

Malthusian Stagnation is Efficient*

JUAN CARLOS CORDOBA[†]

XIYING LIU[‡]

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Abstract

This article studies socially optimal allocations, from the point of view of a benevolent social planner, in environments characterized by fixed resources, endogenous fertility, and full information. Individuals in our environment are fully rational and altruistic toward their descendants. Our model allows for rich heterogeneity of abilities, preferences for children, and costs of raising children. We show that the planner's optimal allocations are efficient in the sense of Golosov et al. (2007). We also show that efficient allocations in the endogenous fertility case differ significantly from its exogenous fertility counterpart. In particular, optimal steady-state population is proportional to the amount of fixed resources and the level of technology while steady-state individual consumption is independent of these variables, a sort of a "Malthusian stagnation" result. Furthermore, optimal allocations exhibit inequality, differential fertility, random consumption, and a higher population density of poorer individuals even when the planner is fully equalitarian, and faces no aggregate risk nor frictions.

Keywords: *P-efficient, A-efficient, efficient population, endogenous fertility, stochastic abilities, inequality.*

JEL Classification: *D04, D10, D63, D64, D80, D91, E10, E60, I30, J13, N00, 011, 040, Q01.*

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[†]Department of Economics, Iowa State University, 279 Heady Hall, Ames IA 50010, e-mail: cordoba@iastate.edu.

[‡]Economics and Management School, Wuhan University, Wuhan, Hubei, China, 430072, e-mail: xiyingliu@whu.edu.cn.

1 Introduction

There is growing interest in understanding the equilibrium and efficiency properties of economies characterized by endogenous fertility (e.g., Golosov et al., 2007; Conde-Ruiz et al., 2010; Hosseini et al., 2013; Schoonbroodt and Tertilt, 2014; Pérez-Nievas et al., 2019). As this literature makes clear, usual notions of efficiency may not apply when population is endogenous. This article contributes to this literature by studying in detail a case of major historical importance: the Malthusian case. In particular, we investigate the properties of socially optimal allocations, from the point of view of a benevolent social planner, in environments characterized by fixed resources and endogenous fertility.

Malthusian models have recently gained renewed interest as part of a larger literature seeking to provide a unified theory of economic growth, from prehistoric to modern times (e.g., Becker et al., 1990; Jones, 1999; Galor and Weil, 2000; Lucas, 2002; Hansen and Prescott, 2002; Doepke, 2004). The focus of this literature has been mostly positive rather than normative: to describe mechanisms for the stagnation of living standards even in the presence of technological progress. But the fundamental issue of efficiency in Malthusian economies, which is key to formulating policy recommendations and a better understanding of the extent to which the "Malthusian trap" could have been avoided, has received scarce attention. This paper seeks to fill this gap.

Our model economy is populated by a large number of finitely-lived fully rational individuals who are altruistic toward their descendants. Individuals are of different types and a type determines characteristics such as labor skills, rate of time preference, and ability to raise children. Types are stochastic and determined at birth. We formulate the problem faced by a utilitarian social planner who cares about the welfare of all *potential* individuals, present and future, as in Golosov et al. (2007). We call the solutions to the planner's problem optimal or efficient solutions.

The planner directly allocates consumption and number of children to individuals in all generations and states subject to aggregate resource constraints, promise-keeping constraints, population dynamics, and a fixed amount of natural resources. For short, we call "land" the fixed resource. The economy is closed, there is no capital accumulation nor migration. Furthermore, there are no underlying frictions, such as private information or moral hazard, so that the focus is on first-best allocations. The rich structure of the model allows us to study questions of aggregate efficiency as well as distributional issues such as optimal inequality, social mobility, and social classes. Distributional considerations can be particularly challenging. Lucas (2002) has shown that inequality is difficult to sustain as an equilibrium in Malthusian economies. In contrast, inequality and randomness arise naturally in our

environment even when the planner weighs everyone in a given generation equally.¹

The following are the main findings of this article. First, we show that the planner's optimal allocations are P-efficient in the sense of Golosov et al. (2007). P-efficiency is a natural extension of Pareto efficiency for the case of endogenous population. It requires to consider the utility of all potential individuals, not only born individuals. If the planner cares only about the initial living agents, the allocation is A-efficient, also in the sense of Golosov et al. (2007). We extend their results by introducing idiosyncratic shocks into the environment. Second, we find that stagnation of the Malthusian type is optimal. Specifically, optimal steady-state consumption is independent of the amount of land and, under general conditions, the level of technology. As a result, land discoveries, such as the ones discussed by Malthus, would lead to more steady-state population but no additional consumption. A similar prediction holds for technological progress as long as the production function is Cobb-Douglas or technological progress is land-augmenting, the type of progress that is more valuable because land is the limiting factor.

The source of the stagnation is a well-known prediction of endogenous fertility models, according to which optimal consumption is proportional to the net costs of raising a child. For example, Becker and Barro find that "*when people are more costly to produce, it is optimal to endow each person produced with a higher level of consumption. In effect, it pays to raise the 'utilization rate' (in the sense of a higher c) when costs of production of descendants are greater*" (Becker and Barro, 1988, pg.10)." We show that this link between optimal consumption and the net cost of raising children also holds for a benevolent planner and under more general conditions. The crux of the proof of stagnation is to show that neither land discoveries nor technological progress alter the steady-state net cost of raising a child. In particular, the steady-state marginal product of labor, which is needed to value both the parental time costs of children and children's marginal output, is unaffected.

Third, we show that efficient allocations exhibit social classes. Only types with the highest rate of time preference have positive population shares and consumption shares in steady-state. Furthermore, unlike the exogenous fertility case, it is generally not efficient to equalize consumption among types, even if their planner's weights are identical, nor to eliminate consumption risk. Efficient consumption is stochastic even in the absence of aggregate risk. These results are further implications of consumption being a function of the net cost of raising children. In an efficient allocation, poor individuals are the ones with the lowest net costs of raising

¹Consumption inequality arises naturally when Pareto weights are different and fertility is exogenous. What is new when fertility is endogenous is that: (i) unequal Pareto weights do not automatically lead to unequal consumption; (ii) Inequality can arise even if Pareto weights are equal and there are no frictions.

children.²

Fourth, there is an inverse relationship between consumption and population size: the lower the consumption of a type, the larger its population share. It is efficient to let individuals with lower net costs of having children reproduce more, but it is also optimal to endow their children with lower consumption – one that is proportional to those net costs. As a result, there are more poor individuals than rich individuals in an efficient allocation. Furthermore, population differences among types are larger than their corresponding consumption differences. The factor controlling the differences depends positively on the elasticity of parental altruism to the number of children, and negatively on the intergenerational elasticity of substitution.³

Fifth, fertility differs among types. Optimal fertility depends on parental types but also on grandparent types. Given grandparent's types, parents with particularly low costs of raising children would have more children than otherwise. Also, given parental types, grandparents with particularly high costs of raising children would have more grandchildren.

Sixth, steady-state allocations, and in particular the land-labor ratio, generally depends on initial the initial distribution of population and planner's weights. This is unlike the Neoclassical Growth model, which features an efficient capital-labor ratio, also called the modified golden rule level of capital, that is independent of initial conditions and planner's weights. Malthusian economies thus do not exhibit a clear separation between efficiency and distribution.

At the light of these results, efficient allocations could rationalize three key aspects of Malthusian economies: (i) stagnation of individual consumption in the presence of technological progress and/or improvements in the availability of land; (ii) social classes, inequality, and widespread poverty; (iii) differential fertilities. These results could also help explain why the so-called Malthusian trap was so pervasive in pre-industrial societies. Even in the best-case scenario of an economy populated by loving rational parents, and governed by an all-powerful benevolent rational planner, stagnation could still naturally arise, as well as social classes and differential fertility. Our results also suggest that it is not irrational animal spirits, as suggested by Malthus, which ultimately explains the stagnation. Stagnation can be the result of a social optimal choice between the quality and quantity of life in the presence of limited natural resources.

Our paper is related to Golosov et al. (2007), who have shown that population

²Córdoba and Liu (2014, 2016) and Córdoba et al. (2016) provide further characterizations of models with endogenous fertility and idiosyncratic risk.

³The intergenerational elasticity of substitution is analogous to the intertemporal elasticity of substitution but applied to different generations rather than different periods. See Córdoba and Ripoll (2019).

is efficient in dynastic altruistic models of endogenous fertility of the Barro-Becker type with fixed land. They do not derive results about stagnation, the distribution of consumption and population, nor differential fertility. Lucas (2002) studies market equilibrium in Malthusian economies populated by altruistic fully rational parents. His focus is on simple representative economies where fertility is equal across groups in the steady state. Lucas shows that stagnation arises under certain conditions and he discusses the difficulties in generating social classes. He can generate classes by assuming heterogeneity in the degree of time preference and binding saving constraints. As a result, the equilibrium with social classes is not efficient in his model. We can generate efficient social classes and differential fertility by allowing individuals to differ in their labor skills and costs of raising children.

Our paper also relates to Dasgupta (2005) who studies the optimal population in an endowment economy with fixed resources. He does not consider the cost of raising children and focuses on the special case of generation-relative utilitarianism. Our model is richer in production, altruism, and the technology of raising children. Nerlove et al. (1986) show that the population in the competitive equilibrium is efficient under two possible externalities. First, a larger population helps to provide more public goods such as national defense. Second, a larger population reduces wages if there is a fixed amount of land. Eckstein et al. (1988) show that population can stabilize and non-subsistence consumption arises in the equilibrium when fertility choices are endogenously introduced into a model with a fixed amount of land. Parents exhibit warm glow altruism while our paper builds on pure altruism. De la Croix (2013) studies sustainable population by proposing non-cooperative bargaining between clans living on an island with limited resources. Children in his model act like an investment good for parents' old-age support.

The rest of the paper is organized as follows. Section 2 illustrates the issues and mechanisms at work using a simple decentralized Barro-Becker model with fixed resources. Section 3 sets up the planner's problem. Section 4 focuses on the deterministic representative agent version of the planner's problem and derives the main stagnation results. Section 5 considers deterministic heterogeneity and derives key results regarding the distribution of population and consumption across types, as well as the importance of initial conditions for the steady state. Section 6 studies the stochastic case and derives the key result for differential fertility, consumption, and population. Section 7 decentralizes the planner's problem by a competitive market. Section 8 concludes. Proofs of the symmetric case are provided in the Appendix.

2 Barro-Becker With Fixed Resources

This section uses a simple market economy to derive a set of baseline results that arise when fertility is endogenous and some factors of production are in fixed supply. The market economy helps motivate the paper and better understand the underlying mechanisms. The remainder of the paper then focuses on social planner solutions rather than market economies.

The economy is populated by Barro-Becker families who have access to a Cobb-Douglas technology in land and labor. Land should be understood more generally as to include all resources in fixed supply.

Households. The economy under consideration has a mass 1 of agents at time 0. Let $i \in [0, 1]$ denote an individual at the beginning of time, who is also the head of dynasty i . Individuals are initially endowed with $k_{i0} (\geq 0)$ units of land. The aggregate amount of land is fixed and given by $K = \int_0^1 k_{i0} di$. Time is discrete: $t = 0, 1, 2, \dots$

Individuals live for two periods, one as children and one as adults. Children do not consume. A time- t adult from dynasty i consumes c_{it} and has n_{it} children. The number of children is a continuous variable taking value in the interval $[0, \bar{n}]$. Let N_{it} be the size of dynasty i at time t and $N_t = \int_0^1 N_{it} di$ be the total population at time t . N_{it} is defined as $N_{it} = \prod_{s=0}^{t-1} n_{is}$ for $i \in [0, 1]$ and $t = 1, 2, \dots$. Let r_t be the rental rate of land and q_t be its price. There are three costs of raising a child: a goods cost of η units per-child, a time cost of λ units of labor per child, and the cost of providing k_{it+1} units of land per child, $q_t k_{it+1}$. Adults are subject to a budget constraint of the form:

$$c_{it} + (\eta + q_t k_{it+1}) n_{it} \leq w_t (1 - \lambda n_{it}) + (r_t + q_t) k_{it} \quad \text{for } t \geq 0,$$

where w_t is the wage rate and $1 - \lambda n_{it}$ is the labor supply.

Parents are assumed to be altruistic toward their children. In particular, the lifetime utility of a time- t adult, U_{it} , takes the Barro-Becker form $U_{it} = c_{it}^\xi / \xi + \beta n_{it}^\psi U_{it+1}$, with $\xi \in (0, 1)$ and $\psi \in (\xi, 1)$. The dynastic problem can then be described as

$$\begin{aligned} & \max_{\{c_{it}, k_{it+1}, N_{it+1}\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t N_{it}^\psi \frac{c_{it}^\xi}{\xi} \\ & \text{s.t. } c_{it} + (\eta + w_t \lambda + q_t k_{it+1}) \frac{N_{it+1}}{N_{it}} \leq w_t + (r_t + q_t) k_{it} \quad \text{for } t \geq 0, \end{aligned}$$

given $k_{i0} > 0$.

Firms. Firms produce using the Cobb-Douglas technology $F(K, L; A) = AK^\alpha L^{1-\alpha}$, which is constant returns in land, K , and labor, L . Parameter A refers to the state of technology. Firms hire labor and rent land in competitive labor markets.

Resource constraints. The following are the economy-wide resource constraints for land, labor and production for $t \geq 0$:

$$\int_0^1 N_{it} k_{it} di \leq K; \quad L_t \leq \int_0^1 N_{it} (1 - \lambda n_{it}) di;$$

and

$$\int_0^1 N_{it} c_{it} di + \eta \int_0^1 N_{it+1} di \leq F(K_t, L_t, A).$$

Definition of equilibrium. Given an initial distribution of land and population, $\{k_{i0}, N_{i0}\}_{i \in [0,1]}$, a competitive equilibrium are sequences of prices, $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ and allocations $\{c_{it}^*, n_{it}^*, k_{it+1}^*, N_{it+1}^*\}_{t=0, i \in [0,1]}^{\infty}$ such that (i) given prices, the allocations solve the dynastic problem; (ii) land, labor, and good markets clear.

Equilibrium. Let $R_{t+1} \equiv (r_{t+1} + q_{t+1})/q_t$ be the gross return. The following four equations characterize the determination of land returns, consumption, fertility and wages for $t > 0$:⁴

$$\left(\frac{c_{it}^*}{c_{it+1}^*} \right)^{\xi-1} = \beta (n_{it}^*)^{\psi-1} R_{t+1} = \beta (n_t^*)^{\psi-1} R_{t+1} \quad \text{for } t \geq 0 \text{ and } i \in [0, 1], \quad (1)$$

$$c_{it+1}^* = c_{t+1}^* = \frac{\xi}{\psi - \xi} [R_{t+1} (\eta + w_t \lambda) - w_{t+1}] \quad \text{for } t \geq 0 \text{ and } i \in [0, 1], \quad (2)$$

$$c_{it}^* = A \left(\frac{K}{L_t^*} \right)^\alpha (1 - \lambda n_{it}^*) - \eta n_{it}^* \quad \text{for } t \geq 0, \quad (3)$$

$$w_t^* = (1 - \alpha) A \left(\frac{K}{L_t^*} \right)^\alpha \quad \text{for } t \geq 0. \quad (4)$$

The first equation is an intergenerational version of the Euler Equation with the special feature that the discount factor, $\beta n_{it}^{\psi-1}$, depends on the number of children. The second equation is derived by combining first-order conditions for fertility and land holdings plus the budget constraint. It states that consumption is equal for all individuals after period 0 and proportional to the net future value costs of raising a child: $R_{t+1} (\eta + w_t \lambda) - w_{t+1}$. Becker and Barro (1988) first derived this result for a representative agent economy while Bosi et al. (2011) extended it to a heterogeneous agent economy.

Equation (3) is the resource constraint in per-capita terms, while Equation (4) defines equilibrium wages. Plugging (2) into (1), it follows that fertility is the same for all individuals in a generation, that is $n_{it} = n_t$ for all $t > 0$. Thus, initial differences in land holding among individuals do not persist through differences in consumption or fertility of their descendants. Instead, they persist through differences in population sizes, N_{it}^* . This is in contrast with the exogenous population

⁴Section 7 provides the solution to a generalized version of this model.

version of the model in which any initial inequality in land holdings would translate into persistent consumption differences.⁵

Steady state. For concreteness we now focus on steady-state solutions. A steady-state population requires fertility to be 1. In that case, (1) reduces to the standard gross return determination, $\beta R^* = 1$, while using (3), equations (2) and (4) can be written as the following system of two equations in two unknowns, c^* and w^* :

$$c^* = \frac{\xi}{\psi - \xi} [\eta/\beta + (\lambda/\beta - 1) w^*], \text{ and} \quad (5)$$

$$w^* = \frac{1 - \alpha}{1 - \lambda} (c^* + \eta). \quad (6)$$

Surprisingly, neither A nor K appear in this system. Consequently, steady-state consumption and wages are independent of the technological level, A , or the fixed amount of resources, K , when fertility is endogenous. In contrast, both consumption and wages increase with A and K when fertility is exogenous (see Footnote 5).

Substituting (6) into (5) and solving for c^* results in:

$$c^* = c \left(\begin{matrix} \eta, \lambda, \beta, \psi, \xi \\ +, +, -, -, +, + \end{matrix} \right) = \frac{1/\beta - (1 - \lambda/\beta) \frac{1-\alpha}{1-\lambda}}{\frac{\psi-\xi}{\xi} + (1 - \lambda/\beta) \frac{1-\alpha}{1-\lambda}} \eta.$$

According to this equation, consumption is a positive function of the goods and time costs of raising children. Use equations (4) and (6) to find the following solution for population:

$$N^* = N \left(\begin{matrix} A, K, \eta, \lambda, \beta, \psi, \xi \\ +, +, -, -, +, +, - \end{matrix} \right) = \left[\frac{(1 - \lambda)^{1-\alpha} A}{c(\eta, \lambda, \beta, \psi, \xi) + \eta} \right]^{1/\alpha} K.$$

Steady-state population responds positively to technology and resource availability and negatively to consumption, and its underlying determinants.

To conclude this section, we highlight three qualitative differences between the endogenous and the exogenous fertility versions of the model. First, steady-state individual consumption is unaffected by technological progress or the discovery of new resources when fertility is endogenous while it fully responds when fertility is exogenous. Second, steady-state population fully responds to technological progress or discoveries of new resources when fertility is endogenous while, by assumption, population is unaffected in the exogenous fertility case. Third, any initial inequality in land holdings vanishes when population is endogenous while it perpetuates when population is exogenous. We now investigate the extent to which the predictions of the canonical model of this section hold more generally, particularly from a planner's perspective.

⁵If $n = 1$ is imposed then the equilibrium would satisfy $k_{it} = k_{i0}$, $c_{it} = c_i = w(1 - \lambda) + rk_{i0} - \eta$, $w = F_L(K, (1 - \lambda)N; A)$ and $r = F_K(K, (1 - \lambda)N; A)$ for all $t \geq 0$.

3 Efficient Allocations

This section considers social optima for more general specifications of preferences, technologies, and rich heterogeneity in earning abilities, ability to raise children and altruism toward children, as well as stochastic intergenerational transmission of abilities and altruism. We focus on planner's solutions rather than market equilibria for at least two reasons. First, characterizing social optima allocations is of interest on its own. One may suspect, for example, that some of the baseline results, such as consumption not responding to technological progress, may be due to some type of market failure. Second, planner's solutions are often simpler to obtain than market solutions, particularly when fertility is endogenous and parents are fully altruistic. In this case, Barro-Becker preferences provide a tractable benchmark but little is known for more general altruistic preferences.

We find that the first two features of the equilibrium solution are generally also properties of the planner's solution: consumption remains unresponsive to discoveries of new natural resources or to technological progress in the Cobb-Douglas case, or to land-augmenting technological progress for a more general formulation of the production function. Long-run equality, however, is not a robust feature. We are able to generate long-run inequality and social mobility in the stochastic version of the model. In this regard, our approach is very different from that of Lucas (2002). He is able to generate two social classes, with no possibility of social mobility, by assuming that the rate of time discount is a non-monotonic function of consumption. This non-standard feature is violated, for example, by the Barro-Becker model. We instead rely on stochastic types which result, even in the full information case, in inequality and social mobility.

3.1 Preliminaries

Consider an infinite horizon stochastic economy with fixed resources. Time is discrete: $t = 0, 1, 2, \dots$. Individuals live for two periods, one as a child and one as an adult, and are altruistic towards their children. Children do not consume. The economy is initially populated by a continuum of agents with identity $i_0 \in P_0 \equiv [0, 1]$. Each person can give birth to a maximum of \bar{n} children. The potential population at time $t \geq 0$ is defined as $P_t \equiv P_0 \times [0, \bar{n}]^t$. A potential individual in period t , a person who could be born in period t , is identified by $i^t = [i_0, i_1, \dots, i_t] \in P_t$.

At time t , each potential agent i^t has a type $\omega_t(i_t) \in \Omega \equiv \{\omega_1, \omega_2, \dots, \omega_K\}$. A type determines an agent's labor skills, rate of time preference, and ability to raise children if born. Specifically, assume that labor supply $l(\omega)$, parental time discounting $\beta(\omega)$, and the goods and time costs of raising a child, $\eta(\omega)$ and $\lambda(\omega)$, are functions of an individual's type. Let $\pi(\omega', \omega)$ denote the probability that a type

ω parent has type ω' child and assume $\pi(\omega', \omega) > 0$ for all $(\omega', \omega) \in \Omega$. Let Ω_0 be the set of types with positive mass at $t = 0$, and $\bar{N}_0(\omega^0) = N_0(\omega_0)$ be the initial mass of population of type $\omega_0 \in \Omega_0$. Define the set of possible types at time t recursively as $\Omega_t \equiv \Omega_{t-1} \times \Omega$. Associated with each potential agent i^t there is a history of shocks of that agent's ancestors, including: $\omega^t(i^t) = [\omega_0(i_0), \omega_1(i_1), \dots, \omega_t(i_t)] \in \Omega_t$. Assuming that a law of large numbers holds, the potential population with family history $\omega^{t+1} \in \Omega_{t+1}$ is given by

$$\bar{N}_{t+1}(\omega^{t+1}) = \pi(\omega_{t+1}, \omega_t) \bar{n} \bar{N}_t(\omega^t) \text{ for } t > 0.$$

where \bar{n} is the maximum number of children possible. Notice that the number of potential individuals is completely exogenous. Moreover, the assumption that $\pi(\omega', \omega) > 0$ guarantees that $\bar{N}_{t+1}(\omega^{t+1})$ is strictly positive for all $\omega^{t+1} \in \Omega_{t+1}$.

A fertility plan, denoted by $\hat{\mathbf{n}} = \left\{ \{n_t(i^t, \omega^t)\}_{\omega^t \in \Omega_t}^{i^t \in P_t} \right\}_{t=0}^{\infty}$, is a description of the number of children born to potential agent i^t , for each possible history of shocks ω^t . Thus, $0 \leq n_t(i^t, \omega^t) \leq \bar{n}$ for all $i^t \in P_t$ and $\omega^t \in \Omega_t$. Each fertility plan implicitly defines a subset, $I_t(n) \subset P_t$, of individuals actually born under the plan n .⁶ An allocation is a fertility plan, $\hat{\mathbf{n}}$, and a consumption plan $\hat{\mathbf{c}} = \left\{ \{c_t(i^t, \omega^t)\}_{\omega^t \in \Omega_t}^{i^t \in I_t(n)} \right\}_{t=0}^{\infty}$. A *symmetric* allocation is an allocation such that for all $i^t \in I_t(n)$, $\omega^t \in \Omega_t$ and $t \geq 0$, $c_t(i^t, \omega^t) = c_t(\omega^t)$, $n_t(i^t, \omega^t) = n_t(\omega^t)$. A symmetric allocation thus provides the same consumption and fertility to born individuals with the same history regardless of their identity i^t . For simplicity we consider only symmetric allocations from now on, as in Phelan and Rustichini (2018). Let $(\mathbf{c}, \mathbf{n}) = \left\{ \{c_t(\omega^t), n_t(\omega^t)\}_{\omega^t \in \Omega_t} \right\}_{t=0}^{\infty}$ denote symmetric allocations for born individuals.⁷

3.2 Resource constraints

The production technology is described by the constant returns to scale function $F(K, L; A)$ with respect to K and L where K is a fixed amount of land, L is labor, and A is a technological parameter. Suppose F is constant returns to scale in K and L . Let $\alpha\left(\frac{K}{L}, A\right) \equiv F_K(K, L; A)K/F(K, L; A)$ denote the land share of output. Aggregate labor supply satisfies

$$L_t = \sum_{\omega^t \in \Omega_t} N_t(\omega^t) l(\omega_t) [1 - \lambda_t(\omega_t) n_t(\omega^t)] \text{ for } t \geq 0, \quad (7)$$

⁶More precisely, let the set $I_t(n)$ be defined recursively by $i_0 \in I_0(n)$ for $i_0 \in P_0$; then $(i_0, i_1) \in I_1(n)$ for $i_0 \in P_0$ and $\omega_0 \in \Omega_0$ if and only if $0 \leq i_1 \leq n_0(i_0, \omega_0)$; and so forth. In particular, $i^{t+1} \in I_{t+1}(n)$ if $0 \leq i_{t+1} \leq n_t(i^t, \omega^t)$ for $i^t \in I_t(n)$ and $\omega^t \in \Omega_t$.

⁷We can show that in a full model, one that does not impose symmetry, optimal allocations are symmetric if the planner's weights are symmetric, parents care about all their born children equally, and optimal solutions are interior. Proof is available upon request.

where $N_t(\omega^t)$ is the *born* population, or just population, with family history ω^t , $l(\omega_t)$ is the labor supply of an individual of type ω_t , $\lambda_t(\omega_t)$ is the cost of raising a child and $l(\omega_t)[1 - \lambda_t(\omega_t)n_t(\omega^t)]$ is the effective labor supply. Population evolves according to:

$$N_{t+1}(\omega^{t+1}) = N_t(\omega^t) n_t(\omega^t) \pi(\omega_{t+1}, \omega_t) \text{ for } t \geq 0 \text{ and } \omega^{t+1} \in \Omega_{t+1}, \quad (8)$$

given $N_0(\omega^0)$ for $\omega^0 = \omega_0 \in \Omega_0$. Let $N_t \equiv \sum_{\omega^t} N_t(\omega^t)$ be the total population at time t . The aggregate resource constraint at time t is then described by

$$F(K, L_t; A) = \sum_{\omega^t \in \Omega_t} N_t(\omega^t) [c_t(\omega^t) + \eta(\omega_t) n_t(\omega^t)] \text{ for } t \geq 0. \quad (9)$$

Definition. A (symmetric) allocation (\mathbf{n}, \mathbf{c}) is feasible if it satisfies (7), (8) and (9) given $[N_0(\omega)]_{\omega \in \Omega_0}$.

3.3 Individual welfare

Parents are assumed to be altruistic toward their children. The lifetime utility of an individual born at time $t \geq 0$ and history ω^t , $U_t(\omega^t)$, is of the expected-utility form:

$$U_t(\omega^t) = u(c_t(\omega^t)) + \beta(\omega_t) \Phi(n_t(\omega^t)) E[U_{t+1}(\omega^{t+1}) | \omega^t] + \beta(\omega_t) (\Phi(\bar{n}) - \Phi(n_t(\omega^t))) \underline{U}, \quad (10)$$

where $u(\cdot)$ is the utility flow from consumption, $\beta(\omega)$ is a time discounting factor, $\Phi(n)$ is the weight that a parent attaches to the welfare of her n born children, $\Phi(\bar{n}) - \Phi(n)$ is the weight attached to the unborn children, $E[U_{t+1}(\omega^{t+1}) | \omega^t]$ is the expected utility of a born child conditional on parental history and \underline{U} is the utility of an unborn child.⁸ Function u satisfies $u' > 0$ and $u'' < 0$.

Equation (10) describes parents as social planners at the family level. This is particularly clear in the special case $\Phi(n) = n$. The more general function $\Phi(\cdot)$ allows for diminishing utility from children. While $\Phi(n)$ is the total weight of the n born children, $\Phi_n(n)$ is the marginal weight assigned to the n -child where $n \in [0, \bar{n}]$. We assume $\Phi_n(n) > 0$ and $\Phi_{nn}(n) \leq 0$ so that parents are altruistic toward each child and marginal altruism is non-increasing. These preferences are discussed in Córdoba and Ripoll (2011), in which they show that (10) satisfies a fundamental axiom of altruism. Specifically, parental utility increases with the number of born children if and only if children are better off born than unborn in expected value, that is, $E[U_{t+1}(\omega^{t+1}) | \omega^t] > \underline{U}$. Let $\bar{U}_t(\omega^t)$ be the utility of a potential agent with

⁸The population ethics literature refers to \underline{U} as the "neutral" utility level, a level above which a life is worth living (Blackorby and Donaldson, 1984, pg.21).

history ω^t . In particular $\bar{U}_t(\omega^t) = U_t(\omega^t)$ if the individual is born and $U_t(\omega^t) = \underline{U}$ if unborn.

Let $\xi(c) \equiv u'(c)c/u(c)$ be the elasticity of the utility flow and $\psi(n) \equiv \Phi'(n)n/\Phi(n)$ be the elasticity of the altruistic function. Barro-Becker preferences are an special case obtained when $u(c) = c^\xi/\xi$, $\beta(\omega) = \beta$, $\Phi(n) = n^\psi$, $\underline{U} = 0$, $\xi \in (0, 1)$ and $\psi \in (\xi, 1)$.

3.4 Social welfare

We consider a social planner that cares about the welfare of all *potential* individuals. In particular, social welfare takes the utilitarian form:

$$SW_1 = \sum_{t=0}^{\infty} \sum_{\omega^t \in \Omega_t} \varphi_t(\omega^t) \bar{N}_t(\omega^t) \bar{U}_t(\omega^t) \quad (11)$$

$\varphi_t(\omega^t)$ is the weight that the planner puts on an individual with history ω^t . It could vary among different types, and it includes time discounting by the planner. When all potential individuals of the same generation are equally weighted, then the weight only represent time discounting, e.g. $\varphi_t(\omega^t) = \delta^t$. Here we focus on the symmetric weight of the planner on people of the same history ω^t . Notice that the planner faces no uncertainty since a law of large numbers is assumed so that all risk is idiosyncratic.

Alternatively we consider a social planner who cares only about the initial generation as well as future generations but only to the extent that the initial generation does. In that case, the planner's objective function becomes

$$SW_2 = \sum_{\omega \in \Omega_0} \varphi_0(\omega) N_0(\omega) U_0(\omega) \quad (12)$$

Different from SW_2 , SW_1 refers to a planner who is more patient than individuals, as in Farhi and Werning (2007). We assume throughout that parameter values are such that social welfare is bounded.

The following assumption bounds the extent to which the planner cares about future generations.

Assumption 1. $\lim_{t \rightarrow \infty} \frac{\varphi_t(\omega^t)}{\prod_{j=0}^{t-1} \beta(\omega_j)} = 0$ for all $\omega^t \in \Omega_t$.

The role of Assumption 1 is tractability. The assumption is not particularly restrictive because it still allows for the planner to care about future generations more than parents do. To see this, consider for a moment the case $\varphi_t(\omega^t) = \delta^t$, $\beta(\omega) = \beta$. In that case, Assumption 1 is satisfied when $\delta < \beta$. In the Appendix we analyze the case $\delta \geq \beta$, which is less tractable but we can still prove that Malthusian stagnation holds in that case (see Section A.2).

Definition Planner's problem. Given an initial distribution of population $\{N_0(\omega)\}_{\omega \in \Omega}$, the planner's problem is to choose a feasible allocation (\mathbf{n}, \mathbf{c}) that maximizes social welfare defined either by (11) or (12). An efficient allocation is one that maximizes social welfare.

The following definitions of P and A efficiency extends those of Golosov et al. (2007) and Pérez-Nievas et al. (2019) to our stochastic environment.⁹

Definition A feasible allocation $(\mathbf{n}^*, \mathbf{c}^*)$ is *P*-efficient if there is no other feasible allocation $(\hat{\mathbf{n}}, \hat{\mathbf{c}})$ such that (i) $\bar{U}_t^{\hat{}}(\omega^t) \geq \bar{U}_t^*(\omega^t)$ for all $\omega^t \in \Omega_t$, and $t \geq 0$ where $\bar{U}_t^*(\omega^t)$ and $\bar{U}_t^{\hat{}}(\omega^t)$ are the utilities of the individual with history ω^t under the allocations (n^*, c^*) and (\hat{n}, \hat{c}) , respectively; and (ii) $\bar{U}_t^{\hat{}}(\omega^t) > \bar{U}_t^*(\omega^t)$ for at least one $\omega^t \in \Omega_t$.

Definition A feasible allocation (n^*, c^*) is *A*-efficient if there is no other feasible allocation (\hat{n}, \hat{c}) such that (i) $\bar{U}_0^{\hat{}}(\omega_0) \geq \bar{U}_0^*(\omega_0)$ for all $\omega_0 \in \Omega_0$ where $\bar{U}_0^*(\omega_0)$ and $\bar{U}_0^{\hat{}}(\omega_0)$ are the utilities of the initial generation of type ω_0 under the allocation $(\mathbf{n}^*, \mathbf{c}^*)$ and $(\hat{\mathbf{n}}, \hat{\mathbf{c}})$, respectively; (ii) $\bar{U}_0^{\hat{}}(\omega_0) > \bar{U}_0^*(\omega_0)$ for at least one $\omega_0 \in \Omega_0$.¹⁰

Next, we show that a solution to the planner's problem SW_1 is *P*-efficient under mild conditions. Similarly, a solution to the planner's problem SW_2 is both *P*- and *A*-efficient.

Proposition 1 (i) A solution to the planner's problem SW_1 is *P*-efficient if $\varphi_t(\omega^t) \geq 0$ for all $\omega^t \in \Omega_t$, $t > 0$. (ii) A solution to the planner's problem SW_2 is *P*-efficient if the solution is unique.

Proposition 2 A solution to the planner's problem SW_2 is *A*-efficient if $\varphi_0(\omega_0) > 0$ for all $\omega_0 \in \Omega_0$ or if the solution is unique.

⁹Golosov et al. (2007) define A and P efficiency for deterministic environments but do not restrict allocations to be symmetric, as we do. Schoonbroodt and Tertilt (2014) consider the possibility of asymmetric allocations in a related deterministic market environment. They find that symmetric allocations are A and P efficient absent market failures (see their Proposition 1). Their results support our focus on symmetric allocations since our environment is frictionless. Asymmetric allocations could improve upon symmetric ones when market failures bind, as they show. See Footnote 7 above for further discussion of the asymmetric case.

¹⁰Golosov et al. (2007) define A-efficiency more generally, by comparing the welfare profiles of agents *alive* in alternative allocations. They show that A-efficiency includes more than dynastic maximization. But Pérez-Nievas et al. (2019) show that "If potential agents are identified by the dates in which they may be born, then A-efficiency reduces to dynastic maximization." (Abstract). Golosov et al. (2007) assume instead that potential agents are identified by their position in their sibling's birth order. Our definition of A-efficiency therefore assumes that potential agents are identified by the birth date criterion.

Proposition 1, and its proof, is analogous to Result 1 and Result 2 in Golosov et al. (2007) but our set up is different in that it allows for randomness and restrict feasible allocations to be symmetric. Under SW_1 , the Proposition does not require uniqueness, as in their Result 1, because the weights of all potential individuals are assumed to be strictly positive. Similarly to their Result 2, using SW_2 requires to assume uniqueness given that planner's weights of future generations are assumed to be zero.

We now proceed to further characterize the optimal solution. We assume parameter values are such that the solution is unique and interior.¹¹ Furthermore, we normalize the utility of the unborn to be zero: $\underline{U} = 0$. In that case social welfare reduces to $\sum_{t=0} \sum_{\omega^t \in \Omega_t} \varphi_t(\omega^t) N_t(\omega^t) U_t(\omega^t)$. Let κ_t , $\gamma_{t+1}(\omega^{t+1})$, μ_t and $\theta_t(\omega^t) N_t(\omega^t)$ be non-negative Lagrangian multipliers associated to restrictions (7), (8), (9) and (10). The first order conditions with respect to $\{U_t(\omega^t), N_{t+1}(\omega^{t+1}), n_t(\omega^t), c_t(\omega^t), L_t\}_{\omega^t \in \Omega_t, t \geq 0}$ are:¹²

$$\theta_0(\omega_0) = \varphi_0(\omega_0), \quad (13)$$

$$\begin{aligned} & \theta_{t+1}(\omega^{t+1}) N_{t+1}(\omega^{t+1}) \\ & = \theta_t(\omega^t) \beta(\omega_t) \Phi(n_t(\omega^t)) N_t(\omega^t) \pi(\omega_{t+1}, \omega_t) + \varphi_{t+1}(\omega^{t+1}) N_{t+1}(\omega^{t+1}), \end{aligned} \quad (14)$$

$$\begin{aligned} & \varphi_{t+1}(\omega^{t+1}) U_{t+1}(\omega^{t+1}) + \gamma_{t+1}(\omega^{t+1}) \\ & + \kappa_{t+1} l(\omega_{t+1}) [1 - \lambda_{t+1}(\omega_{t+1}) n_{t+1}(\omega^{t+1})] \\ & = \mu_{t+1} [c_{t+1}(\omega^{t+1}) + \eta(\omega_{t+1}) n_{t+1}(\omega^{t+1})] \\ & + n_{t+1}(\omega^{t+1}) \sum_{\omega^{t+2} | \omega^{t+1}} \gamma_{t+2}(\omega^{t+2}) \pi(\omega^{t+2}, \omega^{t+1}), \end{aligned} \quad (15)$$

$$\begin{aligned} & \theta_t(\omega^t) \beta(\omega_t) \Phi_n(n_t(\omega^t)) E_t U_{t+1}(\omega^{t+1}) \\ & = \mu_t \eta(\omega_t) + \kappa_t l(\omega_t) \lambda_t(\omega_t) + \sum_{\omega^{t+1} | \omega^t} \gamma_{t+1}(\omega^{t+1}) \pi(\omega^{t+1}, \omega^t), \end{aligned} \quad (16)$$

$$\theta_t(\omega^t) u'(c_t(\omega^t)) = \mu_t, \text{ and} \quad (17)$$

$$\mu_t F_{L,t} = \kappa_t, \quad (18)$$

where

$$F_{L,t} \equiv \left(1 - \alpha \left(\frac{K}{L_t}, A \right) \right) \frac{F(K, L_t; A)}{L_t}. \quad (19)$$

¹¹For example, the Barro-Becker formulation possesses an interior solution under certain parameter restrictions.

¹²To avoid cumbersome notation, we do not introduce new notation to identify optimal allocations. Allocations should be regarded as optimal from now on. The Lagrangian is written in the Proof of Proposition 3.

This system of equations together with (7), (8), (9), (10) and proper transversality conditions fully describe optimal allocations. Equation (13) states that the social value of providing additional utility to an individual of type ω_0 is just the Pareto weight of that individual, $\varphi_0(\omega_0)$. Equation (14) then allows to trace the evolution of $\theta_t(\omega^t)$ which is in fact the effective Pareto weight of an individual in state ω^t . The left hand side of the equation is the marginal cost of providing utility $U_{t+1}(\omega^{t+1})$ while the right hand side is its marginal benefit. It includes a benefit to the altruistic parents plus a direct benefit to the planner, $\varphi_{t+1}(\omega^{t+1})$.

Equation (15) equates marginal benefits to marginal costs of more population. To better understand this expression, suppose for a moment that population is not constrained by (8), for example, because the planner has access to an infinite pool of immigrants. In that case $\gamma_{t+1}(\omega^{t+1}) = 0$ for all t and ω^{t+1} . The marginal benefit of an additional individual of type ω^{t+1} includes her direct effect in social welfare, $\varphi_{t+1}(\omega^{t+1}) U_{t+1}(\omega^{t+1})$ plus her effect in the labor supply, $\kappa_{t+1} l(\omega_{t+1}) [1 - \lambda_{t+1}(\omega_{t+1}) n_{t+1}(\omega^{t+1})]$, while the marginal cost includes the costs of providing consumption and children to the individual, $\mu_{t+1} [c_{t+1}(\omega^{t+1}) + \eta(\omega_{t+1}) n_{t+1}(\omega^{t+1})]$. Adding restriction (8) makes the individual more valuable in the amount $\gamma_{t+1}(\omega^{t+1})$ because it relaxes the population constraint at $t + 1$, but also increases marginal cost because the planner needs to endow the individual with children at $t + 2$.

The condition for optimal fertility is (16). The marginal benefit of a child for an altruistic parent with history ω^t is the expected utility of the child, $E_t U_{t+1}$, times the weight that the parent attaches to the child, $\beta(\omega_t) \Phi_n(n_t(\omega^t))$. The marginal benefit for the planner is this amount times $\theta_t(\omega^t)$, expressed in consumption goods. The corresponding marginal cost of the child for the planner includes good costs, $\mu_t \eta(\omega_t)$, time costs, $\kappa_t l(\omega_t) \lambda(\omega_t)$, and the expected shadow costs of a descendant, $\sum_{\omega^{t+1}|\omega^t} \gamma_{t+1}(\omega^{t+1}) \pi(\omega_{t+1}, \omega_t)$.

To characterize the solution of this system we focus primarily on the steady state and proceed in three steps.¹³ First, we characterize the deterministic case with only one type (Section 4), then the case with multiple but deterministic types (Section 5) and finally the stationary solution with stochastic types. We show that Malthusian stagnation generally arises when technological progress is of the land-augmenting type meaning that steady-state optimal consumption and fertility choices are independent of K and A . We also characterize the optimal composition of population, the potential dependence of the steady-state land-labor ratio on initial conditions, and fertility differentials among types.

¹³Section A.2.2. in the Appendix A provides necessary and sufficient conditions for stability for the deterministic case with $\delta = 0$. As shown below, the steady state of the model with $\delta < \beta$ is similar to that of $\delta = 0$.

4 Deterministic Case With One Type

This section considers the representative agent case with only one type. Assume $\varphi_t(\omega) = \varphi_t$ for all $\omega \in \Omega$. Let $n(\omega) = n$, $\lambda(\omega) = \lambda$, $\eta(\omega) = \eta$, $\beta(\omega) = \beta$ and $l(\omega) = 1$ for simplicity. In this case the resource constraint (9) reduces to:

$$F\left(\frac{K}{N_t}, 1 - \lambda n_t; A\right) = c_t + \eta n_t. \quad (20)$$

Moreover, using (17), (18), and (20), equations (13) to (16) simplify to

$$\theta_0 = \varphi_0,$$

$$\varphi_{t+1} + \theta_t \beta \frac{\Phi(n_t)}{n_t} = \theta_{t+1}, \quad (21)$$

$$\varphi_{t+1} U_{t+1} + \gamma_{t+1} = \mu_{t+1} F_{K,t+1} \frac{K}{N_{t+1}} + n_{t+1} \gamma_{t+2}, \quad (22)$$

and

$$\beta \Phi'(n_t) \frac{U_{t+1}}{u'(c_t)} = \eta + F_{L,t} \lambda + \frac{\gamma_{t+1}}{\gamma_t} \frac{\gamma_t}{\mu_t}. \quad (23)$$

Equation (22) is obtained from (15) after using (18), (20) and the constant returns to scale assumption. Equation (23) is obtained from (16), (17), and (18). The following proposition provides a sharp characterization of consumption for all periods, except period 0, for the special case $\Phi(n) = n^\psi$.

Proposition 3 *Assume $\Phi(n) = n^\psi$, $0 < \psi < 1$, and let $a_m \equiv \varphi_m \beta^{-m} N_m^{1-\psi}$ if $\varphi_m > 0$ or $a_m = 0$ otherwise. Then efficient consumption satisfies:*

$$c_{t+1} = \frac{\xi(c_{t+1})}{\psi - \xi(c_{t+1})} \left[\frac{\mu_t}{\mu_{t+1}} (\eta + F_{L,t} \lambda) - F_{L,t+1} - \frac{a_{t+1}}{\sum_{m=0}^{t+1} a_m} \frac{U_{t+1}}{u'(c_{t+1})} (1 - \psi) \right] \quad (24)$$

This expression is similar, but generalizes equation (2). The term $(\mu_t/\mu_{t+1})(\eta + F_{L,t}\lambda) - F_{L,t+1}$ is the net future cost of raising a child from the planner's perspective. The main difference is the last term in the brackets. This term equals zero when $t \rightarrow \infty$ since $\lim_{t \rightarrow \infty} \varphi_t \beta^{-t} = 0$. In essence, when planner's preferences differ from those of the individuals, because $\varphi_t > 0$ for $t > 0$, then consumption is adjusted downward to free resources in order to expand population. But those adjustments are temporary given that the limit of $\varphi_t \beta^{-t}$ goes to zero is assumed. In the limit, consumption becomes a sole function of the net cost of raising a child.

4.1 Steady state

Consider the steady-state situation in which N , c , U , and L are constant and $n = 1$. The following result holds for general functions F , u and Φ .

Lemma 4 *Steady state consumption satisfies:*

$$c = \frac{\xi(c)}{\psi(1) - \xi(c)} [\eta/\beta + (\lambda/\beta - 1) F_L]. \quad (25)$$

This expression generalizes (5). It shows that consumption is a function of the net costs of raising a child: $(\eta + \lambda F_L)/\beta - F_L$. The parametric restriction $\psi(1) > \xi(c)$ is needed for consumption to be positive. An implication of this result is that immiseration and the Repugnant Conclusion, of $c = 0$ and $N = \infty$, is not optimal unless the net cost of children is zero. Although infinite population is a possibility in P-efficient allocations, this is not the case in our model. Population can only be gradually built up from fertility over time. An arbitrarily large population is avoided by introducing enough time discounting. This is similar to having bounded solutions in the Ramsey model. Besides the time discounting, models with endogenous fertility also require parameters such that children are a net financial costs to keep fertility interior and prevent population from going to infinity.

The resource constraint, equation (20), can be written as:

$$F_L = \frac{1 - \alpha(k; A)}{1 - \lambda} (c + \eta). \quad (26)$$

where $k = K/L$. Equations (25) and (26) form a system of two equations in three unknowns: c , F_L and α . In the case of a Cobb-Douglas production function the land share, α , is a parameter and these two equations can be used to solve for consumption, c , and the marginal product of labor, F_L , independently of K and A , as in Section 2.

In the more general case, equation (19) is needed to close the system. It can be written as:

$$F_L = (1 - \alpha(k; A)) F(k, 1; A). \quad (27)$$

Equations (25), (26) and (27) form a system of three equations in three unknowns: c , F_L and k . Since K does not appear in this system the solutions for c , F_L and k are independent of the amount of land for any constant returns to scale production function. Given k , steady-state population can be solved as $N = K/((1 - \lambda)k)$.

Finally, if A is a land-augmenting parameter then the land share is a function of $\hat{k} = AK/L$. In that case equations (26) and (27) can be written as $F_L = \left[\left(1 - \alpha(\hat{k})\right) / (1 - \lambda) \right] (c + \eta)$ and $F_L = \left(1 - \alpha(\hat{k})\right) F(\hat{k}, 1)$, and the solutions for c , F_L and \hat{k} will be independent of A . Given \hat{k} , steady-state population is given by $N = AK / \left((1 - \lambda) \hat{k} \right)$. The following proposition summarizes these findings.

Proposition 5 *In steady state, efficient consumption is independent of the amount of land while efficient population increases proportionally with the amount of land. Furthermore, if technological progress is land augmenting then efficient consumption is independent of the level of technology while efficient population increases proportionally with the level of technology.*

5 Deterministic Case With Multiple Types

Consider now the case of multiple deterministic types. Specifically, suppose $\omega^t = [\omega, \omega, \omega \dots]$ or just $\omega^t = \omega$ for short. We assume in this section $\varphi_{t+1}(\omega) = 0$ for all $t \geq 0$ and $\omega \in \Omega$. This restriction is without much loss of generality since similar steady state results would be obtained as long as $\lim_{t \rightarrow \infty} \varphi_t(\omega) \beta^{-t} = 0$ for all $\omega \in \Omega$, as shown in the previous section. For tractability we also restrict altruism to be of the Barro-Becker form, $\Phi(n) = n^\psi$, but allow general functional forms for u and F .

We show the following results in this section. First, if $\beta(\omega)$ is different for different types then their population sizes grow at different rates and in steady state only the most patient groups, the ones with highest $\beta(\omega)$, survive. This result implies that efficient social classes cannot be sustained by persistent differences in rates of time preference. Lucas (2002) is able to generate social classes using such a mechanism in a competitive equilibrium with savings constraints which suggests that social classes are not efficient, in the first best sense, in his model.

As an alternative to Lucas (2002), we are able to generate multiple social classes using a more standard mechanism based on heterogeneity in labor skills, $l(\omega)$, and the cost of raising children, $\eta(\omega)$ and $\lambda(\omega)$. This is the second main result of the section. Efficiency requires providing more consumption to individuals with higher costs of raising them. Consumption also increases with labor ability, $l(\omega)$, but only if $\lambda(\omega) > \beta(\omega)$, that is, only if the time costs of raising children are sufficiently high. Otherwise, the efficient allocation involves the high skilled having lower consumption.

Third, we show that the relative population size of a type is inversely related to its relative consumption. Therefore, the population of the poor is larger than the population of the middle class and so on. The planner thus faces a quantity-quality trade-off: she can deliver a certain level of welfare by allocating number of children and/or consumption. If children are particularly costly to raise for a certain group, then the planner optimally delivers welfare more through consumption than through number of children and vice versa.

Fourth, in the deterministic steady state of this section all types have one child and therefore steady-state welfare differences among types only arise from differences in consumption. As a result, types with lower consumption are worse-off than types with higher consumption. All benefits from a larger population accrue only to early members of the dynasty at the expense of later members.

5.1 Dynamics

The following lemma characterizes the evolution of efficient population sizes of different types over time.

Lemma 6 *Let $(\omega, \omega') \in \Omega$. Efficient population sizes satisfy:*

$$\frac{N_t(\omega)}{N_t(\omega')} = \left(\frac{\beta(\omega)}{\beta(\omega')} \right)^{\frac{t}{1-\psi}} \frac{N_0(\omega)}{N_0(\omega')} \left[\frac{u'(c_t(\omega))\varphi_0(\omega)}{u'(c_t(\omega'))\varphi_0(\omega')} \right]^{\frac{1}{1-\psi}}. \quad (28)$$

Lemma 6 provides a partial characterization of the population dynamics of different groups. There are four elements determining relative population sizes. First, more patient types systematically increase their population share relative to less patient types. At $t \rightarrow \infty$ impatient types tend to eventually disappear from the economy. This is because altruistic parents regard children as added longevity, as a way to increase the stream of future utility flows. Impatient individuals discount future streams more heavily and therefore value children less than patient individuals do. As a result, it is efficient for the planner to provide more consumption to impatient individuals in exchange for fewer future family members.¹⁴ Second, the initial distribution of population has a long-lasting impact, one that never dies out since the exponent on $N_0(\omega)/N_0(\omega')$ is equal to 1. Thus, for example, if the initial population is composed mostly of high skilled individuals then this composition persists over time. Third, types assigned with higher Pareto weights by the planner have more population. Finally, types with lower consumption have a higher population share. We characterize the steady state next.

5.2 Steady state

5.2.1 Population distribution

We restrict attention to the set of most patient types, $\Omega_p \subseteq \Omega$, which are the only ones with positive population mass in a steady state. Let $\beta(\omega) = \beta$ for all $\omega \in \Omega_p$ and $\beta \geq \beta(\omega)$ for all $\omega \in \Omega$. Furthermore, consider a steady state in which consumption, population shares and population are constant. This requires $n(\omega) = 1$ for all types. In that case, equation (28) simplifies to:

$$\frac{N(\omega)}{N(\omega')} = \frac{N_0(\omega)}{N_0(\omega')} \left(\frac{u'(c(\omega))\varphi_0(\omega)}{u'(c(\omega'))\varphi_0(\omega')} \right)^{\frac{1}{1-\psi}}, \quad \omega \in \Omega_p. \quad (29)$$

We can now state our next main result which is apparent from this equation and the previous discussion.

Proposition 7 *In an interior steady state: (i) Only the most patient types, the ones with the highest $\beta(\omega)$, have positive mass; (ii) The distribution of population depends*

¹⁴This result also helps qualify a commonly held view according to which the poor are inherently more impatient, less willing to save, and that their large families somehow reflect their impatience. According to our model, if the poor were really impatient, they would have fewer children and their type would eventually disappear from the population.

on the initial distribution and the weights on initial generation; (iii) The relative population size of a particular type is inversely related to its per-capita consumption.

The second part of Proposition 7 states that the steady-state composition of the population depends on the initial composition, a result that is analogous to the dependence of the steady-state wealth distribution on initial conditions in the neoclassical growth model (Chatterjee, 1994). But as shown below, in Proposition 12, this dependence has an important added implication in Malthusian economics because the steady-state aggregate land-labor ratio and steady-state level of population itself depends on initial conditions. This is unlike the Neoclassical Growth model where the golden rule level of capital is independent of initial conditions. Efficiency and distribution are interdependent in Malthusian economies. Notice that the steady-state composition depends but never resembles the initial composition unless planner's weights and consumptions are equal across types, which is not the case in general, as we show below even if planner's weights are the same.¹⁵

The third part of Proposition 7 shows a fundamental prediction of endogenous population models: an inverse relationship between population size and per-capita consumption. The lower the consumption of a type the larger its share of the total population. The reason is that the planner needs to deliver welfare by providing consumption and children to parents. Whenever the planner chooses to use one channel, it downplays the other. We still need to solve for consumption to fully derive the consequences of this inverse relationship.

The following lemma characterizes the steady-state distribution of population in terms of consumptions.

Lemma 8 *Let $g(\omega) \equiv \frac{N(\omega)}{N}$. Then*

$$g(\omega) = \frac{N_0(\omega) [u'(c(\omega))\varphi_0(\omega)]^{1/(1-\psi)}}{\sum_{\omega' \in \Omega_p} N_0(\omega') [u'(c(\omega'))\varphi_0(\omega')]^{1/(1-\psi)}} \quad \text{for all } \omega \in \Omega_p. \quad (30)$$

This lemma provides a simple description of the steady-state distribution of population in terms of the given initial distribution and steady-state consumptions.

5.2.2 Consumption

Similarly to population dynamics, one can show that only the most patient types have positive consumption in a steady state. According to (17), for consumption to be constant $\theta_{t+1}(\omega)/\theta_t(\omega) = \mu_{t+1}/\mu_t$ is required. Otherwise, $\theta_{t+1}(\omega)/\theta_t(\omega) < \mu_{t+1}/\mu_t$ refers to a type for which consumption falls, and vice versa. Therefore,

¹⁵Efficient allocations are not time consistent because re-optimizing starting with an initial steady state distribution of population results in a different steady state distribution.

only the types with the highest ratio $\theta_{t+1}(\omega)/\theta_t(\omega)$ have positive steady-state consumption. Moreover, according to (14), $\theta_{t+1}(\omega)/\theta_t(\omega) = \beta(\omega)$ at steady state. Therefore, $\theta_{t+1}(\omega)/\theta_t(\omega)$ is the highest for all $\omega \in \Omega_p$.

The following lemma provides the solution for consumptions in terms of the marginal product of labor.

Lemma 9 *Efficient consumption satisfies:*

$$c(\omega) = \frac{\xi(c(\omega))/\beta}{\psi - \xi(c(\omega))} [\eta(\omega) + (\lambda(\omega) - \beta) F_L l(\omega)] \text{ for } \omega \in \Omega_p. \quad (31)$$

Equation (31), analogous to (25), shows that consumption is proportional to the net financial cost of a child. In particular, consumption is larger for types with a higher cost of raising children, either a higher goods cost $\eta(\omega)$ and/or a higher time cost $\lambda(\omega)$. The relationship between skills, $l(\omega)$, and consumption is slightly more complicated. If $\lambda(\omega) > \beta$ then efficient consumption is higher for highly skilled individuals. But if $\lambda(\omega) < \beta$, then efficient consumption is actually lower for the high skilled. This feature also characterizes the market economic as shown in Equation (5). The intuition is that the net labor cost of a child is $(\lambda(\omega) - \beta) F_L l(\omega)$ since a child takes time from parents but also adds labor as an adult. The net time cost of a child is thus $(\lambda(\omega) - \beta) F_L l(\omega)$.

We can now state our next main result which follows from (30) and (31)¹⁶.

Proposition 10 *Steady-state efficient allocations exhibit inequality of consumptions and populations. Types with lower net cost of raising children have lower consumption but a larger population.*

Proposition 10 is important for at least three reasons. First, as is discussed by Lucas (2002), obtaining an efficient allocation with heterogeneous social classes in Malthusian economies is not trivial. Lucas's solution, which relies on differences in time discounting, generates inefficient social classes in presence of binding constraints. Different discount factors would still lead to only one social group surviving at steady state in an efficient allocation. Second, the efficient allocation can rationalize a distribution of social classes in which the poor are a larger fraction of the population. Third, the proposition also states that it is not optimal to end a lineage just because it is of lower skill or poorer. This is in contrast to a literature that argues in favor of limiting the fertility of the poor (e.g., Chu and Koo, 1990). Only impatient types disappear from an efficient allocation.

It is possible to obtain a final solution for consumptions and relative population sizes without knowing the marginal product of labor in the following special Barro-Becker case.

¹⁶The proof follows Lemma 7 and Lemma 8 trivially and is hence omitted.

Example 11 Suppose $u(c) = c^\xi/\xi$ with $\xi \in (0, 1)$, $\Phi(n) = n^\psi$, $\psi \in (\xi, 1)$, $\lambda(\omega) = \beta$ and $N_0(\omega) = N_0(\omega')$. Then

$$c(\omega) = \frac{\xi}{\psi - \xi} \frac{\eta(\omega)}{\beta} \text{ and } \frac{N(\omega)}{N(\omega')} = \left(\frac{\eta(\omega')}{\eta(\omega)} \right)^{\frac{1-\xi}{1-\psi}}.$$

In this example, consumption is proportional to the goods cost of raising a child, $\eta(\omega)$, while the exponent $(1 - \xi) / (1 - \psi) \in (1, \infty)$ controls the extent to which consumption inequality translates into population inequality. Since the restriction $\psi > \xi$ is needed for an interior solution, the exponent is larger than 1. Therefore, population inequality is larger than consumption inequality. For example, if consumption of the rich is 5 times that of the poor, $\eta(\omega') / \eta(\omega) = 5$, and $(1 - \xi) / (1 - \psi) = 2$ then the population of the poor is 25 times that of the rich. The planner in this example is more willing to accept a large share of poor individuals when intergenerational substitution of consumption is particularly low (ξ is low) and/or parental altruism does not decrease sharply with family size (ψ is high).

5.2.3 Average output

A full solution requires us to find the marginal product of labor which itself requires a solution for the land-labor ratio. For this purpose, rewrite the steady-state resource constraint as

$$LF(k, 1; A) = N \sum_{\omega} g(\omega) [c(\omega) + \eta(\omega)],$$

where $k = K/L$. Furthermore, total labor supply relative to population is expressed, at steady state, by

$$\frac{L}{N} = \sum_{\omega} g(\omega) l(\omega) [1 - \lambda(\omega)]. \quad (32)$$

Dividing these two equations yields

$$F(k, 1; A) = \frac{\sum_{\omega} g(\omega) [c(\omega) + \eta(\omega)]}{\sum_{\omega} g(\omega) l(\omega) [1 - \lambda(\omega)]}. \quad (33)$$

The system of equations (30), (31) and (33), together with the definition $F_L = (1 - \alpha) F(k, 1; A)$, can then be used to solve for the following unknowns: $g(\omega)$, $c(\omega)$ and k .

5.2.4 Stagnation

Combining (31) and (33), and using the definition of $\alpha(k, A)$, one obtains:

$$\begin{aligned} c(\omega) & \quad (34) \\ &= \frac{\xi(c(\omega))/\beta}{\psi(1 - \xi(c(\omega)))} \left[\begin{array}{c} \eta(\omega) \\ + (\lambda(\omega) - \beta)(1 - \alpha(k, A)) \frac{\sum_{\omega} g(\omega)[c(\omega) + \eta(\omega)]}{\sum_{\omega} g(\omega)l(\omega)[1 - \lambda(\omega)]} l(\omega) \end{array} \right] \end{aligned}$$

Equations (30), (33), and (34) can be used to solve for $g(\omega)$, $c(\omega)$ and k . Since K is not part of the system, it follows that land discoveries do not affect individual consumptions nor the composition of population. Furthermore, if technological progress is land augmenting, the increase in A does not affect $\alpha(k, A)$ which is equal to $\alpha(Ak)$. Hence $c(\omega)$ and $g(\omega)$ are also independent of A in that case. Once k is solved for, aggregate labor supply can be solved as $L = K/k$ while N is solved from (32). The following proposition summarizes these results. The proof is similar to that of Proposition 5 and hence omitted.

Proposition 12 *Suppose $\varphi_{t+1}(\omega) = 0$ for $t \geq 0$ and $\omega \in \Omega$, $\Phi(n) = n^\psi$ and the steady state is interior. Then: (i) in steady-state optimal consumption is independent of the amount of land and optimal population is proportional to the amount of land; (ii) if technological progress is land augmenting then optimal consumption is independent of the level of technology and population increases proportionally with the level of technology; and (iii) optimal allocations depend on the initial distribution of population.*

6 Stochastic Case

The deterministic version of the model considered so far counterfactually predicts equal fertility among different social groups. Malthus, however, observed that fertility rates were higher among the poor. We now show that a version of the model with stochastic types can generate differential fertility. For tractability we once again assume $\varphi_{t+1}(\omega^{t+1}) = 0$ for all $t \geq 0$ ¹⁷, $\beta(\omega) = \beta$ and use the Barro-Becker functional forms: $\Phi(n) = n^\psi$ and $u(c) = c^\xi/\xi$. Equation (14) can be simplified, using (17) and the law of motion for population, (8), as:

$$\beta(\omega_t) \frac{\Phi(n_t(\omega^t))}{n_t(\omega^t)} = \frac{\mu_{t+1}}{\mu_t} \frac{u'(c_t(\omega^t))}{u'(c_{t+1}(\omega^{t+1}))} \text{ for } \omega^{t+1} \in \Omega_{t+1}. \quad (35)$$

Since this equation is the same for all $\omega_{t+1} \in \Omega$ for a given $\omega^t \in \Omega_t$, it follows that all children within a family have the same consumption:

$$c_{t+1}([\omega^t, \omega_{t+1}]) = c_{t+1}(\omega^t) \text{ for all } \omega_{t+1} \in \Omega.$$

This is a standard risk sharing result that is obtained regardless of whether fertility is endogenous or exogenous: absent aggregate uncertainty, smoothing consumption across children, or next period states, is socially optimal. The result, however, does

¹⁷It is represented by SW_2 or a special case of those discussed in SW_1 when the social planner puts positive weights only on initial olds and cares about future generation through the initial generation.

not state that children's consumption is equal to parent's consumption. Consumption smoothing across time is not guaranteed.

The following proposition extends the results of the previous sections by characterizing consumption of a child as proportional to the *expected* net costs of raising that child. A full characterization of efficient allocations is in the Appendix when we prove this proposition. This result implies a strong degree of history independence.

Proposition 13 *Optimal allocations satisfy $c_{t+1}(\omega^{t+1}) = c_{t+1}(\omega_t)$ and $n_t(\omega^t) = n_t(\omega_{t-1}, \omega_t)$. In particular,*

$$c_{t+1}(\omega_t) = \frac{\xi}{\psi - \xi} \left\{ \begin{array}{l} \frac{\mu_t}{\mu_{t+1}} [\eta(\omega_t) + F_L(K, L_t; A)l(\omega_t)\lambda_t(\omega_t)] \\ -F_L(K, L_{t+1}; A)E_t(l(\omega_{t+1})|\omega_t) \end{array} \right\}, \quad (36)$$

and $n_t(\omega_{t-1}, \omega_t)$ satisfies:

$$\beta(\omega_t)\Phi'(n_t(\omega_{t-1}, \omega_t)) = \psi \frac{\mu_{t+1}}{\mu_t} \frac{u'(c_t(\omega_{t-1}))}{u'(c_{t+1}(\omega_t))}$$

for all $\omega_{t-1}, \omega_t \in \Omega$, $t > 0$.

Regarding consumption, the Proposition states that only parental type, ω_t , determines consumption of her children to the extent that it determines the cost of raising the child and the expected labor supply of the child. For consumption to be equalized across parents, the cost of raising children net of children's expected labor supply must be equal. If not, then full consumption smoothing is not socially optimal. Efficient consumption is random. This is in contrast to the exogenous fertility model in which individual consumption is not random as it only depends on ω_0 rather than on the full ancestors' history ω^t .¹⁸ Finally, substituting (36) into (35), it follows that $n_t(\omega^t) = n_t(\omega_{t-1}, \omega_t)$ so that the number of children only depends on parental and grand-parental types.

Another key feature of consumption, stated in the proposition, is the lack of memory, or lack of persistence result. Specifically, $c_{t+1}(\omega^{t+1}) = c_{t+1}(\omega_t)$ means that the only part of history that matter for efficient individual consumption is ω_t while the remaining part, $[\omega_0, \omega_1, \dots, \omega_{t-1}]$, is irrelevant. Thus, the child of lucky parents, parents with a favorable ω_t , will enjoy high consumption even if all the previous ancestors were unlucky and have low consumption. Hosseini et al. (2013) and Alvarez (1999) obtain similar lack of persistence results in incomplete market models.

¹⁸When fertility is exogenously given to be 1, the first order condition with respect to $U_{t+1}(\omega^{t+1})$ derived in the Appendix can be iterated as $\theta_t(\omega^t) = \beta^t \theta_0(\omega^0)$. By the first order condition with respect to consumption, people living in the same generation and coming from the same initial old ancestor (or with the same initial old's weight) have the same level of consumption.

6.1 Steady state

Consider now stationary steady-state allocations in which $n_t(\omega_{t-1}, \omega_t) = n(\omega_{t-1}, \omega_t)$, $c_t(\omega_{t-1}) = c(\omega_{t-1})$, and $N_t = N$. Let $Q_t \equiv \mu_t/\mu_{t+1}$ be the planner's shadow gross return and let $g(\omega_{t-1}, \omega_t) \equiv N(\omega_{t-1}, \omega_t)/N_t$ be the population share with recent history (ω_{t-1}, ω_t) at steady state where $N(\omega_{t-1}, \omega_t) \equiv \sum_{\omega^{t-2}} N_t(\omega^t)$. The following lemma summarizes the system of equations and unknowns describing stationary steady state.

Lemma 14 *Steady state allocations, $c(\omega)$, $n(\omega_{-1}, \omega)$, $g(\omega_{-1}, \omega)$, Q , L and N are solved from the following systems of equations:*

$$c(\omega) = \frac{\xi Q}{\psi - \xi} [\eta(\omega) + F_L l(\omega) \lambda(\omega) - F_L E[l(\omega_{+1})|\omega]/Q], \quad (37)$$

$$n(\omega_{-1}, \omega) = \left[\beta Q \frac{u'(c(\omega))}{u'(c(\omega_{-1}))} \right]^{\frac{1}{1-\psi}}, \quad (38)$$

$$g(\omega, \omega_{+1}) = \sum_{\omega_{-1}} n(\omega_{-1}, \omega) \pi(\omega_{+1}, \omega) g(\omega_{-1}, \omega), \quad (39)$$

$$\sum_{\omega_{-1}} \sum_{\omega} g(\omega_{-1}, \omega) n(\omega_{-1}, \omega) = 1, \quad (40)$$

$$F\left(\frac{K}{L}, 1; A\right) = \frac{N}{L} \sum_{\omega_{-1}} \sum_{\omega} g(\omega_{-1}, \omega) [c(\omega_{-1}) + \eta(\omega) n(\omega_{-1}, \omega)], \quad (41)$$

$$\frac{L}{N} = \sum_{\omega_{-1}} \sum_{\omega} g(\omega_{-1}, \omega) l(\omega) (1 - \lambda(\omega) n(\omega_{-1}, \omega)). \quad (42)$$

where $F_L = (1 - \alpha(K/L, A)) F(K/L, 1; A)$.

Equation (37) shows the consumption of an individual whose parent is of type ω . Consumption is positively associated with parental costs of raising children and parental skills, and it is negatively associated with the expected skills of the child. Equation (38) shows fertility differentials among different types. Optimal fertility depends on parental and grandparent's types. Given grandparent's types, parents with particularly low costs of raising children would have more children than otherwise. Also, given parental types, grandparents with particularly high costs of raising children would have more grandchildren. Equation (40), which in principle serves to solve Q , restricts fertility to be one on average. Equations (41) and (42) are resource constraints of goods and labor.

The next proposition shows that the stagnation property still holds in the stochastic case.

Proposition 15 *Suppose the steady-state is interior. Then, steady-state optimal consumption is independent of the amount of land and optimal population increases*

proportionally with the amount of land. Furthermore, if technological progress is land augmenting then optimal consumption is independent of the level of technology and population increases proportionally with the level of technology.

To summarize, in addition to stagnation, the key properties of the stochastic steady state are differential fertility and heterogeneous social groups. Moreover, all types, or social groups, are represented in a steady state even if their initial population is zero as long as π is non-reducible.

7 Decentralization

This section extends Section 2 to a stochastic environment. We show that when $\varphi_{t+1}(\omega^{t+1}) = 0$ for all $t \geq 0$ the social planner's problem characterized by SW_2 can be decentralized by a competitive market economy with a fixed amount of land¹⁹. The basic environment is the same with the social planner's problem. Parents are altruistic toward children in the form of Barro-Becker. We follow previous notation except for adding a superscript c to allocations of consumption, fertility, population, labor and land to represent competitive equilibrium allocations. Let $k_t^c(\omega^t)$ denote the amount of land each adult living in period t is endowed with when his family history is $\omega^t \in \Omega_t$. It can be regarded as the bequest from parents and can be traded at the price p_t . Land can also be rented at the rental rate r_t . Parents are allowed to sign a contingent contract based on children's type ω_{t+1} , that is buying or selling land for the next generation depending on each one's realization of ability. Let $q_t(\omega_{t+1}, \omega_t)$ be the time t price of one unit of land contingent on the time $t + 1$'s realization of child's ability to be ω_{t+1} and the time t 's realization to be ω_t . People work in a competitive labor market. Let $w_t(\omega_t)$ be the wage of type ω_t at time t .

Initial parents maximize their own dynasty's welfare:

$$\max_{\{c_t(\omega^t), n_t(\omega^t), k_{t+1}(\omega^{t+1})\}_{t=0, \omega^t \in \Omega_t}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \sum_{\omega^t \in \Omega_t} \prod_{j=0}^{t-1} \Phi(n_j(\omega^j)) u(c_t(\omega^t)) \right] \quad (43)$$

Households are subject to the following constraints:

$$\begin{aligned} c_t(\omega^t) + \eta(\omega_t) n_t(\omega^t) + n_t(\omega^t) \sum_{\omega_{t+1}} q_t(\omega_{t+1}, \omega_t) k_{t+1}(\omega^{t+1}) \\ \leq w_t(\omega_t) (1 - \lambda(\omega_t) n_t(\omega^t)) + (r_t + p_t) k_t(\omega^t) \quad \text{for } \omega^t \in \Omega_t, t \geq 0, \end{aligned} \quad (44)$$

¹⁹When $\varphi_{t+1}(\omega^{t+1}) > 0$ for some ω^{t+1} , there is a wedge which makes decentralization with lump-sum instruments unfeasible. This issue arises even if fertility is exogenous and was first carefully studied by Bernheim (1989). But as shown in Section 4, and consistent with Berheim's findings, under Assumption 1 the planner in the limit behaves just like a planner with $\varphi_{t+1}(\omega^{t+1}) = 0$ for all ω^{t+1} .

where initial population and land holding $\{N_0(\omega_0), k_0(\omega_0)\}_{\omega_0 \in \Omega}$ are given. Assume $\Phi(n) = n^\psi$ and $u(c) = c^\xi/\xi$.

Firms hire labor on a competitive labor market, and people rent land to firms on a competitive land market. The competitive equilibrium of this problem is characterized by the following proposition.

Proposition 16 *Given initial levels of land and population, $\{k_0(\omega_0), N_0(\omega_0)\}_{\omega_0 \in \Omega}$, equilibrium allocations and prices are solved by the following system of equations for $t \geq 0$:*

$$r_t = F_K(K, L_t^c; A), \quad w_t(\omega_t) = F_L(K, L_t^c; A)l(\omega_t), \quad q_t(\omega_{t+1}, \omega_t) = p_t \pi(\omega_{t+1}, \omega_t),$$

$$c_{t+1}^c(\omega^{t+1}) = \frac{\xi}{\psi - \xi} \left[\frac{r_{t+1} + p_{t+1}}{p_t} [\eta(\omega_t) + w_t(\omega_t) \lambda(\omega_t)] - E_t[w_{t+1}(\omega_{t+1})] \right], \quad (45)$$

$$\beta \frac{\Phi(n_t^c(\omega^t))}{n_t^c(\omega^t)} = \frac{p_t}{r_{t+1} + p_{t+1}} \frac{u'(c_t^c(\omega^t))}{u'(c_{t+1}^c(\omega^{t+1}))}, \quad (46)$$

$$K = \sum_{\omega^t \in \Omega_t} N_t^c(\omega^t) k_t^c(\omega^t),$$

(44), (7), and (8).

Rewrite (46) and iterate, we obtain

$$p_t = \sum_{j=0}^{\infty} \beta^{j+1} \left(\prod_{k=0}^j \frac{\Phi(n_{t+k}^c(\omega^{t+k}))}{n_{t+k}^c(\omega^{t+k})} \right) \frac{u'(c_{t+j+1}^c(\omega^{t+j+1}))}{u'(c_t^c(\omega^t))} r_{t+j+1}.$$

for any family history $\{\omega^t\}_{t=0}^{\infty}$. p_0 is given by this formula when $t = 0$. Price is the present value of rents discounted by a rate that depends on the standard discount factor in Φ as well as the fertility rates.

Corollary 17 *The competitive equilibrium consumption of every individual depends on parental type but not on one's own type, e.g. $c_{t+1}^c(\omega^{t+1}) = c_{t+1}^c(\omega_t)$. As a result, fertility depends both on parents' and grandparents' types, e.g. $n_t^c(\omega^t) = n_t^c(\omega_{t-1}, \omega_t)^{20}$.*

The proof of this corollary is straightforward according to equations of (45) and (46), and it is omitted. Finally, we show the correspondence of the competitive equilibrium with contingent assets with the planner's problem. The following proposition shows that the competitive equilibrium corresponds to a social planner's problem and it is hence A - and P -efficient.

Proposition 18 *For a given initial distribution of land $\{k_0(\omega_0)\}_{\omega_0 \in \Omega}$, the optimal allocations of the decentralized equilibrium with contingent contracts solve SW_2 for a properly selected planner's weights $\{\varphi_0(\omega_0)\}_{\omega_0 \in \Omega}$ and, hence, it is A - and P -efficient.*

²⁰ $n_0(\omega_{-1}, \omega_0) = n_0(\omega_0)$

Next we show that the problem of the social planner who directly cares only about the initial generation can be decentralized by a competitive equilibrium with an initial land distribution.

Proposition 19 *Given social planner's weights $\{\varphi_0(\omega_0)\}_{\omega_0 \in \Omega}$ on the initial generation with population $\{N_0(\omega_0)\}_{\omega_0 \in \Omega}$, there exists an initial distribution of land $\{k_0(\omega_0)\}_{\omega_0 \in \Omega}$ under which the competitive equilibrium with contingent assets decentralizes the social planner's problem characterized by SW_2 .*

8 Concluding Comments

Existing literature has shown that equilibrium allocations obtained under endogenous fertility differ sharply from those obtained under exogenous fertility. This observation motivates us to better understand social optima when population is endogenous. This article characterizes efficient allocations in fixed resource economies with endogenous fertility.

The pre-industrial world was to a large extent Malthusian. As documented by Ashraf and Galor (2011), periods characterized by improvements in technology or in the availability of land eventually lead to a larger but not richer population. This is remarkable given the diversity of political, social, religious, geographical, cultural, and economic environments they considered, some arguably more advanced than others. We find that stagnation, inequality, high population of the poor and differential fertility can naturally arise as an optimal social choice. Our findings could shed light on why the Malthusian "trap" was so pervasive in pre-industrial societies. We also show that it is not the irrational animal spirit of human beings, as suggested by Malthus, what ultimately explains the stagnation. Stagnation can be the result of an optimal choice between the quality and quantity of life in the presence of limited natural resources.

Finally, we expect that our methodology will further facilitate the study of demographics issues using tools of welfare economics and macroeconomics.

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Appendix

A.1 Proofs of Propositions and Lemmas

Proof of Proposition 1. Let (n^*, c^*) solve the planner's problem. Suppose, by contradiction, that there exist an alternative feasible allocation (\hat{n}, \hat{c}) that P -dominates (n^*, c^*) . Let $\bar{U}_t^{\hat{}}(\omega^t)$ denote the utility of potential agent with history ω^t under the alternative plan (\hat{n}, \hat{c}) and $U_t^*(\omega^t)$ the utility under (n^*, c^*) . For (\hat{n}, \hat{c}) to P -dominate (n^*, c^*) it must be the case that $\bar{U}_t^{\hat{}}(\omega^t) \geq \bar{U}_t^*(\omega^t)$ for all $\omega^t \in \Omega_t$ and all $t \geq 0$ with strict inequality holds for at least one ω^t . (i) Under SW_1 if $\varphi_t(\omega^t) > 0$ then $\varphi_t(\omega^t) \bar{N}_t(\omega^t) > 0$ for all $t \geq 0$ and $\omega^t \in \Omega_t$ so that (11) is larger under $\bar{U}_t^{\hat{}}$ than under \bar{U}_t^* which contradicts the statement that (n^*, c^*) maximizes social welfare. (ii) Look at the social welfare function SW_2 . If $\bar{U}_0^{\hat{}}(\omega_0) \geq \bar{U}_0^*(\omega_0)$ for all $\omega_0 \in \Omega_0$ with strict inequality for some $\omega_0 \in \Omega$, then (n^*, c^*) will not maximize social welfare since $N_0(\omega_0) > 0$ for all $\omega_0 \in \Omega_0$. Social welfare still could be equal under both allocations but this would violate the assumed uniqueness of the maximizer. ■

Proof of Proposition 2. The proof of this proposition is quite similar with that of the P -efficiency. Let (n^*, c^*) solve the planner's problem. Suppose by contradiction that there exist an alternative feasible allocation (\hat{n}, \hat{c}) that A -dominates (n^*, c^*) . Let $\bar{U}_0^{\hat{}}(\omega_0)$ denotes the utility of an agent living in period 0 with type ω_0 under the alternative plan (\hat{n}, \hat{c}) and $U_0^*(\omega_0)$ the utility under (n^*, c^*) . For (\hat{n}, \hat{c}) to A -dominate (n^*, c^*) it must be the case that $\bar{U}_0^{\hat{}}(\omega_0) \geq \bar{U}_0^*(\omega_0)$ for all $\omega_0 \in \Omega_0$ with strict inequality holds for at least one ω^t . If $\varphi_0(\omega_0) > 0$ for all $\omega_0 \in \Omega_0$, then (n^*, c^*) would not maximize social welfare. If $\varphi_0(\omega_0)$ is zero for some ω_0 and the solution to the planner's problem maximizing SW_2 is unique, then it violates the assumed uniqueness of the maximizer. ■

Proof of Proposition 3. The Lagrangian of the planner's problem is

$$\begin{aligned} \mathcal{L} = & \sum_{t=0}^{\infty} \sum_{\omega^t} \varphi_t(\omega^t) N_t(\omega^t) U_t(\omega^t) \\ & + \sum_{t=0}^{\infty} \sum_{\omega^t} \theta_t(\omega^t) N_t(\omega^t) [u(c_t(\omega^t)) + \beta(\omega_t) \Phi(n_t(\omega^t)) E_t U_{t+1}(\omega^{t+1}) - U_t(\omega^t)] \\ & + \sum_{t=0}^{\infty} \sum_{\omega^{t+1}} \gamma_{t+1}(\omega^{t+1}) [N_{t+1}(\omega^{t+1}) - n_t(\omega^t) \pi(\omega_{t+1}, \omega_t) N_t(\omega^t)] \\ & + \sum_{t=0}^{\infty} \mu_t \left[F(K, L_t; A) - \sum_{\omega^t} N_t(\omega^t) [c_t(\omega^t) + \eta(\omega_t) n_t(\omega^t)] \right] \\ & + \sum_{t=0}^{\infty} \kappa_t \left[\sum_{\omega^t} N_t(\omega^t) l(\omega_t) [1 - \lambda_t(\omega_t) n_t(\omega^t)] - L_t \right], \end{aligned}$$

First order conditions with respect to $\{U_0(\omega^0), U_{t+1}(\omega^{t+1}), N_{t+1}(\omega^{t+1}), n_t(\omega^t), c_t(\omega^t), L_t\}_{\omega^t \in \Omega_t, t \geq 0}$ are (13), (14), (15), (16), (17), and (18). In the deterministic

case, the first order conditions with respect to n_t , N_{t+1} , U_t , and c_t can be simplified as

$$\theta_t \beta \Phi_n(n_t) U_{t+1} = \mu_t \eta + \kappa_t l \lambda_t + \gamma_{t+1}, \quad (47)$$

$$\varphi_{t+1} U_{t+1} + \kappa_{t+1} l [1 - \lambda_{t+1} n_{t+1}] + \gamma_{t+1} \quad (48)$$

$$= \mu_{t+1} [c_{t+1} + \eta n_{t+1}] + n_{t+1} \gamma_{t+2}$$

$$\theta_{t+1} N_{t+1} = \varphi_{t+1} N_{t+1} + \theta_t N_t \beta \Phi(n_t) \quad (49)$$

$$\theta_t u'(c_t) = \mu_t$$

Substitute out γ_{t+1} and γ_{t+2} of (48) using (47), and then use (18) and (17) to substitute out κ_{t+1} and μ_t , respectively, we obtain

$$\begin{aligned} & \varphi_{t+1} U_{t+1} + \kappa_{t+1} l [1 - \lambda_{t+1} n_{t+1}] + \theta_t \beta \Phi_n(n_t) U_{t+1} - \mu_t \eta - \kappa_t l \lambda_t \\ & = \mu_{t+1} [c_{t+1} + \eta n_{t+1}] + n_{t+1} [\theta_{t+1} \beta \Phi_n(n_{t+1}) U_{t+2} - \mu_{t+1} \eta - \kappa_{t+1} l \lambda_{t+1}] \end{aligned}$$

so

$$\begin{aligned} & \varphi_{t+1} U_{t+1} + \theta_t \beta \Phi_n(n_t) U_{t+1} \quad (50) \\ & = \theta_{t+1} u'(c_{t+1}) \left(c_{t+1} - F_{L,t+1} l + \frac{\mu_t}{\mu_{t+1}} (\eta + F_{L,t} l \lambda_t) \right) \\ & + \theta_{t+1} \frac{n_{t+1} \Phi_n(n_{t+1})}{\Phi(n_{t+1})} (U_{t+1} - u(c_{t+1})) \end{aligned}$$

Iterate (49), we can get

$$\begin{aligned} \theta_{t+1} & = \varphi_{t+1} + \beta \frac{\Phi(n_t)}{n_t} \left[\varphi_t + \beta \frac{\theta_{t-1} \Phi(n_{t-1})}{n_{t-1}} \right] \\ & = \sum_{m=1}^{t+1} \varphi_m \beta^{t+1-m} \prod_{j=m}^t \frac{\Phi(n_j)}{n_j} + \beta^{t+1} \prod_{j=0}^t \frac{\Phi(n_j)}{n_j} \theta_0 \\ & = \sum_{m=0}^{t+1} \varphi_m \beta^{t+1-m} \prod_{j=m}^t \frac{\Phi(n_j)}{n_j} \end{aligned}$$

Given that the altruism function takes Barro-Becker's form $\Phi(n) = n^\psi$,

$$\theta_{t+1} = \beta^{t+1} N_{t+1}^{\psi-1} \sum_{m=0}^{t+1} \varphi_m \beta^{-m} N_m^{1-\psi}$$

Plug this result into (50),

$$\begin{aligned} & (1 - \psi) \varphi_{t+1} U_{t+1} \\ & = \beta^{t+1} N_{t+1}^{\psi-1} \left(\sum_{m=0}^{t+1} \varphi_m \beta^{-m} N_m^{1-\psi} \right) u'(c_{t+1}) \left(c_{t+1} - F_{L,t+1} l + \frac{\mu_t}{\mu_{t+1}} (\eta + F_{L,t} l \lambda_t) \right) \\ & - u(c_{t+1}) \beta^{t+1} N_{t+1}^{\psi-1} \left(\sum_{m=0}^{t+1} \varphi_m \beta^{-m} N_m^{1-\psi} \right) \psi \end{aligned}$$

Use this equation to express c_{t+1} as

$$c_{t+1} = F_{L,t+1}l - \frac{\mu_t}{\mu_{t+1}} (\eta + F_{L,t}l\lambda_t) + \frac{u(c_{t+1})}{u'(c_{t+1})c_{t+1}} \psi c_{t+1} \\ + \frac{1}{u'(c_{t+1})} \beta^{-(t+1)} N_{t+1}^{1-\psi} \left(\sum_{m=0}^{t+1} \varphi_m \beta^{-m} N_m^{1-\psi} \right)^{-1} (1-\psi) \varphi_{t+1} U_{t+1}$$

Under the assumed form of $\Phi(\cdot)$, $\psi = \frac{n_t \Phi_n(n_t)}{\Phi(n_t)} = \frac{n_{t+1} \Phi_n(n_{t+1})}{\Phi(n_{t+1})}$, we can collect terms and solve for consumption as (24). ■

Proof of Lemma 4. Consider the steady-state situation in which N and c are constant. In that case $n = 1$ and (21) can be written as $\theta_{t+1}/\theta_t = \beta + \varphi_{t+1}/\theta_t \geq \beta$. Under Assumption 1, the ratio φ_{t+1}/θ_t goes to zero in the limit. It is easy to show that the Lagrangian multipliers grow at constant rate at steady state. To see this, first look at (17) and (18), which implies that θ_t , μ_t , and κ_t grow at the same rate at steady state. Using the steady-state version of (16) we can see that

$$\frac{\theta_{t+1}}{\theta_t} = \frac{\theta_{t+1} \Phi_n(1) U}{\theta_t \Phi_n(1) U} = \frac{\frac{\mu_{t+1}}{\mu_t} \mu_t \eta + \frac{\kappa_{t+1}}{\kappa_t} \kappa_t l \lambda + \frac{\gamma_{t+2}}{\gamma_{t+1}} \gamma_{t+1}}{\mu_t \eta + \kappa_t l \lambda + \gamma_{t+1}}$$

so

$$\frac{\theta_{t+1}}{\theta_t} \mu_t \eta + \frac{\theta_{t+1}}{\theta_t} \kappa_t l \lambda + \frac{\theta_{t+1}}{\theta_t} \gamma_{t+1} = \frac{\mu_{t+1}}{\mu_t} \mu_t \eta + \frac{\kappa_{t+1}}{\kappa_t} \kappa_t l \lambda + \frac{\gamma_{t+2}}{\gamma_{t+1}} \gamma_{t+1}$$

Since θ_t , μ_t , and κ_t grow at the same rate at steady state, this equation can be reduced to

$$\frac{\theta_{t+1}}{\theta_t} = \frac{\gamma_{t+2}}{\gamma_{t+1}}$$

Then γ_t grows at the same rate with θ_t , μ_t , and κ_t at the steady state, which implies that γ_t/μ_t is a constant at steady state. Hence γ_{t+1}/γ_t is a constant by (23), and so are other Lagrangian parameters. Therefore at steady state $\theta_{t+1}/\theta_t = \mu_{t+1}/\mu_t = \gamma_{t+1}/\gamma_t = \kappa_{t+1}/\kappa_t = \beta$. Then the steady-state consumption in (24) reduces to (25). ■

Proof of Proposition 5. For the first part of this proposition A is given. Then c , F_L , and K/L are fully determined by the three equations (25), (26) and (27). Hence consumption is independent of the amount of land while efficient population increases proportionally with the amount of land. For the second part when $\hat{F}(AK, L) = F(K, L; A)$,

$$\hat{F}\left(\frac{AK}{N}, 1 - \lambda\right) = F\left(\frac{K}{N}, 1 - \lambda; A\right),$$

then

$$\alpha\left(\frac{K}{L}, A\right) = \frac{K}{L} \frac{F_K\left(\frac{K}{L}, 1; A\right)}{F\left(\frac{K}{L}, 1; A\right)} = \frac{A \hat{F}_1(AK, L) K}{\hat{F}(AK, L)} \\ = \frac{\hat{F}_1(AK/L, 1) AK}{\hat{F}(AK/L, 1) L} = \hat{\alpha}\left(\frac{AK}{L}\right)$$

$$F_L = \frac{1 - \hat{\alpha} \left(\frac{AK}{L} \right)}{1 - \lambda} (c + \eta) \quad (51)$$

where

$$F_L = \left(1 - \hat{\alpha} \left(\frac{AK}{L} \right) \right) \hat{F} \left(\frac{AK}{L}, 1 \right). \quad (52)$$

Then c , F_L and AK/L are solved by (25), (51) and (52), which are all independent of A . ■

Proof of Lemma 6. Let $s_t(\omega) \equiv \theta_t(\omega) N_t(\omega)$. Equation (14), given that $\varphi_t(\omega) = 0$ for all $t > 0$, $\omega \in \Omega$ is assumed, can then be written as $s_{t+1}(\omega) = \beta(\omega) s_t(\omega) \Phi(n_t(\omega))$. Iterate it, we obtain $s_t(\omega) = \beta(\omega)^t s_0(\omega) \prod_{i=0}^{t-1} \Phi(n_i(\omega))$, $t > 0$. Recall that $\Phi(n) = n^\psi$ is assumed in Section 4, it follows that:

$$s_t(\omega) = \theta_0(\omega) (N_0(\omega))^{1-\psi} \beta(\omega)^t (N_t(\omega))^\psi. \quad (53)$$

Now (17) can be written as $\mu_t N_t(\omega) = s_t(\omega) u'(c_t(\omega))$. Therefore

$$\frac{N_t(\omega)}{N_t(\omega')} = \frac{s_t(\omega) u'(c_t(\omega))}{s_t(\omega') u'(c_t(\omega'))}.$$

Substituting (53) into this equation gives

$$\frac{N_t(\omega)}{N_t(\omega')} = \frac{\theta_0(\omega) (N_0(\omega))^{1-\psi} \beta(\omega)^t (N_t(\omega))^\psi u'(c_t(\omega))}{\theta_0(\omega') (N_0(\omega'))^{1-\psi} \beta(\omega')^t (N_t(\omega'))^\psi u'(c_t(\omega'))}.$$

Finally, use (13) to substitute $\theta_0(\omega)$ and solve for $\frac{N_t(\omega)}{N_t(\omega')}$ to obtain (28). ■

Proof of Lemma 8. According to equation (29),

$$\frac{N(\omega')}{N(\omega)} = \frac{N_0(\omega')}{N_0(\omega)} \left(\frac{u'(c(\omega')) \varphi_0(\omega')}{u'(c(\omega)) \varphi_0(\omega)} \right)^{\frac{1}{1-\psi}}.$$

Adding $N(\omega)$ over ω ,

$$\begin{aligned} N &= \sum_{\omega'} N(\omega') \\ &= \frac{N(\omega)}{N_0(\omega) [u'(c(\omega)) \varphi_0(\omega)]^{1/(1-\psi)}} \sum_{\omega'} N_0(\omega') [u'(c(\omega')) \varphi_0(\omega')]^{1/(1-\psi)} \end{aligned}$$

and therefore

$$g(\omega) = \frac{N(\omega)}{N} = \frac{N_0(\omega) [u'(c(\omega)) \varphi_0(\omega)]^{1/(1-\psi)}}{\sum_{\omega \in \Omega_p} N_0(\omega') [u'(c(\omega')) \varphi_0(\omega')]^{1/(1-\psi)}} \text{ for all } \omega \in \Omega_p.$$

■

Proof of Lemma 9. Rewrite (15) using (18) and evaluate it at the steady state:

$$1 = \frac{\mu_{t+1}}{\gamma_{t+1}(\omega)} [c(\omega) + \eta(\omega) - F_L l(\omega) (1 - \lambda(\omega))] + \frac{\gamma_{t+2}(\omega)}{\gamma_{t+1}(\omega)}.$$

Since $\gamma_{t+2}(\omega)/\gamma_{t+1}(\omega)$ is constant in steady state then $\mu_{t+1}/\gamma_{t+1}(\omega)$ needs to be constant for this equation to hold, which means that $\gamma_{t+1}(\omega)/\gamma_t(\omega) = \mu_{t+1}/\mu_t = \beta$. The last equality holds by (17) and (21). Therefore, the previous equation can be written as:

$$\frac{\gamma_t(\omega)}{\mu_t} = \frac{1}{1-\beta} [c(\omega) + \eta(\omega) - F_L l(\omega) (1 - \lambda(\omega))]. \quad (54)$$

This expression states that the value of an immigrant in terms of goods, $\gamma_t(\omega)/\mu_t$, is the net present value of the net cost. In steady state $U \equiv U(c(\omega)) = u(c(\omega))/(1-\beta)$. Use this result, (17) and (18) to rewrite (16) as:

$$\frac{\beta\Phi'(1) u(c(\omega))}{1-\beta u'(c(\omega))} = \eta(\omega) + F_L l(\omega) \lambda(\omega) + \beta \frac{\gamma_t(\omega)}{\mu_t}. \quad (55)$$

One can combine (54) and (55) to solve for consumption as

$$c(\omega) = \frac{\xi(c(\omega))}{\beta\Phi'(1) - \beta\xi(c(\omega))} [\eta(\omega) + (\lambda(\omega) - \beta) F_L l(\omega)].$$

We can obtain the result using $\Phi(n) = n^\psi$. ■

Proof of Proposition 13. The Lagrange of the social planner's problem is

$$\begin{aligned} \mathcal{L} = & \sum_{\omega_0 \in \Omega} \varphi_0(\omega_0) N_0(\omega_0) U_0(\omega_0) \\ & + \sum_{t=0}^{\infty} \sum_{\omega^t} \theta_t(\omega^t) N_t(\omega^t) [u(c_t(\omega^t)) + \beta(\omega_t) \Phi(n_t(\omega^t)) E_t U_{t+1}(\omega^{t+1}) - U_t(\omega^t)] \\ & + \sum_{t=0}^{\infty} \sum_{\omega^{t+1}} \gamma_{t+1}(\omega^{t+1}) [N_{t+1}(\omega^{t+1}) - n_t(\omega^t) \pi(\omega_{t+1}, \omega_t) N_t(\omega^t)] \\ & + \sum_{t=0}^{\infty} \mu_t \left[F(K, L_t; A) - \sum_{\omega^t} N_t(\omega^t) [c_t(\omega^t) + \eta(\omega_t) n_t(\omega^t)] \right] \\ & + \sum_{t=0}^{\infty} \kappa_t \left[\sum_{\omega^t} N_t(\omega^t) l(\omega_t) [1 - \lambda_t(\omega_t) n_t(\omega^t)] - L_t \right], \end{aligned}$$

First order conditions with respect to

$$\{n_t(\omega^t), N_{t+1}(\omega^{t+1}), U_{t+1}(\omega^{t+1}), U_0(\omega_0), c_t(\omega^t), L_t\}_{\omega^t \in \Omega_t, t \geq 0} :$$

$$\begin{aligned} & \theta_t(\omega^t) \beta(\omega_t) \Phi_n(n_t(\omega^t)) E_t U_{t+1}(\omega^{t+1}) \\ & = \mu_t \eta(\omega_t) + \kappa_t l(\omega_t) \lambda_t(\omega_t) + \sum_{\omega^{t+1} | \omega^t} \gamma_{t+1}(\omega^{t+1}) \pi(\omega_{t+1}, \omega_t), \end{aligned}$$

$$\begin{aligned} & \kappa_{t+1} l(\omega_{t+1}) [1 - \lambda_{t+1}(\omega_{t+1}) n_{t+1}(\omega^{t+1})] + \gamma_{t+1}(\omega^{t+1}) \\ & = \mu_{t+1} [c_{t+1}(\omega^{t+1}) + \eta(\omega_{t+1}) n_{t+1}(\omega^{t+1})] \\ & + \sum_{\omega_{t+2} | \omega_{t+1}} n_{t+1}(\omega^{t+1}) \gamma_{t+2}(\omega^{t+2}) \pi(\omega_{t+2}, \omega_{t+1}) \end{aligned}$$

$$\begin{aligned}
\theta_{t+1}(\omega^{t+1}) &= \theta_t(\omega^t) \beta(\omega_t) \frac{\Phi(n_t(\omega^t))}{n_t(\omega^t)} \\
\varphi_0(\omega_0) &= \theta_0(\omega_0) \\
\theta_t(\omega^t) u'(c_t(\omega^t)) &= \mu_t \\
\mu_t F_2(K, L_t; A) &= \kappa_t
\end{aligned}$$

For period 0, using first order conditions with respect to $U_0(\omega_0)$ and $c_0(\omega_0)$, we obtain

$$\frac{u'(c_0(\omega_0))}{u'(c_0(\tilde{\omega}_0))} = \frac{\varphi_0(\tilde{\omega}_0)}{\varphi_0(\omega_0)} \text{ for all } \omega_0, \tilde{\omega}_0 \in \Omega \quad (56)$$

Substitute out κ_t and $\theta_t(\omega^t)$ in first order condition with respect to $n_t(\omega^t)$, we write

$$E_t U_{t+1}(\omega^{t+1}) = \frac{u'(c_t(\omega^t))}{\beta(\omega_t) \Phi_n(n_t(\omega^t))} \left[\begin{array}{l} \eta(\omega_t) + F_2(K, L_t; A) l(\omega_t) \lambda_t(\omega_t) \\ + \frac{1}{\mu_t} \sum_{\omega^{t+1}|\omega^t} \gamma_{t+1}(\omega^{t+1}) \pi(\omega_{t+1}, \omega_t) \end{array} \right]$$

By first order condition with respect to $U_{t+1}(\omega^{t+1})$, and $c_t(\omega^t)$, we obtain the Euler equation as

$$u'(c_t(\omega^t)) = \frac{\mu_t}{\mu_{t+1}} \beta(\omega_t) u'(c_{t+1}(\omega^{t+1})) \frac{\Phi(n_t(\omega^t))}{n_t(\omega^t)}$$

From this Euler equation we can see that $c_{t+1}(\omega^{t+1})$ is independent of ω_{t+1} . Substitute it into $E_t U_{t+1}(\omega^{t+1})$,

$$\begin{aligned}
&E_t U_{t+1}(\omega^{t+1}) \\
&= \frac{\mu_t}{\mu_{t+1}} u'(c_{t+1}(\omega^{t+1})) \frac{\Phi(n_t(\omega^t))}{n_t(\omega^t) \Phi_n(n_t(\omega^t))} \left[\begin{array}{l} \eta(\omega_t) + F_2(K, L_t; A) l(\omega_t) \lambda_t(\omega_t) \\ + \frac{1}{\mu_t} \sum_{\omega^{t+1}|\omega^t} \gamma_{t+1}(\omega^{t+1}) \pi(\omega_{t+1}, \omega_t) \end{array} \right]
\end{aligned}$$

Express the utility function of individual with type history ω^t as,

$$\begin{aligned}
&U_t(\omega^t) \\
&= u(c_t(\omega^t)) + \frac{\Phi(n_t(\omega^t)) u'(c_t(\omega^t))}{\Phi_n(n_t(\omega^t)) n_t(\omega^t)} \left[F_2(K, L_t; A) l(\omega_t) + \frac{1}{\mu_t} \gamma_t(\omega^t) - c_t(\omega^t) \right]
\end{aligned}$$

Forward by one period and take expectation,

$$\begin{aligned}
&E_t U_{t+1}(\omega^{t+1}) \\
&= u(c_{t+1}(\omega^{t+1})) \\
&+ u'(c_{t+1}(\omega^{t+1})) E_t \left[\frac{\Phi(n_{t+1}(\omega^{t+1}))}{\Phi_n(n_{t+1}(\omega^{t+1})) n_{t+1}(\omega^{t+1})} \left[\begin{array}{l} F_2(K, L_{t+1}; A) l(\omega_{t+1}) \\ + \frac{\gamma_{t+1}(\omega^{t+1})}{\mu_{t+1}} - c_{t+1}(\omega^{t+1}) \end{array} \right] \right]
\end{aligned}$$

Equate the two expressions of $E_t U_{t+1}(\omega^{t+1})$, and use the assumptions of the utility and altruism functions, $\frac{n_t(\omega^t) \Phi_n(n_t(\omega^t))}{\Phi(n_t(\omega^t))} = \psi$ and $\frac{u'(c_{t+1}(\omega^{t+1})) c_{t+1}(\omega^{t+1})}{u(c_{t+1}(\omega^{t+1}))} = \xi$, we can solve consumption

$$c_{t+1}(\omega^{t+1}) = \frac{\xi}{\psi - \xi} \left[\begin{array}{l} \frac{\mu_t}{\mu_{t+1}} [\eta(\omega_t) + F_2(K, L_t; A) l(\omega_t) \lambda_t(\omega_t)] \\ - F_2(K, L_{t+1}; A) E_t [l(\omega_{t+1}) | \omega_t] \end{array} \right]$$

for all $t \geq 0$. (56) and the resource constraint at time 0 determines $\{c_0(\omega_0)\}_{\omega_0 \in \Omega_0}$. Next let us derive the transversality condition regarding population. Assume the last period is T . The derivative of the Lagrangian to the last period fertility $n_T(\omega^T)$ is

$$-\mu_T N_T(\omega^T) (\eta(\omega_T) + F_2(K, L_T; A) l(\omega_T) \lambda_T(\omega_T))$$

Iterate first order condition with respect to $N_{T+1}(\omega^{T+1})$ and forward by one period,

$$\begin{aligned} \theta_t(\omega^t) &= \theta_{t-1}(\omega^{t-1}) \beta \frac{\Phi(n_{t-1}(\omega^{t-1}))}{n_{t-1}(\omega^{t-1})} = \dots \\ &= \beta^t \theta_0(\omega^0) \left(\frac{N_t(\omega^t)}{N_0(\omega^0) \pi(\omega^t, \omega^0)} \right)^{\psi-1} \end{aligned}$$

then substitute it into the first order condition with respect to $c_t(\omega^t)$,

$$\mu_t = \varphi_0(\omega^0) \left(\frac{N_t(\omega^t)}{N_0(\omega^0) \pi(\omega^t, \omega^0)} \right)^{\psi-1} \beta^t c_t(\omega^t)^{\xi-1}.$$

The condition becomes

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{\beta^T c_T(\omega^T)^{\xi-1} N_T(\omega^T)^\psi}{\pi(\omega^T, \omega^0)^{\psi-1}} (\eta(\omega_T) + F_2(K, L_T; A) l(\omega_T) \lambda_T(\omega_T)) \\ = 0. \end{aligned}$$

■

Proof of Lemma 14. At steady state (36) becomes (37). (38) can be obtained using (35) and the specified functional forms. The law of motion of population (8) becomes (39). Total population is constant and therefore average fertility is equal to 1 as stated by (40). Equations (41) and (42) are steady-state versions of (9) and (7). ■

Proof of Proposition 15. The proof is similar to that of Proposition 5. For the first part of the proposition A is given. $n(\omega_{-1}, \omega)$, $g(\omega_{-1}, \omega)$, Q , K/L , L/N , $c(\omega)$ and MPL are solved by (38), (39), (40), (41), (42),

$$c(\omega) = \frac{\xi Q}{\psi - \xi} [\eta(\omega) + MPL \cdot (l(\omega) \lambda(\omega) - E[l(\omega_{+1}) | \omega] / Q)],$$

and

$$MPL = \left(1 - \alpha \left(\frac{K}{L}, A \right) \right) F \left(\frac{K}{L}, 1; A \right).$$

A change of land is adjusted only by population and labor. The proof of the second part follows the same way as Proposition 5 and is hence omitted. ■

Proof of Proposition 16. Recall that $\xi \equiv \frac{u'(c_{t+1}(\omega^{t+1})) c_{t+1}(\omega^{t+1})}{u(c_{t+1}(\omega^{t+1}))}$ and $\psi \equiv \frac{n_t(\omega^t) \Phi'(n_t(\omega^t))}{\Phi(n_t(\omega^t))}$. Using (44) to solve for $c_t(\omega^t)$ and plug it into initial parent's objective function (43),

then take first order conditions:

$$\begin{aligned} k_t(\omega^t) &: \beta^{t-1} \prod_{j=0}^{t-2} \Phi(n_j(\omega^j)) u'(c_{t-1}(\omega^{t-1})) n_{t-1}(\omega^{t-1}) q_{t-1}(\omega_t, \omega_{t-1}) \\ &= \beta^t \prod_{j=0}^{t-1} \Phi(n_j(\omega^j)) u'(c_t(\omega^t)) \pi(\omega_t, \omega_{t-1}) (r_t + p_t) \end{aligned}$$

for all $t > 0$. Simplify it,

$$u'(c_{t-1}(\omega^{t-1})) = \beta \frac{\Phi(n_{t-1}(\omega^{t-1}))}{n_{t-1}(\omega^{t-1})} u'(c_t(\omega^t)) \frac{r_t + p_t}{p_{t-1}} \quad \text{for all } t > 0. \quad (57)$$

so $c_{t+1}(\omega^{t+1})$ depends on ω^t but not on ω_{t+1} for all $t \geq 0$, it can be written as (46).

$$\begin{aligned} n_t(\omega^t) &: \beta^t \prod_{j=0}^{t-1} \Phi(n_j(\omega^j)) u'(c_t(\omega^t)) \left[\begin{array}{c} \eta(\omega_t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}, \omega_t) k_{t+1}(\omega^{t+1}) \\ + w_t(\omega_t) \lambda(\omega_t) \end{array} \right] \\ &= E_t \sum_{m=t+1}^{\infty} \beta^m \prod_{j=0}^{m-1} \Phi(n_j(\omega^j)) u(c_m(\omega^m)) \frac{\Phi'(n_t(\omega^t))}{\Phi(n_t(\omega^t))} \end{aligned}$$

for all $t \geq 0$. Cancel out $\beta^t \prod_{j=0}^{t-1} \Phi(n_j(\omega^j))$,

$$\begin{aligned} & u'(c_t(\omega^t)) \left[\begin{array}{c} \eta(\omega_t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}, \omega_t) k_{t+1}(\omega^{t+1}) \\ + w_t(\omega_t) \lambda(\omega_t) \end{array} \right] \\ &= \Phi'(n_t(\omega^t)) E_t \sum_{m=t+1}^{\infty} \beta^{m-t} \prod_{j=t+1}^{m-1} \Phi(n_j(\omega^j)) u(c_m(\omega^m)) \end{aligned}$$

where $\prod_{j=t+1}^t \Phi(n_j(\omega^j)) = 1$. Use first order condition with respect to $k_{t+1}(\omega^{t+1})$ and forward by one period to substitute out $u'(c_t(\omega^t))$,

$$\begin{aligned} & \beta \frac{\Phi(n_t(\omega^t)) u'(c_{t+1}(\omega^{t+1}))}{n_t(\omega^t)} \frac{r_{t+1} + p_{t+1}}{p_t} \left[\begin{array}{c} \eta(\omega_t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}, \omega_t) k_{t+1}(\omega^{t+1}) \\ + w_t(\omega_t) \lambda(\omega_t) \end{array} \right] \\ &= \Phi'(n_t(\omega^t)) E_t \sum_{m=t+1}^{\infty} \beta^{m-t} \prod_{j=t+1}^{m-1} \Phi(n_j(\omega^j)) u(c_m(\omega^m)) \end{aligned} \quad (58)$$

where

$$\begin{aligned} & E_t \sum_{m=t+1}^{\infty} \beta^{m-t-1} \prod_{j=t+1}^{m-1} \Phi(n_j(\omega^j)) u(c_m(\omega^m)) \\ &= E_t \left[u(c_{t+1}(\omega^{t+1})) + \Phi(n_{t+1}(\omega^{t+1})) \sum_{m=t+2}^{\infty} \beta^{m-t-1} \prod_{j=t+2}^{m-1} \Phi(n_j(\omega^j)) u(c_m(\omega^m)) \right] \end{aligned} \quad (59)$$

Forward (58) by one period,

$$\begin{aligned} & \frac{\Phi(n_{t+1}(\omega^{t+1}))}{n_{t+1}(\omega^{t+1})} u'(c_{t+2}(\omega^{t+2})) \frac{r_{t+2} + p_{t+2}}{p_{t+1}} \left[\begin{array}{c} \eta(\omega_{t+1}) \\ + \sum_{\omega_{t+2}} q_{t+1}(\omega_{t+2}, \omega_{t+1}) k_{t+2}(\omega^{t+2}) \\ + w_{t+1}(\omega_{t+1}) \lambda(\omega_{t+1}) \end{array} \right] \\ &= \Phi'(n_{t+1}(\omega^{t+1})) E_{t+1} \sum_{m=t+2}^{\infty} \beta^{m-t-1} \prod_{j=t+2}^{m-1} \Phi(n_j(\omega^j)) u(c_m(\omega^m)) \quad \text{for } t \geq 0. \quad (60) \end{aligned}$$

By (57), (58), (59), (60) and the $(t+1)$ -period budget constraint, we have

$$\begin{aligned} & u'(c_{t+1}(\omega^{t+1})) \frac{r_{t+1} + p_{t+1}}{p_t} \left[\eta(\omega_t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}, \omega_t) k_{t+1}(\omega^{t+1}) + w_t(\omega_t) \lambda(\omega_t) \right] \\ &= \psi u(c_{t+1}(\omega^{t+1})) + u'(c_{t+1}(\omega^{t+1})) E_t \left[\begin{array}{c} w_{t+1}(\omega_{t+1}) + (r_{t+1} + p_{t+1}) k_{t+1}(\omega^{t+1}) \\ - c_{t+1}(\omega^{t+1}) \end{array} \right] \end{aligned}$$

By the actuarially fair price of $q(\omega_{t+1}, \omega_t)$, we are able to cancel the term associated with

$$E_t [(r_{t+1} + p_{t+1}) k_{t+1}(\omega^{t+1})]$$

and solve $c_{t+1}(\omega^{t+1})$ as (45) for all $t \geq 0$. $n_t(\omega^t)$ is given by (46) where $n_0(\omega_{-1}, \omega_0) = n_0(\omega_0)$. Notice that $c_{t+1}(\omega^{t+1})$ depends on ω_t for all t , and hence $n_t(\omega^t)$ depends on ω_t and ω_{t-1} for $t \geq 1$ while $n_0(\omega^0)$ depends on ω_0 . Given prices $\{w_t(\omega^t)\}_{t=0, \omega^t \in \Omega_t}^{\infty}$, $\{r_t\}_{t=0}^{\infty}$, $\{k_0(\omega^0)\}_{\omega^0 \in \Omega}$, and $\{N_0(\omega^0)\}_{\omega^0 \in \Omega}$, in equilibrium

$\{c_0(\omega^0)\}_{\omega^0 \in \Omega}$, $\{c_{t+1}(\omega^{t+1})\}_{t=0, \omega^{t+1} \in \Omega_{t+1}}^{\infty}$, $\{n_t(\omega^t)\}_{t=0, \omega_{t-1}, \omega^t \in \Omega_t}^{\infty}$, $\{k_t(\omega^t)\}_{t=1, \omega^t \in \Omega_t}^{\infty}$, $\{N_{t+1}(\omega^{t+1})\}_{t=0, \omega^{t+1} \in \Omega_{t+1}}^{\infty}$, and $\{L_t\}_{t=0}^{\infty}$ are solved by the system of equations consisting of (45), (46), (44), (8) and (7) where $n_0(\omega_{-1}, \omega_0) = n_0(\omega_0)$. Next let us derive the transversality condition associated with population. Assume the last period is T . The derivative of the objective with respect to $n_T(\omega^T)$ is

$$\begin{aligned} & - \lim_{T \rightarrow \infty} \beta^T \prod_{j=0}^{T-1} \Phi(n_j(\omega^j)) c_T(\omega^T)^{\xi-1} (w_T(\omega_T) \lambda(\omega_T) + \eta(\omega_T)) \pi(\omega^T, \omega_0) \\ &= - \lim_{T \rightarrow \infty} \beta^T \left(\frac{N_T(\omega^T)}{N_0(\omega^0)} \right)^{\psi} \frac{c_T(\omega^T)^{\xi-1}}{\pi(\omega^T, \omega_0)^{\psi-1}} (F_2(K, L_T; A) l_T(\omega_T) \lambda(\omega_T) + \eta(\omega_T)) \end{aligned}$$

so the condition becomes

$$\begin{aligned} & \lim_{T \rightarrow \infty} \beta^T \frac{N_T(\omega^T)^{\psi} c_T(\omega^T)^{\xi-1}}{\pi(\omega^T, \omega_0)^{\psi-1}} (F_2(K, L_T; A) l_T(\omega_T) \lambda(\omega_T) + \eta(\omega_T)) \\ &= 0. \end{aligned}$$

When the No-Ponzi game condition holds, we can use (44) to express the present value budget constraint. The condition is

$$\lim_{T \rightarrow \infty} \sum_{\omega^T | \omega_0} \frac{\prod_{j=0}^{T-2} q_j(\omega_{j+1}, \omega_j) n_j(\omega^j)}{\prod_{j=1}^{T-1} (r_j + p_j)} q_{T-1}(\omega_T, \omega_{T-1}) n_{T-1}(\omega^{T-1}) k_T(\omega^T) = 0$$

Since

$$\begin{aligned} & \frac{\prod_{j=0}^{T-2} q_j(\omega_{j+1}, \omega_j) n_j(\omega^j)}{\prod_{j=1}^{T-1} (r_j + p_j)} \\ &= \beta^{T-1} \frac{N_{T-1}(\omega^{T-1})^\psi}{N_0(\omega^0)^\psi \pi(\omega^{T-1}, \omega_0)^\psi} \frac{u'(c_{T-1}(\omega^{T-1}))}{u'(c_0(\omega^0))} \pi(\omega^{T-1}, \omega_0) \end{aligned}$$

the No-ponzi game condition can be written as

$$\begin{aligned} & \lim_{T \rightarrow \infty} \sum_{\omega^T | \omega_0} \beta^{T-1} \frac{N_{T-1}(\omega^{T-1})^{\psi-1}}{N_0(\omega^0)^\psi \pi(\omega^{T-1}, \omega_0)^{\psi-1}} \frac{u'(c_{T-1}(\omega^{T-1}))}{u'(c_0(\omega^0))} p_{T-1} N_T(\omega^T) k_T(\omega^T) \\ &= 0. \end{aligned}$$

■

Proof of Proposition 18. Substitute the allocation of the competitive equilibrium into (56) and pick a $\tilde{\omega}_0$ to normalize $\varphi_0(\tilde{\omega}_0) = 1$, we can solve for $\varphi_0(\omega_0)$ for all $\omega_0 \in \Omega$. Under this set of planner's weight, we can show that the competitive equilibrium allocation $\{c_t^c(\omega^t), n_t^c(\omega^t)\}_{\omega^t \in \Omega_t}$ and $\left\{ \frac{\mu_{t+1}}{\mu_t} \right\}_{t=0}^\infty$ chosen as follows solve the social planner's problem. For a given series of dynasty history $\{\omega^t\}_{t \geq 0}$, set μ_{t+1}/μ_t by

$$\frac{\mu_{t+1}}{\mu_t} = \frac{1}{\psi} \beta(\omega_t) \Phi'(n_t^c(\omega^t)) \frac{u'(c_{t+1}^c(\omega^{t+1}))}{u'(c_t^c(\omega^t))}.$$

According to the Euler equation (46) $\mu_{t+1}/\mu_t = p_t/(p_{t+1} + r_{t+1})$. Because (46) holds for all $\{\omega^t\}_{t \geq 0}$, this holds for all $\{\omega^t\}_{t \geq 0}$. By the consumption formula of the competitive equilibrium (45), the condition for consumption of the planner's problem, equation (36), is satisfied. Summing up the budget constraints multiplied by population of every type history, using the equilibrium condition for land,

$$K = \sum_{\omega^t \in \Omega_t} N_t^c(\omega^t) k_t^c(\omega^t)$$

and using the homogeneity of the production function, the resource constraint

$$F(K, L_t; A) = \sum_{\omega^t} N_t(\omega^t) [c_t(\omega^t) + \eta(\omega_t) n_t(\omega^t)],$$

is satisfied. By the construction of $\varphi_0(\omega_0)$ the condition (56) characterizing the relative magnitude of the initial consumptions in the planner problem is satisfied. The Euler equation of the planner's problem (35) is satisfied by the proposed value of μ_{t+1}/μ_t . The equations characterizing evolution of population, labor supply and the transversality condition are all the same in the two problems. Hence the competitive equilibrium allocation $\{c_t(\omega^t), n_t(\omega^t)\}_{\omega^t \in \Omega_t}$ and the ratio of the Lagrangian multipliers $\left\{ \frac{\mu_{t+1}}{\mu_t} \right\}_{t=0}^\infty$ set above solve the social planner's problem under the chosen set of $\{\varphi_0(\omega_0)\}_{\omega_0 \in \Omega}$. ■

Proof of Proposition 19. Let $\{c_t^P(\omega^t), n_t^P(\omega^t)\}_{t=0}^\infty$ be the solution of the social planner's problem that maximizes SW_2 . We set up the price system as follows. Fix a dynasty with its history $\{\omega^t\}_{t \geq 0}$,

$$\begin{aligned}
p_{t+1} &= p_t \frac{1}{\beta} \frac{n_t^P(\omega^t)}{\Phi(n_t^P(\omega^t))} \frac{u'(c_t^P(\omega^t))}{u'(c_{t+1}^P(\omega^{t+1}))} - r_{t+1} \\
p_0 &= \sum_{t=0}^{\infty} \beta^{t+1} \left(\prod_{j=0}^t \frac{\Phi(n_j^P(\omega^j))}{n_j^P(\omega^j)} \right) \frac{u'(c_{t+1}^P(\omega^{t+1}))}{u'(c_0^P(\omega^0))} r_{t+1} \\
q_t(\omega_{t+1}, \omega_t) &= p_t \pi(\omega_{t+1}, \omega_t) \\
r_t &= F_K(K, L^P; A) \\
w_t(\omega_t) &= F_L(K, L^P; A) l(\omega_t),
\end{aligned}$$

L_t^P and $N_{t+1}^P(\omega^{t+1})$ are determined by equations (7) and (8) while fertilities are $\{n_t^P(\omega^t)\}_{t=0}^\infty$. Under this set of price and the social planner's allocations, we can solve for $\{k_0(\omega_0)\}_{\omega_0 \in \Omega_0}$ using the present value of the resource constraint

$$\begin{aligned}
& \sum_{t=0}^{\infty} \sum_{\omega^t | \omega_0} \frac{\prod_{j=0}^{t-1} n_j^P(\omega^j) q_j(\omega_{j+1}, \omega_j)}{\prod_{j=0}^{t-1} (r_{j+1} + p_{j+1})} (c_t^P(\omega^t) + \eta(\omega_t) n_t^P(\omega^t)) \\
& \leq (r_0 + p_0) k_0(\omega^0) \\
& + \sum_{t=0}^{\infty} \sum_{\omega^t | \omega_0} \frac{\prod_{j=0}^{t-1} n_j^P(\omega^j) q_j(\omega_{j+1}, \omega_j)}{\prod_{j=0}^{t-1} (r_{j+1} + p_{j+1})} w_t(\omega_t) (1 - \lambda(\omega_t) n_t^P(\omega^t))
\end{aligned}$$

$\{k_{t+1}(\omega^{t+1})\}_{\omega^{t+1} \in \Omega_{t+1}, t \geq 0}$ are solved by (44). Under the initial capital allocation $\{k_0(\omega_0)\}_{\omega_0 \in \Omega_0}$ solved above, we can show that the allocations of the social planner's problem satisfy all the equations characterizing competitive equilibrium in Proposition 16. The set of price constructed implies the return to land in the competitive equilibrium to be equal to the ratio of the resource across periods in the planner's problem, that is $(r_{t+1} + p_{t+1})/p_t = \mu_t/\mu_{t+1}$. This equality and the consumption formulation of the planner's problem leads to (45). Since the Euler equation (35) of the planner's problem holds for all $\omega^t \in \Omega_t$ and all $t \geq 0$, (46) is satisfied by the construction of prices for all $\omega^t \in \Omega_t$ and $t \geq 0$. Budget constraints are satisfied because capital allocations $\{k_{t+1}(\omega^{t+1})\}_{\omega^{t+1} \in \Omega_{t+1}, t \geq 0}$ are chosen based on them. The transversality condition of the two problems coincide. The way we pick for $\{k_0(\omega_0)\}_{\omega_0 \in \Omega_0}$ and $\{k_{t+1}(\omega^{t+1})\}_{\omega^{t+1} \in \Omega_{t+1}, t \geq 0}$ guarantees the No-ponzi game. The planner's allocation satisfies the resource constraints. Summing up the budget constraints among different ω^t in every period and use the resource constraint, we obtain

$$K = \sum_{\omega^t \in \Omega_t} N_t^P(\omega^t) k_t(\omega^t) \text{ for all } \omega^t \in \Omega_t \text{ and } t \geq 0.$$

The prices constructed above are the ones in competitive equilibrium. The initial land distribution $\{k_0(\omega_0)\}_{\omega_0 \in \Omega_0}$ decentralizes the social planner's problem with given weights $\{\varphi_0(\omega_0)\}_{\omega_0 \in \Omega}$. ■

A.2. Deterministic case with one type.

A.2.1. Case $\varphi_t = \delta^t$ and $\delta \geq \beta$.

Proposition 20 *Assume $\eta > 0$. (i) If $\delta > \beta$, a steady state satisfies the following equations:*

$$c = \frac{\xi(c) \eta \left(\frac{\alpha \delta}{1-\delta} - \frac{(1-\alpha)\lambda}{1-\lambda} - 1 \right)}{\frac{\delta-\beta}{(1-\beta)(1-\delta)} - \frac{\beta \Phi'(1)}{1-\beta} - \xi(c) \left(\frac{\alpha \delta}{\delta-1} - \frac{(1-\alpha)\lambda}{1-\lambda} \right)}$$

$$F\left(\frac{K}{N}, 1-\lambda; A\right) = c + \eta \quad (61)$$

and

$$\frac{\theta_{t+1}}{\theta_t} = \frac{\mu_{t+1}}{\mu_t} = \frac{\gamma_{t+1}}{\gamma_t} = \delta.$$

In this case Malthusian stagnation property holds. (ii) If $\delta = \beta$, the steady state does not exist.

Proof. (i) When $\eta > 0$ and N is finite, one can first show that $\theta_{t+1}/\theta_t = \delta$. Otherwise if $\theta_{t+1}/\theta_t > \delta$, then in the limit, according to (21),

$$\theta_{t+1}/\theta_t = \beta + \frac{\delta^{t+1}}{\theta_t} \quad (62)$$

then $\theta_{t+1}/\theta_t = \beta < \delta$, a contradiction. If $\theta_{t+1}/\theta_t < \delta$, then the right hand side of (62) explodes which also leads to a contradiction. (17) then implies that the growth rate of μ_t is the same as that of θ_t , which is δ . Furthermore, (22) at steady state simplifies to:

$$U - \frac{\mu_{t+1}}{\delta^{t+1}} F_K \frac{K}{N} = \frac{\gamma_{t+1}}{\delta^{t+1}} \left(\frac{\gamma_{t+2}}{\gamma_{t+1}} - 1 \right).$$

The left hand side of this equality is constant in steady state since the growth rate of μ is δ . Then for the right hand side to converge to a constant we have the following three possibilities: γ_t grows at a rate smaller than δ , γ_t grows at the rate δ , and γ_t keeps constant over time, e.g. $\gamma_{t+2}/\gamma_{t+1} = 1$.

Consider the first possibility when γ_t grows at a rate smaller than δ , then

$$U = \frac{\mu_{t+1}}{\delta^{t+1}} F_K \frac{K}{N} \quad (63)$$

Express (21) and (17) at steady state,

$$\theta_t = \frac{\delta^{t+1}}{\delta - \beta}$$

$$\mu_t = \theta_t u'(c) = \frac{\delta^{t+1}}{\delta - \beta} u'(c) \quad (64)$$

Plug it into (63) and use $U = u(c) / (1 - \beta)$ at steady state,

$$c = \frac{\delta(1 - \beta)}{\delta - \beta} \xi(c) F_K \frac{K}{N}$$

By the constant returns to scale assumption and the definition of α above, it can be written as

$$\frac{c}{\xi(c)} = \frac{\delta(1 - \beta)}{\delta - \beta} \alpha \frac{F(K, L; A)}{N} \quad (65)$$

which together with (61) and $L = N(1 - \lambda)$ can be used to solve (c, N) . Given that the growth rate of γ_t is smaller than that of μ_t according to (64), we express (23) at steady state as

$$\beta \Phi'(1) \frac{c}{\xi(c)} \frac{1}{1 - \beta} = \eta + (1 - \alpha) \frac{F(K, L; A)}{N} \frac{\lambda}{1 - \lambda} \quad (66)$$

Use $\Phi'(1) = \psi$ and (61) we solve consumption as

$$c = \frac{\xi(c) \frac{1-\beta}{\beta\psi} [1 + (1 - \alpha) \frac{\lambda}{1-\lambda}] \eta}{1 - \xi(c) \frac{1-\beta}{\beta\psi} (1 - \alpha) \frac{\lambda}{1-\lambda}}.$$

(c, N) solved from (65) and (61) does not satisfy (66) in general. Therefore, in the case of δ bigger than β , the steady state with each multiplier growing at a constant rate, in particular γ_t growing at a constant rate smaller than δ , is not the optimal solution except for a knife-edge condition in which (c, N) satisfies (61), (65), and (66) simultaneously.

Next consider the second possibility when γ_t grows at the rate of δ . Express (21), (23) and (17) at steady state, respectively, as

$$\theta_t = \frac{\delta^{t+1}}{\delta - \beta}, \quad (67)$$

$$\gamma_t = \frac{1}{\delta} \mu_t \left[\beta \Phi'(1) \frac{U}{u'(c)} - \eta - F_L \lambda \right], \quad (68)$$

and

$$\mu_t = \theta_t u'(c) = \frac{\delta^{t+1}}{\delta - \beta} u'(c). \quad (69)$$

Plug (68) into the steady-state formula of (22), we get

$$\frac{\delta^{t+1}}{\gamma_{t+1}} U = \frac{\delta}{\beta \Phi'(1) \frac{U}{u'(c)} - \eta - F_L \lambda} F_K \frac{K}{N} + \delta - 1$$

Use (67), (68) and (69) we get

$$(\delta - \beta) U = \delta u'(c) F_K \frac{K}{N} + \frac{\delta - 1}{\delta} \frac{\mu_{t+1}}{\mu_t} u'(c) \left[\beta \Phi'(1) \frac{U}{u'(c)} - \eta - F_L \lambda \right]$$

We have shown that μ_t grows at the rate δ , then

$$\begin{aligned} & \frac{1}{1-\beta} \\ &= \frac{\delta-1}{\delta-\beta} \frac{\xi(c)}{c} \left[\beta \Phi'(1) \frac{U}{u'(c)} - \eta - F_L \lambda + \frac{\delta}{\delta-1} F_K \frac{K}{N} \right] \end{aligned}$$

Note that

$$\begin{aligned} F_L &= \left(\frac{1-\alpha}{1-\lambda} \right) \frac{F}{N} \\ F_K \frac{K}{N} &= \alpha \frac{F}{N}. \end{aligned}$$

So the above equality becomes

$$\frac{\delta-\beta}{(1-\beta)(\delta-1)} = \frac{\xi(c)}{c} \left[\begin{aligned} & \beta \Phi'(1) c \frac{1}{\xi(c)} \frac{1}{1-\beta} - \eta \\ & + F \left(\frac{K}{N}, 1-\lambda; A \right) \left(\frac{\alpha\delta}{\delta-1} - \frac{(1-\alpha)\lambda}{1-\lambda} \right) \end{aligned} \right].$$

which together with (61) solves (c, N) . Substituting out N , c can be solved as

$$c = \frac{\xi(c) \eta \left(\frac{\alpha\delta}{1-\delta} - \frac{(1-\alpha)\lambda}{1-\lambda} - 1 \right)}{\frac{\delta-\beta}{(1-\beta)(1-\delta)} - \frac{\beta\Phi'(1)}{1-\beta} - \xi(c) \left(\frac{\alpha\delta}{\delta-1} - \frac{(1-\alpha)\lambda}{1-\lambda} \right)}$$

This formula of consumption implies Malthusian stagnation property.

For the third possibility $\gamma_{t+1}/\gamma_t = 1$ at steady state, which together with $\mu_{t+1}/\mu_t = \delta < 1$ contradicts with equation (23).

(ii) If $\delta = \beta$, (21) becomes

$$\frac{\theta_{t+1}}{\theta_t} = \frac{\beta^{t+1}}{\theta_t} + \beta \geq \beta.$$

If $\lim_{t \rightarrow \infty} \theta_{t+1}/\theta_t = \beta$, then $\lim_{t \rightarrow \infty} \beta^{t+1}/\theta_t > 0$ and θ_{t+1}/θ_t converges to a number strictly bigger than β which leads to a contradiction. If $\lim_{t \rightarrow \infty} \theta_{t+1}/\theta_t > \beta$, then $\lim_{t \rightarrow \infty} \theta_{t+1}/\theta_t = \lim_{t \rightarrow \infty} \beta^{t+1}/\theta_t = 0$, a contradiction too. Hence steady state with Lagrangian multipliers growing at constant rate over time does not exist in this case.

■

A.2.2. Stability of the steady state

To get some insights about the stability of the steady state, in this section we also focus on the deterministic case with one type and assume $\varphi_0(\omega_0) = 1$ but $\varphi_{t+1}(\omega^{t+1}) = 0$ for all $t > 0$. In that case the social planner cares about future generations to the extent that the initial generation does. Furthermore, assume no time costs of raising children, $\lambda = 0$, Barro-Becker's functional forms $\Phi(n) = n^\psi$, $u(c) = c^\xi/\xi$, a Cobb-Douglas production function $F(K, N_t; A) = AK^\alpha N_t^{1-\alpha}$, and $N_0 = 1$. The restriction $\psi > \xi$ is required for concavity.

Initial parent's utility is then given by

$$U_0 = u(c_0) + \beta n_0^\psi U_1 = \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t-1} n_j^\psi u(c_t) = \sum_{t=0}^{\infty} \beta^t N_t^\psi \frac{1}{\xi} c_t^\xi.$$

The following is the social planner's problem:

$$\max_{\{C_t, N_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t^\psi \frac{1}{\xi} c_t^\xi \text{ subject to } c_t N_t = F(K, L_t; A) - N_{t+1} \eta.$$

Substitute the budget constraint into the objective function,

$$\max_{\{C_t, N_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t^\psi \frac{1}{\xi} \left(\frac{F(K, N_t; A) - N_{t+1} \eta}{N_t} \right)^\xi$$

The optimal choice of population in period t is

$$N_t^{\psi-\xi} C_t^{\xi-1} \eta = \beta N_{t+1}^{\psi-\xi} C_{t+1}^{\xi-1} \left[\left(\frac{\psi}{\xi} - \alpha \right) AK^\alpha N_{t+1}^{-\alpha} - \eta \frac{\psi - \xi}{\xi} \frac{N_{t+2}}{N_{t+1}} \right]$$

where $C_t = c_t N_t$ is the aggregate consumption of all people of generation t . It is convenient to define the variable $X_t \equiv N_t^{\frac{\xi-\psi}{1-\xi}} C_t$, a mix between aggregate consumption and a factor that depends on population. Then the dynamic system can be described by the following two equations:

$$X_t = N_t^{\frac{\xi-\psi}{1-\xi}} [AK^\alpha N_t^{1-\alpha} - N_{t+1} \eta]$$

$$\left(\frac{X_{t+1}}{X_t} \right)^{1-\xi} \eta = \beta \left[\left(\frac{\psi}{\xi} - \alpha \right) AK^\alpha N_{t+1}^{-\alpha} - \eta \frac{\psi - \xi}{\xi} \frac{N_{t+2}}{N_{t+1}} \right]$$

The first equation is the resource constraint and the second is the optimality condition for population. Steady state population, N^* , can be solved as

$$AK^\alpha N^{*\alpha} = \frac{\psi - \xi + \xi/\beta}{\psi - \alpha\xi} \eta.$$

Next we take a first-order Taylor expansion of this system around the steady state to analyze its stability. It is determined by the system of equations (70):

$$W \begin{bmatrix} \frac{dN_{t+2}}{N^*} \\ \frac{dX_{t+1}}{X^*} \end{bmatrix} = G \begin{bmatrix} \frac{dN_{t+1}}{N^*} \\ \frac{dX_t}{X^*} \end{bmatrix} \quad (70)$$

where

$$W = \begin{bmatrix} \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} & 1 \\ \beta \frac{\psi-\xi}{\xi} & 1 - \xi \end{bmatrix}$$

and

$$G = \begin{bmatrix} (\xi - \psi) / (1 - \xi) + (1 - \alpha) \left(1 + \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X} \right) & 0 \\ (1 - \alpha) \beta \frac{\psi-\xi}{\xi} - \alpha & 1 - \xi \end{bmatrix}.$$

The following proposition provides the necessary and sufficient conditions under which the steady state is saddle path stable.

Proposition 21 *The necessary and sufficient conditions for saddle path stability of the steady state are*

$$(2 - \alpha) \frac{\frac{\psi - \xi}{\xi} (1 - 2\xi + (1 - \alpha)\beta) + 2(1 - \alpha)(1 - \xi)}{1/\beta - (1 - \alpha)} > \alpha(1 - 2\xi) - 2(1 - \psi)$$

and

$$\frac{1}{\beta} (\alpha(1 - \xi) - 1 + \psi) \neq (1 - \alpha) \frac{\psi - \xi}{\xi}.$$

Proof. Equation (70) can be written as:

$$\begin{bmatrix} \frac{dN_{t+2}}{N^*} \\ \frac{dX_{t+1}}{X^*} \end{bmatrix} = D \begin{bmatrix} \frac{dN_{t+1}}{N^*} \\ \frac{dX_t}{X^*} \end{bmatrix}.$$

where

$$\begin{aligned} D &\equiv W^{-1}G \\ &= \begin{bmatrix} \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} & 1 \\ \beta \frac{\psi - \xi}{\xi} & 1 - \xi \end{bmatrix}^{-1} \begin{bmatrix} \frac{\xi - \psi}{1 - \xi} + (1 - \alpha) \left(1 + \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} \right) & 0 \\ (1 - \alpha) \beta \frac{\psi - \xi}{\xi} - \alpha & 1 - \xi \end{bmatrix} \\ &= \frac{1}{d} \begin{bmatrix} \left(\begin{array}{c} \xi - \psi + (1 - \xi)(1 - \alpha) \left(1 + \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} \right) \\ + \alpha - (1 - \alpha) \beta \frac{\psi - \xi}{\xi} \end{array} \right) & \xi - 1 \\ -\beta \frac{\psi - \xi}{\xi} \left((1 - \psi) / (1 - \xi) - \alpha \right) - \alpha \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} & \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} (1 - \xi) \end{bmatrix} \end{aligned}$$

with $d = \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} (1 - \xi) - \beta \frac{\psi - \xi}{\xi}$. Let λ_1 and λ_2 denote the eigenvalues of the matrix D . Assume, without loss of generality, that $\lambda_1 > \lambda_2$. They are solved by

$$\begin{aligned} \lambda_1 &= \frac{tr(D) + \sqrt{tr(D)^2 - 4 \det(D)}}{2} \\ \lambda_2 &= \frac{tr(D) - \sqrt{tr(D)^2 - 4 \det(D)}}{2} \end{aligned}$$

where $\det(D)$ and $tr(D)$ can be solved as

$$\begin{aligned} &d^2 \det(D) \\ &= \left\{ \begin{array}{l} \left[\begin{array}{c} \xi - \psi + (1 - \xi)(1 - \alpha) \left(1 + \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} \right) \\ + \alpha - (1 - \alpha) \beta \frac{\psi - \xi}{\xi} \end{array} \right] \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} (1 - \xi) \\ - (1 - \xi) \left[\beta \frac{\psi - \xi}{\xi} \left((1 - \psi) / (1 - \xi) - \alpha \right) + \alpha \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} \right] \end{array} \right\} \\ &d \cdot tr(D) = \left\{ \begin{array}{l} \xi - \psi + (1 - \xi)(1 - \alpha) \left(1 + \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} \right) + \alpha \\ - (1 - \alpha) \beta \frac{\psi - \xi}{\xi} + \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} (1 - \xi) \end{array} \right\} \end{aligned}$$

For saddle-path stability either $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Since $\lambda_1 > \lambda_2$ by assumption this condition can be divided into two cases: (i) $\lambda_1 > 1$ and

$-1 < \lambda_2 < 1$ and (ii) $\lambda_2 < -1$ and $-1 < \lambda_1 < 1$. Let us first consider case (i) which can be reduced to $1 - \text{tr}(D) < -\det(D) < 1 + \text{tr}(D)$, and then to

$$(1 - D_{11})(1 - D_{22}) < D_{12}D_{21} < (1 + D_{11})(1 + D_{22}),$$

where D_{ij} refers to the (i, j) element of matrix D . The condition in case (ii) which can be reduced to $\text{tr}(D) + 1 < -\det(D) < -\text{tr}(D) + 1$, and then to

$$(1 + D_{11})(1 + D_{22}) < D_{12}D_{21} < (1 - D_{11})(1 - D_{22}).$$

Now let us derive terms in these conditions.

$$\begin{aligned} & d^2 D_{12}D_{21} \\ &= \left(-\beta \frac{\psi - \xi}{\xi} \left((1 - \psi) / (1 - \xi) - \alpha \right) - \alpha \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} \right) (\xi - 1) \\ d(1 - D_{11}) &= \psi - (1 + \alpha\xi) + \alpha(1 - \xi) \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} - \alpha\beta \frac{\psi - \xi}{\xi} \\ d(1 - D_{22}) &= -\beta \frac{\psi - \xi}{\xi} \end{aligned}$$

$$\begin{aligned} & d^2 (1 - D_{11})(1 - D_{22}) \\ &= -\beta \frac{\psi - \xi}{\xi} \left(\psi - 1 - \alpha\xi + \alpha(1 - \xi) \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} - \alpha\beta \frac{\psi - \xi}{\xi} \right) \end{aligned}$$

Use the above result, derive the condition $(1 - D_{11})(1 - D_{22}) < D_{12}D_{21}$ in case (i). It holds if and only if

$$\eta(1 - \xi) \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} > \beta \frac{\psi - \xi}{\xi}. \quad (71)$$

At steady state,

$$\eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} = \frac{\psi/\xi - \alpha}{1/\beta - (1 - \alpha)} \quad (72)$$

Substitute out $\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}$ using (72) it becomes

$$\frac{1}{\beta} (-1 + \psi + \alpha(1 - \xi)) < \frac{\psi - \xi}{\xi} (1 - \alpha). \quad (73)$$

$$\begin{aligned} & d^2 (1 + D_{11})(1 + D_{22}) \\ &= \left[(2 - \alpha) \eta \frac{N^{\frac{1-\psi}{1-\xi}}}{X} (1 - \xi) - (2 - \alpha) \beta \frac{\psi - \xi}{\xi} + 1 - \psi + \alpha\xi \right] \left(2\eta \frac{N^{\frac{1-\psi}{1-\xi}}}{X} (1 - \xi) - \beta \frac{\psi - \xi}{\xi} \right) \end{aligned}$$

$D_{12}D_{21} < (1 + D_{11})(1 + D_{22})$ holds if and only if

$$\begin{aligned} & \begin{pmatrix} 2(2 - \alpha) \eta (1 - \xi) \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} \\ -(2 - \alpha) \beta \frac{\psi - \xi}{\xi} + 2(1 - \psi) \\ +\alpha - 2\alpha(1 - \xi) \end{pmatrix} \begin{pmatrix} \eta(1 - \xi) \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} \\ -\beta \frac{\psi - \xi}{\xi} \end{pmatrix} \\ & > 0 \end{aligned}$$

When (71) holds true, this inequality holds if and only if

$$2(2 - \alpha)\eta(1 - \xi) \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} - (2 - \alpha)\beta \frac{\psi - \xi}{\xi} > 2\alpha(1 - \xi) - 2(1 - \psi) - \alpha.$$

Substitute out $\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}$ using (72),

$$(2 - \alpha) \frac{\frac{\psi - \xi}{\xi} (1 - 2\xi + (1 - \alpha)\beta) + 2(1 - \alpha)(1 - \xi)}{1/\beta - (1 - \alpha)} > \alpha(1 - 2\xi) - 2(1 - \psi) \quad (74)$$

Hence the condition for case (i) is (73) and (74). In the same way we can derive that the condition for case (ii) is the following two inequalities

$$\frac{1}{\beta} (\alpha(1 - \xi) - 1 + \psi) > (1 - \alpha) \frac{\psi - \xi}{\xi}$$

and (74). Summarize case (i) and case (ii), the sufficient and necessary condition for saddle path stability is (74) and

$$\frac{1}{\beta} (\alpha(1 - \xi) - 1 + \psi) \neq (1 - \alpha) \frac{\psi - \xi}{\xi}.$$

■

Given $\xi < \psi$, the Barro-Becker's assumption for the concavity of the problem, this condition holds for most sets of parameters. The second condition holds except for a wide range of parameters. In particular, a nice sufficient condition guarantees saddle path stability is $\xi < \frac{1}{2}$ and $\alpha(1 - 2\xi) \leq 2(1 - \psi)$. We summarize it in the following Corollary.

Corollary 22 *A sufficient condition for saddle path stability of the steady state are $\xi < \frac{1}{2}$ and $\alpha(1 - 2\xi) < 2(1 - \psi)$.*

Under this condition, the left hand side of the first part of the condition is positive while its right hand side is negative.