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First-price auctions with budget constraints

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Consider a first-price sealed-bid auction with interdependent valuations and private budget constraints. Focusing on the two-bidder case, we identify new sufficient conditions for the existence of a symmetric equilibrium in pure strategies. In equilibrium, agents may adopt discontinuous bidding strategies that result in a stratification of competition along the budget dimension. Private budgets can simultaneously lead to more aggressive bidding (a high-budget agent leverages his wealth to outbid rivals) and more subdued bidding (competition becomes less intense among bidders at distinct budget levels). The presence of budget constraints may lead to multiple symmetric equilibria in the first-price auction.

Keywords. First-price auction, budget constraints, interdependent values.

JEL classification. D44.

1. Introduction

Private budgets arise in many economic situations, including auctions. Once bidders have multidimensional private information involving a value signal, which informs their preferences, and a budget, which defines their feasible strategy set, even simple bidding games exhibit nontrivial equilibria. Many familiar intuitions require qualification or reassessment.

For concreteness, consider a first-price sealed-bid auction for one item. Bidders simultaneously place bids, the highest bidder wins, and he pays his own bid. Private budgets affect equilibrium bidding in two opposing ways in this familiar setting. First, budgets dampen bids by introducing a spending limit. Che and Gale (1998) characterize this feature in their seminal analysis of the first-price auction with private budget constraints.[1] Assuming private values, they propose sufficient conditions that ensure equilibrium bidding strategies assume the *canonical form*

$$\beta(s_i, w_i) = \min\{\bar{b}(s_i), w_i\}. \tag{1}$$

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[1] Most related to our study is the model found in Section 3 of Che and Gale (1998).

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In (1), s_i is bidder i's private value, w_i is his private budget, and $\bar{b}(\cdot)$ is an increasing, continuous function that defines the bid of an "unconstrained" agent who bids less than his budget.

Second, and less intuitively, budgets can amplify certain agents' equilibrium bids. A higher bid may be disproportionally effective at defeating budget-constrained rivals, thus giving a high-budget bidder a valuable strategic option. Exercising this option can lead to an equilibrium where agents adopt discontinuous bidding strategies, even in a symmetric setting. A stratification of competition between high- and low-budget bidders emerges in equilibrium. Hitherto underappreciated in the literature, these strategic implications complicate equilibrium bidding and affect the auction's efficiency and revenue potential.

To illustrate our argument, it is helpful to consider an example that conveys its central intuition. The following example is an extension of Vickrey's (1961) model.[2]

Example 1. Consider a first-price sealed-bid auction for one item. There are two ex ante identical bidders. Each bidder i observes an independent, uniformly distributed signal $s_i \in [0, 1]$ regarding his value for the item. Signals are private, but their distribution is common knowledge. If a bidder with value signal s_i wins the auction with the bid b_i, his payoff is $s_i - b_i$; otherwise, it is zero. A fair coin flip resolves ties. It is well known that in this auction's Bayesian–Nash equilibrium, both bidders adopt the strategy $s_i/2$.

Suppose, additionally, that each bidder has a budget w_i, which is independent of his value signal. Budgets are private, but their distribution is common knowledge. With probability $1/2$, a bidder's budget is $1/4$; with probability $1/2$, it is $3/4$. A budget is a hard constraint on bids and a type-(s_i, w_i) bidder cannot bid above w_i.

A tempting conjecture is that

$$\min\{s_i/2, w_i\} \tag{2}$$

is a symmetric equilibrium strategy in this enriched setting. In (2), which has the canonical form, a bidder follows the usual equilibrium strategy $s_i/2$ until he exhausts his funds. Though intuitive, (2) cannot be a symmetric equilibrium strategy. If bidder j follows (2), then bidder i has a profitable deviation when $s_i = 1/2 - \varepsilon$ and $w_i = 3/4$. According to (2), he should bid $1/4 - \varepsilon/2$. But if he bids ε more, $1/4 + \varepsilon/2$, his probability of winning jumps from approximately $1/2$ to over $3/4$. At infinitesimal added cost, this deviation is worth it.[3]

Reflecting on the preceding reasoning, we are led to an equilibrium strategy with jump discontinuities. Figure 1 sketches $\beta(s_i, w_i)$, this auction's symmetric equilibrium

[2] Milgrom (2004) presents a related example with a common budget constraint.

[3] Similar reasoning applies whenever the bid distribution has a mass point due to a mass point in the budget distribution. When this occurs, the first-price auction cannot have an equilibrium in continuous strategies.

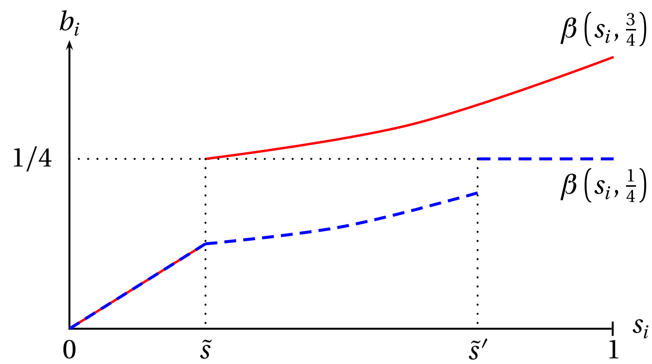


Figure 1. The equilibrium strategy in Example 1. Figure not to scale.

strategy:

$$\beta(s_i, 1/4) = \begin{cases} s_i/2 & \text{if } s_i \in [0, \tilde{s}] \\ \dfrac{1+9s_i^2}{6+18s_i} & \text{if } s_i \in \left(\tilde{s}, \tilde{s}'\right] \\ 1/4 & \text{if } s_i \in \left(\tilde{s}', 1\right] \end{cases} \qquad \beta(s_i, 3/4) = \begin{cases} s_i/2 & \text{if } s_i \in [0, \tilde{s}] \\ \dfrac{5+9s_i^2}{18+18s_i} & \text{if } s_i \in (\tilde{s}, 1], \end{cases}$$

where $\tilde{s} = 1/3$ and $\tilde{s}' = 11/27$.[4]

Both dampening and strategic consequences of private budgets are apparent in this example's equilibrium. The former is obvious. A high-valuation ($s_i > \tilde{s}'$) low-budget ($w_i = 1/4$) bidder hits his spending limit in equilibrium: his bid is capped.

More interesting and consequential are the strategic implications. First, a high-budget bidder's strategy features a prominent jump discontinuity that amplifies his bid at $\tilde{s}$. A high-budget bidder has the option of bidding above 1/4, thus outbidding with certainty a low-budget competitor. Exercising this option becomes worthwhile when his value exceeds $\tilde{s}$. The intuition for the discontinuity at $\tilde{s}'$ is similar. Given that low-budget agents cannot bid above 1/4, there is a chance of a tie at this bid. A low-budget bidder with valuation $s_i > \tilde{s}'$ finds this outcome sufficiently attractive to actually bid 1/4. He would like to bid more to break the (possible) tie in his favor, but lacks the funds to do so.

Second, despite occasional jumps, on the margin bidding is less brazen due to a stratification of competition along the budget dimension. When an agent bids above 1/4, he need only worry about a competing bid from another high-budget bidder, which occurs with probability less than 1/2. Similarly, in equilibrium, a low-budget bidder of

[4]We verify that the defined strategy is a symmetric equilibrium in Appendix A. The working paper Kotowski (2019) examines a generalized version of this example: valuations are distributed according to the continuous cumulative distribution function $H(s)$ on $[0, \bar{s}]$, and budgets assume low ($\underline{w}$) and high ($\overline{w}$) values with probabilities p and $1-p$, respectively. The equilibrium remains qualitatively similar when $\underline{w} < \bar{s} - \int_0^{\bar{s}} H(z)\,dz < \overline{w}$.

type $s_i > \tilde{s}$ competes on the margin only against a low-budget adversary.[5] In each case, there is less incentive to bid aggressively since segregation has reduced competition in the relevant range of bids. ◊

Although the preceding observations appear to be specific to Example 1 or are perhaps an artifact of the discrete budget distribution, this is not the case. As explained in Section 3, the strategic amplification and the competitive stratification can arise even when budgets assume a continuum of values. Necessarily, the construction is more elaborate, but the intuition is the same. Focusing on the two-bidder case, we propose new sufficient conditions for the existence of an equilibrium that satisfies the canonical form (1), even when values are interdependent. Relaxing these assumptions leads to a noncanonical equilibrium that echoes the intuition of Example 1. The resulting empirical confounds can be substantial. For example, a uniform distribution of value signals and budgets may result in a bimodal equilibrium bid distribution (Example 2). Budget constraints may also lead to multiple symmetric equilibria (Example 3). This is because jump discontinuities that arise in equilibrium reflect an underlying coordination problem. An increase in bid to a particular value may only be worthwhile if the other bidder adopts the same strategy.

Our study has three main contributions. First, our results offer guidance concerning bidding in first-price auctions by isolating the strategic implications of private budgets. The importance of the strategic dimension is surprising given the simplicity of the first-price sealed-bid auction format.

Second, we offer a cautionary message for empirical studies. The conflicting dampening and amplifying responses to the budget constraint variable imply that bids need not be skewed in one direction. Local incentive constraints, which underlie many empirical methods for auction analysis, need not fully characterize equilibrium bidding. Even small changes in a bidder's type may lead to large changes in his equilibrium bid. We hope our analysis provides a useful step toward better integrating budgets in empirical analysis of auctions.

Third, our methods may be useful in further investigations of auctions with budget constraints or with other forms of multidimensional private information. The phase-plane analysis employed in Section 3 is both simple and informative of associated economic incentives.

Outline. Section 2 surveys the related literature. Section 3 generalizes Example 1 to allow for interdependent valuations and continuously distributed budgets. We examine both canonical equilibria and noncanonical, discontinuous equilibria. Section 4 presents a discussion of our results, including a comparison with the second-price auction. We also provide an example that demonstrates that budget constraints can lead to multiple symmetric equilibria in the first-price auction. Appendices A–D collect omitted proofs and calculations.

[5]Consider the equilibrium bid of agent i with value signal $s_i \in (\tilde{s}, \tilde{s}')$. When he bids $\beta(s_i, 1/4)$, bidder i wins the auction when $s_j < s_i$ and $w_j = 1/4$ or $s_j < \tilde{s}$ and $w_j = 3/4$. If bidder i bids slightly more, he now wins when $s_j < s_i + \varepsilon$ and $w_j = 1/4$ or $s_j < \tilde{s}$ and $w_j = 3/4$. Agent i's higher bid is effective only against a low-budget adversary.

2. Related literature

Che and Gale (1998) were the first to study standard auctions with private values and private budget constraints.[6] They investigate equilibria that satisfy the canonical form (1). In Section 3.1, we study canonical equilibria in a complementary class of cases, including those with interdependent values. Fang and Perreiras (2002, 2003) study the second-price auction with private budgets and interdependent values. We discuss their results in detail in Section 4. In a similar setting, Kotowski and Li (2014a, 2014b) analyze all-pay auctions. Ghosh et al. (2018) study wars of attrition with private budgets.

In developing our model, we suppress several embellishments commonly associated with budget constraints. Unlike Zheng (2001), Jaramillo (2004), or Rhodes-Kropf and Viswanathan (2005), we model budgets as hard bounds on bids, rather than as imperfections in bid-financing ability. We also abstract from the risk of collusion or the formation of bidding consortia due to limited liquidity (Cho et al. 2002). Burkett (2016) argues that budget constraints solve a principal–agent problem when bidding is delegated to a third party. Budgets are exogenous in our model, as they would be in the absence of delegation.

Interest in the role of budgets in auctions has grown due to their salience in many Internet-related applications (Ashlagi et al. 2010, Balseiro et al. 2015) and spectrum auctions (Cramton 1995, Bulow et al. 2017). Noting these applications, several authors build models to investigate budget constraints in sequential or multi-unit auctions (Pitchik and Schotter 1988, Benoît and Krishna 2001, Pitchik 2009, Brusco and Lopomo 2008, 2009, Ghosh 2015, Kariv et al. 2018). Our model is not a special case of any of them.

The auction design problem following Myerson (1981) that incorporates budget constraints is particularly challenging and has been considered under various guises (Laffont and Robert 1996, Monteiro and Page 1998, Che and Gale 1999, 2000, Maskin 2000, Malakhov and Vohra 2008, Pai and Vohra 2014, Kojima 2014, Baisa 2018, Boulatov and Severinov 2018, Carbajal and Mu'alem 2018, Richter 2019). Auction design with budgets has also spurred an interest among computer scientists.[7] Our analysis takes the auction format as given and examines equilibrium bidding. Che and Gale (2006) examine auction revenue when bidders have multidimensional types, which may include private financial constraints.

3. A symmetric auction with interdependent values

Consider the following adaptation of Milgrom and Weber's (1982) model of a first-price auction. There are two ex ante symmetric bidders. Let $s_i \in [0, 1]$ be the value signal of bidder i. It defines his information about the item for sale. Value signals are independently and identically distributed according to the twice continuously differentiable

[6] The works of Che and Gale (1996a, 1996b) are precursors that examine a common-value setting where only budgets are private information. Bobkova (2019) generalizes these models to an asymmetric setting. Kariv et al. (2018) corroborate the equilibrium predictions of Che and Gale (1996a) with a laboratory experiment.

[7] Borgs et al. (2005) and Dobzinski et al. (2012) are representative contributions.

cumulative distribution function (c.d.f.) $H(s_i)$. (We relax independence in Remark 3 below.) The associated probability density function (p.d.f.) $h(s_i)$ is strictly positive and bounded. Given s_i and s_j, $v(s_i, s_j)$ is bidder i's valuation for the item. The valuation function is continuously differentiable, strictly increasing in the first argument, nondecreasing in the second argument, $v(0,0) = 0$, and $v(1,1) = \overline{v}$. If bidder i wins the auction with the bid b_i, his payoff is $v(s_i, s_j) - b_i$; otherwise, it is 0. Ties are resolved with a uniform randomization.

When budget constraints are absent, Milgrom and Weber (1982) show that there exists a Bayesian–Nash equilibrium where all bidders adopt a common strategy, $\alpha(s_i)$, that solves

$$\alpha'(s_i) = \big(v(s_i, s_i) - \alpha(s_i)\big)\frac{h(s_i)}{H(s_i)} \tag{3}$$

subject to $\alpha(0) = 0$. Henceforth, we reserve the notation $\alpha(\cdot)$ for this specific strategy.

Generalizing the model, let $w_i \in [\underline{w}, \overline{w}]$ be the budget of bidder i. Budgets are independently and identically distributed according to the twice continuously differentiable c.d.f. $G(w_i)$. We assume that $0 \leq \underline{w} < \overline{v} < \overline{w}$ and that the associated p.d.f. $g(w_i)$ is strictly positive and bounded for all $w_i \in [\underline{w}, \overline{w}]$. For convenience, we define $G(w_i) = 0$ if $w_i < \underline{w}$ and $G(w_i) = 1$ if $w_i > \overline{w}$. A bidder's type (s_i, w_i) is private information; otherwise, the environment is common knowledge. We focus on a symmetric Bayesian–Nash equilibrium and we henceforth suppress bidder subscripts.

3.1 *A canonical equilibrium*

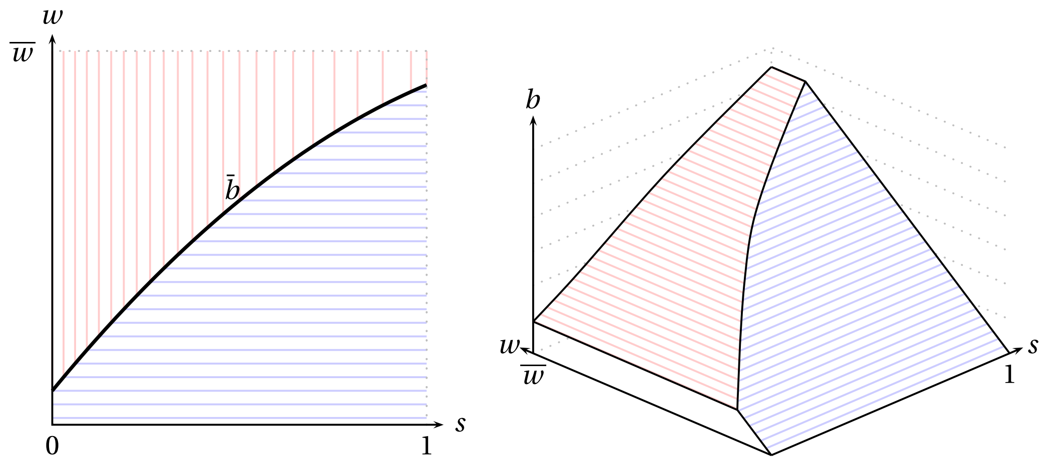
Positing private values, Che and Gale (1998) identify a sufficient condition (see Remark 2 below) that ensures that the first-price auction has an equilibrium where bidders adopt a common canonical strategy, $\beta(s, w) = \min\{\bar{b}(s), w\}$. The function $\bar{b}(s)$, which is strictly increasing and continuous, defines the bid of an "unconstrained" bidder who bids less than his budget in equilibrium. Figure 2 illustrates a representative canonical strategy.

We first investigate conditions that ensure a canonical equilibrium exists in our model. As an initial step, we derive some properties of $\bar{b}(\cdot)$. If bidder j follows a canonical strategy, the expected payoff of a type-(s, w) bidder i when he bids $\bar{b}(x) \leq w$ is

$$\begin{aligned} U\big(\bar{b}(x)|s, w\big) = &\int_0^x \int_0^{\overline{w}} \big(v(s, y) - \bar{b}(x)\big)h(y)g(z)\,dz\,dy \\ &+ \int_x^1 \int_0^{\bar{b}(x)} \big(v(s, y) - \bar{b}(x)\big)h(y)g(z)\,dz\,dy. \end{aligned} \tag{4}$$

The first term in (4) is the expected payoff from defeating an opponent with a value signal less than x. The second term is the expected payoff from defeating an opponent with a value signal greater than x who has a budget less than $\bar{b}(x)$. This term vanishes if $\bar{b}(x) < \underline{w}$. By computing the first-order condition, $dU(\bar{b}(x)|s, w)/dx|_{x=s} = 0$, and defining the expressions

$$\lambda(s) := \frac{h(s)}{1 - H(s)}$$



(a) Level sets of $\beta(s, w)$ (b) Three dimensional sketch of $\beta(s, w)$

FIGURE 2. Illustration of a canonical strategy, $\beta(s, w) = \min\{\bar{b}(s), w\}$, when $\underline{w} = 0$.

$$\gamma(b) := \frac{g(b)}{1 - G(b)}$$

$$\eta(x, s) := \int_x^1 \frac{v(s, y)h(y)}{1 - H(x)}\,dy$$

$$\delta(b, s) := b + \frac{G(b)}{g(b)} + \frac{H(s)}{g(b)\big(1 - H(s)\big)},$$

we can characterize $\bar{b}(s)$ at its points of differentiability as

$$\bar{b}'(s) = \begin{cases} \big(v(s, s) - \bar{b}(s)\big)\dfrac{h(s)}{H(s)} & \text{if } \bar{b}(s) < \underline{w} \\ \dfrac{\lambda(s)}{\gamma\big(\bar{b}(s)\big)}\left(\dfrac{\bar{b}(s) - v(s, s)}{\eta(s, s) - \delta\big(\bar{b}(s), s\big)}\right) & \text{if } \bar{b}(s) > \underline{w}. \end{cases} \tag{5}$$

When $\bar{b}(s) < \underline{w}$, (5) reduces to (3), the differential equation that characterizes the equilibrium strategy in the absence of budget constraints. To interpret (5) when $\bar{b}(s) > \underline{w}$, we expand the equation and rearrange its terms. Omitting the function arguments for readability yields

$$\underbrace{h(1 - G)(v - \bar{b})}_{[A]} + \underbrace{\bar{b}' g(1 - H)(\eta - \bar{b})}_{[B]} = \underbrace{\bar{b}'\big(G(1 - H) + H\big)}_{[C]}. \tag{6a}$$

Terms $[A]$ and $[B]$ are the expected marginal benefit of an infinitesimally higher bid. When the higher bid defeats an opponent with a value signal less than s, the payoff gain is $v(s, s) - \bar{b}(s)$; otherwise, when it defeats an opponent with a budget less than $\bar{b}(s)$, the payoff gain is $\eta(s, s) - \bar{b}(s)$.[8] Term $[C]$ is the expected marginal cost of a higher bid.

[8]The term $\eta(x, s)$ is the expected value of the item to a bidder with value signal s conditional on defeating an opponent with a value signal greater than x.

Further rearrangement of (6a) gives

$$\bar{b}' = \frac{h(1-G)(v-\bar{b})}{\big(G(1-H)+H\big) - g(1-H)(\eta - \bar{b})}. \tag{6b}$$

For $\bar{b}'(s)$ to be positive, as assumed throughout its derivation, both numerator and denominator in (6b) must have the same sign. When the budget distribution is relatively dispersed and say $g(b) \approx 0$, this is not a demanding requirement, as the denominator is positive. If the budget distribution is concentrated near some value and $g(b)$ is relatively large, the denominator of (6b) may approach zero or be negative. Intuitively, when this occurs, the likelihood of defeating a budget-constrained opponent is large and a (high-budget) bidder wishes to increase his bid aggressively to capitalize on this opportunity. In Example 1, the return to a slightly higher bid spiked at the atom in the budget distribution and led to a discontinuous jump in bid. As (6b) reveals, an atom is not necessary to provoke a similar response.

Motivated by the preceding discussion, we introduce a regularity condition that ensures the marginal returns to a slightly higher bid are moderate and well behaved. As explained below, it is a sufficient condition for a canonical equilibrium to exist when $\underline{w} = 0$.

Assumption 1. *For each s, the function $b \mapsto \eta(s,s) - \delta(b,s)$ crosses zero at most one time. Furthermore, if $\eta(s',s') - \delta(b',s') = 0$, then $\partial\delta(b,s')/\partial b|_{b=b'} > 0$.*

Remark 1. Assumption 1 holds when the budget c.d.f. $G(\cdot)$ is concave.

Remark 2. Assumption 1 is implied by Assumption 5 of Che and Gale (1998) when the latter is applied to the case of independent budgets and value signals, as in our model.[9] When value signals and budgets are mutually independent, our contribution beyond Che and Gale's is to show that canonical equilibria continue to obtain in a setting with interdependent values and a weaker restriction on the budget distribution.

Assumption 1 has two parts. The first says that the denominator of (5) satisfies a single-crossing condition. Since $\eta(s,s) - \delta(\overline{w},s) < 0$, any crossing must be from above. Thus, the second part is a technical restriction that ensures the crossing is strict.

Theorem 1. *Suppose Assumption 1 holds and $\underline{w} = 0$. There exists an equilibrium where each bidder adopts a common strategy, $\beta(s,w) = \min\{\bar{b}(s), w\}$. The function $\bar{b}(\cdot)$ is increasing, continuous, and for a.e. s,*

$$\bar{b}'(s) = \frac{\lambda(s)}{\gamma\big(\bar{b}(s)\big)}\left(\frac{\bar{b}(s) - v(s,s)}{\eta(s,s) - \delta\big(\bar{b}(s),s\big)}\right). \tag{7}$$

[9] Che and Gale (1998) invoke their Assumption 5 to prove their Lemma 1. Fang and Perreiras (2002, footnote 7) point out that Che and Gale's lemma may not be true without additional assumptions in their model. When applied to our model, Che and Gale's assumption is sufficient for the existence of an equilibrium in canonical strategies when $\underline{w} = 0$.

Generically, the equilibrium is unique in the class of symmetric equilibria in canonical strategies.

We provide a numerical example that illustrates an equilibrium in Section 3.3.

Identification of $\bar{b}(\cdot)$ and the proof of Theorem 1 To prove Theorem 1, we must first confirm that the function $\bar{b}(\cdot)$ exists with the stated properties. A natural approach is to (try to) solve (7) assuming $\bar{b}(0) = 0$. This is premature. With interdependent values, zero may not be an optimal bid for an unconstrained bidder. A small bid may defeat a budget-constrained opponent with favorable information concerning the item's value. This attenuation of the winner's curse can inflate an agent's bid, even if he observes a low value signal. Moreover, (7) may lack an increasing solution that is defined for all s, as assumed throughout its derivation.

Acknowledging the above complications, we identify $\bar{b}(\cdot)$ indirectly by focusing on the qualitative behavior of solutions to (7). To do so, we use phase-plane analysis to study the closely related plane-autonomous system

$$\dot{s}(s,b) = \gamma(b)\big(\eta(s,s) - \delta(b,s)\big), \qquad \dot{b}(s,b) = \lambda(s)\big(b - v(s,s)\big), \tag{8}$$

where $(s,b) \in [0,1] \times [\underline{w}, \overline{w}]$.[10] In analyzing this system, we are interested in the graphs of its solutions and not in its dynamics. Graphs of its solutions can be identified with integral curves of (7) since, following Birkhoff and Rota (1978, Chapter 5), we observe that

$$\frac{db}{ds} = \frac{db}{dt} \Big/ \frac{ds}{dt} = \frac{\dot{b}(s,b)}{\dot{s}(s,b)} = \bar{b}'(s).$$

Interpreting the problem as proposed lets us use graphical techniques to discern the behavior of candidate solutions over a large domain (Strogatz 1994, Perko 2001). Additionally, our reinterpretation placates the singularities encountered in a direct analysis of (7).

Analysis of system (8) focuses on its nullclines and critical (or fixed) points. The *nullclines* are

$$\psi(s) := \big\{b \mid \eta(s,s) - \delta(b,s) = 0\big\} \quad \text{and} \quad \nu(s) := \big\{b \mid b - v(s,s) = 0\big\}.$$

These graphs characterize points where $\dot{s} = 0$ and $\dot{b} = 0$, respectively. Their economic interpretation is the following. At $b = \psi(s)$, the denominator of (7) changes sign. When $\eta(s,s) - \delta(b,s) > 0$, marginally higher bids are relatively effective (conditional on s) at defeating a budget-constrained opponent. Otherwise, this dimension's marginal contribution to a bidder's payoff is more subdued. By definition, $\nu(s) = v(s,s)$. Thus, $\nu(s)$ is a type-(s,w) bidder's valuation conditional on his opponent observing the same value signal. If a type-(s,w) agent bids more than $\nu(s)$, then he is overbidding conditional on defeating only an opponent with a value signal less than his own. A *critical point* occurs where nullclines intersect.

[10]The functions $\dot{s}$ and $\dot{b}$ are not defined at the boundaries where $b = \overline{w}$ and $s = 1$, respectively. This technical fact is not a concern and we suppress it for expositional clarity. We can perform the analysis on the domain's interior. The identified solution can be extended to the boundary using standard results in the theory of ordinary differential equations.

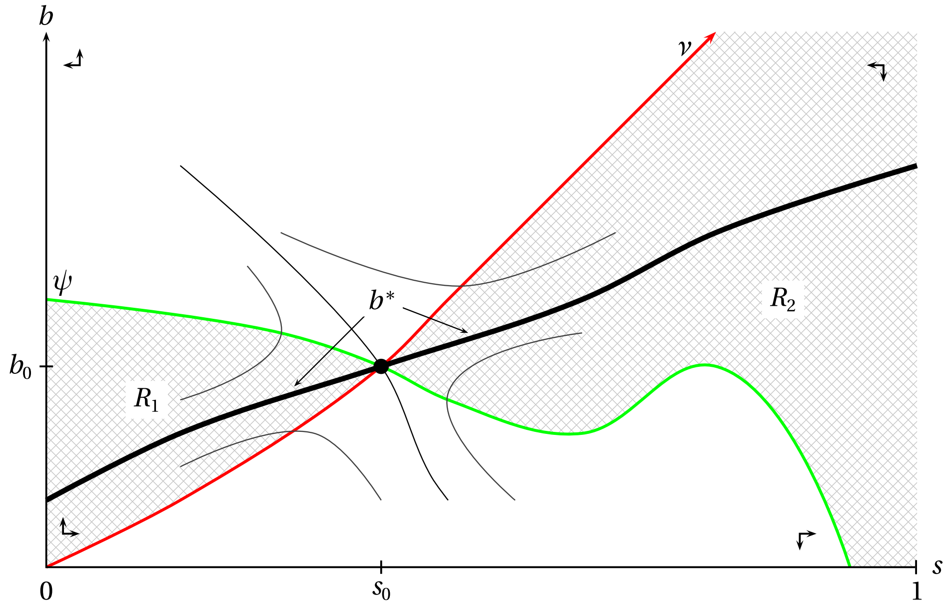


FIGURE 3. A phase portrait of system (8) with one critical point.

Assumption 1 ensures that the analysis of system (8) is routine. Figure 3 illustrates a representative case. Six facts are noteworthy.

(i) The curve $\nu(s)$ runs from the origin to the figure's top-right corner. In the region below $\nu(s)$, $\dot{b} < 0$; otherwise, $\dot{b} > 0$.

(ii) Assumption 1 implies that $\psi(s)$ is at most single-valued. When not empty, $\psi(s)$ is a continuous curve.[11] In the region below $\psi(s)$, $\dot{s} > 0$; otherwise, $\dot{s} < 0$.

(iii) There exists at least one critical point when $\underline{w} = 0$ (Lemma B.2 in Appendix B). For simplicity, Figure 3 shows an instance with one critical point at (s_0, b_0). Multiple critical points do not affect Theorem 1, though an extended analysis is needed to identify $\bar{b}(\cdot)$. We outline the identification of $\bar{b}(\cdot)$ in the case of multiple critical points in the working paper (Kotowski 2019).

(iv) The ratio $\dot{b}/\dot{s}$ is positive at point (s, b) only when b is "in between" $\nu(s)$ and $\psi(s)$. These are regions R_1 and R_2 in Figure 3. In these regions, the solutions of (8) can be identified with increasing functions of s. Elsewhere, solutions have negative slope and, therefore, cannot constitute part of an admissible solution for $\bar{b}(s)$.

(v) Generically, each critical point is either a saddle point or a node (Lemma B.3). When there is one critical point, it is a saddle point.

(vi) When there is one critical point, of particular interest are the stable manifolds that approach it from below and above in regions R_1 and R_2. (The Hartman–

[11]Continuity follows from the implicit function theorem. Lemma B.1 in Appendix B records additional properties of $\psi(s)$, including the fact that $\psi(s) < \overline{v}$.

Grobman theorem implies that these manifolds are generically unique.) In Figure 3, these are the bold curves connecting (s_0, b_0) with the boundaries. Henceforth, we use $b^*(s)$ to denote the union of these solution paths' closures as a function of s. The function $b^*(s)$ is strictly increasing, continuous, and an integral curve of (7).

We are now ready to identify the function $\bar{b}(\cdot)$. This function must be strictly increasing, continuous, and an integral curve of (7). When there is one critical point, as in Figure 3, only the stable solutions of (8) approaching the critical point—the function $b^*(s)$—satisfy the stated requirements. So, in this case, we define $\bar{b}(s) \equiv b^*(s)$ for all s.

The identification of $\bar{b}(s)$ illustrates the competing incentives facing a high-budget bidder. Consider again Figure 3. When $\bar{b}(s)$ is confined to region R_1, a type-(s, w) bidder would like to bid more to capitalize on his opponent's (possible) budget constraint. His incentive to do so is moderated by the fact that he is overbidding conditional on defeating an adversary with a value signal less than his own, i.e., $\bar{b}(s) > v(s, s)$. In region R_2, these incentives flip and $v(s, s) > \bar{b}(s)$.

A standard argument lets us confirm that Theorem 1 characterizes an equilibrium.[12]

Proof of Theorem 1. Suppose bidder j adopts the strategy $\beta(s, w) = \min\{\bar{b}(s), w\}$, where $\bar{b}(s)$ satisfies the conditions stated in Theorem 1. (The preceding discussion has established the existence of this function.) The expected utility of bidder i of type (s, w) who bids $\bar{b}(x)$ is given by (4). Differentiating (4) yields

$$\frac{\partial U\big(\bar{b}(x)|s,w\big)}{\partial x} = g\big(\bar{b}(x)\big)\big(1-H(x)\big)\left[\frac{\lambda(x)}{\gamma\big(\bar{b}(x)\big)}\big(v(s,x)-\bar{b}(x)\big)+\bar{b}'(x)\big(\eta(x,s)-\delta\big(\bar{b}(x),x\big)\big)\right].$$

Suppose $x > s$. Clearly, $\eta(x, x) \geq \eta(x, s)$ and $v(x, x) \geq v(s, x)$. Thus, $\partial U(\bar{b}(x)|s, w)/\partial x \leq \partial U(\bar{b}(x)|x, w)/\partial x = 0$. Hence, bidder i can increase his payoff by bidding less than $\bar{b}(x)$. A parallel argument shows that when $x < s$, bidder i can increase his payoff by bidding more than $\bar{b}(x)$. Together, the two cases imply that bidder i has no incentive to deviate from $\beta(s, w)$ to another bid in the range of $\bar{b}(\cdot)$.[13]

Instead, suppose bidder i bids outside the range of $\bar{b}(\cdot)$. Any bid strictly exceeding $\bar{b}(1)$ is dominated by $\bar{b}(1)$. The bid $b \leq \bar{b}(0)$ yields an expected payoff of

$$U(b|s,w) = \int_0^1 \int_0^b \big(v(s,y)-b\big)h(y)g(z)\,dz\,dy = G(b)\big(\eta(0,s)-b\big). \tag{9}$$

To complete the proof, it is sufficient to show that $\partial U(b|s, w)/\partial b \geq 0$ when $\underline{w} \leq b < \bar{b}(0)$. Differentiating (9), $\partial U(b|s, w)/\partial b = g(b)(\eta(0, s) - b) - G(b) \geq g(b)(\eta(0, 0) - b) - G(b)$. Given the definition of $\bar{b}(\cdot)$, if $\bar{b}(0) > 0$, then $\psi(0) > \bar{b}(0)$. Recalling Assumption 1, $b < \psi(0)$ implies that $\eta(0, 0) > \delta(b, 0) = b + G(b)/g(b)$. Rearranging terms gives $g(b)(\eta(0, 0) - b) - G(b) \geq 0$, which implies the desired conclusion. □

[12] See Fang and Perreiras (2002) and Kotowski and Li (2014a) for similar proofs in the case of the second-price and all-pay auctions with budget constraints.

[13] We have shown that a type-(s, w) bidder can improve his payoff by increasing his bid whenever it is less than $\bar{b}(s)$. If $\beta(s, w) = w \leq \bar{b}(s)$, then he cannot increase it above his budget constraint.

Remark 3. Theorem 1 generalizes to the case of affiliated value signals (Milgrom and Weber 1982). In this case, let $h(s_i, s_j)$ be the strictly positive, bounded, permutation-symmetric, and log-supermodular joint density of value signals.[14] Define $h(x|s)$ and $H(x|s)$ as the conditional p.d.f. and c.d.f., respectively, and let $\lambda(x|s) := h(x|s)/(1 - H(x|s))$. The analogue of Theorem 1 holds if $v(\cdot, x)\lambda(x|\cdot)$ is nondecreasing, a common assumption (Krishna and Morgan 1997, Lizzeri and Persico 2000, Fang and Perreiras 2002). This condition holds when, for example, $v(s_i, s_j) = (s_i + s_j)/2$ and $h(s_i, s_j) = 4(1 + s_i s_j)/5$. Intuitively, the restriction limits the informativeness of s_i concerning s_j relative to its impact on bidder i's own payoff. The supporting analysis is essentially identical to the preceding case and is omitted.

Remark 4. A closed-form expression for $\bar{b}(\cdot)$ is unavailable, but the preceding analysis suggests how to approximate it numerically. First, identify a critical point in (8) through which the function must pass. The stable manifolds of system (8) approach this point with a slope equal to that of the negative eigenvector from the system's Jacobian matrix. An approximate linearized solution can be defined locally at the critical point. This solution can be extended to the domain's remainder with standard numerical methods, thus approximating $\bar{b}(\cdot)$.

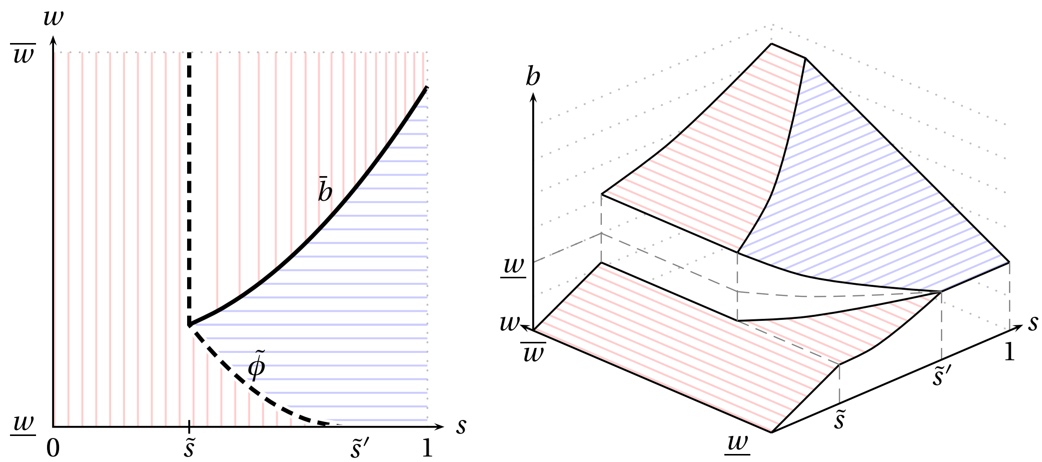
Remark 5. Allowing for $N > 2$ bidders requires amending (4), the expression for an agent's expected payoff. Defeating an opponent with a low value signal or a low budget implies different conclusions concerning the item's value. Each possibility needs to be accounted for on an opponent-by-opponent basis. Kotowski and Li (2014a) present this extension for the all-pay auction with interdependent values. The equilibrium with N bidders in that setting is qualitatively similar to the two-bidder case. Che and Gale (1998) allow for N bidders and study canonical equilibria in the private-values case.

3.2 *A noncanonical equilibrium*

A canonical equilibrium may not exist in the absence of Assumption 1 or if $\underline{w} > 0$. To investigate this possibility, we focus on the case where $\underline{w} > 0$, as the logical parallel with Example 1 is easiest to appreciate. A canonical equilibrium may not exist when $\underline{w} > 0$ because the marginal return to a slightly higher bid changes appreciably at $\underline{w}$. A bid above $\underline{w}$ defeats a budget-constrained opponent while a bid below $\underline{w}$ does not. This change in a bid's marginal return is exaggerated in Example 1 by the mass point in the budget distribution at $1/4$. As shown below, a mass point is not necessary to induce a similar jump discontinuity in the equilibrium strategy of a high-budget bidder.

Consider the strategy illustrated in Figure 4, which translates the intuition from Example 1 to a model with a continuous type space. It has three parts. First, a bidder with a low value signal bids less than $\underline{w}$. Second, a high-budget bidder increases his bid discontinuously when his value signal is $\tilde{s}$. Third, a low-budget bidder cannot match the jump in a high-budget bidder's strategy at $\tilde{s}$. Competition is stratified and low-budget bidders with value signals $s \in (\tilde{s}, \tilde{s}')$ compete only in a low range of bids. A novelty is that the distinction between high- and low-budget bidders is endogenous and changes

[14] These assumptions follow Milgrom and Weber (1982) and are standard in the literature.



(a) Level sets of $\beta(s, w)$. The discontinuity occurs along the dashed curve

(b) Three dimensional sketch of $\beta(s, w)$

FIGURE 4. Illustration of a noncanonical equilibrium strategy $\beta(s, w)$ when $\underline{w} > 0$.

with the agent's value signal. In Figure 4(a), this boundary is given by the function $\tilde{\phi}(\cdot)$. If $w < \tilde{\phi}(s)$, the agent's budget is "low" (conditional on his value signal) and he bids less than $\underline{w}$; otherwise, his budget is "high" and he bids above $\underline{w}$. An important feature of the strategy illustrated in Figure 4 is that there are no gaps in its range, despite $\beta(\cdot, w) : [0, 1] \to \mathbb{R}$ being discontinuous for each $w \in (\underline{w}, \overline{w}]$.

While an equilibrium with the above characteristics *can* arise when $\underline{w} > 0$, its presence is not assured. For some parameters, a canonical equilibrium may apply instead. To characterize all possibilities, we extend Theorem 1 to allow $\underline{w} \geq 0$. Concurrently, we also strengthen Assumption 1 (see Remark 1 above) to simplify the technical analysis.

ASSUMPTION 1′. *The c.d.f. of budget constraints $G(\cdot)$ is concave on $[\underline{w}, \overline{w}]$.*

THEOREM 2. *Suppose Assumption 1′ holds and $\underline{w} \geq 0$. There exist constants $\tilde{s}$ and $\tilde{s}'$, increasing functions $\bar{b} : [0, 1] \to [0, \overline{w}]$ and $\tilde{b} : [\tilde{s}, \tilde{s}'] \to [0, \underline{w}]$, and a nonincreasing function $\tilde{\phi} : [\tilde{s}, \tilde{s}'] \to [\underline{w}, \overline{w}]$ such that the first-price auction has a symmetric equilibrium $\beta(s, w)$ with the following properties:*

(a) For all $s \in [\tilde{s}, \tilde{s}')$ and $w < \tilde{\phi}(s)$, $\beta(s, w) = \tilde{b}(s)$. The function $\tilde{b}(\cdot)$ is the solution of

$$\tilde{b}'(s) = \big(v(s, s) - \tilde{b}(s)\big)\frac{G\big(\tilde{\phi}(s)\big)h(s)}{H(\tilde{s}) + \int_{\tilde{s}}^{s} G\big(\tilde{\phi}(y)\big)h(y)\,dy} \tag{10a}$$

that satisfies the boundary condition $\tilde{b}(\tilde{s}) = \lim_{s\to\tilde{s}^-} \bar{b}(s)$.

(b) Otherwise, $\beta(s, w) = \min\{\bar{b}(s), w\}$. For all $s < \tilde{s}$, the function $\bar{b}(\cdot)$ is continuous, less than $\underline{w}$, and is the solution of

$$\bar{b}'(s) = \big(v(s, s) - \bar{b}(s)\big)\frac{h(s)}{H(s)} \tag{10b}$$

that satisfies the boundary condition $\bar{b}(0) = 0$. *For all* $s > \tilde{s}$, *the function* $\bar{b}(\cdot)$ *is continuous, greater than* $\underline{w}$, *and for a.e.* $s > \tilde{s}$,

$$\bar{b}'(s) = \frac{\lambda(s)}{\gamma(\bar{b}(s))}\left(\frac{\bar{b}(s) - v(s, s)}{\eta(s, s) - \delta(\bar{b}(s), s)}\right). \tag{10c}$$

Remark 6. Recall that (5) characterizes the bid of an unconstrained agent. Acknowledging the similarity between (5), (10b), and (10c), and noting that $\overline{w} > \overline{v}$ (the item's maximal value), $\beta(s, \overline{w}) = \bar{b}(s)$ in the equilibrium defined in Theorem 2.

Theorem 2 allows for both canonical and noncanonical equilibria. Which case applies depends on the economy's parameters. Two qualitatively distinct canonical equilibria are defined solely by part (b). The function $\bar{b}(\cdot)$ in a *type-1 canonical equilibrium* is characterized only by (10c).[15] This class subsumes the analysis in the preceding subsection. As in the preceding subsection, the function $\bar{b}(\cdot)$ coincides with the stable solutions of system (8), the function $b^*(\cdot)$. Given Assumption 1′, a necessary and sufficient condition for a type-1 canonical equilibrium to exist when $\underline{w} \geq 0$ is that $b^*(\cdot)$ exists and has domain $[0, 1]$. Intuitively, such equilibria tend to arise when $\underline{w}$ is close to zero.

A *type-2 canonical equilibrium* is characterized by (10b) and (10c).[16] For low values of s, an agent bids less than $\underline{w}$. In particular, $\bar{b}(s) = \alpha(s)$, the equilibrium bid in the absence of budget constraints and the solution to (10b).[17] If there exists a value $s_\alpha \in (0, 1)$ such that $\alpha(s_\alpha) = \underline{w}$, then $\bar{b}(\cdot)$ makes a *continuous* transition into the range of bids above $\underline{w}$ at $\tilde{s} = s_\alpha$. For $s > \tilde{s}$, $\bar{b}'(s)$ is given by (10c). Given Assumption 1′, a necessary and sufficient condition for a type-2 canonical equilibrium is the following: if $\alpha(s_\alpha) = \underline{w}$, then $\eta(s_\alpha, s_\alpha) - \delta(\underline{w}, s_\alpha) \leq 0$. This condition must hold for (10c) to have an increasing solution at $(s_\alpha, \underline{w})$. Intuitively, type-2 canonical equilibria tend to arise when $\underline{w}$ is relatively large. See the working paper (Kotowski 2019) for an expanded account of type-2 canonical equilibria.

The third case corresponds to the noncanonical equilibrium sketched in Figure 4. In this case, both parts (a) and (b) in Theorem 2 contribute to the definition of the equilibrium strategy. In the remainder of this subsection, we outline the construction of this equilibrium. The associated proofs are relegated to Appendix C. A numerical example is presented in Section 3.3.

Construction of a noncanonical equilibrium Paralleling Section 3.1, the following discussion presumes the existence of one critical point. The absence of a critical point or multiple critical points does not affect Theorem 2, though an extended analysis is required to identify $\bar{b}(\cdot)$. See the working paper (Kotowski 2019) for analysis of these cases.

[15] In this case, $\tilde{s} = 0$ and $\tilde{\phi}(s) = \underline{w}$.

[16] In this case, $\tilde{s} \in (0, 1]$ and $\tilde{\phi}(s) = \underline{w}$.

[17] If $\alpha(s) \leq \underline{w}$ for all s, then $\tilde{s} = 1$ and a type-2 equilibrium reduces to $\beta(s, w) = \alpha(s)$. Though nominally in the model, agents' budgets are always so large that they never bind in equilibrium.

We first introduce an assumption (henceforth maintained) precluding type-1 or type-2 canonical equilibria. It assures the applicability of the noncanonical case given Assumption 1′. Recall that $b^*(\cdot)$ defines the stable solutions to system (8) and $\alpha(\cdot)$ is the first-price auction's equilibrium strategy in the absence of budget constraints.

Assumption 2. *(a) The domain of* $b^*(\cdot)$ *is* $[s_*, 1]$ *and* $s_* > 0$.

(b) There exists an $s_\alpha \in (0, 1)$ *such that* $\alpha(s_\alpha) = \underline{w}$ *and* $\eta(s_\alpha, s_\alpha) - \delta(\underline{w}, s_\alpha) > 0$.

Figure 5 illustrates a case that satisfies Assumption 2. Assumption 2(a) says that $b^*(\cdot)$ is not defined for all s.[18] Assumption 2(b) implies that (10c) does not have an increasing solution at $(s_\alpha, \underline{w})$. In the private-values case, i.e., $v(s_i, s_j) = s_i$, it holds when $(1 - H(s_\alpha)) \int_0^{s_\alpha} H(z)/H(s_\alpha)^2\, dz > 1/g(\underline{w})$. Thus, a sufficient condition for Assumption 2(b) to be satisfied is that the budget distribution is relatively concentrated near $\underline{w}$. In this light, Example 1 can be interpreted as a limiting case where $\underline{w} = 1/4$ is a mass point of the budget distribution.

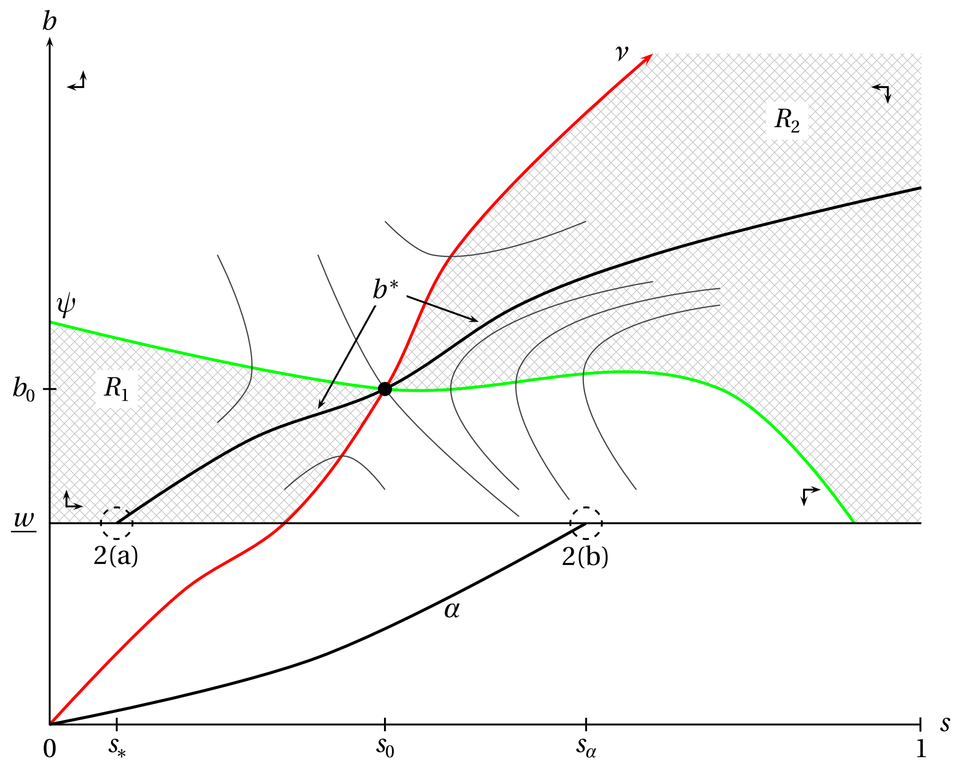
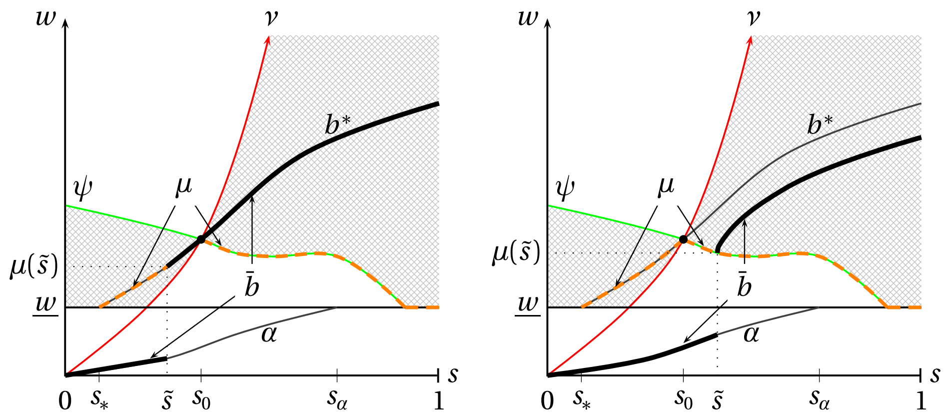


Figure 5. A portrait of system (8) when $\underline{w} > 0$. Assumptions 1′, 2(a), and 2(b) are satisfied.

[18] It can be verified that $b^*(\cdot)$ is defined near the right boundary given Assumption 1′. Thus, $s_* > 0$ is the only possible failure of the full domain assumption when there is one critical point.



(a) Definition of $\bar{b}(\cdot)$ when $\mu(\tilde{s}) = b^*(\tilde{s})$

(b) Definition of $\bar{b}(\cdot)$ when $\mu(\tilde{s}) = \psi(\tilde{s})$

FIGURE 6. Identification of $\bar{b}(\cdot)$ when $\underline{w} > 0$.

Next, we explain how to identify $\tilde{s}$, $\bar{b}(\cdot)$, $\tilde{b}(\cdot)$, and $\tilde{\phi}(\cdot)$.[19] To identify $\tilde{s}$, we first introduce the function $\mu : [s_*, 1] \to [\underline{w}, \overline{w}]$ defined as

$$\mu(s) := \begin{cases} \min\{b^*(s), \psi(s)\} & \text{if } \psi(s) \neq \varnothing \\ \min\{b^*(s), \underline{w}\} & \text{if } \psi(s) = \varnothing. \end{cases}$$

Recall that $\psi(s)$ is the zero of the function $b \mapsto \eta(s, s) - \delta(b, s)$. The function $\mu(\cdot)$ is labeled in Figures 6 and 7. Through each point along this curve passes a solution of system (8) that can be extended as an increasing, continuous function to the boundary.[20] The value $\tilde{s}$ is a solution to

$$\bar{U}\big(\alpha(\tilde{s})|\tilde{s}, w\big) = \bar{U}\big(\mu(\tilde{s})|\tilde{s}, w\big), \tag{11}$$

where

$$\bar{U}(b|s, w) := \int_0^s \big(v(s, y) - b\big)h(y)\,dy + G(b)\int_s^1 \big(v(s, y) - b\big)h(y)\,dy. \tag{12}$$

In words, a bidder with value signal $\tilde{s}$ is indifferent between the bids $\alpha(\tilde{s})$ and $\mu(\tilde{s})$. Lemma C.1 in Appendix C shows that (11) has a solution $\tilde{s} \in [s_*, s_\alpha]$ given the maintained assumptions.

Given $\tilde{s}$ that satisfies (11), $\bar{b}(\cdot)$ is defined as follows. For $s < \tilde{s}$, $\bar{b}(s)$ solves (10b). Of course, this implies $\bar{b}(s) = \alpha(s)$ for $s < \tilde{s}$. For $s \geq \tilde{s}$, $\bar{b}(s)$ coincides with the increasing solution to system (8) passing through point $(\tilde{s}, \mu(\tilde{s}))$, which assures (10c) is satisfied. Figure 6 illustrates this definition in two representative cases. In Figure 6(a), $\mu(\tilde{s}) = b^*(\tilde{s})$ and $\bar{b}(s) = b^*(s)$ for all $s > \tilde{s}$. In Figure 6(b), $\mu(\tilde{s}) = \psi(\tilde{s})$ and $\bar{b}(s) < b^*(s)$ for all $s > \tilde{s}$. There is a jump discontinuity in $\bar{b}(\cdot)$ at $\tilde{s}$.

[19] The value $\tilde{s}'$ is pinned down during the definition of $\tilde{b}(\cdot)$ and $\tilde{\phi}(\cdot)$. It is the largest value for which the domain of these functions can be defined.

[20] This solution may coincide with $b^*(s)$ and pass through the critical point.

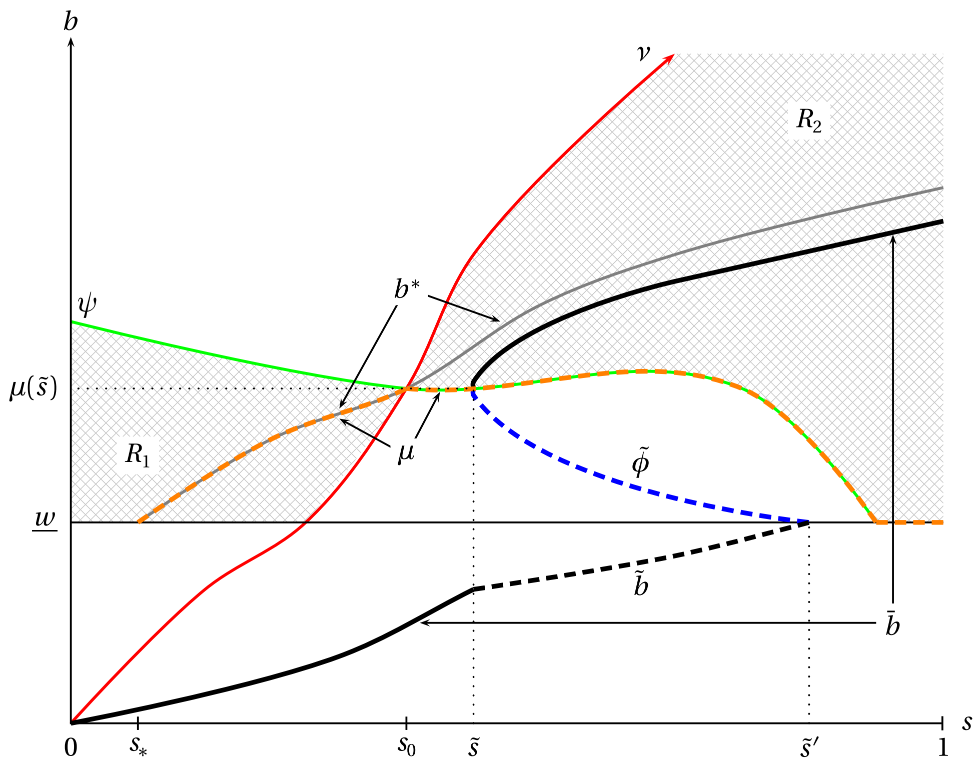


FIGURE 7. The functions $\bar{b}$, $\tilde{\phi}$, and $\tilde{b}$. The function $\bar{b}$ has a jump discontinuity at $\tilde{s}$.

Given $\tilde{s}$ and $\bar{b}(\cdot)$, we can pin down $\tilde{b}(\cdot)$ and $\tilde{\phi}(\cdot)$. The former solves (10a) subject to the boundary condition $\tilde{b}(\tilde{s}) = \lim_{s\to\tilde{s}^-} \bar{b}(s)$. Differential equation (10a) resembles (10b), but accounts for the reduced competition given the jump in a high-budget bidder's strategy. This is captured by the "$G(\tilde{\phi}(\cdot))$ weightings" in the reversed hazard rate term.

The function $\tilde{\phi}(\cdot)$ defines the boundary between high- and low-budget bidders along which $\beta(s, w)$ is discontinuous. Accordingly, it is defined by an indifference condition. A type-$(s, \tilde{\phi}(s))$ bidder is indifferent between the bids $\tilde{b}(s)$ and $\tilde{\phi}(s)$. Thus,

$$\tilde{U}_{\tilde{\phi}}\big(\tilde{b}(s)|s, w\big) = \tilde{U}_{\tilde{\phi}}\big(\tilde{\phi}(s)|s, w\big) \tag{13}$$

for all $s \in [\tilde{s}, \tilde{s}']$, where

$$\begin{aligned}\tilde{U}_{\varphi}(b|s, w) := {}& \int_0^{\tilde{s}} \big(v(s, y) - b\big)h(y)\,dy + \int_{\tilde{s}}^{s} \big(v(s, y) - b\big)G\big(\varphi(y)\big)h(y)\,dy \\ & + G(b)\int_s^1 \big(v(s, y) - b\big)h(y)\,dy.\end{aligned} \tag{14}$$

A subtle complication is that the definitions of $\tilde{b}(\cdot)$ and $\tilde{\phi}(\cdot)$ reference one another. Thus, a fixed point argument is needed to show that these functions exist with the stated properties.

To reinforce intuition for $\tilde{\phi}(\cdot)$, recall Example 1. In that case, a high-budget bidder's strategy is discontinuous in a neighborhood of $\tilde{s}$. A low-budget agent bids less than $1/4$ when $s \in (\tilde{s}, \tilde{s}')$, but his bid increases to $1/4$ once $s > \tilde{s}'$. The function $\tilde{\phi}(\cdot)$ generalizes this sequence of jump discontinuities to a continuum type space.

Verification that Theorem 2 describes an equilibrium mimics the proof of Theorem 1 and is relegated to Appendix C.

Remark 7. Absent Assumption 1′, the auction's equilibrium depends on the specifics of the model. When the function $b \mapsto \eta(s, s) - \delta(b, s)$ crosses zero multiple times, some crossings occur at values where the budget distribution is (relatively) concentrated. Around these values, a bidder has an incentive to increase his bid substantially as the marginal returns to a slightly higher bid rise significantly. Jump discontinuities around such values allow $\bar{b}(s)$, the bid of an unconstrained bidder, to "skip around" regions where (10c) lacks increasing solutions. Multiple discontinuities are possible if $G(\cdot)$ is sufficiently irregular. Despite similar intuition, identifying the analogues of $\tilde{\phi}(\cdot)$ and $\tilde{b}(\cdot)$ becomes challenging. Our proof does not extend immediately, as it relies on the concavity of $G(\cdot)$ for this step.

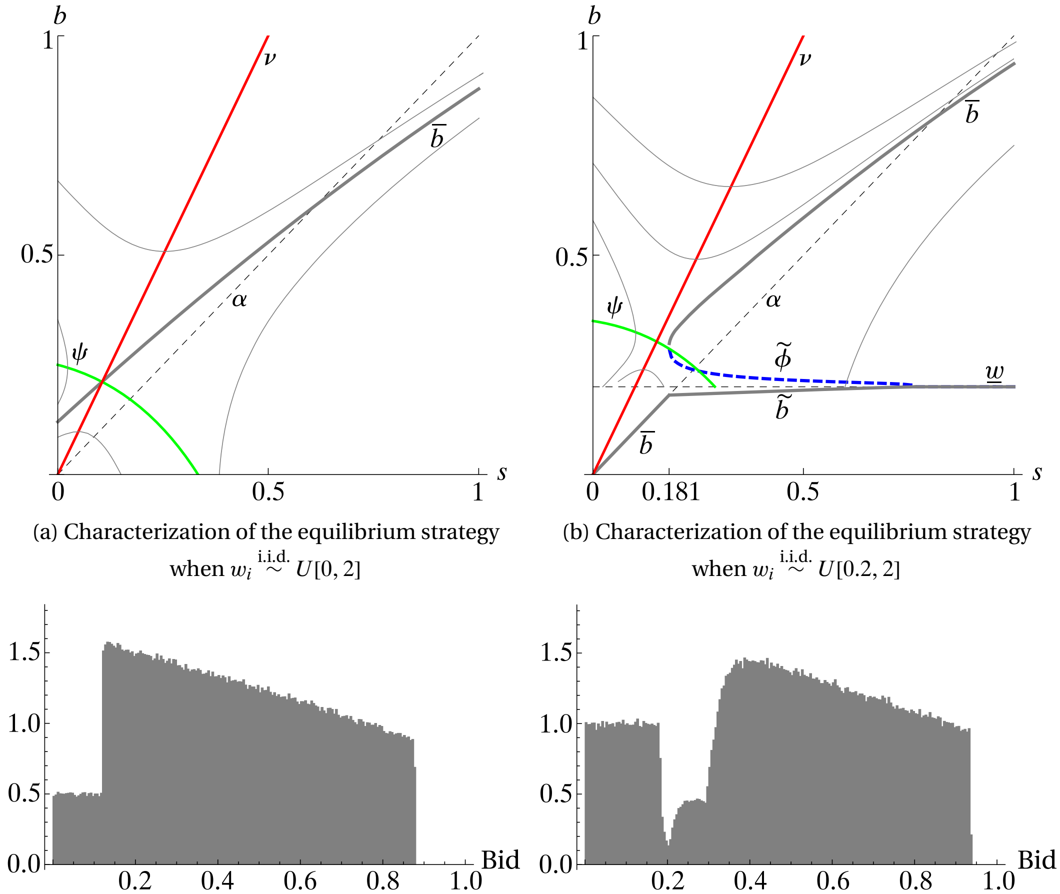
3.3 *An example*

An example can illustrate both canonical and noncanonical equilibria.

Example 2. Suppose $v(s_i, s_j) = s_i + s_j$ and value signals are uniformly distributed on the unit interval. In the absence of budget constraints, $\alpha(s) = s$ is the symmetric equilibrium strategy. Suppose budgets are distributed uniformly on $[\underline{w}, 2]$ and $\underline{w} = 0$. Figure 8(a) shows $\bar{b}(s)$, $\psi(s)$, and $\nu(s)$. There is one critical point at $s_0 \approx 0.105$. Representative solutions from the associated plane-autonomous system and $\alpha(s)$ are graphed for context. Since values are interdependent, even a bidder for whom $s = 0$ bids a positive amount ($\bar{b}(0) > 0$).

Assumption 2 holds when $0.15 < \underline{w} < 0.27$; thus, a noncanonical equilibrium exists. Figure 8(b) illustrates $\bar{b}(s)$, $\psi(s)$, $\nu(s)$, $\tilde{\phi}(s)$, and $\tilde{b}(s)$ when $\underline{w} = 0.2$. The functions $\bar{b}(s)$ and $\alpha(s)$ coincide when s is small, but $\bar{b}(s)$ jumps up at $\tilde{s} \approx 0.181$. A high-budget bidder with that value signal increases his bid from 0.181 to 0.286—an increase of 58 percent. Thus, an agent's strategic response to an opponent's possible budget constraint can be quantitatively large. When $\underline{w} > 0.27$, the equilibrium is again canonical, but of the type-2 variety. The jump discontinuity in $\bar{b}(s)$ disappears.

To highlight the empirical implications, Figure 8(c) and (d) present histograms approximating the equilibrium bid distributions. In the canonical case, the bid distribution has a peak at $\bar{b}(0) \approx 0.12$ for the following reason. In equilibrium, the bid $b < 0.12$ is placed by a type-(s, w) agent if and only if $w = b$. In contrast, the bid $b > 0.12$ is placed by a type-(s, w) agent in two contingencies. In the first, $w = b$ and $s > \bar{b}^{-1}(b)$; thus, the agent is constrained by his budget. In the second, $w > b$ and $s = \bar{b}^{-1}(b)$; thus, the agent is unconstrained but finds it not worthwhile to bid more given his private information. The peak reflects the additional fraction of types covered by the second contingency.



(a) Characterization of the equilibrium strategy when $w_i \overset{\text{i.i.d.}}{\sim} U[0, 2]$

(b) Characterization of the equilibrium strategy when $w_i \overset{\text{i.i.d.}}{\sim} U[0.2, 2]$

(c) Distribution of equilibrium bids when $w_i \overset{\text{i.i.d.}}{\sim} U[0, 2]$. Histogram of 1 million simulated bids (rescaled to unit area)

(d) Distribution of equilibrium bids when $w_i \overset{\text{i.i.d.}}{\sim} U[0.2, 2]$. Histogram of 1 million simulated bids (rescaled to unit area)

Figure 8. Equilibrium bidding in Example 2.

In the noncanonical case, the equilibrium bid distribution is irregular, with a valley between the bids 0.181 and 0.286. No agent bidding less than 0.181 in this example is constrained by his budget. However, it is not in his interest to bid more. Since few bids are placed between 0.181 and 0.286, the probability of winning does not increase commensurably with the payment conditional on winning with a bid in this range. Additionally, a bid above 0.286 is unattractive because the agent's value signal is relatively low. Conversely, no agent bidding above 0.286 wishes to bid less. Slightly lower bids defeat only a budget constrained opponent (who wishes to bid more than he can). In addition, bids below 0.181 are unattractive to a high value signal bidder. The item is sufficiently valuable to merit a higher bid. The indifference conditions (11) and (13) ensure that each type's incentives are balanced in equilibrium. The equilibrium bid distribution would be uniform on [0, 1] in the absence of budget constraints. ◊

4. Discussion

4.1 *The second-price auction*

It is instructive to compare equilibrium bidding in first- and second-price auctions. In the case of private values, i.e., $v(s_i, s_j) = s_i$, the second-price auction has an equilibrium where each type-(s, w) agent bids $\min\{s, w\}$ (Che and Gale 1998). Fang and Perreiras (2002) examine the second-price auction with interdependent values in a two-bidder setting. They identify an equilibrium where each bidder's strategy is $\beta_{II}(s, w) = \min\{\bar{b}_{II}(s), w\}$ and $\bar{b}_{II}(\cdot)$ is an increasing function. When $\bar{b}_{II}(s) < \underline{w}$, then $\bar{b}_{II}(s) = v(s, s)$, which is the equilibrium strategy identified by Milgrom and Weber (1982) in the absence of budget constraints. When $\bar{b}_{II}(s) \geq \underline{w}$, then $\bar{b}_{II}(s)$ solves the differential equation

$$\bar{b}'_{II}(s) = \frac{\lambda(s)}{\gamma(\bar{b}_{II}(s))}\left(\frac{\bar{b}_{II}(s) - v(s, s)}{\eta(s, s) - \bar{b}_{II}(s)}\right) \tag{15}$$

subject to the boundary condition $\bar{b}_{II}(1) = \overline{v}$.[21] There exists a value $\tilde{s}_{II}$ such that $\bar{b}_{II}(s) < \underline{w} \iff s < \tilde{s}_{II}$ and, generally, $\bar{b}_{II}(\cdot)$ may be discontinuous at $\tilde{s}_{II}$.

While the resemblance of (7) and (15) is striking, three differences between the first- and second-price auctions are noteworthy. First, the boundary condition for $\bar{b}_{II}(s)$ is predetermined. Second, the term $\eta(s, s) - b$ appears in the denominator of (15) instead of $\eta(s, s) - \delta(b, s)$. Third, the discontinuity that can arise in the second-price auction is qualitatively different from that identified in the first-price auction. There is no separation among high- and low-budget bidders in the second-price auction. A bidder with value signal $s > \tilde{s}_{II}$ bids above $\underline{w}$ and a gap in the equilibrium bid distribution may arise in the second-price auction. In contrast, the equilibrium bid distribution in the first-price auction has connected support, even in a noncanonical equilibrium. This difference is due to the distinct pricing mechanisms. In the second-price auction, an agent bidding infinitesimally above a gap has no reason to reduce his bid. His probability of winning and his expected payment are unchanged if he bids a bit less. In the first-price auction, an agent bidding infinitesimally above a (hypothetical) gap can significantly reduce his expected payment without reducing his probability of winning. This implies that a gap in the equilibrium bid distribution cannot arise.[22]

Che and Gale (1998, 2006) show that the first-price auction revenue-dominates the second-price auction in an independent private-values setting with financial constraints. It is straightforward to construct examples where this revenue ranking continues to hold when valuations are interdependent but agents' types are independent.[23]

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[21] Equation (15) restates equation (11) from Fang and Perreiras (2002). We specialize it to the case of independent types. Fang and Perreiras (2002) allow for affiliated values.

[22] This reasoning does not apply when there are atoms in the equilibrium bid distribution, as in Example 1. In the example, the bid 1/4 is placed with positive probability in equilibrium. An agent bidding 1/4 would like to bid more but cannot due to his budget constraint. A bid slightly less than 1/4, in contrast, leads to a *discrete* fall in the probability of winning and is, therefore, not beneficial.

[23] In Example 2 when $w_i \overset{\text{i.i.d.}}{\sim} U[0, 2]$, the expected revenue in the first-price auction is approximately 0.57; it is around 0.48 in the second-price auction. When $w_i \overset{\text{i.i.d.}}{\sim} U[0.2, 2]$, these values are 0.64 and 0.58, respectively. In the absence of budget constraints, the first- and second-price auctions are revenue-equivalent in this example. The expected revenue is 2/3.

We conjecture that the ranking applies generally when types are independent, though a proof of this claim is unavailable. If agents' types are affiliated, Milgrom and Weber (1982) prove that the second-price auction is revenue-superior in the absence of budget constraints. Taken together, the preceding observations imply that no revenue ranking exists between first- and second-price auctions that allows for both budget constraints and affiliated values.

4.2 *Equilibrium nonuniqueness*

McAdams (2007) shows that the first-price auction has a unique monotone pure strategy equilibrium when bidders are ex ante symmetric, types are affiliated, and values are interdependent. This conclusion does not extend to a setting with budget constraints. Recall that in a noncanonical equilibrium, an agent discontinuously increases his bid at some value signal $\tilde{s}$ and he conjectures the other bidder does likewise. Of course, this conjecture is correct in equilibrium, but the construction introduces an implicit coordination problem. Different values at which the common bidding strategy "jumps up" may be compatible with different equilibria, as illustrated by the following example.

Example 3. Consider a first-price auction with two bidders. Given (s_i, s_j), each bidder's valuation is $v(s_i, s_j) = s_i + s_j$. Each bidder's value signal is independently distributed according to the c.d.f. $H(s) = \sqrt{s}$ on $[0, 1]$. The symmetric equilibrium bidding strategy in the absence of budget constraints is $\alpha(s) = 2s/3$.

Now suppose bidders face a common budget constraint: $w_i = w_j = 1/3$ with probability 1.[24] There are now two symmetric equilibria with strategies

$$\beta_A(s) = \begin{cases} 2s/3 & \text{if } s \leq 1/9 \\ 1/3 & \text{if } s > 1/9 \end{cases} \quad \text{and} \quad \beta_B(s) = 1/3.$$

We confirm that the stated strategies define equilibria in Appendix D. In the β_A equilibrium, the agents adopt the no-budget-constraints equilibrium strategy when s is small. At $\tilde{s}_A = 1/9$ their bids increase discontinuously from $2/27$ to $1/3$. In the β_B equilibrium, the threshold value signal above which agents bid their entire budget is $\tilde{s}_B = 0$. ◊

4.3 *Comparative statics*

The countervailing incentives associated with budget constraints typically lead to ambiguous comparative statics. For example, consider a "tightening" of financial constraints where the budget distribution is skewed in the sense of likelihood ratio dominance toward a distribution favoring lower budgets.[25] In the second-price auction, this change inflates the bid of an unconstrained bidder (Fang and Perreiras 2002). In the first-price auction, an unconstrained bidder's response may go either way. There is now

[24]We interpret this example as a limiting case of the more general model developed above. Posit, for instance, that $\underline{w} = 1/3$ is the lower limit of the budget distribution's support and its density becomes highly concentrated around this value.

[25]Likelihood ratio dominance implies hazard-rate dominance and first-order stochastic dominance.

less incentive to bid aggressively because competition is reduced at higher bid levels. However, if budgets become more concentrated around particular values, it is easier to exploit the budget dimension for strategic gain. On balance, a bidder may respond with a higher or lower equilibrium bid. Example D.1 in Appendix D documents these phenomena. The comparative static relies on identifying the critical point under various scenarios. Since $\nu(s)$ is unaffected by changes in the budget distribution, movement of the critical point along $\nu(s)$ lets us infer how $\bar{b}(s)$ shifts up and down (locally) in response to changes in the budget distribution.

4.4 *Reserve prices*

Setting the optimal reserve price with budget-constrained bidders is challenging. Reserve prices screen bidders on both value and budget dimensions. Expected revenues may fall if bidders are disproportionately screened along the latter, as the probability of sale is particularly impacted. If agents are unlikely to be budget constrained, imposing the no-budget-constraints revenue-maximizing reserve price can be a sensible policy. For example, in the model of Example 1, if an agent has a low budget ($w = 1/4$) with probability $p < 0.317$, the revenue-maximizing reserve price is $1/2$. This is also the revenue-maximizing reserve price in the absence of budget constraints. Otherwise, the optimal reserve price is less than $1/4$.

5. Concluding remarks

We have examined the first-price auction under the assumption that agents face private budget constraints. Our analysis confirms that private budgets influence bidding in several distinct ways. They cap participants' bids while simultaneously encouraging more aggressive bidding in equilibrium by some agents. Together, these responses can lead to nontrivial equilibrium outcomes. For example, agents may adopt discontinuous bidding strategies; multiple equilibria are possible too. Two directions for further study are especially noteworthy. First, we have precluded many natural embellishments from our model such as resale, collusion, credit and financing, among others. As noted in Section 2, several studies have probed these questions, but much work remains. Second, the empirical implications of private budgets in auctions remain poorly understood. The development of new models or identification strategies to tackle such questions also seems especially promising.

Appendix A: Example 1 (calculations)

In this section, we verify that the function $\beta(s, w)$ defined as

$$\beta(s,1/4)=\begin{cases} s/2 & \text{if } s\in[0,1/3]\\ \dfrac{1+9s^2}{6+18s} & \text{if } s\in(1/3,11/27]\\ 1/4 & \text{if } s\in(11/27,1]\end{cases} \qquad \beta(s,3/4)=\begin{cases} s/2 & \text{if } s\in[0,1/3]\\ \dfrac{5+9s^2}{18+18s} & \text{if } s\in(1/3,1]\end{cases}$$

is a symmetric equilibrium in Example 1. Noting the symmetry of the example, we suppress agent subscripts. Recall that bidders' value signals are distributed uniformly on the unit interval and budgets assume values of 1/4 and 3/4 with equal probability.

We proceed to rule out deviations from $\beta(s, w)$ by each type of bidder given that his opponent adopts $\beta(s, w)$ as his strategy. It is sufficient to rule out deviations to other bids in the range of $\beta(s, w)$. To simplify notation, let $\overline{\beta}(s) := \beta(s, 3/4)$ and $\underline{\beta}(s) := \beta(s, 1/4)$. Similarly, let $\overline{U}(b|s)$ be the expected utility of a type-$(s, 3/4)$ bidder when he bids b and the other bidder follows the strategy defined above. Likewise, let $\underline{U}(b|s)$ be the expected utility of a type-$(s, 1/4)$ bidder when he bids b and the other bidder follows the strategy defined above.

Case 1. Consider a bidder with value signal $s \in [0, 1/3]$ and budget $w = 3/4$. Given that the other agent is bidding according to the strategy $\beta(s, w)$, the bid $\overline{\beta}(s) = s/2$ wins the auction with probability s. Therefore, $\overline{U}(\overline{\beta}(s)|s) = s(s - s/2) = s^2/2$. There are four classes of alternative bids.

(a) A deviation to $\overline{\beta}(x)$, where $x \in [0, 1/3]$, yields an expected payoff of $\overline{U}(\overline{\beta}(x)|s) = x(s - x/2)$. The net gain is $\overline{U}(\overline{\beta}(x)|s) - \overline{U}(\overline{\beta}(s)|s) = -(x - s)^2/2 \leq 0$. Therefore, $\overline{\beta}(x)$, where $x \in [0, 1/3]$, is not a profitable deviation.

(b) A deviation to $\underline{\beta}(x)$, where $x \in (1/3, 11/27]$, yields an expected payoff of

$$\overline{U}\big(\underline{\beta}(x)|s\big) = \left(\frac{1}{6} + \frac{x}{2}\right)\left(s - \frac{1 + 9x^2}{6 + 18x}\right).$$

Observe that $d\overline{U}(\underline{\beta}(x)|s)/dx = (s - x)/2$, which is negative for all $s \leq 1/3$ and $x \in (1/3, 11/27]$. Therefore, the most profitable bid in this range for an agent with value signal $s \leq 1/3$ is $\lim_{x \to 1/3^+} \underline{\beta}(x) = 1/6 = \overline{\beta}(1/3)$. By part (a) above, this bid is not a profitable deviation.

(c) A deviation to $\overline{\beta}(x)$, where $x \in (1/3, 1]$, yields an expected payoff of

$$\overline{U}\big(\overline{\beta}(x)|s\big) = \left(\frac{1}{2} + \frac{x}{2}\right)\left(s - \frac{5 + 9x^2}{18 + 18x}\right).$$

Observe that $d\overline{U}(\overline{\beta}(x)|s)/dx = (s - x)/2 \leq 0$ since $s \leq 1/3 \leq x$. Therefore, the most profitable bid in this range for an agent with value signal $s \leq 1/3$ is infinitesimally above 1/4, i.e., $\lim_{x \to 1/3^+} \overline{\beta}(x) = 1/4$. However,

$$\lim_{x \to 1/3^+} \overline{U}\big(\overline{\beta}(x)|s\big) - \overline{U}\big(\overline{\beta}(s)|s\big) = \frac{2}{3}\left(s - \frac{1}{4}\right) - \frac{s^2}{2} \leq 0$$

for all $s \leq 1/3$. Thus, $\overline{\beta}(x)$, where $x \in (1/3, 1]$, is not a profitable deviation.

(d) The bid 1/4 cannot be a profitable deviation for a high-budget agent. Given $\beta(s, w)$, the bid 1/4 is placed with strictly positive probability by a low-budget agent. However, a high-budget agent can strictly improve upon this bid by bidding infinitesimally more. The slightly higher bid increases his probability of winning by a discrete amount relative to the bid $1/4$, which may result in a tie with positive probability. However, subcase (c) immediately above showed that a bid above $1/4$ is not a profitable deviation for a bidder with value signal $s \le 1/3$.

Case 2. Consider a bidder with value signal $s \in (1/3, 1]$ and budget $w = 3/4$. When he bids $\overline{\beta}(s) = (5 + 9s^2)/(18 + 18s)$, he defeats a low-budget opponent with probability 1 and a high-budget opponent with probability s. Therefore, his expected payoff is

$$\overline{U}\big(\overline{\beta}(s)|s\big) = \left(\frac{1}{2} + \frac{s}{2}\right)\left(s - \frac{5 + 9s^2}{18 + 18s}\right).$$

There are four classes of alternative bids.

(a) A deviation to $\overline{\beta}(x)$, where $x \in (1/3, 1]$, yields an expected payoff of

$$\overline{U}\big(\overline{\beta}(x)|s\big) = \left(\frac{1}{2} + \frac{x}{2}\right)\left(s - \frac{5 + 9x^2}{18 + 18x}\right).$$

The net gain is $\overline{U}(\overline{\beta}(x)|s) - \overline{U}(\overline{\beta}(s)|s) = -(s - x)^2/4 \le 0$. Therefore, $\overline{\beta}(x)$, where $x \in (1/3, 1]$, is not a profitable deviation.

(b) A deviation to $\overline{\beta}(x) = x/2$, where $x \le 1/3$, yields an expected payoff of $\overline{U}(\overline{\beta}(x)|s) = x(s - x/2)$. Since $x \le s$, $\overline{U}(\overline{\beta}(x)|s)$ is increasing in x. Thus, $\overline{U}(\overline{\beta}(x)|s) \le \overline{U}(\overline{\beta}(1/3)|s)$. However, $\overline{U}(\overline{\beta}(1/3)|s) - \overline{U}(\overline{\beta}(s)|s) = (1 - 2s - 3s^2)/12 \le 0$ for all $s > 1/3$. Therefore, $\overline{\beta}(x)$, where $x \le 1/3$, is not a profitable deviation.

(c) A deviation to $\underline{\beta}(x)$, where $x \in (1/3, 11/27]$, yields an expected payoff of

$$\overline{U}\big(\underline{\beta}(x)|s\big) = \left(\frac{1}{6} + \frac{x}{2}\right)\left(s - \frac{1 + 9x^2}{6 + 18x}\right).$$

The net gain is $\Delta(x|s) := \overline{U}(\underline{\beta}(x)|s) - \overline{U}(\overline{\beta}(s)|s) = (4 - 3s(4 + 3s) + 18sx - 9x^2)/36$. Note that $d\Delta(x|s)/ds = (-2 - 3s + 3x)/6 \le 0$. Thus, $\Delta(x|s) \le \Delta(x|1/3) = -(1 - 3x)^2/36 \le 0$. Therefore, $\underline{\beta}(x)$, where $x \in (1/3, 11/27]$, is not a profitable deviation.

(d) A deviation to the bid $1/4$ can be strictly improved upon by a bid infinitesimally above $1/4$. However, that bid was shown to be unprofitable in subcase (a) above.

Case 3. Consider a bidder with value signal $s \in [0, 1/3]$ and budget $w = 1/4$. When he bids $\underline{\beta}(s) = s/2$, his expected payoff is $\underline{U}(\underline{\beta}(s)|s) = s^2/2$. Cases 1(a) and 1(b) confirm that this bidder has no profitable deviation to any other feasible bid strictly less than $1/4$. If this agent bids $1/4$, he defeats a high-budget opponent only if the opponent's value signal is less than $1/3$. He defeats a low-budget opponent with probability 1 if the opponent's value signal is less than $11/27$ and with probability $1/2$ if the opponent's value is more than $11/27$. Thus, the agent's expected payoff is

$$\underline{U}(1/4|s)=\left(\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\left(\frac{11}{27}+\frac{1}{2}\times\frac{16}{27}\right)\right)\left(s-\frac{1}{4}\right)=\frac{14}{27}\left(s-\frac{1}{4}\right).$$

It is simple to verify that $\underline{U}(\underline{\beta}(s)|s)\geq\underline{U}(1/4|s)$ for all $s\leq 1/3$. Therefore, the bid $1/4$ is not a profitable deviation.

Case 4. Consider a bidder with value signal $s\in(1/3,11/27]$ and budget $w=1/4$. When he bids $\underline{\beta}(s)=(1+9s^2)/(6+18s)$, his expected payoff is

$$\underline{U}\big(\underline{\beta}(s)|s\big)=\left(\frac{1}{6}+\frac{s}{2}\right)\left(s-\frac{1+9s^2}{6+18s}\right).$$

There are three classes of alternative bids.

(a) A deviation to $\underline{\beta}(x)=x/2$, where $x\leq 1/3$, yields an expected payoff of $\underline{U}(\underline{\beta}(x)|s)=x(s-x/2)$, which is increasing in x since $s>x$. Thus, $\underline{U}(\underline{\beta}(x)|s)\leq\underline{U}(\underline{\beta}(1/3)|s)$. However, $\underline{U}(\underline{\beta}(1/3)|s)-\underline{U}(\underline{\beta}(s)|s)=-(1-3s)^2/36\leq 0$. Therefore, the bid $\underline{\beta}(x)$, where $x\leq 1/3$, is not a profitable deviation.

(b) A deviation to $\underline{\beta}(x)$, where $x\in(1/3,11/27]$, yields an expected payoff of

$$\underline{U}\big(\underline{\beta}(x)|s\big)=\left(\frac{1}{6}+\frac{x}{2}\right)\left(s-\frac{1+9x^2}{6+18x}\right).$$

The net gain is $\underline{U}(\underline{\beta}(x)|s)-\underline{U}(\underline{\beta}(s)|s)=-(s-x)^2/4\leq 0$. Therefore, the bid $\underline{\beta}(x)$, where $x\in(1/3,11/27]$, is not a profitable deviation.

(c) A deviation to the bid $1/4$ yields an expected payoff of

$$\underline{U}(1/4|s)=\frac{14}{27}\left(s-\frac{1}{4}\right).$$

The net gain from this bid is $\underline{U}(1/4|s)-\underline{U}(\underline{\beta}(s)|s)=(1-s)(27s-11)/108\leq 0$ for all $s\in(1/3,11/27]$. Therefore, the bid $1/4$ is not a profitable deviation.

Case 5. Consider a bidder with value signal $s\in(11/27,1]$ and budget $w=1/4$. When he bids $\underline{\beta}(s)=1/4$, his expected payoff is

$$\underline{U}\big(\underline{\beta}(s)|s\big)=\frac{14}{27}\left(s-\frac{1}{4}\right).$$

There are two classes of alternative bids.

(a) A deviation to $\underline{\beta}(x)$, where $x \in (1/3, 11/27]$, yields an expected payoff of

$$\underline{U}(\underline{\beta}(x)|s) = \left(\frac{1}{6} + \frac{x}{2}\right)\left(s - \frac{1+9x^2}{6+18x}\right),$$

which is increasing in x because $x \leq 11/27 < s$. However, $\underline{U}(\underline{\beta}(11/27)|s) - \underline{U}(1/4|s) = 44/729 - 4s/27 \leq 0$ for all $s > 11/27$. Thus, $\underline{U}(\underline{\beta}(x)|s) \leq \underline{U}(\underline{\beta}(11/27)|s) \leq \underline{U}(1/4|s)$. Hence, the bid $\underline{\beta}(x)$, where $x \in (1/3, 11/27]$, is not a profitable deviation.

(b) A deviation to $\underline{\beta}(x) = x/2$, where $x \leq 1/3$, yields an expected payoff of $\underline{U}(\underline{\beta}(x)|s) = x(s - x/2)$. Since $s > x$, this expression is increasing in x. However, $\underline{U}(\underline{\beta}(1/3)|s) - \underline{U}(1/4|s) = (2 - 5s)/27 \leq 0$ for all $s > 11/27$. Thus, $\underline{U}(\underline{\beta}(x)|s) \leq \underline{U}(\underline{\beta}(1/3)|s) \leq \underline{U}(1/4|s)$. Hence, the bid $\underline{\beta}(x)$, where $x \leq 1/3$, is not a profitable deviation.

Appendix B: Lemmas related to Section 3.1

Lemma B.1. *Suppose Assumption 1 holds.*

(a) If $\psi(s) \neq \varnothing$, then $\psi(s) < \overline{v}$.

(b) Let $\varepsilon > 0$. If $\psi(s') \neq \varnothing$ and $\psi(s) = \varnothing$ for all $s \in (s', s' + \varepsilon)$, then $\psi(s') = \underline{w}$.

(c) Let $\varepsilon > 0$. If $\psi(s') \neq \varnothing$ and $\psi(s) = \varnothing$ for all $s \in (s' - \varepsilon, s')$, then $\psi(s') = \underline{w}$.

(d) If $\underline{w} = 0$, then $\psi(0) \neq \varnothing$.

Proof. Part (a) is immediate since $\delta(\overline{w}, s) > \overline{v} \geq \eta(s, s)$. The proofs of (b) and (c) are similar, so we only prove (b). To derive a contradiction, suppose $\psi(s') = \hat{b} > \underline{w}$. By Assumption 1, there exists $\hat{b}' \in (\underline{w}, \hat{b})$ such that $\eta(s', s') - \delta(\hat{b}', s') > 0$. By continuity, there exists $\hat{s}' \in (s', s' + \varepsilon)$ such that $\eta(s', s') - \delta(\hat{b}', \hat{s}') > 0$. Since $\delta(\overline{w}, \hat{s}') > \eta(\hat{s}', \hat{s}')$, we conclude that $\psi(s') \neq \varnothing$ by the intermediate value theorem—a contradiction. Part (d) is implied by Assumption 1 and the fact that $\eta(0, 0) - \delta(0, 0) \geq 0$. □

Lemma B.2. *Suppose Assumption 1 holds. If $\underline{w} = 0$, then system (8) has at least one critical point.*

Proof. Recall that $\nu(0) = 0$ and $\nu(1) = \overline{v}$. By Lemma B.1(d), $\psi(0) \neq \varnothing$. If $\psi(0) = 0$, then a critical point occurs at the origin. Instead, suppose $\psi(0) > 0$. There are two cases. First, if $\psi(s) \neq \varnothing$ for all s, then by the implicit function theorem, $\psi(s)$ is continuous. Since $\psi(s) \leq \overline{v}$, the intermediate value theorem implies that $\psi(s_0) = \nu(s_0)$ at some $s_0 \in [0, 1]$. Second, if $\psi(s') = \varnothing$ for some s', then there exists an $s'' < s'$ such that $\psi(s)$ is a continuous function on the interval $[0, s'']$ and $\psi(s'') = \underline{w} = 0$. By the intermediate value theorem, there exists some $s_0 \leq s''$ such that $\psi(s_0) = \nu(s_0)$. □

Lemma B.3. *Suppose Assumption 1 holds. If (s_0, b_0) is a critical point of system (8), then it is (generically) either a node or a saddle point.*

Proof. We first show that the eigenvalues of the Jacobian matrix evaluated at a critical point (s_0, b_0),

$$J = \left.\begin{pmatrix} \frac{\partial \dot{s}}{\partial s} & \frac{\partial \dot{s}}{\partial b} \\ \frac{\partial \dot{b}}{\partial s} & \frac{\partial \dot{b}}{\partial b} \end{pmatrix}\right|_{(s_0,b_0)},$$

are real-valued. The eigenvalues will be real if

$$\left(\frac{\partial \dot{s}}{\partial s} - \frac{\partial \dot{b}}{\partial b}\right)^2 + 4\frac{\partial \dot{s}}{\partial b}\frac{\partial \dot{b}}{\partial s}$$

when evaluated at (s_0, b_0) is nonnegative. Thus,

$$\left.\frac{\partial \dot{s}}{\partial b}\right|_{(s_0,b_0)} = \gamma'(b_0)\underbrace{\left(\eta(s_0, s_0) - \delta(b_0, s_0)\right)}_{0} + \gamma(b_0)\left(-\left.\frac{\partial \delta}{\partial b}\right|_{(s_0,b_0)}\right) = -\gamma(b_0)\left.\frac{\partial \delta}{\partial b}\right|_{(s_0,b_0)}.$$

We know that $\gamma(b_0) \neq 0$ and Assumption 1 implies that $\frac{\partial \delta}{\partial b}|_{(s_0,b_0)} > 0$. Hence, $\frac{\partial \dot{s}}{\partial b}|_{(s_0,b_0)} < 0$. Also,

$$\left.\frac{\partial \dot{b}}{\partial s}\right|_{(s_0,b_0)} = \left.\frac{\partial \lambda}{\partial s}\right|_{(s_0,b_0)}\underbrace{\left(b_0 - v(s_0, s_0)\right)}_{0} + \lambda(s_0)\left(-\left.\frac{dv(s,s)}{ds}\right|_{s=s_0}\right) < 0.$$

Therefore, $4\frac{\partial \dot{s}}{\partial b}\frac{\partial \dot{b}}{\partial s} > 0$ as needed.

Since the eigenvalues of J are real, by the Hartman–Grobman theorem, the critical point will be a saddle or a node if the eigenvalues are nonzero. This is equivalent to J having a nonzero determinant at (s_0, b_0):

$$\det J|_{(s_0,b_0)} = \gamma(b_0)\lambda(s_0)\left(\left.\frac{d\eta(s,s)}{ds}\right|_{s=s_0} - \left.\frac{\partial \delta(b_0,s)}{\partial s}\right|_{s=s_0} - \left.\frac{\partial \delta(b,s_0)}{\partial b}\right|_{b=b_0}\left.\frac{d\nu(s)}{ds}\right|_{s=s_0}\right).$$

A perturbation of any of the distributions or of the valuation function is sufficient to ensure that the preceding expression is nonzero. □

Appendix C: Proof of Theorem 2

The proof of Theorem 2 has two parts. The first part concerns the construction of a noncanonical strategy and complements Section 3.2. Construction of a type-1 canonical equilibrium strategy proceeds as in Section 3.1. Type-2 canonical equilibrium strategies are discussed in the working paper (Kotowski 2019). The second part verifies that the proposed strategy is a symmetric equilibrium.

C.1 *Construction of a noncanonical equilibrium strategy*

Outline. The construction of a noncanonical equilibrium strategy with a discontinuity proceeds in several steps. First, Lemma C.1 confirms the existence of the value $\tilde{s}$ introduced in (11). Given $\tilde{s}$, the definition of $\bar{b}(\cdot)$ is provided in Section 3.2 for the case of one critical point. The cases of no critical points or multiple critical points are outlined in the working paper (Kotowski 2019). Then we consider $\tilde{b}(\cdot)$ and $\tilde{\phi}(\cdot)$. These functions are defined with reference to a fixed point of a particular operator, which is introduced in Remark C.4. Definitions C.1–C.4 and Lemma C.2 introduce preliminary concepts used in this definition. Subsequently, Lemmas C.3–C.6 prove some properties of $\tilde{\phi}(\cdot)$. Throughout we assume that Assumptions 1′ and 2 hold.

Lemma C.1. *Consider $\bar{U}(b|\tilde{s}, w)$ defined in (12). There exists $\tilde{s} \in [s_*, s_\alpha]$ such that $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\mu(\tilde{s})|\tilde{s}, w)$. Moreover, $\mu(\tilde{s}) > \underline{w}$.*

Proof. Recall that $G(b) = 0$ if $b < \underline{w}$ and $\mu(s_*) = \underline{w}$. We make two preliminary observations.

(i) Since $G(\alpha(s_*)) = G(\mu(s_*)) = 0$ and $\alpha(s_*) \leq \mu(s_*)$, the inequality $\bar{U}(\alpha(s_*)|s_*, w) \geq \bar{U}(\mu(s_*)|s_*, w)$ follows immediately.

(ii) Assumptions 1′ and 2 imply that $\partial\bar{U}(b|s_\alpha, w)/\partial b = g(b)(1 - H(s_\alpha))(\eta(s_\alpha, s_\alpha) - \delta(b, s_\alpha)) \geq 0$ for all $b \in [\alpha(s_\alpha), \mu(s_\alpha)]$. Hence, $\bar{U}(\alpha(s_\alpha)|s_\alpha, w) \leq \bar{U}(\mu(s_\alpha)|s_\alpha, w)$.

Since $\bar{U}(\alpha(s)|s, w)$ and $\bar{U}(\mu(s)|s, w)$ are continuous functions of s, observations (i) and (ii) imply that there exists $\tilde{s} \in [s_*, s_\alpha]$ such that $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\mu(\tilde{s})|\tilde{s}, w)$. Finally, if $\mu(\tilde{s}) = \underline{w}$, then $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\mu(\tilde{s})|\tilde{s}, w)$ only if $\alpha(\tilde{s}) = \underline{w}$, which implies $\tilde{s} = s_\alpha$. But if $\underline{w} = \mu(s_\alpha) = \alpha(s_\alpha)$, then Assumption 2(b) implies that $\eta(\tilde{s}, \tilde{s}) - \delta(\underline{w}, \tilde{s}) > 0$ while the definition of $\mu(\cdot)$ implies that $\eta(\tilde{s}, \tilde{s}) - \delta(\underline{w}, \tilde{s}) \leq 0$—a contradiction. Hence, $\mu(\tilde{s}) > \underline{w}$. $\square$

Remark C.1. Henceforth, let $\tilde{s} \in [s_*, s_\alpha]$ be a fixed value such that $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\mu(\tilde{s})|\tilde{s}, w)$. To simplify notation, let $\tilde{\mu} := \mu(\tilde{s})$.

Remark C.2. Definitions C.1 and C.2, which follow below, reference the constants $K_1 := \overline{v} \cdot [\sup_{s \geq \tilde{s}} h(s)]/H(\tilde{s})$ and $\tilde{s}_1 := \tilde{s} + (\underline{w} - \alpha(\tilde{s}))/K_1$.

Definition C.1 (The set $\mathcal{B}$). Let $\mathcal{B}$ be the set of all continuous, nondecreasing functions with domain $[\tilde{s}, \tilde{s}_1]$ such that $b(\tilde{s}) = \alpha(\tilde{s})$ and $\forall s, s' \in [\tilde{s}, \tilde{s}_1]$ and $b(\cdot) \in \mathcal{B}$, $|b(s) - b(s')| \leq K_1|s - s'|$.

Definition C.2 (The set $\mathcal{F}$). Let $\mathcal{F}$ be the set of all continuous, nondecreasing functions with domain $[\tilde{s}, \tilde{s}_1]$ such that $F(\tilde{s}) = H(\tilde{s})$ and $\forall s, s' \in [\tilde{s}, \tilde{s}_1]$ and $F(\cdot) \in \mathcal{F}$, $|F(s) - F(s')| \leq (\sup_{s \in [0,1]} h(s))|s - s'|$.

Definition C.3 (The function Δ). For $b(\cdot) \in \mathcal{B}$ and $F(\cdot) \in \mathcal{F}$, let

$$\begin{aligned}\Delta(s, w|b, F) := {}&\big[G(w)\big(1 - H(s)\big)\big(\eta(s, s) - w\big) - wF(s)\big]\\ &- \big[G\big(b(s)\big)\big(1 - H(s)\big)\big(\eta(s, s) - b(s)\big) - b(s)F(s)\big].\end{aligned}$$

Lemma C.2. *Fix $b(\cdot) \in \mathcal{B}$ and $F(\cdot) \in \mathcal{F}$.*

(a) The function $w \mapsto \Delta(s, w|b, F)$ *is strictly concave on* $[\underline{w}, \overline{w}]$.

(b) If $b(s) < \underline{w}$*, then* $\Delta(s, \underline{w}|b, F) < 0$.

(c) We have $\Delta(\tilde{s}, \tilde{\mu}|b, F) = 0$ *and* $\partial\Delta(\tilde{s}, w|b, F)/\partial w|_{w=\tilde{\mu}} \geq 0$.

Proof. Fix $b(\cdot) \in \mathcal{B}$ and $F(\cdot) \in \mathcal{F}$. To simplify notation, let $\Delta(s, w) := \Delta(s, w|b, F)$.

(a) Assumption 1′ says that $G(w)$ is concave. Thus, $G(w)(1 - H(s))(\eta(s, s) - w)$ is strictly concave in w. It follows that $G(w)(1 - H(s))(\eta(s, s) - w) - wF(s)$ is strictly concave; hence $w \mapsto \Delta(s, w|b, F)$ is also strictly concave.

(b) Since $b(s) < \underline{w}$ for all $s \in [\tilde{s}, \tilde{s}_1)$, then $\Delta(s, \underline{w}) = (b(s) - \underline{w})F(s) < 0$.

(c) Recall that $b(\cdot) \in \mathcal{B} \implies b(\tilde{s}) = \alpha(\tilde{s})$ and $F(\cdot) \in \mathcal{F} \implies F(\tilde{s}) = H(\tilde{s})$. Thus, by Lemma C.1, $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\tilde{\mu}|\tilde{s}, w)$, which implies $\Delta(\tilde{s}, \tilde{\mu}) = 0$. Next, observe that

$$\frac{\partial}{\partial w}\Delta(s, w) = g(w)\big(1 - H(s)\big)\underbrace{\left(\eta(s, s) - w - \frac{G(w)}{g(w)} - \frac{F(s)}{g(w)\big(1 - H(s)\big)}\right)}_{A(s,w)}. \tag{C.1}$$

Since $\tilde{\mu} \leq \psi(\tilde{s})$, then $A(\tilde{s}, \tilde{\mu}) \geq 0$, which implies $\partial\Delta(\tilde{s}, w)/\partial w|_{w=\tilde{\mu}} \geq 0$. □

Definition C.4. The mapping $\Phi : [\tilde{s}, \tilde{s}_1] \times \mathcal{B} \times \mathcal{F} \to [\underline{w}, \overline{w}]$ is defined as

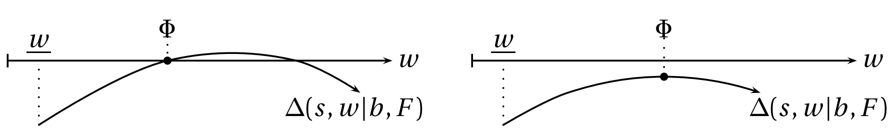
$$\Phi(s|b, F) := \min\Big\{\operatorname*{argmin}_{w \in [\underline{w}, \overline{w}]} \big|\Delta(s, w|b, F)\big|\Big\}.$$

Remark C.3. Figure C.1 illustrates the definition of Φ in two common cases. If $\Delta(s, w|b, F) = 0$ for some w, then Φ picks out the smallest solution (Figure C.1(a)). If $\Delta(s, w|b, F) < 0$ for all w, then Φ picks out its maximizer on $[\underline{w}, \overline{w}]$ (Figure C.1(b)).

Remark C.4 (Definition of $\tilde{\phi}(\cdot)$ and $\tilde{b}(\cdot)$). Define $\Lambda : \mathcal{B} \times \mathcal{F} \to \mathcal{B} \times \mathcal{F}$ as follows. If $(b, F) \in \mathcal{B} \times \mathcal{F}$, then $\Lambda(b, F) = (\check{b}, \check{F})$, where

$$\check{b}(s) = \alpha(\tilde{s}) + \int_{\tilde{s}}^{s} \frac{\big(v(y, y) - b(y)\big)G\big(\Phi(y|b, F)\big)h(s)}{H(\tilde{s}) + \displaystyle\int_{\tilde{s}}^{s} G\big(\Phi(z|b, F)\big)h(z)\,dz}\,dy$$

$$\check{F}(s) = H(\tilde{s}) + \int_{\tilde{s}}^{s} G\big(\Phi(y|b, F)\big)h(y)\,dy.$$



(a) Case 1: Φ identifies the smallest solution of $\Delta(s, w|b, F) = 0$

(b) Case 2: Φ identifies the maximizer of $\Delta(s, w|b, F)$ on the interval $[\underline{w}, \overline{w}]$

Figure C.1. Definition of Φ.

The function Λ is a continuous self-map defined on a compact, convex space. By Schauder's theorem, there exists a fixed point, $(\tilde{b}, \tilde{F}) = \Lambda(\tilde{b}, \tilde{F})$. Given such a fixed point $(\tilde{b}, \tilde{F})$, we define the function

$$\tilde{\phi}(s) := \Phi(s|\tilde{b}, \tilde{F})$$

and we simplify notation by henceforth writing

$$\tilde{\Delta}(s, w) := \Delta(s, w|\tilde{b}, \tilde{F}).$$

The preceding definitions, Lemma C.2, and Definition C.4 imply that

$$\tilde{b}'(s) = \frac{\big(v(s,s) - \tilde{b}(s)\big)G\big(\tilde{\phi}(s)\big)h(s)}{H(\tilde{s}) + \int_{\tilde{s}}^{s} G\big(\tilde{\phi}(y)\big)h(y)\,dy} \quad \text{and} \quad \tilde{F}(s) = H(\tilde{s}) + \int_{\tilde{s}}^{s} G\big(\tilde{\phi}(y)\big)h(y)\,dy.$$

Furthermore, $\tilde{b}(\tilde{s}) = \alpha(\tilde{s})$, $\tilde{b}(s) < \underline{w}$ for all $s \in [\tilde{s}, \tilde{s}_1)$, and $\tilde{\phi}(\tilde{s}) = \tilde{\mu} = \mu(\tilde{s})$.

Remark C.5. By substituting $\tilde{b}$ and $\tilde{\phi}$, we observe that

$$\tilde{\Delta}\big(s, \tilde{\phi}(s)\big) = \tilde{U}_{\tilde{\phi}}\big(\tilde{\phi}(s)|s, w\big) - \tilde{U}_{\tilde{\phi}}\big(\tilde{b}(s)|s, w\big),$$

where $\tilde{U}_{\varphi}(b|s, w)$ is defined in (14).

Lemma C.3. *For all* $s \in [\tilde{s}, \tilde{s}_1]$, $\tilde{\Delta}(s, \tilde{\phi}(s)) = 0$.

Proof. Since $\tilde{\phi}(\tilde{s}) = \tilde{\mu}$ and $\tilde{b}(\tilde{s}) = \alpha(\tilde{s})$, Lemma C.2(c) implies that $\tilde{\Delta}(\tilde{s}, \tilde{\phi}(\tilde{s})) = 0$. Suppose there exists $\hat{s} \in [\tilde{s}, \tilde{s}_1)$ such that $\tilde{\Delta}(\hat{s}, \tilde{\phi}(\hat{s})) = 0$ and for all $\varepsilon > 0$ (sufficiently small), $\tilde{\Delta}(\hat{s} + \varepsilon, \tilde{\phi}(\hat{s} + \varepsilon)) < 0$. (By definition of $\tilde{\phi}(s)$, $\tilde{\Delta}(s, \tilde{\phi}(s))$ cannot be strictly greater than zero.) Thus, for $s \in (\hat{s}, \hat{s} + \varepsilon)$, $\tilde{\phi}(s) = \Phi(s|\tilde{b}, \tilde{F})$ is the unique maximizer of $\tilde{\Delta}(s, \cdot)$ on the interval $[\underline{w}, \overline{w}]$. There are two cases.

Case 1. Suppose $\tilde{\phi}(s) > \underline{w}$. Differentiating $\tilde{\Delta}(s, \tilde{\phi}(s))$, substituting for $\tilde{F}(s)$ and $\tilde{b}'(s)$, and applying the envelope theorem yields

$$\frac{d}{ds}\tilde{\Delta}\big(s, \tilde{\phi}(s)\big) = G\big(\tilde{\phi}(s)\big)\int_{s}^{1} \frac{\partial v(s,y)}{\partial s} h(y)\,dy - G\big(\tilde{b}(s)\big)\int_{s}^{1} \frac{\partial v(s,y)}{\partial s} h(y)\,dy.$$

Recall that $\tilde{\phi}(s) > \tilde{b}(s)$, and so $d\tilde{\Delta}(s, \tilde{\phi}(s))/ds > 0$ for $s \in (\hat{s}, \hat{s} + \varepsilon)$. Since $\tilde{\Delta}(\hat{s}, \tilde{\phi}(\hat{s})) = 0$, this implies $\tilde{\Delta}(s, \tilde{\phi}(s)) > 0$—a contradiction since $\tilde{\Delta}(s, \tilde{\phi}(s))$ is nonpositive.

Case 2. Suppose $\tilde{\phi}(s) = \underline{w}$. By continuity, $\tilde{\phi}(\hat{s}) = \underline{w}$, which implies $\tilde{\Delta}(\hat{s}, \underline{w}) = (\tilde{b}(\hat{s}) - \underline{w})\tilde{F}(\hat{s}) < 0$ since $\tilde{b}(\hat{s}) < \underline{w}$. Thus, we have derived a contradiction.

The above cases exhaust the possibilities; hence, for all $s \in [\tilde{s}, \tilde{s}_1)$, $\tilde{\Delta}(s, \tilde{\phi}(s)) = 0$. By continuity, the preceding conclusion applies to the boundary $s = \tilde{s}_1$ as well. □

Lemma C.4. *The function* $\tilde{\phi}(s)$ *is decreasing.*

Proof. Let $(\hat{s}, \hat{s}') \subset [\tilde{s}, \tilde{s}_1]$. We know that $\tilde{\Delta}(\hat{s}, \tilde{\phi}(\hat{s})) = 0$. Holding $\tilde{\phi}(\hat{s})$ fixed,

$$\frac{\partial}{\partial s}\tilde{\Delta}\big(s, \tilde{\phi}(\hat{s})\big) = G\big(\tilde{\phi}(\hat{s})\big)\int_s^1 \frac{\partial v(s,y)}{\partial s}h(y)\,dy - G\big(\tilde{b}(s)\big)\int_s^1 \frac{\partial v(s,y)}{\partial s}h(y)\,dy > 0.$$

This implies that $\tilde{\Delta}(\hat{s}', \tilde{\phi}(\hat{s})) > 0$. Thus, $\tilde{\phi}(\hat{s}')$ must be the smallest value that solves $\tilde{\Delta}(\hat{s}', \tilde{\phi}(\hat{s}')) = 0$. Since $\tilde{\Delta}(s, \cdot)$ is strictly concave, $\tilde{\phi}(\hat{s}') < \tilde{\phi}(\hat{s})$. □

Lemma C.5. *Suppose $\tilde{\phi}(s) > \underline{w}$. Then $\partial\tilde{\Delta}(s, w)/\partial w|_{w=\tilde{\phi}(s)} = \partial\tilde{U}_{\tilde{\phi}}(b|s, w)/\partial b|_{b=\tilde{\phi}(s)} \geq 0$.*

Proof. The equality follows immediately from Remark C.5. That the derivative is greater than zero when evaluated at $\tilde{\phi}(s)$ follows from the definition of $\tilde{\phi}(s)$. The value $\tilde{\phi}(s)$ is either the smallest solution of $\phi \mapsto \tilde{\Delta}(s, \phi)$ or the maximizer of this function. As the function is concave, its derivative must be nonnegative at $\tilde{\phi}(s)$ when $\tilde{\phi}(s) > \underline{w}$. □

Remark C.6 (Definition of $\tilde{s}'$). Above we defined $\tilde{b}$, $\tilde{F}$, and $\tilde{\phi}$ on the interval $[\tilde{s}, \tilde{s}_1]$. If $\tilde{b}(\tilde{s}_1) < \underline{w}$, then the preceding argument can be repeated inductively with $\tilde{s}_1$ replacing $\tilde{s}$, $\tilde{b}(\tilde{s}_1)$ replacing $\alpha(\tilde{s})$, $\tilde{\phi}(\tilde{s}_1)$ replacing $\tilde{\mu} = \mu(\tilde{s})$, $\tilde{F}(\tilde{s}_1)$ replacing $H(\tilde{s})$, etc. Let $[\tilde{s}, \tilde{s}']$ be the maximal interval on which the functions $\tilde{b}$, $\tilde{F}$, and $\tilde{\phi}$ can be defined inductively in this manner. The construction stops only in the limit. As the functions $\tilde{b}$, $\tilde{F}$, and $\tilde{\phi}$ are monotone, they can be extended to $\tilde{s}'$ by taking the appropriate limit from the left.

Lemma C.6. *Consider $\tilde{b} : [\tilde{s}, \tilde{s}'] \to \mathbb{R}$ and $\tilde{\phi} : [\tilde{s}, \tilde{s}'] \to [\underline{w}, \overline{w}]$ as defined in Remark C.6. Then $\tilde{b}(\tilde{s}') = \underline{w} = \tilde{\phi}(\tilde{s}')$.*

Proof. If $\tilde{b}'(\tilde{s}') = \underline{w}$, then $\tilde{\Delta}(\tilde{s}', \tilde{\phi}(\tilde{s}')) = G(\tilde{\phi}(\tilde{s}'))(1 - H(\tilde{s}'))(\eta(\tilde{s}', \tilde{s}') - \tilde{\phi}(\tilde{s}')) + (\underline{w} - \tilde{\phi}(\tilde{s}'))\tilde{F}(\tilde{s}')$. Recalling the definition of $\tilde{\phi}(\cdot)$ and that $\tilde{\Delta}(\tilde{s}', \tilde{\phi}(\tilde{s}')) = 0$, we conclude that $\tilde{\phi}(\tilde{s}') = \underline{w}$. □

C.2 *Verification of equilibrium*

Let β be the strategy described in Theorem 2. We will verify that an agent does not have an incentive to deviate to an alternative feasible bid in the range of β given that the other agent bids according to β. In a type-1 canonical equilibrium, the proof of Theorem 1 applies. In a type-2 canonical equilibrium, the range of β equals the range of $\bar{b}(\cdot)$ and the argument below applies.[26] Finally, in a noncanonical equilibrium, the range of β is characterized by four segments: (a) the function $\bar{b}(s)$ for $s < \tilde{s}$, (b) the function $\tilde{b}(s)$ for $s \in [\tilde{s}, \tilde{s}')$, (c) the function $\tilde{\phi}(s)$ for $s \in [\tilde{s}, \tilde{s}')$, and (d) the function $\bar{b}(s)$ for $s > \tilde{s}$. We divide our argument accordingly. Writing s^- and s^+ for left- and right-hand limits, respectively, we observe that $\bar{b}(\tilde{s}^-) = \tilde{b}(\tilde{s}^+) \leq \tilde{b}(\tilde{s}'^-) = \underline{w} = \tilde{\phi}(\tilde{s}'^-) \leq \tilde{\phi}(\tilde{s}^+) = \bar{b}(\tilde{s}^+)$.

Case 1. Consider a type-(s, w) bidder where $s < \tilde{s}$. This bidder's expected payoff when bidding $\beta(s, w) = \bar{b}(s)$ is $U(\bar{b}(s)|s, w) = \int_0^s (v(s, y) - \bar{b}(s))h(y)\,dy$.

[26] Only the cases involving a deviation to $\bar{b}(x)$, where $x \in [0, 1]$, are relevant.

(a) For all $s < \tilde{s}$, $\bar{b}(s) = \alpha(s)$, the equilibrium bidding strategy in a first-price auction without budget constraints. Therefore, this bidder has no profitable deviation to any alternative bid $\bar{b}(x)$, where $x < \tilde{s}$ (Milgrom and Weber 1982). By continuity, the bid $\bar{b}(\tilde{s}^-)$ is also not a profitable deviation.

(b) If this agent bids $\tilde{b}(x)$, where $x \in [\tilde{s}, \tilde{s}')$, his payoff is

$$U\big(\tilde{b}(x)|s, w\big) = \int_0^{\tilde{s}} \big(v(s, y) - \tilde{b}(x)\big) h(y)\, dy + \int_{\tilde{s}}^{x} \big(v(s, y) - \tilde{b}(x)\big) G\big(\tilde{\phi}(y)\big) h(y)\, dy. \tag{C.2}$$

Differentiating with respect to x and substituting $\tilde{b}'(x)$ as defined in (10a) gives $dU(\tilde{b}(x)|s, w)/dx = (v(s, x) - v(x, x))G(\tilde{\phi}(x))h(x)$. Since $s < \tilde{s} \leq x$, then $U(\tilde{b}(x)|s, w) \leq U(\bar{b}(\tilde{s}^-)|s, w) \leq U(\bar{b}(s)|s, w)$. Therefore, the bid $\tilde{b}(x)$ is not a profitable deviation.

(c) If this agent bids $\tilde{\phi}(x)$ for $x \in [\tilde{s}, \tilde{s}')$, his payoff is

$$\begin{aligned} U\big(\tilde{\phi}(x)|s, w\big) = &\int_0^{\tilde{s}} \big(v(s, y) - \tilde{\phi}(x)\big) h(y)\, dy \\ &+ \int_{\tilde{s}}^{x} \big(v(s, y) - \tilde{\phi}(x)\big) G\big(\tilde{\phi}(y)\big) h(y)\, dy \\ &+ G\big(\tilde{\phi}(x)\big) \int_x^1 \big(v(s, y) - \tilde{\phi}(x)\big) h(y)\, dy. \end{aligned} \tag{C.3}$$

Subtracting (C.2) from (C.3) yields

$$\begin{aligned} &U\big(\tilde{\phi}(x)|s, w\big) - U\big(\tilde{b}(x)|s, w\big) \\ &\quad = \big(\tilde{b}(x) - \tilde{\phi}(x)\big)\left(H(\tilde{s}) + \int_{\tilde{s}}^{x} G\big(\tilde{\phi}(y)\big) h(y)\, dy\right) \\ &\qquad + G\big(\tilde{\phi}(x)\big) \int_x^1 \big(v(s, y) - \tilde{\phi}(x)\big) h(y)\, dy \\ &\quad \leq \big(\tilde{b}(x) - \tilde{\phi}(x)\big)\left(H(\tilde{s}) + \int_{\tilde{s}}^{x} G\big(\tilde{\phi}(y)\big) h(y)\, dy\right) \\ &\qquad + G\big(\tilde{\phi}(x)\big) \int_x^1 \big(v(x, y) - \tilde{\phi}(x)\big) h(y)\, dy \\ &\quad = U\big(\tilde{\phi}(x)|x, w\big) - U\big(\tilde{b}(x)|x, w\big) = 0. \end{aligned}$$

The inequality follows from $s \leq x$. The final equality follows from the definition of $\tilde{\phi}$. See (13), (14), and Remark C.5 above. Thus, $U(\tilde{\phi}(x)|s, w) \leq U(\tilde{b}(x)|s, w) \leq U(\bar{b}(s)|s, w)$, where the final inequality is implied by subcase (b) above.

(d) If this agent bids $\bar{b}(x)$, where $x \in (\tilde{s}, 1]$, his payoff is

$$U\big(\bar{b}(x)|s, w\big) = \int_0^x \big(v(s, y) - \bar{b}(x)\big)h(y)\,dy + G\big(\bar{b}(x)\big)\int_x^1 \big(v(s, y) - \bar{b}(x)\big)h(y)\,dy.$$

Since $s < x$, by the same argument as in the proof of Theorem 1, we know that $U(\bar{b}(x)|s, w) \le U(\bar{b}(\tilde{s}^+)|s, w)$. Since $\bar{b}(\tilde{s}^+) = \tilde{\phi}(\tilde{s})$, the proof of subcase (c) implies that $U(\bar{b}(\tilde{s}^+)|s, w) \le U(\bar{b}(s)|s, w)$. Therefore, the bid $\bar{b}(x)$, where $x > \tilde{s}$, is not a profitable deviation.

Subcases (a)–(d) show that a bidder with value signal $s < \tilde{s}$ cannot gain by deviating to another bid in the range of β.

Case 2. Consider a type-(s, w) bidder where $s \in (\tilde{s}, \tilde{s}')$ and $w < \tilde{\phi}(s)$. This bidder's expected payoff when bidding $\beta(s, w) = \tilde{b}(s)$ is

$$U\big(\tilde{b}(s)|s, w\big) = \int_0^{\tilde{s}} \big(v(s, y) - \tilde{b}(s)\big)h(y)\,dy + \int_{\tilde{s}}^{s} \big(v(s, y) - \tilde{b}(s)\big)G\big(\tilde{\phi}(y)\big)h(y)\,dy.$$

Deviations to bids of (a) $\bar{b}(x)$, where $x < \tilde{s}$, or (b) $\tilde{b}(x)$, where $x \in [\tilde{s}, \tilde{s}')$, can be shown not to be profitable using an argument paralleling those presented in Cases 1(a) and 1(b). Thus, we consider only the two remaining subcases, (c) and (d).

(c) If this agent bids $\tilde{\phi}(x)$, where $x \in [\tilde{s}, \tilde{s}')$, his payoff is given by (C.3). If $x < s$, then $\tilde{\phi}(x) > w$. Hence, a deviation into this range of bids is not feasible for this bidder. Conversely, if $x > s$, then the argument in Case 1(c) establishes that $U(\tilde{\phi}(x)|s, w) \le U(\tilde{b}(x)|s, w)$. Noting the preceding paragraph, we see that $U(\tilde{\phi}(x)|s, w) \le U(\tilde{b}(s)|s, w)$.

(d) Since $w < \tilde{\phi}(s) \le \bar{b}(\tilde{s}^+)$, the bid $\bar{b}(x)$, where $x > \tilde{s}$, is not feasible.

Case 3. Consider a type-(s, w) bidder where $s \in (\tilde{s}, 1]$ and $w \in [\tilde{\phi}(s), \tilde{\phi}(\tilde{s}^+)]$. There exists $s_w \in [\tilde{s}, s]$ such that $\beta(s, w) = \tilde{\phi}(s_w) = w$ and this bidder's expected payoff is

$$\begin{aligned} U\big(\tilde{\phi}(s_w)|s, w\big) = {} & \int_0^{\tilde{s}} \big(v(s, y) - \tilde{\phi}(s_w)\big)h(y)\,dy \\ & + \int_{\tilde{s}}^{s_w} \big(v(s, y) - \tilde{\phi}(s_w)\big)G\big(\tilde{\phi}(y)\big)h(y)\,dy \\ & + G\big(\tilde{\phi}(s_w)\big)\int_{s_w}^1 \big(v(s, y) - \tilde{\phi}(s_w)\big)h(y)\,dy. \end{aligned} \tag{C.4}$$

First, we consider case (c). Suppose this bidder places the bid $\tilde{\phi}(x)$, where $x \in [s_w, s]$. The agent's expected payoff at this bid is given by (C.4) with x

replacing s_w. Consider the derivative

$$
\begin{aligned}
&\frac{d}{dx}U\big(\tilde{\phi}(x)|s,w\big)\\
&=\tilde{\phi}'(x)\bigg[g\big(\tilde{\phi}(x)\big)\big(1-H(x)\big)\bigg(\eta(s,x)-\tilde{\phi}(x)-\frac{G\big(\tilde{\phi}(x)\big)}{g\big(\tilde{\phi}(x)\big)}-\frac{H(\tilde{s})+\int_{\tilde{s}}^{x}G\big(\tilde{\phi}(y)\big)h(y)\,dy}{g\big(\tilde{\phi}(x)\big)\big(1-H(x)\big)}\bigg)\bigg]\\
&\leq\tilde{\phi}'(x)\underbrace{\bigg[g\big(\tilde{\phi}(x)\big)\big(1-H(x)\big)\bigg(\eta(x,x)-\tilde{\phi}(x)-\frac{G\big(\tilde{\phi}(x)\big)}{g\big(\tilde{\phi}(x)\big)}-\frac{H(\tilde{s})+\int_{\tilde{s}}^{x}G\big(\tilde{\phi}(y)\big)h(y)\,dy}{g\big(\tilde{\phi}(x)\big)\big(1-H(x)\big)}\bigg)\bigg]}_{\frac{\partial}{\partial b}\tilde{U}_{\tilde{\phi}}(b|x,w)|_{b=\tilde{\phi}(x)}}\\
&\leq 0.
\end{aligned}
$$

The first inequality follows from $\tilde{\phi}'(x)\leq 0$ and $\eta(s,x)\geq\eta(x,x)$. The second inequality follows from Lemma C.5, which shows that the term in square brackets is nonnegative. Thus, each of these bids is dominated by $\tilde{\phi}(s_w)=w$.

If $s\leq\tilde{s}'$, then $U(\tilde{\phi}(s_w)|s,w)\geq U(\tilde{\phi}(s)|s,w)=U(\tilde{b}(s)|s,w)$ and the same argument as in Case 2 shows that this bidder has no profitable deviation to any bid below $\tilde{\phi}(s)$ as well.

Similarly, if $s\geq\tilde{s}'$, then $U(\tilde{\phi}(s_w)|s,w)\geq U(\underline{w}|s,w)$. Verifying that bids below $\tilde{b}(\tilde{s}')=\underline{w}$ are not profitable deviations proceeds similarly to the argument in Case 1, subcases (a) and (b). A deviation to the bid $\bar{b}(x)$, where $x>\tilde{s}$, is not feasible.

Case 4. Consider a type-(s,w) bidder where $s\geq\tilde{s}$ and $w\geq\tilde{\phi}(\tilde{s}'^{-})$. The same argument as in the proof of Theorem 1 shows that this agent has no profitable deviation to any bid in the range of $\bar{b}(x)$ for $x\geq\tilde{s}$. In particular, this implies that he prefers $\beta(s,w)$ to the bid $\bar{b}(\tilde{s}^{-})=\tilde{\phi}(\tilde{s})$, and so the same argument as in Case 3 above shows that he cannot gain by deviating to a bid less than $\tilde{\phi}(\tilde{s})$.

The preceding cases show that a type-(s,w) bidder does not have a feasible, profitable deviation from $\beta(s,w)$ given that the other bidder also adopts this strategy.

Appendix D: Examples

Example 3 (calculations)

We confirm that

$$\beta_A(s)=\begin{cases}2s/3 & \text{if } s\leq 1/9\\ 1/3 & \text{if } s>1/9\end{cases}\quad\text{and}\quad\beta_B(s)=1/3$$

are symmetric equilibrium strategies in Example 3. Recall that $v(s_i,s_j)=s_i+s_j$ and value signals are independently distributed according to the c.d.f. $H(s)=\sqrt{s}$ on $[0,1]$. The equilibrium bidding strategy in the absence of budget constraints is $\alpha(s)=2s/3$.

The β_A equilibrium. Suppose bidder j adopts the strategy $\beta_A(s)$. All bids in the range $(2/27,1/3)$ are dominated by the bid $\beta_A(1/9)=2/27$. Thus, to confirm that β_A is an equilibrium strategy, it is sufficient to rule out deviations by bidder i to other bids in the range of β_A.

Consider bidder i with value signal $s \le 1/9$. If he bids $\beta_A(s) = 2s/3$, his expected payoff is $(2/3)s^{3/2}$. Since $2s/3$ is the symmetric equilibrium strategy in the absence of budget constraints, bidder i has no profitable deviation to any bid in the range $[0, 2/27]$. If he bids $1/3$, he wins for sure if his opponent's value signal is less than $1/9$ and he wins with probability $1/2$ if his opponent's value signal is more than $1/9$. Thus, his expected payoff is

$$\sqrt{\frac{1}{9}}\left(s + \mathbb{E}[S_j|S_j \le 1/9] - \frac{1}{3}\right) + \frac{1-\sqrt{1/9}}{2}\left(s + \mathbb{E}[S_j|S_j > 1/9] - \frac{1}{3}\right) = \frac{2}{9}\left(3s - \frac{2}{9}\right).$$

For all $s \le 1/9$, $(2/9)(3s - 2/9) \le (2/3)s^{3/2}$. Thus, $\beta_A(s)$ is an optimal bid for bidder i when $s \le 1/9$.

Now consider bidder i with value signal $s > 1/9$. If he bids $\beta_A(x) = 2x/3$ for some $x \le 1/9$, his expected payoff is $\sqrt{x}(s + \mathbb{E}[S_j|S_j \le x] - 2x/3) = \sqrt{x}(s - x/3) \le (1/3)(s - 1/27)$. Clearly, $(1/3)(s - 1/27) \le (2/9)(3s - 2/9)$ for all $s \ge 1/9$. Thus, he has no profitable deviation from $1/3$ to any bid $b \in [0, 2/27]$.

As the setting is symmetric, we conclude that β_A is a symmetric equilibrium strategy.

The β_B equilibrium. Suppose that bidder j adopts the strategy $\beta_B(s) = 1/3$. The expected payoff of bidder i with value signal s when he bids $1/3$ is

$$\frac{1}{2}\left(s + \mathbb{E}[S_j] - \frac{1}{3}\right) = \frac{1}{2}\left(s + \frac{1}{3} - \frac{1}{3}\right) = \frac{s}{2}.$$

If bidder i bids less than $1/3$, his expected payoff is zero as he loses the auction with certainty. Furthermore, bids above $1/3$ are not feasible. Thus, $1/3$ is a best response for bidder i given bidder j's strategy, and so β_B is a symmetric equilibrium strategy.

Example D.1. Suppose there are two bidders with uniformly distributed value signals on the unit interval. Suppose $v(s_i, s_j) = s_i + s_j$. Consider three budget distributions on $[0, 2]$: $G_0(w) = w/2$, $G_1(w) = \sqrt{w/2}$, and $G_2(w) = w(4 - w)/4$. Each satisfies Assumption 1′. The distribution G_0 likelihood ratio dominates G_1 and G_2. There is one critical point given each budget distribution. Given G_0, the critical point occurs at $(s_0, b_0) = (0.1056, 0.2111)$. Given G_1, the critical point occurs at $(s_1, b_1) = (0.0864, 0.1728)$. Finally, given G_2, the critical point occurs at $(s_2, b_2) = (0.1264, 0.2528)$. By Theorem 1, the function that defines the bid of an unconstrained bidder, $\bar{b}(s)$, passes through the critical point in each case. This implies that "tightening" the budget distribution from G_0 to G_1 induces a *lower* bid by an unconstrained bidder with a value signal $s \approx 0.1$. Conversely, tightening the budget distribution from G_0 to G_2 induces a *higher* bid by an unconstrained bidder with a value signal $s \approx 0.1$. ◊

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