An N-person War of Attrition with the Possibility of a Non-Compromising Type

Shinsuke Kambe* **

Faculty of Economics

Gakushuin University

1–5–1 Mejiro

Toshima-ku, Tokyo 171-8588

Japan

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Abstract

This paper studies an N-person war of attrition which needs one exit for its ending. An N-person war of attrition is qualitatively different from its two-person version. Only in the former, the set of players who are actively engaged in a war of attrition may change over time. We introduce the possibility of a non-compromising type and characterize the unique equilibrium by identifying which players are actively involved in a war of attrition at each moment. We examine who is likely to exit and when the war of attrition ends quickly. As the leading example, we study how a group selects a volunteer in a dynamic setting.

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* Correspondence: Faculty of Economics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan. email: shinsuke.kambe@gakushuin.ac.jp

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1. Introduction

This paper studies an N-person war of attrition which needs one exit for its ending. We introduce the possibility of a non-compromising type and show that the equilibrium is unique. When there are more than two players, a war of attrition becomes qualitatively different from the one with only two players. We say that a player is actively engaged in a war of attrition when he exits with a positive rate. When there are only two players and a war of attrition occurs, both players are actively engaged from beginning to end. When there are more than two players, at least two players need to be actively engaged in a war of attrition, but some may simply wait. Due to this possibility of inactive participation, in a war of attrition with more than two players, we need to identify who are actively engaged at each moment. This issue in an N-person war of attrition has not been adequately studied even though inactive participation is often observed in real wars of attrition and it substantially affects outcomes. Our eventual goal is to examine who is likely to exit and when a war of attrition ends quickly.

As the leading example, we study how a group chooses a volunteer in a dynamic setting. There are many instances of this nature, such as former classmates soliciting the coordinator of their reunion party, an industry group choosing a company to lobby the government, and so on. Our analysis is applicable as long as (1) only one exit is needed and (2) the benefits that players receive at the ending are independent of who exits (though the player who exits has to bear costs to obtain the benefit).

Consider a group that needs one of its members (hereafter players) to take an initiative role for a project. It convenes a meeting and solicits a single volunteer. In this example, exiting a war of attrition is achieved by volunteering. Any player can perform the initiative role equally well. Although the project is valuable to all players, taking the initiative role is costly. Each player prefers that someone else takes the initiative role, which may cause a war of attrition in the selection of a volunteer. We assume that there is a chance that each player is unable to take the initiative role for a private reason, such as illness of a close relative. This type of player is called the non-compromising type. Thus, there is a chance that the other players may not volunteer. In such a situation, a player who is capable of taking the initiative role wants to do so by himself. Even when a player is the non-compromising type, he cannot prove it to the other players. Hence, the players have to guess whether the lack of a volunteer occurs because of strategic waiting or because of inability.

At any moment in equilibrium, even when a player is capable of the initiative role, he does not volunteer with the probability of one (except for possibly just one player in the beginning). If he were supposed to do so, the deviation would bring about a favorable consequence because not volunteering would prove that he is the non-compromising type and would cause the other players to choose a volunteer among themselves. Thus, the players who are able to take the initiative role randomize the timing to volunteer over some interval, which is a war of attrition. Because some players need to volunteer with positive rates during a war of attrition, and there is a chance that each player is the non-compromising type and never volunteers, a war of attrition cannot continue forever. Over some finite time period, all those who can take the initiative role volunteer. Because of this property, the equilibrium is unique and can be constructed backward in time.

When there are two players, both players are actively engaged in a war of attrition. They volunteer with positive rates till the end. When there are more than two players, only a subset of players needs to volunteer with positive rates at each moment. Some players may be passive participants at the beginning of meeting and, after some time, become actively engaged in a war of attrition. Even when some players exhaust the possibility of volunteering, that is, never volunteering after such time, other players may continue a war of attrition. Note that these types of inactive participation cannot occur in the two-person model of a war of attrition.

Because of the possibility of inactive participation, we first need to identify the set of players who are actively engaged in a war of attrition at each moment. Then, using the result, we characterize the identity of the volunteer. We show that the player who is less patient, whose cost of taking the initiative role is lower, and whose net gain from the project is higher tends to volunteer more quickly. We also study the speed of selection and show that the combined rate of volunteering weakly declines over time. When a new player joins the group and even when his characteristics are such that he is likely to be a passive participant at the beginning of meeting, the chance that a volunteer comes forward at the beginning will increase. This property can then be used to obtain the lower bound for the probability that the selection is concluded quickly.

The war of attrition was first studied by Maynard Smith (1974) in the field of biology. It arises when there is more than one alternative and there is a conflict among players as to which alternative should be chosen, and when the players can only concede the right to choose to the others but cannot force their choice on the other players. It captures the essence of many realistic negotiations and is widely applied in economic analyses. (See Bliss and Nalebuff (1984) for an early example.) One of the theoretical difficulties in using the original model of a war of attrition is the multiplicity of equilibria.[1] Based on the idea developed by Kreps et al. (1982), some researchers have introduced the possibility of irrational types, showing that the equilibrium becomes unique in the modified models. See Abreu and Gul (2000), Kambe (1999), and Kornhauser et al. (1989) among others. These analyses have shown that the uniqueness of the equilibrium can be obtained once we introduce sufficient structures into the original model. The current paper follows this line of research. It introduces the possibility that a player is unable to volunteer in order to identify a unique equilibrium. The major difference from the previous literature is the possibility of more than two players.[2] Only in an N-person war of attrition can we investigate how the set of active players changes over time and how the number of players affects the probability of a quick exit.

The remainder of the paper is organized as follows. Section 2 defines the model. In Section 3, we characterize the unique equilibrium and then apply the findings to the example of either two or three players. In Section 4, we further study the property of the equilibrium under a natural assumption. Section 5 concludes. All the proofs are relegated to the Appendix. (The proofs for Lemma 1 and Proposition 2 are omitted as they are obvious from the preceding analyses.)

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[1] The equilibrium outcome may be unique when the situation is not stationary. For this, see Hendricks et al. (1988) for a general treatment.

[2] Bulow and Klemperer (1999) also study a war of attrition with multiple players. Unlike our model, they analyze a model in which more than one exit may be needed before a war of attrition ends. By using the knowledge developed in auction theory, they study who endures a war of attrition. Because they do not introduce the possibility of the non-compromising type and instead introduce a distribution over the valuations, their model predicts that all the remaining players have positive probabilities of concession in a war of attrition, and the timing to exit is determined by the player's valuation. Our model shows that the possibility of the non-compromising type also plays a big role in determining the outcome in a war of attrition. Moreover, our model can explain the possibility of inactive participation and changes in the set of active players in a war of attrition. Our model focuses on the selection of one volunteer. It is interesting to see what happens in our model when the number of required exits is more than one as is supposed by their model.

2. Model

There are N players, where $N \geq 2$. The set of players is given by $I \equiv \{1, \ldots, N\}$, and a typical player is indexed by a lowercase letter, such as i, j, k, or ℓ. There is one project, and the players need to choose one player for the initiative role. They convene a meeting and solicit a volunteer.

Player i's value of the project is publicly known and is given by V_i. Each player is equally effective at the initiative role; this valuation is the same no matter who volunteers. On the other hand, the cost of taking the initiative role for the project is variable and privately known. Either when his private circumstance prevents him from taking the initiative role or when player i needs to spend substantial time to acquire the necessary expertise and is not immediately ready for the current project, the cost is infinitely large. This type of player is called the non-compromising type. It occurs with probability z_i. When player i is able to take the initiative role, the cost is given by c_i. It occurs with probability $1 - z_i$. This value itself is publicly known although whether the cost is finite or not is private information. We assume that $0 < c_i < V_i$ for any $i \in I$. The latter type of player is called a low-cost type. Player i discounts the future payoff by the discount rate $r_i (> 0)$, which is publicly known. When some other player volunteers at time t, the payoff of player i is $e^{-r_i t} V_i$ irrespective of his type. When player i volunteers at time t, his payoff is $e^{-r_i t}\big(V_i - \tilde{c}_i\big)$, where $\tilde{c}_i$ is positive infinity for the non-compromising type and is c_i for the low-cost type. When there is no volunteer in a finite time, the payoff of every player is zero. This implies that the optimal strategy for the non-compromising type is not to volunteer. In the following analysis, when we consider the strategy of a player, we focus on that of the low-cost type.

The game proceeds in continuous time with an infinite time horizon. At each moment, any player can come forward as a volunteer.[3] Once a player volunteers, the meeting ends. The one who volunteers takes the initiative role for the project. When two or more players simultaneously volunteer, one player among them is randomly chosen with equal probability to take the initiative role.

[3] We are considering the situation in which the chance of volunteering is frequent and no player can commit himself not to volunteer. Even when the players take turns to decide whether to volunteer or not in a pre-specified ordering, the current model is a good approximation as long as every player is given frequent chances to volunteer during any interval. This becomes obvious when we derive the rate of volunteering in the following analysis.

Each player chooses a mixed strategy for the timing of volunteering. Let $a_i(t)$ be the (instantaneous) rate of volunteering by player i at time t. We assume that $\lim_{\epsilon\downarrow 0} a_i(t-\epsilon)$ is well defined and $\lim_{\epsilon\downarrow 0} a_i(t+\epsilon) = a_i(t)$. For a technical reason, we assume that $a_i(t)$ is discontinuous at countable points at most. A player may volunteer with a probability mass at some occasion. (We show that this may occur at time 0, but not at any other time.) In the following analysis, we treat volunteering with a probability mass at a particular time separately from volunteering with (instantaneous) rates during an interval. Note that a positive probability mass of volunteering means that the chance of volunteering is concentrated at some instance: no matter how short the interval, the probability of volunteering is bounded from below by a positive value. We use the perfect Bayesian equilibrium as our equilibrium concept.

In the following analysis, it is convenient to categorize the players at each moment based on their behavior strategies. We say that player i is *actively engaged in a war of attrition*, or simply is *active*, at time t when he is volunteering at a positive rate during the period $(t, t+\epsilon)$ for any $\epsilon > 0$: $\int_t^{t+\epsilon} a_i(\tau)d\tau > 0$ for any $\epsilon > 0$. Let A_t be the set of active players at time t. We define a *war of attrition* to be the situation in which at least one player is active. On the other hand, we say that a player is a *passive participant*, or simply is *passive*, at time t when he has no probability of volunteering during the period $(t, t+\epsilon)$ for some $\epsilon > 0$, but has a positive probability of volunteering during some later period. Let B_t be the set of passive players. Alternatively, we can say that $B_t = \{\cup_{\tau>t} A_\tau\}/A_t$. ("/" indicates the subtraction of the elements in its right term from its left term.) When a player becomes no longer active at time t and does not have any chance of volunteering after time t, we say that the player *exhausts the possibility of volunteering* at time t. Let C_t be the set of players who have exhausted the possibility of volunteering by time t and D_t be the set of the players who exhaust the possibility of volunteering exactly at time t. Formally, we define C_t and D_t as follows: $C_t \equiv I/\{\cup_{\tau\geq t} A_\tau\}$ and $D_t \equiv \lim_{\epsilon\downarrow 0}\big(C_t/C_{t-\epsilon}\big)$. We call the combination of passive participants at time t and those who have exhausted the possibility of volunteering by time t *inactive* players at time t. The set of inactive players at time t is given by $B_t \cup C_t$. We define the set of players who are active both at time t and just before it as E_t: $E_t \equiv \{\lim_{\epsilon\downarrow 0} A_{t-\epsilon}\} \cap A_t$. In the following analysis, $t-$ (or $t+$) indicates that we take the limit of τ converging to t from below (or from above, respectively). For example, we write $D_t \equiv C_t/C_{t-}$ and $E_t \equiv A_{t-} \cap A_t$.

3. The Unique Equilibrium

This section derives the unique equilibrium for an N-person war of attrition with the possibility of the non-compromising type. The first subsection studies the general properties of an N-person war of attrition without assuming that there is the possibility of the non-compromising type. It shows that the combined rate of volunteering has to satisfy certain conditions in any N-person war of attrition. The second part of this section studies the equilibrium given the possibility of the non-compromising type. It investigates which players are actively engaged in a war of attrition at each moment, characterizing the unique equilibrium path. Theoretically, this is our main contribution. The third subsection applies the results of the first two subsections to the example of either two or three players.

3.1. General properties of an N-person war of attrition

In this subsection, we suppose that a war of attrition occurs and investigate the properties along the equilibrium path. This subsection does not use the assumption that the non-compromising type exists with positive probabilities: $z_i \geq 0$ for any $i \in I$ in this subsection. The results obtained in this subsection generally hold in the N-person war of attrition which needs one exit for its ending, and the benefits that players receive at the ending are independent of who exits.

The low-cost type waits in a war of attrition because he hopes that some other player volunteers during waiting. Hence, the number of active players cannot be one. Moreover, there is no interval in which there is no possibility of volunteering but after which there is a positive probability of volunteering. (If there were such an interval, the low-cost type who planned to volunteer with a positive probability after the interval would be better off by volunteering before the interval, which would upset the supposition.)

There cannot be a positive probability mass of volunteering except in the beginning. To understand why, let us suppose the contrary. Suppose that a player volunteers with a positive probability at time $t(> 0)$. Then, the other players would want to wait near time t. This would create a period during which the number of active players is less than two. Because there cannot be such a period during a war of attrition, this is a contradiction. Hence, there should not be a positive probability mass of volunteering after time 0. At time 0, a player may volunteer with a positive probability mass, which we call *initial volunteering.*

It is carried out by at most one player. (Otherwise, a player who were supposed to volunteer initially would wait for a moment to see whether the others would do so.)

These properties are well known for wars of attrition in the continuous time. For the purpose of completeness, I summarize them in the next lemma.

Lemma 1

At most one player volunteers initially with a positive probability mass. After time 0, *the following properties hold.*

(i) *The number of active players cannot be one.*

(ii) *Volunteering never occurs with a positive probability mass.*

(iii) *There is no interval in which there is no possibility of volunteering but after which there is a positive probability of volunteering.*

Let $P_i(t)$ denote the ex-ante probability that player i does not volunteer by time t (given that no other player does so before this time). Because $P_i(t)$ is weakly decreasing, it is differentiable almost everywhere. Its derivative at time t is denoted by $a_i(t)$. It gives player i's (instantaneous) rate of volunteering at time t. Lemma 1 shows that no player volunteers with a positive probability mass after time zero. This implies that $P_i(t)$ is continuous after time 0. Hence, by integrating the rate of volunteering, we can compute the ex-ante probability that player i does not volunteer by time τ as $P_i(\tau) = P_i(t)\exp\left\{-\int_{s=t}^{\tau} a_i(s)ds\right\}$ when $0 < t < \tau$. In particular, when the rate of volunteering is constant and given by α during the interval $[t, \tau)$, $P_i(\tau) = P_i(t)e^{-\alpha(\tau-t)}$ holds. Because the non-compromising type never volunteers, we obtain $P_i(t) \geq z_i$.

During a war of attrition, the players' rates of volunteering need to take specific values so that active players are indifferent between volunteering and waiting. Let b_i be the sum of the rates of volunteering by the other players given which player i feels indifferent between volunteering and waiting. Let us compute b_i by comparing the two strategies of player i: immediate volunteering and a short wait before volunteering. (Please note that, as explained in the previous section, we focus on the strategy of the low-cost type. For simplicity, we often omit the reference to the low-cost type hereafter. That is, when we say player i, we actually mean the low-cost type of player i.) When player i volunteers immediately, he obtains the current payoff of $V_i - c_i$. On the other hand, when he waits for the small interval of $\triangle$ and volunteers, there is a chance that one of the other players volunteers during this period. Because this occurs at rate b_i, the probability that none of the other players volunteer during the waiting period is given by $e^{-b_i \triangle}$. Taking into account the discounting, we can compute his current discounted payoff from the latter strategy to be $\int_{\tau=0}^{\triangle} b_i e^{-b_i \tau} e^{-r_i \tau} V_i d\tau + e^{-b_i \triangle} e^{-r_i \triangle}(V_i - c_i)$. Then, the following equation characterizes b_i:

$$V_i - c_i = \int_{\tau=0}^{\triangle} b_i e^{-b_i \tau} e^{-r_i \tau} V_i d\tau + e^{-b_i \triangle} e^{-r_i \triangle}(V_i - c_i).$$

Rearranging the terms in the above equation, we obtain $\left(1 - e^{-b_i \triangle} e^{-r_i \triangle}\right)(V_i - c_i) = \int_{\tau=0}^{\triangle} b_i e^{-b_i \tau} e^{-r_i \tau} V_i d\tau$. By dividing both sides by $\triangle$, setting $\triangle$ to 0, and using L'Hospital's theorem, we can show that $(r_i + b_i)(V_i - c_i) = b_i V_i$. Solving this equation with respect to b_i, we obtain

$$b_i = \frac{r_i(V_i - c_i)}{c_i}.$$

As b_i is the rate at which player i feels indifferent between waiting and immediately volunteering, we refer to b_i as the *threshold rate* of (the low-cost type of) player i. Note that b_i is defined only by the intrinsic parameters of player i and is independent of time. This contrasts with player i's rate of volunteering $a_i(t)$, which may change over time. As we will show shortly, the equilibrium behavior of player i is characterized by the combination of his threshold rate b_i and his probability to be the non-compromising type z_i. Without loss of generality, we assume that b_i is weakly decreasing in i throughout this paper.

Assumption B

We assume that $b_i \geq b_j$ for any $i < j$.

This assumption states that a player with a lower index has a weakly higher threshold rate and thus is indifferent between volunteering and waiting given a weakly higher combined rate of volunteering by the other players. This tends to force him to volunteer earlier during a war of attrition, which we show later to be true.

The rest of this subsection derives the necessary conditions for the combined rate of volunteering $\sum_{j \in I} a_j(t)$ and its relationship with the threshold rates of players in various statuses.

First, let us study the combined rates of volunteering itself.

By the definition of b_i, for any $i \in A_t$, it holds that $\sum_{j \neq i} a_j(t) = b_i$. By summing this equation over the set of active players and noting that $a_j(t) = 0$ for any $j \notin A_t$, we obtain $\sum_{i \in A_t} b_i = \sum_{i \in A_t} \sum_{j \neq i} a_j(t) = (|A_t| - 1) \sum_{j \in I} a_j(t)$. Hence, we obtain

$$\sum_{j \in I} a_j(t) = \frac{\sum_{A_t} b_i}{|A_t| - 1}. \tag{1}$$

This shows that the combined rate of volunteering is determined by the threshold rates of active players. Because $a_i(t) = \sum_I a_j(t) - \sum_{j \neq i} a_j(t)$ and $\sum_{j \neq i} a_j(t) = b_i$ for any $i \in A_t$, the above equation implies that

$$a_i(t) = \frac{\sum_{A_t} b_j}{|A_t| - 1} - b_i \text{ for } \forall i \in A_t. \tag{2}$$

Player i's rate of volunteering is also determined by by the threshold rates of active players.

Equation (1) implies that the combined rate of volunteering changes only when the set of active players changes. It occurs either when all the active players remain active and some passive player becomes active or when some active player stops volunteering. (By Lemma 1, we do not have to consider the case in which no player is active before time t and some player becomes active at time t.) We want to claim that the combined rate of volunteering strictly decreases in either case. Consider the first case. Let player i be the one who is passive before time t and becomes active at time t: $i \in A_t/A_{t-}$. Because he is willing to wait till time t, by the definition of b_i, the combined rate of volunteering just before time t has to be no lower than his threshold rate b_i: $\sum_{j \in I} a_j(t-) \geq b_i$ for any $i \in A_t/A_{t-}$. (Otherwise, he would have preferred not to wait till time t to become active.) Because $\sum_{j \in I} a_j(t-) = \frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1}$ by equation (1), we obtain $\frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1} \geq b_i$ for any $i \in A_t/A_{t-}$. This inequality generically holds strictly. In the proof, we show that when some players change their statuses from being passive to being active at time t, at least one player has to satisfy the above inequality strictly. By equation (1), the rate of volunteering at time t is given by $\sum_{j \in I} a_j(t) = \frac{\left(\sum_{A_{t-}} b_j\right) + \left(\sum_{A_t/A_{t-}} b_j\right)}{|A_{t-}| + |A_t/A_{t-}| - 1}$. Because $\frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1} \geq b_i$ for any $i \in A_t/A_{t-}$ and it holds strictly for some $i \in A_t/A_{t-}$, it is straightforward to show that the right-hand side is smaller than $\frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1}$. Using equation (1), we obtain $\sum_{j \in I} a_j(t-) > \sum_{j \in I} a_j(t)$. Let us now consider the second case and let player i be the player who stops volunteering at time t. If there is no active player at time t, the statement obviously holds. Thus, we consider the case in which there are some active players at time t. Because player i is active before time t, equation (2) shows that $b_i < \frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1}$. Moreover, the fact that player i stops volunteering at time t requires that he is not willing to wait at time t. Thus, his threshold rate has to be no higher than the combined rate of volunteering: $\sum_{j\in I} a_j(t) \le b_i$. Using equation (1), we can show that $\sum_{j\in I} a_j(t) < \sum_{j\in I} a_j(t-)$. Combining the analyses of this paragraph, we can conclude that the combined rate of volunteering never increases over time, and when the set of active players changes, it strictly decreases.

Next, we study the relationship between the combined rate of volunteering and the threshold rates of players in various statuses.

When player i is active, his rate of volunteering has to be positive: $a_i(t) > 0$. Equations (1) and (2) imply that $\sum_{j\in I} a_j(t) = \frac{\sum_{A_t} b_j}{|A_t|-1} > b_i$ for any $i \in A_t$. This shows that the combined rate of volunteering has to be higher than his threshold rate.

If player i is passive ($i \in B_t$), he needs an incentive to wait. Because the combined rate of volunteering is weakly decreasing, waiting for a short period should be preferable to immediately volunteering. Hence, the sum of the rates of volunteering by the other players has to be no lower than b_i: $\sum_{j\neq i} a_j(t) \ge b_i$. Because $a_i(t) = 0$ for any $i \in B_t$, the combined rate of volunteering has to be no lower than his threshold rate: $\sum_{j\in I} a_j(t) = \sum_{j\neq i} a_j(t) \ge b_i$ for any $i \in B_t$.

If player i exhausts the possibility of volunteering at time t, he does not want to delay his volunteering at time t. Thus, the sum of the rates of volunteering by the other players at time t has to be no higher than b_i: $\sum_{j\neq i} a_j(t) \le b_i$. (Otherwise, player i wants to exploit the higher combined rate of volunteering by delaying the volunteering near time t, thus postponing the exhaustion.) Because $a_i(t) = 0$ for any $i \in D_t$, the combined rate of volunteering has to be no greater than his threshold rate: $\sum_{j\in I} a_j(t) = \sum_{j\neq i} a_j(t) \le b_i$ for any $i \in D_t$.

The next lemma summarizes these findings.

Lemma 2

The combined rate of volunteering $\sum_{j\in I} a_j(t)$, as well as player i's rate of volunteering, is determined by the threshold rates of active players: $\sum_{j\in I} a_j(t) = \frac{\sum_{A_t} b_i}{|A_t|-1}$ and $a_i(t) = \frac{\sum_{A_t} b_j}{|A_t|-1} - b_i$ for any $i \in A_t$. The combined rate of volunteering weakly decreases over time *and strictly decreases whenever the set of active players changes: for any s and t such that $s < t$, $\sum_{A_s} a_i(s) \geq \sum_{A_t} a_i(t)$ and $\sum_{A_s} a_i(s) > \sum_{A_t} a_i(t)$ if $A_s \neq A_t$.*

Moreover, at time t, the combined rate of volunteering has to satisfy the following conditions.

(i) For any active player, it has to be higher than his threshold rate: $\sum_{j \in I} a_j(t) > b_i$ for $\forall i \in A_t$.

(ii) For any passive player, it has to be no lower than his threshold rate: $\sum_{j \in I} a_j(t) \geq b_i$ for $\forall i \in B_t$.

(iii) For any player who exhausts the possibility of volunteering exactly at time $t(> 0)$, it has to be no higher than his threshold rate: $\sum_{j \in I} a_j(t) \leq b_i$ for $\forall i \in D_t$.

This lemma plays a crucial role in determining the set of active players in the next subsection.

Before closing this subsection, we perform comparative statics with respect to the rates of volunteering, which can be directly derived from the above lemma.

When player i is active at time t, equation (2) shows that $a_i(t) = \sum_{A_t} a_i(t) - b_i$. When both players i and j are active at time t and player i has the higher threshold rate ($b_i > b_j$), then player i has the lower rate of volunteering ($a_i(t) < a_j(t)$). Thus, among active players, the player with a lower index tends to volunteer with a lower rate.

When player i is active at two different points in time, Lemma 2 in addition to equation (2) implies that his rate of volunteering never increases, and when the set of active players changes, it strictly decreases.

Corollary

Let $a_i(t)$ be the rate of volunteering by player i at time t.

(1) Suppose that both players i and j are active at time t and $i < j$. Then, $a_i(t) \leq a_j(t)$ holds. The strict inequality holds if and only if $b_i > b_j$.

(2) Suppose that player i is active both at time s and at time $t(> s)$. Then, $a_i(s) \geq a_i(t)$ holds. Moreover, if $A_s \neq A_t$, $a_i(s) > a_i(t)$ holds.

3.2. The possibility of the non-compromising type and the unique equilibrium path

Throughout this subsection, we assume that the probability of the non-compromising type is positive for every player: $z_i > 0$ for any $i \in I$. Abreu and Gul (2000) studied the possibility of the non-compromising type in a two-person war of attrition and showed that the equilibrium is unique. This subsection is its extension to the N-person case though the nature of the equilibrium changes substantially as shown shortly.

When there is a possibility that each player is the non-compromising type, the game cannot end by the initial volunteering with the probability of one. If the game does not end at time 0, Lemma 1 shows that no player volunteers with a positive probability mass afterward. Thus, the players randomize their timing of volunteering over time, which is a war of attrition. For a war of attrition to continue, active players need to volunteer at a positive combined rate so that the required condition in Lemma 2 is satisfied for each of these active players. This rate is higher than a certain positive number. Because there is the possibility of the non-compromising type, by some time T, any low-cost type volunteers for sure. At that time, any remaining player exhausts the possibility of volunteering. These properties, which hold for a two-person war of attrition, are carried over to an N-person war of attrition.

Lemma 3

A war of attrition occurs with a positive probability. There exists time T by which time any low-cost type volunteers for sure and consequently any player exhausts the possibility of volunteering.

The combination of Lemma 1 and this lemma implies that, when there are only two players, both are active from time 0 and they exhaust the possibility of volunteering at the same time. On the other hand, when there are more than two players, identifying the set of active players is essential in deriving the equilibrium path. By using the properties derived in the previous subsection, this subsection constructs it through backward induction.

As the first step toward constructing the equilibrium path, we consider the moment when an active player reduces the rate of volunteering to zero. Suppose that player i does so. Then, the total rate of volunteering by the other players just after that time cannot be higher than b_i: $\sum_{j \neq i} a_j(t) \leq b_i$. (Otherwise, player i would not volunteer with a positive rate just before time t.) Using equation (1), we can rewrite the above inequality as $\frac{\sum_{A_t} b_j}{|A_t|-1} \leq b_i$. To understand what will happen at that point, let us consider the generic case in which the strict inequality above holds. (The equality case can be treated in a similar way.) Because the combined rate of volunteering is weakly decreasing, the strict inequality implies that the low-cost type of player i would want to volunteer immediately at time t if he still remains in a war of attrition. Because volunteering does not occur with a positive probability mass after time 0, the low-cost type of player i should have volunteered for sure by time t. This means that player i exhausts the possibility of volunteering at that time. Note that this property implies that, once a player becomes active, he remains active until he exhausts the possibility of volunteering. It also implies that, when there is the possibility of the non-compromising type, exhausting the possibility of volunteering (i.e., never being active) is equivalent to having no chance to be the low-cost type. We use these terms interchangeably from here on.

Because $\sum_{j\neq i} a_j(t) < b_i$ and $a_i(t) = 0$, $\sum_I a_j(t) < b_i \leq b_k$ holds for any $k < i$. Hence, we obtain $\sum_{j\neq k} a_j(t) < b_k$. The low-cost type of player k would want to volunteer immediately at time t if he had not done so by that time. This implies that player k should exhaust the possibility of volunteering at least by time t. That is, when a player exhausts the possibility of volunteering, any player whose threshold rate is no lower exhausts the possibility of volunteering at least by that time. Because of this, we can partition players in a neat way based on their statuses at each moment. Specifically, if k is the highest index among those who exhaust the possibility of volunteering at time t and k' is the lowest among those, we obtain $C_{t-} = \{1, \ldots, k'-1\}$, $D_t = \{k', \ldots, k\}$, and $A_t \cup B_t = \{k+1, \ldots, N\}$.

When player i exhausts the possibility of volunteering at time t, Lemma 2 shows that the combined rate of volunteering is not higher than b_i. Just before time t, player i is active and thus Lemma 2 shows that the combined rate of volunteering has to be higher than b_i. Hence, either there is no active player left at time t or it holds that $\frac{\sum_{A_t} b_j}{|A_t|-1} \leq b_i < \frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1}$. We claim that, for the latter condition to be satisfied, some player who has waited by then needs to start volunteering with a positive rate at time t. Because any player whose index is smaller than i has exhausted the possibility of volunteering by time t, the one who joins the set of active players at time t has an index higher than i. To understand the claim intuitively, let us suppose that only player i exhausts the possibility of volunteering at time t and no player becomes active at time t: $A_t = A_{t-}/\{i\}$. Because $b_i < \frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1}$, a simple computation gives us $b_i < \frac{\sum_{A_{t-}/\{b_i\}} b_j}{|A_{t-}|-2} = \frac{\sum_{A_t} b_j}{|A_t|-1}$. This contradicts the above inequality, which proves that an additional player should become active at time t. By similar argument, even when more than one player exhausts the possibility of volunteering at time t, we can show that the required condition in Lemma 2 cannot be satisfied if no player becomes active at time t. Therefore, some player needs to become active at time t.

The next lemma summarizes these properties about how a set of active players changes along the equilibrium path.

Lemma 4

Suppose that, at time t, player i reduces the rate of volunteering to zero and that there are some players who still have the possibility of the non-compromising type: $P_i(t) > z_i$ for some $i \in I$. Then, the following properties hold.

(1) The low-cost type of player i volunteers for sure by time t, and thus he exhausts the possibility of volunteering at time t. For any $k < i$, player k exhausts the possibility of volunteering at least by time t: $i \in D_t$ and $k \in C_t$ for any $k < i$.

(2) At least one passive player whose index is higher than i becomes active at time t: $k \in A_t$ and $k \notin E_t$ for some $k > i$.

Lemma 4 implies that the set of active players changes either when all the players exhaust the possibility of volunteering or when some player becomes active. The key step in the construction of the unique equilibrium path is to find out whether some player exhausts the possibility of volunteering on these occasions and if so, who does. Let us say the set of active players changes at time t and let D_t be the set of the players who exhaust the possibility of volunteering at time t. We are interested in the characterization of D_t, given the set of continuously active players, E_t. Note that E_t may be empty as when the last remaining players exhaust the possibility of volunteering.

Suppose that the set of active players changes at time t. Let $\overline{C}_t$ denote the largest index among the players who have exhausted the possibility of volunteering by time t: $\overline{C}_t = \max_{j \in C_t} j$. If $\frac{\sum_{j=k}^{\overline{C}_t} b_j + \sum_{E_t} b_j}{\overline{C}_t - k + |E_t|} - b_k \leq 0$ for any $k \in \{1, \ldots, \overline{C}_t\}$, we say that $m(C_t, E_t)$ does not exist. Otherwise, we define the index $m(C_t, E_t)$ to be the smallest one that satisfies the following inequality with respect to k: $\frac{\sum_{j=k}^{\overline{C}_t} b_j + \sum_{E_t} b_j}{\overline{C}_t - k + |E_t|} - b_k > 0$. It can be shown that the above inequality holds for any $k \in \{m(C_t, E_t), \ldots, \overline{C}_t\}$ while $\frac{\sum_{j=k}^{\overline{C}_t} b_j + \sum_{E_t} b_j}{\overline{C}_t - k + |E_t|} - b_k \leq 0$ for any $k < m(C_t, E_t)$. (See Lemma A.1 in the appendix for the proof.) We want to claim that all the players whose index is between $m(C_t, E_t)$ and $\overline{C}_t$ exhaust the possibility of volunteering at time t: $D_t = \{m(C_t, E_t), \ldots, \overline{C}_t\}$. When $m(C_t, E_t)$ does not exist, no player does: $D_t = \emptyset$. Lemma 4 shows that $D_t = \{k, \ldots, \overline{C}_t\}$ for some $k \leq \overline{C}_t$ unless $D_t = \emptyset$. By definition, the set of active players just before time t is the combination of those who exhaust the possibility of volunteering exactly at time t and those who are active both at time t and just before it: $A_{t-} = D_t \cup E_t$. Lemma 2 shows that $\frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1} > b_i$ for any $i \in A_{t-}$. Hence, we know that k cannot be smaller than $m(C_t, E_t)$ because it would violate this inequality. What we are claiming here is that the reverse is true in some sense: the set D_t contains all the players as long as this inequality is satisfied. To understand this, suppose the contrary and assume that k were bigger than $m(C_t, E_t)$. By supposition, it would hold that $\frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1} > b_{k-1}$. Lemma 4 implies that player $k-1$ should have exhausted the possibility of volunteering before time t, say at time s, and no player would exhaust the possibility of volunteering between time s and time t. Thus, the set of active players at time s were a subset of A_{t-}. Because the threshold rates of the players in A_{t-} is no higher than that of player $k-1$, the above inequality implies that $\frac{\sum_{A_s} b_j}{|A_s|-1} > b_{k-1}$. (We prove this claim in Lemma A.1.) This contradicts Lemma 2 as player $k-1$ were supposed to exhaust the possibility of volunteering at time s. Using this logic, the next lemma characterizes the set of the players who exhaust the possibility of volunteering at time t.

Lemma 5

Take time t and suppose that, at that time, either at least one player becomes active or the last remaining low-cost types exhaust the possibility of volunteering.

If $m(C_t, E_t)(\leq \overline{C}_t)$ exists, then the players with the indexes in $\{m(C_t, E_t), \ldots, \overline{C}_t\}$ have to exhaust the possibility of volunteering at time t: $D_t = \{m(C_t, E_t), \ldots, \overline{C}_t\}$. Otherwise, there is no player who exhausts the possibility of volunteering at time t: $D_t = \emptyset$.

Suppose that the set of active players changes at time t. Then, this lemma can identify the set of active players just before time t given the set of continuously active players at time t. This enables us to construct the equilibrium path backward in time. Choose T provisionally as a large number. This is the time at which the last players exhaust the possibility of volunteering.

At $t = T$, we have $C_T = I$ and $E_T = \emptyset$. Let $m(C_T, E_T)$ be the smallest solution to the following inequalities: $\frac{\sum_{j=k}^{N} b_j}{N-k} - b_k > 0$ and $\frac{\sum_{j=k-1}^{N} b_j}{N-k+1} - b_{k-1} \leq 0$. The index $N-1$ satisfies the first inequality. Hence, $m(C_T, E_T)$ exists and $m(C_T, E_T) \leq N-1$ holds.

Lemma 5 shows that, just before time T, the set of active players needs to be given by $\{m(C_T, E_T), \ldots, N\}$.

We now consider the behaviors of the players before time T. Lemma 1 shows that no player volunteers with a positive probability mass after time 0 and Lemma 4 shows that an active player remains active until he exhausts the possibility of volunteering. It implies that a player becomes active at a certain time, and then he keeps randomizing between waiting and volunteering until he exhausts the possibility of volunteering. Note that, unless a player does the initial volunteering, his ex-ante probability of not volunteering by the time when he becomes active is obviously one. (A player who does the initial volunteering volunteers with a positive probability at time 0, and hence his ex-ante probability of not volunteering by the time slightly after time 0 is reduced by that probability from one.) In the following, we see how this occurs for each player by construction. Given the set of active players, we can compute the rate of volunteering for each active player by equation (2). Using this rate backward, we can also compute the ex-ante probability that such a player does not volunteer by each point in time. At some earlier time $t(< T)$, this probability reaches one for some player. This means that he becomes active exactly at this instance. We then apply the procedure in Lemma 5. If $m(C_t, E_t)$ does not exist, Lemma 5 shows that there should be no player who exhausts the possibility of volunteering at that time. Then, we proceed backward in time until we reach the point at which another player becomes active. On the other hand, if $m(C_t, E_t)$ exists, Lemma 5 shows that $A_{t-} = \{m(C_t, E_t), \ldots, \overline{C}_t\} \cup E_t$. Using the modified set of active players, we go back further in time. Lemma 4 shows that the set of active players changes before time T only when a player becomes active. This implies that, when we want to see how the set of active players changes, we only need to look at the time when some player becomes active. Note that a war of attrition can be maintained whenever there is more than one player. Thus, if there is more than one player who either is supposed to exhaust the possibility of volunteering by time t or is continuously active, we always find two active players just before time t. If $|C_t| + |E_t| \geq 2$, the number of active players just before time t is greater than one: $|A_{t-}| \geq 2$.

The process stops at time s when there are not enough players left to maintain a war of attrition prior to time s: $|C_s| + |E_s| < 2$. If there is one remaining player, let i be the index of that player. (There can be at most one player whose ex-ante probability of not volunteering by this time does not reach one. If there were more than one, we could continue the above process with these players.) Let j be that of the player who becomes active at time s. (We choose one arbitrarily if there is more than one player who becomes active at time s.) If there is no remaining player, and thus several players simultaneously become active at time s, we choose players i and j arbitrarily among them. Then, we redefine T by subtracting s from the original one that we chose provisionally. By this change, we make player j become active at time 0. This adjustment is necessary because at most one player does the initial volunteering and also there have to be at least two active players during a war of attrition as shown in Lemma 1. If the ex-ante probability that player i does not volunteer by just after time 0 is less than one, i.e., $P_i(0+) < 1$, its difference from one has to be absorbed by the initial volunteering. In other words, if there is the initial volunteering, it is done by player i and its probability is given by $1 - P_i(0+)$.

The above argument shows that there is only one way to construct the equilibrium path, which proves its uniqueness. The construction is based on the local optimality at each point in time. It can be shown that the constructed strategies are globally optimal. The next proposition summarizes the above arguments and shows that the constructed path forms an equilibrium.

Proposition 1

The equilibrium path is unique and can be constructed as above. In particular, it takes the following form.

(1) *At time* 0, *at most one player volunteers with a positive probability (initial volunteering).*

(2) *During a war of attrition* ($0 < t < T$)*, there are at least two active players and*

(2-i) *the rate of volunteering by player i is given by* $a_i(t) = \frac{\sum_{A_t} b_j}{|A_t|-1} - b_i$ *for any* $i \in A_t$,

(2-ii) *both active players and passive players have higher indexes (i.e., lower threshold rates) than those who have exhausted the possibility of volunteering, and the lower end of those who exhaust the possibility of volunteering exactly at time t is given by the above* $m(C_t, E_t)$: $A_t \cup B_t = \{k+1, \dots, N\}$, $C_t = \{1, \dots, k\}$, *and* $D_t = \{k', \dots, k\}$ *for some k and k' such that* $1 \leq k' \leq k \leq N-2$ *and* $k' = m(C_t, E_t)$, *and*

(2-iii) *if some player exhausts the possibility of volunteering at time t, some passive player becomes active: if* $D_t \neq \emptyset$, $E_t \subset A_t$ *and* $E_t \neq A_t$.

(3) After time T, there is no chance of volunteering.

As this is one of the main findings of the paper, we offer an intuitive explanation. For the proposition to hold, there are three main factors. The first factor is the possibility of the non-compromising type, because of which the players exhaust the possibility of volunteering within a finite time, as shown in Lemma 3. In the above, we construct the equilibrium path backward from the time that the last players do so. The uniqueness of the equilibrium is critically dependent on this property. The second factor is the timing of exhaustion. During a war of attrition, Lemma 2 shows that the threshold rate of a player needs to satisfy a certain condition depending on his status. Using the findings in Lemma 2, Lemma 4 shows that the players with higher threshold rates exhaust the possibility of volunteering no later than those with lower ones. This gives us the ordering structure as to when each player ceases to be active. The third factor is the characterization of the set of active players at each moment. Note that any low-cost type wants to be active at some point because the other players may be the non-compromising type. To answer when he should be active, Lemma 5 proves that the set of active players just before the set of active players changes is uniquely related to the set of continuously active players just after it. Following the equilibrium path backward in time enables us to uniquely identify the set of active players at each moment. The combination of these three factors gives us the above proposition.

3.3. An example: two or three players

We apply the findings of the previous two subsections to the example of either two or three players and demonstrate how the players behave in equilibrium.[4]

For this subsection, we suppose that b_i is strictly decreasing in i, and z_i is positive and weakly increasing in i: $b_1 > b_2 > b_3$ and $0 < z_1 \leq z_2 \leq z_3$. (We discuss the meaning of the latter assumption in the next section.)

3.3.1. Two players

When there are only two players, they exhaust the possibility of volunteering at the same time. Let that time be T. Before time T, both players are active. Based on the analysis in

[4] The case of two players is already analyzed in Abreu and Gul (2000) for the study of bargaining.

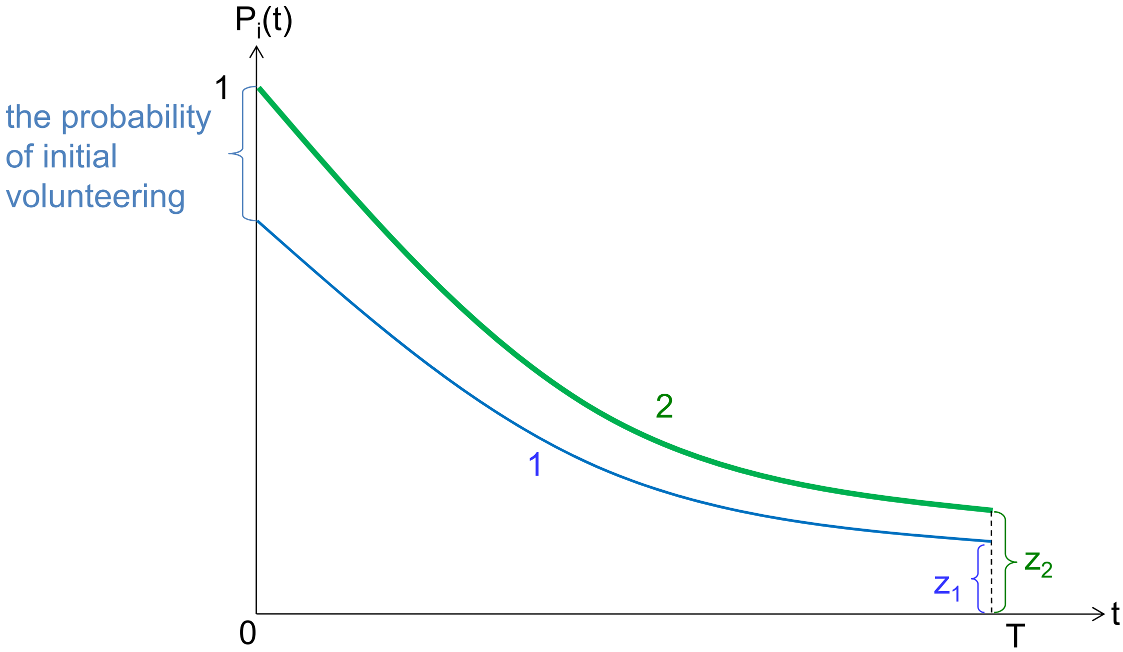
Section 3.1, we know that player 1 volunteers at the constant rate $a_1(t) = b_2$ and player 2 does so at the constant rate $a_2(t) = b_1$. That is, we obtain $a_i(t) \equiv b_{3-i}$ for any $t \in (0, T)$.

Observe that the ex-ante probability that player i has not volunteered by time T is z_i. Hence, $P_i(t) = z_i e^{b_{3-i}(T-t)}$ holds. Because $z_1 \leq z_2$ and $b_1 > b_2$ by the supposition of this subsection, $P_2(t) = z_2 e^{b_1(T-t)} > z_1 e^{b_2(T-t)} = P_1(t)$ for any t such that $0 < t < T$. It implies that the ex-ante probability that player 2 remains in a war of attrition is higher than the one for player 1 before both players exhaust the possibility of volunteering. Figure 1 depicts how the ex-ante probability that each player does not volunteer by time t changes over time. The inequality implies that player 1 does the initial volunteering. It also implies that player 2 exhausts the possibility of volunteering at time T without doing the initial volunteering: $z_2 e^{b_1 T} = 1$. Hence, we have $T = \frac{1}{b_1} \log \frac{1}{z_2}$.

The probability of the initial volunteering by player 1 is given by the difference between one and the ex-ante probability that he does not volunteer by just after time 0: $1 - P_1(0+)$. Using the above formula for T, we can compute it as follows.

$$\begin{aligned} 1 - P_1(0+) &= 1 - z_1 e^{b_2 T} \\ &= 1 - e^{b_2\left(-\frac{-\log z_1}{b_2} + \frac{-\log z_2}{b_1}\right)}. \end{aligned}$$

Figure 1 Two Players



3.3.2. Three players

There are roughly two patterns of equilibrium paths, depending on whether player 1 is active when both players 2 and 3 exhaust the possibility of volunteering.

First, we study the case in which the following condition holds:

$$\frac{\sum_{j \in I} b_j}{2} - b_i > 0 \text{ for } \forall i \in I.$$

Note that this condition is satisfied, for example, when the players have similar threshold rates. Let $\hat{T}$ be the time at which the last players exhaust the possibility of volunteering. When the above inequality holds, Proposition 1 implies that all three players exhaust the possibility of volunteering at time $\hat{T}$. Let a'_i be the rate of volunteering at that time. Equation (2) has shown that $a'_i = \frac{b_1+b_2+b_3}{2} - b_i$. As long as all of the three players are active, the ex-ante probability that player i has not volunteered by time t is given by $z_i e^{a'_i(\hat{T}-t)}$. By the assumption about b_i and z_i, this probability is strictly increasing in i for any $t < T$. Thus, player 3 is not active at time 0 and becomes active at some point, say at time $\hat{t}(> 0)$. At time $\hat{t}$, the ex-ante probability that the player has not volunteered by that time is higher for player 2 than for player 1. Then, applying the above analysis for the example of two players, we can conclude that player 1 has to volunteer initially. (See Figure 2 for the ex-ante probability that each player does not volunteer by time t in this example.)

During the interval $(0, \hat{t})$, the rate of volunteering by player 1 is $a_1(t) = b_2$ and that by player 2 is $a_2(t) = b_1$. During the interval $(\hat{t}, \hat{T})$, the rate of volunteering by player i is given by $a'_i = \frac{b_1+b_2+b_3}{2} - b_i$. For players 1 and 2, we can show that the rates of volunteering decrease at time $\hat{t}$, as predicted by the corollary to Lemma 2. The combined rate of volunteering during the interval $(0, \hat{t})$ is given by $b_1 + b_2$ and the one during the interval $(\hat{t}, \hat{T})$ is given by $\frac{b_1+b_2+b_3}{2}$. As predicted also by the corollary to Lemma 2, the latter is lower than the former even though the number of active players increases.

After some computation, we can derive the probability of the initial volunteering:

$$1 - \exp\left\{ b_2 \left(-\frac{-\log z_1}{b_2} + \left(\frac{a'_1 b_1 - b_2 a'_2}{b_2 b_1} \right) \frac{-\log z_3}{a'_3} + \frac{-\log z_2}{b_1} \right) \right\}.$$

Let us compare this probability with the one in the game in which there are only the first two players. Because it can be shown that $a'_1 b_1 - b_2 a'_2 < 0$, the above is higher than the one for the first two players. The probability of the initial volunteering increases by the addition of the third player with the lowest threshold rate. (This property also holds in general, as shown in Proposition 3.)

Next, we study the case in which the players are fairly asymmetric, and thus

$$\frac{\sum_{j \in I} b_j}{2} - b_1 < 0.$$

This condition is satisfied when the threshold rates of the players are quite different and b_1 is far higher than b_2 and b_3. In this case, the combined rate of volunteering that player 1 needs in a war of attrition is so high that player 1 cannot be in a war of attrition with the other two players. Proposition 1 shows that player 1 has to exhaust the possibility of volunteering before the other players do. (To understand why this is so, let us derive the upper bound for b_1 when all three players are simultaneously engaged in a war of attrition. When player i is engaged in a war of attrition, he is indifferent between volunteering and waiting. Thus, $b_i = a_j(t) + a_k(t)$, where j and k are the indexes of the other two players, respectively. Because $b_2 = a_1(t) + a_3(t)$ and $b_3 = a_1(t) + a_2(t)$, it holds that $b_2 + b_3 > a_2(t) + a_3(t) = b_1$. This implies that, when all of the three players are simultaneously engaged in a war of attrition, it is necessary that b_1 is lower than $b_2 + b_3$. This is what the above inequality requires.) Let $\tilde{T}$ be the time that the last players exhaust the possibility of volunteering in this situation. Proposition 1 prescribes that only player 2 and player 3 are active just before time $\tilde{T}$. By the same argument made above, the ex-ante probability that the player has not volunteered by a given time is higher for player 3 than for player 2. Thus, player 3 cannot be active at time 0 and becomes active at some time, say $\tilde{t}$. At time $\tilde{t}$, player 1 exhausts the possibility of volunteering ahead of the other players. (Figure 3 depicts the ex-ante probability that each player does not volunteer by time t for the case of asymmetric players.) What happens before time $\tilde{t}$ is analogous to what happens in the example of two players. At time 0, player 1 volunteers with a positive probability (initial volunteering). During the interval $(0, \tilde{t})$, player 1 and player 2 are actively engaged in a war of attrition.

During the interval $(0, \tilde{t})$, the rate of volunteering by player 1 is b_2 and that of player 2 is b_1. At time $\tilde{t}$, player 1 exhausts the possibility of volunteering, and then player 3 becomes active. During the interval $(\tilde{t}, \tilde{T})$, the rate of volunteering by player 2 is b_3 and that of player 3 is b_2. As is consistent with the prediction in the corollary to Lemma 2, the rate of volunteering by player 2 decreases from b_1 to b_3 at time $\tilde{t}$, and the combined rate of volunteering decreases from $b_1 + b_2$ to $b_2 + b_3$.

The probability of the initial volunteering can be shown to be

$$1 - \exp\left\{ b_2 \left(-\frac{-\log z_1}{b_2} + \frac{-\log z_2}{b_1} - \frac{b_3}{b_1} \frac{-\log z_3}{b_2} \right) \right\}.$$

This is also higher than the one for the first two players.

An interesting case for the group with fairly asymmetric players occurs when $b_1 > b_2 = b_3$ and $b_1 > b_2 + b_3$. In this special case, it is easy to see that $\tilde{t} = 0$ and that the low-cost type of player 1 volunteers initially with the probability of one, attaining the highest probability of the initial volunteering.[5]

[5] We can extend this property and show the following: when there is a single player with the higher threshold rate (player 1) and sufficiently many identical players with the lower threshold rate, the low-cost type of player 1 volunteers initially with the probability of one. In such a situation, a war of attrition among the players with the lower threshold rate produces a slow combined rate of volunteering. Hence, the low-cost type of player 1 volunteers before it begins.

Figure 2 Three Similar Players

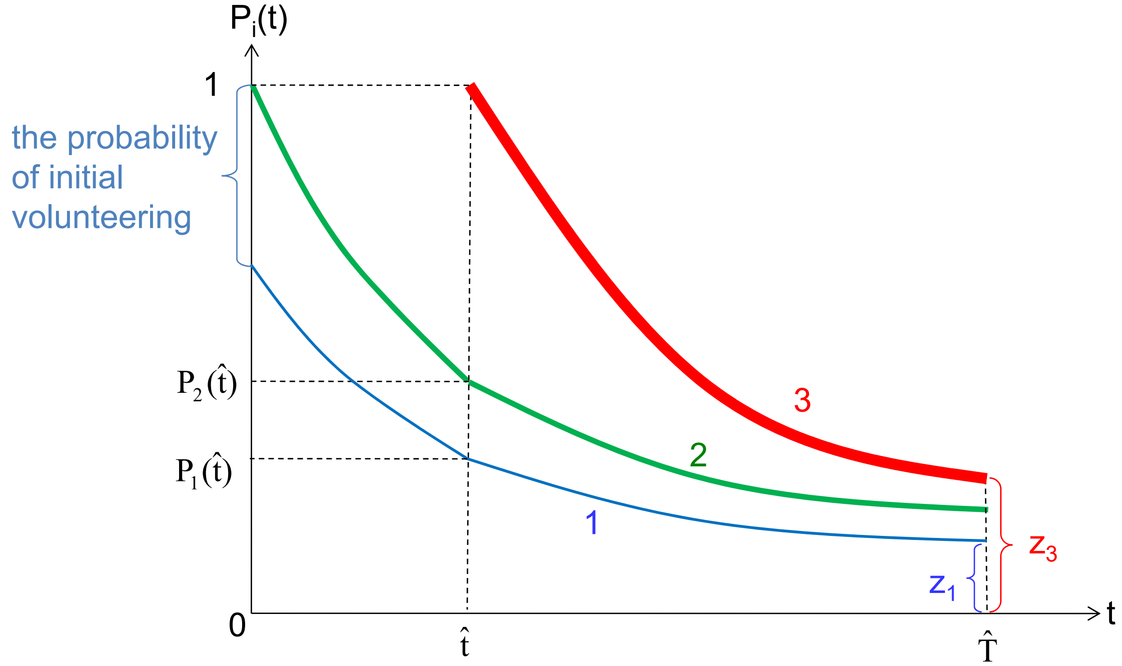
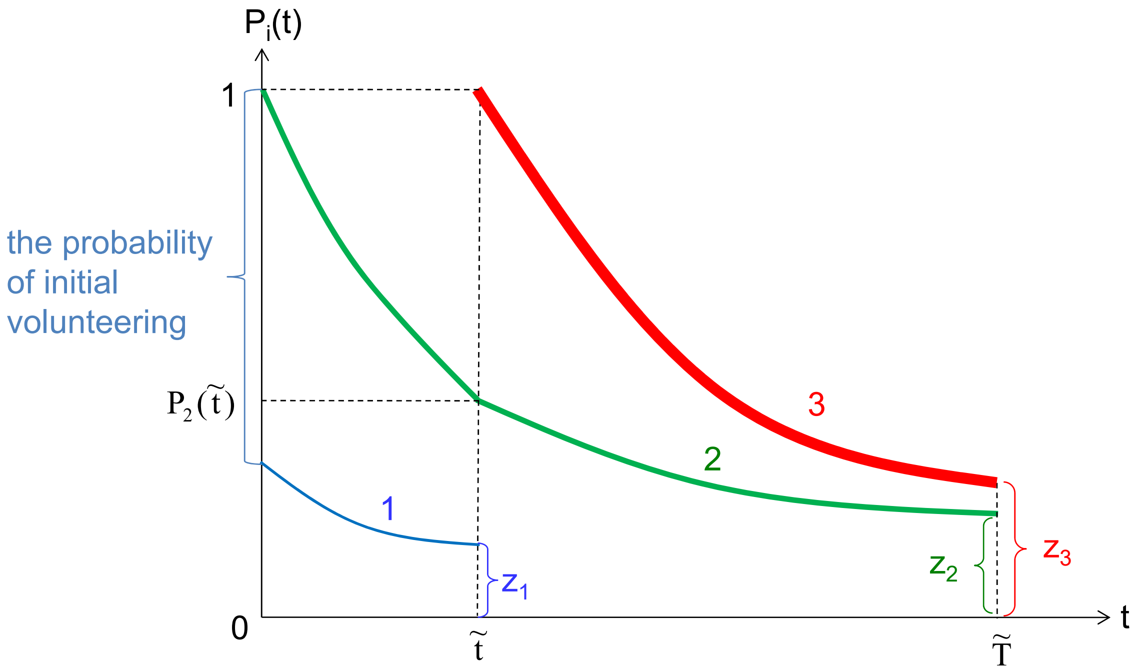


Figure 3 Three Asymmetric Players



4. Initial Volunteering

This section studies who volunteers initially and with what probability. Initial volunteering is an important event for two reasons. First, it improves efficiency, as the delay associated with a war of attrition is avoided. Second, it attains the highest feasible payoff for any player other than the one who does so.

For the analyses in this section, we suppose that the probability of the non-compromising type is weakly increasing in the index.[6] We also assume that the probability of the non-compromising type is positive for every player throughout this section.

Assumption Z

We assume that $0 < z_i \leq z_j$ *for any* $i, j \in I$ *such that* $i < j$.

When the probability of the non-compromising type is higher, given other things are equal, the player tends to exhaust the possibility of volunteering more quickly. As shown in Lemma 4, the player with a lower index (i.e., a weakly higher threshold rate) exhausts the possibility of volunteering no later than the one with a higher index. The corollary to Lemma 2 shows that the rate of volunteering by the player with a lower index is not higher than the one with a higher index. Combining these properties, under Assumption Z, we can conclude that a player with a lower index becomes active and exhausts the possibility of volunteering no later than a player with a higher index. The necessary and sufficient condition that two players become active at the same time is that they have both the same threshold rate and the same probability to be the non-compromising type.

Lemma 6

Suppose that Assumption Z is satisfied. Take $i, j \in I$ *such that* $i < j$. *If player* i *is active during the interval* (t, t') *and player* j *is active during the interval* (τ, τ'), *then* $t \leq \tau$ *and* $t' \leq \tau'$. *When and only when both* $b_i = b_j$ *and* $z_i = z_j$ *hold,* t *is equal to* τ.

[6] This is not an innocuous assumption. For example, when z_2 is substantially lower than any other z_j's, player 2 tends to do the initial volunteering instead of player 1. This is contrary to the conclusion of Proposition 2, which we will obtain shortly under this assumption. We make this assumption because we think it to be natural in a real setting. Consider the situation in which the players can make some ex-ante investment that affects the probability of being the non-compromising type. Suppose that the players are similar in terms of both the discount rate and the cost to take the initial role, but are different in terms of the value from the project. Then, a player with a higher value from the project has a stronger ex-ante incentive to avoid the lack of a volunteer. This implies that a player with a lower index wants to reduce the chance to be the non-compromising type more. In such a situation, this assumption seems natural.

The direct application of this lemma to the start of a war of attrition gives us the following property about the initial volunteering.

Proposition 2

Suppose that Assumption Z is satisfied. The initial volunteering, if any, is made by player 1. It occurs if and only if either $b_1 > b_2$ or $z_1 < z_2$ holds.

This proposition says that, if the initial volunteering occurs, it is made by player 1. Moreover, it shows that, for the initial volunteering to occur, player 1 has to be different from player 2 either in terms of the threshold rate or in terms of the probability to be the non-compromising type. When player 1 and player 2 are identical in both terms, the probability of the initial volunteering is zero.

We now study the probability of the initial volunteering.

As the first inquiry to the probability of the initial volunteering, we analyze the effect of the number of players on it. In particular, we consider the situation in which some player with a lower threshold rate can be brought to the meeting. Though such a player never volunteers initially, we show that his inclusion increases the chance of the initial volunteering by inducing player 1 to do so more often.

Suppose that there are M players originally, and that a player whose threshold rate is lower than any of M original players joins the group. Furthermore, suppose that they are sufficiently similar in threshold rates; even with the added player, all the players exhaust the possibility of volunteering at the same time in equilibrium. To understand the effect of the additional player, let us look at the players' action near the end of a war of attrition. When there are M players, the rate of volunteering by player i is given by $\frac{\sum_{i=1}^{M} b_j}{M-1} - b_i$. When an additional player is brought in, it decreases to $\frac{\sum_{i=1}^{M+1} b_j}{M} - b_i$ because the denominator increases and b_{M+1} is smallest. This decrease is uniform across players. Because the rates of volunteering by the players with higher threshold rates are generally lower than those by the player with lower ones, in terms of proportion, those with higher threshold rates are affected more by this decrease. As a consequence, player 1 never decreases the chance of his initial volunteering and, when $b_1 > b_2$, he increases it.

Proposition 3

Consider $M+1$ players: $\{1, \ldots, M, M+1\}$, where $M \geq 2$. Suppose that Assumption Z is satisfied and that $\frac{\sum_{j=1}^{M+1} b_j}{M} - b_1 > 0$. Then, when $b_1 > b_2$, the probability of the initial *volunteering in the game with the first M players is smaller than the one in the game with all the $M + 1$ players. On the other hand, when $b_1 = b_2$, the former and the latter are the same.*

When the condition in Proposition 3 is not satisfied, the addition of a player with a low threshold rate does not always lead to an increase in the probability of the initial volunteering. For example, suppose that there are originally three players for whom both $b_1 + b_2 + b_3 < 2b_1$ and $b_2 + 2b_3 < 2b_2$ hold. In the original game, player 2 is engaged in a war of attrition against player 1 first and then against player 3. When a player whose threshold rate is identical to that of the third player is brought in, player 2 is engaged in a war of attrition only against player 1. The addition of a fourth player causes the probability of the initial volunteering to decrease. When the players do not exhaust the possibility of volunteering at the same time in the modified game, the addition of the player may decrease the probability of the initial volunteering.

We next derive the lower bound for the probability of the initial volunteering. Using Proposition 3 repeatedly, we can show that the modified game in which only the first two players play gives the lower bound for the probability of the initial volunteering. As the first case, consider the game in which all the players exhaust the possibility of volunteering at the same time in the equilibrium. Because removing the player with the highest index maintains the property that all the players exhaust the possibility of volunteering at the same time, the repeated application of Proposition 3 tells us that the probability of the initial volunteering becomes lower when only the first two players play. As the other case, consider the game in which the first player exhausts the possibility of volunteering before some other players do so. For such a game, we modify the game by truncating it at that point in time. Then, we can apply Proposition 3. (Specifically, consider the modification in which we take only the players who are active just before the time of truncation, and set their probabilities of being the non-compromising type to be equal to the ex-ante probabilities of not volunteering by that time. Then, the equilibrium of the modified game is identical to the original game up to the time of truncation.) In the modified game, the second player may have a higher chance to be the non-compromising type than in the original game, which tends to increase the probability of the initial volunteering. Therefore, in both cases, the probability of the initial volunteering in the modified game with the first two players gives a lower bound for the probability of the initial volunteering.

Proposition 4

Suppose that Assumption Z is satisfied. The probability of the initial volunteering is no smaller than

$$1 - e^{b_2\left(-\frac{-\log z_1}{b_2}+\frac{-\log z_2}{b_1}\right)}.$$

When z_1 is small and $b_1 > b_2$, the above proposition implies that volunteering is likely to occur in the beginning and the meeting is concluded quickly. This is true even when z_2 is also small as long as Assumption Z is satisfied. Under Assumption Z, $(-\log z_2)/(-\log z_1) \leq 1$ holds. When z_1 goes to zero, it is easy to see that $\left(-\frac{-\log z_1}{b_2} + \frac{-\log z_2}{b_1}\right)$ goes to minus infinity. Then, the above proposition implies that the probability of the initial volunteering goes to one. Roughly speaking, when the probability that the player with the highest threshold rate is the non-compromising type is small, and his threshold rate is strictly higher than those of the other players, it is quite likely that he volunteers at the beginning of the meeting.

Consider one of the following four situations: (1) the probability that player 2 is the non-compromising type becomes one ($z_2 \uparrow 1$); (2) player 2 becomes infinitely more patient relative to player 1 ($r_2/r_1 \downarrow 0$); (3) player 1's cost of volunteering becomes infinitesimally small compared with that of player 2 ($c_1/c_2 \downarrow 0$); (4) and player 2's net benefit from the project becomes infinitesimally small compared to that of player 1 ($(V_2 - c_2)/(V_1 - c_1) \downarrow 0$). Note that the exponent part of the exponential function in Proposition 4 can be expressed as $b_2\left(-\frac{-\log z_1}{b_2} + \frac{-\log z_2}{b_1}\right) = \log z_1 - \frac{b_2}{b_1}\log z_2 = \log z_1 - \frac{r_2(V_2-c_2)c_1}{r_1(V_1-c_1)c_2}\log z_2$. Hence, in each of the four cases, when any other variables are fixed, the bound in Proposition 4 converges to $1 - z_1$. Because it is equal to the highest feasible probability for the initial volunteering by player 1, it implies that the probability of the initial volunteering itself converges to $1 - z_1$. That is, at the limit, the low-cost type of player 1 is certain to volunteer at time 0.

Corollary

Suppose that Assumption Z is satisfied.

(1) Suppose that $b_1 > b_2$. When only the probabilities that the players are the non-compromising type change and when $z_1 \downarrow 0$, the probability of the initial volunteering goes to one.

(2) When one of the above four situations occurs without the other variables changing, the probability of the initial volunteering goes to $1 - z_1$.

Observe that the above statement may not hold when some other variables that are supposed to be fixed also change. For example, in the case of the second statement, even when $c_1 \downarrow 0$, the bound does not converge to $1 - z_1$ if $r_1 = \alpha c_1$ for some $\alpha(> 0)$.

5. Concluding Remarks

We have shown that the players' behaviors in an N-person war of attrition are qualitatively different from those in a two-person version. This is because a war of attrition needs more than one, but not all of the players. Hence, in an N-person war of attrition, the set of active players can change over time. This generally leads to a rich variety of equilibria. In the first subsection below, we show that it is actually the case when there is no possibility of the non-compromising type. However, with the possibility of the non-compromising type, this study has shown that we obtain the unique equilibrium path, which enables us to predict who volunteers with what probability at each moment.

A war of attrition is one way to choose a volunteer. From that view point, the initial volunteering is important because the selection occurs without discounting costs. We have shown that it tends to be carried out with a positive probability by the player who is less patient, whose cost of taking the initiative role is lower, and whose net gain from the project is higher. In order to evaluate how efficient this way of selection is, the second subsection compares it with a more formal procedure to choose one player.

5.1. No possibility of the non-compromising type

We have derived a unique equilibrium by assuming that any player is the non-compromising type with a positive probability. For the purpose of comparison, for this subsection only, suppose that any player is always the low-cost type: $z_i = 0$ for any $i \in I$. Without the possibility of the non-compromising type, once a war of attrition starts, there is a positive probability of volunteering for any interval. By extending the arguments made for Lemma 2 to this case, it is easy to see that the following in addition to the conditions in Lemma 1 and Lemma 2 provides the necessary and sufficient condition for the equilibria that involve a war of attrition.

(1) A war of attrition continues forever unless a player volunteers for sure at time 0.

(2) The combined rate of volunteering is no smaller than the threshold rate of any inactive player: for any t, $\frac{\sum_{A_t} b_j}{|A_t|-1} \geq b_i$ for $i \in I/A_t$.

(3) If a player is active at some point, either he remains active from then on or he becomes inactive later when the combined rate of volunteering coincides with his threshold rate: if $i \in A_s$ for some s, it holds either that $i \in A_\tau$ at any time $\tau(> s)$ or that there exists time $\tau(> s)$ such that $i \in A_t$ for any t where $s < t < \tau$, $i \in I/A_\tau$, and $\frac{\sum_{A_\tau} b_j}{|A_\tau|-1} = b_i$.

(4) If player i does the initial volunteering with a probability between zero and one, either he is active at time 0 or his threshold rate is equal to the combined rate of volunteering at time 0: $i \in A_0 \cup \{i \notin A_0 : \frac{\sum_{A_0} b_j}{|A_0|-1} = b_i\}$.

This observation implies that, without the possibility of the non-compromising type, the set of equilibria is large and who volunteers initially with what probability is indeterminate. (There are also equilibria where any one of the players volunteers in the beginning with the probability of one and there is no war of attrition.) Moreover, once a war of attrition starts, any active player generically remains active forever. (The condition for him to stop volunteering is given by the equality and thus is not satisfied generically.) One consequence of this property is that only a subset of players may participate in a war of attrition and some player never becomes active. Thus, the equilibrium behaviors are also quite different from those in the model with the possibility of the non-compromising type.

To illustrate these differences, let us consider a situation in which the threshold rates are similar among players. Then, it is easy to see that the following forms an equilibrium: any given combination of players is active at the beginning, any one of them volunteers initially with any given positive probability, and any player can join the set of active players at any time.

5.2. Comparison with the predetermined assignment

We have studied the meeting in which any player can volunteer at any time. This particular format is meant to capture both the democratic nature and the flexible procedure of a casual meeting. On the other hand, we can consider a more formal meeting in which the selection for the initiative role is done according to a formal rule. This subsection briefly compares these two different procedures. Given the possibility that each player is the non-compromising type and thus is unable to take the initiative role, the purpose of the comparison is to find out how efficiently each procedure selects a low-cost type as a volunteer.

First, as a typical example of a more formal procedure, we consider the procedure in which the players take turns to decide whether to volunteer or not within finite rounds in a pre-specified order.[7] Suppose that the probabilities of the non-compromising types are not so large; $(1-\max_j z_j)V_i > V_i - c_i$ for any $i \in I$. Under this supposition, in any finite ordering of the players' chances to volunteer, all players but the last one are certain to decline to volunteer. Let player j be the last player to decide. He volunteers when and only when he is the low-cost type. Hence, his expected payoff is given by $(1 - z_j)(V_j - c_j)$. The expected payoff of player $i(\neq j)$ is given by $(1 - z_j)V_i$. (We ignore the discounting for the analysis of this procedure.) The sum of the expected payoffs is given by $(1 - z_j)\left[\left(\sum_i V_i\right) - c_j\right]$.

Next consider the procedure that is studied in this paper. Let player k be the one who volunteers initially with a positive probability and let P_k be its probability. When player k volunteers initially, player $i(\neq k)$ obtains the payoff of V_i. Otherwise, the expected payoff of the low-cost type of player i just after time 0 is $V_i - c_i$ when he is active at time 0, and that is between $V_i - c_i$ and V_i when he is not active at time 0. The non-compromising type of player i obtains a lower payoff due to the inability to volunteer. Hence, when player i is active at time 0, his expected payoff is no smaller than $P_k V_i + (1 - z_i)(1 - P_k)(V_i - c_i)$ and is no higher than $P_k V_i + (1 - P_k)(V_i - c_i)$. When player i is not active at time 0, his expected payoff is no smaller than $P_k V_i + (1 - z_i)(1 - P_k)(V_i - c_i)$ and is no higher than V_i. By the same argument, the expected payoff of player k is no smaller than $P_k(V_k - c_k) + (1 - P_k - z_k)(V_k - c_k)$ and is no higher than $P_k(V_k - c_k) + (1 - P_k)(V_k - c_k)$. Summing these, we can say that the total expected payoff is between $P_k[(\sum_i V_i) - c_k] + (1 - P_k)\sum_{i\neq k}(1 - z_i)(V_i - c_i)$ and $P_k[(\sum_i V_i) - c_k] + (1 - P_k)[\sum_{A_0}(V_i - c_i) + \sum_{I/A_0} V_i]$.

[7] This procedure is more general than the one in which one particular player is singled out as the sole candidate for the initiative role. As argued in the main text, it turns out that these two procedures are equivalent in that only one player effectively decides whether to volunteer or not. Note that, in each of these procedures, the rejection by the last player or the designated player respectively is supposed to result in the failure of the project. However, in reality, it will often prompt the resumption of the search process. If the selection process continued after rejections, the situation would become similar to what we study in this paper. That is, we can interpret our model as the one in which the players cannot commit themselves not to terminate the meeting unless they are certain that there is no low-cost type.

For the sake of comparison, suppose that the selected player in the former procedure is player 1 and that the player who volunteers initially with a positive probability in the latter procedure is also player 1. Moreover, suppose that player 2 is active at time 0 in the latter procedure. (By Lemma 1, there have to be at least two active players during a war of attrition.) Note that the sum of the expected payoffs from the former procedure is given by $(1 - z_1) \left[\left(\sum_i V_i \right) - c_1 \right]$. When the low-cost type of player 1 does the initial volunteering almost certainly (i.e., $P_1 \approx 1 - z_1$), the lower bound of the total expected payoff under the latter procedure is approximated by $(1 - z_1) \left[\left(\sum_i V_i \right) - c_1 \right] + z_1 \sum_{i \neq 1} (1 - z_i)(V_i - c_i)$. The first term is equal to the total expected payoff under the former procedure. Thus, when the low-cost type of player 1 volunteers initially with a probability close to one, the latter procedure with the possibility of a war of attrition can achieve higher efficiency. One such example is presented at the end of Section 3, where there is strong asymmetry among players and the low-cost type of player 1 volunteers initially with the probability of one. On the other hand, when the low-cost type of player 1 almost never does the initial volunteering (i.e., $P_1 \approx 0$), the higher bound of the total expected payoff under the latter procedure is approximated by $\sum_{i=1}^{2} (V_i - c_i) + \sum_{i=3}^{N} V_i = \left(\sum_i V_i \right) - c_1 - c_2$. (To obtain the higher bound, we suppose that only players 1 and 2 are active at time 0.) When z_1 is small and c_2 is significant, this tends to be lower than the expected total payoff under the former procedure. In such a situation, it is likely that the former procedure achieves the higher efficiency. When the set of active players in the beginning includes more players, the payoff from the latter procedure is even more unfavorable. That is the case when the players are symmetric. Given the symmetry, it is easy to see that there is no initial volunteering and every player is active from the beginning. This implies that the total expected payoff under the latter procedure is at most $\sum_i (V_i - c_i)$. In this example, the costs of all the players are subtracted in the computation of the total expected payoff instead of just that of player 1 as in the former procedure. When there are many symmetric players, the former procedure tends to attain the higher total expected payoff.

We conjecture that some stochastic mechanisms achieve higher efficiency. It would be interesting to examine which procedures are used in real meetings and study their adoption in each circumstance.

Appendix. The Proofs

The appendix first proves a technical lemma and then provides all the proofs for the lemmas and the propositions in the main text.

Lemma 2 claims that, depending on the status of player i, the difference $\frac{\sum_{A_t} b_j}{|A_t|-1} - b_i$ has to take a particular sign. The next lemma tells us whether this difference changes the signs or not when the set of active players is expanded (or reduced). The first part says that, when the difference has the non-negative sign and when the players with higher indexes are removed from the set, it still has the non-negative sign. This is because the players with higher indexes have lower b_j's. The second part is its converse. When the difference has the non-positive sign and when the players with lower b_j's are added, it has the non-positive sign. (The lemma also states the condition under which the inequality holds strictly.)

Lemma A.1

Let $I', I'' \subset I$ and $I' \cap I'' = \emptyset$. Take $i \in I$ such that $b_i \geq b_j$ for any $j \in I''$.

If $\frac{\sum_{I' \cup I''} b_j}{|I' \cup I''|-1} - b_i \geq 0$, it holds that $\frac{\sum_{I'} b_j}{|I'|-1} - b_i \geq 0$. Moreover, either if the inequality in the condition is strict or if $b_i > b_j$ for some $j \in I''$, the inequality in the statement holds strictly.

Conversely, if $\frac{\sum_{I'} b_j}{|I'|-1} - b_i \leq 0$, it holds that $\frac{\sum_{I' \cup I''} b_j}{|I' \cup I''|-1} - b_i \leq 0$. Moreover, either if the inequality in the condition is strict or if $b_i > b_j$ for some $j \in I''$, the inequality in the statement holds strictly.

(proof)

Because $b_j \leq b_i$ for any $j \in I''$, we have

$$\begin{aligned}
0 \leq \frac{\sum_{I' \cup I''} b_j}{|I' \cup I''| - 1} - b_i &= \frac{\sum_{I'} b_j + \sum_{I''} b_j - (|I'| + |I''| - 1) b_i}{|I'| + |I''| - 1} \\
&= \frac{\sum_{I'} b_j - \big(|I'| - 1\big) b_i + \sum_{I''} \big(b_j - b_i\big)}{|I'| + |I''| - 1} \\
&\leq \frac{\sum_{I'} b_j - \big(|I'| - 1\big) b_i}{|I'| + |I''| - 1} \\
&= \frac{|I'| - 1}{|I'| + |I''| - 1} \left(\frac{\sum_{I'} b_j}{|I'| - 1} - b_i \right).
\end{aligned}$$

This proves the weak part of the first statement. When the inequality in the condition is strict, the first inequality above becomes strict. If $b_i > b_j$ for some $j \in I''$, the second inequality above becomes strict. In either case, the inequality in the statement holds strictly.

Using the above formula in the reverse way (except the first inequality), we obtain the second statement.

Q.E.D.

A.1. Proof of Lemma 2

We prove this lemma part by part.

(a) the condition for active players

For active players, the analysis in the main text shows that equations (1) and (2) have to hold. In particular, for player i to be active at time t, equation (2) implies that $a_i(t) = \frac{\sum_{A_t} b_j}{|A_t|-1} - b_i > 0$. It also means that, if the sets of active players are same, the rates of volunteering for any active players are same.

(b) the condition for the players who exhaust the possibility of volunteering at time t

If player i exhausts the possibility of volunteering at time $t(> 0)$, we claim that the combined rate of volunteering at time t should not be larger than the threshold rate of player i: $\sum_{j \in I} a_j(t) \leq b_i$.

Suppose not. We would have $\sum_{j \in I} a_j(t) > b_i$. When we evaluate the expected payoff of player i at time t from waiting for a short period and then volunteering, we would have the following inequality by the definition of b_i:

$$\int_{\tau=0}^{\triangle} Be^{-B\tau} e^{-r_i \tau} V_i d\tau + e^{-B\triangle} e^{-r_i \triangle} (V_i - c_i) > V_i - c_i,$$

where $\triangle > 0$, and $B = \sum_{j \in I} a_j(t)$. Because volunteering always gives $V_i - c_i$ to player i at that point, this inequality implies that volunteering near time t would be dominated by waiting a bit more after time t. Then, he would never volunteer during the interval $(t - \epsilon, t)$ for some $\epsilon > 0$. This contradicts the supposition that he exhausts the possibility of volunteering at time t. Therefore, it has to hold that $\sum_{j \in I} a_j(t) \leq b_i$.

(c) the combined rate of volunteering

Take any time $t(> 0)$ such that there exist $t'(< t)$, and $t''(> t)$ for which the following properties hold: $A_\tau = A_{t-}$ for any $\tau \in [t', t)$, $A_\tau = A_t$ for any $\tau \in [t, t'')$, and moreover $A_{t-} \neq A_t$. That is, we consider two adjacent intervals between which the set of active players changes. (By the assumption made in Section 2, the players' strategies are well-behaved, and thus we can always find such intervals.) We prove that the combined rate of volunteering strictly decreases at time t: $\sum_{A_{t-}} a_j(t-) > \sum_{A_t} a_j(t)$. We prove it in two steps.

First, consider the case in which any player who is active just before time t remains active, and some passive player becomes active at time t: $A_{t-} \subset A_t$, and $A_{t-} \neq A_t$. Then, we claim that the combined rate of volunteering strictly decreases. Consider player i who is not active just before time t and becomes active at time t: $i \notin A_{t-}$, and $i \in A_t$. The expected payoff of player i at time $t - \triangle$ from waiting till time t and then volunteering is given by

$$\int_{\tau=0}^{\triangle} B' e^{-B'\tau} e^{-r_i \tau} V_i d\tau + e^{-B'\triangle} e^{-r_i \triangle}(V_i - c_i).$$

where $\triangle \in (0, t - t')$, and $B' = \sum_{A_{t-}} a_j(t-)$. Because player i prefers waiting to immediately volunteering before time t, it should be no smaller than $V_i - c_i$. By the definition of b_i, it means that the combined rate of volunteering should be no smaller than the threshold rate of player i: $b_i \leq \sum_{A_{t-}} a_j(t-)$. By equation (1), this inequality can be written as $b_i \leq \frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1}$. Using equation (1), we compare $\sum_{A_{t-}} a_j(t-)$ with $\sum_{A_t} a_j(t)$:

$$\begin{aligned}
&\sum_{A_{t-}} a_j(t-) - \sum_{A_t} a_j(t) \\
&= \frac{\sum_{A_{t-}} b_j}{|A_{t-}| - 1} - \frac{\sum_{A_t} b_j}{|A_t| - 1} \\
&= \frac{\sum_{A_{t-}} b_j}{|A_{t-}| - 1} - \frac{\sum_{A_{t-}} b_j + \sum_{A_t/A_{t-}} b_j}{|A_{t-}| + |A_t/A_{t-}| - 1} \\
&= \frac{(|A_{t-}| + |A_t/A_{t-}| - 1)\sum_{A_{t-}} b_j - (|A_{t-}| - 1)\sum_{A_{t-}} b_j - (|A_{t-}| - 1)\sum_{A_t/A_{t-}} b_j}{(|A_{t-}| - 1)(|A_{t-}| + |A_t/A_{t-}| - 1)} \\
&= \frac{1}{|A_{t-}| + |A_t/A_{t-}| - 1} \sum_{i \in A_t/A_{t-}} \left(\frac{\sum_{j \in A_{t-}} b_j}{|A_{t-}| - 1} - b_i \right).
\end{aligned}$$

If $b_i = \frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1}$ for any $i \in A_t/A_{t-}$, the above is zero. It means that $\frac{\sum_{A_{t-}} b_i}{|A_{t-}|-1} = \frac{\sum_{A_t} b_i}{|A_t|-1}$. Equation (2) then implies that $a_i(t) = \frac{\sum_{A_t} b_i}{|A_t|-1} - b_i = 0$. This could not happen because player i is supposed to be active at time t. Hence, $b_i < \frac{\sum_{A_{t-}} b_i}{|A_{t-}|-1}$ for some $i \in A_t/A_{t-}$, Thus, the above has to be positive, which proves the claim.

Second, we consider the case in which some player who is active just before time t is not active at time t. Then, at least a player, say player i, is included in A_{t-} but not in A_t.

Supposing that $\sum_{A_{t-}} a_j(t-) \leq \sum_{A_t} a_j(t)$, we derive contradictions. Because $i \in A_{t-}$, it holds that $b_i = \left(\sum_{A_{t-}} a_j(t-)\right) - a_i(t-)$. Combining this with the supposition, we have

$$b_i < \sum_{j \in A_{t-}} a_j(t-) \leq \sum_{j \in A_t} a_j(t).$$

When we evaluate the expected payoff of player i at time t from waiting for a short period and then volunteering, we have the following inequality by the definition of b_i:

$$\int_{\tau=0}^{\triangle} Be^{-B\tau}e^{-r_i\tau}V_i d\tau + e^{-B\triangle}e^{-r_i\triangle}(V_i - c_i) > V_i - c_i.$$

where $\triangle \in (0, t'')$, and $B = \sum_{A_t} a_j(t)$. This implies that volunteering near time t is dominated by waiting for some time after time t. This contradicts the fact that player i is active till time t. Therefore, the combined rate of volunteering has to decrease strictly at time t: $\sum_{A_{t-}} a_j(t-) > \sum_{A_t} a_j(t)$.

Because the above two cases cover all the possibilities of the change in the sets of active players, they prove that the combined rate of volunteering strictly decreases when the set of active players changes between adjacent intervals.

Because the combined rate of volunteering is determined by the set of active players, and it never increases, the set of active players changes only finite times. Hence, by using the above argument repeatedly, we can prove that the combined rate of volunteering weakly decreases and strictly does so when the set of active players changes.

(d) the condition for passive players

Suppose that player i is passive at time t. By supposing $\sum_{j \neq i} a_j(t) - b_i < 0$, we derive a contradiction. The above analysis has shown that the combined rate of volunteering is weakly decreasing. Thus, the sum of the rates of volunteering by other players will never be higher than b_i. (This is the case even when player i would become active in the future.) That is, it holds that $\sum_{j \neq i} a_j(\tau) - b_i < 0$ for any $\tau \geq t$. By the definition of b_i, it implies that he would strictly prefer volunteering to waiting at any time after time t. Because player i is supposed to be passive at time t, this is a contradiction. Therefore, it has to hold that $\sum_{j \neq i} a_j(t) - b_i \geq 0$. Because $a_i(t) = 0$, we have $\sum_{j \in I} a_j(t) = \sum_{j \neq i} a_j(t) \geq b_i$.

A.2. Proof of Lemma 3

Because the non-compromising type cannot volunteer, at most one player volunteers initially with a positive probability mass, and the low-cost type prefers volunteering to waiting perpetually, any equilibrium involves a positive probability of volunteering after time 0. As shown in Lemma 1, no player volunteers with a probability mass after time 0. Hence, there has to be the possibility of a war of attrition in any equilibrium.

To prove that all the players exhaust the possibility of volunteering within a finite time, let us suppose the opposite. For a war of attrition to continue, there have to be at least two active players. Hence, there are more than one player who never exhausts the possibility of volunteering. Let F be the set of such players. There exists time S after which only those in this set can be active. After time S, for player $i \in F$ to have an incentive to wait, it has to hold that $\sum_{F/\{i\}} a_j(t) \geq b_i$. Summing this inequality over the set F, we obtain $(|F|-1)\sum_F a_j(t) \geq \sum_F b_i$. Thus, after time S, it holds that $\sum_F a_j(t) \geq \frac{\sum_F b_i}{|F|-1}$. Let β denote the right hand side. Observe that $\beta > 0$.

Let $P(t)$ denote the ex-ante probability that none of the players in the set F volunteers by time t. Because it decreases by the rate of $\sum_F a_j(t)$, it holds that
$P(t) = P(S)\exp\left(-\int_{\tau=S}^{t}\sum_F a_j(\tau)d\tau\right)$. We obtain $P(t) \leq P(S)\exp\left(-\int_{\tau=S}^{t}\beta d\tau\right)$ because $\sum_F a_j(t) \geq \beta$. As $t \to \infty$, the right hand side converges to zero, and thus the left hand side also converges to zero. However, this contradicts the fact that $P(t) \geq \Pi_F z_j$, which holds because the non-compromising types never volunteer. Therefore, the players cannot be active forever.

Note that a low-cost type stays in a war of attrition because of the possibility that the other players volunteer with positive rates. Hence, when the players exhaust the possibility of volunteering, there should be no possibility of the low-cost type.

A.3. Proof of Lemma 4

(1) Note that a war of attrition continues at time t because there are some players who still have the possibility of the non-compromising type. The fact that player i is active till time t means that he does not strictly prefer waiting to volunteering at time t. By the same argument used in the proof of Lemma 2, we know that $\sum_{j\neq i} a_j(t) \leq b_i$. (If this inequality is violated, player i does not want to volunteer near time t because his expected payoff from waiting at time t is higher than $V_i - c_i$ by the definition of b_i.) From equation (1), it holds that $\sum_{j\in I} a_j(t) = \frac{\sum_{j\in A_t} b_j}{|A_t|-1}$. Because $a_i(t) = 0$, the above inequality is equivalent to $\frac{\sum_{A_t} b_j}{|A_t|-1} \leq b_i$.

First, suppose that $\sum_{j \neq i} a_j(t) < b_i$. Then, because the combined rate of volunteering is weakly decreasing, the low-cost type of player i has no incentive to wait during the interval $(t, t+\epsilon)$ for some $\epsilon > 0$ and strictly prefers volunteering at time t. Because volunteering with a probability mass cannot happen, he should have volunteered for sure by time t. Consider player k such that $k < i$. For any $k < i$, it holds that $\sum_{j \neq k} a_j(t) \leq \sum_{j \in I} a_j(t) < b_i \leq b_k$ because $b_k \geq b_i$. By the same reason, the low-cost type of payer k should have volunteered for sure at least by time t.

Next, suppose that $\sum_{j \neq i} a_j(t) = b_i$. For any $k < i$ such that $b_k > b_i$, the above argument can be used to show that the low-cost type of player k needs to volunteer for sure at least by time t. Now, we look at the player whose index k satisfies the following equation: $\sum_{j \neq k} a_j(t) = b_k$. (Note that player i is such a player.) Let T be the time by which all the players exhaust the possibility of volunteering. If there is no change in the set of active players between time t and time T, the low-cost type of player k needs to volunteer for sure at least by time t because he is not active during that period. Suppose that there is a change in the set of active players between time t and time T. Let S be the first time when that happens. Lemma 2 shows that the combined rate of volunteering strictly decreases at time S. Hence, we have $\sum_{A_S} a_j(S) < \sum_{A_t} a_j(t) = \sum_{j \neq k} a_j(t) = b_k$. This implies that player k strictly prefers volunteering to waiting at time S because the combined rate of volunteering is weakly decreasing over time. Because there cannot be volunteering by a probability mass during a war of attrition (Lemma 1), the low-cost type of player k needs to volunteer for sure at least by time t.

In either case, the low-cost type of player k volunteers for sure by time t and thus exhausts the possibility of volunteering by that time. In particular, player i exhausts the possibility of volunteering at time t as the probability that he is the low-cost type becomes zero at that time.

(2) We suppose that no passive player becomes active at time t and derive a contradiction.

By supposition, the set of the players who are active just before time t comprises of the active players at time t in addition to those who exhaust the possibility of volunteering at time t. Hence, its set is given by $A_t \cup D_t$. Because any player in this set is active just before time t, Lemma 2 implies that

$$\frac{\sum_{A_t \cup D_t} b_j}{|A_t \cup D_t| - 1} - b_i > 0 \text{ for any } i \in A_t \cup D_t.$$

Let k be the smallest number among D_t. Then, for any $j \in D_t$, it holds that $b_j \leq b_k$. By Lemma A.1, from the above inequality, we obtain

$$\frac{\sum_{A_t} b_j}{|A_t| - 1} - b_k > 0.$$

Because $k \in D_t$, and $\frac{\sum_{A_t} b_j}{|A_t|-1} = \sum_{j \in I} a_j(t)$ by Lemma 2, this contradicts the condition in Lemma 2 for the player who exhausts the possibility of volunteering at time t.

A.4. Proof of Lemma 5

Using Lemma 4, we know that, if some player exhausts the possibility of volunteering at time t, the set of these players is given by $D_t = \{k, \ldots, \overline{C}_t\}$ for some $k(\leq \overline{C}_t)$. Then, we have $A_{t-} = \{k, \ldots, \overline{C}_t\} \cup E_t$.

First, suppose that $m(C_t, E_t)$ does not exist, and yet some player exhausts the possibility of volunteering at time t. By supposition, it holds that $\frac{\sum_{j=k}^{\overline{C}_t} b_j + \sum_{E_t} b_j}{\overline{C}_t - k + |E_t|} - b_k \leq 0$. It, however, contradicts the condition in Lemma 2 for an active player because $k \in A_{t-}$, and $A_{t-} = \{k, \ldots, \overline{C}_t\} \cup E_t$. Hence, if $m(C_t, E_t)$ does not exist, no player exhausts the possibility of volunteering at time t.

Second, suppose that $m(C_t, E_t)$ exists, and also $k < m(C_t, E_t)$. By supposition, it holds that $\frac{\sum_{j=k}^{\overline{C}_t} b_j + \sum_{E_t} b_j}{\overline{C}_t - k + |E_t|} - b_k \leq 0$. By the same reason as above, it contradicts the condition in Lemma 2 for an active player. Hence, if $m(C_t, E_t)$ exists, it has to hold that $k \geq m(C_t, E_t)$.

Third, suppose that $m(C_t, E_t)$ exists, and also $k > m(C_t, E_t)$. By construction, it holds that $\frac{\sum_{j=k-1}^{\overline{C}_t} b_j + \sum_{E_t} b_j}{\overline{C}_t - k + |E_t| + 1} - b_{k-1} > 0$. At some time s before time t, player $k-1$ exhausts the possibility of volunteering. From Lemma 4, we know that A_s comprises of the players whose indexes are included in the set $\{k, \ldots, \overline{C}_t\} \cup E_t$. Because $b_{k-1} \geq b_j$ for any $j \in \{k, \ldots, \overline{C}_t\} \cup E_t$, by Lemma A.1, it holds that $\frac{\sum_{A_s} b_j}{|A_s|-1} - b_{k-1} > 0$. Because player $k-1$ is supposed to exhaust the possibility of volunteering at time s, and thus $\frac{\sum_{A_s} b_j}{|A_s|-1} = \sum_{j \in I} a_j(s)$ by Lemma 2, this contradicts the condition in Lemma 2 for the player who exhausts the possibility of volunteering. Hence, if $m(C_t, E_t)$ exists, it has to hold that $k \leq m(C_t, E_t)$.

Combining the above arguments, we can conclude that the statement in the lemma has to hold.

A.5. Proof of Proposition 1

By the way that the construction is conducted, we know that, if there is an equilibrium, it has to take the unique form specified before and in Proposition 1. In the following, we show that the constructed strategies form an equilibrium.

We examine three different statuses in turn: being active, being passive, and having exhausted the possibility of volunteering. We want to show that, for player i, the corresponding continuation payoff is no smaller than $V_i - c_i$ in the passive status, it is equal to that in the active status, and it is no larger than that when he has exhausted the possibility of volunteering. Observe that the statuses change only in this ordering though some player may not experience the passive status. When the above condition is satisfied, player i has no incentive to deviate from the constructed strategy.

Because the continuation payoff is determined by the combined rate of volunteering by the other players, using equation (1), we can restate the above condition as follows: $\frac{\sum_{A_t} b_j}{|A_t|-1} - b_i$ is non-negative when player i is passive at time t, it is positive when player i is active at time t, and it is non-positive when player i has exhausted the possibility of volunteering by time t. We show that this condition is satisfied along the constructed path.

(i) Being active

We look at two possible situations in which the set of active players changes and prove that the required condition holds just before those times.

First, suppose that some player exhausts the possibility of volunteering at time t. By the property of $m(C_t, E_t)$, the following equation holds: $\frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1} > \max_{A_{t-}} b_j$. Because $b_i \leq \max_{A_{t-}} b_j$ for any $i \in A_{t-}$, we have $\frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1} > b_i$ for any $i \in A_{t-}$.

Next, suppose that some player becomes active without any player exhausting the possibility of volunteering at time s. Suppose that no player exhausts the possibility of volunteering between time s and time $t-$. Then, it holds that $A_{s-} \subset A_{t-}$. As is discussed above, we have $\frac{\sum_{A_{t-}} b_j}{|A_{t-}|-1} > \max_{A_{t-}} b_j$, and $\max_{A_{t-}} b_j \geq b_i$ for any $i \in A_{t-}$. Then, Lemma A.1 implies that $\frac{\sum_{A_{s-}} b_j}{|A_{s-}|-1} > \max_{A_{t-}} b_j$. Because $\max_{A_{t-}} b_j \geq b_i$ for any $i \in A_{t-}$, it holds that $\frac{\sum_{A_{s-}} b_j}{|A_{s-}|-1} > b_i$ for any $i \in A_{s-}$.

Because the rates of volunteering change only when the set of active players changes on these occasions, and the construction is made backward in time with $A_T = \emptyset$, the above analyses prove that $\frac{\sum_{A_\tau} b_j}{|A_\tau|-1} > b_i$ for any $i \in A_\tau$ on the constructed equilibrium path.

(ii) Being passive

Suppose that player i becomes active at time $u(>0)$. We prove that player i satisfies the property required for being passive on the constructed path at any time t such that $t<u$. We examine two situations, depending on whether some active player exhausts the possibility of volunteering by time u or not.

First, suppose that an active player at time t exhausts the possibility of volunteering by time u (time u included). Let player k be such a player. Because player k exhausts the possibility of volunteering by time u, by construction, it holds that $k<i$. Because player k is active at time t, the above argument for the active status has shown that $\frac{\sum_{A_t} b_j}{|A_t|-1} > b_k$ on the constructed path. Because $b_k \geq b_i$, we have $\frac{\sum_{A_t} b_j}{|A_t|-1} > b_i$ on the constructed path.

Next, consider the case in which any active player at time t does not exhaust the possibility of volunteering by time u (time u included). Then, it holds that $A_t \subset A_u$. The above argument for the active status has proven that $\frac{\sum_{A_u} b_j}{|A_u|-1} > \max_{A_u} b_j$ as the player with the index $\max_{A_u} b_j$ is active at time u. Because $A_t \subset A_u$, and $\max_{A_u} b_j \geq b_k$ for any $k \in A_u$, Lemma A.1 implies that $\frac{\sum_{A_t} b_j}{|A_t|-1} > \max_{A_u} b_j$. Because $i \in A_u$ by supposition, we have $\max_{A_u} b_j \geq b_i$. Therefore, it holds that $\frac{\sum_{A_t} b_j}{|A_t|-1} > b_i$.

Combining the above arguments, we can conclude that $\frac{\sum_{A_t} b_j}{|A_t|-1} > b_i$ on the constructed path whenever player i is passive at time t.

(iii) Having exhausted the possibility of volunteering

Suppose that player i exhausts the possibility of volunteering at time t, and there are still active players at that time. Define $k=\overline{C}_t$. This means that player k has the highest index among those who exhaust the possibility of volunteering at time t. Let u be the next time at which some player exhausts the possibility of volunteering. Let player ℓ be such a player. (We choose one player randomly when there are multiple such players.) By construction, it holds that $i \leq k < \ell$.

Take s such that $t<s\leq u$. Observe that player k exhausts the possibility of volunteering at time t when the path is constructed backward. Hence, when the set of active players changes at time s, the construction does not include player k in the set of the players who exhaust the possibility of volunteering at that point. Hence, it holds that $\frac{\left(\sum_{A_{s-}} b_j\right)+b_k}{|A_{s-}|} \leq b_k$. After simple computation, this inequality can be transformed to $\frac{\sum_{A_{s-}} b_j}{|A_{s-}|-1} \leq b_k$. It implies that $\frac{\sum_{A_\tau} b_j}{|A_\tau|-1} \leq b_k$ for any $\tau \in [t,u)$. Because $b_k \leq b_i$, we have $\frac{\sum_{A_\tau} b_j}{|A_\tau|-1} \leq b_i$ for any $\tau \in [t,u)$.

When $A_u \neq \emptyset$, we can apply the above argument and can show that $\frac{\sum_{A_\tau} b_j}{|A_\tau|-1} \leq b_\ell$ where τ is between time u (included) and the time just before the next exhaustion occurs (not included). Because $\ell > i$, during this period also, we have $\frac{\sum_{A_\tau} b_j}{|A_\tau|-1} \leq b_i$.

By repeating this process toward time T at which all the remaining players exhaust the possibility of volunteering, we can show that the desired property holds along the constructed path.

A.6. Proof of Lemma 6

Take player i and player j such that $i < j$. By Assumption B, we have $b_i \geq b_j$, and, by Assumption Z, we have $z_i \leq z_j$. From Lemma 4, we already know that $t' \leq \tau'$. Let $P_i(t)$ be the ex-ante probability that player i does not volunteer by time t, and let S be the time when player j becomes active. In order to prove the rest of the statement, we compute $P_i(S)$. Note that S is defined by $P_j(S) = 1$. We need to show that $P_i(S) = 1$ when both $b_i = b_j$ and $z_i = z_j$ hold, and that $P_i(S) < 1$ when either $b_i > b_j$ or $z_i < z_j$ holds.

(i) *Either $b_i > b_j$ or $z_i < z_j$ holds.*

We want to show that $P_i(S) < 1$. Let T_i denote the time that player i exhausts the possibility of volunteering, and let T_j denote the one for player j. Lemma 4 shows that $T_i \leq T_j$. We examine two cases in turn depending on whether this inequality holds strictly or not.

First, consider the case in which $T_i = T_j$.

When $z_i < z_j$, we have $P_i(T_i) < P_j(T_j)$ as $P_k(T_k) = z_k$ for any $k \in I$ by definition. Because $b_i \geq b_j$, equation (2) implies that $a_i(t) \leq a_j(t)$ whenever both players are active. Computing backward in time, we can conclude that $P_i(t) < P_j(t)$ holds for any $t \in (S, T_i)$. (Observe that $P_k(t)$ is an exponential function whose rate of decline is given by $a_k(t)$.) It implies that $P_i(S) < 1$.

When $z_i = z_j$, we have $P_i(T_i) = P_j(T_j)$. Moreover, by supposition, it holds that $b_i > b_j$. Then, equation (2) implies that $a_i(t) < a_j(t)$ whenever both players are active. Computing backward in time, we can conclude that $P_i(t) < P_j(t)$ holds for any $t \in (S, T_i)$. It implies that $P_i(S) < 1$.

Second, consider the case in which $T_i < T_j$.

When player j is not active at time T_i, Lemma 4 implies that $T_i < S$. Then, it holds that $P_i(S) = z_i < 1$.

When player j is active at time T_i, we have $P_j(T_i) > z_j \geq z_i$. Then, when we compute both $P_i(t)$ and $P_j(t)$ backward from time T_i, we can apply the above argument for the case in which $z_i < z_j$. Hence, we have $P_i(S) < 1$ for this case as well.

Therefore, when either $b_i > b_j$ or $z_i < z_j$ holds, it holds that $P_i(S) < 1$.

(ii) *Both $b_i = b_j$ and $z_i = z_j$ hold.*

We want to show that $P_i(S) = 1$.

Because $b_i = b_j$, Lemma 4 implies that both player i and player j exhaust the possibility of volunteering at the same time. Let time T' be such time. From equation (2), we know that $a_i(t) = a_j(t)$ as long as both players are active. Because $P_i(T') = z_i = z_j = P_j(T')$, we can compute both $P_i(t)$ and $P_j(t)$ backward in time and can conclude that $P_i(t) = P_j(t)$ holds for any $t \leq T'$. Therefore, we have $P_i(S) = 1$ as S is defined by $P_j(S) = 1$.

A.7. Proof of Proposition 3

We suppose that there are M players ($\{1, \ldots, M\}$) in the original situation and compare that situation with the modified situation where player $M+1$ is added. Because $\frac{\sum_{j=1}^{M+1} b_j}{M} - b_1 > 0$ by supposition, and $b_1 \geq b_{M+1}$, by Lemma A.1, it holds that $\frac{\sum_{j=1}^{M} b_j}{M-1} - b_1 > 0$. Lemma 5 shows that, both in the original situation and in the modified situation, all the players exhaust the possibility of volunteering at the same time. Let T be the time by which all the players exhaust the possibility of volunteering in the original situation and T' be the one in the modified situation.

Let $a_i(t)$ be the rate of volunteering by player i at time t in the original situation, and let $a'_i(t)$ be the one in the modified situation. Let S_i be the time that player i becomes active for the first time in the original situation and S'_i be the one in the modified situation. Under Assumption Z, Lemma 6 has shown that both $S_i \leq S_j$ and $S'_i \leq S'_j$ for any $i \leq j$, and moreover that the equality holds when and only when both $b_i = b_j$ and $z_i = z_j$ hold. This implies that, when $S'_i < S'_{M+1}$, the set of active players at time S_i in the original situation is identical to the one at time S'_i in the modified situation. (The following analysis holds even when $S'_i = S'_{i+1}$ for some $i(\leq M)$.)

For $i \in \{i, \ldots, M-1\}$, let $P_i(S_{i+1})$ be the ex-ante probability that player i has not volunteered by time S_{i+1} in the original situation, and let $P'_i(S'_{i+1})$ be the ex-ante probability that player i has not volunteered by time S'_{i+1} in the modified situation. We compute these probabilities backward in time:

$$\begin{aligned} P_i(S_{i+1}) = z_i \exp\{&a_i(S_M)(T - S_M) + a_i(S_{M-1})(S_M - S_{M-1}) + \ldots \\ &+ a_i(S_{i+2})(S_{i+3} - S_{i+2}) + a_i(S_{i+1})(S_{i+2} - S_{i+1})\}, \end{aligned}$$

and

$$\begin{aligned} P'_i(S'_{i+1}) = z_i \exp\{&a'_i(S'_{M+1})(T' - S'_{M+1}) \\ &+ a'_i(S'_M)(S'_{M+1} - S'_M) + a'_i(S'_{M-1})(S'_M - S'_{M-1}) + \ldots \\ &+ a'_i(S'_{i+2})(S'_{i+3} - S'_{i+2}) + a'_i(S'_{i+1})(S'_{i+2} - S'_{i+1})\}. \end{aligned}$$

Lemma 2 shows that the rate of volunteering depends only on the set of active players. Moreover, for any $k < M$, Lemma 6 shows that $S_k = S_{k+1}$ if and only if $b_k = b_{k+1}$, and $z_k = z_{k+1}$. Hence, it holds that $a_i(S_k) = a'_i(S'_k)$ for any k such that $S'_k < S'_{M+1}$ because the set of active players is identical at the corresponding times. In addition, when $S'_k = S'_{M+1}$, it holds that $S'_{k+1} - S'_k = 0$. Hence, for these k's, changing $a'_i(S'_k)$ to $a_i(S_k)$ does not affect the above equality. Therefore, the above equality holds when we change $a'_i(S'_k)$ to $a_i(S_k)$ for any $k \leq M$. The difference of the arguments of the exponential function in the above expressions (the inside parts of the curly brackets) is denoted by Γ_i. Substituting $a_i(S_k)$ for $a'_i(S'_k)$ for any $k \leq M$, we obtain

$$\begin{aligned} \Gamma_i = \;& a_i(S_M)(T - S_M) + a_i(S_{M-1})(S_M - S_{M-1}) + \ldots \\ &+ a_i(S_{i+2})(S_{i+3} - S_{i+2}) + a_i(S_{i+1})(S_{i+2} - S_{i+1}) \\ &- \Big(a'_i(S'_{M+1})(T' - S'_{M+1}) + a_i(S_M)(S'_{M+1} - S'_M) + a_i(S_{M-1})(S'_M - S'_{M-1}) + \ldots \\ &+ a_i(S_{i+2})(S'_{i+3} - S'_{i+2}) + a_i(S_{i+1})(S'_{i+2} - S'_{i+1})\Big) \\ = \;& a_i(S_M)(T - S_M) - a_i(S_M)(S'_{M+1} - S'_M) - a'_i(S'_{M+1})(T' - S'_{M+1}) \\ &+ a_i(S_{M-1})\Delta_{M-1} + \ldots + a_i(S_{i+2})\Delta_{i+2} + a_i(S_{i+1})\Delta_{i+1}, \end{aligned}$$

where $\Delta_k = S_{k+1} - S_k - (S'_{k+1} - S'_k)$, and $i \in \{1, \ldots, M-1\}$. Define δ_a by $\delta_a = a_i(S_M) -$

$a_i'(S'_{M+1})$. It holds that

$$\begin{aligned}\delta_a = a_i(S_M) - a_i'(S'_{M+1}) &= \frac{\sum_{j=1}^{M} b_j}{M-1} - b_i - \left(\frac{\sum_{j=1}^{M+1} b_j}{M} - b_i\right) \\ &= \frac{\sum_{j=1}^{M} b_j}{M-1} - \frac{\sum_{j=1}^{M+1} b_j}{M} \\ &= \frac{\sum_{j=1}^{M} b_j - (M-1)b_{M+1}}{(M-1)M} > 0.\end{aligned}$$

The last inequality holds because $b_j \geq b_{M+1}$ for any $j \in I$. We define S_{M+1} to satisfy $T - S_{M+1} = T' - S'_{M+1}$ as a convention. (Because it is an artificial term, it may take a negative value.) Using this convention and substituting the above formula for δ_a, we can rewrite Γ_i as follows.

$$\begin{aligned}\Gamma_i &= a_i(S_M)(T - S_{M+1} + S_{M+1} - S_M) \\ &\quad - a_i(S_M)(S'_{M+1} - S'_M) - a_i'(S'_{M+1})(T' - S'_{M+1}) \\ &\quad + a_i(S_{M-1})\Delta_{M-1} + \ldots + a_i(S_{i+2})\Delta_{i+2} + a_i(S_{i+1})\Delta_{i+1} \\ &= \big(a_i(S_M) - a_i'(S'_{M+1})\big)(T - S_{M+1}) + a_i(S_M)\Delta_M \\ &\quad + a_i(S_{M-1})\Delta_{M-1} + \ldots + a_i(S_{i+2})\Delta_{i+2} + a_i(S_{i+1})\Delta_{i+1} \\ &= \delta_a(T - S_{M+1}) + a_i(S_M)\Delta_M \\ &\quad + a_i(S_{M-1})\Delta_{M-1} + \ldots + a_i(S_{i+2})\Delta_{i+2} + a_i(S_{i+1})\Delta_{i+1}.\end{aligned}$$

There have to be at least two active players in a war of attrition, and thus $S_2 = 0$. The probability of the initial volunteering is given by the difference between one and the ex-ante probability that player 1 does not volunteered by time $S_2 (= S'_2 = 0)$. Hence, the difference between the ex-ante probability that player 1 does not volunteer by time S_2 in the original situation and the ex-ante probability that player 1 does not volunteer by time S'_2 in the modified situation has the same absolute value as the difference in the probabilities of the initial volunteering, and the former has the opposite sign from the latter. By construction, the difference in the ex-ante probabilities that player 1 does not volunteer by the time that the second player becomes active has the same sign as Γ_1. Hence, when Γ_1 is positive, the probability of the initial volunteering increases by the addition of player $M+1$.

In order to evaluate Γ_i's, we look at the time that player $i+1$ becomes active for the first time, where $1 \leq i \leq M-1$. At that point in time, the ex-ante probability that the player has not volunteered is equal to one: $1 = P_{i+1}(S_{i+1}) = P'_{i+1}(S'_{i+1})$. Using the same kind of computation as above, we can rewrite this equation as follows.

$$
\begin{aligned}
1 &= z_{i+1}\exp\{a_{i+1}(S_M)(T - S_M) + a_{i+1}(S_{M-1})(S_M - S_{M-1}) + \dots \\
&\quad + a_{i+1}(S_{i+2})(S_{i+3} - S_{i+2}) + a_{i+1}(S_{i+1})(S_{i+2} - S_{i+1})\} \\
&= z_{i+1}\exp\{a'_{i+1}(S'_{M+1})(T' - S'_{M+1}) + a'_{i+1}(S'_M)(S'_{M+1} - S'_M) \\
&\quad + a'_{i+1}(S'_{M-1})(S'_M - S'_{M-1}) + \dots \\
&\quad + a'_{i+1}(S'_{i+2})(S'_{i+3} - S'_{i+2}) + a'_{i+1}(S'_{i+1})(S'_{i+2} - S'_{i+1})\}.
\end{aligned}
$$

Substituting $a_i(S_k)$ for $a'_i(S'_k)$ for any $k \leq M$, from the second equality, we have

$$
\begin{aligned}
&a_{i+1}(S_M)(T - S_M) + a_{i+1}(S_{M-1})(S_M - S_{M-1}) + \dots \\
&\quad + a_{i+1}(S_{i+2})(S_{i+3} - S_{i+2}) + a_{i+1}(S_{i+1})(S_{i+2} - S_{i+1}) \\
&= a'_{i+1}(S'_{M+1})(T' - S'_{M+1}) + a_{i+1}(S_M)(S'_{M+1} - S'_M) \\
&\quad + a_{i+1}(S_{M-1})(S'_M - S'_{M-1}) + \dots \\
&\quad + a_{i+1}(S_{i+2})(S'_{i+3} - S'_{i+2}) + a_{i+1}(S_{i+1})(S'_{i+2} - S'_{i+1}).
\end{aligned}
$$

Rearranging the terms and substituting the derived formulas for both δ_a and Δ_i's, we obtain the following equation:

$$
\begin{aligned}
&\delta_a(T - S_{M+1}) + a_{i+1}(S_M)\Delta_M \\
&\quad + a_{i+1}(S_{M-1})\Delta_{M-1} + \dots + a_{i+1}(S_{i+2})\Delta_{i+2} + a_{i+1}(S_{i+1})\Delta_{i+1} = 0, \qquad (A-1)
\end{aligned}
$$

for $i \in \{1, \dots, M-1\}$. The analysis in Section 3.1 has shown that $a_i(S_j) - a_{i+1}(S_j) = b_{i+1} - b_i$ for any $j \geq i+1$. Subtracting the left-hand side of equation (A-1) from Γ_i and using this property, we evaluate Γ_i: for $i \in \{1, \dots, M-1\}$,

$$
\begin{aligned}
\Gamma_i &= \big(a_i(S_M) - a_{i+1}(S_M)\big)\Delta_M + \big(a_i(S_{M-1}) - a_{i+1}(S_{M-1})\big)\Delta_{M-1} \\
&\quad + \dots + \big(a_i(S_{i+2}) - a_{i+1}(S_{i+2})\big)\Delta_{i+2} + \big(a_i(S_{i+1}) - a_{i+1}(S_{i+1})\big)\Delta_{i+1} \\
&= (b_{i+1} - b_i)\big(\Delta_M + \Delta_{M-1} + \dots + \Delta_{i+2} + \Delta_{i+1}\big). \qquad (A-2)
\end{aligned}
$$

We identify the sign of Γ_1 by mathematical induction. The induction hypothesis is as follows. For $i \in \{1, \dots, M-1\}$,

(i) $\Gamma_i \geq 0$, where the inequality holds strictly when and only when $b_i > b_{i+1}$, and

(ii) $\Delta_M + \ldots + \Delta_{i+1} < 0$.

We proceed backward from $M-1$ to 1.

(1) $i = M-1$.

Applying equation (A-1) to this case, we obtain

$$\delta_a(T - S_{M+1}) + a_M(S_M)\Delta_M = 0.$$

Because $\delta_a > 0$, we have $\Delta_M < 0$, which proves the second part of the induction hypothesis. Applying equation (A-2) to this case, we obtain

$$\Gamma_{M-1} = (b_M - b_{M-1})\Delta_M.$$

Because $\Delta_M < 0$, we obtain the first part of the induction hypothesis.

(2) $i < M-1$.

Suppose that the induction hypothesis holds for $i+1$. Then, Γ_{i+1} is non-negative. Thus, we have $P_{i+1}(S_{i+2}) \geq P'_{i+1}(S'_{i+2})$. Because $P_{i+1}(S_{i+1}) = P'_{i+1}(S'_{i+1}) = 1$, and $a_{i+1}(S_{i+1}) = a'_{i+1}(S'_{i+1})$, $S_{i+1} - S_{i+2} \leq S'_{i+1} - S'_{i+2}$ holds. It implies that $\Delta_{i+1} \leq 0$. Because $\Delta_M + \Delta_{M-1} + \ldots + \Delta_{i+2} < 0$ by the induction hypothesis, we have $\Delta_M + \Delta_{M-1} + \ldots + \Delta_{i+1} < 0$, proving the second part of the induction hypothesis. Then, applying this inequality to equation (A-2), we obtain the first part of the induction hypothesis.

This proves that $\Gamma_1 \geq 0$, where the inequality holds strictly when and only when $b_1 > b_2$. This implies the statement in the proposition.

A.8. Proof of Proposition 4

When both $b_1 = b_2$ and $z_1 = z_2$ hold, the probability given in the statement of the proposition becomes zero. Then, the proposition is trivially true. (In fact, Proposition 2 shows that there is no initial volunteering in this case.) In the following, we assume that either $b_1 > b_2$ or $z_1 < z_2$ holds, and player 1 volunteers initially with a positive probability.

Proposition 2 implies that player 1 is the one who volunteers initially, and that player 2 is active at time 0. Let T be the time by which all the players exhaust the possibility of volunteering, S be the time when player 1 exhausts the possibility of volunteering, and K be the largest index among the players who are active just before time S: $K = \max A_{S-}$. (When player 1 exhausts the possibility of volunteering at the same time as all the others do, we set $S = T$ and $K = I$.) Because player 1 and player K are both active just before time S, and K is the largest index among those active at that time, Lemma 6 implies that the set of active players just before time S is given by $\{1, \ldots, K\}$. Consider the modified game where there are only the first K players, and their chances of being the non-compromising type are given by $(z_1, P_2(S), \ldots, P_K(S))$, where $P_j(S)$ is the ex-ante probability that player j has not volunteered by time S. Then, by construction, the equilibrium path till time S is the unique equilibrium of this modified game.

When $1 \leq j < k$, the corollary to Lemma 2 shows that $a_j(t) \leq a_k(t)$ for any t. Note that $z_i = P_i(S)\exp\left\{-\int_{t=S}^{T} a_i(t)dt\right\}$. Because $z_j \leq z_k$ by Assumption Z, we have $P_j(S) = z_j\exp\left\{\int_{t=S}^{T} a_j(t)dt\right\} \leq z_k\exp\left\{\int_{t=S}^{T} a_k(t)dt\right\} = P_k(S)$. This shows that Assumption Z is satisfied for the modified game. For any $j \in \{1, \ldots, K\}$, player j is active just before time S. It implies that the modified game satisfies another condition supposed in Proposition 3. Proposition 3 shows that the removal of the player whose b_j is lowest never increases the chance of the initial volunteering. Moreover, because $\frac{\sum_{j=1}^{K} b_j}{K-1} > b_1$, and $b_1 \geq b_j$ for any $j \in I$, Lemma A.1 implies that $\frac{\sum_{j=1}^{k} b_j}{k-1} > b_1$ for any $k \leq K$. Proposition 3 is applicable even after the removal of such a player. Hence, we can remove the player with the highest index repeatedly and then, applying Proposition 3, can conclude that the probability of the initial volunteering is not smaller than the one in the game where there are only player 1 and player 2, and their probabilities to be the non-compromising types are given by $(z_1, P_2(S))$. The probability of the initial volunteering is increasing in z_2 for the two-player case as shown in Subsection 3.3.1, and $P_2(S) \geq z_2$. Hence, the probability of the initial volunteering in the game in which there are only the first two players serves as the lower bound.

A.9. Proof of the observation in Section 5.1

We first prove the necessity of the conditions one by one.

(1) Lemma 1 shows that there is no gap in a war of attrition in terms of the possibility of volunteering once it starts. Hence, if a war of attrition occurs, it either ends in a finite time or forever. When there is no non-compromising type, the certain ending would have to be carried out by one player volunteering for sure. However, it contradicts Lemma 1 as it would entail volunteering with a probability mass. Hence, once a war of attrition starts, it continues forever.

(2) By the same argument made in Lemma 2 for passive players, we can prove the statement.

(3) Suppose that player i becomes inactive at time τ: $i \in A_t$ for any t such that $s \leq t < \tau$, and $i \in I/A_\tau$. Player i's expected payoff from waiting for a short period at time τ has to be equal to $V_i - c_i$. If it were higher than the latter, then he would not volunteer just before τ. If it were lower, he would want to volunteer for sure at time τ. Both are contradictions. For the equality to hold, his threshold rate has to be equal to the combined rate of volunteering at time τ. This proves the third part.

(4) Let us suppose the opposite. That is, suppose that $\frac{\sum_{A_0} b_j}{|A_0|-1} \neq b_i$, and $i \notin A_0$. When $\frac{\sum_{A_0} b_j}{|A_0|-1} > b_i$, player i would obtain the payoff higher than $V_i - c_i$ in a war of attrition just after time 0. Then, he would never volunteer initially. When $\frac{\sum_{A_0} b_j}{|A_0|-1} < b_i$, player i would obtain the payoff lower than $V_i - c_i$ in a war of attrition just after time 0. Because the combined rate of volunteering is weakly decreasing, it implies that he would prefer volunteering initially with the probability of one. In either case, we had a contradiction. Therefore, the forth statement has to hold.

We next show that the conditions are sufficient.

First, observe that when either of the equality conditions in the third or in the fourth part holds, the path that satisfies the required conditions entails a war of attrition by the same set of active players from that time. When there is a change in the set of active players, Lemma 2 requires that the combined rate of volunteering strictly decreases. However, Lemma 2 and the additionally required conditions in this observation show that the threshold rates of both active and inactive players are no higher than the combined rate. Hence, when there is a player whose threshold rate is equal to the combined rate of volunteering at some point, there cannot be the change in the set of active players after that time.

Second, let us look at the players' strategies for the initial volunteering. Let player i be the player who does the initial volunteering with a probability between zero and one. If he is active at time 0, his expected payoff is equal to $V_i - c_i$. Hence, he has no incentive to deviate. If he is not active at time 0, his threshold rate needs to be equal to the combined rate of volunteering at time 0: $\frac{\sum_{A_0} b_j}{|A_0|-1} = b_i$. The above analysis shows that the combined rate of volunteering stays the same throughout a war of attrition. By the definition of the threshold rate, it implies that his expected payoff at time 0 is equal to $V_i - c_i$, which shows that he has no incentive to deviate. We claim that any player who is not supposed to do the initial volunteering has no incentive to deviate. Let player j be such a player. He has no incentive to do the initial volunteering when player $i(\neq j)$ does the initial volunteering with a positive probability. Consider the case in which no player is supposed to do the initial volunteering. When player j is active at time 0, his expected payoff at that time is $V_j - c_j$. When he is inactive at time 0, by the second condition of this observation, it holds that $\frac{\sum_{A_0} b_j}{|A_0|-1} \geq b_j$. It implies that the expected payoff from waiting at time 0 is at least as large as $V_j - c_j$. Because the initial volunteering gives him the payoff of $V_j - c_j$, he has no strict incentive to do the initial volunteering.

Finally, consider the players' strategies during a war of attrition. The above analysis has shown that, when player i is active just before time τ and becomes inactive at time τ, his continuation payoff is equal to $V_i - c_i$. Lemma 2 requires that the rates of volunteering is set so that, when player i is active, his continuation payoff is equal to $V_i - c_i$. Thus, once a player becomes active, his continuation payoff is equal to his net benefit. It implies that he is indifferent between volunteering and waiting. Thus, once a player becomes active, he has no incentive to deviate from then on. As to the inactive players, the second condition of this observation requires that $\frac{\sum_{A_t} b_j}{|A_t|-1} \geq b_j$. It implies that he weakly prefers waiting to volunteering at time t. Hence, he has no way to improve his payoff. Therefore, once a war of attrition starts, no player has an incentive to deviate.

These show that the conditions in the observation constitute the sufficient conditions.

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