Iterated weak dominance and interval-dominance supermodular games*

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Abstract

This paper extends Milgrom and Robert's treatment of supermodular games in two ways. It points out that their main characterization result holds under a weaker assumption. It refines the arguments to provide bounds on the set of strategies that survive iterated deletion of weakly dominated strategies. I derive the bounds by iterating the best-response correspondence. I give conditions under which they are independent of the order of deletion of dominated strategies. The results have implications for equilibrium selection and dynamic stability in games. *Journal of Economic Literature* Classification Numbers: C72, D81.

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1 Introduction

Milgrom and Roberts [17] and Vives [26] provide useful analyses of the class of supermodular games introduced by Topkis [24]. In a supermodular game, each player's strategy set is partially ordered and there are strategic complementarities that cause a player's best response to be increasing in opponents' strategies. Milgrom and Roberts and Vives describe many applications of the games in the class. Perhaps the leading example is the linear Cournot duopoly.

Milgrom and Roberts [17] and Vives [26] demonstrate that supermodular games have a largest and smallest equilibrium. These equilibria necessarily are pure-strategy Nash equilibria. Milgrom and Roberts demonstrate that these extreme equilibria can be obtained by iterating the best-response correspondence and characterize the set of strategies that survive iterated deletion of strictly dominated strategies. They show that pure-strategy Nash equilibria exist in supermodular games. In addition, they provide useful results about comparative statics and dynamic stability. These results enable models to make more precise and more confident predictions for games with complementarities. Predictions are more precise because of the bounds; in general, largest and smallest Nash equilibria need not exist. Predictions are more confident because they may not require equilibrium assumptions. Milgrom and Roberts give examples (including Cournot duopoly) in which the largest equilibrium is equal to the smallest equilibrium. This observation not only guarantees uniqueness of equilibrium, but also implies that the unique equilibrium is the only outcome that survives iterated deletion of strictly dominated strategies. Hence the prediction does not depend on the assumption of equilibrium. What is more, the argument guarantees that a best-response dynamic arrives at the equilibrium, again suggesting that the equilibrium prediction has a strong behavioral foundation. When the game has multiple equilibria, the existence of lower and upper bounds still offers useful limits on predictions. Further, tools of monotone comparative statics enable me to make statements about how the set of equilibria responds to changes in parameters.

Not all games are supermodular. The current paper shows how to modify the techniques pioneered by Milgrom and Roberts and Vives to a broader class. The extension has two parts. First, I show that Milgrom and Roberts's main results extend without modification to a slightly broader class of games. This extension is a small one both logically and substantively. The logical extension is small because one can prove the result with little modification to Milgrom and Roberts's argument. The substantive extension is small because I do not have an economic applications in which the more general result provides a novel insight. The second extension is more substantial. I establish results that parallel those obtained by Milgrom and Roberts using a stronger solution concept, deleting weakly dominated strategies rather than strongly dominated strategies. That is, I enlarge the class of supermodular games and describe the sets of strategies that survive iterated deletion of weakly dominates strategies.[1] Once again, this set will be bounded by a largest and smallest equilibrium. Analogs of existing results on dynamics and comparative statics also hold. This extension requires a modification of the existing

[1] In related work, Kultti and Salonen [11] and [12] study supermodular games in which some weakly dominated strategies are removed. I discuss these papers in Section 4.

proof technique (so the paper makes a technical contribution) and allows one to apply the results to a class of games that includes a familiar model of communication that fails to be supermodular. The second extension is substantive because there are games that satisfy the generalized definition of supermodularity and have large sets of strategies that survive iterated strong dominance, but smaller sets that survive iterated weak dominance.

My extension is useful precisely because there are games with strategic complementarities in which strong dominance has little power to eliminate strategies, but weak dominance is effective. It is not hard to generate games in this class. Imagine a game obtained by adding an initial round to a supermodular game to create a two-stage game. The initial stage might involve an investment choice, communication, or an attempt to learn about the environment. When viewed as a strategic-form game, there will typically be weakly dominated strategies (in complete information games, these strategies may involve choosing second-stage actions that would be eliminated by an application of subgame perfection). Elimination of strictly dominated strategies will generally lack the power to reduce the strategy set, but eliminating weakly dominated strategies may be effective. I apply the methods of this paper to cheap-talk games. I show that the set of strategies that survive iterated deletion of weakly dominated strategies has nice features (upper and lower bounds and attractive dynamic stability properties) and is strictly smaller than the set of strategies that survive iterated deletion of strictly dominated strategies.

The analysis leverages two things. Discarding strategies that are weakly dominated instead of strongly dominated has the potential to make the set of predictions stronger. Broadening the definition of supermodular games by weakening an assumption has the potential to enlarge the class of games covered by the argument. I expand the class of supermodular games by replacing an increasing-difference condition used by Milgrom and Roberts and Vives with a weaker condition, interval dominance, introduced by Quah and Strulovici [22]. The central property used in the literature is that best response correspondences are increasing. Increasing differences guarantees monotonic best replies, but interval dominance, a weaker condition, also implies the critical monotonicity property.

Section 3 points out a small generalization of the basic result of Milgrom and Roberts characterizing the set of strategies that survive iterated deletion of strictly dominated strategies. Section 4 extends the results to weak dominance. Section 5 discusses the implications of the characterization result for comparative statics and dynamics.

Section 6 discusses cheap-talk games and games involving competition in persuasion. I demonstrate that analogs of the methods introduced to study supermodular games can refine the set of predictions in these games. These games are not interval-dominance supermodular, but they satisfy a weaker condition under which the main characterization result applies. Specifically, Section 6.1 studies cheap-talk games and demonstrates that, when a monotonicity assumption holds, the babbling equilibrium fails to survive iterated deletion of weakly dominated strategies whenever an informative equilibrium exists. Section 6.2 describes a game introduced by Gentzkow and Kamenica [9] to study Bayesian persuasion with multiple informed parties. This game typically has multiple equilibria that are Pareto-ranked from the perspective of the informed players. I point out that only the Pareto-efficient equilibrium survives iterated deletion of weakly dominated strategies.

I place definitions of standard concepts in Appendix A (in order to make the paper self contained). Appendix B contains proofs omitted from the main text. Appendix C contains auxiliary results about interval-dominance conditions. Appendix D contains arguments that support claims made in Section 6.2

2 Preliminaries

There is a finite set of players. I denotes the player set. Each player has a strategy set X_i with typical element x_i. $X = \Pi_{i \in I} X_i$ is the set of strategy profiles. I denote by x_{-i} the strategies of Player i's opponents. Each strategy set is partially ordered by $\geq_i$; $\geq$ denotes the product order derived from the $\geq_i$ (so that $x \geq x'$ if and only if $x_i \geq_i x_i$ for all i). Denote Player i's utility function by $u_i(x_i, x_{-i})$. Denote by $u = (u_i)_{i \in I}$ the set of utility functions. A game in ordered-normal form is $\Gamma = (I, X, u, \geq)$.

Consider a set X with a partial order $\geq$ that is transitive, reflexive, and antisymmetric. I place standard definitions (lattice, chain, order continuity, supermodularity, strong set order) in Appendix A.

The paper uses weaker versions of basic single-crossing properties. I review the basic ideas and then discuss the role they play in studying games in ordered-normal form.

Definition 1. *Given two lattices X and Y, a function $f : X \times Y \to \mathbb{R}$ has increasing differences in its two arguments x and y if for all $x'' \geq x'$, the difference $f(x'', y) - f(x', y)$ is nondecreasing in y.*

This paper replaces increasing differences with weaker assumptions. There are several ways to weaken the increasing-differences property. The next definition is standard.

Definition 2. *Given two lattices X and Y, a function $f : X \times Y \to \mathbb{R}$ satisfies the single-crossing property in its two arguments x and y if for all $y'' > y'$, $x'' > x'$,*

$$f(x'', y') \geq (>) f(x', y') \implies f(x'', y'') \geq (>) f(x', y''). \tag{1}$$

Single crossing is also more restrictive than necessary.

Definition 3. *Given two lattices X and Y. A function $f : X \times Y \to \mathbb{R}$ satisfies the interval-dominance (ID) property in its two arguments x and y if for all $y'' > y'$, $x'' > x'$, (1) holds whenever $f(x'', y') \geq f(x, y')$ for all $x \in [x', x'']$.*

Quah and Strulovici [22] introduces Condition (ID) and derives basic properties. Quah and Strulovici [21] contains additional results, including detailed discussion of the implications of (ID) when X is multidimensional. It is apparent that increasing differences implies single crossing which in turn implies interval dominance. It is straightforward to confirm that the converse implication does not hold.

The current paper introduces and uses variations on Condition (ID) to study an application. I defer these discussions to when they are needed in Section 6.

Definition 4. *The game $\Gamma = (I, X, u, \geq)$ is an interval-dominance supermodular (ID-supermodular) game if, for each $i \in I$:*

(A1) *X is a complete lattice;*

(A2) *$u_i : X \to \mathbb{R}$ is order upper semi-continuous in x_i for fixed x_{-i}; u_i order upper semi-continuous in x_{-i} for fixed x_i; and u_i is bounded above;*

(A3) *u_i is supermodular in x_i for fixed x_{-i};*

(A4) *u_i satisfies the interval-dominance property in x_i and x_{-i} on all interval sublattices of X.*

The distinction between supermodular and ID-supermodular games is that (A4) replaces the condition that u_i has increasing differences.

A useful preliminary observation is Topkis's Monotonicity Theorem.

Fact 1. *Let X be a lattice and Y a partially ordered set. Let $f(x, y) : X \times Y \to \mathbb{R}$. Suppose that $f(\cdot)$ is supermodular in x for fixed y. For any sublattice $X' \subset X$, let $M(X') \equiv \arg\max_{z \in X'} f(z, y)$. $M(X')$ is a sublattice of X. If, furthermore, X' is complete and f is order upper semi-continuous in x for fixed y, then $M(X')$ is a complete sublattice of X.*

In the context of games, Fact 1 states that the set of best replies forms a sublattice when the payoff function is supermodular in a player's strategy. This result is part of the Topkis Monotonicity Theorem as stated in Milgrom and Roberts [17].

One important property of supermodular games is monotonicity of the best-reply correspondence.[2]

Fact 2. *Let Γ be an ID-supermodular game. Let $J = J_1 \times \cdots \times J_I$ be an interval sublattice of X. If $x''_{-i} \geq x'_{-i}$, then*

$$\arg\max_{x_i \in J_i} u_i(x_i, x''_{-i}) \geq_i \arg\max_{x_i \in J_i} u_i(x_i, x'_{-i}).$$

Fact 2 generalizes a result of Milgrom and Shannon [18, Theorem 4] that assumes the single-crossing property rather than (ID) and a result of Quah and Strulovici [22, Theorem 1] that assumes that X is a subset of $\mathbb{R}$. Quah and Strulovici [21, Theorem 1] proves Fact 2. As Milgrom and Shannon note, the lemma holds if one replaces the assumption of supermodularity with the weaker assumption of quasi-supermodularity.[3]

Another important fact is Tarski's Fixed-Point Theorem.

Fact 3. *If T is a complete lattice and $f : T \to T$ is a nondecreasing function, then $f(\cdot)$ has a fixed point. Moreover, the set of fixed points has $\sup\{x \in T : f(x) \geq x\}$ as its largest element and $\inf\{x \in T : f(x) \leq x\}$ as its smallest element.*

[2] Fact 2 states that the set $\arg\max_{x_i \in J_i} u_i(x_i, x''_{-i}) \geq_i \arg\max_{x_i \in J_i} u_i(x_i, x'_{-i})$. Here "$\geq$" represents dominance in the strong-set order (defined in Appendix A).

[3] A function is quasi-supermodular if $f(x) \geq f(x \wedge y)$ implies $f(x \vee y) \geq f(y)$ and $f(x) > f(x \wedge y)$ implies $f(x \vee y) > f(y)$.

One can use Facts 2 and 3 to establish the existence of pure-strategy Nash equilibrium. Consider mappings $\underline{\Phi}, \overline{\Phi} : X \to X$ defined by

$$\underline{\Phi}(x) = (\min \arg\max_{x_1'} u_1(x_1', x_{-1}), \ldots, \min \arg\max_{x_n'} u_n(x_n', x_{-n}))$$

and

$$\overline{\Phi}(x) = (\max \arg\max_{x_1'} u_1(x_1', x_{-1}), \ldots, \max \arg\max_{x_n'} u_n(x_n', x_{-n})).$$

By Facts 1 and Fact 2 these mappings are well defined and nondecreasing. Consequently, they have fixed points. It is straightforward to show that these fixed points are pure-strategy Nash equilibria. The (unique) fixed point of $\underline{\Phi}$ is the lowest Nash equilibrium while the (unique) fixed point of $\overline{\Phi}$ is the largest Nash equilibrium. Theorem 1 in Section 3 shows the existence of these equilibria using a direct argument that does not invoke Fact 3.

3 Iterated Deletion of Strictly Dominated Strategies

This section presents a small generalization of Milgrom and Roberts [17, Theorem 5].

In order to formulate the result, let $\hat{X} \subset X$. Define a mapping Z from subsets of X to subsets of X by:

$$Z_i(\hat{X}) = \{x_i \in X_i : \text{ for all } x_i' \in X_i \text{ there exists } \hat{x} \in \hat{X} \text{ such that } u_i(x_i, \hat{x}_{-i}) \geq u_i(x_i', \hat{x}_{-i})\},$$

and $Z(\hat{X}) = \{(z_1, \ldots, z_I) : z_i \in Z_i(\hat{X})\}$. Strategies in $Z_i(\hat{X})$ are not dominated in $\hat{X}_{-i}$. Let $\overline{Z}(\hat{X})$ denote the interval $[\inf(Z(\hat{X})), \sup(Z(\hat{X}))]$. The process of iteratively deleting strictly dominated strategies starts with $X^0 = X$ and lets $X^t = Z(X^{t-1})$. A strategy $x_i \in X_i$ is **serially undominated** if $x_i \in Z_i(X^t)$ for all t.

Theorem 1. *Let Γ be an ID-supermodular game. For each player i, there exist largest and smallest serially undominated strategies, $\underline{x}_i$ and $\overline{x}_i$. Moreover, the strategy profiles $\{\underline{x}_i : i \in I\}$ and $\{\overline{x}_i : i \in I\}$ are pure Nash equilibrium profiles.*

Theorem 1 is Milgrom and Roberts's Theorem 5 under the assumption of interval dominance rather than increasing differences. The theorem follows from the next lemma. I include a proof of the lemma to identify precisely where I relax Milgrom and Roberts's condition.

Let $\underline{B}_i(x)$ and $\overline{B}_i(x)$ denote the smallest and largest best responses for i to $x \in X$, and let $\underline{B}(x)$ and $\overline{B}(x)$ denote the collections $\underline{B}_i(x)$ and $\overline{B}_i(x)$, $i \in I$. Fact 1 guarantees that these sets are well defined.

Lemma 1. *Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$. Then $\sup Z([\underline{z}, \overline{z}]) = \overline{B}(\overline{z})$ and $\inf Z([\underline{z}, \overline{z}]) = \underline{B}(\underline{z})$, and $\overline{Z}([\underline{z}, \overline{z}]) = [\underline{B}(\underline{z}), \overline{B}(\overline{z})]$.*

Proof of Lemma 1. The largest and smallest best responses are well defined by Fact 1. By definition, $\underline{B}(\underline{z})$ and $\overline{B}(\overline{z})$ are in $Z([\underline{z}, \overline{z}])$, and thus $[\underline{B}(\underline{z}), \overline{B}(\overline{z})] \subset \overline{Z}([\underline{z}, \overline{z}])$. Suppose

$z \notin [\underline{B}(\underline{z}), \overline{B}(\overline{z})]$ and, in particular, suppose $z_i \ngeq_i z_i^* \equiv \underline{B}_i(\underline{z})$. I claim that $z_i \notin Z_i([\underline{z}, \overline{z}])$ because z_i is strongly dominated by $z_i \vee z_i^*$. For any $x_i \in [z_i, z_i \vee z_i^*)$,

$$u_i(x_i \vee z_i^*, \underline{z}_{-i}) - u_i(x_i, \underline{z}_{-i}) \geq u_i(z_i^*, \underline{z}_{-i}) - u_i(x_i \wedge z_i^*, \underline{z}_{-i}) > 0, \tag{2}$$

where the first inequality follows from supermodularity and the second from the definition of z_i^*.

It follows from (2) that for any $x_i \in [z_i, z_i \vee z_i^*)$,

$$u_i(x_i \vee z_i^*, \underline{z}_{-i}) > u_i(x_i, \underline{z}_{-i}). \tag{3}$$

Furthermore, if $x_i \in [z_i, z_i \vee z_i^*)$, then $x_i \vee z_i^* = z_i \vee z_i^*$ and inequality (3) implies that for $x_i \in [z_i, z_i \vee z_i^*)$,

$$u_i(z_i \vee z_i^*, \underline{z}_{-i}) > u_i(x_i, \underline{z}_{-i}). \tag{4}$$

It follows from (ID) and (4) that if $z_i \ngeq_i z_i^*$ then

$$u_i(z_i \vee z_i^*, z_{-i}) > u_i(z_i, z_{-i}) \text{ for all } z_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]. \tag{5}$$

An analogous argument applies to show that if $z_i \nleq_i \overline{B}_i(\overline{z})$, then z_i is strictly dominated. ■

It is straightforward to show that (3) follows from quasi-supermodularity when $x_i \ngeq_i z_i^*$, so the lemma holds if the weaker assumption of quasi-supermodularity replaces (A3) in the definition of (ID)-supermodular games.

Milgrom and Roberts [17, Theorem 5] state and prove this result for supermodular games. The proof above follows their proof. They derive inequality (2) and then complete the proof by pointing out that increasing differences implies

$$u_i(z_i \vee \hat{z}_i, z_{-i}) - u_i(z_i, z_{-i}) \geq u_i(z_i \vee \hat{z}_i, \underline{z}_{-i}) - u_i(z_i, \underline{z}_{-i}) \tag{6}$$

provided that $z_{-i} \geq \underline{z}_{-i}$. The lemma follows from (2) and (6). I simply point out that the (ID) condition is sufficient for the result.

Milgrom and Roberts use the lemma to prove the theorem. Their proof goes through without modification.

Later in the paper I discuss economically interesting games in which the mathematical extensions I propose lead to sharper predictions. I conclude this section with the example of a game that satisfies the assumptions of Theorem 1 but does not satisfy the assumptions of Milgrom and Roberts's theorem.

Example 1. *There are a finite number N of players, strategies are elements of $[0, M]$, $M > 0$. The payoff to player i, $u_i(x_i, x_{-i}) = \alpha g(x_i) \sum_{j \neq i} x_j - C(x_i)$, for $\alpha > 0$. If $g(\cdot)$ is strictly increasing, then $u_i(\cdot)$ satisfies increasing differences; if $g(\cdot)$ is positive, then $u_i(\cdot)$ satisfies single crossing; but $u_i(\cdot)$ satisfies interval dominance without any assumptions on $g(\cdot)$. The game still may have strategic complementaries. There are parameter values in which the game has multiple, Pareto-ranked pure-strategy equilibria. For example, if $C(x) \equiv x^3$ and*

$$g(x) = \begin{cases} x & \text{if } x \in [0, 1], \\ 2 - x & \text{if } x \in (1, 3), \\ x - 4 & \text{if } x \in [3, M], \end{cases}$$

then there will be a range of values for $\alpha(N-1)$ in which there is an equilibrium in which $x_i = 1$ for all i (this equilibrium exists, given that other players set $x_i = 1$, player j prefers to set $x_j = 1$ than any other value) and another (non disjoint) interval in which $x_i = \alpha(N-1)/3 > 4$ for all i.

4 Iterated Deletion of Weakly Dominated Strategies

Modifications of the proofs of Lemma 1 and Theorem 1 allow us to establish descriptions of the set of strategies that survive iterated deletion of weakly dominated strategies.

Definition 5. *Given a game $\Gamma = (I, X, u, \geq)$ and subsets $X_i' \subset X_i$, with $X' = \Pi_{i \in I} X_i'$, player i's strategy $x_i \in X_i'$ is weakly dominated relative to X' if there exists $z_i \in X_i'$ such that $u_i(x_i, x_{-i}) \leq u_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}'$, with strict inequality for at least one $x_{-i} \in X_{-i}'$.*

Weak dominance will typically delete more strategies than strong dominance. Hence it has the potential to provide more restrictive predictions. I analyze the implications of applying iterated deletion of weakly dominated strategies instead of iterated deletion of strongly dominated strategies. This section studies **iterated interval deletion of weakly dominated strategies**. The procedure iteratively removes weakly dominated strategies beginning with a game $\Gamma^0 = (I, X^0, u, \geq)$ in which $\inf X^0 = \underline{x}^0$ and $\sup X^0 = \overline{x}^0]$ and constructs games $\Gamma^k = (I, X^k, u, \geq)$ where $\inf X^k = \underline{x}^k$ and $\sup X^k = \overline{x}^k$ is the smallest set such that all strategies in $X^{k-1} \setminus [\underline{x}^k, \overline{x}^k]$ are weakly dominated with respect to X^{k-1}. I will describe the set of strategies that survive this process, that is, the set of strategies that are in X^k for all k. It is possible that different ways of deleting weakly dominated strategies will lead to different limit sets. I reference results that identify games in which the order of deletion is essentially unimportant.

The procedure that iteratively deletes dominated strategies works by assuming that existing strategies are in an interval and then finding a (potentially smaller) interval of strategies that are undominated. It is possible that some strategies are weakly dominated but not strictly dominated. If this happens, then the process of iterated deletion of weakly dominated strategies will lead to a smaller set of surviving strategies. In this section, I point out how to modify Milgrom and Robert's arguments to apply to weak dominance. In Section 6, I discuss how weak dominance is, in fact, more selective than strong dominance in cheap-talk games and that is possible to use the arguments of supermodular games to characterize a refined set of equilibria. Before stating and proving the extension of Theorem 1 to weak dominance, I provide an example that illustrates the value of the result.

Example 2. *Consider the following game:*

	L	R
U	$3,3$	$0,0$
M	$1,2$	$1,1$
D	$0,0$	$1,1$

This game is supermodular. The arguments of Milgrom and Roberts guarantee that there is a smallest and largest Nash Equilibrium. These are (D, R) and (U, L) respectively. It However, D is weakly dominated. Applying iterated deletion of weakly dominated strategies leaves only the (U, L) equilibrium. The selection seems plausible in the example. I would like to know whether it is possible to rule weakly dominated strategies and still preserve the structure identified in Theorem 1. The example suggests a possibility. Milgrom and Roberts obtain a lower bound to Row's strategies by taking the smallest best reply. In the example, D is the smallest best reply. Because best replies are monotonic, the smallest best reply will be a best reply to Column's smallest strategy. In the example, this strategy is R. Note that R has two best replies. What if, instead of taking the smallest best reply to R (D) as a lower bound, one takes the largest best reply? (M). Because M is larger than D, it must do at least as well as D against all of Column's strategies. That is, by selecting the largest best response to the smallest strategy of Column, Row eliminates weakly dominated strategies. This idea forms the basis of the proof of the next result.

Theorem 2. *Let Γ be a finite ID-supermodular game. For each Player i, there exist largest and smallest strategies that survive iterated interval deletion of weakly dominated strategies, $\underline{x}_i$ and $\overline{x}_i$. Moreover, the strategy profiles $\{\underline{x}_i : i \in I\}$ and $\{\overline{x}_i : i \in I\}$ are pure Nash equilibrium profiles.*

Theorem 2 extends Theorem 1 to weak dominance. I have added the assumption that Γ is finite. I explain the importance of this assumption after the proof.

The theorem requires two preliminary results.

Let $X' = X'_1 \times \cdots \times X'_I \subset X$ and

$$E_i(x_i; X') = \{z_i \in X'_i : u_i(x_i, z_{-i}) = u_i(z_i, z_{-i}) \text{ for all } z_{-i} \in X'_{-i}\}$$

be the set of strategies that give the same payoff to i against all strategies in X'_{-i}.

Lemma 2. *Let Γ be an ID-supermodular game. Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$. There exist largest and smallest strategies that are not weakly dominated relative to $[\underline{z}, \overline{z}]$. These strategies are, respectively, the largest element in $E_i(\underline{B}_i(\overline{z}); [\underline{z}, \overline{z}])$ and the smallest element in $E_i(\overline{B}_i(\underline{z}); [\underline{z}, \overline{z}])$.*

The way to construct the smallest strategy that is not weakly dominated for Player i is to consider the set of strategies that are best responses to the lowest strategy in $[\underline{z}, \overline{z}]$. If there are multiple best responses, the interval-dominance property suggests that the largest of the best responses performs at least as well as other best responses against higher strategies. This observation makes the largest best response to the smallest strategy a candidate for smallest strategy that is not weakly dominated. In fact, there may be other, smaller, strategies that are equivalent to the largest best response to $\underline{z}_{-i}$ in the sense that these strategies yield identical payoffs against all strategies in $[\underline{z}_{-i}, \overline{z}_{-i}]$. The proof of Lemma 2 shows that there exists a smallest strategy that is equivalent to the largest best response to $\underline{z}_{-i}$ and that this strategy is the smallest strategy that is not weakly dominated. The details are in Appendix B.

Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$. Let $\overline{E}_i(x; [\underline{z} \leq \overline{z}])$ denote the sup of $E_i(x)$ and $\underline{E}_i(x; [\underline{z} \leq \overline{z}])$ denote the inf of $E_i(x; [\underline{z} \leq \overline{z}])$. Let $\underline{E}(x; [\underline{z} \leq \overline{z}]) = (\underline{E}_1(x; [\underline{z} \leq \overline{z}]), \ldots, \underline{E}_I(x; [\underline{z} \leq \overline{z}]))$ and $\overline{E}(x; [\underline{z} \leq \overline{z}]) = (\overline{E}_1(x; [\underline{z} \leq \overline{z}]), \ldots, \overline{E}_I(x; [\underline{z} \leq \overline{z}]))$. Define

$$\underline{s}_i = \inf\{x_i \in [\underline{z}_i, \overline{z}_i] : x_i \text{ is not weakly dominated in } [\underline{z}, \overline{z}]\}$$

and

$$\overline{s}_i = \sup\{x_i \in [\underline{z}_i, \overline{z}_i] : x_i \text{ is not weakly dominated in } [\underline{z}, \overline{z}]\}.$$

Now let $Z_i^w([\underline{z}, \overline{z}]) = [\underline{s}_i, \overline{s}_i]$ and $Z^w([\underline{z}, \overline{z}]) = (Z_1^w([\underline{z}, \overline{z}]), \ldots, Z_I^w([\underline{z}, \overline{z}])$. Finally let $\overline{Z}^w([\underline{z}, \overline{z}])$ denote the interval $[\inf(Z^w([\underline{z}, \overline{z}]), \sup(Z^w([\underline{z}, \overline{z}]))]$.

Lemma 2 implies the following result.

Lemma 3. *Let Γ be an ID-supermodular game. Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$. Then $\overline{E}(\underline{B}(\overline{z}); [\underline{z}, \overline{z}])$ and $\underline{E}(\overline{B}(\underline{z}); [\underline{z}, \overline{z}])$ exist,* $\sup Z^w([\underline{z}, \overline{z}]) = \overline{E}(\underline{B}(\overline{z}); [\underline{z}, \overline{z}])$ *and* $\inf Z^w([\underline{z}, \overline{z}]) = \underline{E}(\overline{B}(\underline{z}); [\underline{z}, \overline{z}])$, *and* $\overline{Z}([\underline{z}, \overline{z}]) = [\underline{E}(\overline{B}(\underline{z}); [\underline{z}, \overline{z}]), \overline{E}(\underline{B}(\overline{z}); [\underline{z}, \overline{z}])]$.

Lemma 3 parallels Lemma 1. The first difference is that if $z_i \not\geq_i z_i^* \equiv \overline{E}(\underline{B}(\overline{z}); [\underline{z}, \overline{z}])$, there is no guarantee that $z_i \vee z_i^*$ strictly dominates z_i. It is possible that $z_i \wedge z_i^*$ is a best response to $\underline{z}_{-i}$. Hence the second inequality in (2) could be weak. The second difference is that one can use weak dominance rather than strict dominance to delete a strategy. So one need only establish that $u_i(z_i \vee z_i^*, z_{-i}) > u_i(z_i, z_{-i})$ for some $z_{-i} \in [\underline{z}, \overline{z}]$. This follows from the definition of z_i^*.

Proof of Theorem 2. The proof of the theorem follows the proof of Theorem 1. Let $\underline{y}_i^1$ be equal to the smallest element in $E_i(\overline{B}_i(\underline{z}); [\underline{z}, \overline{z}])$ and $\overline{y}_i^1$ be equal to the largest element of $E_i(\underline{B}_i(\overline{z}); [\underline{z}, \overline{z}])$. Lemma 3 implies that $\underline{y}_i^1$ and $\overline{y}_i^1$ are well defined and are respectively the smallest and greatest strategies that are not weakly dominated relative to $[\underline{z}, \overline{z}]$. It follows that $\underline{z} \leq \underline{y}^1 \leq \overline{y}^1 \leq \overline{z}$. Continuing inductively one can construct sequences $\{\underline{y}^k\}$ and $\{\overline{y}^k\}$ such that $\underline{y}^k \leq \underline{y}^{k+1} \leq \overline{y}^{k+1} \leq \overline{y}^k$ and every strategy outside of $[\underline{y}^{k+1}, \overline{y}^{k+1}]$ is weakly dominated relative to $[\underline{y}^k, \overline{y}^k]$. By monotonicity, $\lim_{k\to\infty} \underline{y}^k$ and $\lim_{k\to\infty} \overline{y}^k$ exist. Denote the limits by $\underline{y}$ and $\overline{y}$ respectively. It is straightforward to show that these limits are Nash Equilibrium profiles. In finite games (where the process of deleting strategies terminates after a finite number of iterations), it follows by construction that $\underline{y}$ and $\overline{y}$ are not weakly dominated by any strategy in $[\underline{y}, \overline{y}]$. From Lemma 3, it follows that anything that survives iterated deletion of weakly dominated strategies must be inside the interval.

The process described only removes strategies outside of the interval $[\underline{y}^k, \overline{y}^k]$. Consequently, it is possible that there are strategies in the interval $[\underline{y}, \overline{y}]$ that are weakly dominated. When the strategy set is finite, it must be the case that $\underline{y}_i$ and $\overline{y}_i$ remain undominated even if additional strategies are deleted. To see this notice that, by construction $\underline{y}_i$ is a best response to $\underline{y}_{-i}$ and the only other best responses to $\underline{y}_{-i}$ in $[\underline{y}_i, \overline{y}_i]$ are equivalent to $\underline{y}_i$. Consequently, $\underline{y}_i$ can only be weakly dominated if $\underline{y}_j$ is deleted for $j \neq i$. Hence no procedure can delete $\underline{y}_i$. Similarly, $\overline{y}_i$ cannot be deleted. This completes the proof of Theorem 2. ■

Theorem 2 uses the assumption that strategy sets are finite. This assumption guarantees that the iterated deletion process terminates in a finite number of steps and, consequently, that $\underline{y}$ and $\overline{y}$ are not weakly dominated. The next example demonstrates that the bounds obtained through the process may be weakly dominated in games in which X_i are infinite.

Example 3. *Consider a three player game in which* $X_1 = [0, 1]$ *and* $X_i = [0, 2]$ *for* $i = 2, 3$*;* $u_1(x) = x_1(x_2 - 1)$*,* $u_i(x) = x_1x_2x_3 - x_i^3/3$ *for* $i = 2, 3$*. In this case* $\overline{y}^k = (1, 2^{2^{-k}}, 2^{2^{-k}})$ *and* $\underline{y}^k = (0, 0, 0)$*. It follows that* $\overline{y} = (1, 1, 1)$ *and* $\underline{y} = (0, 0, 0)$*. Both* $\underline{y}$ *and* $\overline{y}$ *are Nash equilibria, but* $\overline{y}$ *is weakly dominated with respect to strategies in* $[\underline{y}, \overline{y}]$*.*

Theorem 2 applies to a particular procedure for removal of weakly dominated strategies. Unlike iterated deletion of strictly dominated strategies, the outcome of iterated deletion of weakly dominated strategies may depend on the procedure.[4] On the other hand, for some interesting classes of games, deletion of weakly dominated strategies is essentially independent of the procedure.

Marx and Swinkels [16] show that if a game satisfies the transfer of decision maker indifference (TDI) property, then two "full"[5] procedures for deleting weakly dominated strategies are the same up to the addition or removal of redundant strategies and a renaming of strategies. The TDI property states that if (given the behavior of the other players) Player i is indifferent between two strategies, then all other players are also indifferent between Player i's choice of strategies. TDI is restrictive, but can be shown to hold in interesting applications including (generically) the examples described in Section 6.

Two papers of Kultti and Salonen take a different approach to the concern that iterated deletion of weakly dominated strategies may be order dependent. Kultti and Salonen study undominated equilibria in supermodular games.[6] An undominated equilibrium is a Nash equilibrium in which no player's equilibrium strategy is weakly dominated by another pure strategy. Börgers [2] and Dekel and Fudenberg [4] identify properties that make undominated equilibria an attractive refinement of Nash Equilibrium. Kultti and Salonen [11] show that in supermodular games there exist a least and greatest undominated equilibrium in pure strategies. Example 3 demonstrates that the bounds that I have constructed may be dominated. Hence my result does not include theirs.

Iterated deletion of weakly dominated strategies yields stronger predictions in interesting applications. Kultti and Salonen [12] study a process in which players eliminate all weakly dominated strategies in the first step and subsequently iteratively remove all strictly dominated strategies. They present conditions under which this process identifies the lower and upper bounds of the set of equilibrium payoffs.

[4] Dufwenberg and Stegeman [6] show that iterated deletion of strictly dominated strategies may be order dependent in infinite games if payoff functions and strategy spaces do not satisfy regularity conditions.

[5] A "full" procedure stops only if it reaches a stage where there are no weakly dominated strategies.

[6] In fact, Kultti and Salonen study quasi-supermodular games in which the (weaker) assumption that u_i is quasi-supermodular replaces (A3) in Definition 4.

5 Additional Properties

5.1 Dynamics

Milgrom and Roberts [17] show that there is relationship between adaptive dynamics and supermodular games. To do this, they consider a time-dependent strategy profile $x(t)$. They let $P(T,t)$ denote the strategies played between times T and t: $P(T,t) = \{x(s) : s \in [T,t)\}$ and say that $\{x(t)\}$ is a process **consistent with adaptive dynamics** if for all T there exists $T' > T$ such that for all $t > T'$, $x(t) \in \overline{Z}([\inf P(T,t), \sup P(T,t)])$. They define $\underline{x} = \inf X$, $\overline{x} = \sup X$, $\underline{B}^k(x) = \underline{B}(\underline{B}^{k-1}(x))$, and $\overline{B}^k(x) = \overline{B}(\overline{B}^{k-1}(x))$ and show (in Theorem 8) that whenever $\{x(t)\}$ is a process consistent with adaptive dynamics in a supermodular game, for all k there exists T_k such that for all $t > T_k$, $x(t) \in [\underline{B}^k(\underline{x}), \overline{B}^k(\overline{x})]$.

The condition that a process is consistent with adaptive dynamics guarantees that strategies played at time t are best replies to strategies played in the not-too-distant past. The conclusion of the theorem is that any process consistent with adaptive dynamics must eventually stop playing strictly dominated strategies and therefore converge to the interval of strategies with lower bound equal to the smallest Nash equilibrium and upper bound equal to the largest Nash equilibrium. This result is a direct consequence of Lemma 1 and holds for ID-supermodular games. It is straightforward to modify the result to conclude that a more restrictive class of adaptive dynamics converges to the smaller set of strategies identified in Theorem 2.

The process $\{x(t)\}$ is **consistent with cautious adaptive dynamics** if for all T there exists T' such that for all $t > T'$, $x(t) \in \overline{Z}^w([\inf P(T,t), \sup P(T,t)])$.[7] Let $\underline{H}^1(x) = \underline{E}(\overline{B}(x); [\underline{x}, \overline{x}])$, $\overline{H}^1(x) = \overline{E}(\underline{B}(x); [\underline{x}, \overline{x}])$, and $\underline{H}^k(x) = \underline{E}(\overline{B}(\underline{H}^{k-1}(x); [H^{k-1}(\underline{x}, H^{k-1}(\overline{x})])$; and $\overline{H}^k(x) = \overline{E}(\underline{B}(\overline{H}^{k-1}(\overline{x}); [H^{k-1}(x, H^{k-1}(\overline{x})])$.

Theorem 3. *If $\{x(t)\}$ is a process consistent with cautious adaptive dynamics in an ID-supermodular game, then for all k there exists T_k such that for all $t > T_k$, $x(t) \in [\underline{H}^k(\underline{x}), \overline{H}^k(\overline{x})]$.*

Theorem 3 is a direct consequence of Lemma 3.

Echenique [7] presents a modification of the procedure used to find upper and lower bounds in the proofs of Theorems 1 and 2 to provide an algorithm that finds all pure-strategy Nash equilibria in supermodular games. One can interpret the algorithm as a dynamic process. Consequently, there exist adaptive processes that reach Nash equilibria that do not survive iterated deletion of weakly dominated strategies. This result does not contradict Theorem 3. Instead it indicates that procedures that reach Nash equilibria that do not survive iterated deletion of weakly dominated strategies are not cautious. A critical issue is whether it is plausible to restrict attention to cautious dynamics. I believe that the correct answer is "it depends." On one hand, Cabrales and Ponti [3] and Gale, Binmore, and Samuelson [8] present examples of plausible evolutionary dynamics that converge to outcomes that use weakly dominated strategies. On the other

[7] I use "cautious" in the sense of cautious rationalizability of Pearce [20]. The notion is that the adaptive process is a best response to beliefs that place positive probability on all "recently" used strategies.

hand, Dubey, Haimanko, and Zapechelnyuk [5] introduce **pseudo-potential games**. A pseudo-potential game is a game for which there exists function $\phi : X \to \mathbb{R}$ such that $\arg\max_{x_i \in X_i} \phi(x_i, x_{-i}) \subset \arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$. Dubey, Haimanko, and Zapechelnyuk [5] give conditions under which games with complementarities are pseudo-potential games. Their results imply that finite, two-player (ID) supermodular games are pseudo-potential games. Dubey, Haimanko, and Zapechelnyuk identify several properties of pseudo-potential games, including the property that there are no best-response cycles in generic, finite pseudo-potential games. This property guarantees convergence of best reply dynamics. Weak dominance has interesting implications only for games with non-generic payoffs.[8] Cautiously adaptive dynamics provide a way to extend these results to non-generic games.

5.2 Comparative Statics

In order to ask comparative statics questions, assume that there is a partially ordered set of parameters P and there is a family of games $\{\Gamma(p)\}_{p \in P}$ where $\Gamma(p) = \{I, X, u(\cdot; p), \geq\}$ where $u : X \times P \to \mathbb{R}^I$.

Theorem 4. *If $\{\Gamma(p)\}_{p \in P}$ is a family of ID-supermodular games and u_i satisfies interval dominance in x_i and p for fixed x_{-i} then the largest and smallest strategies that survive iterated interval deletion of weakly dominated strategies, $\underline{x}_i(p)$ and $\overline{x}_i(p)$, are nondecreasing functions of p.*

The proof of this result is a straightforward modification of Theorem 6 in Milgrom and Roberts [17]. The proof, which is in Appendix B, requires verification that $\underline{H}$ and $\overline{H}$ are monotonic.

Milgrom and Roberts [17, Theorem 7] gives conditions under which it is possible to compare payoffs of different equilibria.

Theorem 5. *Let $\Gamma = (I, X, u, \geq)$ be an ID-supermodular game. Let $\underline{x}_i$ and $\overline{x}_i$ denote the smallest and largest elements of X_i, and suppose y and z are two equilibria with $y \geq z$. (1) If $u_i(\underline{x}_i, x_{-i})$ is increasing in x_{-i}, then $u_i(y) \geq u_i(z)$. (2) If $u_i(\overline{x}_i, x_{-i})$ is decreasing in x_{-i}, then $u_i(y) \leq u_i(z)$. If the condition in (1) holds for some subset of players I_1 and the condition in (2) holds for the remainder $I \setminus I_1$, then the largest equilibrium is the most preferred equilibrium for the players in I_1, and the least preferred for the remaining players.*

This result holds in my setting, but one variation is worth noting. If Condition 1 in the theorem holds, then the largest Nash Equilibrium is Pareto dominant (in the set of Nash Equilibria). It is possible that strategies used in this equilibrium do not survive iterated deletion of weakly dominated strategies. The upper bound in Theorem 2 may therefore not be the Pareto dominant Nash Equilibrium. Instead it will be (in finite games), the Pareto-dominant Nash Equilibrium in strategies that survive iterated deletion of weakly dominated strategies. Milgrom and Roberts discuss an interesting

[8] The applications I study have non-generic normal-form payoffs because they are derived from games with a fixed dynamic structure.

classes of games (games with positive spillovers) in which equilibria are Pareto ranked. The literature treats the largest Nash Equilibrium as salient in these games. For typical specifications of these games, the largest equilibrium is also an equilibrium that survives iterated deletion of weakly dominated strategies.[9]

5.3 Quasisupermodularity

This paper concentrates on weakening the monotonicity condition (increasing differences) used by Milgrom and Roberts. Theorem 1 merely replaces increasing differences with interval dominance. Theorem 2 extends the result – again with the weaker condition – to iterated weak dominance. In the same way, one can replace the supermodularity assumption with quasi-supermodularity. They compare values of two quantities, which are in turn the difference between a function evaluated at a higher and a lower point. Supermodularity and increasing differences require that the first quantity is greater than the second. Quasi-supermodularity and single crossing (interval dominance) require the weaker condition that the first quantity is non-negative (positive) whenever the first one is non-negative (positive). It is the second implication that is needed for the main results. That is, Theorems 1 and 2 hold if payoff functions are quasi-supermodular. I chose not to state the more general results because I know of no application in which payoffs are quasi-supermodular but not supermodular.[10]

5.4 Identification

There is a literature that estimates supermodular games. For example, Uetake and Watanabe [25] use the bounds constructed in Milgrom and Roberts [Theorem 5][17] to generate moment inequalities. I believe that the same techniques would apply to estimate strategies that satisfy the refinement (surviving iterated deletion of weakly dominated strategies). The bounds constructed in Theorem 2 would replace those in Theorem 1.[11] This kind of study would be consistent with research by Aradillas-Lopez and Tamer [1], which compares the identification power of rationalizability to Nash Equilibria and Molinari and Rosen [19] who estimate level-k rationality in a supermodular game.

There is an econometric literature that tries to identify and test monotone comparative statics in supermodular games. There are two basic approaches. The first approach (for example, Lazzati [13], and Uetake and Watantabe [25]) is to impose monotonicity and study the restrictions imposed by a solution concept (Nash equilibrium or rationalizability) on data. One could ask this question instead requiring the solution only use strategies that survive iterated deletion of weakly dominated strategies. Theorem 2

[9]Nevertheless, Theorem 5 suggests that in more general settings the Pareto-efficient Nash equilibrium may fail to survive iterated deletion of weakly dominated strategies.

[10]Quah and Strulovici [21, Theorem 1] recognize that it is possible to obtain comparative-statics results with a weaker version of supermodularity. They demonstrate that an interval-dominance version of the condition is sufficient for basic results.

[11]One limitation of the approach is that Uetake and Watanabe focus on one-dimensional strategies spaces. The (ID)-supermodular games that I identify in Section 6 in which weak dominance has selection power involve multidimensional strategy spaces.

suggests new bounds on strategies that would replace the restrictions the literature has provided for rationalizability.

Another approach imposes no a priori restrictions and asks when a data set is consistent with equilibrium behavior in a supermodular game. Lazzati, Quah, and Shirai [14] provides a necessary and sufficient condition for a data set to be consistent with Nash equilibrium behavior in a supermodular game with a one-dimensional strategy space. A natural modification of the question is to ask whether the data set is consistent with equilibrium behavior in weakly undominated strategies in a ID-supermodular game.

6 Applications

Extending the results about supermodular games from strong to weak dominance is more than a curiosity only if there exist interesting games under which the assumptions of the previous section hold and the arguments reduce the set of predictions. An ideal application would be a ID-supermodular game that is not supermodular, in which weak dominance arguments have more power to refine the set of equilibria than strong dominance arguments, and in which insights about the structure of equilibria available from the results in this paper have substantive interest.

Examples 1 and 2 provide some evidence of the usefulness of the approach, but these examples primarily illustrate technical points and are somewhat artificial. Sobel [23] shows how iterated weak dominance has power to select outcomes in games with preplay communication about intentions. These games are not ID-supermodular games, but the selection arguments use partial ordering on strategies and monotonicity properties that are similar to the methods in the current paper.

This section applies the ideas to cheap-talk games and games with competition in persuasion.

6.1 Cheap Talk

Cheap-talk games add a round of strategic behavior to an underlying game. This kind of game is a natural place to expect weak dominance to play a role as weak dominance can place restrictions on off-the-path behavior.

Cheap-talk games are not supermodular, but have some of the structure of supermodular games. Strong dominance arguments do not restrict the predictions. The application is imperfect because the game is not ID-supermodular. I must extend the theory somewhat.

In a cheap-talk game, nature selects $t \in T$; one player, the Sender (S), learns t and sends a message $m \in M$; the other player, the Receiver (R), takes an action $a \in A$ in response to m. A strategy for S is a mapping $\sigma : T \to M$. A strategy for R is a mapping $\alpha : M \to A$. Assume that M is a finite, ordered, set and that A and T are equal to the unit interval. Assume that there is a prior distribution on types; for convenience assume that the prior is finitely supported and $p(t)$ is the probability that the type is t. Payoffs depend only on a and t. The payoff to Player i when t is the Sender's type and a is the action of the Receiver is $U^i(a, t)$. Assume that $U^i(\cdot)$ is twice continuously differentiable, with negative second derivative with respect to a and positive cross partial. With this structure, order R strategies in the natural way:[12] $\alpha'' \geq_R \alpha'$ if $\alpha''(m) \geq \alpha'(m)$ for all m. Order S strategies "backwards" so that $\sigma'' \geq_S \sigma'$ if and only if $\sigma''(t) \leq \sigma'(t)$ for all t.[13] The payoff functions for the cheap-talk game are $u_S(\sigma, \alpha) = EU^S(\alpha(\sigma(t)), t)$ and $u_R(\alpha, \sigma) = EU^R(\alpha(\sigma(t), t))$, where the expectation is taken using the prior on types. It is straightforward to check that this game satisfies the TDI condition of Marx and Swinkels [16].

I describe several properties of this class of games and show how the general results provide some insight into the structure of their equilibria.

Lemma 4. *For $i = S, R$, $u_i(\cdot)$ is supermodular in x_i for fixed x_{-i} in cheap-talk games.*

Lemma 4 follows from a straightforward argument, which appears in Appendix B.

Without further assumptions best responses will not have any monotonicity properties in the basic cheap-talk game. For example, suppose that $U^R(a, t) = -(a - t)^2$, and the prior is uniform on $\{0, 1/N, \ldots, k/N, \ldots, 1\}$ for some even number N. Assume that M contains messages m_0 and m_1 with $m_0 < m_1$. If the Sender always sends m_0, then it is a best response for the Receiver to respond to m_0 with .5 and all other messages with 0. Denote this strategy by α^{**}. Let

$$\sigma(t) = \begin{cases} m_1 & \text{if } t \in [0, .5] \\ m = m_0 & \text{if } t \in (.5, 1] \end{cases} \text{ and } \alpha(m) = \begin{cases} 1 & \text{if } m_0 \\ 0 & \text{otherwise} \end{cases}.$$

The Receiver prefers $\alpha \wedge \alpha^{**}$ to α when S always sends m_0, but R's preferences reverse when S plays σ. Consequently interval dominance does not hold for R. One can also confirm that S's preferences violate interval dominance and that the violations do not depend on the choice of order over S's strategies.

Best response correspondences do have some monotonicity properties for a restricted version of the cheap-talk game. Henceforth consider a **monotonic restriction** of the cheap-talk game. In the monotonic restriction, the Sender and Receiver are restricted to monotonic strategies (σ is monotonic if $t'' > t'$ implies $\sigma(t'') \geq \sigma(t')$; α is monotonic if $m'' > m'$ implies that $\alpha(m'') \geq \alpha(m')$). See Kartik and Sobel [10] for a justification of the monotonic restriction. I call the monotonic restriction of a cheap-talk game a **monotone cheap-talk game**.

Even with the restriction to monotonic strategies, the cheap-talk game does not satisfy increasing differences.

To see that the Sender's payoff does not satisfy increasing differences, let $\sigma(t) \equiv 0$ and $\sigma'(t) \equiv 1$ so that $u_S(\sigma, \alpha) - u_S(\sigma', \alpha) = E[U^S(\alpha(0), t) - U^S(\alpha(1), t)]$. The right-hand side is not monotonic in $\alpha(0)$ (or in $\alpha(1)$), so the increasing difference condition does not hold.

To see that the Receiver's payoff does not satisfy increasing differences, let $\alpha'(t) \equiv 1$.

[12] Note that I use $\geq$ to denote both the standard order on the real numbers and the order on strategies.

[13] This ordering guarantees that R's best response on the equilibrium path increases when S's strategy increases.

Hence $u_R(\alpha', \sigma)$ does not depend on σ. Fix a message $\tilde{m}$ and let

$$\alpha(m) = \begin{cases} 0 & \text{if } m \leq \tilde{m} \\ 1 & \text{if } m > \tilde{m} \end{cases}$$

so that $u_R(\alpha, \sigma) - u_R(\alpha', \sigma) = E_{\sigma(t) \leq \tilde{m}}[U^R(t, 0) - U^R(t, 1)]$. Increasing σ can increase or decrease this quantity.

In general, the Receiver's preferences do not satisfy (ID). To see this, let m_0 denote the lowest message and suppose that $\sigma'(t) > m_0$ for all t, whereas $\sigma''(t) \equiv m_0$. It follows that $\sigma'' \geq_S \sigma'$. It is straightforward to construct σ', α and α^* such that $u^R(\alpha \vee \alpha^*, \sigma') > u^R(\alpha, \sigma')$ but $u^R(\alpha \vee \alpha^*, \sigma'') < u^R(\alpha, \sigma'')$. For example, let σ' be a separating strategy; let α^* be a best response to σ'; and let $\alpha(m) = \arg\max \sum_t U_R(a, t) p(t)$ for all m.

Consequently, the general results about ID-supermodular games do not apply to this example. In order to use the characterization results, I must weaken the (ID) property.

Definition 6. *Let X and Y be lattices. A function $f : X \times Y \to \mathbb{R}$ satisfies the weak generalized interval-dominance property (WID) in its two arguments on the set $X \times Y$ if for all $y'' > y'$,*

$$f(x' \vee t, y') \geq f(x', y') \implies$$
$$\exists \tilde{t} \leq t, f(x' \vee \tilde{t}, y') \geq f(x', y'), \quad \text{such that } f(x' \vee \tilde{t}, y'') \geq f(x', y'') \tag{7}$$

and

$$f(x' \wedge t, y'') \leq f(x', y'') \implies$$
$$\exists \tilde{t} \geq t, f(x' \wedge \tilde{t}, y'') \leq f(x', y''), \quad \text{such that } f(x' \wedge \tilde{t}, y') \leq f(x', y'). \tag{8}$$

The (WID) condition is weaker than (ID). Appendix C proves this result and introduces related concepts. One way to get an intuition for (WID) is to compare it to single crossing, which requires Condition (7) and (8) to hold when $\tilde{t} = t$.[14]

(ID) and (WID) are both conditions that relate to how solutions to $\max_x u_i(x, y)$ change with the parameter y. Fact 2 states that in an ID-supermodular game, Player i's set of best responses are increasing in x_{-i}, where "increasing" is interpreted in the sense of the strong set order. If $u_i(\cdot)$ satisfies (WID), then best responses are increasing in a weaker sense.

The next result describes a property of (WID). The proposition uses the following notation: $x^{**} \in \arg\max f(x, y'')$, $x^* \in \arg\max f(x, y')$, $\overline{x}^{**} = \max \arg\max f(x, y'')$, $\underline{x}^* = \min \arg\max f(x, y')$.

Proposition 1. *Let X and Y be lattices. If the function $f : X \times Y \to \mathbb{R}$ satisfies (WID), then for $y'' > y'$,*

$$x^{**} \vee \underline{x}^* \in \arg\max f(x, y'') \text{ and } x^* \wedge \overline{x}^{**} \in \arg\max f(x, y'). \tag{9}$$

[14] The analog to (8) in the definition of single crossing is implied by Condition 1.

Appendix C contains a proof of Proposition 1. The conclusion of Proposition 1 certainly holds when $\arg\max u_i(\cdot, x_{-i})$ is increasing in the strong set order (provided that there exist solutions to the maximization problems). It is straightforward to confirm that monotonicity property in the proposition is actually weaker.

I say that a game $\Gamma = (I, X, u, \geq)$ is WID-supermodular if it satisfies Conditions (A1)-(A3) in Definition 4 and Condition (A4) is replaced by the requirement that u_i satisfies (WID) in x_i and x_{-i} on all interval sublattices of X.

The class of WID-supermodular games is interesting because monotone cheap-talk games are WID-supermodular and because the equilibria of these games have some of the important properties of ID-supermodular games. The remainder of this subsection reports results that confirm these claims. I show first that monotone cheap-talk games are WID-supermodular. I conclude the subsection (Theorem 6) with the observation that equilibria that survive iterated deletion of weakly dominated strategies have nice bounds in WID-supermodular games.

Lemma 5. *Receiver's preferences in a monotone cheap-talk game satisfy (WID).*

Similarly, the Sender's preferences also satisfy (WID) but not (ID).

Lemma 6. *Sender's preferences in a monotone cheap-talk game satisfy (WID).*

Lemmas 4, 5, and 6 combine to establish the following proposition.

Proposition 2. *Monotone cheap-talk games are (WID)-supermodular.*

Proposition 2 is useful because it is possible to extend Theorem 2. Although I am unable to prove an analog to Lemma 2 for WID-supermodular games, the following result holds for WID-supermodular games.

Let $\inf X = \underline{x}^0$ and $\sup X = \overline{x}^0$.

Theorem 6. *Let Γ be a WID-supermodular game. For each player i, there exist pure Nash equilibrium profiles $\underline{x}_i$ and $\overline{x}_i$ such that all strategies that survive iterated interval deletion of weakly dominated strategies are contained in $[\underline{x}_i, \overline{x}_i]$. Moreover, there exist an increasing sequences $\{\underline{y}^n\}_{n=1}^{\infty}$ and a decreasing sequence $\{\overline{y}^n\}_{n=1}^{\infty}$ where $\underline{y}^0 = \underline{x}^0$ and $\overline{y}^0 = \overline{x}^0$; for $n \geq 1$, $\underline{y}^n = \underline{B}(\underline{y}^{n-1})$ and $\overline{y}^n = \overline{B}(\overline{y}^{n-1})$; and $\underline{x} = \lim_{n\to\infty} \underline{y}^n$, and $\overline{x} = \lim_{n\to\infty} \overline{y}^n$.*

Theorem 6 combines elements of Theorem 1 and Theorem 2. All three results identify extreme, pure-strategy Nash equilibria. The bounds in Theorem 6 are the same as the bounds in Theorem 1. At each stage of the deletion process, the lower (upper) bound is the smallest best response to the smallest (largest) remaining strategy of the opponent. Theorem 6 uses a weaker assumption on preferences (weak interval dominance rather than interval dominance), but it is not a generalization of Theorem 1 because in order to obtain the bounds in Theorem 6, I must delete weakly dominated strategies. The argument is Theorem 2 deletes strictly dominated strategies. Theorem 6 shares with Theorem 2 the focus on eliminating weakly dominated strategies. Compared to Theorem 2, Theorem 6 applies to a larger class of games (every WID-supermodular game is also ID-supermodular), but delivers a less restrictive conclusion because the bounds derived in Theorem 2 may define a strictly larger set than the bounds in Theorem 6.

Replacing the interval-dominance assumption with the weak interval-dominance assumption means that it is no longer possible to guarantee that best response correspondences are increasing in the weak set order.[15] Consequently, the argument sketched at the end of Section 2 that establishes existence of pure-strategy Nash equilibria using Tarski's Fixed-Point Theorem does not apply. Existence of pure-strategy Nash equilibrium does follow from iterating best replies.

Appendix B provides details of the modifications of earlier arguments needed to prove Theorem 6. The proof parallels the proof of Theorem 1. The only difference is that when WID replaces ID, the argument in Lemma 1 that establishes that any z_i not greater than a putative lower bound is strictly dominated uses ID. If I replace ID by WID, then the identical argument only guarantees that z_i is weakly dominated.

Example 3 applies to WID-supermodular games, so that the profiles $\underline{x}$ and $\overline{x}$ need not survive iterated deletion of weakly dominated strategies. If the original game is finite, then the bounds must survive iterated deletion of weakly dominated strategies. Provided that a parameterized family of WID-supermodular games satisfies WID in own strategy and parameter, the monotonicity of the bounds in Theorem 6 with respect to changes in the parameter holds for WID-supermodular games. Similarly, Theorem 3 extends to WID-supermodular games.

Theorem 6 is useful: simple cheap-talk games are an example of a game that is WID-supermodular, but not ID-supermodular. In Example 4, the upper and lower bounds provided in Theorem 6 are equal, providing a selection result even when multiple Nash equilibria exist.

Example 4. *Assume that there are two players, S and R; two equally likely states, 1 and 3; three actions, 1, 2, and 3; and two messages, L and H. A strategy for S is a pair m_1m_2, where $m_i \in \{L, H\}$ is the message sent when the Sender observes state i. A strategy for R is a pair a_1a_2 where $a_j \in \{1, 2, 3\}$ is the action takes when the Receiver receives the message j. Assume that $U^S(a,t) = -2(a-t-b)^2$ and $U^R(a,t) = -2(a-t)^2$, where $b \geq 0$ is a parameter that measures the conflict of interest between S and R.*

The following table describes the expected payoffs.[16] I have deleted strategies that are not monotonic.

	LL	*LH*	*HH*
11	$-4, -b^2-(2+b)^2$	$-4, -b^2-(2+b)^2$	$-4, -b^2-(2+b)^2$
12	$-4, -b^2-(2+b)^2$	$-1, -b^2-(1+b)^2$	$-2, -(1-b)^2-(1+b)^2$
13	$-4, -b^2-(2+b)^2$	$0, -2b^2$	$-4, -b^2-(2-b)^2$
22	$-2, -(1-b)^2-(1+b)^2$	$-2, -(1-b)^2-(1+b)^2$	$-2, -(1-b)^2-(1+b)^2$
23	$-2, -(1-b)^2-(1+b)^2$	$-1, -(1-b)^2-b^2$	$-4, -b^2-(2-b)^2$
33	$-4, -b^2-(2-b)^2$	$-4, -b^2-(2-b)^2$	$-4, -b^2-(2-b)^2$

This is not an ID-supermodular game. A straightforward way to see this is to note that the best response correspondence is not monotonic. Specifically, the Receiver's best responses to LL, LH, and HH are $\{22, 23\}$, $\{13\}$, and $\{12, 22\}$ respectively.

[15] Best response correspondences are increasing in the weaker sense described in Proposition 1.

[16] Rows represent R's strategies; Columns represent S's strategies; cells contain (Row's payoffs, Column's payoffs).

The first round of deletion of weakly dominated strategies yields:

	LL	*LH*	*HH*
12	$-4, -b^2 - (2+b)^2$	$-1, -b^2 - (1+b)^2$	$-2, -(1-b)^2 - (1+b)^2$
13	$-4, -b^2 - (2+b)^2$	$0, -2b^2$	$-4, -b^2 - (2-b)^2$
22	$-2, -(1-b)^2 - (1+b)^2$	$-2, -(1-b)^2 - (1+b)^2$	$-2, -(1-b)^2 - (1+b)^2$
23	$-2, -(1-b)^2 - (1+b)^2$	$-1, -(1-b)^2 - b^2$	$-4, -b^2 - (2-b)^2$

The second round of deletion of weakly dominated strategies yields:

	LH	*HH*
12	$-1, -b^2 - (1+b)^2$	$-2, -(1-b)^2 - (1+b)^2$
13	$0, -2b^2$	$-4, -b^2 - (2-b)^2$
22	$-2, -(1-b)^2 - (1+b)^2$	$-2, -(1-b)^2 - (1+b)^2$
23	$-1, -(1-b)^2 - b^2$	$-4, -b^2 - (2-b)^2$

The third round of deletion yields:

	LH	*HH*
12	$-1, -b^2 - (1+b)^2$	$-2, -(1-b)^2 - (1+b)^2$
13	$0, -2b^2$	$-4, -b^2 - (2-b)^2$

The rest of the analysis depends on the value of b. If $b \in [0, 1/2]$, then removing weakly dominated strategies yields the single outcome $(13, LH)$; if $b \geq 1$, then removing weakly dominated strategies yields the single outcome $(12, HH)$; if $b \in (1/2, 1)$, then no more strategies can be deleted. The game has two Nash Equilibria: the pure strategy equilibrium $(12, HH)$ and a completely mixed equilibrium.

Several things are worth noting. The profile $(12, HH)$ corresponds to the babbling (no-communication) equilibrium. It is a Nash Equilibrium of the original game (without deleting strategies), but when b is small, it is removed. Hence the procedure reduces the set of predicted payoffs. Another strategy profile, $(22, LL)$, also supports the babbling outcome. Iterated deletion of weakly dominated strategies removes this profile. Consequently, the procedure not only selects payoffs, it selects the relationship between types and messages that support the equilibrium. When $b > 0$, the Sender has an upward bias. The procedure predicts that S will "exaggerate" and avoid her lowest message.

Theorem 6 specifies a particular order in which one deletes strategies. This order preserves the lattice structure. (I did not remove all weakly dominated strategies in the first stage.)

Finally, payoffs satisfy increasing differences in own strategy and the parameter b. Hence Theorem 4 applies: when b is smaller, the "largest" equilibrium increases. Kartik and Sobel [10] studies the implications of applying iterated deletion of weakly dominated strategies to monotonic cheap-talk games in more detail.

6.2 Competition in Persuasion

Gentzkow and Kamenica [9] study a model of persuasion in which informed agents simultaneously announce "information structures" to a decision maker. The decision maker then obtains a signal induced by the join of the information structures and makes a decision. (For example, if each agent selects a partition, then the decision maker learns that the state of the world is in the intersection of the two partition elements.) Gentzkow and Kamenica use the model to investigate how competition between agents influences the amount of information available to the decision maker. There always exists a full disclosure equilibrium in which two or more agents announce the finest feasible disclosure policy. Any other equilibrium is preferred by all agents to the full-disclosure equilibrium. One can model this situation in reduced-form as a game between the agents in which their strategies are information structures and payoffs are the expect value assuming that the decision maker makes optimal decision given available information.

Formally, let $\Gamma = (I, X, u, \geq)$ be such that $u_i(x) = U_i(x_1 \vee x_2 \vee \cdots \vee x_I)$. u_i does not satisfy the interval-dominance property in x_i and x_{-i}. To see this, let $I = 2$, $X_i \subset \mathbb{R}$ and assume that $x_2'' > x_1'' > x_1' > x_2'$. It follows that $u_1(x_1'', x_2'') = u_1(x_1', x_2'')$ but it could be that $u_1(x_1'', x_2') > u_1(x_1, x_2')$ for all $x \in (x_1', x_1'')$. Nevertheless, $u_i(\cdot)$ does satisfy WID. To see this, let $\tilde{t} = t \wedge x'$ (for the first part of the definition) and $\tilde{t} = t \vee x'$ (for the second part of the definition). Hence this game will be a WID-supermodular game provided that each $U_i(\cdot)$ satisfies the necessary supermodularity and continuity properties. (These will certainly hold if X_i is a bounded subset of $\mathbb{R}$ and $U_i(\cdot)$ is continuous for all i.) It follows that Theorem 6 applies to this game.

These games typically have multiple, Pareto-ranked equilibria, but iterated weak dominance makes a selection. To see this clearly, consider the (generic) case in which $U_i(\cdot)$ is one-to-one. The most preferred equilibrium is defined by a disclosure level π^* defined by

$$\pi^* = \min\{\pi : U_i(\pi) > U_i(x_i) \text{ for all } x_i > \pi \text{ and all } i\}.$$

Proposition 3. *Suppose that X_i is a finite subset of $\mathbb{R}$ independent of i and that $U_i(x) \neq U_i(x')$ if $x \neq x'$. If x is a strategy profile that survives iterated deletion of weakly dominated strategies,* $\max\{x_1, \ldots x_I\} = \pi^*$.

I provide a proof of Proposition 3 and additional results in Appendix D.[17]

This game fails to satisfy the Marx and Swinkels [16] TDI property. If $U_i(x) = U_i(x')$ for $x \neq x'$, then it is possible for the set of strategies that survive IDWDS to depend on the order. Even in this case, however, the full-disclosure equilibrium will never survive if there exists another equilibrium that strictly dominates it. The supplement provides a full characterization.

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Appendix A: Definitions

Following Milgrom and Roberts, I define several basic concepts.

Definition 7. *Given $T \subset X$, $\bar{b} \in X$ is called an upper bound for T if $\bar{b} \geq x$ for all $x \in T$; it is the supremum of T (denoted $\sup(T)$) if it is an upper bound and for all upper bounds b of T, $b \geq \bar{b}$. Lower bounds and infimums are defined analogously. A point x is a maximal element of X if there is no $y \in X$ such that $y > x$ (that is, no y such that $y \geq x$ but not $x \geq y$); it is the largest element of X if $x \geq y$ for all $y \in X$. Minimal and smallest elements are defined similarly.*

Definition 8. *The set X is a lattice if for each two point set $\{x, y\} \subset X$, there is a supremum for $\{x, y\}$ (denoted $x \vee y$ and called the join of x and y) and an infimum (denoted $x \wedge y$ and called the meet of x and y) in X. The lattice is complete if for all nonempty subsets $T \subset X$, $\inf(T) \in X$ and $\sup(T) \in X$. An interval is a set of the form $[x, y] \equiv \{z : y \geq z \geq x\}$.*

Definition 9. *A sublattice T of a lattice X is a subset of X that is closed under $\wedge$ and $\vee$. An interval sublattice T of a lattice X is a sublattice of X of the form $[\underline{x}, \overline{x}]$ for some $\underline{x}, \overline{x} \in X$, $\underline{x} \leq \overline{x}$. A complete sublattice T is a sublattice such that the infimum and supremum of every subset of T is in T.*

Definition 10. *A chain $C \subset X$ is a totally ordered subset of X, that is, for any $x \in C$ and $y \in C$, $x \geq y$ or $y \geq x$.*

Definition 11. *Given a complete lattice X, a function $f : X \to \mathbb{R}$ is order continuous if it converges along every chain C (in both the increasing and decreasing directions), that is, if $\lim_{x \in C, x \downarrow \inf C} f(x) = f(\inf(C))$ and $\lim_{x \in C, x \uparrow \sup C} f(x) = f(\sup(C))$. It is order upper-semicontinuous if $\limsup_{x \in C, x \downarrow \inf C} f(x) \leq f(\inf(C))$ and $\limsup_{x \in C, x \uparrow \sup C} f(x) \leq f(\sup(C))$.*

Definition 12. *A function $f : X \to \mathbb{R}$ is supermodular if for all $x, y \in X$,*

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y). \tag{10}$$

Definition 13. *The set S'' dominates S' in the strong set order (written $S'' \geq S'$) if $x^* \in S'$ and $x^{**} \in S''$ imply that $x^* \wedge x^{**} \in S'$ and $x^* \vee x^{**} \in S''$.*

Appendix B: Proofs

The Appendix contains proofs that did not appear in the main text.

Let $\underline{w}_i = \overline{B}_i(\underline{z})$ be the largest best response of Player i to the smallest strategy profile.

Proof of Lemma 2. It follows from (ID) that any $x_i \leq_i \underline{w}_i$ is either weakly dominated by $\underline{w}_i$ or equivalent to $\underline{w}_i$ in the sense that $u_i(x_i, x_{-i}) = u_i(\underline{w}_i, x_{-i})$ for all $x_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]$.

Note that $\underline{w}_i$ is not weakly dominated. To see this, suppose that w_i is a best response to $\underline{z}_{-i}$. On one hand, because $\underline{w}_i$ is the largest best response to $\underline{z}_{-i}$, $w_i \leq_i \underline{w}_i$. Consequently w_i is equivalent to $\underline{w}_i$ or weakly dominated by $\underline{w}_i$. Consequently, no strategy that best responds to $\underline{z}_{-i}$ can weakly dominate $\underline{w}_i$. On the other hand, any strategy that weakly dominates $\underline{w}_i$ must be a best response to $\underline{z}_{-i}$. It follows that $\underline{w}_i$ is not weakly dominated.

Hence any strategy in $E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$ (a strategy equivalent to $\underline{w}_i$) is not weakly dominated. I claim that $E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$ is a lattice. If $x_i, x_i' \in E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$, then $x_i \vee x_i'$ and $x_i \wedge x_i'$ are best responses to $\underline{z}_{-i}$. Hence $x_i \vee x_i' \leq \underline{w}_i$ by the definition of $\underline{w}_i$. Consequently, by (ID), $x_i \vee x_i' \in E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$. Furthermore, $u_i(x_i \wedge x_i', z_{-i}) \leq u_i(x_i \vee x_i', z_{-i})$ for all $z_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]$ by (ID). It follows that

$$2u_i(x_i \vee x_i', z_{-i}) \geq u_i(x_i \vee x_i', z_{-i}) + u_i(x_i \wedge x_i', z_{-i}) \geq u_i(x_i, z_{-i}) + u_i(x_i', z_{-i}) = 2u_i(x_i \vee x_i', z_{-i}), \tag{11}$$

where the second inequality follows from supermodularity and the equation follows because $x_i, x_i', x_i \vee x_i' \in E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$. Consequently the first inequality in (11) must be an equation and $x_i \wedge x_i' \in E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$ by supermodularity.

Because $E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$ is a lattice and u is order upper semicontinuous, it is a complete lattice and has a smallest element, w_i^*. I claim that w_i^* is the smallest strategy that is not weakly dominated. We know that w_i^* is not weakly dominated. Take any $z_i \not\geq_i w_i^*$. Note that for all $x_i \in [w_i^*, z_i \vee w_i^*]$

$$u_i(x_i \vee w_i^*, \underline{z}_{-i}) - u_i(x_i, \underline{z}_{-i}) \geq u_i(w_i^*, \underline{z}_{-i}) - u_i(x_i \wedge w_i^*, \underline{z}_{-i}) \geq 0, \tag{12}$$

where the first inequality follows by supermodularity and the second follows because w_i^* is a best response to $\underline{z}_{-i}$. Furthermore, (ID) and (12) imply that

$$u_i(z_i \vee w_i^*, z_{-i}) - u_i(z_i, z_{-i}) \geq u_i(w_i^*, z_{-i}) - u_i(z_i \wedge w_i^*, z_{-i}) \geq 0, \tag{13}$$

for all $z_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]$. Inequality (13) cannot always hold as an equation, because that would imply $z_i \wedge w_i^* \in E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$, which cannot be true because $w_i^* >_i z_i \wedge w_i^*$ is the smallest element in $E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$. Consequently (13) implies that $z_i \vee w_i^*$ weakly dominates z_i. It follows that w_i^* is the smallest of $[\underline{z}_i, \overline{z}_i]$ that is not weakly dominated.

A similar argument demonstrates that there is a largest element of $[\underline{z}_i, \overline{z}_i]$ that is not weakly dominated. ■

Proof of Theorem 4. Let $\underline{H}(x, p)$ be the smallest strategy that is equivalent to the largest best response to x; $\underline{H}(x, p) = w^*$ exists because $E_i(\underline{w}_i; [\underline{z}_i, \overline{z}_i])$ is a complete lattice. I claim that $\underline{H}(x, p)$ is nondecreasing in p. To do this, I will show that if $z_i \not\geq w_i^*$, then z_i is weakly dominated relative to $[\underline{z}_i, \overline{z}_i]$ (in the p' game). Note that for all $x_i \in [w_i^*, z_i \vee w_i^*]$

$$u_i(x_i \vee w_i^*, \underline{z}_{-i}, p) - u_i(x_i, \underline{z}_{-i}, p) \geq u_i(w_i^*, \underline{z}_{-i}, p) - u_i(x_i \wedge w_i^*, \underline{z}_{-i}, p) \geq 0, \tag{14}$$

where the first inequality follows by supermodularity and the second follows because w_i^* is a best response to $\underline{z}_{-i}$. Furthermore, (ID) and (14) imply that for $x_i \in [w_i^*, z_i \vee w_i^*]$,

$$u_i(x_i \vee w_i^*, z_{-i}, p) - u_i(x_i, z_{-i}, p) \geq u_i(w_i^*, z_{-i}, p) - u_i(x_i \wedge w_i^*, z_{-i}, p) \geq 0, \tag{15}$$

for all $z_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]$. (ID) implies that

$$u_i(x_i \vee w_i^*, z_{-i}, p) - u_i(x_i, z_{-i}, p) \geq 0, \tag{16}$$

for all $z_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]$. Because $x_i \wedge w_i^* < w_i^*$, it follows from the definition of w_i^* that $x_i \wedge w_i^*$ is not equivalent to w_i^*. Consequently, the second inequality in (15) holds strictly for some z_{-i}. Expression (15) therefore implies that $z_i \vee w_i^*$ weakly dominates z_i for preferences $u_i(\cdot, p)$. Consequently (15), (16), and (ID) implies that $z_i \vee w_i^*$ weakly dominates z_i for preferences $u_i(\cdot, p')$ for $p' > p$. This establishes that $\underline{H}_i(x, p)$ (and hence $\underline{H}(x, p)$) is nondecreasing. Because if $z_i \not\geq \underline{H}_i(x, p)$ is weakly dominated, every Nash equilibrium that survives iterated deletion of weakly dominated strategies satisfies $\underline{H}(x, p) \leq x$. By Tarksi's Fixed Point Theorem, $\underline{x}(p) = \inf\{x : \underline{H}(x, p) \leq x\}$ is a fixed point of $\underline{H}(\cdot, p)$, so it is the smallest Nash equilibrium. A similar argument applies to the largest equilibrium. ■

Proof of Lemma 4.

$$\begin{aligned} u_S(\sigma \vee \sigma', \alpha) + u_S(\sigma \wedge \sigma', \alpha) &= E[U^S(\alpha(\min\{\sigma(t), \sigma'(t)\}), t) + U^S(\alpha(\max\{\sigma(t), \sigma'(t)\}), t)] \\ &= E[U^S(\alpha(\sigma(t)), t) + U^S(\alpha(\sigma'(t)), t)] \\ &= u_S(\sigma, \alpha) + u_S(\sigma', \alpha) \end{aligned}$$

$$\begin{aligned} u_R(\alpha \vee \alpha', \sigma) + u_R(\alpha \wedge \alpha', \sigma) &= E[U^R(\max\{\alpha(\sigma(t)), \alpha'(\sigma(t))\}, t) + U^R(\min\{\alpha(\sigma(t)), \alpha'(\sigma(t))\}, t)] \\ &= E[U^R(\alpha(\sigma(t)), t) + U^R(\alpha'(\sigma(t)), t)] \\ &= u_R(\alpha, \sigma) + u_R(\alpha', \sigma) \end{aligned}$$

■

Proof of Lemma 5. Assume $u_R(\alpha \vee \alpha^*, \sigma') \geq u_R(\alpha, \sigma')$. Let $\tilde{\alpha}^* = \min \arg\max_\alpha u_R(\alpha, \sigma')$ be the smallest best response to σ'. From Proposition 5 (Appendix C), it sufficient to show that if $\sigma'' \geq_S \sigma'$, then $u_R(\alpha \vee \tilde{\alpha}^*, \sigma'') \geq u_R(\alpha, \sigma'')$. Let $\mu_\sigma(\cdot \mid m)$ be the posterior distribution over t given $\sigma(t) = m$. The posterior is well defined if there exists t such that $\sigma(t) = m$. It suffices to prove that, for all m in the image of $\sigma''(\cdot)$,

$$\sum_t U^R(\max\{\alpha(m), \tilde{\alpha}^*(m)\}, t)\mu_{\sigma''}(t \mid m) \geq \sum_t U^R(\alpha(m), t)\mu_{\sigma''}(t \mid m). \tag{17}$$

I divide the argument into four cases depending on whether $\sigma'(t) < m$ for all t; $\sigma'(t) = m$ for some t; $\sigma'(t) > m$ for all t; or $\sigma'(t) \neq m$ for all t and there exist t' and t'' such that $\sigma'(t') < m < \sigma'(t'')$.

If $\sigma'(t) < m$ for all t, then $\sigma''(t) < m$ for all t (recall that $\sigma'' \geq_S \sigma'$ implies $\sigma''(t) \leq \sigma'(t)$ for all t) so m is not in the image of $\sigma''(\cdot)$.

If there exists t such that $\sigma'(t) = m$, then because m is in the image of $\sigma''(\cdot)$ and $\sigma'' \geq_S \sigma'$, $\mu_{\sigma''}(\cdot \mid m)$ (weakly) stochastically dominates $\mu_{\sigma'}(\cdot \mid m)$. Because $\tilde{\alpha}^*(m)$ solves $\max_a \sum_t U^R(a, t)\mu_{\sigma'}(t \mid m)$, it follows from the supermodularity of $u_R(\cdot)$ that the solution to $\max_a \sum_t U^R(a, t)\mu_{\sigma''}(t \mid m)$ is greater than $\tilde{\alpha}^*(m)$ and by concavity of $U_R(\cdot, t)$ that inequality (17) holds.

If $\sigma'(t) > m$ for all t, then $\tilde{\alpha}^*(m) = 0$ by definition and inequality (17) holds.

It remains to consider the case in which there does not exist t such that $\sigma'(t) = m$, but $\sigma'(t') < m < \sigma(t'')$ for some t' and t''. In this case, define $\underline{m}$ to be

$$\max\{m' < m : \text{ there exists } t \text{ such that } \sigma'(t) = m'\}.$$

It follows that $\tilde{\alpha}^*(m)$ solves $\max_a \sum_t U^R(a,t)\mu_{\sigma'}(t \mid \underline{m})$. Let $\bar{t} = \max\{t : \sigma'(t) \leq \underline{m}\}$. Because $\sigma'(t) = \underline{m}$ for some t, $\bar{t}$ is well defined. Furthermore, $\tilde{\alpha}^*(m) \leq \arg\max U^R(a, \bar{t})$. Because $\sigma'' \geq_S \sigma'$, $\mu_{\sigma''}(t \mid m) = 0$ if $t < \bar{t}$. Hence

$$\tilde{\alpha}^*(m) \leq \arg\max U^R(a, \bar{t}) \leq \arg\max \sum_t U^R(a,t)\mu_{\sigma''}(t \mid m)$$

and so (17) holds.

A symmetric argument establishes that if $\sigma'' \geq_S \sigma'$, $u_R(\alpha \wedge \alpha^{**}, \sigma'') \geq u_R(\alpha, \sigma'')$, then $u_R(\alpha \wedge \tilde{\alpha}^{**}, \sigma') \geq u_R(\alpha, \sigma')$ (when $\tilde{\alpha}^{**}$ is the largest best response to σ''). ■

Proof of Lemma 6. Assume that $u_S(\alpha', \sigma^* \vee \sigma) \geq u_S(\alpha', \sigma)$. Let $\tilde{\sigma}^* = \min \arg\max u_S(\alpha', \sigma)$ be the smallest best response to α'. From Proposition 5, it suffices to show that if $\alpha'' \geq_R \alpha'$, then $u_S(\alpha'', \sigma \vee \tilde{\sigma}^*) \geq u_S(\alpha'', \sigma)$. It suffices to show that, for all t, $\sigma(t) < \tilde{\sigma}^*(t)$ implies that $U^S(\alpha''(\tilde{\sigma}^*(t)), t) \geq U^S(\alpha''(\sigma(t)), t)$. If $\sigma(t) < \tilde{\sigma}^*(t)$, then by definition of $\tilde{\sigma}^*$, $U^S(\alpha'(\tilde{\sigma}^*(t)), t) > U^S(\alpha'(\sigma(t)), t)$. The inequality must be strict because $\tilde{\sigma}^*$ is the smallest best response (so type t sends the highest message that leads to the maximum available payoff) and $\tilde{\sigma}^*(t) > \sigma(t)$. It follows from concavity of $U^S(\cdot, t)$ that $U^S(\alpha''(\tilde{\sigma}^*(t)), t) \geq U^S(\alpha''(\sigma(t)), t))$. This inequality may be weak (if $\alpha''(\tilde{\sigma}^*(t)) = \alpha''(\sigma(t))$) so that (WID) does not hold.

A symmetric argument establishes that if $\alpha'' \geq_R \alpha'$, $u_S(\alpha'', \sigma^{**} \wedge \sigma) \geq u_S(\alpha'', \sigma)$ implies that $u_S(\alpha'', \tilde{\sigma}^{**} \wedge \sigma) \geq u_S(\alpha'', \sigma)$, where $\tilde{\sigma}^{**} = \max \arg\max u_S(\alpha'', \sigma)$. ■

Proof of Theorem 6. Let $\underline{x}^0$ be the smallest strategy profile and $\overline{x}^0$ be the largest strategy profile. I claim that the set of strategies that are not weakly dominated is contained in $[\underline{y}^0, \overline{y}^0]$. Suppose $z \notin [\underline{y}^0, \overline{y}^0]$ and, in particular, suppose $z_i \not\geq_i \underline{y}_i^0$. I claim that z_i is weakly dominated by $z_i \vee \underline{y}_i^0$.

Observe that

$$u_i(z_i \vee \underline{y}_i^0, \underline{x}_{-i}^0) - u_i(z_i, \underline{x}_{-i}^0) \geq u_i(\underline{y}_i^0, \underline{x}_{-i}^0) - u_i(z_i \wedge \underline{y}_i^0, \underline{x}_{-i}^0) > 0, \tag{18}$$

where the first inequality follows from supermodularity and the second from the definition of $\underline{y}_i^0$.

It follows from (18) that

$$u_i(z_i \vee y_i^0, \underline{x}_{-i}^0) > u_i(z_i, \underline{x}_{-i}^0). \tag{19}$$

Because $\underline{y}_i^0$ is the smallest best response to $\underline{x}_{-i}^0$, it follows from (WID) and (19) that if $z_i \not\geq_i \underline{y}_i^0$, then

$$u_i(z_i \vee \underline{y}_0^1, z_{-i}) \geq u_i(z_i, z_{-i}) \text{ for all } z_{-i} \in [\underline{x}_{-i}^0, \overline{x}_{-i}^0]. \tag{20}$$

Because inequality (20) holds strictly when $z_{-i} = \underline{x}^0_{-i}$ by inequality (18), it follows that z_i is weakly dominated by $z_i \vee \underline{y}^0_i$. An analogous argument applies to show that if $z_i \not\leq_i \overline{x}^0_i$, then z_i is weakly dominated.

It is straightforward to continue the argument by induction to obtain a nested sequence of intervals $[\underline{y}^k, \overline{y}^k]$ and to conclude that the limiting interval has the desired properties. ■

Appendix C

This appendix clarifies the connection between the WID and ID conditions. I begin by introducing a new concept and then I show its relationship to (ID). I then introduce another concept and show that it is equivalent to (WID). The new definitions are transparently nested, making it clear that (ID) implies (WID). Finally, I prove that (WID) implies that best responses are monotonic in a way that is implied by (ID). Throughout I will assume that X and Y are lattices, $f(\cdot)$ is a function $f : X \times Y \to \mathbb{R}$, and $\arg\max_{x \in J} f(x, y)$ is nonempty for all intervals $J \subset X$ and $y \in Y$.

Definition 14. *Assume $x^* \in \arg\max_{x\in X} f(x, y')$, and $x^{**} \in \arg\max_{x\in X} f(x, y'')$. A function $f : X \times Y \to \mathbb{R}$ satisfies the revised interval-dominance property (RID) in its two arguments on the set $X \times Y$ if for all $y'' \geq y'$,*

$$f(x' \vee x^*, y') \geq f(x', y') \implies f(x' \vee x^*, y'') \geq f(x', y'')$$

and

$$f(x' \wedge x^{**}, y'') \geq f(x', y'') \implies f(x' \wedge x^{**}, y') \geq f(x', y').$$

(RID) is an awkward condition because it relies on conditions defined in terms of x^*. It is a useful formulation for some of the arguments in Appendix B. Letting $x'' = x' \vee x^*$, it follows that $x'' \geq x'$ and therefore the conditions in Definition 14 are implied by single crossing. Definition 14 imposes the condition less often than single crossing. The next result demonstrates that (RID) is a reformulation of (ID).[18]

Proposition 4. *Let X and Y be lattices. A supermodular function $f : X \times Y \to \mathbb{R}$ satisfies (ID) if and only if it satisfies (RID) on all intervals $[x', x''] \subset X$.*

Proof of Proposition 4. First I show that (RID) implies (ID). If $f(x'', y') \geq f(x, y')$ for all $x \in [x', x'']$, then $x'' \in \arg\max_{x\in[x',x'']} f(x, y')$. It follows that $f(x''\vee x, y') \geq f(x, y')$ and so (RID) implies that $f(x'', y'') \geq f(x, y'')$ for $x \in [x', x'']$. It remains to show that if $f(x'', y') > f(x, y')$, then $f(x'', y'') > f(x', y'')$. But if $f(x', y'') \geq f(x'', y'')$, then $x' \in \arg\max_{x\in[x',x'']} f(x, y'')$ so (RID) implies that $f(x', y') \geq f(x'', y')$. Consequently, if $f(x'', y') > f(x', y')$ then $f(x'', y'') > f(x', y'')$. It follows that if (RID) holds on all intervals, then (ID) holds.

[18] I owe this argument to an anonymous referee.

Next I show that (ID) implies (RID). Fix an interval $[x', x''] \subset X$. Let $x^* \in \arg\max_{x \in [x',x'']} f(x, y')$ and $x^{**} \in \arg\max_{x \in [x',x'']} f(x, y'')$.

Let $f(\hat{x} \vee x^*, y') \geq f(\hat{x}, y')$ for some $\hat{x} \in [x', x'']$.

It follows from supermodularity of $f(\cdot)$ that for any $x \in X$,

$$f(x \vee x^*, y') + f(x \wedge x^*, y') \geq f(x, y') + f(x^*, y'). \tag{21}$$

Because $x, x^* \in [x', x'']$ implies that $x \wedge x^* \in [x', x'']$, it follows from the definition of x^* that $f(x^*, y') \geq f(x \wedge x^*, y')$ for all $x \in [x', x'']$. Inequality (21) implies that

$$f(x \vee x^*, y') \geq f(x, y') \tag{22}$$

for all $x \in [x', x'']$. Because $\hat{x} \in [x', x'']$, (22) implies

$$f(x \vee x^*, y') \geq f(x, y') \tag{23}$$

for all $x \in [\hat{x}, \hat{x} \vee x^*]$. Because $x \in [\hat{x}, \hat{x} \vee x^*]$ implies that $x \vee x^* = \hat{x} \vee x^*$, it follows that $f(x \vee x^*, y') = f(\hat{x} \vee x^*, y')$. Consequently (23) implies that $f(\hat{x} \vee x^*, y') \geq f(x, y')$ for all $x \in [\hat{x}, \hat{x} \vee x^*]$ and therefore, by (ID), $f(\hat{x} \vee x^*, y'') \geq f(\hat{x}, y'')$.

A similar argument establishes the symmetric implication. ■

The next definition parallels (RID).

Definition 15. *Assume $y'' > y'$, $x^* \in \arg\max_{x \in X} f(x, y')$, and $x^{**} \in \arg\max_{x \in X} f(x, y'')$. A function $f : X \times Y \to \mathbb{R}$ satisfies the revised weak interval-dominance property (RWID) in its two arguments on the set $X \times Y$ if*

$$f(x' \vee x^*, y') \geq f(x', y') \implies$$

$$\exists \tilde{x}^* \in \arg\max_{x \in X} f(x, y'),\ \tilde{x}^* \leq x^*,\ \textit{such that } f(x' \vee \tilde{x}^*, y'') \geq f(x', y'') \tag{24}$$

and

$$f(x' \wedge x^{**}, y'') \leq f(x', y'') \implies$$

$$\exists \tilde{x}^{**} \in \arg\max_{x \in X} f(x, y''),\ \tilde{x}^{**} \geq x^{**},\ \textit{such that } f(x' \wedge \tilde{x}^{**}, y') \leq f(x', y'). \tag{25}$$

It is clear that (RID) implies (RWID). The next result shows that (WID) and (RWID) are equivalent. Propositions 4 and 5 imply that (ID) implies (WID).

Proposition 5. *Let X and Y be lattices. Assume $y'' > y'$. A supermodular function $f : X \times Y \to \mathbb{R}$ satisfies the revised weak interval-dominance property (RWID) in its two arguments on the set $X \times Y$ if and only if it satisfies (WID) in its two arguments on the set $X \times Y$.*

Proof of Proposition 5. If (WID) holds, then (RWID) clearly holds. I want to show that if $f(x \vee z, y') \geq f(x, y')$, then $f(x \vee \tilde{z}, y'') \geq f(x, y'')$ for $\tilde{z} \leq z$. Let $z^* = \min \arg\max_{w \in [x \wedge z, z]} f(w, y')$. It follows that $x \wedge z^* \in [x \wedge z, z]$ so $f(z^*, y') \geq f(x \wedge z^*, y')$. It follows from supermodularity that $f(x \vee z^*, y') \geq f(x, y')$. Hence (RWID) implies that $f(x \vee z^*, y'') \geq f(x, y'')$. Because $z^* \leq z$ and $f(z^*, y') \geq f(z, y')$, it follows that (WID) holds. ■

Proposition 1 (stated in the text) shows that (WID) implies that solutions to parameterized optimizations are increasing in a sense that is weaker than the strong set order. The proposition uses the following notation: $x^{**} \in \arg\max f(x, y'')$, $x^* \in \arg\max f(x, y')$, $\overline{x}^{**} = \max \arg\max f(x, y'')$, $\underline{x}^* = \min \arg\max f(x, y')$.

Proof of Proposition 1. By definition, $f(\underline{x}^*, y') \geq f(\underline{x}^* \wedge x^{**}, y')$ and hence, by supermodularity, $f(\underline{x}^* \vee x^{**}, y') \geq f(x^{**}, y')$. It follows from (RWID) that $f(\underline{x}^* \vee x^{**}, y'') \geq f(x^{**}, y'')$. A similar argument shows that when (RWID) holds, $\overline{x}^{**} \wedge x^* \in \arg\max f(x, y')$. ■

Proposition 1 is a variation on Fact 2. Both results demonstrate how assumptions of $f(\cdot)$ make it possible to evaluate how the set of solutions to the parameterized optimization problem $\max_{x \in J} f(x, y)$ change with the parameter y. Fact 2 demonstrates that supermodularity and (ID) combine to guarantee that maximizers are increasing with respect to the strong set order. Proposition 1 demonstrates that supermodularity and (WID) combine to guarantee that maximizers are increasing in the weaker sense captured by (9).[19]

LiCalzi and Veinott [15] present several variations on single-crossing conditions. Corollary 11 contains results that demonstrate different ways in which these conditions can lead to monotone comparative statics with respect to different ways to order sets. These results are in the spirit of Proposition 1 but are distinct.

Appendix D

This Appendix provides a proof of Proposition 3.

There is a finite set of players. I denotes the player set. The strategy set for each player is X_i, a finite subset of the real line. For $y = (y_1, \dots, y_I)$, $y_i \in \mathbb{R}$, let $M(y) = \max\{y_1, \dots, y_I\}$. Payoffs are given by $u_i(x) = U_i(M(x))$, where $U_i(\cdot) : X_i \to \mathbb{R}$ are arbitrary.

[19] The relationship induced by the conditions in (9) need not be transitive. That is, it is possible for $\arg\max f(x, y_1)$ to be distinct from $\arg\max f(x, y_2)$ and for (9) to hold both when $(y_1, y_2) = (y', y'')$ and when $(y_1, y_2) = (y'', y')$.

Definition 16. *The* **smallest strict Pareto disclosure** *is*

$$\pi^* = \min\{\pi : U_i(\pi) > U_i(x_i) \text{ for all } x_i > \pi \text{ and all } i\}.$$

Definition 17. *The* **smallest weak Pareto disclosure** *is*

$$\tilde{\pi}^* = \min\{\pi : U_i(\pi) \geq U_i(x_i) \text{ for all } x_i > \pi \text{ and all } i\}.$$

Because the game is finite, it is clear that π^* and $\tilde{\pi}^*$ are well defined and that $\pi^* \geq \tilde{\pi}^*$. Equality will hold if $U_i(\cdot)$ is one-to-one for each player.

Any strategy profile x satisfies $x_i \leq \pi$ and at least two $x_j = \pi^*$ is a Nash Equilibrium for $\pi = \pi^*$ or $\tilde{\pi}^*$.

Full disclosure is always a Nash equilibrium in this game, but there are typically other Nash Equilibria. It is straightforward to show that pure-strategy Nash Equilibria are Pareto ranked. If x^* and x^{**} are both Nash Equilibria and $M(x^*) \leq M(x^{**})$, then $U_i(x^*) \geq U_i(x^{**})$ for all i.

Denote the set of strategies that survive IDWDS by S.

Lemma 7. *For all $x \in X$ and every i, there exists $x_i \in S_i$ that is a best response to x relative to the X.*

Proof. The result is clear if the best response to x has not yet been deleted. If the best response to x has been deleted, then it was deleted by a strategy that weakly dominates it. This strategy must be a best reply to x. ■

Lemma 8. *There exists a strategy profile $x \in S$ such that* $\max\{x_1, \ldots, x_I\} \leq \pi^*$.

Proof. Suppose that after k iterations, there exists a strategy profile x satisfying the condition in the lemma. In the next iteration, every agent must have a strategy that is a best response to x. The best response must do at least as well as disclosing π^* by Lemma 7. By definition of π^*, no strategy $x_i > \pi^*$ can do at least as well as π^* against x. Hence a strategy less than or equal to π^* must remain. ■

Lemma 9. *There exists no strategy profile $x \in S$ such that $M(x) < \tilde{\pi}^*$.*

Proof. Let $\tilde{x}$ be a strategy profile that minimizes $M(x')$ subject to x' surviving IDWDS. This strategy profile exists and, by Lemma 8, $M(\tilde{x}) \leq \pi^*$. Let $\tilde{\pi} \equiv M(\tilde{x})$. We wish to show that $\tilde{\pi} \geq \tilde{\pi}^*$. In order to reach a contradiction, assume that $\tilde{\pi} < \tilde{\pi}^*$. By the definition of $\tilde{\pi}^*$, it must be the case that for some i,

$$\text{there exists } x_i \text{ with } \tilde{\pi} < x_i \leq \tilde{\pi}^* \text{ such that } U_i(x_i) > U_i(\tilde{\pi}), \tag{26}$$

because otherwise π^* would not be the least strict Pareto disclosure. Lemma 7 guarantees that Player i has a best response to $\tilde{x}$ relative to the original strategy set that survives IDWDS. Denote such a strategy by y_i. By the definition of π^*, $y_i \leq \pi^*$. It follows from (26) that $y_i > \tilde{\pi}$. We claim that y_i weakly dominates $\tilde{x}_i$. For any x such that $M(x_{-i}) \geq y_i$, $u_i(y_i, x_{-i}) = u_i(\tilde{x}_i, x_{-i})$. For any x such that $M(x_{-i}) < y_i$,

$$U_i(y_i) = u_i(y_i, x_{-i}) > U_i(\tilde{\pi}) \geq U_i(x_i) \tag{27}$$

for $\tilde{\pi} \leq x_i \leq y_i$, where the strict inequality follows from (26) and the definition of y_i and the weak inequality follows from the definition of $\tilde{\pi}$. It follows that y_i is weakly better than $\tilde{x}_i$. Inequality (27) guarantees that y_i is strictly better than $\hat{x}_i$ when $x_{-i} = \tilde{x}_{-i}$. Hence y_i weakly dominates $\tilde{x}_i$ as claimed. By definition $\tilde{x}$ survived IDWDS. Hence we have a contradiction. ∎

Lemma 10. *No strategy $z_i > \pi^*$ survives IDWDS.*

Proof. Fix a stage in the process of deleting strategies and let S_i^k be the set of strategies remaining for player i. Let $z_i = \min\{\arg\min_{x_i \geq \pi^*, x_i \in S_i^k} U_i(x_i)\}$. If $z_i = \pi^*$, then all strategies greater than π^* have been deleted because $U_i(\pi^*) > U_i(x_i)$ for all $x_i > \pi_i^*$. Otherwise, we claim that z_i is weakly dominated by π^*. For any x such that $M(x_{-i}) \geq z_i$, z_i and π^* earn the same payoff. For any x such that $M(x_{-i}) < z_i$, Player i's utility using z_i is $U_i(z_i)$, while Player i's utility using π^* is $U_i(y_i)$ for $y_i = M(\pi^*, x_{-i})$. It follows that $\pi^* \leq y_i < z_i$. It follows from the definition of y_i that π^* does strictly better than z_i for Player i against strategies such that $M(x_{-i}) < z_i$.

Because there always exists a strategy in which $M(x_{-i}) < z_i$ by Lemma 8, π^* must be strictly better than z_i against one strategy profile that survives IDWDS. Consequently, π^* weakly dominates z_i. It is possible that $\pi^* \notin S_i^k$, but in this case there must remain a strategy $x_i^* \in S_i^k$ such that $u_i(x_i^*, x_{-i}) \geq u_i(\pi^*, x_{-i})$ for all $x_{-i} \in S_{-i}^k$. Therefore, x_i^* weakly dominates z_i and so z_i must eventually be deleted. ∎

Proposition 6. *If x is a strategy profile that survives IDWDS, then $M(x) \in [\tilde{\pi}^*, \pi^*]$. If x is a Nash equilibrium strategy profile that survives IDWDS, then $u_i(x) \geq \pi^*$ for all i.*

Proof. Lemma 9 guarantees that there are no strategy profiles with maximum less than $\hat{pi}^*$. Lemma 10 guarantees that no higher strategy survives. This establishes the first part of the Proposition. Given any surviving strategy x, it follows from Lemma 7 that each player has a surviving strategy that is a best response to x_{-i} relative to the full strategy set. Since $x_i = \pi^*$ leads to payoff $U_i(\pi^*)$ for Player i against any surviving strategy by Lemma 10, the second part of the proposition follows. ∎

Corollary 1. *If $\pi^* = \hat{\pi}^*$, then for all x that survives IDWDS, $M(x) = \pi^*$.*

Corollary 1 follows directly from Proposition 6. It follows from the definition of π^* and $\hat{\pi}^*$ that $\pi^* = \hat{\pi}^*$ will hold provided that $U_i(\cdot)$ is one-to-one (no ties) for all i. Consequently Proposition 3 follows directly from Corollary 1.