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Subgame-perfect equilibrium in games with almost perfect information: Dispensing with public randomization

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Harris, Reny, and Robson (1995) added a public randomization device to dynamic games with almost perfect information to ensure existence of subgame perfect equilibria (SPE). We show that when Nature's moves are atomless in the original game, public randomization does not enlarge the set of SPE payoffs: any SPE obtained using public randomization can be "decorrelated" to produce a payoff-equivalent SPE of the original game. As a corollary, we provide an alternative route to a result of He and Sun (2020) on existence of SPE without public randomization, which in turn yields equilibrium existence for stochastic games with weakly continuous state transitions.

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JEL classification. C72, C73.

1. Introduction

A seminal result of Harris, Reny, and Robson (1995) (henceforth, HRR) ensures existence of subgame perfect equilibrium (SPE) in dynamic games with almost perfect information by augmenting such games with a public randomization device. That is, they assume that in addition to Nature's moves in the original game, players observe a uniformly distributed, payoff-irrelevant public signal in every stage. This convexifies equilibrium probabilities over continuation paths in the extended game and allows them to use limiting arguments to deduce existence of a SPE; in the original game, their construction corresponds to a generalized strategy profile in which players' actions are marked by a form of correlation. We focus on the subclass of *games with atomless moves by Nature*, and our main contribution is that in such games, one can dispense with public signals in HRR's result: each SPE obtained by augmenting the original game to allow public signals can be "decorrelated" to produce a payoff-equivalent SPE of the original game involving no correlation or public signals. Therefore, for a large class of dynamic games, public randomization is without loss of generality; or put differently, in a world

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with atomless moves by Nature, introducing "sunspots" does not enlarge the set of subgame perfect equilibrium payoffs. In particular, it cannot enlarge the empty set to a nonempty set, so a direct corollary is existence of SPE in games with atomless moves by Nature, providing an alternative route to existence of SPE to that recently taken by He and Sun (2020).

The class of games with atomless moves by Nature is general enough to capture many applications of interest; in particular, it subsumes stochastic games with weakly continuous, atomless state transitions—in some respects, going well beyond the analysis in the classical literature. More formally, a stochastic game is played among n players; in each period, a state variable z is publicly observed; players then simultaneously choose actions y_i; and given the state z and action profile $y = (y_1, \dots, y_n)$, next period's state is drawn from a transition probability $\mu(\cdot|y, z)$. This process is repeated in discrete time over an infinite horizon. In their classic analysis, Mertens and Parthasarathy (2003) did not assume that states are atomlessly distributed, but their conditions for existence of SPE impose the strong condition of norm-continuity on the state transition. Letting Z denote the set of states and $\Delta(Z)$ the set of probability measures over states, the assumption of Mertens and Parthasarathy (which is standard in the literature) is that the mapping $(y, z) \mapsto \mu(\cdot|y, z)$ is jointly measurable and continuous in y with the total variation norm on $\Delta(Z)$. This norm-continuity assumption precludes the possibility that states have a component that depends in a deterministic way on a continuous action.

In contrast, our decorrelation approach does not impose norm-continuity; rather, adding atomless transitions to HRR's framework, we require only that state transitions are weakly continuous, i.e., the mapping $(y, z) \mapsto \mu(\cdot|y, z)$ is continuous with the weak* topology on $\Delta(Z)$. Thus, a component of the state is permitted to vary in a deterministic, continuous way with respect to states and actions. Our analysis is also more general than that of Mertens and Parthasarathy in that it permits the players' payoffs and Nature's moves to depend on the entire history of play, but our payoff structure is less general in one respect: this dependence has to be jointly continuous, whereas Mertens and Parthasarathy allow for payoffs and moves by Nature that are continuous in the current action profile but merely measurable in the current state.[1]

Our analytical approach proceeds as follows. Given a game with atomless moves by Nature, HRR show that there is a SPE in the extended game, which adds payoff-irrelevant public signals in each period. For any such SPE strategy profile, we exploit nonatomicity to "decorrelate" the SPE in each period via repeated application of Mertens' (2003) "measurable 'measurable choice' theorem." In the first period, there is no previous public signal, so the players' actions in the SPE are trivially uncorrelated. In the second and later periods of the extended game, however, players can condition on the public signal in the first period. We use Mertens' theorem to replace conditioning of moves on the first-period public signal (which is payoff-irrelevant) in the *extended game* with conditioning on Nature's move (which is nonatomically distributed and payoff-relevant) in the first period of the *original game*, in which public signals are unavailable. The key

[1] We also derive a result on existence of a *correlated* SPE that allows the transition probability to have atoms and assumes only weakly continuous transitions.

is to do so in a way that is measurable and preserves the expected discounted payoff of each profile of the players' actions in the first period, thereby maintaining equilibrium conditions on the players' first-period choices.

This step renders the first-period public signal moot, but in the third and later periods of the extended game, SPE strategies can still condition on the public signal in the second period. We then use Mertens' theorem to replace conditioning on the second-period public signal with conditioning on Nature's moves in the original game, again preserving expected payoffs from profiles of the players' actions in the first two periods and thereby maintaining equilibrium conditions. By repeating this procedure, we inductively construct a new profile of strategies such that players' actions after any history do not depend on the previous public signals, and such that expected payoffs in the first period are preserved from the original SPE. As a consequence, the new profile of strategies is a SPE of the original game that is payoff-equivalent to the SPE of the extended game. Thus, we dispense with HRR's public randomization for a large class of games of interest, and we conclude that for the purpose of characterizing SPE payoffs of such games, the inclusion of an external public randomization device is without loss of generality.

We emphasize that public randomization is payoff-irrelevant, whereas moves by Nature are payoff relevant, so that the substitution of conditioning on Nature's moves in place of correlation requires delicate arguments. When removing correlation via the period t public signal, we must specify new equilibrium strategies in all future periods; moreover, this "cleansing" of correlation must be iterated an infinite number of times, once for each period t.[2]

Our existence corollary is consistent with an example of Luttmer and Mariotti (2003), in which there is no SPE in a game of perfect information with moves by Nature that are not always atomless. It is also obtained in Proposition 1 of He and Sun (2020), who establish existence of SPE without resorting to public signals under the assumption that Nature's moves are atomless; their focus is on existence, and they do not provide results on "decorrelation" of equilibria. Our route to existence is comparatively short, but theirs is more direct, in that they do not rely on the results of HRR, but instead follow a backward-then-forward induction argument similar to that of HRR. Their use of Mertens' (2003) theorem is in ensuring the existence of a suitable jointly measurable selection from next period's equilibrium payoff correspondence in the forward induction argument, whereas we use Mertens' (2003) theorem to replace conditioning on public signals with conditioning on moves by Nature.[3] Methodologically, this decorrelation approach based on nonatomicity is related to our previous work on dynamic games, where

[2]Observe that although one can work with payoffs or distributions over infinite histories in the extended game Mariotti (2000), the process of decorrelation involves a substitution of equilibrium play after the first period: we maintain the players' moves in the first period, and we preserve payoffs of the original SPE evaluated at the beginning of the game, but players' moves in subsequent periods are otherwise unrelated to the original profile. Thus, decorrelation generally produces a distinct distribution over paths of play. Of course, this is unavoidable, because correlation is removed by adding extra conditioning on the realizations of Nature's moves.

[3]He and Sun (2020) covered games of perfect information, and they give results for a class of dynamic games that drops continuity of Nature's moves in exchange for absolute continuity with respect to a fixed, atomless measure.

payoff correspondences with convex values are often needed on technical grounds, but where correlation can be difficult to justify on economic grounds. Similar methods are used in Duggan (2012) to prove existence of stationary Markov perfect equilibria in noisy stochastic games: a correlated equilibrium is deduced from Nowak and Raghavan (1992), and a version of Mertens' theorem is used to construct a payoff-equivalent equilibrium in which correlation is replaced by conditioning on the noise component of the state.[4] Barelli and Duggan (2014) used this approach to prove that for every stationary correlated equilibrium in the Nowak–Raghavan sense, there is a payoff-equivalent stationary semi-Markov equilibrium obtained by replacing correlation with conditioning on the state and actions in the previous period.[5]

The remainder of the paper is organized as follows. In Section 2, we set forth the framework of games of almost perfect information used by HRR, and we specialize this to games with atomless moves by Nature. In Section 3, we present our main result, which shows that we can dispense with public randomization in any game with atomless moves by Nature. In Section 4, we apply our result to stochastic games, and in Section 5, we prove the decorrelation theorem. Section 6 concludes, and the Appendix shows the equivalence of geometric discounting (which we use) and the payoff formulation of HRR.

2. Games with atomless moves by nature

We adopt the framework of HRR but for two modifications, both inconsequential. First, we omit the set Y_0 of starting points of the game, which was used by HRR to establish upper hemicontinuity of equilibria, whereas our focus is on de-correlation of equilibria. Second, we use a representation of payoff functions in terms of stage payoffs and geometric discounting, whereas HRR use continuous payoff functions defined on infinite histories. This formulation is convenient for our analysis, and we show in Proposition A.1, in the Appendix, that the discounting approach is without loss of generality; given the widespread use of geometric discounting in applied work, this result may be of independent interest.

Data. The data of the HRR framework (with our two modifications) are as follows.

- There is a finite, nonempty set $N = \{1, \ldots, n\}$ of active players, indexed by i or j, and a passive player, "Nature," denoted 0. Let $N_0 = N \cup \{0\}$.

- There is a countably infinite set $T = \{1, 2, \ldots\}$ of time periods, indexed by s or t. Let $T_0 = T \cup \{0\}$.

[4] See also He and Sun (2017) for an alternative approach: instead of decomposing states into two components as in Duggan (2012), He and Sun (2017) focused on two sigma-algebras on states, and, assuming that transitions are measurable with respect to the (much) coarser of the two, apply ideas from Dynkin and Evstigneev (1976) to obtain the required convexity.

[5] Similar techniques are also applied in Barelli and Duggan (2015a) to show that stationary Markov perfect equilibria in noisy stochastic games are payoff-equivalent to equilibria that select only extreme points of equilibrium payoffs in induced games, and they are applied to purification of Bayes Nash equilibria in Barelli and Duggan (2015b).

- For each $t \in T_0$ and each $i \in N$, there is a nonempty, complete, separable metric space $Y_{t,i}$ of actions denoted $y_{t,i}$. Set $Y_t = \times_{i \in N} Y_{t,i}$, with elements $y_t = (y_{t,i})_{i \in N}$, and set $Y = (Y_t)_{t \in T_0}$.
- For each $t \in T_0$, there is a nonempty, complete, separable metric space Z_t of Nature's actions denoted z_t. Set $Z = (Z_t)_{t \in T_0}$.
- For each $t \in T$, there is a nonempty, closed subset $X_t \subseteq \times_{s=1}^{t} (Y_s \times Z_s)$ of possible t-period histories with typical element of X_t denoted x_t (the structure of X_t is elaborated below). Designate a fixed $x_0 \in Y_0 \times Z_0$ as the initial history at which the game begins, and set $X_0 = \{x_0\}$. Finally, set $X = (X_t)_{t \in T_0}$.
- For each $t \in T$ and each $i \in N$, there is continuous correspondence $A_{t,i} \colon X_{t-1} \rightrightarrows Y_{t,i}$ of feasible actions with nonempty, compact values. In addition, there is a continuous correspondence $A_{t,0} \colon X_{t-1} \rightrightarrows Z_t$ with nonempty, closed values. Set $A_t = \times_{i \in N_0} A_{t,i}$ and $A = (A_t)_{t \in T}$. Consistent with the interpretation of X_t as the set of possible t-period histories, we assume that for each $t \in T$, $X_t = \mathrm{graph}(A_t)$. In particular, the projection of X_t on $Y_t \times Z_t$ is a product of sets for each player and Nature.
- For each $t \in T$, there is a continuous mapping $\varphi_t \colon X_{t-1} \to \Delta(Z_t)$, where $\Delta(\cdot)$ represents the set of Borel probability measures endowed with the weak* topology. Assume that for each $x_{t-1} \in X_{t-1}$, the support of $\varphi_t(x_{t-1})$ is contained in $A_{t,0}(x_{t-1})$. Set $\varphi = (\varphi_t)_{t \in T}$.
- For each $t \in T$ and each $i \in N$, there is a bounded, continuous stage payoff function $u_{t,i} \colon X_t \to \mathbb{R}$. Set $u_t = (u_{t,i})_{i \in N}$ and $u = (u_t)_{t \in T}$.
- For each $i \in N$, there is a discount factor $\delta_i \in [0, 1)$. Let $\delta = (\delta_i)_{i \in N}$.

These elements describe a *game of almost perfect information* (or simply, a *game*), denoted $G = (N_0, Y, Z, X, x_0, A, \varphi, u, \delta)$, in which players and Nature move simultaneously in each period. Given any history x_t, we define the *subgame* at x_t, denoted $G(x_t)$, in the obvious way.

Infinite histories. For a sequence $x \in \times_{t \in T_0} (Y_t \times Z_t)$ of action profiles in each period, given any $t \in T_0$, we denote by x_t the *truncation* of x, which projects x onto its first $t+1$ coordinates. An *infinite history* is a sequence $x \in \times_{t \in T_0} (Y_t \times Z_t)$ such that for each $t \in T_0$, we have $x_t \in X_t$. Let X_∞ denote the set of infinite histories, which we endow with relative topology inherited from the product topology on $\times_{t \in T_0} (Y_t \times Z_t)$, along with the measurable structure generated by finite cylinder sets. Then $\Delta(X_\infty)$ is the set of probability measures ξ on infinite histories. Given any $t \in T$ and history $x_t \in X_t$, let $H_t(x_t) = \{x' \in X_\infty \mid x'_t = x_t\}$ denote the set of *continuation histories* at x_t. Since A_t is continuous for each $t \in T$, it follows that $H_t \colon X_t \rightrightarrows X_\infty$ is a continuous correspondence. Since Y_t and Z_t are metric spaces for each $t \in T$, it follows that the product topology on X_∞ is metrizable (Theorem 3.36, Aliprantis and Border (2006)).

Stage payoffs. Assume that for all players $i \in N$, the discounted sum of stage payoffs, $\sum_{t \in T} \delta_i^{t-1} u_{t,i}(x_t)$, is bounded and continuous on X_∞.[6] An alternative formulation of

[6]Continuity of discounted streams of payoffs would follow from continuity of each $u_{t,i}$ if stage payoffs were uniformly bounded across t, but we do not impose the latter assumption.

payoffs, used by HRR, is to assume a bounded, continuous payoff function $u_i \colon X_\infty \to \mathbb{R}$ over infinite histories for each player. In the Appendix, we exploit history dependence of stage payoffs to show that the two formulations are equivalent; in fact, we show that it is without loss of generality to assume a common, positive discount factor in games of almost perfect information.[7]

Strategies. A *strategy* for player $i \in N$ is a sequence $f_i = (f_{t,i})_{t \in T}$ of Borel measurable mappings $f_{t,i} \colon X_{t-1} \to \Delta(Y_{t,i})$ such that $f_{t,i}(A_{t,i}(x_{t-1})|x_{t-1}) = 1$ for each $t \in T$ and each $x_{t-1} \in X_{t-1}$. Define $f_t = (f_{t,i})_{i \in N} \colon X_{t-1} \to \times_{i \in N} \Delta(Y_{t,i})$ as the profile of mappings for each player in period t. A *strategy profile* is an ordered n-tuple $f = (f_i)_{i \in N}$. Let F_i denote the set of strategies for player i, and let $F = \times_{i \in N} F_i$ denote the set of strategy profiles.

Continuation payoffs. Given $f \in F$, $t \in T$, $i \in N$, and $x_{t-1} \in X_{t-1}$, player i's *continuation payoff* is the expected discounted payoff in the remainder of the game, following history x_{t-1}, defined recursively by

$$\begin{aligned} U_{t,i}(x_{t-1}, f) = \int_y \biggl[\int_z \bigl[& u_{t,i}(x_{t-1}, y, z) \\ & + \delta_i U_{t+1,i}\bigl((x_{t-1}, y, z), f\bigr) \bigr] \varphi_t(x_{t-1})(dz) \biggr] \biggl(\bigotimes_{i \in N} f_{t,i}(x_{t-1}) \biggr)(dy), \end{aligned}$$

where we integrate over players' and Nature's actions in period t. This is a standard construction using dynamic programming techniques to establish the existence of mappings $U_t \colon X_{t-1} \times F \to \mathbb{R}^n$ for each t such that, for each f, the continuation payoff $U_t(x_{t-1}, f)$ is Borel measurable as a function of x_{t-1}.[8]

Equilibrium. A *subgame perfect equilibrium (SPE)* is a strategy profile f such that for each $t \in T$, each $i \in N$, each $x_{t-1} \in X_{t-1}$, and each $\tilde{f}_i \in F_i$,

$$U_{t,i}(x_{t-1}, f) \geq U_{t,i}\bigl(x_{t-1}, (\tilde{f}_i, f_{-i})\bigr).$$

Clearly, a strategy profile f is a SPE of G if and only if for every history x_t, the strategies restricted to this subgame form a SPE of the subgame $G(x_t)$.

Auxiliary games. Given a game G, any $t \in T$, any $x_{t-1} \in X_{t-1}$, and any bounded, Borel measurable function $V \colon Y_t \times Z_t \to \mathbb{R}^n$, the *auxiliary game* induced by V at t given x_{t-1} is the strategic form game $G_t(x_{t-1}, V) = (N, (A_{t,i}(x_{t-1}))_{i \in N}, (\pi_i)_{i \in N})$ with player set N, strategy sets $A_{t,i}(x_{t-1})$ with mixed strategies $\sigma_i \in \Delta(A_{t,i}(x_{t-1}))$, and payoff functions

$$\pi_i(y) = \int_z \bigl[u_{t,i}(x_{t-1}, y, z) + \delta_i V_i(y, z) \bigr] \varphi_t(x_{t-1})(dz).$$

[7]This is not true in a stationary stochastic game, because stage payoffs are history-independent in that setting.

[8]Specifically, given x_{t-1}, f, and φ, a standard argument (see, e.g., Bertsekas and Shreve (1996)) yields a unique probability measure $P^{f,\varphi}(\cdot|x_{t-1})$ over the Borel sets of $\times_{s \geq t} X_s$ such that the mapping $x_{t-1} \mapsto P^{f,\varphi}(\cdot|x_{t-1})$ is Borel measurable and with the appropriate marginals over the factors; in particular, the marginal over $Z_t \times Y_t$ is $\bigotimes_{i \in N_0} f_{t,i}(x_{t-1})$. Then i's continuation payoff at t given f is $U_{t,i}(x_{t-1}, f) = \mathbb{E}[\sum_{s \geq t} \delta_i^{s-1} u_{s,i}]$, where $\mathbb{E}$ denotes expectation with respect to $P^{f,\varphi}(\cdot|x_{t-1})$. Observe that $U_{t,i}$ is Borel measurable on x_{t-1}, and the recursive definition above follows immediately.

Here, the values $V(y, z)$ stand in for the players' expected future payoffs given action profile (y, z); note that $U_{t+1}((x_{t-1}, \cdot), f)$ in the definition of SPE plays the same role as $V(\cdot)$ in the definition of auxiliary game.

One-shot deviation principle. Let $N_t(x_{t-1}, V)$ denote the set of mixed strategy Nash equilibria of the auxiliary game $G_t(x_{t-1}, V)$, i.e., $\sigma = (\sigma_i)_{i \in N} \in N_t(x_{t-1}, V)$ if and only if for each $i \in N$ and each $y'_i \in A_{t,i}(x_{t-1})$, we have

$$\int_y \pi_i(y)\left(\bigotimes_{j \in N} \sigma_j\right)(dy) \geq \int_{y_{-i}} \pi_i(y'_i, y_{-i})\left(\bigotimes_{j \neq i} \sigma_j\right)(dy_{-i}).$$

By the one-shot deviation principle, a strategy profile f is a SPE of G if and only if for all $t \in T$ and all $x_{t-1} \in X_{t-1}$, the profile $f_t(x_{t-1}) = (f_{t,i}(x_{t-1}))_{i \in N}$ is a mixed strategy equilibrium of the auxiliary game $G_t(x_{t-1}, U_{t+1}((x_{t-1}, \cdot), f))$.[9]

Extended games. Given a game $G = (N_0, Y, Z, X, x_0, A, f_0, u, \delta)$, the *extension* of G is the game $\hat{G}$ such that $\hat{N} = N$, and for each $t \in T$, (i) for each $i \in N$, $\hat{Y}_{t,i} = Y_{t,i}$, (ii) $\hat{Z}_t = Z_t \times [0, 1]$, (iii) possible t-period histories are as in the original game with the addition of a signal $\omega_s \in [0, 1]$ in each period $t \in T$, i.e.,

$$\hat{X}_t = \left\{\left((y_0, z_0), (y_1, z_1, \omega_1), \dots, (y_t, z_t, \omega_t)\right) \mid x_t \in X_t, (\omega_s)_{s=1}^t \in [0, 1]^t\right\},$$

writing elements as $\hat{x}_t = (x_t, \omega_1, \dots, \omega_t)$, (iv) for each $\hat{x}_t \in \hat{X}_{t-1}$, the marginal of $\hat{f}_{t,0}(\hat{x}_{t-1})$ on Z_t is $\varphi_t(x_{t-1})$, and ω_t is drawn independently from the uniform distribution on $[0, 1]$, that is, $\hat{\varphi}_t(\hat{x}_{t-1}) = \varphi_t(x_{t-1}) \otimes \lambda$, where λ is the uniform measure on $[0, 1]$, (v) for each $i \in N_0$ and each $\hat{x}_{t-1} \in \hat{X}_{t-1}$, we have $\hat{A}_{t,i}(\hat{x}_{t-1}) = A_{t,i}(x_{t-1})$, and (vi) for each $i \in N$ and each $t \in T$, we have $\hat{u}_{t,i}(\hat{x}_t) = u_{t,i}(x_t)$. Thus, ω_t is a payoff-irrelevant public signal. Without risk of confusion, given $\hat{V}: \hat{Y}_t \times \hat{Z}_t \to \mathbb{R}^n$, we shall use $\hat{G}_t(\hat{x}_{t-1}, \hat{V})$ to denote the corresponding auxiliary game at period $t \in T$, and $\hat{U}_t(\hat{x}_{t-1}, \hat{f})$ to denote continuation payoffs at $t \in T$.

We now present the definition of games with atomless moves by Nature. The natural definition would be to have $\varphi_t(x_{t-1})$ atomless for all $t \in T$ and all $x_{t-1} \in X_{t-1}$, but we will use a weaker definition that allows for the possibility that the distribution of Nature's actions has an atom, as long as the active players have only trivial moves and Nature's moves are atomless in the next period.[10] This definition admits games in which the

[9]The one-shot deviation principle applies because the game is "continuous at infinity" (Blackwell (1965)).

[10]He and Sun (2020) assumed the stronger version that $\varphi_t(x_{t-1})$ is atomless for all $t \in T$ to obtain their Theorem 1, and then give their Assumption 2, which allows atoms in periods with a single active player, to obtain their Proposition 1. Their class of games satisfying Assumption 2 is closely related to our games with atomless moves by Nature. Any game with atomless moves by Nature can be reformulated as one that satisfies He and Sun's Assumption 2: if Nature's move has an atom in period t in our setting, then period $t+1$ could be collapsed into period t to satisfy Assumption 2 in their framework; this transformation relies on the possibility that Nature's move in a given period depends on the actions of players in that period, which He and Sun allow. Conversely, any game in which Nature's moves are atomless and depend on players' actions within a period can be transformed into a game with atomless moves by Nature by staggering Nature's moves into a subsequent period. However, we do not allow for sequences of perfect information moves, as described in part (i) of their Assumption 2.

active players and Nature move in alternate periods, a fact that allows us to apply our results to the class of stochastic games, in Section 4.

Games with atomless moves by Nature. A game G is a *game with atomless moves by Nature* if for all $t \geq 2$, and all $x_t \in X_t$, the following holds:

$$\begin{array}{c}\varphi_t(x_{t-1}) \text{ has} \\ \text{an atom}\end{array} \quad \Rightarrow \quad \begin{array}{c}\text{for all } i \in N, \left|A_{t+1,i}(x_t)\right| = 1 \\ \text{and } \varphi_{t+1,0}(x_t) \text{ is atomless.}\end{array}$$

Clearly, G is a game with atomless moves by Nature if $\varphi_t(x_{t-1})$ is atomless for all t and all histories x_{t-1}, but our condition is strictly weaker than this requirement.

3. Dispensing with public randomization

Theorem 4 of HRR establishes the following equilibrium existence result.

Theorem 1 (Harris, Reny, and Robson). *For each game G, the extension $\hat{G}$ admits a subgame perfect equilibrium.*

HRR consider several approaches to establishing existence of SPE in the original game G, without public randomization. They note that their theorem implies existence of SPE in games with finite action sets and in zero-sum games, where Nature's role in the extended game is not crucial.[11] We use the HRR theorem to analyze games without public randomization by imposing nonatomicity structure on Nature's moves. The main result of this paper is the following theorem: for a game with atomless moves by Nature, every SPE $\hat{f}$ of the extended game can be "decorrelated" to produce a SPE f of the original game that preserves the players' expected discounted payoffs from all action profiles at the initial history. Formally, we say f is *payoff-equivalent* to $\hat{f}$ if for all $i \in N$ and all $y \in \times_{i\in N} A_{1,i}(x_0)$, we have

$$\begin{aligned}&\int_z \big[u_{1,i}(x_0, y, z) + \delta_i U_{2,i}\big((x_0, y, z), f\big)\big]\varphi_1(x_0)(dz) \\ &\quad = \int_z \left[u_{1,i}(x_0, y, z) + \delta_i \int_\omega \hat{U}_{2,i}\big((x_0, y, z), \omega, \hat{f}\big)\lambda(d\omega)\right]\varphi_1(x_0)(dz),\end{aligned}$$

where $U_{2,i}$ is the continuation payoff after period 1 in the original game and $\hat{U}_{2,i}$ is the continuation payoff after period 1 in the extended game.[12]

Theorem 2. *In a game G with atomless moves by Nature, every subgame perfect equilibrium of the extended game $\hat{G}$ is payoff equivalent to a subgame perfect equilibrium of G.*

[11] HRR also claimed that existence of SPE in games of perfect information is a consequence of their main result, but Luttmer and Mariotti (2003) provide a counterexample to this claim.

[12] In the proof of Theorem 2, we define a notion of payoff equivalence at an arbitrary history. Note, however, that the equivalence established in the theorem holds for payoffs calculated at the beginning of the game; in general, the process of de-correlation can change equilibrium play and payoffs in later subgames.

Thus, when Nature's moves are already atomless, the addition of uniformly distributed payoff-irrelevant signals is without loss of generality, and can be interpreted simply as a technical device in HRR. From Theorems 1 and 2, we obtain the obvious existence result, which is also obtained in Proposition 1 of He and Sun (2020).

Corollary 1. *Every game with atomless moves by Nature admits a subgame perfect equilibrium.*

Here, we present the main ideas of the proof of Theorem 2, which is located in Section 5. By the theorem of HHR, the extended game $\hat{G}$ has a SPE $\hat{f}$. The proof of Theorem 2 consists of transforming $\hat{f}$ to a SPE of G via a "triangular" sequence $(f^{t-1})_{t\in T}$ of SPE profiles in the extended game of the following form:

$$\begin{aligned}
f^0 &= (\hat{\boldsymbol{f}}_{\mathbf{1}}, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{f}_5, \dots), \\
f^1 &= (f_1^0, \boldsymbol{f}_{\mathbf{2}}^{\mathbf{1}}, f_3^1, f_4^1, f_5^1, \dots), \\
f^2 &= (f_1^0, f_2^1, \boldsymbol{f}_{\mathbf{3}}^{\mathbf{2}}, f_4^2, f_5^2, \dots), \\
f^3 &= (f_1^0, f_2^1, f_3^2, \boldsymbol{f}_{\mathbf{4}}^{\mathbf{3}}, f_5^3, \dots), \\
&\vdots
\end{aligned} \tag{1}$$

That is, the first profile f^0 is just $\hat{f}$. To define f^1, we leave period 1 strategies unchanged, i.e., $f_1^0 = \hat{f}_1$, and we update strategies in all periods $t = 2, 3, \dots$. To define f^2, we leave strategies in the first two periods unchanged, and we update strategies in all periods $t = 3, 4, \dots$, and so on. At the end of the construction, we select the "diagonal" moves to arrive at a SPE $f^\infty = (f_t^{t-1})_{t\in T}$, indicated in boldface above, of the extended game. Each component f_t^{t-1} is constructed so as to be independent of the public signals $\omega_1, \dots, \omega_{t-1}$ and to preserve expected payoffs from action profiles in period $t-1$ (and thus all previous periods). With the profile f^∞ in hand, since players' actions do not depend on the payoff-irrelevant public signals, we then define the SPE f of the game with atomless moves by Nature in the obvious way: for each period t, we project f^∞ onto the set X_{t-1} of histories of the original game. The formal proof is complicated by the fact that a game with atomless moves by Nature allows for atomic moves by Nature, under specific circumstances. Here, we convey the approach of the proof for the simpler case in which for each period t and each history $x_t \in X_t$, Nature's move $\varphi_t(x_t)$ is atomless.

The idea of the construction of f^1 is to select, in a measurable way, a SPE of the subgame $\hat{G}(x_1, \omega_1)$ for each $(x_1, \omega_1) \in X_1 \times [0, 1]$, and to do so in a way that is independent of ω_1, yet preserves the active players' expected discounted payoffs from every feasible action profile $y_1 \in \times_{i\in N} A_{1,i}(x_0)$ in period 1. More precisely, letting $\hat{U}_2$ denote the continuation payoffs in the extended game, the selection we construct must preserve the expected payoff

$$\int_z \int_\omega \hat{U}_2\big((x_0, y_1, z, \omega), \hat{f}\big)\lambda(d\omega)\varphi_1(x_0)(dz)$$

for each feasible action profile y_1. This is possible for two reasons. First, the public signal ω_1 is payoff irrelevant, so that the set of subgame perfect equilibria in subgame $\hat{G}(x_1, \omega_1)$ is in fact independent of ω_1. Thus, we can write $E_2(x_1)$ for the set of SPE payoffs in the subgame $\hat{G}(x_1, \omega_1)$. By Lyapunov's theorem, it follows that the integral $\int_\omega E_2(x_1)\lambda(d\omega)$ is just the convex hull of $E_2(x_1)$. Second, the assumption (for the present discussion) that Nature's move $\varphi_1(x_0)$ is atomless allows us to apply the "measurable 'measurable choice' theorem" of Mertens (2003) to measurably select SPE payoffs from $E_2(\cdot)$, while preserving expected discounted payoffs following action profiles in the first period.[13]

To elaborate on this, we use the first observation above to write:[14]

$$\int_z \int_\omega \hat{U}_2\big((x_0, y_1, z, \omega), \hat{f}\big)\lambda(d\omega)\varphi_1(x_0)(dz) \in \int_z \operatorname{co} E_2(y_1, z)\varphi_1(x_0)(dz). \tag{2}$$

Moreover, because Nature's moves $\varphi_1(x_0)$ are atomless, any payoff obtained by integrating Nature's moves over the convex hull $\operatorname{co} E_2(x_1)$ can be obtained by integrating Nature's moves over $E_2(x_1)$, i.e.,

$$\int_z \operatorname{co} E_2(y_1, z)\varphi_1(x_0)(dz) = \int_z E_2(y_1, z)\varphi_1(x_0)(dz). \tag{3}$$

Mertens' theorem establishes the existence of a measurable mapping $g(y_1, p, z_1)$ defined on $(\operatorname{graph} \int_z E_2(\cdot, z)\varphi_1(x_0)(dz)) \times Z_1$ and such that both $g(y_1, p, z_1) \in E_2(y_1, z_1)$ and

$$p = \int_z g(y_1, p, z)\varphi_1(x_0)(dz). \tag{4}$$

That is, $g(y_1, p, \cdot)$ measurably selects from SPE payoffs of the subgames $\hat{G}(x_1, \omega_1)$ in way that generates payoffs p, while conditioning only on z_1, and not on ω_1. Combining (2) and (3), we have

$$\left(y_1, \int_z \int_\omega \hat{U}_2\big((x_0, y_1, z, \omega), \hat{f}\big)\lambda(d\omega)\varphi_1(x_0)(dz)\right) \in \operatorname{graph} \int_z E_2(\cdot, z)\varphi_1(x_0)(dz)$$

and, therefore, we can define the composite mapping Ψ by

$$\Psi(x_1) = g\left(y_1, \int_z \int_\omega \hat{U}_2\big((x_0, y_1, z, \omega), \hat{f}\big)\lambda(d\omega)\varphi_1(x_0)(dz), z_1\right)$$

to obtain a selection of SPE payoffs from $E_2(\cdot)$ that by (4) preserves the players' expected payoffs from each action profile y_1 in the first period.

Once this is done, we must translate the selection Ψ of SPE payoffs in each subgame $\hat{G}(x_1, \omega_1)$ into a SPE of the subgame that is measurable as a function of x_1 and

[13] In general, using the full strength of Mertens' theorem, purification can be performed as long as players do not correlate their choices following any atoms.

[14] Here, "co" denotes the convex hull, and the integral of a correspondence refers to the set consisting of integrals of all Borel measurable selections of the correspondence.

independent of ω_1. For this, we rely on Proposition 10 of HRR, which yields the desired strategies $f_2^1, f_3^1, \dots$ in periods following period 1. Note that the originally specified moves $\hat{f}_1(x_0) = f_1^0(x_0)$ still form a mixed strategy equilibrium of the induced game $\hat{G}_1(x_0, \hat{U}_2((x_0, \cdot), f^1))$ in the first period, and thus we can specify that the players use $\hat{f}_1$ in the first period to obtain a SPE f^1 of the extended game in which payoffs in the first period are maintained, and actions in future periods are not conditioned on the public signal ω_1.

At this point in the argument, continuation payoffs from action profiles in the first period are preserved by f^1, but payoffs after histories x_t with $t \geq 2$ need have no relation to continuation payoffs induced by $\hat{f}$; indeed, Proposition 10 of HRR specifies new SPE strategies that are a measurable function of x_1 and preserve continuation payoffs for each action profile in the first period, but the play of the game in periods $t \geq 2$ is otherwise independent of the original equilibrium. In the second iteration of the argument, we repeat the process to specify new subgame perfect equilibria in subgames $\hat{G}(x_2, \omega_1, \omega_2)$ in a way that is measurable as a function of x_2, independent of ω_2 (in addition to ω_1), and for each feasible action profile $y_2 \in \times_{i \in N} A_{2,i}(x_1)$, preserves the expected payoff from (x_1, y_2) in period 2. Importantly, we do not preserve continuation payoffs of the original equilibrium $\hat{f}$, but rather we preserve the payoffs $\int_z \int_\omega \hat{U}_3((x_1, y_2, z, \omega), f^1, \omega)\lambda(d\omega) f_{2,0}(x_1)(dz)$ for the equilibrium f^1 produced in the first application of Proposition 10 of HRR.

Delving further into the second iteration, we construct f^2 by selecting, in a measurable way, SPE payoffs of the subgames $\hat{G}(x_2, \omega_1, \omega_2)$ for each $(x_2, \omega_1, \omega_2) \in X_2 \times [0, 1]^2$, substituting conditioning on z_2 for conditioning on ω_2 in later periods $t \geq 3$ and preserving the expected payoff from each (x_1, y_2) in period 2. We leave moves in the first two periods unchanged, so that $f_1^2 = f_1^1 = \hat{f}_1$ and $f_2^2 = f_2^1$, and we use Proposition 10 of HRR to obtain desired SPE strategies $f_3^2, f_4^2, \dots$ in later periods. The construction continues in this way, iteratively generating a sequence $f^0, f^1, f^2, \dots$, and we construct a SPE $f^\infty = (f_t^{t-1})_{t \in T}$ of the extended game by selecting moves along the "diagonal" of this sequence. Since f_t^{t-1} is independent of public signals prior to period t, we project each f_t^{t-1} onto X_{t-1} to produce a SPE f of the original game with atomless moves by Nature that is payoff-equivalent to $\hat{f}$.

The heart of the proof of Theorem 2 is a recursive construction in which each iteration consists of three steps, taking as given a SPE f^{t-1} of the extended game $\hat{G}$ that is independent of public signals in the first $t-1$ periods and preserves the expected payoff of each action profile in period $t-1$. In Step 1, using Mertens' theorem, we construct a measurable selection $\Psi_{t+1}(x_t)$ of SPE payoffs in period $t+1$ from subgames $\hat{G}(\hat{x}_t)$ that does not depend on public signals in period t or earlier, and that preserves the expected payoff from each (x_{t-1}, y_t) in period t, where the expectation is taken with respect to Nature's move in period t. Hence, the "correlation" induced by dependence on the period t public signal is substituted for conditioning on Nature's move in that period.[15] In Step 2, we use Proposition 10 of HRR to construct a SPE f^t of $\hat{G}$ that maintains

[15]In the formal proof, Step 1 is broken into two parts to address the possibility that Nature's move in period $t-1$ has an atom, in which case the active players' moves in period t are exogenously fixed in a game with atomless moves by Nature.

$(f_1^{t-1}, \ldots, f_t^{t-1})$ in the first t periods, and that generates the payoffs $\Psi_{t+1}(x_t)$ from Step 1. In Step 3, we verify that in period t, f^t is independent of public signals and preserves the expected payoff of each (x_{t-1}, y_t). Steps 1, 2, and 3 are repeated inductively to arrive at the sequence $f^0, f^1, f^2, \ldots$ of SPE profiles in (1). The final Step 4 is constructing the SPE $f^\infty = (f_t^{t-1})_{t\in T}$ of the extended game, and then projecting each f_t^{t-1} onto X_{t-1} to produce the desired SPE f of G. Note that the final Step 4, unlike Steps 1, 2, and 3, is carried out only once.

4. Application to stochastic games

A classical (discounted) *stochastic game* is a dynamic game played in discrete time among n players such that: in each period, a state $z \in Z$ is publicly observed; each player i chooses an action y_i from a feasible set $A_i(z) \subseteq Y_i$; payoffs $u_i(y, z)$ are realized; a new state is drawn from the transition probability $\mu(\cdot|y, z) \in \Delta(Z)$; and the process is repeated. The standard assumptions are that each Y_i is a compact metric space with its Borel structure, Z is a measurable space, the feasible action correspondences A_i are lower measurable,[16] with nonempty, compact values, and each $u_i(y, z)$ is jointly measurable and continuous in y. Typically, the transition probability $\mu\colon Y \times Z \to \Delta(Z)$ is assumed to be measurable and, with an appropriate topology on $\Delta(Z)$, continuous in y; general existence results for stochastic games have imposed the total-variation norm topology on $\Delta(Z)$, which makes the continuity assumption quite restrictive. We refer to this condition as *norm continuity of transitions*. Note that feasible action correspondences depend only on the current state, and payoff functions depend only on the current state and profile of actions. Finally, payoffs are computed using the discounted sum of stage payoffs using discount factors $\delta_i \in [0, 1)$ for each player i.

We subsume the classical stochastic game framework within the class of games of almost perfect information as follows. First, we specify that for all $t \in T$, we have $Z_t = Z$ and $Y_{t,i} = Y_i$. Second, to capture the timing of a stochastic game, we have active players move in *odd periods* and Nature in *even periods* by adding dummy moves: in even periods, active players' moves are fixed at some given $\bar{y} \in Y = \times_{i\in N} Y_i$, and in odd periods, Nature's move is fixed at some given $\bar{z} \in Z$. Formally, we specify that for each odd period t and each history x_{t-1}, player i's feasible set $A_{t,i}(x_{t-1}) = A_i(z_{t-1})$ is given by the action correspondence from the stochastic game; and in even periods, the action set is the singleton $A_{t,i}(z_{t-1}) = \{\bar{y}_i\}$. For Nature's moves, for each even period $t \geq 2$ and each history x_{t-1}, we set $\varphi_t(x_{t-1}) = \mu(\cdot|y_{t-1}, z_{t-2})$, consistent with the transition probability μ using the players' actions and state in the previous period; and in odd periods, we set $\varphi_t(x_{t-1})$ to be equal to the point-mass at $\bar{z}$. Thus, Nature's move z_t in an even period determines the state for the following period, in which players move. Finally, we define stage payoffs so that for each odd period t and each history x_t, $u_{t,i}(x_t) = u_i(y_t, z_{t-1})$, so that player i's payoff is determined by the current action profile and state; and in each even period t, we set $u_{t,i} = 0$. This alternation of moves implies that stage payoffs from odd periods are subject to double discounting, so to preserve dynamic preferences in the

[16]Given a measurable space S with sigma-algebra Σ, and given a topological space X, a correspondence $\varphi\colon S \rightrightarrows X$ is lower measurable if for each open set $G \subseteq X$, we have $\varphi^\ell(G) = \{s \in S : \varphi(s) \cap G \neq \emptyset\} \in \Sigma$.

stochastic game, we specify that player i's discount factor is $\sqrt{\delta_i}$ in the game of almost perfect information.

By the above argument, any stochastic game can be represented as a game G of almost perfect information as long as: (i) Z is a complete, separable metric space, (ii) each A_i is continuous, (iii) each u_i is continuous, and (iv) the transition probability $\mu\colon Y \times Z \to \Delta(Z)$ is weak* continuous. It should now be apparent why we formulated games with atomless moves by Nature as we have. Our definition allows for atoms in moves of Nature in periods preceding a "no play" period; this accommodates stochastic games, which have atoms in odd periods (in fact, Nature's move is completely atomic) and no atoms in even periods. Thus, a stochastic game can be viewed as a game with atomless moves by Nature if, in addition to (i)–(iv), we have: (v) the distribution over states, $\mu(\cdot|y, z)$, is atomless for all $(y, z) \in Y \times Z$. In this case, we refer to G as a *stochastic game with atomless moves by Nature.*

Our results for games with atomless moves by Nature have immediate application to stochastic games. Clearly, Theorem 1 implies that given any stochastic game G satisfying (i)–(iv), the extension $\hat{G}$ admits a SPE. In turn, Theorem 2 shows that if the transition probability is atomless, then every SPE of the extended game corresponds to a SPE of the original stochastic game.

Corollary 2. *In a stochastic game G with atomless moves by Nature, every subgame perfect equilibrium of the extended game $\hat{G}$ is payoff equivalent to a subgame perfect equilibrium of G. In particular, every stochastic game satisfying (i)–(v) admits a subgame perfect equilibrium.*

The existence of SPE in stochastic games satisfying (i)–(v) is also obtained in Proposition 2 of He and Sun (2020).[17] This existence result is not logically nested with Theorem 1 of Mertens and Parthasarathy (2003), as the latter authors allow feasible action sets, payoffs, and the state transition to depend on the state in a measurable way, whereas we assume continuous dependence.[18] However, as we mentioned above, Mertens and Parthasarathy require norm continuity of transitions, precluding the possibility that the state has a component that varies deterministically as a function of players' continuous actions. Such deterministic dependence is natural in many applications, such as the strategic growth model in Example 1 below, and thus Corollary 2 has comparatively broad applicability.

Example 1 (Strategic growth and autocorrelated shocks). In an infinite-horizon, discrete-time model of growth with two agents, let k_i represent the capital stock for agent $i = 1, 2$, and let c_i be the level of consumption of agent i. Given capital stock

[17] Whereas we capture stochastic games by effectively having players and Nature alternate moves, He and Sun (2020) assumed players and Nature move simultaneously, but they let players' actions at t influence Nature's move at t, which stands in for the state in $t+1$.

[18] See also Theorems 3 and 4 of He and Sun (2020), which allow measurable dependence on the state and weak continuity with respect to actions. On the other hand, their results assumes that Nature's moves are absolutely continuous with respect to a fixed, atomless measure, precluding deterministic transition probabilities.

levels k_1 and k_2 at the end of the previous period and realized depreciation rates $r_1, r_2 \in (0, 1)$, the agents simultaneously choose consumption levels c_i subject to $0 \leq c_i \leq F_i(k_1, k_2) + (1 - r_i)k_i$, reflecting externalities in production given by F_i and a reduction in capital stocks due to depreciation. Consumption choices in the current period, in turn, leave capital stock levels at $k_i' = F_i(k_1, k_2) + (1 - r_i)k_i - c_i$. Assume that: the production functions F_i are bounded and jointly continuous; utility from consumption is bounded, continuous, and discounted over time; and that depreciation rates between periods are subject to random shocks, $r_i' = r_i + \epsilon_i$, where the shocks (ϵ_1, ϵ_2) have density $f(\epsilon_1, \epsilon_2 | c_1, c_2, k_1, k_2, r_1, r_2)$ that is jointly continuous in consumption levels, capital stocks, and shocks, thereby permitting general correlation across time.[19] This is an example of a stochastic game in which the state variable (k_1, k_2, r_1, r_2), has a deterministic component, namely (k_1, k_2), that depends on continuous actions, namely the consumption levels of the agents. As such, existence of SPE in this strategic version of the standard growth model does not follow from Mertens and Parthasarathy (2003).[20] It does, however, follow directly from Corollary 2 above. $\diamondsuit$

An advantage of our approach is that we can also deduce existence of a *correlated* SPE in stochastic games with weakly continuous transitions, even if the transition probability has atoms. The decorrelation result of the preceding corollary relies on the application of Theorem 1 to a stochastic game G satisfying (i)–(v), and in particular, the extended game $\hat{G}$ is defined by adding public signals in every period, including even periods in which Nature moves, but also odd periods in which the active players move. Thus, $\hat{G}$ requires that the active players in an odd period observe *two* public signals, ω_{t-2} and ω_{t-1}, between moves—in effect, duplicating the public signal. The standard interpretation of a correlated equilibrium (e.g., Nowak and Raghavan (1992)) has players observing only one signal for a given play of the stage game, complicating the interpretation of equilibria in $\hat{G}$ as correlated equilibria of the original game. It is possible to recover the standard interpretation by having both public signals be drawn in the same period, so that players in effect observe only one, albeit two-dimensional, public signal between moves; then the signal can be reduced to one dimension via a measure-preserving bijection between the unit square and the unit interval. Instead, we follow a more direct approach using Theorem 2 to establish existence of a SPE in correlated strategies, a result recently alluded to by Jaśkiewicz and Nowak (2017).

Given a stochastic game G satisfying (i)–(iv), we define the *associated game with public randomization*, denoted $\tilde{G}$, by adding a payoff-irrelevant public signal, drawn from the uniform distribution λ on $[0, 1]$, in every *even period* t. Denoting histories in the associated game by $\tilde{x}_t$, Nature's move in every even period t is then $\tilde{\varphi}_t(\tilde{x}_{t-1}) = \varphi_t(x_{t-1}) \otimes \lambda$. Regardless of the structure of the original game, $\tilde{\varphi}_t$ is atomless, and thus the associated game $\tilde{G}$ with public randomization is a game with atomless moves by Nature. Thus, we have the following additional corollary of Theorem 2.

[19] To bound action sets of the agents, we assume there is $\underline{r} > 0$ such that $r_i \in [\underline{r}, 1)$.

[20] Dutta and Sundaram (1992) studied a version of a strategic growth model allowing for weakly continuous transitions as well. They establish existence of a stationary Markov perfect equilibrium under strong concavity/differentiability assumptions.

Corollary 3. *For every stochastic game G satisfying (i)–(iv), the associated game $\tilde{G}$ with public randomization admits a subgame perfect equilibrium.*

By Corollary 3, any stochastic game satisfying (i)–(iv) admits a form of correlated SPE, complementing the well-known result of Nowak and Raghavan (1992), which establishes existence of a *stationary* correlated equilibrium under norm continuity of transitions.[21] Importantly, by allowing for subgame perfect equilibria in which strategies are history-dependent, we extend this existence result to stochastic games G in which the transition probability may have atoms and is merely weak* continuous. For example, it may be that the transition probability in G is a deterministic, continuous function of the previous state and actions. Returning to Example 1, we can remove noise from the depreciation rates, simply fixing them at $\bar{r}_1$ and $\bar{r}_2$, so that in any period, the capital stock levels, k_1 and k_2, and consumption levels, c_1 and c_2, determine new capital stock levels $k_i' = F_i(k_1, k_2) + (1 - \bar{r}_i)k_i - c_i$ in a deterministic way. Such deterministic transitions violate norm continuity, but Corollary 3 delivers a SPE in the associated game, where the agents observe a single, one-dimensional public signal between periods.

5. Proof of Theorem 2

Let G be any game with atomless moves by Nature. Before proceeding to the proof, we extend the concept of payoff equivalence to any period $t \in T$ and history $\hat{x}_{t-1} \in \hat{X}_{t-1}$ as follows. Given strategy profiles $\hat{f}$ and $\hat{f}'$ in the extended game $\hat{G}$, we say that $\hat{f}'$ is *payoff-equivalent to $\hat{f}$ at $\hat{x}_{t-1}$* if for all $i \in N$ and all $y \in \times_{i \in N} A_{t,i}(x_{t-1})$, we have

$$\int_z \left[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}\big((x_{t-1}, y, z, \omega), \hat{f}'\big)\lambda(d\omega) \right] \varphi_t(x_{t-1})(dz)$$
$$= \int_z \left[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}\big((x_{t-1}, y, z, \omega), \hat{f}\big)\lambda(d\omega) \right] \varphi_t(x_{t-1})(dz),$$

so that the expected discounted payoffs from $\hat{f}'$ and $\hat{f}$, calculated at $\hat{x}_{t-1}$, from every action profile are the same for every active player.[22] In addition, as in HRR, let $E_{t+1}(x_t)$ be the set of SPE payoffs in any subgame $\hat{G}(\hat{x}_t)$ of $\hat{G}$ such that the history of actions in $\hat{x}_t$ is x_t. Because the signals $\omega_1, \ldots, \omega_t$ are payoff irrelevant, this set is well-defined, and by Theorem 5 of HRR, the correspondence $E_{t+1}\colon X_t \rightrightarrows \mathbb{R}^n$ so-defined has a closed graph, and thus is lower measurable (Theorem 18.20, Aliprantis and Border (2006)).

To prove Theorem 2, fix a SPE $\hat{f}$ of $\hat{G}$ and set $f^0 = \hat{f}$. In general, for a recursive construction, for each $t \in T$, we take f^{t-1} as a given SPE of $\hat{G}$ satisfying the following conditions:[23]

[21] They also assume that the transitions are absolutely continuous with respect to a fixed measure. See also Jaśkiewicz and Nowak (2017) for an overview of the literature on existence of stationary equilibria in discounted stochastic games.

[22] Note that the initial definition of payoff equivalence compares a strategy profile f in G with a profile $\hat{f}$ in $\hat{G}$, whereas payoff equivalence at a history compares to profiles in $\hat{G}$.

[23] These conditions are vacuously satisfied by f^0 when $t = 1$, and (C1$_t$) is vacuously satisfied when $t = 2$.

($C1_t$) for all $s \in T$ and all $\hat{x}_{s-1} \in \hat{X}_{s-1}$, $f_s^{t-1}(\hat{x}_{s-1})$ is independent of $\omega_1, \dots, \omega_{t-2}$,

($C2_t$) for all $s \geq t$ and all $\hat{x}_{s-1} \in \hat{X}_{s-1}$ such that $\varphi_{t-1}(x_{t-2})$ is atomless, $f_s^{t-1}(\hat{x}_{s-1})$ is independent of $\omega_1, \dots, \omega_{t-1}$,

($C3_t$) for all $\hat{x}_{t-2} \in \hat{X}_{t-2}$, f^{t-1} is payoff-equivalent to f^{t-2} at $\hat{x}_{t-2}$.

That is, the strategy profile f^{t-1} does not depend on public signals in periods up to and including $t-1$, unless Nature's move $\varphi_{t-1}(x_{t-2})$ has an atom at history x_{t-2}, in which case strategies are still independent of public signals in the first $t-2$ periods. Moreover, expected discounted payoffs from action profiles calculated in period $t-1$ are the same for f^{t-1} and f^{t-2}.

We will construct a strategy profile f^t in $\hat{G}$ satisfying ($C1_{t+1}$)–($C3_{t+1}$) such that f^t shares the first t moves with f^{t-1}, i.e.,

$$\left(f_1^t, \dots, f_t^t\right) = \left(f_1^{t-1}, \dots, f_t^{t-1}\right).$$

Later moves $(f_{t+1}^t, f_{t+2}^t, \dots)$ will be independent of $\omega_1, \dots, \omega_{t-1}$; and if $\varphi_t(x_{t-1})$ is atomless, they will be independent of ω_t as well. The later moves will also preserve expected payoffs at each history $\hat{x}_{t-1}$ from each action profile $y_t \in \times_{i\in N} A_{t,i}(x_{t-1})$. In fact, if Nature's move $\varphi_t(x_{t-1})$ has an atom at any history $\hat{x}_{t-1}$, then we simply specify that $(f_{t+1}^t, f_{t+2}^t, \dots)$ restricted to $\hat{G}(\hat{x}_{t-1})$ is identical to f^{t-1}, i.e., for all $s \geq t+1$ and all $\hat{x}'_{s-1} \in \hat{X}_{s-1}$ with $\hat{x}'_{t-1} = \hat{x}_{t-1}$, $f_s^t(\hat{x}'_{s-1}) = f_s^{t-1}(\hat{x}'_{s-1})$. Importantly, if Nature's move $\varphi_t(x_{t-1})$ is atomless at history $\hat{x}_{t-1}$, then $(f_{t+1}^t, f_{t+2}^t, \dots)$ is chosen so that it forms a SPE in the subgame $\hat{G}(\hat{x}_{t-1})$, and these later moves will generally differ from those in f^{t-1}. Then strategies restricted to earlier subgames remain subgame perfect by payoff equivalence. Because f^{t-1} does not depend on public signals prior to $t-1$, by ($C1_t$), we suppress the realizations of public signals prior to period $t-1$ and write $f_t^{t-1}(\hat{x}_{t-1})$ as $f_t^{t-1}(x_{t-1}, \omega_{t-1})$. When $\varphi_{t-1}(x_{t-2})$ is atomless, ($C2_t$) allows us to write this simply as $f_t^{t-1}(x_{t-1})$.

The proof consists of a recursive construction in Steps 1–3, and a final Step 4 that produces the desired SPE of the original game G: Steps 1 and 2 perform the inductive construction, and Step 3 verifies that the inductively constructed profile of strategies satisfies ($C1_{t+1}$)–($C3_{t+1}$); Step 4 is an independent verification step, taking place after Steps 1–3 have been repeated countably many times, and establishing that the construction in fact yields an SPE. For each $t \in T$, let X_t° consist of histories $x_t \in X_t$ of the original game G such that $f_{t+1,0}(x_t)$ is atomless. Note that the set of atomless probability measures is Borel measurable, and since $f_{t+1,0}$ is Borel measurable, the set X_t° is itself Borel measurable. In case $x_{t-1} \notin X_{t-1}^\circ$, as mentioned in the preceding paragraph, we specify f^t so that at all subsequent histories, play proceeds according to f^{t-1}. To confirm that our conditions are satisfied in this case, note that since $f_t(x_{t-1})$ has an atom, the definition of game with atomless moves by Nature implies that Nature's move $f_{t-1}(x_{t-2})$ prior to that is atomless, and then ($C2_t$) implies for all $s \geq t$, $f_s^{t-1}(\hat{x}_{t-1})$ is independent of $\omega_1, \dots, \omega_{t-1}$, so that our specification of $f^t = f^{t-1}$ in later subgames satisfies ($C1_{t+1}$).[24]

[24]The specification vacuously satisfies ($C2_{t+1}$) and ($C3_{t+1}$), since $\varphi_{t-1}(x_{t-2})$ has an atom.

Thus, our arguments focus on histories $\hat{x}_{t-1} \in X^{\circ}_{t-1} \times [0,1]^{t-1}$ at which Nature's moves are atomless.

Step 1 in the construction, next, is broken into two parts. The task is to disconnect continuation payoffs in period $t+1$ from public signals in preceding periods. In the first part, we consider a history $\hat{x}_{t-1} \in X^{\circ}_{t-1} \times [0,1]^{t-1}$ such that Nature's move in the previous period was also atomless, in which case, by (C2$_t$), we need address only dependence on public signals ω_t, in period t. In the second part, we consider a history such that Nature's move $\varphi_{t-1}(x_{t-2})$ in period $t-2$ has an atom. In this case, (C1$_t$) delivers independence from Nature's moves in the first $t-2$ periods, but dependence on both ω_{t-1} and ω_t must be addressed.

Step 1.1: Disconnecting continuation payoffs after atomless moves in period $\mathbf{t-1}$. Let $X^{1.1}_{t-1} = \{x_{t-1} \in X^{\circ}_{t-1} \mid x_{t-2} \in X^{\circ}_{t-2}\}$ consist of $(t-1)$-period histories in the original game such that Nature's moves, $\varphi_t(x_{t-1})$ and $\varphi_{t-1}(x_{t-2})$, in periods t and $t-1$ are atomless. Consider a history $\hat{x}_{t-1} \in X^{1.1}_{t-1} \times [0,1]^{t-1}$. Since f^{t-1} is a SPE of the extended game $\hat{G}$, the profile $(f^{t-1}_{t,i}(\hat{x}_{t-1}))_{i\in N}$ is a Nash equilibrium of the auxiliary game $\hat{G}_t(\hat{x}_{t-1}, \hat{U}_{t+1}((\hat{x}_{t-1},\cdot), f^{t-1}))$. By (C2$_t$), the strategies f^{t-1}_s in periods $s \geq t+1$ are independent of public signals $\omega_1, \dots, \omega_{t-1}$, so we can write player i's continuation payoff at $\hat{x}_t$, namely $\hat{U}_{t+1,i}(\hat{x}_t, f^{t-1})$, as $\hat{U}_{t+1,i}((x_t, \omega_t), f^{t-1})$. Then player i's expected payoff at $\hat{x}_{t-1}$ from an action profile $y \in \times_{i\in N} A_{t,i}(x_{t-1})$ simplifies to

$$\int_z \left[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}\big((x_{t-1}, y, z, \omega), f^{t-1}\big)\lambda(d\omega)\right] \varphi_t(x_{t-1})(dz). \tag{5}$$

For all histories $x_{t-1} \in X^{1.1}_{t-1}$ and all $(y,z) \in A_t(x_{t-1})$, the continuation payoff $\hat{U}_{t+1}((x_{t-1}, y, z, \omega), f^{t-1})$ belongs to $E_{t+1}(x_{t-1}, y, z)$ and, therefore,

$$\begin{aligned}\int_\omega \hat{U}_{t+1}\big((x_{t-1}, y, z, \omega), f^{t-1}\big)\lambda(d\omega) &\in \int_\omega E_{t+1}(x_{t-1}, y, z)\lambda(d\omega)\\ &= \operatorname{co} E_{t+1}(x_{t-1}, y, z),\end{aligned}$$

where the equality follows from Lyapunov's theorem (cf. part 2 of the theorem of Mertens (2003)). Integrating over z, we have

$$\begin{aligned}\int_z \int_\omega \hat{U}_{t+1}\big((x_{t-1}, y, z, \omega), f^{t-1}\big)\lambda(d\omega) &\in \int_z \operatorname{co} E_{t+1}(x_{t-1}, y, z)\varphi_t(x_{t-1})(dz)\\ &= \int_z E_{t+1}(x_{t-1}, y, z)\varphi_t(x_{t-1})(dz),\end{aligned}$$

where the equality follows from Lyapunov's theorem, since $\varphi_t(x_{t-1})$ is atomless and $E_{t+1}(x_{t-1}, y, z)$ is closed.

By part 3 of the theorem of Mertens (2003), there is a Borel measurable mapping $\Psi_{t+1} \colon \{x_t \in X_t \mid x_{t-1} \in X^{1.1}_{t-1}\} \to \mathbb{R}^n$ such that for all $(x_{t-1}, y) \in X^{\circ}_{t-1} \times Y_t$ with $x_{t-2} \in X^{\circ}_{t-2}$ and $y \in \times_{i\in N} A_{t,i}(x_{t-1})$, the mapping $\Psi_{t+1}(x_{t-1}, y, \cdot)$ is a selection from $E_{t+1}(x_{t-1}, y, \cdot)$, and

$$\int_z \Psi_{t+1}(x_{t-1}, y, z)\varphi_t(x_{t-1})(dz) = \int_z \int_\omega \hat{U}_{t+1}\big((x_{t-1}, y, z, \omega), f^{t-1}\big)\lambda(d\omega).$$

This gives us a selection of SPE payoffs that is independent of $\omega_1, \dots, \omega_t$ and such that after each history $\hat{x}_{t-1} \in X^{1.1}_{t-1} \times [0,1]^{t-1}$, player i's expected payoff from each action profile $y \in \times_{i\in N} A_{t,i}(x_{t-1})$ is

$$\begin{aligned}
&\int_z [u_{t,i}(x_{t-1}, y, z) + \delta_i \Psi_{t+1,i}(x_{t-1}, y, z)\varphi_t(x_{t-1})(dz) \\
&\quad = \int_z \biggl[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}\bigl((x_{t-1}, y, z, \omega), f^{t-1}\bigr)\lambda(d\omega)\biggr]\varphi_t(x_{t-1})(dz), \qquad (6)
\end{aligned}$$

which is just (5). Thus, the selection Ψ_{t+1} preserves the expected payoff from f^{t-1} for each action profile y in period t following histories $\hat{x}_{t-1}$ with $x_{t-1} \in X^{1.1}_{t-1}$.

Step 1.2: Disconnecting continuation payoffs after moves with atoms in period $\mathbf{t-1}$. Let $X^{1.2}_{t-1} = \{x_{t-1} \in X^\circ_{t-1} \mid x_{t-2} \notin X^\circ_{t-2}\}$ consist of $(t-1)$-period histories in the original game such that Nature's move $\varphi_t(x_{t-1})$ in period t is atomless, but its move $\varphi_{t-1}(x_{t-2})$ in period $t-1$ is not atomless. Consider a history $\hat{x}_{t-1} \in X^{1.2}_{t-1} \times [0,1]^{t-1}$, so that $\varphi_{t-1}(x_{t-2})$ has an atom. By definition of a game with atomless moves by Nature, it follows that the active players' moves are trivial in period t at history $\hat{x}_{t-1}$, in the sense that their actions are predetermined at this history. By (C1$_t$), for all $s \geq t-1$ and all histories $\hat{x}'_s$ with $\hat{x}'_{t-2} = \hat{x}_{t-2}$, $f^{t-1}_s(\hat{x}'_s)$ is independent of $\omega_1, \dots, \omega_{t-2}$, but future actions may depend on the public signal ω_{t-1} in period $t-1$. Since $\varphi_{t-1}(x_{t-2})$ has an atom, the arguments from Step 1.1 cannot be used to eliminate dependence of f^{t-1}_s on ω_{t-1}, but we can use a similar argument to replace dependence on (ω_{t-1}, ω_t) with an appropriate selection of equilibrium payoffs as a function of z_t, the distribution of which, namely $\varphi_t(x_{t-1})$, is atomless. Because action sets are singleton at $\hat{x}_{t-1}$, this selection may be constructed without concern for equilibrium incentives in period t, but we must preserve expected payoffs from action profiles in period $t-1$. Player i's continuation payoff at $\hat{x}_{t-2}$ from action profile $y \in \times_{i\in N} A_{t-1,i}(x_{t-2})$ is

$$\begin{aligned}
&\int_z \biggl[u_{t-1,i}(x_{t-2}, y, z) + \delta_i \int_{z'} \biggl[u_{t,i}\bigl(x_{t-2}, y, z, y', z'\bigr) \\
&\quad + \delta_i \int_{(\omega,\omega')} \hat{U}_{t+1,i}\bigl((x_{t-2}, y, z, \omega, y', z', \omega'), f^{t-1}\bigr)\lambda^2\bigl(d(\omega,\omega')\bigr)\biggr]\varphi_t(x_{t-2}, y, z)\bigl(dz'\bigr)\biggr] \\
&\quad \times \varphi_{t-1}(x_{t-2})(dz), \qquad (7)
\end{aligned}$$

where y' is the unique feasible action profile at (x_{t-2}, y, z), and λ^2 is Lebesgue measure on the unit square $[0,1]^2$. We can write y' explicitly as a function $\alpha_t(x_{t-1})$ of history; and because the feasible action correspondences $A_{t,i}$ are continuous, the mapping $\alpha_t \colon X^{1.2}_{t-1} \to Y_t$ is continuous.

Since the public signal is payoff irrelevant, it follows that for all (ω, ω'), we have

$$\hat{U}_{t+1,i}\bigl((x_{t-2}, y, z, \omega, y', z', \omega'), f^{t-1}\bigr) \in E_{t+1}\bigl(x_{t-1}, y, z, y', z'\bigr),$$

where again $y' = \alpha_t(x_{t-2}, y, z)$, and thus

$$\int_{(\omega,\omega')} \hat{U}_{t+1,i}\bigl((x_{t-2}, y, z, \omega, y', z', \omega'), f^{t-1}\bigr)\lambda^2\bigl(d(\omega,\omega')\bigr) \in \operatorname{co} E_{t+1}\bigl(x_{t-2}, y, z, y', z'\bigr),$$

by Lyapunov's theorem. Then the integral

$$\int_{z'}\int_{(\omega,\omega')}\hat{U}_{t+1,i}\big((x_{t-2},y,z,\omega,y',z',\omega'),f^{t-1}\big)\lambda^2\big(d(\omega,\omega')\big)\varphi_t(x_{t-2},y,z)\big(dz'\big)$$

belongs to

$$\begin{aligned}&\int_z \operatorname{co} E_{t+1}\big(x_{t-2},y,z,y',z'\big)\varphi_t(x_{t-2},y,z)\big(dz'\big)\\&\quad=\int_z E_{t+1}\big(x_{t-2},y,z,y',z'\big)\varphi_t(x_{t-2},y,z)\big(dz'\big),\end{aligned}$$

where the equality follows by Lyapunov's theorem from the assumption that $\varphi_t(x_{t-2}, y, z)$ is atomless.

By Mertens (2003), there is a Borel measurable mapping $\Psi_{t+1}\colon \{x_t \in X_t \mid x_{t-1} \in X^{1.2}_{t-1}\} \to \mathbb{R}^n$ such that for all $(x_{t-1}, y') \in X^\circ_{t-1} \times Y_t$ with $x_{t-2} \notin X^\circ_{t-2}$ and $y' = \alpha_t(x_{t-1})$, the mapping $\Psi_{t+1}(x_{t-1}, y', \cdot)$ is a selection from $E_{t+1}(x_{t-1}, y', \cdot)$ and

$$\begin{aligned}&\int_z \Psi_{t+1}(x_{t-1},y,z)\varphi_t(x_{t-1})(dz)\\&\quad=\int_{z'}\int_{(\omega,\omega')}\hat{U}_{t+1,i}\big((x_{t-2},y,z,\omega,y',z',\omega'),f^{t-1}\big)\lambda^2\big(d(\omega,\omega')\big)\varphi_t(x_{t-2},y,z)\big(dz'\big).\end{aligned}$$

This gives us a selection of SPE payoffs that is independent of $\omega_1, \dots, \omega_t$ and such that after each history $\hat{x}_{t-2} \notin X^\circ_{t-2} \times [0, 1]^{t-2}$, player i's expected payoff from each action profile $y \in \times_{i\in N} A_{t-1}(x_{t-2})$ is

$$\begin{aligned}&\int_z\bigg[u_{t-1,i}(x_{t-2},y,z)+\delta_i\int_{z'}u_{t,i}\big(x_{t-2},y,z,y',z'\big)\varphi_t(x_{t-2},y,z)\big(dz'\big)\bigg]\varphi_{t-1}(x_{t-2})(dz)\\&\quad+\delta_i^2\int_z \Psi_{t+1}(x_{t-1},y,z)\varphi_t(x_{t-1})(dz),\end{aligned}\tag{8}$$

where $y' = \alpha_t(x_{t-1})$, which is equivalent to (7). Thus, the selection Ψ_{t+1} preserves payoffs from f^{t-1} for each action profile y in period $t-1$ following histories $\hat{x}_{t-2}$ with $x_{t-2} \notin X^\circ_{t-2}$, as required.

Having removed dependence of continuation payoffs in period $t+1$ on Nature's previous moves, we next describe equilibrium behavior that supports those payoffs. To this end, we assign a SPE of every subgame $\hat{G}(\hat{x}_t)$ with $x_t \in X^\circ_t$ to generate payoffs $\Psi_{t+1}(x_t)$, and we must do so in a way that is measurable and independent of the history of public signals. We apply Proposition 10 of HRR, reproduced below for the reader's convenience, to construct the desired equilibrium selection, say $\tilde{f}^t$.[25] Since actions are pinned down at any history $\hat{x}_t$ with $x_t \notin X^\circ_t$, we then splice these together with $\tilde{f}^t$ to arrive at the desired strategy profile f^t.

[25] Most of the notation in the proposition will be clear from its statement and from our explanations below, but C_{t+1} is an arbitrary upper hemicontinuous correspondence from X_t to $\mathbb{R}^n$ with nonempty, closed values contained in U^n, where U is a compact set that contains the range of each player's discounted payoffs. Then ΨC_{t+1} is the set of payoff vectors for the period t stage game when continuation payoffs are chosen from C_{t+1}.

Theorem 3 (Proposition 10 of HRR). *Suppose that $c_t : \hat{X}_t \to \mathbb{R}^n$ is a Borel measurable selection from ΨC_{t+1}. Then there exist Borel measurable mappings $\tilde{f}_{t,i} : \hat{X}_{t-1} \to \Delta(Y_{t,i})$ for all $i \in N$ and a Borel measurable random selection $c_{t+1} : \hat{X}_t \to \mathbb{R}^n$ from C_{t+1}, such that, for all $\hat{x} \in \hat{X}_{t-1}$:*

(i) $(\tilde{f}_{t,i}(\hat{x})(\cdot))_{i\in N}$ is a Nash equilibrium of the stage game when continuation payoff vectors are given by $c_{t+1}(\hat{x}, \cdot) : \hat{A}_t(\hat{x}) \to \mathbb{R}^n$; and

(ii) $c_t(\hat{x})$ is the payoff vector of this Nash equilibrium.

Step 2: From continuation payoffs to actions. To apply Proposition 10 of HRR, we set $t = 1$ in their result, and we consider a game of almost perfect information with set of starting points equal to our X_t°, and with subgames determined by each starting point $x_t \in X^\circ$ identical to the subgame $G(x_t)$ in our original game, G. We identify their correspondence ΨC_2 with our E_{t+1}, their mapping c_1 with our Ψ_{t+1}, and their 1-period histories with our set $\{x_{t+1} \in X_{t+1} \mid x_t \in X_t^\circ\} \times [0, 1]$ of $(t+1)$-period histories such that Nature's move is atomless in period t, together with a public signal in period t. Hence, $c_1 = \Psi_{t+1}$ maps each $x_t \in X_t^\circ$ to SPE payoff vectors in $(\Psi C_2)(x_t) = E_{t+1}(x_t)$. By HRR's Proposition 10, there exist Borel measurable mappings $\tilde{f}^t_{t+1} \colon X_t^\circ \to \Delta(Y_{t+1,i})$ for each $i \in N$ and a Borel measurable selection $c_2 \colon \{x_{t+1} \in X_{t+1} \mid x_t \in X_t^\circ\} \times [0, 1] \to \mathbb{R}^n$ from E_{t+2} such that for all $x_t \in X_t^\circ$, (i) $\tilde{f}^t_{t+1}(x_t) = (\tilde{f}^t_{t+1,i}(x_t))_{i\in N}$ is a mixed strategy equilibrium of the auxiliary game $\hat{G}_1(x_t, c_2)$, and (ii) equilibrium payoffs from $\tilde{f}^t_{t+1}$ in the auxiliary game are $c_1(x_t) = \Psi_{t+1}(x_t)$. Note that the domain of the mapping $\tilde{f}^t_{t+1}$ is the set of starting points $x_t \in X_t^\circ$, and so it is manifestly independent of public signals $\omega_1, \ldots, \omega_t$. However, we want to use $\tilde{f}^t_{t+1}$ to describe moves in the extended game $\hat{G}$, and so we extend it to the domain $X_t^\circ \times [0, 1]^t$ in the obvious way: for all $\hat{x}_t \in \hat{X}_t$ with $x_t \in X_t^\circ$, we set the value of $\tilde{f}^t_{t+1}$ at $\hat{x}_t$ to be equal to $\tilde{f}^t_{t+1}(x_t)$. This introduces nominal dependence on public signals that will be removed in Step 4.

Applying HRR's Proposition 10 recursively (as in the proof of Lemma 18 of HRR), we obtain a sequence $\tilde{f}^t_{t+1}, \tilde{f}^t_{t+2}, \ldots$ such that for all $s \in T$ with $s > t$, the mapping $\tilde{f}^t_s$ is defined on histories $\hat{x}_{s-1} \in \hat{X}_{s-1}$ such that Nature's move is atomless in period t, i.e., $x_{t-1} \in X^\circ_{t-1}$. Moreover, $\tilde{f}^t_s(\hat{x}_{s-1})$ is independent of public signals $\omega_1, \ldots, \omega_t$, and the profile $\tilde{f}^t_s(\hat{x}_{s-1})$ is a mixed strategy equilibrium of the auxiliary game with continuation payoffs generated by $\tilde{f}^t_{s+1}, \tilde{f}^t_{s+2}, \ldots$. We then define the strategy profile $f^t = (f^t_s)_{s\in T}$ so that in any period $s > t$, given any history $\hat{x}_{s-1} \in \hat{X}_{s-1}$ such that Nature's move is atomless in period t, players use the strategies obtained via HRR's Proposition 10; and otherwise, the players follow their strategies in f^{t-1}. Formally, for all $s \in T$ and all $\hat{x}_{s-1} \in \hat{X}_{s-1}$, (i) if $s > t$ and $x_{t-1} \in X^\circ_{t-1}$, then $f^t_s(\hat{x}_{s-1}) = \tilde{f}^t_s(\hat{x}_{s-1})$; (ii) if $s > t$ and $x_{t-1} \notin X^\circ_{t-1}$, then $f^t_s(\hat{x}_{s-1}) = f^{t-1}_s(\hat{x}_{s-1})$; and (iii) if $s \le t$, then $f^t_s(\hat{x}_{s-1}) = f^{t-1}_s(\hat{x}_{s-1})$. Because X°_{t-1} is Borel measurable, the mappings f^t_s so-defined are Borel measurable for all $s \in T$. Later, we will use the fact that for all $\hat{x}_t \in \hat{X}_t$ such that $x_{t-1} \in X^\circ_{t-1}$,

$$\hat{U}_{t+1}(\hat{x}_t, f^t) = \Psi_{t+1}(x_t), \tag{9}$$

which follows from (ii), above.

We claim that for all $s \in T$ and all $\hat{x}_{s-1} \in \hat{X}_{s-1}$, the players' moves $f_s^t(\hat{x}_{s-1})$ are independent of public signals $\omega_1, \ldots, \omega_{t-1}$ in the first $t-1$ periods. Indeed, in case (i), above, this is implied by the HRR construction. In cases (ii) and (iii), f_s^t specifies the same actions as f_s^{t-1}, and (C1$_t$) implies that actions are independent of $\omega_1, \ldots, \omega_{t-2}$. Clearly, the claim also holds for ω_{t-1} if $s < t$, so consider $s \geq t$. If Nature's move $\varphi_{t-1}(x_{t-2})$ is atomless in period $t-1$, then the claim follows from (C2$_t$); and otherwise, the definition of game with atomless moves by Nature implies that Nature's move is atomless in period t, and thus case (i) applies. This establishes the claim, and for the remainder of Step 2, *we omit dependence of strategies and subgames on public signals in the first* $t-1$ *periods for notational simplicity.*

To verify that f^t is a SPE, note that the construction of HRR implies that $(f_{t+1}^t, f_{t+2}^t, \ldots)$ forms a SPE in each subgame $\hat{G}(x_{t-1})$ such that Nature's move is atomless in period t, i.e., $x_{t-1} \in X_{t-1}^\circ$. As well, for all $x_{t-1} \notin X_{t-1}^\circ$, play in the subgame $\hat{G}(x_{t-1})$ proceeds according to f^{t-1}, which is a SPE. For subgames starting at histories $x_{t-2} \in X_{t-2}$, we consider two cases. First, if Nature's move $\varphi_{t-1}(x_{t-2})$ in period $t-1$ is atomless, then in Step 1.1, equation (6) implies that for all $i \in N$ and all $y \in \times_{i \in N} A_{t,i}(x_{t-1})$,

$$
\begin{aligned}
&\int_z [u_{t,i}(x_{t-1}, y, z) + \delta_i \hat{U}_{t+1,i}\big((x_{t-1}, y, z), f^t\big)\varphi_t(x_{t-1})(dz) \\
&\quad = \int_z \left[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}\big((x_{t-1}, y, z, \omega), f^{t-1}\big)\lambda(d\omega) \right] \varphi_t(x_{t-1})(dz).
\end{aligned}
$$

Using $f_t^t(x_{t-1}) = f_t^{t-1}(x_{t-1})$, we can integrate both sides over action profiles y to obtain $\hat{U}_{t,i}(x_{t-1}, f^t) = \hat{U}_{t,i}(x_{t-1}, f^{t-1})$. This, in turn, implies that the players' payoff functions in the auxiliary games $\hat{G}_{t-1}(x_{t-2}, \hat{U}_t((x_{t-2}, \cdot), f^t))$ and $\hat{G}_{t-1}(x_{t-2}, \hat{U}_t((x_{t-2}, \cdot), f^{t-1}))$ are identical. Because f^t specifies the same mixtures over action as f^{t-1} at x_{t-2}, and f^{t-1} is a SPE, it follows that $f_{t-1}^t(x_{t-2}) = f_{t-1}^{t-1}(x_{t-2})$ is a mixed strategy equilibrium of the auxiliary game. Second, if Nature's move $\varphi_{t-1}(x_{t-2})$ in period $t-1$ has an atom, then in Step 1.2, equality of (6) and (7), together with the fact that the players' moves are pinned down in period t, again implies that $\hat{U}_{t,i}(x_{t-1}, f^t) = \hat{U}_{t,i}(x_{t-1}, f^{t-1})$, which implies that the auxiliary games are identical. Since f^{t-1} is a SPE, it follows that $f_{t-1}^t(x_{t-2}) = f_{t-1}^{t-1}(x_{t-2})$ is a mixed strategy equilibrium of the auxiliary game.

In general, for any period $s < t$, suppose that for all $x_s \in X_s$, we have $\hat{U}_{s+1,i}(x_s, f^t) = \hat{U}_{s+1,i}(x_s, f^{t-1})$. This implies that for all $x_{s-1} \in X_{s-1}$ and all $y \in \times_{i \in N} A_{s,i}(x_{s-1})$, we have

$$
\begin{aligned}
&\int_z [u_{s,i}(x_{s-1}, y, z) + \delta_i \hat{U}_{s+1,i}\big((x_{s-1}, y, z), f^t\big)\varphi_s(x_{s-1})(dz) \\
&\quad = \int_z \left[u_{s,i}(x_{s-1}, y, z) + \delta_i \int_\omega \hat{U}_{s+1,i}\big((x_{s-1}, y, z, \omega), f^{t-1}\big)\lambda(d\omega) \right] \varphi_s(x_{s-1})(dz).
\end{aligned}
$$

Using $f_s^t(x_{s-1}) = f_s^{t-1}(x_{s-1})$, we then integrate both sides over action profiles y to obtain $\hat{U}_{s,i}(x_{s-1}, f^t) = \hat{U}_{s,i}(x_{s-1}, f^{t-1})$. By induction, it follows that for all $s = 0, 1, \ldots, t-1$ and all $x_{s-1} \in X_{s-1}$, we have $\hat{U}_{s,i}(x_{s-1}, f^t) = \hat{U}_{s,i}(x_{s-1}, f^{t-1})$, implying coincidence of the auxiliary games $\hat{G}_s(x_{s-1}, \hat{U}_s((x_{s-1}, \cdot), f^t))$ and $\hat{G}_s(x_{s-1}, \hat{U}_s((x_{s-1}, \cdot), f^{t-1}))$, and since

f^{t-1} is a SPE, $f_s^t(x_{s-1}) = f_s^{t-1}(x_{s-1})$ is a mixed strategy equilibrium of the auxiliary game. We conclude that f^t is a SPE, as required.

Next, we verify that the strategy profile f^t specified in Step 2 satisfies conditions $(\mathrm{C1}_{t+1})$–$(\mathrm{C3}_{t+1})$.

Step 3: Completing the induction. In Step 2, we have already shown that for all $s \in T$ and all $\hat{x}_{s-1} \in \hat{X}_{s-1}$, the players' moves $f_s^t(\hat{x}_{s-1})$ are independent of public signals $\omega_1, \ldots, \omega_{t-1}$, verifying $(\mathrm{C1}_{t+1})$. Now consider any period $s \geq t+1$ and history $\hat{x}_{s-1} \in \hat{X}_{s-1}$ such that $x_{t-1} \in X_{t-1}^\circ$, so that Nature's move in period t, namely $\varphi_t(x_{t-1})$, is atomless. Then $f_s^t(\hat{x}_{s-1}) = \tilde{f}_s^t(\hat{x}_{s-1})$, and the latter is independent of $\omega_1, \ldots, \omega_t$, by the HRR construction, fulfilling $(\mathrm{C2}_{t+1})$. To verify payoff equivalence, consider any $\hat{x}_{t-1} \in \hat{X}_{t-2}$ and any $y \in \times_{i\in N} A_{t,i}(x_{t-1})$. If Nature's move has an atom in period t, i.e., $x_{t-1} \notin X_{t-1}^\circ$, then f^t coincides with f^{t-1} in the subgame $\hat{G}(\hat{x}_t)$, and thus f^t is payoff equivalent to f^{t-1} at $\hat{x}_{t-1}$. If Nature's move is atomless, i.e., $x_{t-1} \in X_{t-1}^\circ$, then

$$
\begin{aligned}
&\int_z \left[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}\big((x_{t-1}, y, z, \omega), f^t\big)\lambda(d\omega) \right] \varphi_t(x_{t-1})(dz) \\
&\quad = \int_z \big[u_{t,i}(x_{t-1}, y, z) + \delta_i \Psi_{t+1,i}(x_{t-1}, y, z) \big] \varphi_t(x_{t-1})(dz) \\
&\quad = \int_z \left[u_{t,i}(x_{t-1}, y, z) + \delta_i \int_\omega \hat{U}_{t+1,i}\big((x_{t-1}, y, z, \omega), f^{t-1}\big)\lambda(d\omega) \right] \varphi_t(x_{t-1})(dz),
\end{aligned}
$$

where the first equality follows from (9) and the second equality from (6). This yields $(\mathrm{C3}_{t+1})$, as required.

Finally, after Steps 1–3 have been performed countably many times, we construct a SPE of the original game with atomless moves by Nature, G, that is payoff-equivalent to $\hat{f}$.

Step 4: Construction of f. Let $f^\infty = (f_t^{t-1})_{t\in T}$ be the strategy profile in $\hat{G}$ such that play in each period t is determined by f^{t-1}. For each $t \in T$, consider any $\hat{x}_{t-1} \in \hat{X}_{t-1}$. If Nature's move is atomless in period $t-1$, i.e., $x_{t-2} \in X_{t-2}^\circ$, then $(\mathrm{C2}_t)$ implies that $f_t^\infty(\hat{x}_{t-1}) = f_t^{t-1}(\hat{x}_{t-1})$ is independent of $\omega_1, \ldots, \omega_{t-1}$. Otherwise, if Nature's move has an atom in period $t-1$, then $(\mathrm{C1}_t)$ implies that $f_t^\infty(\hat{x}_{t-1}) = f_t^{t-1}(\hat{x}_{t-1})$ is independent of $\omega_1, \ldots, \omega_{t-2}$, and the definition of game with atomless moves by Nature implies that the players' actions are pinned down by the history x_{t-1}, so that they are independent of ω_{t-1}, as well. Thus, we remove the nominal dependence of strategies on public signals by simply projecting each f_t^{t-1} onto X_{t-1}. Formally, for all $i \in N$, we define $f_{t,i}\colon X_{t-1} \to \Delta(Y_{t,i})$ as follows: for all $x_{t-1} \in X_{t-1}$, choose an arbitrary $\hat{x}_{t-1} \in \hat{X}_{t-1}$ such that the history of actions is x_{t-1}, and set $f_{t,i}(x_{t-1}) = f_{t,i}^\infty(\hat{x}_{t-1})$. Finally, we define $f_t = (f_{t,i})_{i\in N}$ and $f = (f_t)_{t\in T}$.

To see that f is a SPE of G, recall that for each $t \in T$, f_t^{t-1} is a SPE of $\hat{G}$. Thus, for each $\hat{x}_{t-1} \in \hat{X}_{t-1}$, the profile $f_t^{t-1}(\hat{x}_{t-1})$ is a mixed strategy equilibrium of the auxiliary game $\hat{G}_t(\hat{x}_{t-1}, \hat{U}_{t+1}((\hat{x}_{t-1}, \cdot), f^{t-1}))$. For each $s > t$, (C_{s+1}) implies that f^s is payoff equivalent to f^{s-1} at all $\hat{x}_{s-1} \in \hat{X}_{s-1}$. This means that for all $i \in N$ and all $y \in \times_{i\in N} A_{s,i}(x_{s-1})$, we have

$$
\int_z \left[u_{s,i}(x_{s-1}, y, z) + \delta_i \int_\omega \hat{U}_{s+1,i}\big((x_{s-1}, y, z, \omega), f^s\big)\lambda(d\omega) \right] \varphi_s(x_{s-1})(dz)
$$

$$= \int_z \left[u_{s,i}(x_{s-1}, y, z) + \delta_i \int_\omega \hat{U}_{s+1,i}\big((x_{s-1}, y, z, \omega), f^{s-1}\big)\lambda(d\omega) \right] \varphi_s(x_{s-1})(dz).$$

Arguing as in Step 2, we can integrate both sides by $f_s^s(x_{s-1}) = f_s^{s-1}(x_{s-1})$ to obtain $\hat{U}_s(x_{s-1}, f^s) = \hat{U}_s(x_{s-1}, f^{s-1})$. This, in turn, implies that f^s is payoff equivalent to f^{s-1} at each $\hat{x}_{s-2} \in \hat{X}_{s-2}$. By (C3$_s$), f^{s-1} is payoff equivalent to f^{s-2} at $\hat{x}_{s-2}$, and thus we similarly obtain

$$\hat{U}_{s-1}\big(x_{s-2}, f^s\big) = \hat{U}_{s-1}\big(x_{s-2}, f^{s-1}\big) = \hat{U}_{s-1}\big(x_{s-2}, f^{s-2}\big).$$

Continuing in this way, we conclude that $\hat{U}_t(x_{t-1}, f^s) = \hat{U}_t(x_{t-1}, f^t)$. Taking the limit as $s \to \infty$, continuity of payoffs implies that for all $x_{t-1} \in X_{t-1}$, we have $\hat{U}_t(x_{t-1}, f^\infty) = \hat{U}_t(x_{t-1}, f^t)$, which implies that f^∞ is payoff equivalent to f^{t-1} at $\hat{x}_{t-1}$. Therefore, $f_t^{t-1}(\hat{x}_{t-1})$ is, in fact, a mixed strategy equilibrium of the auxiliary game $\hat{G}_t(\hat{x}_{t-1}, \hat{U}_{t+1}((\hat{x}_{t-1}, \cdot), f^\infty))$. By the one-shot deviation principle, we conclude that f^∞ is a SPE of $\hat{G}$. Since the strategy profile f^∞ is independent of the payoff-irrelevant public signals, we conclude that f is a SPE of G. Finally, setting $t = 2$ in the above discussion, we obtain $\hat{U}_2(x_1, f^\infty) = \hat{U}_2(x_1, f^1)$, and then (C2$_2$) implies that f^∞ is payoff-equivalent to $\hat{f}$ at $\hat{x}_0$ and, therefore, f is payoff equivalent to $\hat{f}$.

6. Conclusions and variations

For the class of dynamic games with atomless moves by Nature, we have shown that any SPE obtained in the extended game with public randomization is payoff-equivalent to a SPE of the original game. This has several implications. First, HRR's public randomization device can be invoked without any loss of generality, as SPE payoffs are unaffected when we allow players to correlate their choices. Second, HRR's public randomization can be viewed as a step in a proof scheme to ensure existence of an SPE of the original game. Third, although we fix the initial history x_0 in the analysis, our decorrelation result allows us to invoke HRR's closed graph result for the extended game (Proposition 34, p. 537) to deduce a closed graph of SPE payoffs of the original game: if we vary the initial action so that $x_0^m \to x_0$, and if we select a corresponding convergent sequence of SPE payoffs, then HRR show that the limit of those payoffs is a SPE payoff of the extended game at x_0, and our de-correlation result implies that it is, in fact, a SPE payoff of the game at x_0.

In turn, this has implications for continuity of SPE payoffs when action sets vary in a continuous way, or when we approximate an infinite-horizon game by a sequence of finite-horizon games. It is known that SPE payoffs are not generally upper hemicontinuous in action sets, as correlation may be required in the limiting game to support the limit of SPE payoffs (Börgers (1991)), but this concession is unneeded in games with atomless moves by Nature. For example, in Remark 1 below we illustrate that if we impose a finite "grid" on the action sets of the players (possibly including Nature) and compute SPE payoffs of the game as the grid becomes fine, then the sequence of payoffs will approach a SPE payoff of the original game with continuous actions.

Remark 1. Let $G = (N_0, Y, Z, X, x_0, A, \varphi, u, \delta)$ be a game with atomless moves by Nature, and let $\{G^m\}$ be a sequence with $G^m = (N_0, Y, Z, X, x_0, A^m, \varphi, u, \delta)$ such that for all $t \in T$, all $i \in N_0$, and all $x_{t-1} \in X_{t-1}$, the sequence $\{A^m_{t,i}(x_{t-1})\}$ converges to $A_{t,i}(x_{t-1})$ in the Hausdorff metric. For each m, let p^m be a SPE payoff of $G^m(x_0)$, and assume $p^m \to p$. Then p is a SPE payoff of G at x_0.

The result follows from HRR's Proposition 34 by formulating a game of almost perfect information, $\overline{G}$, in their framework with the same primitives as G, but specifying their set of initial histories as a subset of the real line $\overline{X}_0 = \{\frac{1}{m} \mid m = 1, 2, \dots\} \cup \{0\}$ with the relative topology, and specifying their feasible action correspondences as follows: for all $t \in T$, all $i \in N_0$, and all $(t-1)$-period histories $\overline{x}_{t-1}$,

$$\overline{A}_{t,i}(\overline{x}_{t-1}) = \begin{cases} A^m_{t,i}(x_{t-1}) & \text{if } \overline{x}_0 = \dfrac{1}{m}, \\ A_{t,i}(x_{t-1}) & \text{if } \overline{x}_0 = 0, \end{cases}$$

where x_{t-1} is the history of our game that coincides with $\overline{x}_{t-1}$ in periods $1, 2, \dots, t-1$. By the assumption that feasible action sets converge in the Hausdorff metric, the feasible action correspondences in $\overline{G}$ satisfy the continuity assumption of HRR, and their proposition applies. Beyond merely technical interest, our remark can facilitate computation of SPE payoffs via finite approximation of games with atomless moves by Nature, without the need to invoke public randomization to support the limiting payoff.

Appendix: Comparison of payoff formulations

This Appendix compares the payoff formulation of HRR, which assumes payoffs defined directly on infinite histories, to the payoff formulation with geometric discounting, which we use in this paper. Given the prevalence of geometric discounting in applications, the next proposition, which establishes that the two payoff formulations are equivalent, may be of independent interest. The first part of the proposition, which states that discounting is a special case of the HRR formulation, is immediate; the contribution of the result is the converse.

Of note, the proof of the converse direction relies on the nontrivial step that for a given player i and any period t, we can choose a path of play starting from each history x_t such that player i's expected payoff is a continuous function of the starting point. This step could be replaced by a theorem of the maximum argument if moves by Nature were restricted to compact sets, but we do not assume the sets $A_{t,0}(x_{t-1})$ are compact.[26] Since Nature's moves are exogenously given, however, this noncompactness is not critical. The problem essentially boils down to continuity of the optimal value in a general, nonstationary Markov decision process (viewing the history x_t as a parameter of the problem). For this, we in fact formulate the decision problem as a two-player, zero-sum

[26]It would actually suffice if there were a compact $K_t \subset Z_t$ for each t such that $K_t \cap \operatorname{int} A_{t,0}(x_{t-1}) \neq \emptyset$ for all x_{t-1}. This assumption is quite weak, but we do not impose it.

game, and we invoke Proposition 39 of HRR on continuity of the equilibrium payoffs in this class, which packages the needed technical machinery in a simple way.[27]

Proposition A.1. *Let G by any game. Given any $\delta_i \in [0, 1)$ and any bounded, continuous mappings $u_{t,i}\colon X_t \to \mathbb{R}$, $t \in T$, if*

$$\sup_{x\in X_\infty}\left|\sum_{t\in T}\delta_i^{t-1}u_{t,i}(x_t)\right| < \infty, \tag{10}$$

and $\sum_{t\in T}\delta_i^{t-1}u_{t,i}(x_t)$ is continuous on X_∞ with the product topology, then there are bounded, continuous mappings $u_i\colon X_\infty \to \mathbb{R}$ and $\underline{u}_{0,i} : X_0 \to \mathbb{R}$ such that

$$u_i(x) = \underline{u}_{0,i}(x_0) + \sum_{t\in T}\delta_i^{t-1}u_{t,i}(x_t). \tag{11}$$

Conversely, for every bounded, continuous mapping $u_i\colon X_\infty \to \mathbb{R}$ and every $\delta_i \in (0, 1)$, there are bounded, continuous mappings $\underline{u}_{0,i} : X_0 \to \mathbb{R}$ and $u_{t,i}\colon X_t \to \mathbb{R}$, $t \in T$, satisfying (10) such that $\sum_{t\in T}\delta_i^{t-1}u_{t,i}(x_t)$ is continuous in X_∞ and (11) holds.

An implication of the proposition is that the two payoff formulations are equivalent, as (11) implies that we have

$$\int_x u_i(x)\,d\xi > \int_x u_i(x)\,d\xi' \quad\Leftrightarrow\quad \int_x\sum_{t\in T}\delta_i^{t-1}u_{t,i}(x_t)\,d\xi > \int_x\sum_{t\in T}\delta_i^{t-1}u_{t,i}(x_t)\,d\xi',$$

for all $\xi, \xi' \in \Delta(X_\infty)$. Another implication is that our formulation of the HRR framework, which allows for heterogeneous discount factors, can be further simplified to assume a common discount factor.[28]

The first part of the proposition is immediate, as we simply set $u_i(x) = \sum_{t\in T}\delta_i^{t-1}u_{t,i}(x_t)$ and $\underline{u}_{0,i}(x_0) \equiv 0$. To prove the converse, consider any bounded, continuous function $u_i\colon X_\infty \to \mathbb{R}$ and discount factor $\delta_i \in (0, 1)$. For this direction, we define an associated two-player, zero-sum game $\tilde{G}$ in the HRR framework. We assume two players, $\tilde{N} = \{1, 2\}$, in addition to Nature, where player 1's choices determine payoffs for the two players, and player 2 is passive. Intuitively, the environment is as in the original game G, but player 1 chooses an entire action profile in G after every history. Formally, player 1's feasible actions in $\tilde{G}$ in period t after history x_{t-1} are $\tilde{A}_{t,1}(x_{t-1}) = \times_{i\in N}A_{t,i}(x_{t-1})$, so that an action $\tilde{y}_1 = (y_1, \ldots, y_n)$ in $\tilde{G}$ specifies actions for all players in the original game G. Nature's moves are defined as in the original game.

[27]Note that Proposition 39 of HRR makes reference to Proposition 36 of HRR, which is known to contain a false claim, namely, that of convexity of the values of the continuation payoff correspondences—this is established by the counterexample of Luttmer and Mariotti (2003). But the reference to Proposition 36 used in Proposition 39 is only related to the forward step, which does not require such convexity, and thus the proof of Proposition 39 in HRR is unaffected by the counterexample.

[28]Note that (10) is strictly weaker than the requirement that $\sum_{t\in T}\delta_i^{t-1}\sup_{x_t\in X_t}|u_{t,i}(x_t)| < \infty$, which is sufficient, but not necessary, for the infinite sum $\sum_{t\in T}\delta_i^{t-1}u_{t,i}(x_t)$ to be well-defined, a condition we assume in our framework.

Finally, we specify payoffs as a function of infinite histories in $\tilde{G}$ as follows: $\tilde{u}_1(x) = u_i(x)$ and $\tilde{u}_2(x) = -u_i(x)$. This game is obviously zero-sum, with player 2 playing a trivial accounting role; the purpose is to permit the application of Proposition 39 of HRR, which neatly packages the needed continuity arguments. The latter establishes continuity of the value in zero-sum games: the function $\tilde{E}_1\colon X_0 \to \mathbb{R}$ giving the unique SPE payoff $\tilde{E}_1(x_0)$ of player 1 in $\tilde{G}$ is continuous in x_0. In the original game G, given any history x_t, note that $\tilde{E}_1(x_t)$ has the interpretation of being the highest possible payoff for player 1 following x_t, with the maximization being over the strategies $f = (f_1, \dots, f_n)$ used by the n players, with Nature's moves determined by φ.

Next, for all $t \in T$, define $\underline{u}_{t,i}\colon X_t \to \mathbb{R}$ by $\underline{u}_{t,i}(x_t) = \tilde{E}_1(x_t)$. We first claim that for all $x \in X_\infty$, we have $\lim_{t\to\infty} \underline{u}_{t,i}(x_t) = u_i(x)$. Indeed, for each t, there are infinite histories $\underline{x}^t, \overline{x}^t \in H_t(x_t)$ such that

$$u_i(\underline{x}^t) \le \underline{u}_{t,i}(x_t) \le u_i(\overline{x}^t), \tag{12}$$

and continuity of u_i yields

$$\lim_{t\to\infty} u_i(\underline{x}^t) = \lim_{t\to\infty} u_i(\overline{x}^t) = u_i(x).$$

Taking limits in (12), we obtain $\underline{u}_{t,i}(x_t) \to u_i(x)$, as claimed.

Now, define stage payoffs $u_{t,i}\colon X_t \to \mathbb{R}$ so that $u_{t,i}(x_t) = \delta_i^{1-t}(\underline{u}_{t,i}(x_t) - \underline{u}_{t-1,i}(x_{t-1}))$. Obviously, $u_{t,i}$ inherits boundedness of u_i, and continuity of $u_{t,i}$ follows from Proposition 39 of HRR. For (10), we next observe that for every infinite history $x \in X_\infty$, we have

$$\begin{aligned}\sum_{t\in T} \delta_i^{t-1} u_{t,i}(x_t) &= \lim_{t\to\infty} \sum_{s=1}^{t} \delta_i^{s-1} u_{s,i}(x_s) \\ &= \lim_{t\to\infty} \sum_{s=1}^{t} \delta_i^{s-1}\delta_i^{1-s}\big(u_{s,i}(x_s) - u_{s-1,i}(x_{s-1})\big) \\ &= \lim_{t\to\infty}\big(\underline{u}_{t,i}(x_t) - \underline{u}_{0,i}(x_0)\big) \\ &= u_i(x) - \underline{u}_{0,i}(x_0),\end{aligned}$$

where the second equality follows from specification of $u_{t,i}$, and the fourth follows from the above claim. Then (10) follows from boundedness of u_i. To verify continuity of the discounted sum, consider any sequence $\{x^m\}$ of infinite histories converging to $x \in X_\infty$, and note that

$$\begin{aligned}\lim_{m\to\infty} \sum_{t\in T} \delta_i^{t-1} u_{t,i}(x_t^m) &= \lim_{m\to\infty}\big(u_i(x^m) - \underline{u}_{0,i}(x_0)\big) \\ &= u_i(x) - \underline{u}_{0,i}(x_0) \\ &= \sum_{t\in T} \delta_i^{t-1} u_{t,i}(x_t),\end{aligned}$$

where the first and third equalities follow from the observation above, and the second follows from continuity of u_i. We conclude that the mappings $(u_{t,i})_{t \in T}$ and $\underline{u}_{0,i}$ satisfy the conditions of the proposition.

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