule  $F$  does not satisfy Axiom 1 or Axiom 2, there is an induced hedonic game that does not belong to  $\mathcal{D}_{\text{WPA}}$ .

First, assume that rule  $F$  does not satisfy Axiom 1. Let  $N \in \mathcal{N}$  be a set of agents and let  $C \in 2^N \setminus \{\emptyset\}$ . Let  $(d_C, E(C))$  and let  $(d_C, E'(C))$  be two rationing problems such that  $E'(C) > E(C) > 0$ . Let  $F$  be a rule such that  $F(d_C, E(C)) = x_C$  and  $F(d_C, E'(C)) = x'_C$ . The lack of monotonicity of rule  $F$  for these two rationing problems implies that there exists at least one agent in  $C$ , say agent  $j$ , such that  $x_j > x'_j$  and, consequently, there exists at least another agent, say agent  $k$ , such that  $x_k < x'_k$ .

If  $|C| > 2$ , construct rationing problem  $(d_{\{j,k\}}, x_j + x_k)$  such that  $F(d_{\{j,k\}}, x_j + x_k) = (y_j, y_k)$ . Two cases are distinguished:

- (i) We have  $(y_j, y_k) = (x_j, x_k)$ . In this case  $y_j > x'_j$  and  $y_k < x'_k$ . Now define a coalition formation problem with claims  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$ , with outputs  $E'(C)$  and  $x_j + x_k$  for coalitions  $C$  and  $\{j, k\}$ , respectively. Then agents  $j$  and  $k$  order coalitions  $C$  and  $\{j, k\}$  in opposite ways. Hence, the hedonic game induced by rule  $F$  from  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$  does not belong to  $\mathcal{D}_{\text{WPA}}$ .
- (ii) We have  $(y_j, y_k) \neq (x_j, x_k)$ . In this case,  $y_j > x_j$  and  $y_k < x_k$  or vice versa. Now define a coalition formation problem with claims  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$ , with outputs  $E(C)$  and  $x_j + x_k$  for coalitions  $C$  and  $\{j, k\}$ , respectively. Then agents  $j$  and  $k$  order coalitions  $C$  and  $\{j, k\}$  in opposite ways. Hence, the hedonic game induced by rule  $F$  from  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$  does not belong to  $\mathcal{D}_{\text{WPA}}$ .

If  $|C| = 2$ , without loss of generality (w.l.o.g.), let  $C = \{j, k\}$ , and construct rationing problem  $(d_{C'}, x_j + x_k)$  such that  $\{j, k\} \subset C' \in 2^N \setminus \{\emptyset\}$  with  $d_i = 0$  for each  $i \neq j, k$ . Let  $F_{\{j,k\}}(d_{C'}, x_j + x_k) = (y_j, y_k)$ . Note that any other agent in  $C'$  receives 0. Reasoning as in cases (i) and (ii), we get that agents  $j$  and  $k$  order coalitions  $\{j, k\}$  and  $C'$  in opposite ways. Hence, the hedonic game induced by rule  $F$  from  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$  does not belong to  $\mathcal{D}_{\text{WPA}}$ .

<sup>16</sup>Note that the proof is done for any value of  $d_i \in \mathbb{R}_+$ . If  $(d_i)_{i \in C \setminus C} = 0$ , then  $\bar{E}(C') = E(C)$  and the proof is trivial.

Second, assume that rule  $F$  does not satisfy consistency. Let  $N \in \mathcal{N}$  be a set of agents and let  $C, C' \in 2^N \setminus \{\emptyset\}$  be two coalitions such that  $C \subset C'$ . Let  $(d_{C'}, E(C'))$  be a rationing problem and let  $F$  be a rule such that  $F(d_{C'}, E(C')) = x_{C'}$ . Consider rationing problem  $(d_C, \sum_{i \in C} x_i)$ . As rule  $F$  is not consistent, there exist at least two agents in  $C$ , say agents  $j$  and  $k$ , such that  $F_j(d_C, \sum_{i \in C} x_i) > x_j$  and  $F_k(d_C, \sum_{i \in C} x_i) < x_k$  or vice versa. Now define a coalition formation problem with claims  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$  with outputs  $E(C')$  and  $\sum_{i \in C} x_i$  for coalitions  $C'$  and  $C$ , respectively. In this case, either agent  $j$  ranks  $C$  over  $C'$  and agent  $k$  ranks  $C'$  over  $C$  or vice versa. Hence, the hedonic game induced by rule  $F$  from  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$  does not belong to  $\mathcal{D}_{WPA}$ .

On the other hand, we prove that if rule  $F$  satisfies Axioms 1 and 2, then it induces hedonic games in  $\mathcal{D}_{WPA}$ .

Let  $N \in \mathcal{N}$  be a set of agents and let  $C, C' \in 2^N \setminus \{\emptyset\}$  be two coalitions such that agents  $j, k \in C \cap C'$ . Let  $(d_C, E(C)), (d_{C'}, E(C'))$  be two rationing problems. Assume that a resource monotonic and consistent rule  $F$  gives  $x_C = F(d_C, E(C))$  and  $x'_{C'} = F(d_{C'}, E(C'))$ .

Next consider two auxiliary rationing problems  $(d_{\{j,k\}}, (x_j + x_k))$  and  $(d_{\{j,k\}}, (x'_j + x'_k))$ . If rule  $F$  satisfies Axiom 2, then

$$x_{\{j,k\}} = F(d_{\{j,k\}}, (x_j + x_k)) \quad \text{and} \quad x'_{\{j,k\}} = F(d_{\{j,k\}}, (x'_j + x'_k)).$$

As rule  $F$  is resource monotonic, one of the following two cases is satisfied:

- (i) We have  $x_j + x_k \geq x'_j + x'_k$ :

$$x_{\{j,k\}} = F(d_{\{j,k\}}, (x_j + x_k)) \geq F(d_{\{j,k\}}, (x'_j + x'_k)) = x'_{\{j,k\}}.$$

Since  $x_{\{j,k\}} = F_{\{j,k\}}(d_C, E(C))$  and  $x'_{\{j,k\}} = F_{\{j,k\}}(d_{C'}, E(C'))$ ,

$$F_{\{j,k\}}(d_C, E(C)) \geq F_{\{j,k\}}(d_{C'}, E(C')).$$

Therefore, agents  $j$  and  $k$  weakly prefer coalition  $C$  to coalition  $C'$ , so they do not rank them in opposite ways.

- (ii) We have  $x_j + x_k \leq x'_j + x'_k$ . Proceeding as in case (i), agents  $j$  and  $k$  weakly prefer coalition  $C'$  to coalition  $C$ . Therefore, they do not rank these two coalitions in opposite ways.

Thus, we have shown that either  $F_{\{j,k\}}(d_C, E(C)) \geq F_{\{j,k\}}(d_{C'}, E(C'))$  or  $F_{\{j,k\}}(d_C, E(C)) \leq F_{\{j,k\}}(d_{C'}, E(C'))$ , i.e., agents  $j$  and  $k$  cannot rank coalitions  $C$  and  $C'$  in opposite ways. As this argument follows for each two coalitions  $C, C' \in 2^N \setminus \{\emptyset\}$  and each two agents  $i, j \in C \cap C'$ , the hedonic game induced from  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$  belongs to  $\mathcal{D}_{WPA}$ . □

The next example shows that there are hedonic games satisfying weak pairwise alignment that do not admit stable coalition structures.

EXAMPLE 3. Let  $N = \{1, 2, 3\}$  and let  $\succsim_N$  be

$$\begin{aligned} \{12\} &\succ_1 \{13\} \succ_1 \{123\} \succ_1 \{1\}, \\ \{23\} &\succ_2 \{12\} \succ_2 \{123\} \succ_2 \{2\}, \\ \{13\} &\succ_3 \{23\} \succ_3 \{123\} \succ_3 \{3\}. \end{aligned}$$

In this game, which satisfies weak pairwise alignment, there is no stable coalition structure due to the existence of ring  $\{\{13\}, \{12\}, \{23\}\}$ .  $\diamond$

The following lemma discards hedonic games in which weak pairwise alignment and rings coexist as games induced by any rule that satisfies the two properties under analysis.

LEMMA 3. *If a rule satisfies Axioms 1 and 2, then the induced hedonic game does not contain rings.*

PROOF. Let  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$  be a coalition formation problem with claims and let  $F$  be a rule that satisfies Axioms 1 and 2. Assume that  $F$  induces a hedonic game with ring  $\{C_1, \dots, C_k\}$ ,  $k > 2$ .

By the definition of ring, there exists at least one agent, say agent  $j_{i+1} \in C_{i+1} \cap C_i$ , such that  $C_{i+1} \succ_{j_{i+1}} C_i$  for each  $i = 1, \dots, k$  (subscripts modulo  $k$ ). Observe that by transitivity of individual preferences, it cannot happen that  $j_1 = \dots = j_k$ . Now we claim that  $C^* = \bigcup_{i=1}^k C_i$  does not belong to ring  $\{C_1, \dots, C_k\}$ , i.e.,  $C^* \neq C_i$  for each  $i = 1, \dots, k$ . Assume to the contrary that  $C^* = C_i$  for some  $i \in \{1, \dots, k\}$ . Without loss of generality, let  $C^* = C_1$ . Then  $C_2 \succ_{j_2} C^*$  and  $C^* \succ_{j_1} C_k$ . Since  $C_3 \succ_{j_3} C_2$  and, by weak pairwise alignment,  $C_2 \succ_{j_3} C^*$ , then  $C_3 \succ_{j_3} C^*$ . Likewise,  $C_i \succ_{j_i} C^*$  when  $i > 3$ . Now if  $j_k = j_1$ , then  $C^* \succ_{j_1} C_k$  and  $C_k \succ_{j_1} C^*$ , and, therefore, agent  $j_1$ 's preference is not transitive and  $F$  does not induce a hedonic game. If  $j_k \neq j_1$ , then  $C^* \succ_{j_1} C_k$  and  $C_k \succ_{j_k} C^*$ , and, therefore, the weak pairwise aligned property is not satisfied and, by Lemma 2,  $F$  does not satisfy Axioms 1 and 2.

Next, from the coalition formation problem with claims  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$ , construct a new problem  $(d_N, E'(C)_{C \in 2^N \setminus \{\emptyset\}})$  in which only the output of coalition  $C^*$  is modified from  $E(C^*)$  to  $E'(C^*)$ . This new output is derived as follows: Consider the rationing problem  $(d_{C_1}, E(C_1))$  with  $F(d_{C_1}, E(C_1)) = x_{C_1}$ . As  $C_1 \subset C^*$ , by Lemma 1, there exists a rationing problem  $(d_{C^*}, E'(C^*))$  with  $E'(C^*)$  such that  $F_{C_1}(d_{C^*}, E'(C^*)) = x_{C_1}$ . Then, by construction, the agents in  $C_1$  receive the same payoffs in  $C_1$  as in  $C^*$ , and, therefore, they are indifferent between these two coalitions. Hence,  $C^*$  would take part in ring  $\{C^*, \dots, C_k\}$ , which is impossible from the previous argument. Therefore, ring  $\{C_1, \dots, C_k\}$  cannot be induced by a resource monotonic and consistent rule  $F$ .  $\square$

Finally, Lemmas 2 and 3 jointly with Theorem 1 prove the characterization result.

THEOREM 2. *There is a stable coalition structure for each hedonic game induced by rule  $F$  if and only if  $F$  is resource monotonic and consistent.*

### 3.1 Parametric rules induce stability

In rationing problems, there are many rationing rules that are resource monotonic and consistent. The most important ones are the “parametric” rules. The payoffs of an agent given by a rule in this class can be obtained by a function that only depends on her individual claim and a parameter,  $\lambda$ , common to all agents, that is chosen so that the endowment is exhausted. The fact that a common parameter intervenes in the division of the endowment may be interpreted as a way to equalize the payoffs that agents would receive by considering only their own claims. Young (1987) characterizes symmetric parametric rules using *bilateral consistency* and symmetry.<sup>17</sup> Recently, there has been growing interest in studying parametric rules (see, for instance, Kaminski 2006, Stovall 2014, and Erlanson and Flores-Szwagrzak 2015). Stovall characterizes the family of symmetric and asymmetric parametric rules using continuity, resource monotonicity, bilateral consistency, and two additional axioms. Therefore, this family of rules induces stable hedonic games. A simple proof of this result is given below.

A parametric rule is defined as follows:

Let  $f$  be a collection of functions  $\{f_i\}_{i \in N}$ ,<sup>18</sup> where each  $f_i : \mathbb{R}_+ \times [a, b] \rightarrow \mathbb{R}_+$  is continuous and weakly increasing in  $\lambda$ ,  $\lambda \in [a, b]$ ,  $-\infty \leq a < b \leq \infty$ , and for each  $i \in N$  and  $d_i \in \mathbb{R}_+$ ,  $f_i(d_i, a) = 0$  and  $f_i(d_i, b) = d_i$ . Hence, for each  $f$ , a rule  $F$  for rationing problem  $(d, E)$  is defined as follows.

For each  $i \in N$ ,

$$F_i(d, E) = f_i(d_i, \lambda) \quad \text{where } \lambda \text{ is chosen so that } \sum_{i \in N} f_i(d_i, \lambda) = E.$$

Thus,  $f$  is said to be a *parametric representation of  $F$* .

The proportional, constrained equal awards, constrained equal losses, and the Talmud and reverse Talmud rules are symmetric parametric rules, while the dictatorial rule with strict priority is an asymmetric parametric rule.<sup>19</sup>

**PROPOSITION 1.** *A parametric rule induces hedonic games that have at least one stable coalition structure.*

**PROOF.** (i) First, we show that parametric rules induce hedonic games that satisfy the weak pairwise aligned property.

Consider a coalition formation problem with claims  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$ . If a preference profile over coalitions does not satisfy the weak pairwise aligned property, then there exist  $i, j \in C \cap C'$ , where  $C, C' \in 2^N \setminus \{\emptyset\}$  such that  $C \succ_i C'$  and  $C' \succ_j C$ .

Let  $F$  be a parametric rule that induces a hedonic game solving the problem above. By definition of  $F$ , for each  $C \in 2^N \setminus \{\emptyset\}$  and each  $i \in C$ , there exist a collection of functions  $f$  and a parameter  $\lambda$  such that  $x_i = f_i(d_i, \lambda)$ . For the sake of convenience, we denote the value of  $\lambda$  for coalition  $C$  by  $\lambda(C)$ . Let  $x_i = f_i(d_i, \lambda(C))$ , and let  $y_i = f_i(d_i, \lambda(C'))$

<sup>17</sup>Two agents with equal claims should receive equal payoffs.

<sup>18</sup>When the rule is symmetric,  $f_i$  is the same for all agents.

<sup>19</sup>Moulin (2000) characterizes a class of asymmetric rules using consistency and other properties.

be the allocations given by rule  $F$  to agent  $i$  and similarly for agent  $j$ . If  $C \succ_i C'$ , then  $x_i > y_i$  and  $\lambda(C) > \lambda(C')$ . Alternatively, if  $C' \succ_j C$ , then  $y_j > x_j$  and, therefore,  $\lambda(C') > \lambda(C)$ , which is a contradiction.

(ii) Second, we show that parametric rules do not induce hedonic games with rings.

Consider a coalition formation problem with claims  $(d_N, E(C)_{C \in 2^N \setminus \{\emptyset\}})$  and a parametric rule  $F$ . Assume that the induced hedonic game contains ring  $\{C_1, \dots, C_k\} \subset 2^N \setminus \{\emptyset\}$ . Let  $\{S_1, \dots, S_k\}$  be the sets of agents such that  $S_j = C_j \cap C_{j+1}$  (subscripts modulo  $k$ ).

By definition of parametric rule, for each coalition  $C_j$  in the ring and each agent  $i$  in  $C_j$ , there exist a collection of functions  $f$  and a parameter  $\lambda$  such that  $x_i = f_i(d_i, \lambda)$ . For the sake of convenience, we denote the payoff of agent  $i$  in coalition  $C_j$  by  $x_i(C_j)$  and denote the value of  $\lambda$  associated to coalition  $C_j$  by  $\lambda(C_j)$ .

Consider  $C_{j+1} \succ_{S_j} C_j$ , where  $S_j = C_j \cap C_{j+1}$  with at least one agent in  $S_j$  having strict preference. Let  $i \in S_j$  be such an agent. Observe that no other agent at the intersection opposes agent  $i$ 's strict preference of  $C_{j+1}$  over  $C_j$ . Therefore,

$$\begin{aligned} C_{j+1} \succ_{S_j} C_j &\implies C_{j+1} \succ_i C_j \implies x_i(C_{j+1}) > x_i(C_j) \\ &\implies f_i(d_i, \lambda(C_{j+1})) > f_i(d_i, \lambda(C_j)) \implies \lambda(C_{j+1}) > \lambda(C_j). \end{aligned}$$

Since this is true for  $\{C_1, \dots, C_k\}$  (subscripts modulo  $k$ ), a contradiction is reached.

Finally, considering (i) and (ii) together with [Theorem 1](#), it can be stated that parametric rules induce at least one stable coalition structure.  $\square$

Parametric rules, however, are not the only rules that verify our results. There are continuous nonparametric rules that induce stability in hedonic games, such as the following example borrowed from [Stovall \(2014\)](#) shows.

**EXAMPLE 4.** For  $i \neq 1$ , let  $f_i(d_i, \lambda) = \lambda d_i$  be  $i$ 's parametric function on  $[0, 1]$ . For  $i = 1$ ,  $f_1$  is not a function, but is a correspondence on  $[0, 1]$  defined by

$$f_1(d_1, \lambda) = \begin{cases} 0 & \text{for } \lambda < \frac{d_1}{1+d_1}, \\ [0, d_1] & \text{for } \lambda = \frac{d_1}{1+d_1}, \\ d_1 & \text{for } \lambda > \frac{d_1}{1+d_1}. \end{cases}$$

Observe that for any  $(d_C, E(C))$ , there is a unique  $\lambda$  such that  $E(C) \in \sum_{i \in N} f_i(d_i, \lambda)$ . So define a rule  $F$  as follows: For  $i \neq 1$ ,

$$F_i(d_C, E(C)) = f_i(d_i, \lambda) = \lambda d_i,$$

and for  $i = 1$ ,

$$F_1(d_C, E(C)) = E(C) - \sum_{i \in N \setminus \{1\}} F_i(d_C, E(C)),$$

where  $\lambda$  is chosen so that  $E(C) \in \sum_{i \in N} f_i(d_i, \lambda)$ .  $\diamond$

Stovall (2014) shows that  $F$  has no parametric representation but nevertheless satisfies continuity, consistency, and resource monotonicity. This is a simplified exposition of Stovall's example in our context that justifies the characterization of rules beyond the class of parametric rules. Such a characterization is given in [Theorem 2](#).

#### 4. CONCLUDING REMARKS

In this paper, we introduce a coalition formation problem with claims that links the literatures on rationing problems and hedonic games. We start with a group of agents, each one pondering whether to join coalitions to produce an output. Each coalitional output is rationed among its members by a rule, which takes the agent's claims over the outputs as input. Thus, every agent orders the coalitions that she can join according to the payoffs proposed by the rule. The orderings define a hedonic game. It turns out that only resource monotonic and consistent rationing rules (among continuous rules) induce noncircular hedonic games that are core-stable.

In our approach, claims and outputs are assumed to be exogenous and independent of each other. Other assumptions about claims and outputs may also be realistic: Each agent's claim may well depend on the identity of the members of the coalition she can join, and outputs may be a function of the size of the coalition or may be contingent on how the remaining agents are organized.<sup>20</sup> These and related considerations offer potential for future research.

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<sup>20</sup>Some of these modifications may generate externalities across coalitions and call for the analysis of partition function form games (see Thrall and Lucas 1963).

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