A Dominant Strategy, Double Clock Auction with Estimation-Based Tâtonnement*

Simon Loertscher[†] Claudio Mezzetti[‡]

October 16, 2020

Abstract

The price mechanism is fundamental to economics but difficult to reconcile with incentive compatibility and individual rationality. We introduce a double clock auction for a homogeneous good market with multi-dimensional private information and multi-unit traders that is deficit-free, ex post individually rational, constrained efficient, and makes sincere bidding a dominant strategy equilibrium. Under a weak dependence and an identifiability condition, our double clock auction is also asymptotically efficient. Asymptotic efficiency is achieved by estimating demand and supply using information from the bids of traders that have dropped out and following a tâtonnement process that adjusts the clock prices based on the estimates.

Keywords: Deficit free, dominant strategy mechanisms, double clock auctions, individual rationality, multi-dimensional types, privacy preservation, reserve prices, VCG mechanism.
JEL Classification: C72, D44, D47, D82.

*We thank the editor and two anonymous referees for comments that have helped us to improve the paper substantially. The paper has also benefited from comments by and discussions with Larry Ausubel, Fuhito Kojima, Anton Kolotilin, Leslie Marx, Ellen Muir, Marek Pycia, Ilya Segal, Utku Ünver, Takuro Yamashita, and Steve Williams. We also would like to thank audiences at NYU-Abu Dhabi Advances in Mechanism Design Conference 2016, SMU Singapore Mechanism Design Workshop 2016, Auctions and Market Design Workshop Vienna, 2016, ASSA 2015 in Boston, EEA 2014 in Toulouse, AMES 2013 in Singapore, ESAM Sydney 2013, AETW 2013 in Queensland and at the following universities: ANU, Auckland, Bonn, CUHK, Deakin, Duke/UNC, Essex, HKUST, Monash, Paris (Seminar of Game Theory), Queensland, UNSW, and UTS for valuable comments and feedback. This is a substantial revision superseding the previous version, which circulated under the title "A Multi-Unit Dominant Strategy Double Auction". Financial support from the Australian Research Council Discovery Project grants DP190102904 and DP200103574 and from the Samuel and June Hordern Endowment is also gratefully acknowledged. Bing Liu provided outstanding research assistance.

[†]Department of Economics, University of Melbourne, Level 4, FBE Building 433, 111 Barry Street, Parkville, Victoria 3010, Australia. Email: simonl@unimelb.edu.au.

[‡]School of Economics, University of Queensland, Level 6, Colin Clark Building 39, Brisbane St. Lucia, Queensland, 4072, Australia. Email: c.mezzetti@uq.edu.au.

1 Introduction

The study of price formation and market making with variable demand and supply and a focus on the efficient allocation of resources has a long tradition in economics. Walras (1874) proposed a procedure, called *tâtonnement*, in which buyers and sellers quote their demands and supplies at a given price to an auctioneer that increases the price if there is excess demand and decreases it if there is excess supply, with transactions only taking place when equilibrium is reached. One important problem with the Walrasian tâtonnement is that agents do not have an incentive to indicate truthfully their demand and supply schedules, as their bidding affects the final price.[1] In his landmark paper, Vickrey (1961) showed that it is possible to elicit the true demands and supplies and implement the efficient allocation, using a generalization of the static auction that bears his name. Observing that it runs a deficit and hence must be financed by an outside source, Vickrey was skeptical about the practical relevance of the market mechanism he proposed, calling it "inordinately expensive" for the market maker. Vickrey did not see an easy way to modify it so as to avoid the deficit, preserve the truth telling property and achieve an approximately efficient allocation, noting (Vickrey, 1961, p.13-14):

> It is tempting to try to modify this scheme in various ways that would reduce or eliminate this cost of operation while still preserving the tendency to optimum resource allocation. However, it seems that all modifications that do diminish the cost of the scheme either imply the use of some external information as to the true equilibrium price or reintroduce a direct incentive for misrepresentation of the marginal-cost and marginal-value curves. To be sure, in some cases the impairment of optimum allocation would be small relative to the reduction in cost, but, unfortunately, the analysis of such variations is extremely difficult; ...

In this paper, we propose a novel double clock auction that induces price taking behavior by all buyers and sellers at all times and hence elicits revelation of the true quantities demanded and supplied, without running a deficit. We do so for a general environment in which traders have multi-unit demands and supplies and multi-dimensional private information about their marginal values and costs. We view our double clock auction as a possible solution to the challenges identified by Vickrey. Under mild regularity conditions, we show that our double clock auction generates an outcome converging to the efficient allocation at rate $1/n$ as the number n of traders grows.

As emphasized by Ausubel (2004), two fundamental prescriptions for practical auction design derived from the auction literature are that the prices paid by an agent ought to be as

[1] The substantial impact on social welfare of strategic behavior in tâtonnement mechanisms was discussed by Babaioff et al. (2014). As they pointed out, tâtonnement mechanisms "are used, for example, in the daily opening of the New York Stock Exchange and the call market for copper and gold in London."

independent as possible from her own bids, and that the auction should be structured in an open, dynamic fashion, so as to convey clear price information to bidders and to preserve the privacy of the winners' valuations. Under the latter property, market participants are protected from hold up by the designer, because they do not reveal their willingness to pay on units they trade, and the designer is protected from the often substantial political and public risk of ex post regret – not knowing the agents' willingness to pay makes it difficult if not impossible to claim that there was "money left on the table."[2]

Our double clock auction (DCA) satisfies both prescriptions. It consists of a descending clock price for sellers and an ascending clock price for buyers. At every point in the DCA, traders indicate the number of units they are active, or bid, on, with activity meaning that this is the number of units they supply (demand) if they are sellers (buyers). There is a monotone activity rule that stipulates that in the course of the auction a trader can only *decrease* her activity. Once an agent's activity has dropped to zero, the agent is said to have dropped out (or exited). Based on information obtained only from agents who have exited, the DCA *estimates* supply and demand and, at any point in the process, sets target prices that are such that estimated excess demand is zero. If a given target price is reached without any additional exits, this target price becomes the reserve price. If an additional exit occurs before the target is reached, supply and demand are estimated anew, the target price is adjusted, and the DCA proceeds as before until the earliest of two points in time – both clock prices reach the target price, or an additional trader drops out.

Once both clock prices reach the target price, this price becomes the *reserve*, and the quantities supplied and demanded by all remaining *active* traders are used to determine whether buyers or seller are on the long side of the market at the reserve. If aggregate quantity demanded equals aggregate quantity supplied at the reserve, then all trades are executed at this price. Otherwise, agents on the long side participate in an Ausubel (2004) auction, starting at the reserve.

We show that sincere bidding by each agent is a dominant strategy equilibrium in the DCA. By construction, it never runs a deficit. It is ex post individually rational and satisfies *constrained efficiency*, that is, the units traded in the DCA are procured at minimum cost and allocated to buyers to maximize value. Constrained efficiency eliminates post-allocation gains from trade on each side of the market and thereby reduces scope for resale and, related, bid shading. Moreover, like privacy preservation, it eliminates political fall-out due to discrimination that arises when, say, a buyer who submitted a lower bid for a unit is served while a

[2]The practice of mechanism design and historical experience with auctions offer plenty of examples of such public outcry. The 1990 spectrum license auction in New Zealand is one famous example of political risk due to ex post regret (see, for example, McMillan, 1994, or Milgrom, 2004). That static, sealed bid, mechanisms are prone to the bidders' hold-up problem was known by stamp collectors before the middle of the 20th century (Lucking-Reiley, 2000).

buyer with a higher value is not served. We also provide conditions under which the DCA is asymptotically efficient. Asymptotic efficiency obtains, for example, in the *order statistics model* (Burdett and Woodward, 2020), according to which each buyer (seller) draws a number of values (costs) independently from the same distribution equal to its maximum demand (capacity).

Our paper relates to the literature on dominant strategy mechanisms in the tradition of Vickrey (1961), Clarke (1971) and Groves (1973). There are particularly close connections to papers that develop deficit-free dominant strategy mechanisms such as Hagerty and Rogerson (1987) and McAfee (1992). We provide a detailed discussion of these in Section 4, after we have formally introduced our double clock auction and derived its key properties. The paper, and the double clock auction we design, draws inspiration from the extended body of research that has emphasized advantages of clock auctions in a one-sided environment, such as Ausubel (2004, 2006), Ausubel et al. (2006), Perry and Reny (2005), Bergemann and Morris (2007), Levin and Skrzypacz (2016), Li (2017), Sun and Yang (2009, 2014), and Milgrom and Segal (2020).[3] Perry and Reny (2005, p.568), for example, argue that "simultaneous auction formats tend to treat information as if it were costless to collect and costless to provide" while dynamic auctions economize on the information collected.

The paper also relates to the recent and growing literature on mechanism design with estimation initiated by Baliga and Vohra (2003) and Segal (2003).[4] In that literature, the designer's objective is profit-maximization, and hence the objects to be estimated are hazard rates and virtual types. In contrast, our market maker's objective is social surplus, without running a deficit, and so the object to be estimated is the Walrasian price. An alternative route to estimating Walrasian prices is taken by Kojima and Yamashita (2017) in a paper that assumes interdependent values and was developed in parallel to the present one. Like ours, Kojima and Yamashita (2017) use an Ausubel auction on the long side. Their mechanism randomly splits traders into different submarkets and uses reports from all other submarkets to estimate the market clearing price in a given submarket; it assumes single dimensional types, thereby escaping from the impossibility results that plague ex post implementation (see, e.g., Jehiel et al., 2006) when two-stage mechanisms as in Mezzetti (2004) are not allowed. Because traders are randomly split in mechanisms in the tradition of Baliga and Vohra (2003), these mechanisms are not constrained efficient and, with single-unit traders, fail to be envy-free. Moreover, none of the papers above has two-sided clock auctions with multi-unit demands and multi-dimensional information.

[3]Ausubel's 2004 proposed a clock implementation of the VCG mechanism for the case of homogeneous goods. For subsequent generalizations to the case of heterogeneous objects, see Ausubel (2006), and Sun and Yang (2009, 2014).

[4]See also Loertscher and Marx (2020), who develop a prior-free clock auction that is asymptotically profit-maximizing in an environment with single-unit traders and independently distributed types.

Of course, the very idea of a tâtonnement process to discover market clearing prices dates back to Walras (1874), and so our paper is also tightly connected to the literature on the decentralized micro-foundations of competitive equilibrium, such as Satterthwaite and Williams (1989, 2002), Rustichini et al. (1994), and Cripps and Swinkels (2006) as well as to Reny and Perry (2006), who study the related question of the foundations of rational expectation equilibrium.[5] Our double clock auction can be viewed as providing a centralized micro-foundation in which the "Walrasian" auctioneer does the heavy lifting while endowing agents with dominant strategies. Rather than getting rid of the Walrasian auctioneer, it fills her role with substance.

The remainder of the paper is organized as follows. Section 2 provides the setup. In Section 3, we introduce the DCA and derive its key properties. Section 4 provides a comparison of different mechanisms in the small and a discussion of the most closely related literature. Section 5 introduces conditions under which the double clock auction is asymptotically efficient, and Section 6 concludes the paper.

2 The Setup

There is a set $\mathcal{N} = \{1, ..., N\}$ of buyers, and a set $\mathcal{M} = \{1, ..., M\}$ of sellers of a homogeneous good. In Section 5, to study convergence to efficiency, we will proportionally expand the sets of buyers and sellers to $\mathcal{N} = \{1, ..., nN\}$ and $\mathcal{M} = \{1, ..., nM\}$ and we will let n go to infinity.

Denote by $\boldsymbol{v}^b = \left(v_1^b, ..., v_{k_B}^b\right)$ the valuation, or type, of buyer $b \in \mathcal{N}$, where $v_k^b \in [0, 1]$ is buyer b's marginal value for the k-th unit of the good and k_B is an upper bound on each buyer's demand. Denote by $\boldsymbol{c}^s = \left(c_1^s, ..., c_{k_S}^s\right)$ the cost, or type, of seller $s \in \mathcal{M}$, where $c_k^s \in [0, 1]$ is seller s's cost for producing, or giving up the use of, the k-th unit and k_S is an upper bound on each seller's capacity.[6] Let $\boldsymbol{v} = \left(\boldsymbol{v}^1, ..., \boldsymbol{v}^N\right) = \left(\boldsymbol{v}^b, \boldsymbol{v}^{-b}\right)$ be the profile of valuations, $\boldsymbol{c} = \left(\boldsymbol{c}^1, ..., \boldsymbol{c}^M\right) = (\boldsymbol{c}^s, \boldsymbol{c}^{-s})$ be the profile of costs, and $\boldsymbol{\theta} = (\boldsymbol{v}, \boldsymbol{c}) = (\boldsymbol{v}^b, \boldsymbol{\theta}^{-b}) = (\boldsymbol{c}^s, \boldsymbol{\theta}^{-s})$. We assume diminishing marginal values and increasing marginal costs; that is, for all $b \in \mathcal{N}$, all $k \in \{1, .., k_B - 1\}$, we have $v_k^b \geqslant v_{k+1}^b$ and, for all $s \in \mathcal{M}$, all $k \in \{1, .., k_S - 1\}$, we have $c_k^s \leqslant c_{k+1}^s$. A buyer b receiving q goods at unit prices $p_1^b, ..., p_q^b$ obtains payoff $\sum_{k=1}^q \left(v_k^b - p_k^b\right)$; a buyer receiving no units and making no payments has zero payoff. Similarly, a seller s selling q goods at prices $p_1^s, ..., p_q^s$ obtains payoff $\sum_{k=1}^q (p_k^s - c_k^s)$; a seller receiving no payments and selling no units has zero payoff. The payoff functions and the upper bounds on traders' capacities are common knowledge, but marginal values and marginal costs are private information of each trader.[7]

[5] Other papers on the convergence to competitive equilibrium in the single-unit case include Gresik and Satterthwaite (1989) who looked at optimal trading mechanisms, Yoon (2001) who studied a double auction with participation fees and Tatur (2005), who introduced a double auction with a fixed fee. For the multi-unit case, Yoon (2008) introduced the participatory Vickrey-Clarke-Groves mechanism.

[6] The assumption that values and costs are in $[0, 1]$ is just a normalization.

[7] Our results remain valid when traders have complete information about all marginal values and costs. This

The mechanism we propose has an open bid, clock format. As ours is a setting with active buyers and sellers (as opposed to a one-sided auction), the mechanism is a *double clock auction*; that is, it will be run with an ascending clock on the buyers' side and a descending clock on the sellers' side. This implies that the mechanism is privacy preserving; that is, it does not reveal the marginal values or marginal costs of the units that are traded.[8] Our mechanism is robust in the sense of Bergemann and Morris (2005), because it satisfies *dominant strategy* incentive compatibility, so that agents do not need well specified beliefs about the other agents' types in order to bid optimally.

Denote the individualized price vector of agent i by $\boldsymbol{p}^i(\boldsymbol{\theta}^{-i}) = \left(p_0^i(\boldsymbol{\theta}^{-i}), ..., p_{k_i}^i(\boldsymbol{\theta}^{-i})\right)$, where $p_k^i\left(\boldsymbol{\theta}^{-i}\right)$ is the price buyer i must pay (seller i is paid) for the k-th unit of the good.[9] Using the convention $v_0^b = c_0^s = 0$ for all b and s, let the quantities traded by each buyer $b \in \mathcal{N}$ and seller $s \in \mathcal{M}$ at their personalized prices be:

$$q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}), \boldsymbol{v}^b) = \underset{0 \leqslant q \leqslant k_B}{\arg\max} \sum_{k=0}^{q} \left(v_k^b - p_k^b(\boldsymbol{\theta}^{-b})\right) \quad \text{and}$$

$$q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}), \boldsymbol{c}^s) = \underset{0 \leqslant q \leqslant k_S}{\arg\max} \sum_{k=0}^{q} \left(p_k^s(\boldsymbol{\theta}^{-s}) - c_k^s\right)$$

Let $q_B(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}), \boldsymbol{v}^b)$ be the total quantity acquired by buyers and $q_S(\boldsymbol{\theta}) = \sum_{s \in \mathcal{M}} q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}), \boldsymbol{c}^s)$ be the total quantity sold by sellers.

A mechanism is *feasible* if for every $\boldsymbol{\theta}$, $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$.[10]

Given that the outside option has zero value for every agent, a mechanism satisfies *ex post individual rationality* if for all b, $\boldsymbol{\theta} = \left(\boldsymbol{v}^b, \boldsymbol{\theta}^{-b}\right)$ and for all s, $\boldsymbol{\theta} = (\boldsymbol{c}^s, \boldsymbol{\theta}^{-s})$:

$$p_0^b(\boldsymbol{\theta}^{-b}) \leqslant 0\,; \qquad p_0^s(\boldsymbol{\theta}^{-s}) \geqslant 0\,.$$

The profit a mechanism generates at $\boldsymbol{\theta}$ is:

$$\Pi(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} \sum_{q^b=0}^{q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}))} p_{q^b}^b(\boldsymbol{\theta}^{-b}) - \sum_{s \in \mathcal{M}} \sum_{q^s=0}^{q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}))} p_{q^s}^s(\boldsymbol{\theta}^{-s})\,;$$

a mechanism is *deficit free* if for all $\boldsymbol{\theta}$, $\Pi(\boldsymbol{\theta}) \geqslant 0$.

The performance of any allocation mechanism that targets welfare maximization must be evaluated in term of its efficiency level. In our setting, full ex post efficiency, which implies

is because in the DCA we introduce traders have the dominant strategy of bidding sincerely.

[8] See Engelbrecht-Wiggans and Kahn (1991), Naor et al. (1999), Ausubel (2004) and Milgrom and Segal (2020) for discussions of the importance of this requirement.

[9] By the taxation and revelation principles (see Rochet, 1985, and Myerson, 1979), any dominant strategy mechanism is strategically equivalent to a "direct" price mechanism that sets an individualized marginal price vector for each agent as a function of the other agents' types and lets each agent decide how many units to trade at the specified prices.

[10] If there was free disposal, we could weaken the feasibility condition to $q_B(\boldsymbol{\theta}) \leqslant q_S(\boldsymbol{\theta})$, but this would not help in any substantial way in the design of our DCA.

feasibility, requires that for all possible type profiles the buyers with the highest marginal valuations trade with the sellers with the lowest marginal costs and that the total quantity traded is $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta}) = q_{CE}(\boldsymbol{\theta})$, where $q_{CE}(\boldsymbol{\theta})$ is a Walrasian (competitive equilibrium) quantity associated with $\boldsymbol{\theta}$:[11]

$$\max\left\{q \in \{0, ..., K\} : v_{(q)} > c_{[q]}\right\} \leqslant q_{CE}(\boldsymbol{\theta}) \leqslant \max\left\{q \in \{0, ..., K\} : v_{(q)} \geqslant c_{[q]}\right\}.$$

In our setting, dominant strategy incentive compatibility and ex post efficiency are satisfied if and only if the mechanism is a Groves mechanism (e.g., see Holmström, 1979) and ex post individual rationality and deficit minimization further restrict the mechanism to be a VCG mechanism. The VCG mechanism is not deficit free. Indeed, in Loertscher and Mezzetti (2019), we have shown that in the setting of a market for a homogeneous good the two-sided VCG auction runs a deficit on each trade and the total deficit does not vanish as the number of traders grows large. While it is not possible to construct a mechanism that is ex post efficient and deficit free, efficiency is an important feature of an allocation mechanism. Thus, we require our double clock auction to satisfy two efficiency properties, constrained efficiency and asymptotic efficiency.

A mechanism is *constrained efficient* if, given the total quantity traded $q(\boldsymbol{\theta}) = q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$, the trades completed are the most valuable ones – those associated with the $q(\boldsymbol{\theta})$-th highest marginal values and the $q(\boldsymbol{\theta})$-th lowest marginal costs. Constrained efficiency is an appealing property of the price mechanism in competitive and oligopolistic markets.

The total welfare at $\boldsymbol{\theta}$ generated by a mechanism is given by the gains of trade:

$$W(\boldsymbol{\theta}) = \sum_{b\in\mathcal{N}} \sum_{q^b=0}^{q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}))} v^b_{q^b}(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b})) - \sum_{s\in\mathcal{M}} \sum_{q^s=0}^{q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}))} c^s_{q^s}(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s})).$$

Let $q^b_{CE}(\boldsymbol{\theta})$ and $q^s_{CE}(\boldsymbol{\theta})$ be the quantity traded by buyer b and seller s in a Walrasian equilibrium. Under a fully efficient allocation, total welfare at $\boldsymbol{\theta}$ is:

$$W_{CE}(\boldsymbol{\theta}) = \sum_{b\in\mathcal{N}} \sum_{q^b=0}^{q^b_{CE}(\boldsymbol{\theta})} v^b_{q^b}(\boldsymbol{\theta}) - \sum_{s\in\mathcal{M}} \sum_{q^s=0}^{q^s_{CE}(\boldsymbol{\theta})} c^s_{q^s}(\boldsymbol{\theta}),$$

Thus, the percentage welfare loss at $\boldsymbol{\theta}$ is $\mathcal{L}(\boldsymbol{\theta}) = 1 - \frac{W(\boldsymbol{\theta})}{W_{CE}(\boldsymbol{\theta})}$. Let $\mathbb{P}_{\phi*}$ be the probability measure determining the true marginal values and costs (i.e., $\boldsymbol{\theta}$) and $\mathbb{E}_{\phi*}$ be the expectation operator with respect to $\mathbb{P}_{\phi*}$.[12] For $\rho > 0$, we say that a mechanism is *asymptotically efficient*

[11] Given a vector $\boldsymbol{x}$, we denote by $x_{(i)}$ its i-th highest element and by $x_{[i]}$ its i-th lowest element. Thus, $x_{(q)} = x_{[m+1-q]}$ if the vector contains m elements. We also adopt the notational convention that $v_{(0)} = 1$ and $c_{[0]} = 0$, which implies that $q_{CE}(\boldsymbol{\theta})$ is well defined.

[12] The true probability distribution is not known by the auctioneer or by the traders. There is a set Φ indexing the possible probability measures $\mathbb{P}_\phi$, with $\phi \in \Phi$. See Section 5 for details.

at rate $1/n^\rho$ if the expected percentage welfare loss converges to zero at rate $1/n^\rho$ as the size of the market n goes to infinity; that is, if there is a constant $L > 0$ such that for all n: $\mathbb{E}_{\phi_*}[\mathcal{L}(\boldsymbol{\theta})] \leqslant L/n^\rho$. Our double clock auction will be constrained efficient and asymptotically efficient at rate $1/n$.

3 The Dominant Strategy Double Clock Auction

There are two clock prices in our DCA, one for buyers and one for sellers. At the start, the prices for buyers and sellers are $p_0^B = 0$, $p_0^S = 1$. The buyers' clock price increases or stays constant and the sellers' clock price decreases or stays constant. Each buyer starts the DCA with a quantity demanded equal to k_B and each seller starts with a quantity supplied equal to k_S. Buyers can only take an action when their clock price increases and sellers can only take an action when their clock price decreases; they may reduce their quantity demanded or supplied by any non-negative integer.

The DCA is composed of two phases. Phase 1 permits estimated demand and supply to adjust using the revealed marginal values of the traders that drop out – that is, reduce their demand or supply to zero – as the buyers' clock price increases and the sellers' clock price decreases. The estimation procedure (to be explained below) determines which of three possible states prevails at each point during Phase 1: a buyers' clock state (*state BC*), in which only the buyers' clock price changes; a sellers' clock state (*state SC*), in which only the sellers' clock price changes; a double clock state (*state DC*), in which both clock prices change. Phase 1 ends when the two clocks reach a common price p_t^T.

Before describing the precise mechanics of the price adjustment that takes place in Phase 1, which is done in Subsection 3.1, we now complete the specification of how the allocation is determined in Phase 2.

Phase 2 of the DCA begins by setting the reserve price r. Then it computes the revealed aggregate demand and supply at the reserve price r of the buyers and sellers who have not dropped out and determines which side is the long side – the buyers' side if demand exceeds supply, the seller's side if supply exceeds demand at r. The DCA allocates to all traders on the short side the units they demand or supply and charges the reserve price r for each such unit. Let $q(r)$ be the total quantity demanded or supplied at r on the short side. To ration units on the long side, the DCA runs an Ausubel auction for $q(r)$ units with a clock price starting at r.[13] The Ausubel auction ends when the available $q(r)$ units are assigned to traders on the

[13] An Ausubel auction is a clock version of a Vickrey auction in which each trader pays or is paid the Vickrey price on each unit it buys or sells. If it is run for sellers as sellers are on the long side, then the clock price decreases starting from r and sellers decide if and when to reduce their supply. A seller s is paid the current clock price each time she clinches an additional unit, which happens when the residual supply of the other sellers has decreased below the number of demanded units $q(r)$ minus the units already clinched by s. If it is run for buyers, the Ausubel auction works similarly, with a clock price increasing starting from the reserve price r.

long side.

3.1 Description of Phase 1 of the DCA

Let $\mathcal{N}_{\mathcal{O}}(p^B)$ be the set of buyers whose quantity demanded is zero when the buyers' clock price reaches p^B and $\mathcal{M}_{\mathcal{O}}(p^S)$ be the set of sellers whose quantity supplied is zero when the sellers' clock price reaches p^S. These two sets are the traders who have irrevocably dropped out of the DCA; these traders cannot re-enter and will trade zero units. The only active traders after the clock prices have reached p^B and p^S are the buyers in the set $\mathcal{N}_{\mathcal{A}}(p^B) = \mathcal{N}\backslash\mathcal{N}_{\mathcal{O}}(p^B)$ and the sellers in the set $\mathcal{M}_{\mathcal{A}}(p^S) = \mathcal{M}\backslash\mathcal{M}_{\mathcal{O}}(p^S)$.

In Phase 1 of the DCA, estimation takes place in discrete rounds. A new estimation round is entered when a trader drops out of the DCA and the sellers' clock price is higher than the buyers' clock price; let $p_t^B < p_t^S$ be the clock prices for buyers and sellers in round t. In estimation round 0, at the start of the DCA, the auctioneer has prior estimates of the aggregate demand and supply functions. In any other round $t > 0$ the auctioneer estimates the aggregate demand and supply functions using the marginal values of the traders that have already dropped out (i.e., the traders in $\mathcal{N}_{\mathcal{O}}(p_t^B)$ and $\mathcal{M}_{\mathcal{O}}(p_t^S)$), as revealed from the history of their demand and supply reductions.[14] Let $\mathbb{E}\left[D^{\mathcal{N}}(p^B)\right]$ be estimated demand at price p^B and $\mathbb{E}\left[S^{\mathcal{M}}(p^S)\right]$ be estimated supply at price p^S. While the only public information available to traders are the state and current clock prices, if traders reduce demand and supply when their prices reach their marginal values, this history reveals to the auctioneer all the values and costs of the traders who have become inactive.

Minimum distance estimation will be used to prove the asymptotic efficiency of the DCA; it will be explained in detail in Section 5.[15] However, the precise estimation approach used in an estimation round is not important for the purpose of establishing the main result of this section, Theorem 1. Many other methods (e.g., Bayesian estimation, maximum likelihood, OLS and also ad hoc interpolation methods) would work equally well.[16]

In Phase 1, following each trader dropout, *estimation* rounds determine whether there is estimated excess demand or supply, lead to a transition of the DCA to one of three states (either a buyers' clock state, a sellers' clock state, or a double clock state) and set the target prices in such states. The aim is to balance estimated demand and supply subject to the monotonicity

[14] Although our mechanism is different, the idea of using only information from losing bidders is not novel, as it is the basis for price formation in a single-unit English auction and its strategic equivalence with the second-price auction. Brooks (2013) also exploits this idea.

[15] The index that minimizes the distance between the demand and supply functions associated to that index and the empirical demand and supply functions obtained from the traders that have dropped out of the DCA is chosen from a set of indexes distinguishing different random processes generating the traders' valuations. The chosen index is then used to infer the estimated demand and supply functions.

[16] Indeed, we will use a combination of OLS and simple interpolation to perform estimation in the order statistics model that we discuss in Section 4.

constraints imposed by clock auctions. We next describe Phase 1 algorithmically.

Estimation round t.

- If estimated demand at p_t^B exceeds estimated supply at p_t^S (i.e., $\mathbb{E}\left[D^{\mathcal{N}}(p_t^B)\right] > \mathbb{E}\left[S^{\mathcal{M}}(p_t^S)\right]$), then Phase 1 of the DCA transitions into a buyers' clock state (state BC) and:
  - The auctioneer sets the buyers' target price, $p_{t+1}^{TB} = \min\{p_t^{EB}, p_t^S\}$; where p_t^{EB} is the price at which estimated demand equals estimated supply at p_t^S (i.e., $\mathbb{E}\left[D^{\mathcal{N}}(p_t^{EB})\right] = \mathbb{E}\left[S^{\mathcal{M}}(p_t^S)\right]$).[17] (The target price serves as an upper bound on the adjustment of the buyers' clock price during the BC state.)
- If estimated supply at p_t^S exceeds estimated demand at p_t^B (i.e., $\mathbb{E}\left[D^{\mathcal{N}}(p_t^B)\right] < \mathbb{E}\left[S^{\mathcal{M}}(p_t^S)\right]$), then Phase 1 of the DCA transitions into a sellers' clock state (state SC) and:
  - The auctioneer sets the sellers' target price, $p_{t+1}^{TS} = \max\{p_t^{ES}, p_t^B\}$; where p_t^{ES} is the price at which estimated supply equals estimated demand at p_t^B (i.e., $\mathbb{E}\left[D^{\mathcal{N}}(p_t^B)\right] = \mathbb{E}\left[S^{\mathcal{M}}(p_t^{ES})\right]$). (The target price serves as a lower bound on the adjustment of the sellers' clock price during the SC state.)
- If estimated demand at p_t^B equals estimated supply at p_t^S (i.e., $\mathbb{E}\left[D^{\mathcal{N}}(p_t^B)\right] = \mathbb{E}\left[S^{\mathcal{M}}(p_t^S)\right]$), then Phase 1 of the DCA transitions into a double clock state (state DC) and:
  - The auctioneer sets as the target price for both buyers and sellers in state DC the estimated market clearing price p_{t+1}^T at which estimated supply equals estimated demand, $\mathbb{E}\left[D^{\mathcal{N}}(p_{t+1}^T)\right] = \mathbb{E}\left[S^{\mathcal{M}}(p_{t+1}^T)\right]$.

From an estimation round t, the DCA transitions to one of the following three states:

State Bc: Buyers' clock state.

- The buyers' clock price p^B increases continuously starting from p_t^B.
- At any price p^B, each active buyer $b \in \mathcal{N}_{\mathcal{A}}(p^B)$ decides whether to reduce her demand.
  - If the demand of an active buyer becomes zero at price $p^B \leqslant p_{t+1}^{TB}$, then the auctioneer sets $p_{t+1}^B = p^B$ and $p_{t+1}^S = p_t^S$ and the DCA goes to estimation round $t+1$.
  - If the clock price p^B reaches the target price $p_{t+1}^{TB} < p_t^S$ with no buyer dropping out, then the DCA transitions into state DC with target price p_{t+1}^T, the price at which estimated demand equals estimated supply, $\mathbb{E}\left[D^{\mathcal{N}}(p_{t+1}^T)\right] = \mathbb{E}\left[S^{\mathcal{M}}(p_{t+1}^T)\right]$.

[17] The target price for buyers is the buyers' clock price at which estimated demand equals estimated supply evaluated at the current sellers' clock price, unless such a price is higher than the current sellers' clock price, in which case the latter becomes the buyers' target price.

- If the clock price p^B reaches the target price $p^{TB}_{t+1} = p^S_t$ with no buyer dropping out, then the auctioneer sets $r = p^{TB}_{t+1}$ as the reserve price and the DCA transition into Phase 2.

State Sc: Sellers' clock state.

- The sellers' clock price p^S decreases continuously starting from p^S_t.
- At any price p^S, each active seller $s \in \mathcal{M}_{\mathcal{A}}(p^S)$ decides whether to reduce her supply.
  - If the supply of an active seller becomes zero at price $p^S \geqslant p^{TS}_{t+1}$, then the auctioneer sets $p^S_{t+1} = p^S$ and $p^B_{t+1} = p^B_t$ and the DCA goes to estimation round $t+1$.
  - If the clock price p^S reaches the target price $p^{TS}_{t+1} > p^B_t$ with no seller dropping out, then the DCA transitions into state DC with target price p^T_{t+1}, the price at which estimated demand equals estimated supply, $\mathbb{E}\left[D^{\mathcal{N}}(p^T_{t+1})\right] = \mathbb{E}\left[S^{\mathcal{M}}(p^T_{t+1})\right]$.
  - If the clock price p^S reaches the target price $p^{TS}_{t+1} = p^B_t$ with no seller dropping out, then the auctioneer sets $r = p^{TS}_{t+1}$ as the reserve price and the DCA transitions into Phase 2.

State Dc: Double clock state.

- The buyers' clock price p^B increases continuously starting from p^B_t and the sellers' clock price p^S decreases continuously starting from p^S_t in such a way that equality of estimated demand and supply is maintained at all points in time (i.e., at all p^B, p^S it is $\mathbb{E}\left[D^{\mathcal{N}}(p^B)\right] = \mathbb{E}\left[S^{\mathcal{M}}(p^S)\right]$), so that, if no trader drops out, p^B and p^S reach the target price p^T_{t+1} simultaneously.
- At any price p^B, each active buyer decides whether to reduce her demand; at any price p^S, each active seller decides whether to reduce her supply.
  - If the demand of one of the active buyers or the supply of one of the active sellers becomes zero at prices $p^B < p^T_{t+1}$, $p^S > p^T_{t+1}$, then the auctioneer sets $p^B_{t+1} = p^B$, $p^S_{t+1} = p^S$ and the DCA goes to estimation round $t+1$ (since estimation is triggered whenever a trader drops out).
  - If prices p^B and p^S reach the target price p^T_{t+1}, then the auctioneer sets $r = p^T_{t+1}$ as the reserve price and the DCA transitions into Phase 2.

3.2 Properties of the DCA

At all points in the DCA the only information available to the active traders are the phase, state and current clock prices. We say that an agent engages in *sincere bidding* if she expresses her quantity demanded or supplied truthfully. That is, buyer b bids sincerely if for any buyers' clock price p^B her demand is q^b such that $v^b_{q^b} \geqslant p^B \geqslant v^b_{q^b+1}$ and seller s bids sincerely if for any sellers' clock price p^S her supply q^s is such that $c^s_{q^s} \leqslant p^S \leqslant c^s_{q^s+1}$.[18]

Theorem 1 shows that the DCA is deficit free and constrained efficient – that is, conditional on the volume of trade, the trades realised are the ones yielding the highest total welfare. Two points are worth making. First, the DCA typically makes a positive revenue, as buyers pay at least the reserve price r for each unit they acquire and sellers are paid at most r on each unit they sell. Second, the DCA is not fully efficient, some profitable trades may be missed because the reserve price r is set so as to equate estimated aggregate supply and demand, which may be different from the true realized supply and demand.

Theorem 1. *Sincere bidding by each agent is a dominant strategy equilibrium in the DCA. The DCA is also feasible, deficit free, ex post individually rational and constrained efficient.*

Proof. By construction, the DCA is feasible as the quantity traded is determined by the short side of the market at the reserve price, and it is deficit free since the minimum price paid by buyers (the reserve price r) is equal to the maximum price paid to sellers (also the reserve price r). Ex post individual rationality holds since each trader may guarantee herself the outside option payoff by dropping out of the bidding. Constrained efficiency holds because, under sincere bidding, for any given quantity to be traded q, the trades that are completed are those associated with the q highest marginal values and the q lowest marginal costs.

Because of the symmetry of buyers and sellers, to save space we will just argue that sincere bidding is a dominant strategy for each buyer b.

<u>Case 1.</u> The first case arises if by bidding sincerely buyer b ends up dropping out and not buying any unit. In such a case, the reserve price r is at least as high as her marginal valuation for the first item. No alternative strategy could increase buyer b's payoff, as if she did not drop out and instead stayed longer in the DCA, so as to acquire at least one unit, then the reserve price could only be higher and she would end up acquiring units at a price above their marginal value.

<u>Case 2.</u> The second case arises when by bidding sincerely buyer b acquires at least one unit. In such a case the reserve price r does not depend on any of buyer b's marginal values and the buyer makes a non-negative payoff on any unit she acquires. This implies the following first conclusion: (i) no strategy that leads buyer b to drop out before price r is reached and thus acquire zero units is a profitable deviation from sincere bidding.

[18] There are two reasons why no bid information about the other agents is revealed to a trader. First, it makes the bidding environment straightforward; much like in the Walrasian analysis of competitive markets, all information that an agent has is the price she faces. Second, as show in Theorem 1, it makes sincere bidding by all agents a dominant strategy equilibrium. As in Ausubel (2004), if we allowed either full or aggregate bid information, then sincere bidding would be an ex post perfect equilibrium.

Now recall that the DCA uses the revealed demands and supplies of the active traders at the reserve price r to determine which side of the market is the short side – buyers if supply exceeds demands, sellers if demands exceeds supply. Suppose first that buyers are on the short side. Then buyer b pays the reserve price r on all the units she acquires and by bidding sincerely she acquires all units with marginal values above r. There are two possible kinds of deviations from sincere bidding under which buyer b continues to acquire some units. The first type leads to b remaining on the short side of the market. Such a deviation cannot be profitable, because either b ends up acquiring extra units that she values less than the reserve price r or she gives up acquiring units that she values above r. The second type of deviation requires buyer b to increase her demand to the point that buyers end up on the long side of the market. As such a deviation does not change the reserve price r, it cannot be profitable, as it leads b possibly to acquire units valued less than the reserve price r and paying at least r on all units. Thus we have shown that: (ii) there is no profitable deviation from sincere bidding for buyer b when buyers are on the short side of the market.

Now suppose buyers are on the long side of the market. Again, there are two possible kinds of deviations from sincere bidding under which buyer b continues to acquire some units. The first type leads to b remaining on the long side of the market. To see that such a deviation cannot be profitable, recall that on the long side of the market units are allocated using an Ausubel auction. If there were a profitable deviation from sincere bidding, it would imply that such a deviation is profitable in the Ausubel auction, but we know that bidding sincerely is a dominant strategy in such an auction. The second type of deviation (which may or may not be feasible) requires buyer b to reduce her demand to the point that buyers end up on the short side of the market. It leads b to acquire a smaller number of units at the reserve price r. Let q^d be the number of units buyer b acquires under this deviation and Δ^d her demand reduction relative to sincere bidding; that is, under sincere bidding she obtains $q^d + \Delta^d$ units. This then implies that under sincere bidding buyer b clinches at least q^d units at the reserve price r in the Ausubel auction. Thus, the deviation that leads to buyers being on the short side is weakly dominated by the deviation of staying on the long side and bidding the reserve price r on all units $q^d + \Delta^d$, but such a deviation is in turn dominated by sincere bidding, as it is a dominant strategy in the Ausubel auction. This concludes the proof, as it shows that: (iii) there is also no profitable deviation from sincere bidding for buyer b when buyers are on the long side of the market. $\square$

4 Discussion and Comparisons

Vickrey (1961) first noted that developing mechanisms for two-sided allocation problems that minimize inefficiencies, do not run a deficit and require no prior information about the true equilibrium price is "extremely difficult". For a bilateral trade setting à la Myerson and Satterthwaite (1983) with the buyer and seller drawing their value and cost independently from distributions with overlapping support, Hagerty and Rogerson (1987) showed that the best the market maker can do subject to dominant strategy incentive compatibility, ex post individual rationality and budget balance is to post an exogenously given price and let the buyer and seller decide if they want to trade at that price. With only one agent on each side of the market, there is simply no way of endogenizing the price at which trade occurs without giving up on dominant strategies (see also Copič and Ponsatí, 2016, and Copič, 2017).

McAfee (1992) proposed a double auction that embeds this insight in a setup with multiple *single-unit* traders, whose values and costs are elements of the $[0, 1]$ interval, and that endogenizes the posted price of Hagerty and Rogerson. There are two clock prices: a decreasing sellers' clock price p^S and an increasing buyers' clock price p^B. In any round t starting with the same number of buyers and sellers and clock prices p^B and p^S, the auctioneer posts a price $p_t^T = \frac{p^B+p^S}{2}$.[19] Both clocks are then run and if no agent exits by the time both clocks reach p_t^T (i.e., by the time $p^B = p^S = p_t^T$), then all active agents trade at p_t^T. If the numbers of buyers and sellers are not the same, either at the outset or after a trader drops out, then only the clock price on the long side moves until the number of active agents is the same on both sides of the market. If this happens when the buyers' clock price p^B is lower than the sellers' clock price p^S, then a new posted price in the middle of the interval (p^B, p^S) is selected and both clocks run again. If equality in the number of buyers and sellers is reached when $p^B > p^S$, then the remaining active traders trade at those prices; buyers pay p^B sellers receive p^S.

McAfee's double auction endows traders with dominant strategies and it either implements trading of the efficient quantity, which happens if trade occurs at a posted price p_t^T, or it just excludes the single least efficient trade, which happens if trade occurs at prices $p^B > p^S$. Although McAfee does not refer to *estimation*, we argue below that it is natural to interpret his double auction as using estimates of demand and supply. McAfee's double auction can also be viewed as entering an Ausubel auction phase when $p^B = p^S$ and the number of traders on the long side exceeds the number of traders on the short side (with probability one only by one trader). With single-unit traders, the single clock, Ausubel auction on the long side is simply a clock implementation of the second-price Vickrey auction, determining the trading price at the drop-out price of the first trader that exits, the most competitive losing bid.

Our DCA can be viewed as an extension of McAfee's (1992) double auction to traders with *multi-unit* demand and supply for a specific estimation procedure.[20] It is worth recalling

[19] Any choice of a posted price equal to $\lambda_t p^B + (1 - \lambda_t) p^S$, with $\lambda_t \in [0, 1]$ would work equally well with regards to the incentive compatibility and individual rationality constraints.

[20] McAfee also proposed a simultaneous bid version of his mechanism, which has been extended in the operation research and computer science literature (see Chu, 2009, and Segal-Halev et al., 2017, for recent contributions and references). None of these extensions has considered a setting with multi-dimensional types, or has used a

that in standard auction formats multi-unit buyers and sellers have an incentive to reduce their demands and supplies so as to manipulate the prices at which they trade (e.g.,see Ausubel et al., 2014). Unlike the DCA, many apparently intuitive generalizations of McAfee's double auction, that rely on counting the number of drop-outs or on excluding the least efficient trades, in general either give traders incentives to misrepresent their marginal values or do not guarantee asymptotic efficiency.

To facilitate comparison with alternative approaches to market clearing and estimation, and to make the connection between estimation in the DCA and McAfee's double auction most transparent, in the remainder of this section we consider a special case of our model, the *order statistics model*, in which each buyer has k_B independent value draws from an unknown distribution F with density f and each seller has k_S independent cost draws from an unknown distribution G with density g. If the distributions F and G were known, then expected market demand and supply at prices p^B and p^S would be, respectively,

$$Nk_B(1 - F(p^B)) \quad \text{and} \quad Mk_S G(p^S). \tag{1}$$

If $k_B = k_S = 1$, then $|\mathcal{N}_{\mathcal{O}}(p^B)|/N$ – the fraction of buyers that have dropped out when p^B is reached by the buyers' clock (that starts at 0) – is the empirical distribution of the draws that are below p^B out of N draws from F; it is thus an estimate of $F(p^B)$. Similarly, $\mathcal{M}_{\mathcal{A}}(p^S)|/M = 1 - |\mathcal{M}_{\mathcal{O}}(p^S)|/M$ – the fraction of sellers that are active when p^S is reached by the sellers' clock (which starts at 1) – is an estimate of $G(p^S)$, as it is the empirical distribution of the draws that are below p^S out of M draws from G. Thus, using (1) estimated demand and supply at p^B and p^S are $Nk_B(1 - |\mathcal{N}_{\mathcal{O}}(p^B)|/N) = N - |\mathcal{N}_{\mathcal{O}}(p^B)|$ and $Mk_S\left(1 - |\mathcal{M}_{\mathcal{O}}(p^S)|/M\right) = M - |\mathcal{M}_{\mathcal{O}}(p^S)|$, which are precisely the true, and McAfee's "estimated," demand and supply at p^B and p^S.

When $k^B > 1$, the fraction of buyers that have dropped out at p^B is instead the empirical distribution of the draws that are below p^B out of N draws from $F_{(1)}$ – the distributions of the highest draw out of k_B for each buyer. Similarly, if $k^S > 1$, then the fraction of sellers that are active at p^S is the empirical distribution of the draws below p^S out of M draws from $G_{[1]}$ – the distribution of the lowest draw out of k_S for each seller. This is because the probability that a given buyer has dropped out at price p^B is $F_{(1)}(p^B)$ and the probability that a given seller is active at price p^S is $G_{[1]}(p^S)$. The distribution of these order statistics are:

$$F_{(1)}(v) = F(v)^{k_B} \quad \text{and} \quad G_{[1]}(c) = 1 - (1 - G(c))^{k_S}. \tag{2}$$

Substituting (2) into (1) after replacing $F_{(1)}(p^B)$ and $G_{[1]}(p^S)$ with their empirical distributions, it follows that estimated demand and supply at p^B and p^S using the fractions of active

double clock format.

traders are:

$$\mathbb{E}\left[D^{\mathcal{N}}(p^B)\right] = Nk_B\left(1-\left(\frac{|\mathcal{N}_{\mathcal{O}}(p^B)|}{N}\right)^{\frac{1}{k_B}}\right) \quad \text{and} \quad \mathbb{E}\left[S^{\mathcal{M}}(p^S)\right] = Mk_S\left(1-\left(\frac{|\mathcal{M}_{\mathcal{O}}(p^S)|}{M}\right)^{\frac{1}{k_S}}\right). \tag{3}$$

Using these estimates in the DCA while the number of bidders increases to infinity (keeping N/M constant) leads to setting a reserve price r approximately equal to the Walrasian price p^W at which $\mathbb{E}\left[D^{\mathcal{N}}(p^W)\right] = \mathbb{E}\left[S^{\mathcal{M}}(p^W)\right]$; the DCA is asymptotically efficient. On the contrary, a "naive" application of McAfee's estimates of demand and supply would lead the DCA to set as reserve price the solution p^* to $N - |\mathcal{N}_{\mathcal{O}}(p^*)| = M - |\mathcal{M}_{\mathcal{O}}(p^*)|$; the DCA would be asymptotically efficient only if $N = M$ and $k_B = k_S$.

To complete the description of the DCA for the order statistics model, one is only left to specify the estimated demand and supply for prices $p \in (p^B, p^S)$, which are used to determine the price at which the adjustments stop if no further exits occur (i.e., the target price) and the speed of price adjustments in case both clock prices move simultaneously. The approach followed by McAfee can be interpreted as taking the estimated demand and supply functions to be linear functions starting from the estimated demand and supply at the current clock prices p^B and p^S; that is, estimated demand at price $p \geqslant p^B$ is $\mathbb{E}\left[D^{\mathcal{N}}(p^B)\right] - \lambda_D\left(p - p^B\right)$ and estimated supply at price $p \leqslant p^S$ is $\mathbb{E}\left[S^{\mathcal{M}}(p^S)\right] + \lambda_S\left(p - p^S\right)$ for some arbitrary $\lambda_D, \lambda_S > 0$. In addition, McAfee implicitly assumes $\lambda_D = \lambda_S$, which implies that estimated demand equals estimated supply at the target price $p_t^T = \frac{p^B+p^S}{2}$. In the numerical simulations of the order statistic model in this section, we will instead estimate the coefficients λ_D and λ_S via OLS using as data the revealed marginal values of all buyers, respectively sellers, that have become inactive when the prices p^B and p^S are reached by the DCA's clocks.[21]

An alternative approach to estimation and market clearing while respecting incentive compatibility and individual rationality constraints is to randomly assign traders to different sub-markets and use reports from all other sub-markets to estimate the market clearing price in any given sub-market. For an interdependent values model with multi-unit demands and supplies and one-dimensional types, Kojima and Yamashita (2017) develop a mechanism that estimates market clearing prices in this *random splitting* fashion. After the agents' reports from the other sub-markets have been used to determine a given market's reserve price the double auction of Kojima and Yamashita, like our DCA, uses an Ausubel auction on the long side.

We now briefly discuss the pros and cons of our DCA and the random splitting mechanism, and provide a few insightful numerical comparisons. We maintain this paper's assumption that the setting is one of private values and confine the discussion and comparisons to this environment.

[21]In Section 5, we will assume more generally that the distribution functions from which buyers' and sellers' values are drawn belong to a parameterized family, and estimates of demand and supply will be obtained via a minimum distance approach.

First, observe that unlike the DCA random splitting mechanisms are not constrained efficient. More generally, the random splitting approach has the downside that it sacrifices the superior sorting or matching properties that larger markets afford. This is particularly relevant with small numbers of traders. To appreciate the relevance of sorting, consider the case of single-unit traders who draw their types independently from the uniform with $N = M = 2$. Under the random splitting mechanism, welfare is bound above by 1/3 because it cannot possibly exceed two times expected welfare in a bilateral trade problem. In contrast, the expected welfare from only executing the trade between the buyer with the higher value and the seller with the lower cost in an integrated market is 11/30.[22]

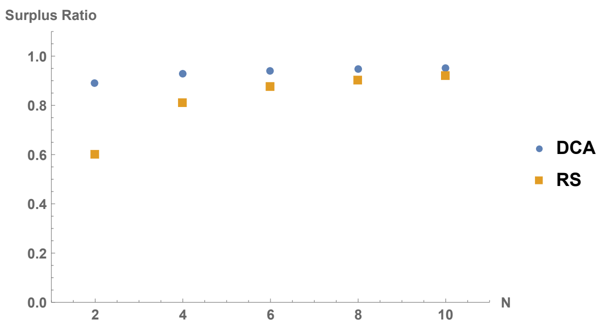
On the other hand, random splitting mechanisms are detail free in the sense of Wilson (1987) (i.e., make no use of a priori information about traders' types and beliefs) and have the advantage of not relying on the informativeness of the behaviour of agents who become inactive, let alone on the existence of such agents. For an extreme example, if quantities were continuous variables and traders' payoff functions satisfied the Inada conditions, then no agent would be inactive under the efficient allocation. This suggests that in the order statistics model the DCA will sacrifice more surplus relative to ex post efficiency, for a fixed number of traders, as the agents' capacities increase.

Thus, intuitively, one would expect the random splitting approach to be outperformed by McAfee's mechanism or the DCA when the number of traders is small; matching is important; and capacities are small. These intuitions are corroborated by our numerical simulations. Figure 1 displays simulation results for the DCA and random splitting mechanisms with two submarkets. The simulations assume that $k_B = k_S = k$, $N = M$, and that all values and costs are drawn from the uniform distribution, but the results are qualitatively the same if all values and costs are drawn instead from the same Beta-distribution with a symmetric density. Panel (a) shows that, keeping capacity fixed, the performance of both mechanisms relative to ex post efficiency improves as N increases. It also shows that with a small number of traders and relatively small capacities k, the DCA outperforms the random splitting mechanisms. Panel (b) shows that, as capacities increase while N is kept fixed, the benefits of the DCA relative to random splitting diminish.

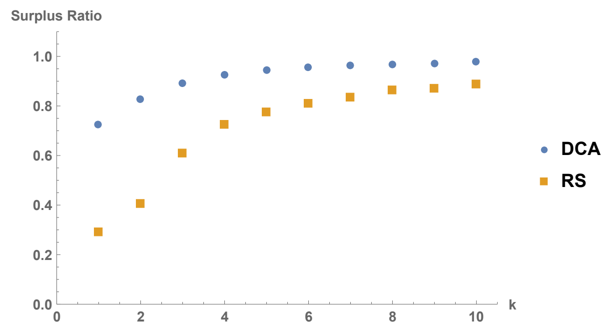
Both the DCA and the random splitting mechanism perform better as k increases. For the random splitting mechanism, this is as expected because in the order statistics model increases in k increase the data that can be used for estimation. In contrast, in the DCA the opposite is the case—as k increases, less data is available for estimation because the probability that each agent is active at a given price increases in k, which means that the probability that

[22]For $N = M = 2$, $k_B = k_S = 1$, and buyers and sellers drawing their types independently from the same distribution – not necessarily uniform – Loertscher and Mezzetti (2020) show that McAfee's double auction, and hence our DCA, generates a higher expected welfare than the random sampling mechanism.

the DCA reaches the initial target price increases in k. So if the initial target price is equal to the Walrasian price in the large market limit, which is the case with symmetric densities, the performance of the DCA improves as k increases because, in a sense, estimation becomes less important. (Indeed, our simulations—not displayed—show that, for asymmetric densities, random splitting eventually outperforms the DCA as k increases.)



(a) N varies



(b) k_B varies

Figure 1: Comparisons of DCA and random splitting mechanism with two submarkets. Panel (a): $k = 3$, $N \in \{2, 4, 6, 8, 10\}$, uniform distributions. Panel (b): $N = 2$, $k \in \{1, \ldots, 10\}$, uniform distributions

5 Asymptotic Efficiency of the DCA

To prove the asymptotic efficiency of the DCA, we now endow the auctioneer with a model of the random process generating traders' valuations, allowing the number of traders to grow large. Thus, as foreshadowed in Section 2, the sets of buyers $\mathcal{N}$ and sellers $\mathcal{M}$ now contain, respectively, nN and nM elements, and we study the limit equilibrium outcome as $n \to \infty$.

Given an integer n, we assume that the marginal values and costs of the agents are drawn from one of the feasible probability measures $\mathbb{P}^n_\phi$. The set of indexes Φ determines the set of feasible measures and the index $\phi \in \Phi$ specifies an element of the set. We assume that Φ is a compact subset of a metric space and that $\mathbb{P}^n_\phi$ is continuous as a function of ϕ (an assumption that is trivially satisfied if Φ is a finite set). We make three assumptions about the probability measures $\mathbb{P}^n_\phi$. First, we require the expected per capita demand and supply functions they induce to be strictly monotone. Second, we allow traders' valuations to be correlated but require a form of weak dependence of individual demands and supplies which guarantees that the law of large numbers holds. Third, we impose an identifiability condition that guarantees that minimum distance estimation can identify the stochastic process generating traders' valuations. Note that all three conditions are satisfied by the order statistic model described in the last section, by just postulating that the possible distribution functions from which buyers' and sellers' values are drawn belong to a parameterized family, e.g., $F_{\phi_B}(v) = v^{\phi_B}$ and $G_{\phi_S}(v) = v^{\phi_S}$ with $\phi = (\phi_B, \phi_S) \in \Phi = [\underline{\phi}, \overline{\phi}]^2$. They are also satisfied in a conditionally independent generalization of the order statistics model, in which a state is drawn first from a distribution from some parameterized family and then buyers' and sellers' values are drawn, conditional on the state, from distributions from another parameterized family.

To state our three assumptions formally, let $\mathbf{1}(\cdot)$ be the indicator function and define the true demand and supply for the k-th unit by buyer b and seller s at price p as: $D_k^b(p) = \mathbf{1}\left(v_k^b \geqslant p\right)$ and $S_k^s(p) = \mathbf{1}\left(c_k^s \leqslant p\right)$. We denote the demand at price p for the k-th unit of the buyers who are still active at price p^B by $D_k^{\mathcal{N}_{\mathcal{A}}(p^B)}(p)$, and of those who have dropped out by $D_k^{\mathcal{N}_{\mathcal{O}}(p^B)}(p)$. Adding the two we obtain the aggregate demand for the k-th unit $D_k^{\mathcal{N}}(p)$, which allows us to define aggregate demand at price p as $D^{\mathcal{N}}(p) = \sum_{k=1}^{k_B} D_k^{\mathcal{N}}(p)$. Similarly, we denote the supply at price p for the k-th unit of the active sellers at price p^S by $S_k^{\mathcal{M}_{\mathcal{A}}(p^S)}(p)$ and of those who have dropped out by $S_k^{\mathcal{M}_{\mathcal{O}}(p^S)}(p)$; aggregate supply for the k-th unit is denoted by $S_k^{\mathcal{M}}(p)$ and aggregate supply at price p is $S^{\mathcal{M}}(p) = \sum_{k=1}^{k_S} S_k^{\mathcal{M}}(p)$.

Given any probability measure $\mathbb{P}_\phi^n$, any possible event $\mathcal{Z}$ describing the information obtained from buyers and sellers that have dropped out when prices p^B, p^S are reached, and any random variable X which is measurable with respect to such drop-outs information, let $\mathbb{E}_\phi^n\big[X \mid \mathcal{Z}\big]$ be the conditional expectation of X and $\mathbb{E}_\phi^n\big[\mathbb{E}_\phi^n\big[X \mid \mathcal{Z}\big]\big] = \mathbb{E}_\phi^n\big[X\big]$ be the unconditional expectation. Thus, for example, $\mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p) \mid \mathcal{Z}\big]$ is expected aggregate demand at price p conditional on $\mathcal{Z}$ and $\mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p)\big]$ is unconditional expected aggregate demand at price p.

We are now ready to state our assumption that the expected per capita demand and supply functions are strictly monotone.

Assumption 1. (*Monotonicity of Demand and Supply*) *There exist w and W with $0 < w < W$ such that:*

(i) *For all $p \in [0,1]$, all $\epsilon \in [0, 1-p]$, all n, and all $\phi \in \Phi$, we have:*

$$wn\epsilon \leqslant \mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p)\big] - \mathbb{E}_\phi^n\big[D^{\mathcal{N}}(p+\epsilon)\big] \leqslant Wn\epsilon\,. \tag{4}$$

(ii) *For all $p \in [0,1]$, all $\epsilon \in [0, p]$, all n, and all $\phi \in \Phi$, we have:*

$$wn\epsilon \leqslant \mathbb{E}_\phi^n\big[S^{\mathcal{M}}(p)\big] - \mathbb{E}_\phi^n\big[S^{\mathcal{M}}(p-\epsilon)\big] \leqslant Wn\epsilon\,. \tag{5}$$

Assumption 1 holds in the order statistics since there $\mathbb{E}_\phi^n[D^{\mathcal{N}}(p)] = nNk_B[1 - F_{\phi_B}(p)]$ and $\mathbb{E}_\phi^n[S^{\mathcal{M}}(p)] = nMk_S G_{\phi_S}(p)$.[23] More generally, a sufficient condition for Assumption 1 to hold

[23] The numerical simulations we discussed in Section 4 are based on and use for estimation the specificity of the order statistics model, but the DCA satisfies asymptotic efficiency in more general settings.

is that the probability measures $\mathbb{P}^n_\phi$ are absolutely continuous with respect to Lebesgue measure and their Radon-Nikodym derivatives (densities) are bounded away from zero and finite.[24]

Our second assumption allows traders' values to be correlated and borrows the concept of weak independence from the statistical literature (e.g., see Bradley, 2005, and Dedecker et al., 2007); it requires that, for any given index $\phi \in \Phi$, the covariances among the marginal values of two traders vanish as the distance between them, as measured by their position in an ordered list, grows large.[25] Given a probability measure $\mathbb{P}^n_\phi$, consider the following covariances:

$$\alpha^{ij}_k(p;\phi) = \mathbb{P}^n_\phi\left(D^i_k(p) = D^j_k(p) = 1\right) - \mathbb{P}^n_\phi\left(D^i_k(p) = 1\right)\mathbb{P}_\phi\left(D^j_k(p) = 1\right) ;$$
$$\beta^{ij}_k(p;\phi) = \mathbb{P}^n_\phi\left(S^i_k(p) = S^j_k(p) = 1\right) - \mathbb{P}^n_\phi\left(S^i_k(p) = 1\right)\mathbb{P}_\phi\left(S^j_k(p) = 1\right) .$$

Note that $\alpha^{ij}_k(p;\phi)$ and $\beta^{ij}_k(p;\phi)$ are bounded above by $1/4$ and below by $-1/4$. If the individual demands at p of buyers i and j are independent as in the order statistic model, or if individual demands are deterministic, then $\alpha^{ij}_k(p;\phi) = 0$; similarly, if the individual supplies of sellers i and j at p are independent conditional on ϕ, or if individual supplies are deterministic, then $\beta^{ij}_k(p;\phi) = 0$. In both cases Assumption 2 holds.

Assumption 2. (*Weak Dependence of Individual Demands and Supplies*)

(i) *There exists $\Delta_B < \infty$ and a permutation $b \to i$ of the buyers' names such that, for all $p \in (0,1)$, all $k \in \{1, ..., k_B\}$, all n, all $i \in \mathcal{N}$ and all $\phi \in \Phi$:*

$$\sum_{j \in \mathcal{N},\ j > i} \alpha^{ij}_k(p;\phi) \leqslant \Delta_B . \tag{6}$$

(ii) *There exists $\Delta_S < \infty$ and a permutation $s \to i$ of the sellers' names such that, for all $p \in (0,1)$, all $k \in \{1, ..., k_S\}$, all n, all $i \in \mathcal{M}$ and all $\phi \in \Phi$:*

$$\sum_{j \in \mathcal{M},\ j > i} \beta^{ij}_k(p;\phi) \leqslant \Delta_S . \tag{7}$$

The bite of Assumption 2 comes as the number of buyers and sellers grows large; it requires that there is a listing of buyers, and one of sellers, under which the covariance between the demands of any buyer b and buyer $b + \tau$, and seller s and $s + \tau$, vanishes as the distance τ between the position in the list of the two buyers, and the two sellers, grows large. To see in an example what Assumption 2 requires, suppose that, given $\phi_B \in [\underline{\phi}, \overline{\phi}]$, the marginal values of

[24] The requirement that $wn\epsilon \leqslant \mathbb{E}^n_\phi\left[D^{\mathcal{N}}(p)\right] - \mathbb{E}_\phi\left[D^{\mathcal{N}}(p+\epsilon)\right]$ and $wn\epsilon \leqslant \mathbb{E}^n_\phi\left[S^{\mathcal{M}}(p)\right] - \mathbb{E}_\phi\left[S^{\mathcal{M}}(p-\epsilon)\right]$ is essentially the same as the assumption of No Asymptotic Gaps in Cripps and Swinkels (2006), while the requirement that $\mathbb{E}^n_\phi\left[D^{\mathcal{N}}(p)\right] - \mathbb{E}_\phi\left[D^{\mathcal{N}}(p+\epsilon)\right] \leqslant Wn\epsilon$ and $\mathbb{E}^n_\phi\left[S^{\mathcal{M}}(p)\right] - \mathbb{E}_\phi\left[S^{\mathcal{M}}(p-\epsilon)\right] \leqslant Wn\epsilon$ is the counterpart of their No Asymptotic Atoms assumption.

[25] Cripps and Swinkels (2006)) and Peters and Severinov (2006) use different assumptions that are inspired by the related statistical literature on mixing conditions.

buyer 1 are independently drawn from the probability distribution $F_{\phi_B}(v)$, while the marginal values of traders $i > 1$ are independently drawn from $F_{\phi_B}(v)$ with probability $0 < \lambda < 1$ and with the remaining probability they are either: (A) identical to the marginal values drawn by trader $i-1$, or (B) identical to the marginal values drawn by trader 1. In case (A) Assumption 2 holds for buyers, while in case (B) it fails.[26]

Before formally stating the third assumption, recall that when the buyers' clock price in the DCA is p^B and the sellers' clock price is p^S, to estimate the parameter ϕ the data available to the auctioneer are the true demands and supplies of the traders that have dropped out of the DCA, that is, of the buyers and sellers in the sets $\mathcal{N}_{\mathcal{O}}(p^B)$ and $\mathcal{M}_{\mathcal{O}}(p^S)$. We assume that the auctioneer computes the parameter ϕ that minimizes the integrated square distance between true and expected per capita demand and supply of the traders that have dropped out; that is, she solves the minimum distance problem:[27]

$$\min_{\phi\in\Phi}\left(\int_0^{p^B}\left(\frac{D^{\mathcal{N}_{\mathcal{O}}(p^B)}(p)-\mathbb{E}_\phi\big[D^{\mathcal{N}_{\mathcal{O}}(p^B)}(p)\big]}{n}\right)^2 dp+\int_{p^S}^1\left(\frac{S^{\mathcal{M}_{\mathcal{O}}(p^S)}(p)-\mathbb{E}_\phi\big[S^{\mathcal{M}_{\mathcal{O}}(p^S)}(p)\big]}{n}\right)^2 dp\right). \tag{8}$$

For any given event $\mathcal{Z}$ describing the information obtained from the traders that have dropped out when the DCA has reached prices p^B and p^S, let $\phi(\mathcal{Z})$ be the solution of the minimum distance problem.[28] Estimated demand and supply then are $\mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(p)\mid\mathcal{Z}\big]$ and $\mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}}(p)\mid\mathcal{Z}\big]$.

Convergence to efficiency requires that the estimation procedure be informative about the stochastic process generating the data (i.e., marginal values and costs). Thus, like in any statistical or econometric model, we need an identifiability assumption on the admissible probability measures. In our setting, this is Assumption 3 below. It guarantees, loosely speaking, that the data available are sufficient to determine the value of ϕ. Let $\mathbb{P}^n_{\phi*}$ be the probability measure from which values and costs are drawn. In combination with the operation of the DCA, $\mathbb{P}^n_{\phi*}$ determines the distribution of the reserve price. Indeed, the reserve price only depends on the event $\mathcal{Z}$ describing the information obtained from traders that have dropped out of the DCA; to emphasize this dependency and the fact that the reserve price is a random variable, we will now denote by $R_{\mathcal{Z}}$ the reserve price when the event is $\mathcal{Z}$.

[26] We would obtain the same conclusion if the marginal values of buyer 1 where drawn from a known distribution F and then we took the realized values as the index ϕ; in case (A) Assumption 2 would hold, but it would not in case (B) when each trader has the same values as buyer 1 with probability $1-\lambda$.

[27] The mean square distance is the "right" distance because to prove Theorem 2 we will use the convergence in mean square to their expectations of aggregate demand and supply at the reserve price.

[28] Since Φ is a compact subset of a metric space and $\mathbb{P}^n_\phi$ is a continuous function of ϕ, the minimizer $\phi(\mathcal{Z})$ of (8) exists. The size of the set Φ does not matter for our results, but it would affect the computability of the estimator $\phi(\mathcal{Z})$. As long as the probability measures are well behaved functions of ϕ, for the purpose of computation Φ could be approximated by a finite grid.

Assumption 3. (*Identifiability*) *Suppose the vectors of valuations of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_{\phi_*}$. Let $\mathcal{Z}$ be the event describing the information obtained from the traders that have dropped out when the DCA has reached the reserve price $R_{\mathcal{Z}}$. For $\phi \in \Phi$ and $\phi \neq \phi_*$, let $\mathbb{P}^n_{\phi}$ be any other feasible probability measure. There exists $\zeta > 0$ such that:*

$$\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big] - \mathbb{E}^n_{\phi}\big[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]}{n}\right)^2 + \left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big] - \mathbb{E}^n_{\phi}\big[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]}{n}\right)^2$$

$$\leqslant \zeta \cdot \mathbb{E}^n_{\phi_*}\Bigg[\int_0^{R_{\mathcal{Z}}} \left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(p)\big] - \mathbb{E}^n_{\phi}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(p)\big]}{n}\right)^2 dp$$

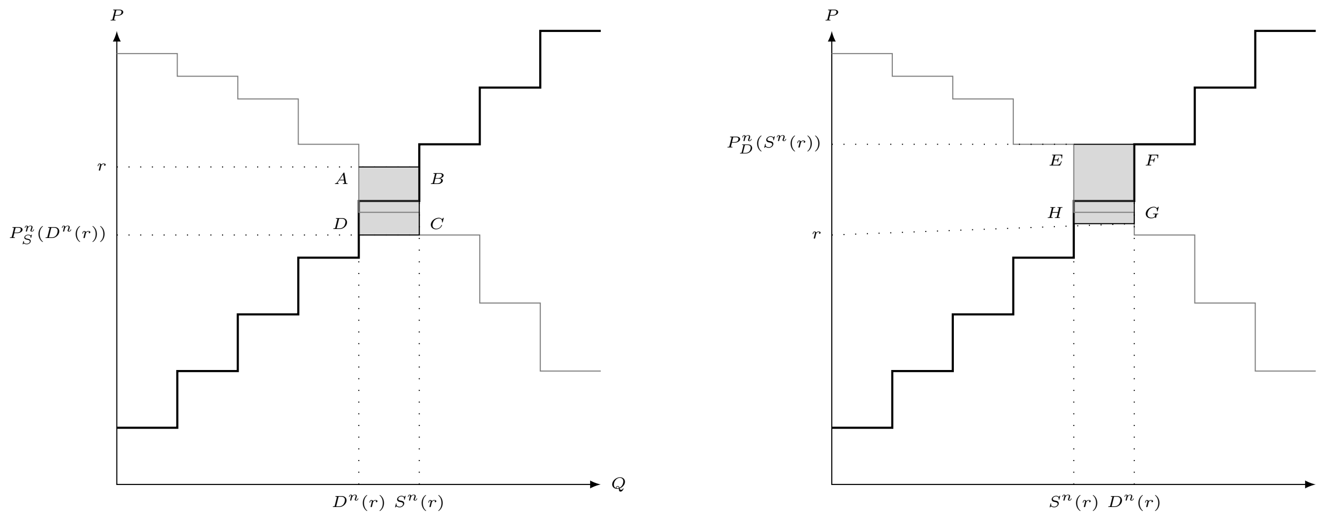
$$+ \int_{R_{\mathcal{Z}}}^1 \left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(p)\big] - \mathbb{E}^n_{\phi}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(p)\big]}{n}\right)^2 dp\Bigg] \tag{9}$$

The identifiability condition requires that the difference between true expected demand and supply and expected demand and supply according to a different probability measure at the reserve price $R_{\mathcal{Z}}$, conditional on the event $\mathcal{Z}$, is bounded by some multiple of the expected demand and supply distance of the buyers and sellers that have dropped out at $R_{\mathcal{Z}}$.

To gain some intuition about Assumption 3 in the simplest setting, consider the order statistics model in which there is a unique, known, distribution from which the sellers' costs are drawn, so that the second terms on both sides of (9) vanish, while the buyers' values are drawn from a distribution belonging to a parameterized family. Let $|\mathcal{N}_{\mathcal{A}}(R_{\mathcal{Z}})|$ be the number of buyers still active at the reserve price $R_{\mathcal{Z}}$. Then, conditional on the event $\mathcal{Z}$, all active buyers demand at least one unit plus an additional number of units equal to the number of the other $k_B - 1$ independent draws that are above $R_{\mathcal{Z}}$. In other words, expected demand conditional on $\mathcal{Z}$ by an active buyer when the index is ϕ_* is: $1 + (k_B - 1)[1 - F_{\phi^*}(R_{\mathcal{Z}})]$. It then follows that the left hand side of (9) is: $\left(\frac{|\mathcal{N}_{\mathcal{A}}(R_{\mathcal{Z}})|}{n}(k_B - 1)[F_{\phi}(R_{\mathcal{Z}}) - F_{\phi_*}(R_{\mathcal{Z}})]\right)^2$, which is less than $N^2(k_B-1)^2[F_{\phi}(R_{\mathcal{Z}}) - F_{\phi_*}(R_{\mathcal{Z}})]^2$. We may follow the same approach to compute the integrand on the right hand side of (9), after first noting that there is no conditioning on the event $\mathcal{Z}$ apart from the number $|\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})|$ of buyers who have dropped out before the reserve price is reached. Thus, the integrand on the right hand side of (9) is: $\left(\frac{|\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})|}{n} k_B[F_{\phi}(p) - F_{\phi_*}(p)]\right)^2$. It follows that Assumption 3 holds if $\phi \neq \phi_*$ implies that for all $p \in (0,1]$, we have $F_{\phi}(v) \neq F_{\phi_*}(v)$ for a positive Lebesgue measure set of values $v \in (0,p]$.[29]

The first, trivial, way in which Assumption 3 would fail is if for all feasible probability measures, all buyers had the highest possible value for the first unit, $v^b_1 = 1$ for all b, and all sellers had the lowest possible cost for the first unit $c^s_1 = 0$ for all s. In such a case there would be no drop-outs at any interior reserve price and the right hand side of (9) would always equal

[29]Note that the expected number of buyers that drop out by the time any reserve price r is reached is $nNF_{\phi_*}(r)^{k_B}$



(a) Buyers on short side

(b) Sellers on short side

Figure 2: Illustration of bounds on welfare losses. Panel (a): Buyers are on short side at r. Panel (b): Sellers are on short side at r. Generically, r will not be equal to any agent's value or cost while $P_S^n(D^n(r))$ and $P_D^n(S^n(r))$ are, by construction, equal to a cost or a value, respectively.

zero.[30] More generally, for Assumption 3 to fail the active traders at the reserve price must be unpredictably different from the inactive traders. Thus, in a similar vein to the example just discussed, suppose for simplicity that all sellers' values are independently drawn from the same distribution G while buyers are first drawn to be *weak* or *strong*; weak buyers draw all their values independently from the same distribution F, while strong buyers value the first unit at 1 and the marginal values for all other units are independently drawn from a distribution F_ϕ, with $\phi \in \Phi$. Now there will be traders that drop out (both sellers and weak buyers), but their values provide no information about the values of the strong buyers.

We are now ready to prove the asymptotic efficiency of the DCA. Denote by $P_B^n(q) = \{\min p : D^{\mathcal{N}}(p') \leqslant q \leqslant D^{\mathcal{N}}(p) \text{ for all } p' > p\}$ the inverse realized market demand and by $P_S^n(q) = \{\max p : S^{\mathcal{M}}(p') \leqslant q \leqslant S^{\mathcal{M}}(p) \text{ for all } p' < p\}$ the inverse realized market supply. Consider the demand and supply diagram in Figure 2, with r being the realized reserve price. When buyers are on the short side of the market – i.e., when $D^{\mathcal{N}}(r) < S^{\mathcal{M}}(r')$ for some $r' < r$, as in Panel (a) – the quantity traded in the DCA is $q(r) = D^{\mathcal{N}}(r)$; let $P_S^n\left(D^{\mathcal{N}}(r)\right) < r$ be the price at which supply is equal to $D^{\mathcal{N}}(r)$. The difference between efficient and realized welfare, $W_{CE}(\boldsymbol{\theta}) - W(\boldsymbol{\theta})$, is bounded above by the area of the shaded rectangle ABCD. Thus, the welfare difference is at most the area of this rectangle; that is, $[r - P_S^n\left(D^{\mathcal{N}}(r)\right)] \cdot [S^{\mathcal{M}}(r) - D^{\mathcal{N}}(r)]$. Similarly, when sellers are on the short side of the market – i.e., when $S^{\mathcal{M}}(r) < D^{\mathcal{N}}(r')$ for some $r' > r$ as

[30]Note however that if the auctioneer knows this information, she could allocate the first unit from sellers to buyers at an arbitrary price and then run the DCA.

in Panel (b) of Figure 2 – the quantity traded is $q(r) = S^{\mathcal{M}}(r)$; let $P_B^n\left((S^{\mathcal{M}}(r)\right)$ be the price at which demand would be equal to $S^{\mathcal{M}}(r)$. The welfare difference is now bounded above by $[P_B^n\left(S^{\mathcal{M}}(r)\right) - r] \cdot [D^{\mathcal{N}}(r) - S^{\mathcal{M}}(r)]$, the area of the rectangle EFGH.

In the proof of Theorem 2 we show that the ratio of the area of the rectangle ABCD (or EFGH) to total welfare, and hence the expected percentage welfare loss, converges to zero at rate $1/n$.

Theorem 2. *Under Assumptions 1, 2 and 3, the expected percentage welfare loss in the DCA converges to zero at rate $1/n$ as $n \to \infty$.*

To prove Theorem 2 we need to establish that the expected distance between demand and supply at the reserve price r reached by the DCA is "small". The proof strategy is to observe that an upper bound on the expected distance between demand and supply is given by a multiple of the highest of three expected distances, all of which are small. The first is the expected distance between demand and expected demand at r (given the true index ϕ_*). The second is the expected distance between supply and expected supply at r (again, given the index ϕ_*). The proof that these two expected distances are small appeals to the law of large numbers, Corollary 1 in the Appendix, and only requires monotonicity and weak dependence of demand and supply, that is, Assumptions 1 and 2. The third expected distance is the expected distance between estimated demand and estimated supply; that is, the expected magnitude of estimated excess demand. The claim that this expected distance is small is in Lemma 2 in the Appendix. This is the only part of the proof of Theorem 2 that requires our identifiability condition, Assumption 3.

The rate of convergence to efficiency in Theorem 2 is $1/n$ because the auctioneer uses the empirical distribution of values and costs of the traders that have dropped out to estimate demand and supply, and the empirical distribution converges to the true distribution at rate $1/\sqrt{n}$. In McAfee (1992, see his Remark 3 on p.444) the rate of convergence is $1/n^2$ as the gap between demand and supply is never more than one unit; with single unit traders there is no need to use the values and costs of the traders that have dropped out to estimate demand and supply at the current prices. Thus, in McAfee's mechanism not only the percentage welfare loss, but also the total welfare loss goes to zero as the number of traders increases. The literature on the k-double auction (see Rustichini et al., 1994, and Cripps and Swinkels, 2006) has also obtained convergence to efficiency at rate $1/n^2$. In that literature, no estimation procedure is needed as the auctioneer is passive and the *traders know the true distribution* of values and costs when computing their equilibrium strategies. The k-double auction literature puts the burden of aggregating information on the traders' knowledge of the true distribution and their ability to compute and coordinate on an equilibrium.[31] In contrast, our DCA puts the burden

[31] In the case of unit demand and supply it is well known that there are a continuum of equilibria. In the

of aggregating information on the auctioneer, making the traders' strategy straightforward. Finally, the random splitting mechanism in Kojima and Yamashita (2017, see their Remark 3 on p.1424) converges to efficiency at rate $1/n^{1/6}$. Their and our rate of convergence, however, are not not easily comparable, as the models are different; they assume interdependent values and single dimensional types determining the shape of a trader's valuation.

6 Conclusions

Progress in research is made one step at a time; in this paper, we have proposed an estimation-based market design for a homogeneous good market which targets efficiency. In contrast to Walrasian tâtonnement, from which it draws inspiration, it maintains dominant strategy incentive compatibility throughout by making all agents price-takers at all times.

Importantly, our DCA achieves this while accommodating multi-unit traders, which is of relevance in practice. Of course, it can be criticized on the ground that it does not perform well (e.g., in the large) in environments different from those we have studied. This is however true of any mechanism; for example, McAfee's double auction has nice incentive and asymptotic properties with single-unit traders, but these properties do not hold in "naive" extensions of the double auction if one introduces multi-unit traders with multi-dimensional types. Similarly, our DCA may be vulnerable to shill bidding if the designer cannot prevent agents from registering under multiple identities, because it estimates target prices and eventually the reserve price based on the revealed values and costs of the traders that have dropped out. By registering multiple times and dropping out early on the shills an agent may be able to affect the price at which she trades in her favor. Robustness to shill bidding is not a problem specific to our mechanism but applies to the entire literature on mechanism design with estimation.

Our DCA design is quite flexible, and can be modified in several ways, depending on the goals and constraints facing the designer, while preserving the property that sincere bidding is a dominant strategy equilibrium.

First, the DCA generates a budget surplus, because it runs an Ausubel auction on the long side of the market. While in many practical applications (e.g., double auctions run by governments or public agencies) running a surplus is acceptable, or even desirable, a budget surplus could be avoided by using a rationing procedure instead of an Ausubel auction. Suppose that after selecting the reserve price the auctioneer randomly selects a priority order of the traders on the long side of the market and fulfills their demands or supplies according to the drawn priority, up to the quantity determined on the short side of the market. All traders are charged or paid the reserve price for each unit they receive or provide. This modification does not change the incentive properties of the DCA, as no trader can affect the reserve price

case of multi-unit demands and supplies it is only known that a mixed strategy equilibrium exists, but no such equilibrium has yet been found.

unless they drop out. They also cannot profitably affect the quantity traded. Thus, Theorem 1 continues to hold and the modified DCA balances the budget. However, this comes at the cost of giving up constrained efficiency and slowing the convergence to efficiency as the number of traders grows. Consider the case when buyers are on the long side of the market. The number of efficient trades that are not completed is still given by the difference between demand and supply at the random reserve price $R_{\mathcal{Z}}$, $D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})$. However, as the non-completed trades are randomly selected among buyers with marginal values above $R_{\mathcal{Z}}$, the upper bound on the welfare loss is now (some multiple of) $D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})$. Thus, an upper bound on the expected percentage efficiency loss is $\mathbb{E}^n_{\phi*}\left[\frac{\sqrt{(D^{\mathcal{N}}(R_{\mathcal{Z}})-S^{\mathcal{M}}(R_{\mathcal{Z}}))^2}}{n}\right]$. Since Lemma 2 in Appendix A proves that $\mathbb{E}^n_{\phi*}\left[\mathbb{E}^n_{\phi*}\left[\frac{D^{\mathcal{N}}(R_{\mathcal{Z}})-S^{\mathcal{M}}(R_{\mathcal{Z}})}{n} \mid \mathcal{Z}\right]^2\right]$ converges to zero at rate $1/n$, it follows from Jensen's inequality that the expected percentage efficiency loss of the DCA with rationing converges to zero at rate $1/\sqrt{n}$.

Second, our DCA can be modified to allow for the incorporation of constraints on the aggregate quantities subsets of bidders may be allocated or may procure, such as a cap on the number of units a subgroup of buyers may acquire in total. Quantity constraints like these may arise for a number of reasons, such as antitrust concerns or technological constraints.

Third, in the DCA the auctioneer selects target prices in each estimation round and the clock state is determined so as to achieve equality of estimated demand and supply; the goal is to reach a unique reserve price at which estimated excess demand is zero. A profit maximizing intermediary could instead set target prices and clock states so as to target equality of estimated marginal cost and marginal benefit (derived from estimated demand and supply), with the goal of reaching two different reserve prices, one for buyers and one for sellers, at which estimated marginal revenue equals estimated marginal cost and estimated demand equals estimated supply. Call *maximum profit* the profit that would be generated if the profit maximizing intermediary knew demand and supply, but was constrained to select two prices, a uniform price for buyers and a uniform price for sellers.[32] With an additional monotonicity assumption on marginal cost and benefit, we conjecture that such a modified DCA would be asymptotically profit maximizing; that is, the percentage profit loss relative to maximum profit would convergences to zero as the number of traders grows. We leave a proper investigation to future research.

In future research, it would also be important to expand the setup to allow for heterogenous commodities or incorporate versions of the assignment model.[33] The latter is simpler than what

[32]Extending Myerson's (1981) optimal single-side auction in a Bayesian setting to the case of buyers with multi-unit demand is still an open problem; a fortiori, we do not know the Bayesian mechanism (or, for that matter, the dominant strategy mechanism) that maximizes the intermediary profit in a setting with multi-unit demands and supplies and no restrictions on the prices the intermediary may charge.

[33]See Ausubel (2006) and Shapley and Shubik (1972), respectively.

we have studied here, insofar as agents trade at most one unit, but the challenges arise because there is no natural ordering of agents according to their types. One could also depart from the two-sided setup we considered here by studying an asset market model in which every agent is endowed with some units while having demand for more units. This setup takes away the market maker's ability to separate traders a priori into buyers and sellers.

Appendix

Lemma 1. *Suppose Assumptions 1 and 2 hold and the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_\phi$. Then there exists $\Delta < \infty$ such that, for all $p, p^B, p^S \in (0,1)$, $k \in \{1, ..., k_B\}$ or $k \in \{1, ..., k_S\}$, and all $\phi \in \Phi$:*[34]

$$\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}_k(p) - \mathbb{E}_\phi\big[D^{\mathcal{N}}_k(p)\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}, \tag{10}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}_{\mathcal{O}}(p^B)}_k(p) - \mathbb{E}_\phi\big[D^{\mathcal{N}_{\mathcal{O}}(p^B)}_k(p)\big]}{n_O(p^B)}\Big)^2\Big] \leqslant \frac{\Delta}{n_O(p^B)}, \tag{11}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{S^{\mathcal{M}}_k(p) - \mathbb{E}_\phi\big[S^{\mathcal{M}}_k(p)\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}, \tag{12}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{S^{\mathcal{M}_{\mathcal{O}}(p^S)}_k(p) - \mathbb{E}_\phi\big[S^{\mathcal{M}_{\mathcal{O}}(p^S)}_k(p)\big]}{m_O(p^S)}\Big)^2\Big] \leqslant \frac{\Delta}{m_O(p^S)}. \tag{13}$$

Proof. We will only prove (10), as the the proofs of (11)−(13) are analogous. We have:

$$\begin{aligned}
&\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}_k(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}}_k(p)\big]}{n}\Big)^2\Big] \\
&= \frac{1}{n^2}\mathbb{E}^n_\phi\Big[\Big(\sum_{i\in\mathcal{N}}\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)\Big)^2\Big] \\
&= \frac{1}{n^2}\cdot\sum_{i\in\mathcal{N}}\Bigg(\mathbb{E}^n_\phi\Big[\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)^2\Big] \\
&\qquad + 2\sum_{j\in\mathcal{N},\, j>i}\mathbb{E}^n_\phi\Big[\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)\Big(D^j_k(p) - \mathbb{E}^n_\phi\big[D^j_k(p)\big]\Big)\Big]\Bigg) \\
&= \frac{1}{n^2}\cdot\sum_{i\in\mathcal{N}}\Bigg(\mathbb{E}^n_\phi\Big[\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)^2\Big] \\
&\qquad + 2\sum_{j\in\mathcal{N},\, j>i}\Big(\mathbb{E}^n_\phi\big[D^i_k(p)D^j_k(p)\big] - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\mathbb{E}^n_\phi\big[D^j_k(p)\big]\Big)\Bigg) \\
&= \frac{1}{n^2}\cdot\sum_{i\in\mathcal{N}}\Bigg(\mathbb{E}^n_\phi\Big[\Big(D^i_k(p) - \mathbb{E}^n_\phi\big[D^i_k(p)\big]\Big)^2\Big] + 2\sum_{j\in\mathcal{N},\, j>i}\alpha^{ij}_k(p;\phi)\Bigg) \\
&\leqslant \frac{1}{n^2}\cdot\Big(n + 2n\Delta_B\Big),
\end{aligned}$$

where the inequality follows from Assumption 2. Setting $\Delta = 1 + 2\Delta_B$ concludes the proof. □

[34] Expectations in (11) and (13) are taken given the identities of the inactive traders in $\mathcal{N}_{\mathcal{O}}(p^B)$ and $\mathcal{M}_{\mathcal{O}}(p^S)$.

Corollary 1. *Suppose the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_\phi$ and Assumptions 1 and 2 hold, then there exists $\Delta < \infty$ such that, for all $p, p^B, p^S \in (0,1)$, and all $\phi \in \Phi$:*

$$\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p)\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{14}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}_{\mathcal{O}}(p^B)}(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}_{\mathcal{O}}(p^B)}(p)\big]}{n_O(p^B)}\Big)^2\Big] \leqslant \frac{\Delta}{n_O(p^B)}\,, \tag{15}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{S^{\mathcal{M}}(p) - \mathbb{E}^n_\phi\big[S^{\mathcal{M}}(p)\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{16}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{S^{\mathcal{M}_{\mathcal{O}}(p^S)}(p) - \mathbb{E}^n_\phi\big[S^{\mathcal{M}_{\mathcal{O}}(p^S)}(p)\big]}{m_O(p^S)}\Big)^2\Big] \leqslant \frac{\Delta}{m_O(p^S)}\,. \tag{17}$$

Proof. Define $Y^{\mathcal{N}}_k(p) = D^{\mathcal{N}}_k(p) - \mathbb{E}_\phi\big[D^{\mathcal{N}}_k(p)\big]$. It is:

$$\begin{aligned}
&\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p)\big]}{n}\Big)^2\Big] \\
&\quad= \mathbb{E}^n_\phi\Big[\Big(\sum_{k=1}^{k_B}\frac{D^{\mathcal{N}}_k(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}}_k(p)\big]}{n}\Big)^2\Big] \\
&\quad= \mathbb{E}^n_\phi\Big[\Big(\sum_{k=1}^{k_B}\frac{Y^{\mathcal{N}}_k(p)}{n}\Big)^2\Big] \\
&\quad= \sum_{k=1}^{k_B}\mathbb{E}^n_\phi\Big[\Big(\frac{Y^{\mathcal{N}}_k(p)}{n}\Big)^2\Big] + 2\sum_{k=1}^{k_B}\sum_{h=k+1}^{k_B}\mathbb{E}^n_\phi\Big[\Big(\frac{Y^{\mathcal{N}}_k(p)}{n}\Big)\Big(\frac{Y^{\mathcal{N}}_h(p)}{n}\Big)\Big] \\
&\quad\leqslant \sum_{k=1}^{k_B}\mathbb{E}^n_\phi\Big[\Big(\frac{Y^{\mathcal{N}}_k(p)}{n}\Big)^2\Big] + \underbrace{2\sum_{k=1}^{k_B}\sum_{h=k+1}^{k_B}\mathbb{E}^n_\phi\Big[\Big(\frac{Y^{\mathcal{N}}_k(p)}{n}\Big)^2\Big]^{1/2}\cdot\mathbb{E}^n_\phi\Big[\Big(\frac{Y^{\mathcal{N}}_h(p)}{n}\Big)^2\Big]^{1/2}}_{\text{by Hölder's inequality}} \\
&\quad\leqslant (k_B)^2\cdot\max_{k\in\{1,\dots,k_B\}}\Big\{\mathbb{E}_\phi\Big[\Big(\frac{Y^{\mathcal{N}}_k(p)}{n}\Big)^2\Big]\Big\} \\
&\quad= (k_B)^2\cdot\max_{k\in\{1,\dots,k_B\}}\Big\{\mathbb{E}_\phi\Big[\Big(\frac{D^{\mathcal{N}}_k(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}}_k(p)\big]}{n}\Big)^2\Big]\Big\}.
\end{aligned}$$

Then (14) follows from Lemma 1. The proofs of (15) – (17) are analogous. □

Corollary 2. *Suppose the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_\phi$ and Assumptions 1 and 2 hold, then there exists $\Delta < \infty$ such that, for all $p, p^B, p^S \in (0,1)$, all events $\mathcal{Z}$ determining the set of buyers and sellers that* *have dropped out when prices are* p^B *and* p^S *and all* $\phi \in \Phi$*:*

$$\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p) \,|\, \mathcal{Z}\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{18}$$

$$\mathbb{E}^n_\phi\Big[\Big(\frac{S^{\mathcal{M}}(p) - \mathbb{E}^n_\phi\big[S^{\mathcal{M}}(p) \,|\, \mathcal{Z}\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,. \tag{19}$$

Proof. We only prove (18), as the the proof of (19) is analogous. It is:

$$\begin{aligned}
&\mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p) \,|\, \mathcal{Z}\big]}{n}\Big)^2\Big] \\
&\quad = \mathbb{E}^n_\phi\Big[\frac{1}{n^2}\cdot\Big(D^{\mathcal{N}}(p)^2 + \mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p) \,|\, \mathcal{Z}\big]^2 - 2D^{\mathcal{N}}(p)\mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p) \,|\, \mathcal{Z}\big]\Big)\Big] \\
&\quad = \frac{1}{n^2}\cdot\Big(\mathbb{E}^n_\phi\Big[D^{\mathcal{N}}(p)^2\Big] + \mathbb{E}^n_\phi\Big[\mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p) \,|\, \mathcal{Z}\big]^2\Big] - \underbrace{2\mathbb{E}^n_\phi\Big[\mathbb{E}^n_\phi\Big[D^{\mathcal{N}}(p)\mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p) \,|\, \mathcal{Z}\big] \,|\mathcal{Z}\Big]\Big]}_{\text{by iterated expectations}}\Big) \\
&\quad = \frac{1}{n^2}\cdot\Big(\mathbb{E}^n_\phi\Big[D^{\mathcal{N}}(p)^2\Big] - \mathbb{E}^n_\phi\Big[\mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p) \,|\, \mathcal{Z}\big]^2\Big]\Big) \\
&\quad \leqslant \mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}(p)}{n}\Big)^2\Big] - \underbrace{\mathbb{E}^n_\phi\Big[\frac{D^{\mathcal{N}}(p)}{n}\Big]^2}_{\text{by Jensen's inequality}} \\
&\quad = \mathbb{E}^n_\phi\Big[\Big(\frac{D^{\mathcal{N}}(p) - \mathbb{E}^n_\phi\big[D^{\mathcal{N}}(p)\big]}{n}\Big)^2\Big] \leqslant \underbrace{\frac{\Delta}{n}}_{\text{by Corollary 1.}}
\end{aligned}$$

□

Proof of Theorem 2. Take ϕ_* to be the true index; that is, take $\mathbb{P}^n_{\phi_*}$ to be the probability measure determining values and costs. Define δ_S as $\delta_S = [r - P^n_S\left(D^{\mathcal{N}}(r)\right)]$, where r is the realized reserve price. By Assumption 1, $nw\delta_S \leqslant \mathbb{E}^n_{\phi_*}[S^{\mathcal{M}}(r) - S^{\mathcal{M}}(r-\delta_S)] = \mathbb{E}^n_{\phi_*}[S^{\mathcal{N}}(r) - D^{\mathcal{N}}(r)]$; thus, when buyers are on the short side, we have that $\mathbb{E}^n_{\phi_*}\left[\left|D^{\mathcal{N}}(r) - S^{\mathcal{M}}(r)\right|\right]^2/nw$ is an upper bound of the area of the rectangle ABCD in Fig. 2(a), which in turn is an an upper bound of the expected welfare loss when the reserve price is r. Similarly, define $\delta_B = [P^n_B\left(S^{mathcalM}(r)\right) - r]$. By Assumption 1, $nw\delta_B \leqslant \mathbb{E}^n_{\phi_*}[D^{\mathcal{N}}(r) - D^{\mathcal{N}}(r+\delta_B)] = \mathbb{E}^n_{\phi_*}[D^{\mathcal{N}} - S^{\mathcal{M}}(r)]$; thus, when sellers are on the short side of the market, $\mathbb{E}^n_{\phi_*}\frac{1}{nw}\left[\left|D^{\mathcal{N}}(r) - S^{\mathcal{M}}(r)\right|\right]^2$ is also an upper bound of the expected welfare loss, as it is an upper bound of the area of the rectangle EFGH in Fig. 2(b).

Recall that the reserve price $R_{\mathcal{Z}}$ only depends on the event $\mathcal{Z}$ describing the information obtained from traders that have dropped out of the DCA. The percentage welfare loss is

$\mathcal{L}(\boldsymbol{\theta}) = \frac{(W_{CE}(\boldsymbol{\theta}) - W(\boldsymbol{\theta}))/n}{W_{CE}(\boldsymbol{\theta})/n}$ and the per capita efficient welfare is finite, has finite variance and its expectation converges to a finite level as $n \to \infty$. By Assumption 1, we may then conclude that to prove that the expected percentage efficiency loss $\mathbb{E}^n_{\phi_*}[\mathcal{L}(\boldsymbol{\theta})]$ converges to zero at rate $1/n$, it is sufficient to prove that the expectation of the numerator of $\mathcal{L}(\boldsymbol{\theta})$, which is bounded above by $\frac{1}{n}\mathbb{E}^n_{\phi_*}\frac{1}{n}\left[\mathbb{E}^n_{\phi_*}\left[|D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})| \mid \mathcal{Z}\right]\right]^2$, converges to zero at rate $1/n$, where the inside expectation is taken over demand and supply conditional on $\mathcal{Z}$ and the outside expectation is over events $\mathcal{Z}$ and hence the random reserve price $R_{\mathcal{Z}}$. That is, we must prove that $\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\frac{|D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})|}{n} \mid \mathcal{Z}\right]\right]^2 \leqslant \frac{L}{n}$ for some constant $L > 0$ and all n. By Jensen's inequality: $\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\frac{|D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})|}{n} \mid \mathcal{Z}\right]\right]^2 \leqslant \mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})}{n}\right)^2 \mid \mathcal{Z}\right]\right]$ and hence it suffices to show that for some $L > 0$ and all n: $\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})}{n}\right)^2 \mid \mathcal{Z}\right]\right] \leqslant \frac{L}{n}$.

For all $r \in [0,1]$, define expected excess demand at r as $X^{\mathcal{N}}_{\phi*}(r;\mathcal{Z}) = \mathbb{E}^n_{\phi*}\left[D^{\mathcal{N}}(r) - S^{\mathcal{M}}(r) \mid \mathcal{Z}\right]$. Note that for all $\mathcal{Z}$ and $R_{\mathcal{Z}} \in [0,1]$:

$$
\begin{aligned}
&\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})}{n}\right)^2 \,\Big|\, \mathcal{Z}\right]\right] \\
&= \mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi*}[D^{\mathcal{N}}(R_{\mathcal{Z}})|\mathcal{Z}]}{n} - \frac{S^{\mathcal{M}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}})|\mathcal{Z}]}{n} + \frac{X^{\mathcal{N}}_{\phi*}(R_{\mathcal{Z}};\mathcal{Z})}{n}\right)^2 \,\Big|\, \mathcal{Z}\right]\right] \\
&\leqslant 9\max\left\{\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - \mathbb{E}^n_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})|\mathcal{Z}]}{n}\right)^2 \,\Big|\, \mathcal{Z}\right]\right],\right. \\
&\qquad \left.\mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{S^{\mathcal{M}}(R_{\mathcal{Z}}) - \mathbb{E}^n_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \mid \mathcal{Z}]}{n}\right)^2 \,\Big|\, \mathcal{Z}\right]\right], \mathbb{E}^n_{\phi_*}\left[\mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z})}{n}\right)^2\right] \,\Big|\, \mathcal{Z}\right]\right\}.
\end{aligned}
$$

To conclude the proof we only need to show that there exists $\Delta < \infty$ such that each of the three terms in the max is smaller than Δ/n. For the first two terms, this follows immediately from Corollary 1, as the inequality holds for all realizations of $R_{\mathcal{Z}}$. Lemma 2 below shows that the third term, which equals $\mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z})}{n}\right)^2\right]$ is also smaller than $\frac{\Delta}{n}$. □

Lemma 2. *Suppose the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_{\phi_*}$ and Assumptions 1, 2 and 3 hold, then there exists $\Delta < \infty$ such that:* $\mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z})}{n}\right)^2\right] \leqslant \frac{\Delta}{n}$.

Proof. We first need to establish two preliminary lemmas.

Lemma 3. *Suppose the valuations and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_{\phi_*}$ and Assumptions 1, 2 and 3 hold. Let $\mathcal{Z}$ be the event containing the information from the dropped-out traders. There exists $\Delta < \infty$ such that:*

$$\mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z}) - \mathbb{E}^n_{\phi(\mathcal{Z})}\left[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \mid \mathcal{Z}\right]}{n}\right)^2\right] \leqslant \frac{\Delta}{n}.$$

Proof. Note that:

$$
\begin{aligned}
&\Big(X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z})) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]\Big)^2 \\
&\quad = \Big(\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]\Big)^2 \\
&\quad \leqslant 2\Big(\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]\Big)^2 + 2\Big(\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]\Big)^2
\end{aligned} \tag{20}
$$

Taking the expectation with respect to $\mathbb{P}^n_{\phi_*}$, by Assumption 3 there exists a $\zeta > 0$ such that:

$$
\begin{aligned}
&\frac{1}{2\zeta}\cdot \mathbb{E}^n_{\phi_*}\left[\left(\frac{X^{\mathcal{N}}_{\phi_*}(R_{\mathcal{Z}};\mathcal{Z}) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}\big]}{n}\right)^2\right] \\
&\leqslant \mathbb{E}^n_{\phi_*}\left[\int_0^{R_{\mathcal{Z}}}\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right. \\
&\qquad\left. + \int_{R_{\mathcal{Z}}}^1\left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right] \\
&= \mathbb{E}^n_{\phi_*}\left[\int_0^{R_{\mathcal{Z}}}\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) + D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right. \\
&\qquad\left. + \int_{R_{\mathcal{Z}}}^1\left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) + S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right] \\
&\leqslant 2\mathbb{E}^n_{\phi_*}\left[\int_0^{R_{\mathcal{Z}}}\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)}{n}\right)^2 dt + \int_0^{R_{\mathcal{Z}}}\left(\frac{D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right. \\
&\qquad\left. + \int_{R_{\mathcal{Z}}}^1\left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)}{n}\right)^2 dt + \int_{R_{\mathcal{Z}}}^1\left(\frac{S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t) - \mathbb{E}^n_{\phi(\mathcal{Z})}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big]}{n}\right)^2 dt\right] \\
&\leqslant 4\mathbb{E}^n_{\phi_*}\left[\int_0^{R_{\mathcal{Z}}}\left(\frac{\mathbb{E}^n_{\phi_*}\big[D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - D^{\mathcal{N}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)}{n}\right)^2 dt + \int_{R_{\mathcal{Z}}}^1\left(\frac{\mathbb{E}^n_{\phi_*}\big[S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)\big] - S^{\mathcal{M}_{\mathcal{O}}(R_{\mathcal{Z}})}(t)}{n}\right)^2 dt\right]
\end{aligned}
$$

where the first inequality follows from Assumption 3, the second from simple algebra, and the third from the definition of $\phi(\mathcal{Z})$ in (8) as the minimum distance estimation index.

Applying Corollary 1 concludes the proof of Lemma 3, as for some $\Delta > 0$ two terms in the square brackets on the right hand side are both less than $\frac{\Delta}{n\zeta}$ for all realization $R_{\mathcal{Z}}$. □

Lemma 4. *Suppose the valuation and costs of the nN buyers and nM sellers are drawn according to the probability measure $\mathbb{P}^n_{\phi_*}$ and Assumptions 1, 2 and 3 hold. Let $\mathcal{Z}$ be the event containing the information from the dropped-out traders. There exists $\Delta < \infty$ such that:*

$$
\mathbb{E}^n_{\phi_*}\left[\left(\frac{\mathbb{E}^n_{\phi(\mathcal{Z})}\big[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,\big|\,\mathcal{Z}\big]}{n}\right)^2\right] \leqslant \frac{\Delta}{n}.
$$

Proof. Recall that given an event $\mathcal{Z}$ estimated demand and estimated supply at $R_{\mathcal{Z}}$ are given by $\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}]$ and $\mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}]$. There are three cases, or set of events, to consider depending on whether estimated demand is greater, equal or smaller than estimated supply. The case of equality is trivial, as it obviously implies that the term in brackets in the inequality in the lemma is less than $\frac{\Delta}{n}$ for any $\Delta > 0$. The cases of estimated excess demand and estimated excess supply are mirror images of one another and we will thus only consider one of them.

Thus, take events $\mathcal{Z}$ for which $\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}] > \mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}]$, so that the last clock state of the DCA, the state when the reserve price is reached, is a buyers' clock state. This implies that the state preceding the last clock state is either a double clock or a sellers' clock state and there was a sequence of clock prices along which the sellers' price decreased until it reached $R_{\mathcal{Z}}$ and the buyers' clock price stayed constant or increased and stopped at $R_{\mathcal{Z}} - \epsilon_B$. Along that price sequence conditional estimated supply must be at least as large as conditional estimated demand and the sequence must end with either a seller or a buyer dropping out of the DCA. Denote by $\mathcal{Z}_-$ the event that occurred before the state preceding the last; this is the event used to estimate demand and supply in the the second to last state. As we have argued, it must be $\mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B)\,|\,\mathcal{Z}_-] \leqslant \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}_-]$ and hence:

$$
\begin{aligned}
&\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}]\Big)^2 \qquad (21)\\
&\quad\leqslant \Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}] - \mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}] + \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B)\,|\,\mathcal{Z}_-]\Big)^2\\
&\quad\leqslant \Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}] \underbrace{-\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}] + \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}]}_{=0} - \underbrace{D^{\mathcal{N}}(R_{\mathcal{Z}}) + D^{\mathcal{N}}(R_{\mathcal{Z}})}_{=0}\\
&\quad\underbrace{-\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}_-] + \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B)\,|\,\mathcal{Z}_-]}_{\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B)\,|\,\mathcal{Z}_-] \geqslant \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}_-]} - \ \mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B)\,|\,\mathcal{Z}_-]\\
&\qquad - \mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}] \underbrace{+\mathbb{E}_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}] - \mathbb{E}_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}]}_{=0} \underbrace{+S^{\mathcal{M}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}})}_{=0}\\
&\qquad \underbrace{+\mathbb{E}_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}_-] - \mathbb{E}_{\phi_*}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}_-]}_{=0} + \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}_-]\Big)^2\\
&\leqslant 8\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}] - \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}] - D^{\mathcal{N}}(R_{\mathcal{Z}})\Big)^2\\
&\quad + 8\Big(D^{\mathcal{N}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}})\,|\,\mathcal{Z}_-]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B)\,|\,\mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B)\,|\,\mathcal{Z}_-]\Big)^2
\end{aligned}
$$

$$+ 8\Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]\Big)^2 + 8\Big(\mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - S^{\mathcal{M}}(R_{\mathcal{Z}})\Big)^2$$

$$+ 8\Big(S^{\mathcal{M}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]\Big)^2 + 8\Big(\mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]\Big)^2$$

Taking the expectation with respect to $\mathbb{P}^n_{\phi*}$, we obtain:

$$\begin{aligned}
&\frac{1}{8} \cdot \mathbb{E}_{\phi*}\bigg[\Big(\frac{\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2\bigg] \\
&\leqslant \mathbb{E}_{\phi*}\bigg[\Big(\frac{\mathbb{E}_{\phi*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - D^{\mathcal{N}}(R_{\mathcal{Z}})}{n}\Big)^2\bigg] + \mathbb{E}_{\phi*}\bigg[\Big(\frac{D^{\mathcal{N}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]}{n}\Big)^2\bigg] \\
&+ \mathbb{E}_{\phi*}\bigg[\Big(\frac{\mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - S^{\mathcal{M}}(R_{\mathcal{Z}})}{n}\Big)^2\bigg] + \mathbb{E}_{\phi*}\bigg[\Big(\frac{S^{\mathcal{M}}(R_{\mathcal{Z}}) - \mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]}{n}\Big)^2\bigg] \\
&+ \frac{1}{n^2}\mathbb{E}_{\phi*}\bigg[\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi*}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]\Big)^2 + \Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]\Big)^2\bigg] \\
&+ \frac{1}{n^2}\mathbb{E}_{\phi*}\bigg[\Big(\mathbb{E}_{\phi*}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[D^{\mathcal{N}}(R_{\mathcal{Z}} - \epsilon_B) \,|\, \mathcal{Z}_-]\Big)^2 \\
&\qquad + \Big(\mathbb{E}_{\phi*}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-] - \mathbb{E}_{\phi(\mathcal{Z}_-)}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}_-]\Big)^2\bigg]
\end{aligned}$$

By Corollary 2 the first four terms on the right hand side of the last expression are bounded from above by Δ/n for some $\Delta < \infty$. The first of the remaining two terms is equal to half the right hand side of (20), while the second is equal to half the right hand side of (20) conditional on $\mathcal{Z}_-$ rather than $\mathcal{Z}$ and with the demand evaluated at $R_{\mathcal{Z}} - \epsilon_B$ instead of $R_{\mathcal{Z}}$. Following the same argument as in the proof of Lemma 3 we conclude that there exists a $\Delta < \infty$ such that Δ/n is an upper bound for the two terms. This conclude the proof of Lemma 4 since, as claimed above, the case of events with expected excess supply at $R_{\mathcal{Z}}$ (i.e., such that $\mathbb{E}_{\phi(\mathcal{Z})}[S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}] > \mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]$) can be dealt analogously to the case of expected excess demand we just considered. □

We now conclude the proof of Lemma 2. For all events $\mathcal{Z}$ is it is:

$$\begin{aligned}
\Big(\frac{X^{\mathcal{N}}_{\phi*}(R_{\mathcal{Z}};\mathcal{Z})}{n}\Big)^2 &= \Big(\frac{X^{\mathcal{N}}_{\phi*}(R_{\mathcal{Z}};\mathcal{Z}) - \mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n} + \frac{\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2 \\
&\leqslant 2\Big(\frac{X^{\mathcal{N}}_{\phi*}(R_{\mathcal{Z}};\mathcal{Z}) - \mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2 + 2\Big(\frac{\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2 \\
&\leqslant 4\max\bigg\{\Big(\frac{X^{\mathcal{N}}_{\phi*}(R_{\mathcal{Z}};\mathcal{Z}) - \mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2, \Big(\frac{\mathbb{E}_{\phi(\mathcal{Z})}[D^{\mathcal{N}}(R_{\mathcal{Z}}) - S^{\mathcal{M}}(R_{\mathcal{Z}}) \,|\, \mathcal{Z}]}{n}\Big)^2\bigg\}.
\end{aligned}$$

Taking the expectation $\mathbb{E}^n_{\phi*}$ on both sides of the inequality, Lemma 2 follows from Lemmas 3 and 4, stating that the expectation of each term in curly brackets is less than Δ/n for some $\Delta > 0$. □

References

Ausubel, L. (2004): "An Efficient Ascending-Bid Auction for Multiple Objects," *American Economic Review*, 94, 1452–1475.

——— (2006): "An Efficient Dynamic Auction for Heterogeneous Commodities," *American Economic Review*, 96, 602–629.

Ausubel, L., P. Cramton, and P. Milgrom (2006): "The Clock-Proxy Auction: A Practical Combinatorial Auction Design," in *Combinatorial Auctions*, ed. by P. Cramton, Y. Shoham, and R. Steinberg, Cambridge and London: Cambridge University Press, 115–138.

Ausubel, L., P. Cramton, M. Pycia, and M. Weretka (2014): "Demand Reduction and Inefficiency in Multi-Unit Auctions," *Review of Economic Studies*, 81, 1366–1400.

Babaioff, M., B. Lucier, N. Nisan, and R. Paes Leme (2014): "On the Efficiency of the Walrasian Mechanism," *ACM*, 9, 18.

Baliga, S. and R. Vohra (2003): "Market Research and Market Design," *Advances in Theoretical Economics*, 3, Article 5.

Bergemann, D. and S. Morris (2005): "Robust Mechanism Design," *Econometrica*, 73, 1771–1813.

——— (2007): "An Ascending Auction for Interdependent Values: Uniqueness and Robustness to Strategic Uncertainty," *American Economic Review, Papers and Proceedings*, 97, 125–30.

Bradley, R. (2005): "Properties of Strong Mixing Conditions. A Survey and Some Open Questions," *Probability Surveys*, 2, 107–44.

Brooks, B. A. (2013): "Revenue Sharing in Second-Price Auctions," University of Chicago.

Burdett, J. and K. Woodward (2020): "Uniform price auctions with a last accepted bid pricing rule," *Journal of Economic Theory*.

Chu, L. (2009): "Truthful Bundle/multi-unit Double Auctions," *Management Science*, 55, 1184–1198.

Clarke, E. (1971): "Multipart Pricing of Public Goods," *Public Choice*, 11, 17–33.

Copič, J. (2017): "Optimal robust bilateral trade: bid-ask spread," Mimeo.

Copič, J. and C. Ponsatí (2016): "Optimal Robust Bilateral Trade: Risk neutrality," *Journal of Economic Theory*, 276–287.

Cripps, M. W. and J. M. Swinkels (2006): "Efficiency of Large Double Auctions," *Econometrica*, 74, 47–92.

Dedecker, J., P. Doukhan, a. L. J. Lang, G., S. Louhichi, and C. Prieur (2007): *Weak Dependence: with Examples and Applications*, New York, NY: Springer.

Engelbrecht-Wiggans, R. and C. Kahn (1991): "Protecting the Winner: Second Price vs Ascending Bid Auctions," *Economic Letters*, 35, 243–248.

Gresik, T. and M. Satterthwaite (1989): "The Rate at which a Simple Market Converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms," *Journal of Economic Theory*, 48, 304–332.

Groves, T. (1973): "Incentives in Teams," *Econometrica*, 41, 617–631.

Hagerty, K. and W. Rogerson (1987): "Robust Trading Mechanisms," *Journal of Economic Theory*, 42, 94–107.

Holmström, B. (1979): "Groves' Scheme on Restricted Domains." *Econometrica*, 47, 1137 – 1144.

Jehiel, P., B. M. Meyer-ter Vehn, M., and W. Zame (2006): "The Limits of Ex Post Implementation," *Econometrica*, 74, 585–610.

Kojima, F. and T. Yamashita (2017): "Double Auction with Interdependent Valuation: Incentives and Efficiency," *Theoretical Economics*, 1393–1438.

Levin, J. and A. Skrzypacz (2016): "Properties of the Combinatorial Clock Auction," *American Economic Review*, 106, 2528–2551.

Li, S. (2017): "Obviously Strategy-Proof Mechanisms," *American Economic Review*, 107, 3257–3287.

Loertscher, S. and L. M. Marx (2020): "Asymptotically Optimal Prior-Free Clock Auctions," *Journal of Economic Theory*, 187.

Loertscher, S. and C. Mezzetti (2019): "The Deficit on Each Trade in a Vickrey Double Auction Is at Least as Large as the Walrasian Price Gap," *Journal of Mathematical Economics*, 84, 101–106.

——— (2020): "A Tale of Two Mechanisms," Working Paper, University of Melbourne.

Lucking-Reiley, D. (2000): "Vickrey Auctions in Practice: From Nineteenth-Century Philately to Twenty-First-Century E-Commerce," *Journal of Economic Perspectives*, 14, 183–192.

McAfee, R. P. (1992): "A Dominant Strategy Double Auction," *Journal of Economic Theory*, 56, 434–450.

McMillan, J. (1994): "Selling Spectrum Rights," *Journal of Economic Perspectives*, 8, 145–162.

Mezzetti, C. (2004): "Mechanism Design with Interdependent Valuations: Efficiency." *Econometrica*, 72, 1617–1626.

Milgrom, P. (2004): *Putting Auction Theory to Work*, Cambridge University Press.

Milgrom, P. and I. Segal (2020): "Clock Auctions and Radio Spectrum Reallocation," *Journal of Political Economy*, 128, 1–31.

Myerson, R. (1979): "Incentive Compatibility and the Bargaining Problem," *Econometrica*, 47, 61–73.

Myerson, R. and M. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265–281.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Naor, M., B. Pinkas, and R. Sumner (1999): "Privacy Preserving Auctions and Mechanism Design," *Proceedings of the 1st ACM Conference on Electronic Commerce*, 129–139.

Perry, M. and P. J. Reny (2005): "An Efficient Multi-Unit Ascending Auction," *Review of Economic Studies*, 72, 567–592.

Peters, M. and S. Severinov (2006): "Internet Auctions with Many Traders," *Journal of Economic Theory*, 220–245.

Reny, P. J. and M. Perry (2006): "Toward a Strategic Foundation for Rational Expectations Equilibrium," *Econometrica*, 74, 1231–1269.

Rochet, J. (1985): "The Taxation Principle and Multi Time Hamilton-Jacobi Equations," *Journal of Mathematical Economics*, 14, 113–128.

Rustichini, A., M. A. Satterthwaite, and S. R. Williams (1994): "Convergence to Efficiency in a Simple Market with Incomplete Information," *Econometrica*, 62, 1041–1063.

Satterthwaite, M. and S. R. Williams (1989): "The Rate of Convergence to Efficiency in the Buyer's Bid Double Auction as the Market Becomes Large," *Review of Economic Studies*, 56, 477–489.

Satterthwaite, M. A. and S. R. Williams (2002): "The Optimality of a Simple Market Mechanism," *Econometrica*, 70, 1841–1863.

Segal, I. (2003): "Optimal Pricing Mechanisms with Unknown Demand," *American Economic Review*, 93, 509–529.

Segal-Halev, E., A. Hassidim, and Y. Aumann (2017): "MIDA: A Multi Item-type Double-Auction Mechanism," Working Paper, Bar-Ilan University

Shapley, L. and M. Shubik (1972): "The Assignment Game I: The Core," *International Journal of Game Theory*, 1, 111–130.

Sun, N. and Z. Yang (2009): "A Double-Track Adjustment Process for Discrete Markets with Substitutes and Complements," *Econometrica*, 77, 933–952.

——— (2014): "An Efficient and Incentive Compatible Dynamic Auction for Multiple Complements," *Journal of Political Economy*, 122, 422–466.

Tatur, T. (2005): "On the Trade off between Deficit and Inefficiency and the Double Auction with a Fixed Transaction Fee," *Econometrica*, 73, 517–570.

Vickrey, W. (1961): "Counterspeculation, Auction, and Competitive Sealed Tenders," *Journal of Finance*, 16, 8–37.

Walras, L. (1874): *'Eléments d'Économie Politique Pure; ou Théorie de la Richesse Sociale*, Guillaumin, Paris.

Wilson, R. (1987): "Game-Theoretic Analyses of Trading Processes," in *Advances in Economic Theory, Fifth World Congress*, ed. by T. Bewley, Cambridge and London: Cambridge University Press, 33–70.

Yoon, K. (2001): "The Modified Vickrey Double Auction." *Journal of Economic Theory*, 101, 572–584.

——— (2008): "The Participatory Vickrey-Clarke-Groves Mechanism." *Journal of Mathematical Economics*, 44, 324–336.