Production Priorities in Dynamic Relationships *

Jean Guillaume Forand[†] Jan Zápal[‡]

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Abstract

We characterise optimal contracts in a dynamic principal-agent model of joint production in which project opportunities are heterogenous, utility from these projects is non-transferable and the agent has the option to quit the relationship at any time. In order to demand the production of projects that benefit her but not the agent, the principal must commit to produce projects that benefit the agent in the future. Production at all stages of the relationship is ordered by projects' *cost-effectiveness*, which is their efficiency in transferring utility between the principal and the agent: cost-effective demands impose relatively low costs on the agent, and cost-effective compensation imposes relatively low costs on the principal. Over time, optimal contracts become more generous towards the agent by adding commitments to less cost-effective compensation. In turn, because this new compensation cannot be profitably exchanged against less cost-effective demands, the principal narrows the scope of her demands.

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[†]Department of Economics, University of Waterloo, Hagey Hall of Humanities, Waterloo, Ontario, Canada N2L 3G1. Email: jgforand@uwaterloo.ca. Website: http://arts.uwaterloo.ca/~jgforand.

[‡]CERGE-EI, a joint workplace of Charles University and the Economics Institute of the Czech Academy of Sciences, Politickych veznu 7, 111 21 Prague, Czech Republic. Email: j.zapal@cerge-ei.cz. Website: https://sites.google.com/site/jzapal.

1 Introduction

Productive relationships generate a variety of joint project opportunities over their lifetimes. This raises two related questions: what criteria guide decisions to produce some opportunities and pass up others, and how does project selection evolve over time? In this paper, we address these questions in a dynamic principal-agent model in which (a) heterogenous project opportunities arrive according to an arbitrary stochastic process, (b) utility from these projects is non-transferable (although transferable utility is a special case of our model), and (c) the principal is contractually committed to production decisions but the agent can walk away from the relationship at any time. We characterise optimal contracts in this setting and detail the dynamics of the principal's *demand* and *supply* of projects (i.e., the production of projects that benefit the principal but are costly for the agent, and vice-versa).

Although we model a canonical principal-agent relationship, for the remainder of the Introduction we fix ideas by focusing on a manager-worker pair within a larger firm. The simplest model of their interaction features two project opportunities: at each stage, the manager demands effort from the worker and supplies a wage (Mirrlees, 1976). We allow for a rich set of productive activities that arise randomly over the course of this relationship. The manager can make demands on the worker that differ in their benefits for the manager and their costs to the worker. For example, the manager may need the worker to deal with an emergency, like a failure in the firm's server, or she may ask the worker to complete a routine project that is less time-sensitive, like writing a plan for the firm's IT infrastructure. Similarly, the manager can supply a number of projects to the worker in the form of both financial and non-monetary compensation. For example, the manager can recommend the worker for a bonus, offer perks like travel opportunities, accommodations for family issues and better office space, or tilt task allocations towards those that benefit the worker's career (e.g., involving training programs). More broadly the (stochastic) dynamics of project opportunities within the relationship can be driven by the business cycle, industry trends, or human capital accumulation by both the manager and the worker.

In the case in which the manager demands effort and supplies money, it is well known (Lazear, 1981) that she benefits from delaying the worker's compensation: whereas current payments are sunk when the manager makes future demands for effort, committing to pay the worker in the future motivates both current and future effort. Not surprisingly, the optimal contracts in our model will also feature backloaded compensation, but our central task is to determine how the manager selects the projects she uses to reward the worker. If, for example, the worker prefers increased flexibility in his schedule to access to a job training program, will the manager prioritise the former type of compensation over the latter? The answer, in general, is no, because focusing only on the workers' preferences neglects the manager's costs from supplying projects. If job training increases the worker's productivity, then its net cost for the firm can be small relative to the cost of scheduling flexibility, which offers less countervailing benefits to the firm. If the worker does not value the two types of compensation too differently, then the manager always benefits from substituting job training for schedule flexibility. Therefore, she will commit to sending the worker to all available job training programs before making any promises about future scheduling flexibility.

This intuition underlies our main result characterising optimal contracts: we show that the manager always prioritises her supply of projects according to their *cost-effectiveness*: their benefit for the worker relative to their cost for the manager. At any point in the relationship, the optimal contract identifies a threshold supply project and projects that are more cost-effective than this threshold are supplied whenever they arrive. Projects that are less cost-effective than the threshold are not supplied unless (a) the manager makes a new demand and (b) the worker's participation constraint requires fresh commitments to future compensation. Because the manager adds new supply commitments through cost-effectiveness, the threshold project transitions to less cost-effective projects over time. How does the growth in the scope of the worker's compensation affect the manager's demand for projects? We show that the manager only demands those projects that are more cost-effective than the threshold supply project (where cost-effectiveness for demand projects measures the benefit for the manager relative to the cost for the worker). Therefore, the manager's accumulation of increasingly less cost-effective supply commitments is tied to rationing of her demands on the worker, which become concentrated on the most cost-effective projects.

To further illustrate our results, suppose that the manager can demand emergency or routine projects, with the effort cost being the same for both types of projects but emergency projects being more important for the manager. Therefore, emergency projects are more cost-effective. Suppose also that all demands are more cost-effective than supplying the worker with job training but that only emergency projects are more cost-effective than supplying the worker with schedule flexibility. Early in the relationship, the manager demands both emergency and routine projects and only commits to supply job training:[1] because the manager benefits from trading both emergency and routine projects against promises of job training, she will not pass up any demand until all future training opportunities have been promised. Later in the relationship, the manager demands only emergency projects and supplies both job training and schedule flexibility: because the manager prefers to scale back her demands for routine projects

[1] This illustrates typical dynamics of optimal contracts, the details of which will depend on, among other things, the process driving project opportunities. We revisit this example in Section 4.

to avoid promising schedule flexibility, she will pass over the former once she must supply the latter to incentivise the worker to take up emergency projects. The inefficiency generated by the worker's inability to commit to remain in her job is captured by the fact that production decisions for the same project can differ over time: both parties could be made better off ex ante if the manager could use training opportunities that are passed over early in the relationship to incentivise demands for routine projects that are passed over later on. In fact, we show that *ex ante Pareto-efficient contracts* involve a time-invariant threshold project.

Cost-effectiveness pins down project priorities, but not the exact dynamics of production. For example, for how long can the manager keep demanding routine projects? Answering such questions requires determining the worker's value from the relationship at any point in time, which sets the level of his participation constraint. This value, which is endogenous, incorporates the worker's utility from producing projects, his time preferences and the availability of projects in the future: when the process driving project opportunities is arbitrary, the value has little structure. In Section 5, we specialise the model to the case of Markov project processes and construct optimal contracts directly. In doing so, we rank the manager's demands by how *expensive* they are for her: more expensive demands require that a broader scope of projects be supplied to the worker.

Because we study how future opportunities provide incentives for current production, our work has connections to the literature on informal risk-sharing in the presence of stochastic endowment shocks (Thomas and Worrall, 1988; Kocherlakota, 1996; Dixit, Grossman, and Gul, 2000). Important generalisations of this work incorporate hidden information, about endowment shocks or utility from production, as well as sequential actions. The former literature analyses chips mechanisms (Möbius, 2001; Hauser and Hopenhayn, 2008) and dynamic contracts with and without commitment (Guo and Hörner, 2015; Lipnowski and Ramos, 2016; Li, Matouschek, and Powell, 2017). The latter literature studies hold-up situations (Thomas and Worrall, 1994, 2018; Board, 2011) and has close links to the relational contracts literature (Levin, 2003). Furthermore, our work is related to the literature on dynamic principal-agent interactions (Lazear, 1981; Rogerson, 1985; Spear and Srivastava, 1987; Sannikov, 2008). Our focus on selection from heterogenous project opportunities is the key difference between these contributions and ours. Furthermore, we assume that players are risk-neutral, so that risk-sharing plays no role in our results; we do not rely on transfers; we place no restrictions on the process driving project opportunities, as opposed to the standard *iid* or Markov assumptions;[2] and we abstract from information asymmetries and hold-up problems.

[2]From a technical point of view, this rules out standard recursive approaches to characterising optimal dynamic contracts (Spear and Srivastava, 1987; Thomas and Worrall, 1988; Abreu, Pearce, and Stacchetti, 1990). In contrast, our proofs rely on direct arguments.

Three papers are most closely related to ours. First, Ray (2002) shows that any optimal principal-agent relationship backloads the agent's compensation: by increasing the agent's continuation value the principal relaxes the agent's no-deviation constraint, so that she can make the agent work harder and improve efficiency. In our model this logic is one of the forces that drive the backloading of the agent's utility: by promising to supply the threshold project in the future, the principal gains the ability to demand projects that are more cost-effective than this threshold. The other reason is the rationing in the principal's demands stemming from her accumulation of increasingly less cost-effective supply commitments. This latter reason has no analog in Ray (2002), which features a repeated stage game and hence no heterogeneity in future production opportunities.[3]

Second, Bird and Frug (2019a) study project production in a closely related dynamic principal-agent model, in which project arrivals follow independent Poisson processes and are privately observed by the agent. Like us, they highlight the criterion of cost-effectiveness for prioritising project production. Unlike us, they show that the principal frontloads the agent's compensation, in that she might make less cost-effective supply commitments before exhausting all more cost-effective supply commitments. Informational asymmetries are the key to understanding why our results differ from theirs. In their environment, the principal's only tool to incentivise the agent to disclose the arrival of a demand project is the *growth* in the agent's continuation value. Therefore, frontloading the agent's compensation allows the principal to free up incentives for future disclosures, and the principal trades off prioritising cost-effective projects against future flexibility.[4] In our model with commonly observed project opportunities, it is the *level* of the principal's future commitments that underpin the agent's incentives. Consequently, the agent's compensation is backloaded because the principal always follows cost-effectiveness when making supply commitments, and the (inefficient) variability in the set of projects that the principal demands and supplies vanishes in the long run.

Third, in a contemporaneous paper, Samuelson and Stacchetti (2017) study the role of transfers in a version of our model with two-sided lack of commitment and an *iid* process driving project opportunities. They show that the principal uses variation in either continuation values or transfers to generate incentives when transfers are either absent or present, respectively. Their model, however, does not admit a simple description of the relationship's dynamics. In our model, we can capture transfers to the agent or to the principal through suitably defined supply and demand projects. Because the principal follows cost-effectiveness when committing

[3]See also Bird and Frug (2019b) who study conditions under which principal-agent relationships increasingly favour the agent over time.

[4]See also Hopenhayn, Llobet, and Mitchell (2006), in which a planner rations a fixed (expected discounted) stock of rights to future monopoly power to retain the ability to reward a sequence of competing innovators.

to supply projects, our results imply that the principal will not start paying the agent until she has exhausted more cost-effective means to reward him. Moreover, the relationship dynamics in our model initially favour the principal and eventually favour the agent. This implies that, when available, transfers flow towards the principal early in the relationship and towards the agent later in the relationship.

2 Model

A principal and an agent participate in a long-lived relationship in which a joint project opportunity arises in each period $t = 1, 2, \ldots$. Specifically, let $\mathcal{U} \subset \mathbb{R}^2$ be a finite set and let $u = \{u_t\}_{t \geq 1}$ be a $\mathcal{U} \cup \{(0,0)\}$-valued stochastic process that describes the arrival of projects over time, where $u_t = (0,0)$ denotes the absence of a project at t. Let $u^t = (u_1, \ldots, u_t)$ denote a project history at t, and let $\mathcal{H}$ denote the set of all such histories for all times t. Because optimal contracts are indeterminate at histories that occur with zero probability, we assume that $\mathbb{P}_0(u^t) > 0$ for all project histories u^t. This is the only assumption that we impose on the project process u for our main results, and we do so mainly to ease the exposition.[5]

Given a project u_t at time t, the principal and the agent simultaneously decide whether or not to participate in the production of the project, and project u_t is produced if and only if both players agree to produce it. We let $u_t = (u_{P,t}, u_{A,t})$ denoted the payoffs to the principal and the agent if project u_t is produced, and we normalise each player's payoff from no production to 0. For simplicity, we assume that the players' stage preferences over the production of projects are strict, that is, that $u_{A,t} \neq 0$ and $u_{P,t} \neq 0$ for all projects $u_t \in \mathcal{U}$. Therefore, player i (myopically) prefers to participate in the production of project u_t if $u_{i,t} > 0$ and prefers not to participate if $u_{i,t} < 0$. Finally, the players discount future payoffs with common factor $\delta \in (0,1)$. We model projects parsimoniously, but we can accommodate projects which are more complicated ventures with uncertain outcomes: in this case, u_t is interpreted as the expected utilities to the principal and the agent from these richer lotteries. Similarly, production of project u_t might generate payoffs in periods beyond t: in this case, u_t is the present value to the principal and the agent of that payoff flow.

Project histories and production decisions, and hence all players' payoffs, are publicly observed and verifiable. A *contract* $\kappa : \mathcal{H} \to [0,1]$ maps project histories into production probabilities. Given a project history u^t at time t, $\kappa(u^t)$, henceforth κ_t for short with history u^t understood, is the probability with which contract κ specifies that the project at t is produced.

[5] Any process with zero-probability events can be expressed as the limit of a sequence of processes without such events, and the limit of the corresponding sequence of optimal contracts is an optimal contract for the limiting process.

Let $\mathcal{K}$ denote the set of all contracts. We make the strong assumption that production decisions can be verifiably conditioned on a public randomisation device. However, our model also admits an interpretation in which all production decisions are deterministic. Specifically, we can reinterpret κ_t as specifying the intensity with which project u_t is produced. In this view, interior production probabilities represent reducing the scale of a project's implementation.[6]

Given a contract κ and a history u^t at time t, let

$$U_{i,t} = \mathbb{E}_t \sum_{t'=t}^{\infty} \delta^{t'-t} \kappa_{t'} u_{i,t'},$$

denote the associated expected discounted sum of payoffs to player i starting from t. The expectation is taken conditional on the information contained in project history u^t, but, as for contracts, we leave the history-dependence of payoffs implicit to lighten notation. Notice that the linearity of stage utilities in production probabilities implies that intertemporal smoothing of production decisions due to risk-aversion plays no role in our results.

We assume that the principal commits to contracts. Meanwhile, the agent has the option to irreversibly quit the relationship at the beginning of every period t, after the arrival of project u_t but before the realisation of the contract's production decision (determined by κ_t). If the agent remains in the relationship, then he is committed to following the outcome of the public randomisation device for that period. Quitting yields a payoff of 0 to both players, which is the payoff they receive when no project is ever produced. It follows that an *optimal contract* κ^* is a solution to the problem

$$\begin{aligned} &\max_{\kappa \in \mathcal{K}} \; \mathbb{E}_0 U_{P,1} \\ &\text{subject to } U_{A,t} \geq 0 \text{ for all project histories } u^t. \end{aligned} \tag{$IR_{A,t}$}$$

In words, an optimal contract maximises principal's ex ante utility from the relationship subject to being *individually rational* for the agent following all project histories.[7] As we show below, production probabilities in optimal contracts are often bang-bang, in which case our restriction to ex ante individual rationality constraints for the agent (i.e., prior to the realisation of the production decision) is not constraining. However, following some histories the agent's ex post individual rationality constraint could fail if the public randomisation device calls for some

[6] Allowing the contract to depend on a richer notion of histories, which record past outcomes of the randomisation in production, would not change any of our results. By using this randomisation the principal can offer the agent random continuation utility, but this randomisation can only (weakly) hurt the principal due to the convexity of the underlying utility possibility set.

[7] Standard arguments establish the existence of an optimal contract (e.g., Dixit et al., 2000).

project to be produced. In such cases, our results will exploit the fact that the principal can provide adequate incentives to the agent ex ante by committing to interior production probabilities.

In any period and given any project over which the preferences of the principal and the agent are aligned, optimal contracts must specify jointly optimal production decisions.[8]

Lemma 1. *If contract κ^* is optimal, then*

1. *if $u_{P,t}, u_{A,t} > 0$, then $\kappa^*_t = 1$, and*

2. *if $u_{P,t}, u_{A,t} < 0$, then $\kappa^*_t = 0$.*

Common interest projects contribute to the value of the relationship, but Lemma 1 confirms that optimal contracts can be identified with the production decisions they prescribe for those projects on which the principal and the agent disagree. To this end, define the sets $\mathcal{D} = \{u \in \mathcal{U} : u_P > 0 > u_A\}$ and $\mathcal{S} = \{u \in \mathcal{U} : u_A > 0 > u_P\}$ and assume, to avoid trivialities, that $\mathcal{D}$ and $\mathcal{S}$ are both non-empty. Given a contract κ, we say that the principal *demands a project with probability κ_t at t* whenever $u_t \in \mathcal{D}$, and conversely that the principal *supplies a project with probability κ_t at t* whenever $u_t \in \mathcal{S}$. The decomposition of an optimal contract into the demand and supply of projects turns out to be useful for describing project selection and its dynamics. To simplify notation, we denote a typical element of $\mathcal{D}$ by v and a typical element of $\mathcal{S}$ by w, and any statement referring to *demand project* v (respectively, *supply project* w) should be read as being restricted to projects $u \in \mathcal{D}$ (respectively, $u \in \mathcal{S}$).

3 Benchmark: Ex Ante Pareto-Efficiency

A useful benchmark is that of *ex ante Pareto-efficient contracts*, in which the agent can commit to production decisions. These contracts maximise the principal's ex ante utility subject to a lower bound $\overline{u}$ on the agent's ex ante utility. An efficient contract κ^e is a solution to

$$\max_{\kappa \in \mathcal{K}} \mathbb{E}_0 U_{P,1} \text{ subject to } \mathbb{E}_0 U_{A,1} \geq \overline{u}.$$

Efficient contracts resolve many of the same tradeoffs as optimal contracts. Therefore, we introduce and discuss these key properties in this simpler setting, and in Section 4 we detail how they are affected when the agent must be continually incentivised to support production.

Define an ordering of projects in $\mathcal{D} \cup \mathcal{S}$ such that $u \succ u'$ if and only $|{}^{u_P}/_{u_A}| > |{}^{u'_P}/_{u'_A}|$. In words, if $\overline{v} \succ \underline{v}$, then project $\overline{v}$ is *more cost-effective* to demand than project $\underline{v}$ for the principal: in this

[8]The proofs of all results are in the Appendix.

case the ratio $\overline{v}_P/|\overline{v}_A|$, the principal's benefit per util cost to the agent, measures the productivity of project $\overline{v}$ as a tool for extracting utility from the agent. Conversely, if $\underline{w} \succ \overline{w}$, then project $\overline{w}$ is *more cost-effective* to supply than project $\underline{w}$ for the principal: in this case the ratio $|\overline{w}_P|/\overline{w}_A$, the principal's cost per util benefit to the agent, measures the productivity of project $\overline{w}$ as a tool for providing utility to the agent. Notice that more cost-effective demands are ranked higher by $\succ$ while more cost-effective supplies are ranked lower by $\succ$. This is illustrated in Figure 1, where points in the plane represent projects, projects in the northwestern quadrant can be demanded by the principal, projects in the southeastern quadrant can be supplied, and more cost-effective projects are represented by larger dots. For simplicity, we assume that the ordering $\succ$ is complete on $\mathcal{D} \cup \mathcal{S}$, i.e., that all project pairs are ranked strictly by cost-effectiveness.

Our first result shows that the principal's demand and supply of projects in efficient contracts are determined by cost-effectiveness.

Proposition 1. *Fix any ex ante Pareto-efficient contract κ^e and any time t. The principal demands and supplies projects that are more cost-effective than some threshold: there exists a project U^e such that*

$$\kappa_t^e = \begin{cases} 1 & \textit{if } v_t \succ U^e, \\ 0 & \textit{if } U^e \succ v_t, \end{cases} \qquad \textit{and} \qquad \kappa_t^e = \begin{cases} 1 & \textit{if } U^e \succ w_t, \\ 0 & \textit{if } w_t \succ U^e.^{9} \end{cases} \tag{1}$$

Efficient production decisions can be represented by a history-independent threshold project: at any point in the relationship, those projects that are more cost-effective than U^e (these lie above the thick solid line in Figure 1, which is drawn for the case when U^e is a supply project) are produced and those that are less cost-effective than U^e (these lie below the thick solid line) are not. What drives this result is that the principal can always profitably reallocate production decisions that do not follow cost-effectiveness. If the principal supplied a less cost-effective project $\underline{w}$ following some project history but declined to supply a more cost-effective project $\overline{w}$ following another history, then she could gain by shifting some production probability from $\underline{w}$ to $\overline{w}$ while keeping the agent's ex ante utility fixed. The same logic applies if the principal demanded a less cost-effective project $\underline{v}$ while some opportunities to demand a more cost-effective project $\overline{v}$ following some histories were still available: she would gain by exhausting all opportunities to demand project $\overline{v}$ before making any demands for $\underline{v}$.[10] This implies that the

[9]Recall that, by our notational convention, statements like $v_t \succ U^e$ should be read as applying only to histories with $u_t \in \mathcal{D}$, and statements like $U^e \succ w_t$ should be read as applying only to histories u^t with $u_t \in \mathcal{S}$.

[10]Two features of our model are critical for these intertemporal reallocation arguments: the players' common discount factor and the ability to commit to follow the outcomes of the public randomisation device, which allows for the production of projects with interior probability.

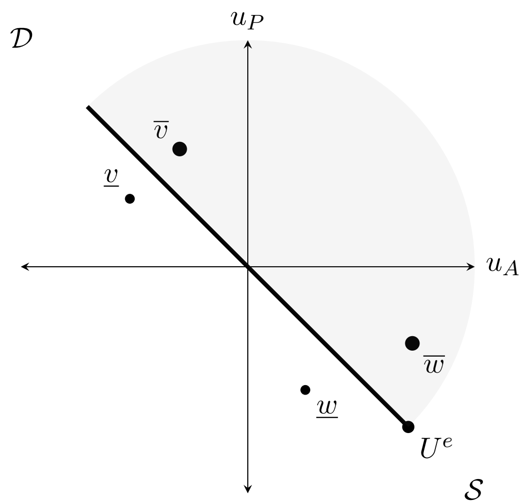

Figure 1: Ex ante Pareto-efficient contracts. Here, the threshold U^e is a supply project. The more cost-effective projects $\overline{v}$ and $\overline{w}$ are always produced, and the less cost effective $\underline{v}$ and $\underline{w}$ are never produced.

principal only demands projects more cost-effective than some threshold demand project and only supplies projects more cost-effective than some threshold supply project.

These threshold demand and supply projects are connected by a straightforward cost-benefit calculation. Returning to Figure 1, suppose that project U^e is supplied with interior probability following some history so that, by our arguments above, $\overline{w}$ is always supplied and $\underline{w}$ is never supplied. The threshold supply U^e identifies both the principal's incentive cost of additional demands and her potential savings from reduced demands. Therefore, the principal cannot pass over any opportunity to demand project $\overline{v}$, which is more cost-effective than U^e: she could gain by increasing production of both $\overline{v}$ and U^e while keeping the agent's ex ante utility fixed. Also, the principal can never demand project $\underline{v}$, which is less cost-effective than U^e: in this case she could gain by decreasing production of both $\underline{v}$ and U^e. Therefore, if the principal ever supplies U^e with interior probability, then she always demands the more cost-effective $\overline{v}$ and never demands the less cost effective $\underline{v}$.[11] If instead the threshold U^e is a demand project, then it identifies both the principal's return from supplying more projects to the agent and her opportunity cost to scaling back the agent's compensation. Therefore, for the same reason as above, the principal must supply all projects more cost-effective that U^e with probability 1, and no less cost-effective project is ever supplied.

Our description of ex ante Pareto-efficient contracts appears to ignore important factors like the scale of project opportunities (i.e., the absolute values of u_P and u_A), the players'

[11]The proof of Proposition 1 deals with the corner case when no production probability is ever interior.

time preferences (i.e., the common discount factor δ) and the dynamics of project opportunities (i.e., the properties of the project process u). On the one hand, Proposition 1 states that the principal establishes production priorities among heterogenous projects by relying only on the cost-effectiveness criterion. On the other hand, any factor that hinders the principal's ability to reward the agent by supplying projects throughout the relationship will restrict her ability to make demands, and hence affect the level of the threshold U^e. Put differently, if, for example, all demands that can be made of the agent impose a high stage cost on him, if players are impatient or if supplied projects only arrive late in the relationship, then ex ante Pareto-efficient contracts will be more generous towards the agent: the threshold project U^e will be ranked higher by $\succ$ which, in turn, means that the principal's demands will be restricted to a smaller set of more cost-effective projects, while her supplies will include a larger set of less cost-effective projects.

Proposition 1 does not specify production decisions at the threshold project U^e. This is due to payoff-irrelevant multiplicity in efficient contracts: the agent's ex ante utility constraint identifies the total (expected discounted) quantity of production at the threshold project (and hence also the principal's ex ante payoff), but the linearity of the players' payoffs in production probabilities allows that production to be distributed arbitrarily across project histories. Because efficient production is organised by cost-effectiveness, this indeterminacy is restricted to the threshold project: given two ex ante Pareto-efficient contracts κ^e and κ'^e with the same ex ante utility to the agent, we have $U^e = U'^e$.

4 Optimal Contracts

In an optimal contract, the agent must have incentives to participate in all production decisions. However, because the agent benefits from projects that are supplied, his individual rationality constraint can only bind when a project is demanded. Correspondingly, our paper's main result below shows that the supply of projects in optimal contracts follows a threshold rule analogous to the one from efficient contracts. In contrast, the demand for projects does not follow a simple threshold rule. This is because the principal can be prevented from reallocating production from less cost-effective to more cost-effective demands by the agent's individual rationality constraints. Instead, mirroring the cost-benefit calculation from efficient contracts, we describe the set of demanded projects from the threshold supply project. Finally, because the principal overcomes binding individual rationality constraints by committing to supply more projects in the future, the threshold supply project can change over time.

Proposition 2. *Fix any optimal contract* κ^* *and any project history* u^{t-1}.

(i) The principal supplies projects that are more cost-effective than some threshold: there exists a supply project W^*_{t-1} such that

$$\kappa^*_t = \begin{cases} 1 & \text{if } W^*_{t-1} \succ w_t, \\ 0 & \text{if } w_t \succ W^*_{t-1}. \end{cases} \tag{2}$$

(ii) The principal demands projects that are more cost-effective than those that are supplied:

$$\kappa^*_t \begin{cases} > 0 & \text{if } v_t \succ \min_{\succ}\{w \succ W^*_{t-1}\}, \\ = 0 & \text{if } \max_{\succ}\{W^*_{t-1} \succ w\} \succ v_t. \end{cases} \tag{3}$$

(iii) Over time, the principal increases her supply of projects and decreases her demands: if u^{t-1} is a subhistory of $u^{t'-1}$, then either $W^*_{t'-1} = W^*_{t-1}$ or $W^*_{t'-1} \succ W^*_{t-1}$.[12]

The principal makes different sequences of demands along different project histories u^{t-1}, and she provides incentives for these demands by adjusting her commitments to supply projects in the future. Part (*i*) says that the supply commitments inherited from u^{t-1} can be captured by a threshold supply project W^*_{t-1}, which in turn determines current supply decisions: at t, projects more cost-effective than W^*_{t-1} are supplied and projects less cost-effective than W^*_{t-1} are not. The supply of the threshold project (i.e., when $w_t = W^*_{t-1}$) may involve interior production probabilities, which we discuss below.

Part (*ii*) says that the threshold supply project also describes the principal's demand for projects: loosely speaking, following history u^{t-1} the principal only demands projects that are more cost-effective than W^*_{t-1}. More precisely, let the supply project $\underline{W}_{t-1} = \min_{\succ}\{w \succ W^*_{t-1}\}$ be the most cost-effective project among those that, from part (*i*), we know that the principal has not yet committed to supply. It follows that if the principal were to make additional demands on the agent at t, project $\underline{W}_{t-1}$ is an upper bound on the incentive costs of these demands. Therefore, the principal cannot pass over an opportunity to demand some project v_t more cost-effective than $\underline{W}_{t-1}$: the principal could gain by increasing production of both v_t and $\underline{W}_{t-1}$ while keeping the agent's ex ante utility fixed. We discuss this further below, but notice for now that (3) requires only that v_t is demanded with positive probability. In Figure 2, we illustrate the threshold supply project $W^*_{t-1} = \overline{w}$ following history u^{t-1}. Parts (*i*) and (*ii*) then imply that the set of produced projects at t is contained in the shaded area. To complete the description of part (*ii*), let the supply project $\overline{W}_{t-1} = \max_{\succ}\{W^*_{t-1} \succ w\}$ be the least cost-effective project among those that, from part (*i*), the principal is committed to always supply.

[12] History u^{t-1} is a subhistory of $u^{t'-1}$ if $u^{t'-1} = (u^{t-1}, u_t, \ldots, u_{t'-1})$.

The project $\overline{W}_{t-1}$ is a lower bound on the savings that the principal could achieve by curtailing her demands, so that any demand v_t that is less cost-effective than $\overline{W}_{t-1}$ is surplus-destroying for the principal: the principal could gain by decreasing production of both v_t and $\overline{W}_{t-1}$ while keeping the agent's ex ante utility fixed. Finally, the demand for projects $\underline{W}_{t-1} \succ v_t \succ \overline{W}_{t-1}$ depends in part on production decisions at the threshold supply project, as we discuss below.

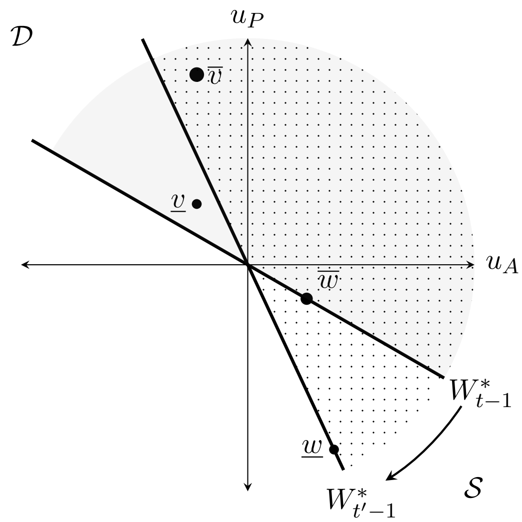

Figure 2: Optimal contracts. Between times t and $t' > t$, the threshold supply project transitions from $\overline{w}$ to the less cost-effective $\underline{w}$.

Not only does the optimal supply threshold W^*_{t-1} vary across project histories at any given time, but part (*iii*) says that, along any given history, it becomes more generous towards the agent over time by transitioning to less cost-effective projects. Combined with part (*ii*), it follows that the principal's demands become concentrated on successively smaller sets of more cost-effective projects. This is illustrated in Figure 2 in which, given $t' > t$, a shift in the threshold from $W^*_{t-1} = \overline{w}$ to the less cost-effective $W^*_{t'-1} = \underline{w}$ leads to a clockwise shift in the region containing produced projects, which is now the dotted area. Why can the threshold supply project never become more cost-effective? Because, as in the case of efficient contracts, the principal never supplies any less cost-effective project before all her future opportunities to supply more cost-effective projects have been exhausted: otherwise, the principal could commit to substitute the future supply of more cost-effective projects against the current supply of less cost-effective projects while maintaining the agent's incentives.[13] Why can the threshold supply project become less cost-effective? Because new demands may violate the agent's individual rationality constraint if his rewards remain determined by the current

[13] Contrary to the case of efficient contracts, such intertemporal reallocations of production must be constructed within-histories and in such a way that no intervening individual rationality constraints are violated.

supply threshold. In this case, the principal must commit to supply more projects in the future, and these additional commitments are typically less cost-effective than the current threshold because of the limited (expected discounted) production possibilities of the threshold project. In particular, the threshold moves in the direction of the agent only if a binding individual rationality has been met: if any history u^t is such that $W_t^* \succ W_{t-1}^*$, then we have that the principal made a demand at t and that $U_{A,t}^* = 0$.

The production dynamics from Proposition 2 have close connections to the seminal results of Thomas and Worrall (1988), who study the wages paid by a risk-neutral firm to a risk-averse worker who receives *iid* outside offers each period. There, optimal wages are increased when some individual rationality constraint for the worker binds, while between updates the optimal contract tracks some efficient contract. In Thomas and Worrall (1988), wage updates are driven by the exogenous arrival of outside offers and efficiency requires constant wages between updates due to risk-aversion. In our model, contract updates take the form of decreasing the cost-effectiveness of the threshold supply project, they are driven endogenously by the principal's decision to demand a project, and efficiency requires that the threshold project be constant between updates. Our results are not driven by risk aversion, so that production decisions and the agent's utility need not be constant between updates.

Outside of special cases, the threshold project W_{t-1}^* will not be constant (either within or across histories), and optimal contracts will not be efficient. In efficient contracts, the principal's total (expected discounted) supply of projects provides incentives for the agent to respect the principal's total (expected discounted) demands, irrespective of the timing of the supplies and demands. When the agent can quit the relationship at any time, past rewards are sunk and provide no incentives for his current production decisions. Therefore, relative to efficient contracts, the principal undersupplies projects early in the relationship and underdemands projects late in the relationship. This is illustrated in Figure 2: the revision of W_{t-1}^* to $W_{t'-1}^*$ means that the principal may demand $\underline{v}$ early but would never demand it late, while she does not supply $\underline{w}$ early but may supply it late. Nevertheless, the inefficiencies associated with any optimal contract eventually vanish. Because the threshold project W_{t-1}^* is monotone along any project history and there is a finite number of projects, W_{t-1}^* converges to the threshold associated to some ex ante Pareto-efficient contract.[14]

Proposition 2 does not specify supply decisions at the threshold project W_{t-1}^*. As in the case of efficient contracts, there is scope for payoff-irrelevant multiplicity in production decisions at the threshold project: in particular, given two optimal contracts κ^* and κ'^*, $W_{t-1}^* = W_{t-1}'^*$

[14] Note that any supply project w is the threshold project of an ex ante Pareto-efficient contract with appropriately chosen $\bar{u}$ in the optimisation problem defining ex ante Pareto-efficient contracts.

for all histories u^{t-1}. It would be possible to resolve this multiplicity through an appropriate selection from the set of optimal contracts. The cost of this approach is that it requires a characterisation that is substantially more intricate than that of Proposition 2 while yielding few additional insights.[15] Its benefit is that we could more thoroughly describe the principal's demand for those projects that have a cost-effectiveness near the threshold. To be more concrete, return to Figure 2. Fix history u^{t-1} and suppose that, as illustrated, the threshold project is the most cost-effective supply project (i.e., $W^*_{t-1} = \overline{w}$). Suppose further that the principal has an opportunity to demand a project at t (i.e., $v_t \in \{\overline{v}, \underline{v}\}$). Part ($i$) of Proposition 2 subsumes two possibilities: either (a) the principal is committed to supply the threshold project $\overline{w}$ whenever it occurs following u^t, or (b) some future production opportunities for project $\overline{w}$ have not yet been promised. In case (a), the principal can provide incentives for demands at t only by committing to produce the less cost-effective project $\underline{w}$. Therefore, because $\overline{v} \succ \underline{w} \succ \underline{v}$, the principal can profitably demand $\overline{v}$ but not $\underline{v}$. In case (b), the principal can still provide additional incentives by committing to supply the threshold project $\overline{w}$. Therefore, because $\overline{v} \succ \underline{v} \succ \overline{w}$, the principal can demand both $\overline{v}$ and $\underline{v}$. Only project $\overline{v}$ is demanded in both cases (a) and (b), and this demand is financed, at worst, by committing to project $\underline{w}$. This is captured by part (ii) of Proposition 2 because $\underline{W}_{t-1} = \underline{w}$, but by construction this neglects the possibility of a demand for $v_t = \underline{v}$ in case (b). Clearly, when the set of projects is rich, the substantive impact of our simplified characterisation is small.

Proposition 2 shows that, in contrast to efficient contracts, optimal contracts treat the demand and supply of projects asymmetrically. In particular, part (i) says that supply projects above the threshold are produced with probability 1, whereas part (ii) says only that demand projects above the (adjusted) threshold must be produced with positive probability. To understand this better, return to Figure 2. Fix history u^{t-1} and suppose that, as illustrated, the threshold project is the most cost-effective supply project (i.e., $W^*_{t-1} = \overline{w}$). Suppose further that the principal is committed to producing the threshold project if it is available at t (as in case (a) from the previous paragraph). What if $\overline{w}$ is very costly for the principal? This does not matter because in this case its high rank in cost-effectiveness means that $\overline{w}$ is very valuable to the agent. The principal would benefit from scaling back her supply of $\overline{w}$ at t, but she committed to producing it in exchange for some demand at some time prior to t because that promise was the most profitable way to reward the agent at that time. Now suppose that

[15] In contrast to the case of efficient contracts, multiplicity in optimal contracts cannot be trivially resolved because arbitrary distributions of production decisions at the threshold may violate some of the agent's individual rationality constraints. In our working paper (Forand and Zápal, 2017), we show that it is without loss of generality for optimal payoffs to consider contracts represented by time thresholds that frontload demands and backload supplies.

the most cost-effective demand $\overline{v}$ is available at t. Because $\overline{v} \succ \underline{w} = \underline{W}_{t-1}$, part ($ii$) says that $\overline{v}$ cannot be passed over. Because the agent cannot commit, the scale of the principal's demand for $\overline{v}$ will depend on the stringency of the agent's individual rationality constraint. Therefore, if $\overline{v}$ is very costly for the agent, if $\underline{w}$ is not very beneficial to the agent, or if the relationship following $(u^{t-1}, \overline{v})$ offers sufficiently few opportunities for the principal to reward the agent, then the agent cannot be provided with incentives to produce $\overline{v}$ with probability 1.[16]

Example. We return to the manager-worker application from the Introduction, in which potential projects are $\mathcal{U} = \{\overline{v}, \underline{v}, \overline{w}, \underline{w}\}$ and $\overline{v} \succ \underline{w} \succ \underline{v} \succ \overline{w}$: the manager can demand emergency and routine projects ($\overline{v}$ and $\underline{v}$ respectively); supply job training or schedule flexibility ($\overline{w}$ and $\underline{w}$ respectively); emergency projects are more cost-effective than routine projects; and because some of the firm's costs of providing job training are offset by the benefits of having higher-skilled workers, training is more cost-effective than job flexibility, which delivers more targeted benefits to the worker. These projects are illustrated in Figure 2. Consider two scenarios in which job training opportunities are either growing or declining over time. Suppose, for example, that the firm employing the manager and the worker operates in an industry which is young and innovative. In this case, expansion in the industry's technological frontier would increase the benefits of continued investments in human capital both for the worker's career prospects and for the firm's productivity. These opportunities would be fewer and less valuable in a maturing industry with a stable production process. We will show that there are stark differences in production dynamics and efficiency across these two scenarios.

To make the example as simple as possible, suppose that the two project processes are identical except for the availability of job training opportunities, and that furthermore the arrival of non-training projects is history-independent: given any time $t > 1$, any project history u^{t-1} and any project $u \in \{\overline{v}, \underline{v}, \underline{w}\}$, we assume that $u_t = u$ with probability p^u in both the growth and decline scenarios. We also assume that the arrival of job training opportunities depends on histories only through time index t. Specifically, if training opportunities are growing, then given any time $t > 1$ and any project history u^{t-1}, we assume that $u_t = \overline{w}$ with probability γ_t, where $\gamma_{t+1} > \gamma_t$. On the other hand, if training opportunities are declining, then given any time $t > 1$ and any history u^{t-1}, we assume that $u_t = \overline{w}$ with probability β_t, where $\beta_{t+1} < \beta_t$.[17] In this case, we also assume that training opportunities become exceedingly rare over time: $\lim_{t\to\infty} \beta_t = 0$. To isolate the effect of how job training opportunities are distributed over time,

[16]For similar reasons, a property similar to (2) fails for demand projects: it is possible that the optimal contract prescribes $\kappa_t^* \in (0,1)$ for two histories (u^{t-1}, v_t) and (u^{t-1}, v_t') with $v_t \succ v_t'$.

[17]Recall that project processes take values in $\mathcal{U} \cup \{(0,0)\}$, so that the growth and decline scenarios will have different probabilities of having no project arrive at any given time. For simplicity we leave these and other feasibility constraints on the two processes implicit.

we fix their (expected discounted) quantity across both scenarios: $\sum_{t=2}^{\infty} \delta^{t-1}\gamma_t = \sum_{t=2}^{\infty} \delta^{t-1}\beta_t$. This means that when training opportunities are growing the manager has few of these to offer the worker initially relative to the case of decline, but that the opposite is true later in the relationship.

We make further assumptions to tighten the link between optimal and efficient contracts in this example. First, because optimal contracts are conditioned on the arrival of a first demand by the manager and ex ante Pareto-efficient contracts depend on the workers' individual rationality constraint evaluated prior to the realisation of an initial project, we assume that the project at time $t = 1$ is $\overline{v}$. Second, because optimal contracts deliver all ex ante surplus to the manager, we focus on those efficient contracts yielding expected utility $\overline{u} = 0$ to the worker. Notice that given our assumption that the ex ante quantities of all projects are fixed whether training opportunities are growing or declining, efficient contracts are identical in both scenarios. Finally, suppose that the efficient contract is such that the manager always demands both emergency and routine projects and always supplies job training but never supplies schedule flexibility.

If training opportunities are growing, the manager's ability to reward the worker increases over time and her potential demands are time-invariant: for any time $t > 2$, $\sum_{t'=t}^{\infty} \delta^{t'-1}\gamma_{t'} > \sum_{t'=2}^{\infty} \delta^{t'-1}\gamma_{t'}$ and $\sum_{t'=t}^{\infty} \delta^{t'-1}p^u = \sum_{t'=2}^{\infty} \delta^{t'-1}p^u$ for all $u \in \{\overline{v}, \underline{v}, \underline{w}\}$. Therefore, in this case the efficient contract satisfies the worker's individual rationality constraint at all times $t \geq 1$ and is optimal. In contrast, optimal contracts in the declining scenario cannot be efficient. First, because $\overline{v} \succ \underline{w}$, the manager must eventually supply schedule flexibility in exchange for continued demands for emergency projects when her commitments to job training no longer provide meaningful incentives to the worker. Second, because $\underline{w} \succ \underline{v}$, the manager must stop demanding routine projects when she starts rewarding the worker through commitments to schedule flexibility. In the long run, production in the declining scenario is essentially reduced to exchanging emergency projects against scheduling flexibility (although job training is provided in the rare cases when it is available).

After observing a manager extend both job training and schedule flexibility to a worker, it would be natural to interpret this as a sign that the firm has a plentiful supply of rewards to offer its workforce. Our results suggest the opposite interpretation: an increase in the scope of the worker's non-monetary compensation points to scarcity in those rewards that are most effective from the firm's perspective. Because the manager benefits from substituting job training for schedule flexibility, observing the latter means that the bound on the availability of training opportunities is binding. Similarly, it would be natural to conjecture that a worker receiving non-monetary compensation from a variety of sources would produce more for the firm. Again, our results predict the opposite: diversified rewards are tied to rationing in the worker's tasks. Because the manager benefits from substituting decreased schedule flexibility for routine projects, a worker that is rewarded by job flexibility cannot be profitably asked to work on routine projects.

5 Markov Project Processes

From Section 4, we know that the optimal supply threshold can only transition to less cost-effective supply projects if the principal makes additional demands. However, in general, we cannot identify which project histories, and in particular which demands, lead to changes in the threshold supply project. In this section, we sharpen our results by assuming that the project process u is Markov.

Proposition 3. *Suppose that the project process is Markov, and fix any optimal contract κ^*. For all demands v, there exists a supply project W^v such that, given any history u^t with $u_t = v$, the optimal threshold project W^*_t is the least cost-effective of projects W^*_{t-1} and W^v:*

$$W^*_t = \max_{\succ}\{W^*_{t-1}, W^v\}.$$

In the Markov case, the updating rule for the optimal supply threshold has a simple form: following any history (u^{t-1}, v), the supply threshold is updated to W^v if this project is less cost-effective than W^*_{t-1}, while it remains at W^*_{t-1} otherwise. We provide more details about the relationship between the threshold W^v and its associated demand v below: here we note that W^v captures the minimal level of future supply commitments (and, correspondingly, the maximal level of future demands) that provide incentives for a demand for v. Therefore, if W^*_{t-1} is more cost-effective than W^v, then the agent's individual rationality constraint at t binds as a result of principal's demand for v and the optimal contract must become more generous towards the agent by adding commitments to less cost-effective projects (and, correspondingly, dropping less cost-effective demands). Furthermore, because the project process is Markov and the threshold W^v is history-independent, the continuation contracts following all histories at which a binding individual rationality constraint is met at demand v are identical. This stationary updating rule and corresponding "amnesia property" for optimal contracts are analogous to well-known results in related models, notably those of Thomas and Worrall (1988) and Kocherlakota (1996) for the *iid* case and Ligon, Thomas, and Worrall (2002) for the Markov case.

We can use Proposition 3 to order demands by the scale of the incentives that must be supplied to the agent in order to produce them. If demand projects $\overline{v}$ and $\underline{v}$ are such that $W^{\overline{v}}$ is less cost-effective than $W^{\underline{v}}$, then we say that demand $\overline{v}$ is *more expensive* for the principal than demand $\underline{v}$: in return for $\overline{v}$, the principal must commit to supply more projects to (and demand less from) the agent in the future. An important note is that the expensiveness of a demand v is different from its cost-effectiveness: the latter is the ratio of the principal's benefit from v to the agents' cost, while the former is a measure of the stringency of the agent's individual rationality constraint following v. Intuitively, expensiveness depends on two potentially countervailing factors: the agent's stage cost from producing v (given by $|v_A|$); and the value to the agent of future project opportunities conditional on having reached project v, which depends on both the discount factor δ and the project process. However, if the project process is *iid*, then the relationship's future production opportunities are history-independent. In that case, a demand's expensiveness is determined solely by its cost to the agent.

Corollary 1. *Suppose that the project process is iid. If demands $\overline{v}$ and $\underline{v}$ are such that $|\overline{v}_A| \geq |\underline{v}_A|$, then $\underline{v}$ cannot be more expensive for the principal than $\overline{v}$.*

This is analogous to a result from Thomas and Worrall (1988). In their model, the agent's opportunity cost from outside offers (which are *iid*) stands for the cost of the principal's demand in that state, and the level of the corresponding optimal wage stands for the expensiveness of this demand for the principal. In line with Corollary 1, their Proposition 3 shows that optimal wages are non-decreasing in outside market wages. For the general Markov case, we derive the ranking of demands by expensiveness through our construction of optimal contracts in the proof of Proposition 3, and it tracks the solutions to a recursive sequence of reduced problems: v^1, the most expensive demand for the principal, has the least cost-effective threshold W^{v^1}, over all $v \in \mathcal{D}$, associated to the problem of finding an optimal contract subject only to (a) the initial project being v (i.e., $u_1 = v$) and (b) the individual rationality constraint for the agent at time 1 (i.e., $U_{A,1} \geq 0$); then v^2, the second most expensive demand for the principal, has the least cost-effective threshold W^{v^2}, over all $v \in \mathcal{D} \setminus \{v^1\}$, to the problem of finding an optimal contract for the principal subject only to (a) $u_1 = v$, (b) $U_{A,1} \geq 0$ and (c) the fact that the threshold transitions to W^{v^1} whenever v^1 arrives; and so on. As opposed to the *iid* case, the ranking of demands by expensiveness depends on the project process. For example, the most expensive demand v^1 might impose a low cost on the agent if the continuation process following v^1 provides few opportunities to reward him.

Our final goal in this section is to derive sufficient conditions for ranking demands by their expensiveness in the non-*iid* case. To this end, fix two demands $\overline{v}$ and $\underline{v}$. First, *Condition 1* for $\overline{v}$ to be more expensive than $\underline{v}$ is that it is more costly for the agent: $|\overline{v}_A| \geq |\underline{v}_A|$. Second, note that the thresholds associated to both demands $\overline{v}$ and $\underline{v}$ must take into account future transitions to thresholds associated to demands that are more expensive than both of them (as in constraint (c) of the reduced problem defining v^2 in the previous paragraph). Correspondingly, *Condition*

2 is that the distributions over these future transitions are identical following $\overline{v}$ and $\underline{v}$: for all $t > 1$, $\mathbb{P}_{\overline{v}}[u_t = u] = \mathbb{P}_{\underline{v}}[u_t = u]$ for all $u \notin \mathcal{S}$, where $\mathbb{P}_v$ stands for the distribution of process u conditional on $u_1 = v$.[18] Third, the expensiveness of demand $\overline{v}$ should capture the idea that the principal has worse opportunities to supply projects following $\overline{v}$. Because production decisions are ordered by cost-effectiveness, this requires that the project process puts less weight on more cost-effective supply projects following $\overline{v}$. However, such a condition would not be sufficient on its own, as less cost-effective supplies may yield high stage benefits to the agent, so that the effect on the agent's utility following $\overline{v}$ would be ambiguous. Correspondingly, *Condition 3* is a joint restriction on the cost-effectiveness and the stage benefit of supply opportunities following $\overline{v}$ and $\underline{v}$: for all $t > 1$ and all $c \geq 0$,

$$\mathbb{P}_{\overline{v}}\left[{}^{|w_{P,t}|}/_{w_{A,t}} \leq c\right]\mathbb{E}_{\overline{v}}\left[w_{A,t}\middle|{}^{|w_{P,t}|}/_{w_{A,t}} \leq c\right] \leq \mathbb{P}_{\underline{v}}\left[{}^{|w_{P,t}|}/_{w_{A,t}} \leq c\right]\mathbb{E}_{\underline{v}}\left[w_{A,t}\middle|{}^{|w_{P,t}|}/_{w_{A,t}} \leq c\right]. \tag{4}$$

Condition 3 says that, given any fixed supply threshold, the agent's expected rewards are higher following $\underline{v}$ than following $\overline{v}$.

Corollary 2. *Suppose that the project process is Markov. If demands $\overline{v}$ and $\underline{v}$ satisfy Conditions 1-3, then $\underline{v}$ cannot be more expensive for the principal than $\overline{v}$.*

Conditions 1-3 are clearly stringent, but our preceding remarks highlight that this is to some degree by necessity. Notice that *Conditions 2* and *3* are satisfied if the project process is *iid*. Returning to our Example, what would be needed to conclude that routine projects cannot be more expensive for the manager than emergency projects? We have assumed that the agent is indifferent between producing both types of projects, so that *Condition 1* is satisfied. Ignoring *Condition 2*, *Condition 3* can be satisfied if job training programs are more likely to arrive following a routine project, so that the worker can expect to be compensated more often with the manager's preferred supply project. But because the agent prefers schedule flexibility to job training, the former cannot arrive too rarely following a routine project. Otherwise, the agent's expected compensation could be lower following routine projects when the principal rewards the agent with both job training and schedule flexibility.

6 Conclusion

We recap our main results by discussing their relationship to two key assumptions of our model: that no transfers are available to support production and that the principal can commit pro-

[18] For related reasons, *Condition 2* also requires that the distribution of projects that are neither supply nor demand projects (i.e., projects in $(\mathcal{U} \cup \{(0,0)\}) \setminus (\mathcal{D} \cup \mathcal{S})$) are also identical following $\overline{v}$ and $\underline{v}$.

duction decisions.

Transfers. While we have not explicitly allowed for monetary payments between the principal and the agent, models with transfers are special cases of our model. Indeed, a transfer of $\overline{k}$ dollars to the agent can be represented by a supply project $m^S = (-\overline{k}, \overline{k}) \in \mathcal{S}$, while a transfer of $\overline{k}$ dollars to the principal can be represented by a demand project $m^D = (\overline{k}, -\overline{k}) \in \mathcal{D}$, where m^S and m^D are equally cost-effective. The flexibility of the project process allows for different specifications of transfer opportunities. On the one hand, if all non-monetary projects are followed by transitions to both m^S and m^D and $\overline{k}$ is large, then transfers are always available and essentially unrestricted in size. On the other hand, if m^S arrives at fixed intervals, then the principal has infrequent but regular opportunities to pay a bonus to the agent. While our results apply to all models with transfers, they provide specific implications for the use of money in the dynamic relationships captured by our environment. First, the principal's ability to use transfers to reward the agent does not crowd out supply through production: the principal will not start paying the agent until she has committed to supply projects that are more cost-effective than money in all their future occurrences. Furthermore, in an optimal contract the principal may even supply projects that are less cost-effective than money, if the availability of future transfer opportunities is sufficiently constrained. However, if $\overline{k}$ is large and transfer opportunities are frequent, then the principal would always use money instead of less cost-effective projects. Second, the direction of the flow of money between the principal and the agent varies over the relationship's lifetime: the principal demands transfers from the agent early in the relationship, and supplies transfers to the agent later in the relationship.

No commitment for the principal. If the principal cannot commit to production decisions, then the model must be augmented with history-dependent individual rationality constraints for the principal which cap her supply of projects. Cost-effectiveness still drives project selection decisions, but with an important qualification: if the principal supplies a less cost-effective project, then she must also supply more cost-effective projects in all succeeding histories *in which none of her individual rationality constraints have been binding.* This implies that some characterisation of the optimal contract in terms of threshold supply projects would still be possible without commitment by the principal, but that pinning down general properties of optimal contracts' dynamics would be difficult. Recall that if both sides can commit to production decisions, then the threshold supply project is fixed over time, and if only the principal has commitment power, then the threshold becomes more favourable to the agent over time to incentivise demands. If the principal cannot commit either, then she must have incentives to supply projects, which would imply a threshold that becomes less favourable to the agent following some histories. Therefore, in contrast with our results, the optimal contract will typically not stabilise in the long-run.

Absence of commitment power for the principal would generate an inefficiency closely related to the one we discussed in Section 4: the principal and the agent would be better off if past demands could incentivise the principal's current supply of projects, but without commitment these can only be supported by future demands.

A Appendix

Proof of Lemma 1. Suppose, towards a contradiction, that κ^* is optimal and that, for some project history u^t such that $u_{P,t}, u_{A,t} > 0$, we have that $\kappa^*_t < 1$. Fix a contract $\tilde{\kappa}$ that is identical to κ^* except that $\tilde{\kappa}_t = 1$ at u^t. It follows that $\tilde{\kappa}$ is individually rational because κ^* is individually rational. Furthermore, $\tilde{U}_{P,t} > U^*_{P,t}$, yielding the desired contradiction. The proof for the case of u^t such that $u_{P,t}, u_{A,t} < 0$ is similar, and is omitted. □

We prove Proposition 2 before proving Proposition 1 in order to avoid repeating several arguments that simplify in the context of Proposition 1.

Proof of Proposition 2. We proceed in a number of steps.
Step 1. Fix an optimal contract κ^*, project history u^t, its superhistories $u^{t'}$ and $u^{t''}$, and projects $\underline{w} \succ \overline{w}$. Suppose that (i) $u_{t'} = \overline{w}$ and (ii) $u_{t''} = \underline{w}$ and $\sum_{s=t+1}^{t''-1} \kappa^*_s \mathbb{I}_{u_s \in \mathcal{D}} \mathbb{I}_{U^*_{A,s}=0} = 0$.[19] We show that

$$\text{if } \kappa^*_{t'} < 1, \text{ then } \kappa^*_{t''} = 0.$$

To see this suppose, towards a contradiction, that $\kappa^*_{t'} < 1$ at $u^{t'}$ and that $\kappa^*_{t''} > 0$ at $u^{t''}$. Now consider an alternative contract $\tilde{\kappa}$, identical to κ^* except that (i) $\kappa^*_{t'} < \tilde{\kappa}_{t'} \leq 1$ at $u^{t'}$, (ii) $0 \leq \tilde{\kappa}_{t''} < \kappa^*_{t''}$ at $u^{t''}$, (iii)

$$\tilde{U}_{A,t} - U^*_{A,t} = \delta^{t'-t}\mathbb{P}_t(u^{t'})[\tilde{\kappa}_{t'} - \kappa^*_{t'}]\overline{w}_A - \delta^{t''-t}\mathbb{P}_t(u^{t''})[\kappa^*_{t''} - \tilde{\kappa}_{t''}]\underline{w}_A = 0, \tag{5}$$

and (iv) $U^*_{A,r} + \delta^{t''-r}\mathbb{P}_r(u^{t''})[\tilde{\kappa}_{t''} - \kappa^*_{t''}]\underline{w}_A \geq 0$ for any history u^r that is a proper superhistory of u^t and a proper subhistory of $u^{t''}$ (i.e., with $t+1 \leq r \leq t''-1$) and such that $\kappa^*_r > 0$ and $u_r \in \mathcal{D}$. Because $U^*_{A,r} > 0$ for any history u^r in (iv), such a contract always exists. Furthermore, $\tilde{\kappa}$ is individually rational for the agent. To see this, first note that, because $\tilde{U}_{A,t} = U^*_{A,t} \geq 0$, we have that $\tilde{\kappa}$ satisfies $(IR_{A,r})$ for all times $r \leq t$. Second, because $\tilde{U}_{A,t'} > U^*_{A,t'} \geq 0$, it follows that

[19] Throughout, $\sum_{s=t+1}^{t''-1} \kappa^*_s \mathbb{I}_{u_s \in \mathcal{D}} \mathbb{I}_{U^*_{A,s}=0} = 0$ denotes that, given history u^t and its superhistory $u^{t''}$, for any history u^s that is a proper superhistory of u^t and a proper subhistory of $u^{t''}$ (i.e., with $t+1 \leq s \leq t''-1$), either $\kappa^*_s = 0$, or $u_s \notin \mathcal{D}$, or $U^*_{A,s} > 0$.

given any time $r > t$ and history u^r that is not a subhistory of $u^{t''}$, we have that $\tilde{U}_{A,r} \geq U^*_{A,r} \geq 0$. Third, even though we have that $\tilde{U}_{A,t''} < U^*_{A,t''}$, because $\tilde{\kappa}_{t''} u_{A,t''} = \tilde{\kappa}_{t''} \underline{w}_A \geq 0$ it also follows that

$$\begin{aligned} \tilde{U}_{A,t''} &\geq \delta \mathbb{E}_{t''} U^*_{A,t''+1} \\ &\geq 0. \end{aligned}$$

Finally, consider history u^r that is a proper superhistory of u^t and a proper subhistory of $u^{t''}$ (i.e., with $t+1 \leq r \leq t''-1$). Suppose $\tilde{\kappa}$ satisfies $(IR_{A,r+1})$ for u^{r+1} that is a subhistory of $u^{t''}$. If $\tilde{\kappa}_r u_{A,r} \geq 0$, then it follows that

$$\begin{aligned} \tilde{U}_{A,r} &\geq \delta \mathbb{E}_r \tilde{U}_{A,r+1} \\ &\geq 0. \end{aligned}$$

If $\tilde{\kappa}_r u_{A,r} < 0$, then $\tilde{\kappa}_r = \kappa^*_r > 0$ and $u_r \in \mathcal{D}$ and from (iv) it follows that

$$\tilde{U}_{A,r} = U^*_{A,r} + \delta^{t''-r} \mathbb{P}_r(u^{t''})[\tilde{\kappa}_{t''} - \kappa^*_{t''}] \underline{w}_A \geq 0.$$

It thus follows recursively that $\tilde{\kappa}$ satisfies $(IR_{A,r})$ for all times $t+1 \leq r \leq t''-1$. It remains only to note that, by (5), we have

$$\begin{aligned} \tilde{U}_{P,t} - U^*_{P,t} &= -\delta^{t'-t} \mathbb{P}_t(u^{t'})[\tilde{\kappa}_{t'} - \kappa^*_{t'}]|\overline{w}_P| + \delta^{t''-t} \mathbb{P}_t(u^{t''})[\kappa^*_{t''} - \tilde{\kappa}_{t''}]|\underline{w}_P| \\ &= \delta^{t''-t} \mathbb{P}_t(u^{t''})[\kappa^*_{t''} - \tilde{\kappa}_{t''}]|\underline{w}_P| \left[1 - \frac{|\overline{w}_P| / \overline{w}_A}{|\underline{w}_P| / \underline{w}_A}\right] \\ &> 0, \end{aligned}$$

where the inequality follows because $\underline{w} \succ \overline{w}$, contradicting the optimality of κ^*.

Step 2. Step 1 implies that to any optimal contract κ^* corresponds a history-dependent threshold project mapping $W^* : \mathcal{H} \to \mathcal{S}$ such that, for all times t and histories u^t,

$$\kappa^*_t = \begin{cases} 1 & \text{if } W^*_t \succ w_t, \\ 0 & \text{if } w_t \succ W^*_t, \end{cases}$$

where for simplicity we denote $W^*(u^t)$ by W^*_t, with the project history understood. For any history $u^t = (u^{t-1}, u_t)$, the threshold is given by

$$W^*(u^t) = \max_{\succ} \left\{ \{W^*(u^{t-1})\} \cup \left\{ w : \mathbb{P}_t(\kappa^*_{t'} > 0, u_{t'} = w, \sum\nolimits_{s=t+1}^{t'-1} \kappa^*_s \mathbb{I}_{u_s \in \mathcal{D}} \mathbb{I}_{U^*_{A,s}=0} = 0) > 0 \right\} \right\},$$

where we set $W^*(u^0) = \min_{\succ} \mathcal{S}$.[20] By construction, W^*_t is non-decreasing with respect to $\succ$, and is such that, given any history u^t and its superhistory $u^{t'}$, $W^*_{t'} = W^*_t$ if $\sum_{s=t+1}^{t'} \kappa^*_s \mathbb{I}_{u_s \in \mathcal{D}} \mathbb{I}_{U^*_{A,s}=0} = 0$. This proves Parts ($i$) and ($iii$) of Proposition 2.

Step 3. Fix an optimal contract κ^* and project history $u^t = (u^{t-1}, v_t)$ where $v_t \succ \min_{\succ}\{w \succ W^*_{t-1}\}$. We show that $\kappa^*_t > 0$. To see this suppose, towards a contradiction, that $\kappa^*_t = 0$. Because $\kappa^*_t = 0$, we have $W^*_{t-1} = W^*_t$. Hence, because $v_t \succ \min_{\succ}\{w \succ W^*_{t-1}\}$, there exists $u^{t+1} = (u^{t-1}, v_t, w_{t+1})$ such that $v_t \succ w_{t+1} \succ W^*_t$, where $w_{t+1} \succ W^*_t$ and Part (i) of Proposition 2 imply $\kappa^*_{t+1} = 0$. Now consider an alternative contract $\tilde{\kappa}$, identical to κ^* except that (i) $\kappa^*_t < \tilde{\kappa}_t \leq 1$ at u^t, (ii) $\kappa^*_{t+1} < \tilde{\kappa}_{t+1} \leq 1$ at u^{t+1}, and (iii)

$$\tilde{U}_{A,t} - U^*_{A,t} = [\tilde{\kappa}_t - \kappa^*_t] v_{A,t} + \delta \mathbb{P}_t(u^{t+1}) [\tilde{\kappa}_{t+1} - \kappa^*_{t+1}] w_{A,t+1} = 0. \tag{6}$$

Such a contract always exists. Furthermore, $\tilde{\kappa}$ is individually rational for the agent: because $\tilde{U}_{A,t} = U^*_{A,t} \geq 0$, $\tilde{\kappa}$ satisfies $(IR_{A,r})$ for all times $r \leq t$, and because $\tilde{U}_{A,t+1} \geq U^*_{A,t+1} \geq 0$, $\tilde{\kappa}$ satisfies $(IR_{A,r})$ for all times $r \geq t+1$. It remains only to note that, by (6), we have

$$\begin{aligned} \tilde{U}_{P,t} - U^*_{P,t} &= [\tilde{\kappa}_t - \kappa^*_t] v_{P,t} + \delta \mathbb{P}_t(u^{t+1}) [\tilde{\kappa}_{t+1} - \kappa^*_{t+1}] w_{P,t+1} \\ &= [\tilde{\kappa}_t - \kappa^*_t] v_{P,t} \left[1 - \frac{|w_{P,t+1}| / w_{A,t+1}}{v_{P,t} / |v_{A,t}|} \right] \\ &> 0, \end{aligned}$$

where the inequality follows because $v_t \succ w_{t+1}$, contradicting the optimality of κ^*.

Step 4. Fix an optimal contract κ^* and project history $u^t = (u^{t-1}, v_t)$ where $\max_{\succ}\{W^*_{t-1} \succ w\} \succ v_t$. We show that $\kappa^*_t = 0$. To see this suppose, towards a contradiction, that $\kappa^*_t > 0$. We have either $W^*_t = W^*_{t-1}$ or $W^*_t \succ W^*_{t-1}$. Hence, because $\max_{\succ}\{W^*_{t-1} \succ w\} \succ v_t$, there exists $u^{t+1} = (u^{t-1}, v_t, w_{t+1})$ such that $W^*_t \succ w_{t+1} \succ v_t$, where $W^*_t \succ w_{t+1}$ and Part (i) of Proposition 2 imply $\kappa^*_{t+1} = 1$. Now consider an alternative contract $\tilde{\kappa}$, identical to κ^* except that (i) $0 \leq \tilde{\kappa}_t < \kappa^*_t$ at u^t, (ii) $0 \leq \tilde{\kappa}_{t+1} < \kappa^*_{t+1}$ at u^{t+1}, and (iii)

$$U^*_{A,t} - \tilde{U}_{A,t} = [\kappa^*_t - \tilde{\kappa}_t] v_{A,t} + \delta \mathbb{P}_t(u^{t+1}) [\kappa^*_{t+1} - \tilde{\kappa}_{t+1}] w_{A,t+1} = 0. \tag{7}$$

[20] Throughout, $\mathbb{P}_t(\kappa^*_{t'} > 0, u_{t'} = w, \sum_{s=t+1}^{t'-1} \kappa^*_s \mathbb{I}_{u_s \in \mathcal{D}} \mathbb{I}_{U^*_{A,s}=0} = 0)$ denotes $\mathbb{P}_t(u^{t'})$ of a superhistory $u^{t'}$ of u^t with $\kappa^*_{t'} > 0$, $u_{t'} = w$ and $\sum_{s=t+1}^{t'-1} \kappa^*_s \mathbb{I}_{u_s \in \mathcal{D}} \mathbb{I}_{U^*_{A,s}=0} = 0$.

Such a contract always exists. Furthermore, $\tilde{\kappa}$ is individually rational for the agent. To see this, first note that, because $\tilde{U}_{A,t} = U^*_{A,t} \geq 0$, $\tilde{\kappa}$ satisfies $(IR_{A,r})$ for all times $r \leq t$. Second, even though we have that $\tilde{U}_{A,t+1} < U^*_{A,t+1}$, because $\tilde{\kappa}_{t+1} w_{A,t+1} \geq 0$ it also follows that

$$\begin{aligned}\tilde{U}_{A,t+1} &\geq \delta \mathbb{E}_{t+1} U^*_{A,t+2} \\ &\geq 0,\end{aligned}$$

and hence $\tilde{\kappa}$ satisfies $(IR_{A,r})$ for all times $r \geq t+1$. It remains only to note that, by (7), we have

$$\begin{aligned}U^*_{P,t} - \tilde{U}_{P,t} &= [\kappa^*_t - \tilde{\kappa}_t] v_{P,t} + \delta \mathbb{P}_t(u^{t+1}) [\kappa^*_{t+1} - \tilde{\kappa}_{t+1}] w_{P,t+1} \\ &= [\kappa^*_t - \tilde{\kappa}_t] v_{P,t} \left[1 - \frac{|w_{P,t+1}|/w_{A,t+1}}{v_{P,t}/|v_{A,t}|}\right] \\ &< 0,\end{aligned}$$

where the inequality follows because $w_{t+1} \succ v_t$, contradicting the optimality of κ^*. Steps 3 and 4 jointly imply Part (*ii*) of Proposition 2. □

Proof of Proposition 1. We specialise the results of Proposition 2 for optimal contracts to establish our results for efficient contracts. Recall that efficient contracts are solutions to

$$\max_{\kappa \in \mathcal{K}} \mathbb{E}_0 U_{P,1} \quad \text{subject to} \quad \mathbb{E}_0 U_{A,1} \geq \overline{u}.$$

Fix any efficient contract κ^e and consider histories u^t and $u^{t'}$. First, arguments closely mirroring those of Step 1 in the proof of Proposition 2 show that

$$\text{if } \kappa^e_t < 1, \text{ then } \kappa^e_{t'} = 0 \text{ if either } w_{t'} \succ w_t \text{ or } v_t \succ v_{t'}.$$

Second, arguments mirroring those of Steps 3 and 4 show that

$$\begin{aligned}&\text{if } \kappa^e_t < 1, \text{ then } \kappa^e_{t'} = 1 \text{ if } v_{t'} \succ w_t, \text{ and} \\ &\text{if } \kappa^e_t > 0, \text{ then } \kappa^e_{t'} = 0 \text{ if } w_t \succ v_{t'}.\end{aligned}$$

In fact, all the arguments from the previous steps are simplified because the only individual rationality constraint for the agent is the ex ante one. Third, define the following sets of projects:

$$\begin{aligned}
&\mathcal{S}_1^e = \{w : \kappa_t^e = 1 \text{ for any } u^t \text{ with } u_t = w\}, && \mathcal{D}_1^e = \{v : \kappa_t^e = 1 \text{ for any } u^t \text{ with } u_t = v\},\\
&\mathcal{S}_0^e = \{w : \kappa_t^e = 0 \text{ for any } u^t \text{ with } u_t = w\}, && \mathcal{D}_0^e = \{v : \kappa_t^e = 0 \text{ for any } u^t \text{ with } u_t = v\},\\
&\mathcal{S}_i^e = \mathcal{S} \setminus (\mathcal{S}_1^e \cup \mathcal{S}_0^e), && \mathcal{D}_i^e = \mathcal{D} \setminus (\mathcal{D}_1^e \cup \mathcal{D}_0^e).
\end{aligned}$$

Note that from our preceding results we have that

$$\begin{aligned}
&\text{given any } u \in \mathcal{D}_0^e \cup \mathcal{D}_i^e \cup \mathcal{S}_1^e \cup \mathcal{S}_i^e\text{: } u \succ v \text{ implies } v \in \mathcal{D}_0^e \text{ and } u \succ w \text{ implies } w \in \mathcal{S}_1^e,\\
&\text{given any } u \in \mathcal{D}_i^e \cup \mathcal{D}_1^e \cup \mathcal{S}_i^e \cup \mathcal{S}_0^e\text{: } v \succ u \text{ implies } v \in \mathcal{D}_1^e \text{ and } w \succ u \text{ implies } w \in \mathcal{S}_0^e.
\end{aligned} \tag{8}$$

Therefore, $\mathcal{D}_i^e$ and $\mathcal{S}_i^e$ each include at most one project and at most one of these sets is non-empty so that even $\mathcal{D}_i^e \cup \mathcal{S}_i^e$ includes at most one project. Therefore, we can set

$$U^e \begin{cases} \in \mathcal{D}_i^e \cup \mathcal{S}_i^e & \text{if } \mathcal{D}_i^e \cup \mathcal{S}_i^e \neq \varnothing,\\ = \max_{\succ} \mathcal{D}_0^e \cup \mathcal{S}_1^e & \text{if } \mathcal{D}_i^e \cup \mathcal{S}_i^e = \varnothing \text{ and } \mathcal{D}_0^e \cup \mathcal{S}_1^e \neq \varnothing,\\ = \min_{\succ} \mathcal{D} \cup \mathcal{S} & \text{if } \mathcal{D}_i^e \cup \mathcal{S}_i^e = \varnothing \text{ and } \mathcal{D}_0^e \cup \mathcal{S}_1^e = \varnothing. \end{cases}$$

Proposition 1 then follows from (8).

Notice that our results above do not pin down production probabilities at projects in $\mathcal{D}_i^e \cup \mathcal{S}_i^e$. However, a simple selection from the set of efficient contracts allows a complete characterisation of κ^e. Specifically, given the linearity of payoffs in production probabilities, it is immediate that it is without of loss of generality for optimal payoffs to assume that κ^e prescribes equal production probability at all histories u^t with $u_t \in \mathcal{D}_i^e \cup \mathcal{S}_i^e$: we can restrict attention to contracts such that $\kappa_t^e = k_D^* \in [0,1]$ for any history u^t with $u_t \in \mathcal{D}_i^e$ and $\kappa_t^e = k_S^* \in [0,1]$ for any history u^t with $u_t \in \mathcal{S}_i^e$. Now define threshold project $u^* = U^e \in \mathcal{D} \cup \mathcal{S}$ along with threshold production probability $k^* \in [0,1]$ as follows:

$$k^* = \begin{cases} 1 - k_D^* & \text{if } u^* \in \mathcal{D}_i^e,\\ k_S^* & \text{if } u^* \in \mathcal{S}_i^e,\\ 1 & \text{if } \mathcal{D}_i^e \cup \mathcal{S}_i^e = \varnothing \text{ and } \mathcal{D}_0^e \cup \mathcal{S}_1^e \neq \varnothing,\\ 0 & \text{if } \mathcal{D}_i^e \cup \mathcal{S}_i^e = \varnothing \text{ and } \mathcal{D}_0^e \cup \mathcal{S}_1^e = \varnothing. \end{cases}$$

The reason for expressing $k^* = 1 - k_D^*$ when $u^* \in \mathcal{D}_i^e$ will become clear in the proof of Proposition 3 below, where we apply our results on efficient contracts to characterise optimal contracts when the project process is Markov. There, we order demands $v \in \mathcal{D}$ by their expensiveness to the principal, so that whether the threshold u^* is a demand or a supply project, our formulation of the threshold production probability k^* always identifies the scale of the principal's costs. Finally, by (8), it follows that, given any history u^t, we have that

$$\kappa_t^e = \begin{cases} 1 & \text{if } v_t \succ u^*, \\ 1-k^* & \text{if } v_t = u^*, \\ 0 & \text{if } u^* \succ v_t, \end{cases} \tag{9}$$

and that

$$\kappa_t^e = \begin{cases} 1 & \text{if } u^* \succ w_t, \\ k^* & \text{if } w_t = u^*, \\ 0 & \text{if } w_t \succ u^*. \end{cases} \tag{10}$$

□

Proof of Proposition 3. Our characterisation of optimal contracts with Markov project processes in Proposition 3 shows how to define the cutoff supply project from Proposition 2 through a recursive rule involving fixed threshold $\{W^v\}_{v\in\mathcal{D}}$ associated to all demand projects. Our proof of this result will follow from the construction of an optimal contract. We proceed in a number of steps.

Step 1. Fix project $v' \in \mathcal{D}$ and suppose that $u_1 = v'$. We define the reduced problem

$$\max_{\kappa\in\mathcal{K}} U_{P,1} \quad \text{subject to} \quad U_{A,1} \geq 0. \tag{11}$$

Notice that problem (11) is a special case of the problem solved by efficient contracts. Therefore, as in the proof of Proposition 1 we can conclude that the solution κ^* to (11) can be characterised by threshold project u^* and production probability k^*, as described in (9) and (10).

Step 2. We can rank the solutions to (11) for various $v' \in \mathcal{D}$ for which $u_1 = v'$ in terms of how expensive they are to the principal. Specifically, fix $\overline{v}, \underline{v} \in \mathcal{D}$ and consider the associated solutions $\overline{\kappa}^*$ and $\underline{\kappa}^*$ to the problem (11) with $u_1 = \overline{v}$ and $u_1 = \underline{v}$, respectively. If either $\overline{u}^* \succ \underline{u}^*$ or if $\overline{u}^* = \underline{u}^*$ and $\overline{k}^* > \underline{k}^*$, then we say that the contract $\overline{\kappa}^*$ is *more expensive for the principal* than contract $\underline{\kappa}^*$. In words, when these conditions are met, then $\overline{\kappa}^*$ demands less of every project $v \in \mathcal{D}$, and supplies more of every project $w \in \mathcal{S}$, than $\underline{\kappa}^*$. Formally, this is a different definition of expensiveness for demand projects as that in the text, and all references to expensiveness in the remainder of the proof refer to this definition. The notion of expensiveness in the text, which is based on the thresholds W^v defined in this proof, is easily seen to be a consequence of the notion defined here. Fix any project u and some history u^t such that $u_t = u$, and let $\overline{U}_i$ denote the payoff to i from contract $\overline{\kappa}^*$ starting from u^t, and $\underline{U}_i$ denote the payoff to i from contract $\underline{\kappa}^*$ starting from u^t: these payoffs are history-independent because u is Markov and both contracts $\overline{\kappa}^*$ and $\underline{\kappa}^*$ are stationary. Furthermore, it follows that if $\overline{\kappa}^*$ is more expensive for the principal than $\underline{\kappa}^*$, then we have that $\overline{U}_A \geq \underline{U}_A$. An implication is that, for any t, contract $\overline{\kappa}^*$ must still satisfy $(IR_{A,t})$ if $u_t = \underline{v}$, but that contract $\underline{\kappa}^*$ does not in general satisfy $(IR_{A,t})$ if $u_t = \overline{v}$.

Step 3. Let $v^1 \in \mathcal{D}$ be the project for which the solution κ^{1*} to problem (11) with $u_1 = v^1$ is the most expensive for the principal among all solutions to (11) with $u_1 = v'$ for some $v' \in \mathcal{D}$. Suppose that $\kappa_1^{1*} = 0$. We show that, for all demand projects $v' \in \mathcal{D}$, the solution $\kappa^{v'*}$ to (11) given $u_1 = v'$ has $\kappa_t^{v'*} = 0$ at all histories u^t. First, if $\kappa_1^{1*} = 0$, then (i) there is no project $u \in \mathcal{U}$ such that $u_P, u_A > 0$ and (ii) $\kappa_t^{1*} = 0$ for all histories (u^{t-1}, w_t). Suppose, towards a contradiction, that $u_P, u_A > 0$ for some $u \in \mathcal{U}$ or that $\kappa_t^{1*} > 0$ for some history (u^{t-1}, w_t). Then there exists history u^t such that $\kappa_t^{1*} u_{A,t} > 0$, and, because contract κ^{1*} is stationary, there exists history u^2 such that $\kappa_2^{1*} u_{A,2} > 0$. Because κ^{1*} is the most expensive for the principal among all solutions to (11) with $u_1 = v'$ for some $v' \in \mathcal{D}$, $U_{A,t}^{1*} \geq 0$ at any u^t. Hence, $\kappa_1^{1*} = 0$ and $\kappa_2^{1*} u_{A,2} > 0$ imply $U_{A,1}^{1*} > 0$, a contradiction because then there exists an alternative contract $\tilde{\kappa}$ identical to κ^{1*} except that $\tilde{\kappa}_1 > 0$, $\tilde{U}_{A,1} \geq 0$ and $\tilde{U}_{P,1} > U_{P,1}^{1*}$. Second, because $\kappa_t^{1*} = 0$ for all histories (u^{t-1}, w_t) and because κ^{1*} is the most expensive for the principal among all solutions to (11) with $u_1 = v'$ for some $v' \in \mathcal{D}$, we have, for all $v' \in \mathcal{D}$, $\kappa_t^{v'*} = 0$ for all histories (u^{t-1}, w_t) and hence, because there are no projects $u \in \mathcal{U}$ with $u_P, u_A > 0$, $\kappa_t^{v'*} = 0$ also for all histories (u^{t-1}, v_t). In this case, which arises if and only if $u_P, u_A > 0$ for no project $u \in \mathcal{U}$ and $w \succ v$ for all $w \in \mathcal{S}$ and $v \in \mathcal{D}$, our construction of the optimal contract is completed and the optimal contract calls for no production for all projects $u \in \mathcal{D} \cup \mathcal{S}$. Therefore, to proceed to the next step we can assume that $\kappa_1^{1*} > 0$.

From Step 2, we know that following any history u^t with $t \geq 2$, contract κ^{1*} satisfies $(IR_{A,t})$. Also, note that by the construction of problem (11), no individually rational contract delivers a higher payoff to the principal than κ^{1*} at any history u^t with $u_t = v^1$.

Step 4. Define the set of projects $V^1 = \{v^1\}$ with associated set of contracts $K^1 = \{\kappa^{1*}\}$. Now, inductively, fix a set of projects $V^{n-1} = \{v^1, \ldots, v^{n-1}\}$ and associated set of contract $K^{n-1} = \{\kappa^{1*}, \ldots, \kappa^{n-1*}\}$. Assume that (i) each κ^{j*} is individually rational following all histories, and that (ii) no individually rational contract delivers a higher payoff to the principal than κ^{j*} following any history u^t with $u_t = v^j$. Further assume that (iii) the contracts in K^{n-1} are ordered by their expensiveness for the principal (with κ^{1*} the most expensive and κ^{n-1*} the least expensive). Fix any project $v' \in \mathcal{D} \setminus V^{n-1}$ and suppose that $u_1 = v'$. We define the reduced problem

$$\begin{aligned} &\max_{\kappa \in \mathcal{K}} U_{P,1} \text{ subject to } \quad U_{A,1} \geq 0, \text{ and} \\ &\qquad \kappa = \kappa^{j*} \text{ following all histories with } v_t = v^j \in V^{t-1} \text{ and } u_{t'} \notin V^{n-1} \text{ for all } t' < t. \end{aligned} \tag{12}$$

This problem corresponds closely to the problem (11), with the additional requirement that the contract κ^{j*} be adopted following the first arrival of an opportunity to demand project $v^j \in V^{n-1}$ (knowing that, following that history, the agent's individual rationality constraint binds). As in Step 1 for problem (11), it is clear that the arguments from the proof of Proposition 1 can be adapted to show that there exists a solution κ^* to (12) which, for all histories u^t with $u_{t'} \notin V^{n-1}$ for all $t' \leq t$, is characterised by threshold project u^* and production probability k^*. In words, these thresholds are valid until the contract transitions to κ^{j*} for some $j \in \{1, \ldots, n-1\}$. Furthermore, given simple adaptations of the arguments in Step 2 for problem (11), it can be shown that solutions to (12) for projects $v' \in \mathcal{D} \setminus V^{n-1}$ can be ranked according to how expensive they are for the principal.

Step 5. Let $v^n \in \mathcal{D} \setminus V^{n-1}$ be the project for which the solution κ^{n*} to problem (12) with $u_1 = v^n$ is the most expensive for the principal among all solutions to (12) with $u_1 = v'$ for some $v' \in \mathcal{D} \setminus V^{n-1}$. By an argument along the lines of Step 3, it can be shown that if $\kappa_1^{n*} = 0$, then for all demand projects $v' \in \mathcal{D} \setminus V^{n-1}$, the solution $\kappa^{v'*}$ to (12) has $\kappa_t^{v'*} = 0$ for any history (u^{t-1}, w_t) and (u^{t-1}, v_t) prior to the transition to some contract in K^{n-1}. In this case, which arises if and only if $u_P, u_A > 0$ for no project $u \in \mathcal{U}$ and $w \succ v$ for all $w \in \mathcal{S}$ and $v \in \mathcal{D} \setminus V^{n-1}$, our construction of the optimal contract is completed and this contract calls for no production for all projects $u \in (\mathcal{D} \setminus V^{n-1}) \cup \mathcal{S}$ until a transition to some contract in K^{n-1} occurs. Therefore, to proceed to the next step we can assume that $\kappa_1^{n*} > 0$.

Arguments as in Step 3 for problem (11) establish that contract κ^{n*} is such that (i) it satisfies $(IR_{A,t})$ following any history u^t and (ii) no individually rational contract delivers a higher payoff to the principal than κ^{n*} at any history u^t with $u_t = v^n$.

Step 6. It remains to be verified that the contracts κ^{n-1*} and κ^{n*} are ranked by their expensiveness for the principal (with κ^{n-1*} being more expensive). Let $\underline{v} = v^n$ and $\overline{v} = v^{n-1}$. First, note that, by construction, the contract $\underline{\kappa}^{n-1*}$ which solved the version of problem (12) at stage $n-1$ (given sets V^{n-2} and K^{n-2}) with $u_1 = \underline{v}$ was less expensive than $\overline{\kappa}^{n-1*}$. Second, given $u_1 = \underline{v}$, problem (12) at stage n differs from the version of this problem at stage $n-1$ only through the additional constraint the contract transitions to $\overline{\kappa}^{n-1*}$ following all histories with $u_t = \overline{v}$. Because following all such histories the agent's payoff is higher under $\overline{\kappa}^{n-1*}$ than under

$\underline{\kappa}^{n-1*}$, it follows that $\underline{\kappa}^{n*}$ is less expensive for the principal than $\underline{\kappa}^{n-1*}$, which in turn is less expensive than $\overline{\kappa}^{n-1*}$, yielding the desired result.

The previous step concludes the inductive construction of the optimal contract. The final issue is to relate this contract to its characterisation in Proposition 3. Given any $v \in \mathcal{D}$ for which the construction above assigns some stage j at which v is the most expensive demand for the principal, let u^{j*} denote the threshold project characterising κ^{j*}. In this case, define $W^v = u^{j^*}$ if $u^{j*} \in \mathcal{S}$ and $W^v = \max_{\succ} \{\max_{\succ} \{u^{j*} \succ w\}, \min_{\succ} \mathcal{S}\}$ if $u^{j*} \in \mathcal{D}$. Given any other demand project v, define $W^v = \min_{\succ} \mathcal{S}$. $\square$

Proof of Corollary 1. Suppose that the project process u is *iid*, fix some history u^t along with a contract κ, consider the agent's payoff

$$U_{A,t} = \kappa_t u_{A,t} + \delta \mathbb{E}_t U_{A,t+1},$$

and note that $\mathbb{E}_t U_{A,t+1}$ is independent of u_t. It follows that if $|\overline{v}_A| \geq |\underline{v}_A|$ then the solution to problem (11) with $u_1 = \underline{v}$ cannot be more expensive for the principal than the solution to problem (11) with $u_1 = \overline{v}$. The same property holds for solutions to problem (12). The threshold projects $\overline{u}^*$ and $\underline{u}^*$ associated to projects $\overline{v}$ and $\underline{v}$ thus satisfy either $\overline{u}^* = \underline{u}^*$ or $\overline{u}^* \succ \underline{v}^*$ and hence either $W^{\overline{v}} = W^{\underline{v}}$ or $W^{\overline{v}} \succ W^{\underline{v}}$. $\square$

Proof of Corollary 2. The result follows by showing that if *Conditions 1-3* are satisfied, then given any stage n of the construction of the optimal contract in Proposition 3, the solution to problem (12) with $u_1 = \underline{v}$ cannot be more expensive for the principal than the solution to problem (12) with $u_1 = \overline{v}$ (we omit the simpler argument showing the same property for solutions to problem (11)). To this end, fix a set of projects V^{n-1} with $\overline{v}, \underline{v} \notin V^{n-1}$ and consider the solution from problem (12) given $u_1 = \overline{v}$, which specifies production probability $\overline{\kappa}_1^*$ at $t = 1$ along with threshold $\overline{u}^*$ and production probability $\overline{k}^*$ at all times $t > 1$ (prior to reaching some project $v \in V^{n-1}$). Let $\overline{U}_{A,1}^*$ denote the agent's utility from this contract conditional on $u_1 = \overline{v}$, with $\underline{U}_{A,1}$ the corresponding expression for the same contract conditional on $u_1 = \underline{v}$. Given any time $t > 1$, let $H_{\mathcal{S}}^t$ be the set of histories of length t in which only supply projects have occurred between times 2 and t: that is, $H_{\mathcal{S}}^t = \{u^t : u_{t'} \in \mathcal{S} \text{ for all } 2 \leq t' \leq t\}$. By *Condition 2* and the fact that the project process is Markov, it follows that the difference between $\underline{U}_{A,1}$ and $\overline{U}_{A,1}^*$

depends only on histories in $H^t_{\mathcal{S}}$:

$$\begin{aligned}
\underline{U}_{A,1} - \overline{U}^*_{A,1} = \overline{\kappa}^*_1 \left[|\overline{v}_A| - |\underline{v}_A|\right] + \sum_{t>1} \delta^t \Bigg[& \mathbb{P}_{\underline{v}} \left[\overline{u}^* \succ w_t, H^t_{\mathcal{S}}\right] \mathbb{E}_{\underline{v}} \left[w_{A,t} \middle| \overline{u}^* \succ w_t, H^t_{\mathcal{S}}\right] \\
& - \mathbb{P}_{\overline{v}} \left[\overline{u}^* \succ w_t, H^t_{\mathcal{S}}\right] \mathbb{E}_{\overline{v}} \left[w_{A,t} \middle| \overline{u}^* \succ w_t, H^t_{\mathcal{S}}\right] \\
& + \left[\mathbb{P}_{\underline{v}} \left[w_t = \overline{u}^*, H^t_{\mathcal{S}}\right] - \mathbb{P}_{\overline{v}} \left[w_t = \overline{u}^*, H^t_{\mathcal{S}}\right]\right] \overline{k}^* \overline{u}^*_A \Bigg] \\
\geq \overline{k}^* \mathbb{I}_{\overline{u}^* \in \mathcal{S}} \sum_{t>1} \delta^t \Bigg[& \mathbb{P}_{\underline{v}} \left[{}^{|w_{P,t}|}\!/_{w_{A,t}} \leq \left|{}^{\overline{u}^*_P}\!/_{\overline{u}^*_A}\right|, H^t_{\mathcal{S}}\right] \mathbb{E}_{\underline{v}} \left[w_{A,t} \middle| {}^{|w_{P,t}|}\!/_{w_{A,t}} \leq \left|{}^{\overline{u}^*_P}\!/_{\overline{u}^*_A}\right|, H^t_{\mathcal{S}}\right] \\
& - \mathbb{P}_{\overline{v}} \left[{}^{|w_{P,t}|}\!/_{w_{A,t}} \leq \left|{}^{u^*_P}\!/_{u^*_A}\right|, H^t_{\mathcal{S}}\right] \mathbb{E}_{\overline{v}} \left[w_{A,t} \middle| {}^{|w_{P,t}|}\!/_{w_{A,t}} \leq \left|{}^{\overline{u}^*_P}\!/_{\overline{u}^*_A}\right|, H^t_{\mathcal{S}}\right] \Bigg] \\
& + (1 - \overline{k}^* \mathbb{I}_{\overline{u}^* \in \mathcal{S}}) \sum_{t>1} \delta^t \Bigg[\mathbb{P}_{\underline{v}} \left[{}^{|w_{P,t}|}\!/_{w_{A,t}} < \left|{}^{\overline{u}^*_P}\!/_{\overline{u}^*_A}\right|, H^t_{\mathcal{S}}\right] \mathbb{E}_{\underline{v}} \left[w_{A,t} \middle| {}^{|w_{P,t}|}\!/_{w_{A,t}} < \left|{}^{\overline{u}^*_P}\!/_{\overline{u}^*_A}\right|, H^t_{\mathcal{S}}\right] \\
& - \mathbb{P}_{\overline{v}} \left[{}^{|w_{P,t}|}\!/_{w_{A,t}} < \left|{}^{\overline{u}^*_P}\!/_{\overline{u}^*_A}\right|, H^t_{\mathcal{S}}\right] \mathbb{E}_{\overline{v}} \left[w_{A,t} \middle| {}^{|w_{P,t}|}\!/_{w_{A,t}} < \left|{}^{\overline{u}^*_P}\!/_{\overline{u}^*_A}\right|, H^t_{\mathcal{S}}\right] \Bigg],
\end{aligned}$$

where the inequality follows from *Condition 1*. To show that $\underline{U}_{A,1} - \overline{U}^*_{A,1} \geq 0$, and hence that the contract associated to project $\underline{v}$ in problem (12) cannot be more expensive for the principal than the one associated to project $\overline{v}$, we show that for all $c \geq 0$ inequality (4) in *Condition 3* implies the inequality

$$\begin{aligned}
& \mathbb{P}_{\overline{v}} \left[{}^{|w_{P,t}|}\!/_{w_{A,t}} \leq c, H^t_{\mathcal{S}}\right] \mathbb{E}_{\overline{v}} \left[w_{A,t} \middle| {}^{|w_{P,t}|}\!/_{w_{A,t}} \leq c, H^t_{\mathcal{S}}\right] \\
& \qquad \leq \mathbb{P}_{\underline{v}} \left[{}^{|w_{P,t}|}\!/_{w_{A,t}} \leq c, H^t_{\mathcal{S}}\right] \mathbb{E}_{\underline{v}} \left[w_{A,t} \middle| {}^{|w_{P,t}|}\!/_{w_{A,t}} \leq c, H^t_{\mathcal{S}}\right]. \qquad (13)
\end{aligned}$$

Note that the versions of (4) and (13) with the strict inequality ${}^{|w_{P,t}|}\!/_{w_{A,t}} < c$ must hold if (4) and (13), respectively, hold because $\mathcal{U}$ is finite. It is straightforward to compute that (13) follows from (4) if, for any time $t > 1$ and any $w \in \mathcal{S}$,

$$\mathbb{P}_{\underline{v}} \left[(u^{t-1}, w) \notin H^t_{\mathcal{S}}\right] = \mathbb{P}_{\overline{v}} \left[(u^{t-1}, w) \notin H^t_{\mathcal{S}}\right],$$

and this latter property can be shown by induction. If $t = 2$, then the claim follows because $\mathbb{P}_{\underline{v}} [(\underline{v}, w) \notin H^2_{\mathcal{S}}] = \mathbb{P}_{\overline{v}} [(\overline{v}, w) \notin H^2_{\mathcal{S}}] = 0$ for all $w \in \mathcal{S}$. If the claim holds for all $t-1$ with $t > 2$, then note that, for any $w \in \mathcal{S}$ and any $v \in \{\overline{v}, \underline{v}\}$,

$$\begin{aligned}\mathbb{P}_v\left[(u^{t-1}, w) \notin H^t_{\mathcal{S}}\right] = &\sum_{u \in \mathcal{U}\setminus\mathcal{S}} \mathbb{P}_v\left[u_{t-1} = u\right] \mathbb{P}_v\left[u_t = w \middle| u_{t-1} = u\right] \\ &+ \sum_{w' \in \mathcal{S}} \mathbb{P}_v\left[(u^{t-2}, w') \notin H^{t-1}_{\mathcal{S}}\right] \mathbb{P}_v\left[u_t = w \middle| u_{t-1} = w'\right].\end{aligned}$$

The conclusion follows from the facts that $\mathbb{P}_{\overline{v}}\left[u_{t-1} = u\right] = \mathbb{P}_{\underline{v}}\left[u_{t-1} = u\right]$ for all $u \notin \mathcal{S}$ by *Condition 2*, that $\mathbb{P}_v\left[u_t = w \middle| u_{t-1} = u\right]$ and $\mathbb{P}_v\left[u_t = w \middle| u_{t-1} = w'\right]$ are independent of v because the project process is Markov, and that $\mathbb{P}_v\left[(u^{t-2}, w') \notin H^{t-1}_{\mathcal{S}}\right]$ is independent of v by the induction hypothesis. □

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