Short-Term Investments and Indices of Risk

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Abstract

We study various decision problems regarding short-term investments in risky assets whose returns evolve continuously in time. We show that in each problem, all risk-averse decision makers have the same (problem-dependent) ranking over short-term risky assets. Moreover, in each problem, the ranking is represented by the same risk index as in the case of CARA utility agents and normally distributed risky assets.

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1 Introduction

We study various decision problems regarding investments in risky assets (henceforth, gambles), such as whether to accept a gamble, or how to choose the optimal capital allocation. To rank the desirability of gambles with respect to the relevant decision problem, it is often helpful to use an objective riskiness index that is independent of any specific subjective utility. For example, an objective riskiness index is needed when pension funds are required not to exceed a stated level of riskiness (see, e.g., the discussion in Aumann & Serrano, 2008, p. 812).

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We analyze four decision problems that are important in economic settings. In general, different risk-averse agents rank the desirability of gambles differently. However, our main result shows that in each of these problems, all risk-averse agents have the same (problem-dependent) ranking over short-term investments in risky assets whose returns evolve continuously. Moreover, in each problem, the ranking is represented by the same risk index obtained in the commonly used mean-variance preferences (e.g., Markowitz, 1952), which are induced by CARA utility agents and normally distributed gambles.

Brief Description of the Model We consider an agent who has to make an investment decision related to a gamble. We think of a gamble as the additive return on a financial investment. We assume that the agent has (1) an initial wealth w, and (2) a von Neumann–Morgenstern utility u that is increasing and risk-averse (i.e, $u' > 0$ and $u'' < 0$). We assume that a gamble is represented by a random variable with (1) positive expectation, and (2) some negative values in its support. For each problem the agents' choices are modeled by a *decision function* that assigns a number to each agent and each gamble, where a higher number is interpreted as the agent finding the gamble to be more attractive (i.e., less risky) for the relevant decision problem.

We study four decision problems in the paper: (1) *acceptance/rejection*, in which the agent faces a binary choice between accepting and rejecting the gamble (e.g., Hart, 2011); (2) *capital allocation*, in which the agent has a continuous choice of how much to invest in the gamble (e.g., Markowitz, 1952; Sharpe, 1964); (3) the *optimal certainty equivalent*, in which the agent evaluates how much an opportunity to invest in the gamble (according to the optimal investment level) is worth to the agent (e.g., Hellman & Schreiber, 2018); and (4) *risk premium,* in which the agent evaluates how much investing in the gamble is inferior to obtaining the gamble's expected payoff (Arrow, 1970).[1]

A *risk index* is a function that assigns to each gamble a nonnegative number, which is interpreted as the gamble's riskiness. We say that a risk index is *consistent* with a decision function f over some set of agents and gambles, if each agent in the set ranks all gambles in the set according to that risk index; that is, f assigns for each agent a higher value for gamble g than for gamble g' iff the risk index assigns a lower value to g. A *risk-aversion* index is a function that assigns to each agent a non-negative number, which is interpreted as the agent's risk aversion. We say that a risk-aversion index is *consistent*

[1]We use "risk premium" in its common acceptation in the economic literature since Arrow (1970). In the financial literature (and in practice), the "risk premium" of a security commonly has a somewhat different meaning, namely, the security return less the risk-free interest rate (e.g., Cochrane, 2009).

with a decision function over some set of agents and gambles, if for each gamble and each pair of agents in the set, the agent with the higher index of risk-aversion invests less in the gamble than the other agent. Observe that different decision functions may correspond to different concepts of risks, and, may induce different indices of risk and of risk aversion.

Summary of Results Agents, typically, have heterogeneous rankings of gambles, and, thus, no risk index (nor risk-aversion index) can be consistent with the rankings of all agents, unless one restricts the set of gambles. Our main result restricts the set of gambles to assets whose returns evolve continuously in time, where the local uncertainty is induced by a Wiener process. Specifically, we focus on Ito processes which are continuous-time Markov processes. The class of Ito processes, is commonly used in economic and financial applications and includes, in particular, the geometric Brownian motion and mean-reverting processes (e.g., Merton, 1992).[2]

Our main result shows that in each of the four decision problems discussed above, all agents rank all gambles in the same (problem-dependent) way when they have to decide on short-term investments in gambles whose returns evolve continuously in time. Moreover, the risk indices that are consistent with these decision functions are the same as in the classic model of agents with CARA (exponential) utilities and normally distributed gambles. Specifically, we show that: (1) the variance-to-mean index $Q_{VM}(g) = \frac{\sigma^2[g]}{\boldsymbol{E}[g]}$ is consistent with both the capital allocation function and the acceptance/rejection function, (2) the inverse Sharpe index $Q_{IS}(g) = \frac{\sigma[g]}{\boldsymbol{E}[g]}$ is consistent with the optimal certainty equivalent function, and (3) the standard deviation index $Q_{SD}(g) = \sigma[g]$ is consistent with the risk premium function. Finally, we adapt the classic results of Pratt (1964) and Arrow (1970) to the present setup, and show that the local Arrow–Pratt coefficient of absolute risk aversion $\rho(u, w) = \frac{-u''(w)}{u'(w)}$ is consistent with all four decision functions.

Related Literature and Contribution Aumann & Serrano (2008) and Foster & Hart (2009) presented two "objective" indices of riskiness of gambles, which are independent of the subjective utility of the agent. These indices are either based on reasonable axioms that an index of risk should satisfy (e.g., Artzner *et al.*, 1999; Aumann & Serrano, 2008; Cherny & Madan, 2009; Foster & Hart, 2013; Schreiber, 2014; Hellman & Schreiber, 2018; see also the recent survey of Föllmer & Weber, 2015), or they are based on an "operative" criterion such as an agent never going bankrupt when relying on an index of risk in

[2]Sec. 4.5 demonstrates that our results *cannot* be extended to continuous-time processes with jumps.

deciding whether to accept a gamble (Foster & Hart, 2009; and see also Meilijson, 2009, for a discussion of operative implication of Aumann & Serrano's index of risk).[3]

We argue that risk is a multidimensional attribute that crucially depends on the investment problem. Different aspects of risk are relevant when an agent has to decide whether to accept a gamble, compared with a situation in which an agent has to choose how much to invest in a gamble, or has to evaluate the certainty equivalent of the optimal investment. Many existing papers focus on a single decision function. By contrast, we suggest a framework for studying various decision problems, and associate each such problem with its relevant index of risk. We believe that this general framework may be helpful in future research on risk indices.

In general, different agents make different investment decisions, based on the subjective utility of each agent. Thus, a single risk index cannot be consistent with the choices of all agents, which, arguably, limits the index's objectiveness (even when the index satisfies appealing axioms or some operative criterion for avoiding bankruptcy). However, our main result shows that in various important decision problems, all agents rank all gambles in the same way when deciding on short-term investments in gambles whose returns evolve continuously in time. This finding enables us to construct objective risk indices that are consistent with the short-term investment decisions of risk-averse agents.

There are pairs of gambles for which all risk-averse agents agree on which one of the gambles is more desirable. This happens if one gamble second-order stochastically dominates the other gamble. However, the well-known order of stochastic dominance (Hadar & Russell, 1969; Hanoch & Levy, 1969; Rothschild & Stiglitz, 1970) is only a partial order and "most" pairs of gambles are incomparable. Interestingly, even if one gamble second-order stochastically dominates another gamble, it is not sufficient for a uniform ranking among all risk-averse agents in every decision problem (see, e.g., the analysis of capital allocation decisions in Landsberger & Meilijson, 1993).

A large body of literature uses the classic mean-variance capital asset pricing model (Markowitz, 1952; see Smetters & Zhang, 2013; Kadan *et al.*, 2016 for recent extensions). A well-known critique is that in a discrete-time setup the mean-variance preferences are consistent with expected utility maximization only under severe restrictions, such as CARA utilities and normally distributed gambles (see, e.g., Borch, 1969; Feld-

[3] Aumann & Serrano's (2008) and Foster & Hart's (2009) indices of risk have been extended to gambles with an infinite support (Homm & Pigorsch, 2012; Schulze, 2014; Riedel & Hellmann, 2015) and to gambles with unknown probabilities (Michaeli, 2014). These indices have been applied to study real-life investment strategies in Kadan & Liu (2014); Bali *et al.* (2015); Anand *et al.* (2016); Leiss & Nax (2018).

stein, 1969; Hakansson, 1971). By contrast, the seminal results of Robert Merton (as summarized and discussed in Merton, 1975, 1992) show that in a continuous-time model with log-normally distributed asset prices mean-variance preferences are consistent with the optimal portfolio allocation of all risk-averse agents. Merton's results present an important theoretical foundation for the classic model.

The present paper extends this idea by showing that the equivalence between the decisions of agents with CARA utilities with respect to normally distributed gambles and the decisions of risk-averse agents with respect to short-term investments holds more broadly: (1) it holds with respect to various decision functions beyond optimal portfolio allocation, and (2) it holds with respect to a broad class of continuous-time processes beyond log-normally distributed asset prices. On the other hand, our analysis is less general than Merton's in that we analyze situations in which the agent acts only at the beginning, at time zero, and cares about his wealth at a single future nearby point $t > 0$, rather than allowing the agent to act continuously in time.

Our paper is also related to the literature on local risks. This literature focuses on discrete-time gambles, rather than continuous-time returns (which are the focus of the present paper). Pratt (1964) shows that if the distribution of the returns is sufficiently concentrated, i.e., the third absolute central moment is sufficiently small relative to variance, then for any agent, the magnitude of the risk premium depends on the local level of the agent's risk aversion. Samuelson (1970) shows that classic mean-variance analysis (Markowitz, 1952), applies approximately to all utility functions in situations that involve what he calls "compact" distributions. More recently, Schreiber (2015) shows that if one gamble is riskier than another gamble according to the Aumann & Serrano's index of risk, then every decision maker who is willing to accept a small proportion of the riskier gamble is also willing to accept the same proportion of the less risky gamble.

In this context, two papers are close to the present paper: Shorrer (2014) and Schreiber (2016). Shorrer shows that there exist risk indices that are consistent with the acceptance/rejection decisions of all risk-averse agents with respect to bounded discrete gambles with sufficiently small support. This result is similar to our characterization of risk indices that are consistent with various short-term investment decisions of all risk-averse agents with respect to assets whose (possibly, unbounded) returns evolve continuously in time. Shorrer's main result shows that by adding a few additional axioms, one can uniquely choose Aumann & Serrano's index among all the indices that are consistent with agents' acceptance/rejection with respect to small discrete gambles.[4] In principle, one could apply a similar axiomatic method to our three other decision functions; we leave this interesting research direction for future research (for further discussion see Sec. 5). Unlike the other papers mentioned above, Schreiber (2016) deals with returns in the continuous-time setup. Specifically, he analyzes acceptance and rejection of short-term investments. The key contributions of the present paper with respect to Schreiber (2016) consists in, first, extending the analysis to the other three decision functions (namely, capital allocation, optimal certainty equivalent, and risk premium) and, second, showing in all four cases an equivalence to the indices in the exponential-normal setup.

Structure In Section 2 we present our model. In Section 3 we analyze the benchmark setup of CARA utilities and normally distributed gambles. In Section 4 we adapt the model to study risky assets whose returns evolve continuously in time, and present our main result. We conclude with a discussion in Section 5. Appendix A extends our model to multiplicative gambles. The formal proofs are presented in Appendix B.

2 Model

We consider an agent who has to make an investment decision related to a risky asset. We begin by defining each of these components: agent, risky asset, and investment decision.

A *decision maker* (or *agent*) is modeled as a pair (u, w), where $u : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable von Neumann–Morgenstern utility function over wealth satisfying $u' > 0$ (i.e., utility is increasing in wealth) and $u'' < 0$ (i.e., risk aversion), and $w \in \mathbb{R}$ is an initial wealth level. Let $\mathcal{DM}$ denote the set of all such decision makers.

A *gamble* g is a real-valued random variable with a positive expectation and some negative values (i.e., $0 < \boldsymbol{E}[g]$, and $\boldsymbol{P}[g < 0] > 0$). We think of a gamble as the additive return on a risky investment; for example, if the initial investment is x dollars and the random payoff from the investment is y dollars, then the additive return $g \equiv y - x$ is a gamble. Let $\mathcal{G}$ denote the set of all such gambles.

A *decision function* $f : \mathcal{DM} \times \mathcal{G} \to \mathbb{R}$ is a function that assigns to each agent and each gamble a nonnegative number, where a higher value is interpreted as the agent finding

[4] Shorrer (2014) further applies analogous axioms in the related setup in which an agent has to accept/reject an option to allocate a certain amount of money in a multiplicative gamble, and other interesting setups that deal with acceptance/rejection of cash flows and information transactions.

the gamble to be more attractive (i.e., less risky) for the relevant investment decision.

2.1 Decision Functions

We study four decision functions in the paper:

1. *Acceptance/rejection*: We consider a situation in which an agent faces a binary choice between accepting and rejecting the gamble. Specifically, the acceptance function $f_{AR} : \mathcal{DM} \times \mathcal{G} \to \{0, 1\}$ is given by

$$f_{AR}\left(\left(u,w\right),g\right) = \begin{cases} 1 & \mathbb{E}\left[u\left(w+g\right)\right] \geq u\left(w\right) \\ 0 & \mathbb{E}\left[u\left(w+g\right)\right] < u\left(w\right). \end{cases}$$

   That is, $f_{AR}\left(\left(u,w\right),g\right)$ is equal to one if accepting the gamble yields a weakly higher expected payoff than rejecting it, and it is equal to zero otherwise. The acceptance function has been used to study risk indices in various papers (e.g., Foster & Hart, 2009, 2013). In particular, our analysis of this decision function extends the analysis of Schreiber (2016), by showing the similarity between this function in the mean-variance setup and the corresponding decision function in the continuous-time setup.

2. *Capital allocation*: Second, we study a situation in which an agent has a continuous choice of how much to invest in the gamble. Specifically, the capital (or asset) allocation function $f_{CA} : \mathcal{DM} \times \mathcal{G} \to \mathbb{R}^+ \cup \{\infty\}$ is given by

$$f_{CA}\left(\left(u,w\right),g\right) = \arg\max_{\alpha \in \mathbb{R}^+} \mathbb{E}\left[u\Big(w+\alpha g\Big)\right]; \tag{1}$$

   if (1) does not admit of a solution (i.e., $\mathbb{E}\left[u\Big(w+\alpha g\Big)\right]$ is increasing for all α-s), then we set $f_{CA}\left(\left(u,w\right),g\right) = \infty$. That is, $f_{CA}\left(\left(u,w\right),g\right)$ is the optimal level the agent $\left(u,w\right)$ chooses to invest in gamble g. An investment level of zero is interpreted as no investment in the gamble. An investment level in the interval $(0,1)$ is interpreted as a partial investment in the gamble. An investment level of one is interpreted as a total investment in the gamble (without leverage). Finally, an investment level strictly greater than one is interpreted as a more than total investment in the gamble (achieved, for example, through high leverage). The capital allocation function is prominent in classic analyses of riskiness of assets (e.g., Markowitz, 1952; Sharpe, 1964), and, more recently, it has been used to derive an *incomplete* ranking over the riskiness of gambles (Landsberger & Meilijson, 1993).

3. *The optimal certainty equivalent:* Third, we study a situation in which an agent has to assess how much an opportunity to invest in the gamble g is worth to him (where we allow the agent to choose his optimal investment level). Specifically, the optimal certainty equivalent function $f_{CE} : \mathcal{DM} \times \mathcal{G} \to \mathbb{R}^+ \cup \{\infty\}$ is defined implicitly as the unique solution to the equation

$$u\left(w + f_{CE}\right) = \max_{\alpha \in \mathbb{R}^+} \mathbb{E}\left[u\Big(w + \alpha \cdot g\Big)\right]; \tag{2}$$

   if (2) does not admit of a solution (which happens when $\mathbb{E}\left[u\Big(w + \alpha \cdot g\Big)\right]$ is increasing for all α-s), then we set $f_{CE} = \infty$. That is, $f_{CE}\left(\left(u, w\right), g\right)$ is interpreted as the certain gain for which the decision maker is indifferent between obtaining this gain for sure and having an option to invest in the gamble g, when the agent is allowed to optimally choose his investment level in g. Observe that one can express the RHS in (2) in terms of f_{CA} and obtain the following equivalent definition of f_{CE} as the unique solution to the equation $u\left(w + f_{CE}\right) = \mathbb{E}\left[u\Big(w + f_{CA}\left(\left(u, w\right), g\right) \cdot g\Big)\right]$. The function f_{CE} has been studied axiomatically in Hellman & Schreiber (2018).

4. *Risk premium*: Lastly, we study a situation in which the agent has to decide between investing in the gamble and obtaining a certain amount that is less than the gamble's expected payoff. Specifically, the risk premium function $f_{RP} : \mathcal{DM} \times \mathcal{G} \to \mathbb{R}^- \cup \{-\infty\}$ is defined implicitly as the unique solution to the equation

$$\mathbb{E}\left[u\left(w + g\right)\right] = u\left(w + E\left[g\right] + f_{RP}\right); \tag{3}$$

   if such a solution does not exist then we set $f_{RP}\left(\left(u, w\right), g\right) = -\infty$. That is, $f_{RP}\left(\left(u, w\right), g\right)$ is interpreted as the negative amount that has to be added to the expected value of the gamble, to make the agent indifferent between investing in the gamble, and obtaining the gamble's expected payoff plus this negative amount. Here we use the common acceptation of risk premium in the economic literature (Arrow, 1970; see Kreps, 1990, Section 3.2, for a textbook definition), which has a somewhat different meaning in some of the finance literature (see Footnote 1).

In the main text we study additive gambles, in which the gamble's realized outcome is added to the initial wealth. In Appendix A we extend our model to multiplicative gambles, in which the realized outcome of the gamble is interpreted as the per-dollar return.

2.2 Risk Indices

We define a *risk index* as a function $Q : \mathcal{G} \to \mathbb{R}^{++}$ that assigns to each gamble a positive number, which is interpreted as the gamble's riskiness. We study three risk indices:

1. The *variance-to-mean index* $Q_{VM}(g)$ is the ratio of the variance to the mean:

$$Q_{VM}(g) = \frac{\sigma^2[g]}{\boldsymbol{E}[g]}, \qquad \text{where } \sigma^2[g] \equiv E\left[(g - E[g])^2\right].$$

2. The *inverse Sharpe index* $Q_{IS}(g)$ is the ratio of the standard deviation to the mean:

$$Q_{IS}(g) = \frac{\sigma[g]}{\boldsymbol{E}[g]}.$$

3. The *standard deviation index* $Q_{SD}(g)$ is equal to: $Q_{SD}(g) = \sigma[g]$.

We say that a risk index is consistent with a decision function over a domain of agents and gambles, if: each agent in the domain finds gamble g less attractive than g' with respect to the relevant decision function iff the risk index of g is higher than in g'. Formally:

Definition 1. Risk index Q is *consistent* with f over the domain $DM \times G \subseteq \mathcal{DM} \times \mathcal{G}$ if

$$Q(g) > Q(g') \Leftrightarrow f((u,w), g) < f((u,w), g')$$

for each agent $(u, w) \in DM$ and each pair of gambles $g, g' \in G$.

Our definition of consistency is restrictive and for a given domain of gambles and agents it may not apply at all. In particular observe that a domain $DM \times G \subseteq \mathcal{DM} \times \mathcal{G}$ admits a consistent risk index iff *all agents have the same ranking over gambles*, i.e., if

$$f((u,w), g) < f((u,w), g') \Leftrightarrow f((u',w'), g) < f((u',w'), g')$$

for each pair of agents $(u, w), (u', w') \in DM$ and each pair of gambles $g, g' \in G$.

Note that consistency is an ordinal concept; i.e., a consistent risk index is unique up to monotone transformations; if risk index Q is *consistent* with function f over the domain

$DM \times G$, then risk index Q' is consistent with f over this domain iff there exists a strictly increasing mapping $\theta : Q(G) \to Q'(G)$, s.t. $Q'(g) = \theta(Q(g))$ for each gamble $g \in G$.

2.3 Risk-Aversion Indices

We define a *risk-aversion index* as a function $\phi : \mathcal{DM} \to \mathbb{R}^{++}$ that assigns to each agent a non-negative number, which is interpreted as the agent's risk aversion. We mainly study one risk index in the paper, the Arrow–Pratt coefficient of absolute risk aversion, denoted by $\rho : \mathcal{DM} \to \mathbb{R}^{++}$, which is defined as follows:

$$\rho(u, w) = \frac{-u''(w)}{u'(w)}.$$

We say that a risk-aversion index is consistent with a decision function over a domain of agents and gambles, if, for each gamble and each pair of agents in the domain, the agent with the higher index chooses a lower value for his investment decision in the gamble.

Definition 2. Risk-aversion index ϕ is *consistent* with f over $DM \times G \subseteq \mathcal{DM} \times \mathcal{G}$ if

$$\phi(u, w) > \phi(u', w') \Leftrightarrow f((u, w), g) < f((u', w'), g)$$

for each pair of agents $(u, w), (u', w') \in DM$ and each gamble $g \in G$.

Here again, the definition of consistency is restrictive and for a given domain of gambles and agents it may not apply at all. Specifically, a domain $DM \times G \subseteq \mathcal{DM} \times \mathcal{G}$ admits a consistent risk-aversion index iff all *gambles induce the same ranking over agents*, i.e., if

$$f((u, w), g) < f((u', w'), g) \Leftrightarrow f((u, w), g') < f((u', w'), g')$$

for each pair of agents $(u, w), (u', w') \in DM$ and each pair of gambles $g, g' \in G$. Further, the consistency of a risk-aversion index is unique up to a strictly monotone transformation.

3 Normal Distributions and CARA Utilities

3.1 Result

We begin by presenting a claim, which summarizes known results for normal distributions and CARA utilities. Specifically, we show that in each of the decision functions described above, all agents with CARA utilities have the same ranking over all normally distributed gambles, and that each of these rankings is consistent with one of the risk indices presented above. Moreover, all normally distributed gambles induce the same ranking over all agents with CARA utilities, which is consistent with the Arrow–Pratt coefficient.

Formally, let $DM_{CARA} \subseteq \mathcal{DM}$ be the set of decision makers with CARA utilities:[5]

$$DM_{CARA} = \left\{ (u, w) \in DM \,|\exists \rho > 0,\ s.t.\ u(x) = 1 - e^{-\rho \cdot x} \right\},$$

and let $\mathcal{G}_N \subseteq \mathcal{G}$ be the set of normally distributed gambles with positive expectations:

$$G_N = \{ g \in \mathcal{G} | g \sim Norm(\mu, \sigma),\ \text{for some } \mu, \sigma > 0 \}.$$

Claim 1. Let u be a CARA utility with parameter ρ (i.e., $u(x) = 1 - e^{-\rho \cdot x}$). Then:

1. $f_{RP}((u, w), g) = -0.5 \cdot \rho \cdot \sigma^2$, which implies that the standard deviation index Q_{SD} is consistent with the risk premium function f_{RP} in the domain $DM_{CARA} \times G_N$.

2. (I)$f_{AR}((u, w), g) = 1$ iff $\frac{2}{\rho} \cdot \frac{\mu}{\sigma^2} \geq 1$ (and $f_{AR}((u, w), g) = 0$ otherwise), (II) $f_{CA}((u, w), g) = \frac{1}{\rho} \cdot \frac{\mu}{\sigma^2}$, which imply that the variance-to-mean index Q_{VM} is consistent with both the acceptance/rejection function f_{AR} and the capital allocation function f_{CA}.

3. $f_{CE}((u, w), g) = \frac{1}{2 \cdot \rho} \cdot \left(\frac{\mu}{\sigma} \right)^2$, which implies that the inverse Sharpe index Q_{IS} is consistent with the optimal certainty equivalent function f_{CE}.

4. The Arrow–Pratt coefficient of absolute risk aversion ρ is consistent with all four decision functions f_{CA}, f_{AR}, f_{CE}, and f_{RP} in the domain $DM_{CARA} \times G_N$.

For completeness, we present the proof of Claim 1 in Appendix B.1.

3.2 Discussion

Each agent with CARA utility is described by two parameters (initial wealth w, and Arrow–Pratt coefficient ρ). Similarly, each normal gamble is described by two parameters (expectation μ and standard deviation σ). This implies that any decision function can be expressed as a function $g(w, \rho, \mu, \sigma)$ of these four parameters.

[5] Clearly, one can extend the definition of DM_{CARA} (without affecting any of the results) by allowing the utilities to differ from $1 - e^{-\rho \cdot x}$ by adding a constant and multiplying by a positive scalar.

Other Consistent Indices of Risk Aversion CARA utilities have the well-known property that the initial wealth does not affect expected utility calculations with respect to investments in gambles. Thus, whenever the investment decision is made by choosing the option that maximizes the agent's expected utility (such as in all four of the decision functions analyzed above), then the decision function is independent of w, which implies that the parameter ρ is a consistent risk-aversion index. By contrast, for investment decisions that are not determined by maximizing the agent's expected utility, there might be different risk-aversion consistent indices. For instance, Foster & Hart (2009) analyze a situation in which an agent accepts or rejects gambles while his goal is to avoid bankruptcy. The index of risk aversion that is consistent with their decision function is the wealth level.

Separability Condition for Having a Consistent Risk Index The decision functions analyzed above have the additional separability property that each function f can be represented as a product of two functions: one that depends only on the parameters describing the agent (w and ρ), and one that depends only on the parameters of the gamble, i.e., $f\left(\left(u,w\right),g\right)=\tilde{f}\left(w,\rho,\sigma,\mu\right)=h\left(w,\rho\right)\cdot\nu\left(\mu,\sigma\right)$. This separability implies that all agents with CARA utilities have the same ranking over normal gambles (as this ranking depends only on $\nu\left(\mu,\sigma\right)$, which does not depend on the agent's parameters), which, in turn, implies that there exists a consistent risk index. Similarly, the separability implies that all normal gambles induce the same ranking over agents (as this ranking depends only on $h\left(w,\rho\right)$, which does not depend on the parameters of the normal gamble).

Other decision functions might not satisfy this separability property. One example of such a non-separable decision function is the standard certainty equivalent of a continuous gamble f_{SCE} (as opposed to the certainty equivalent of the optimal allocation of the gamble $f_{CE}\left(\left(u,w\right),g\right)$ discussed above), which is implicitly defined by

$$\mathbb{E}\left[u(w+g)\right]=u(w+f_{SCE}).$$

The definitions of f_{SCE} and f_{RP} imply that $f_{SCE}=\mathbb{E}\left[g\right]+f_{RP}$. Substituting $f_{RP}=-0.5\cdot\rho\cdot\sigma^2$ (which is proven in Appx. B.1) yields: $f_{SCE}=\mathbb{E}\left[g\right]+f_{RP}=\mu-0.5\cdot\rho\cdot\sigma^2$, which is a non-separable function of ρ,μ,σ. The non-separability implies that agents with different CARA utilities have different rankings for normal gambles and, therefore, no risk index can be consistent with these decisions.

4 Short-Term Investments in Continuous Gambles

In what follows we adapt our model to short-term investment decisions regarding assets whose value follows a continuous random process. Our description of the continuous-time setup follows Shreve (2004).

4.1 Continuous-Time Random Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a Brownian motion W_t is defined,[6] with an associated filtration $\mathcal{F}(t)$. Let the process g be described by the following stochastic differential equation:

$$dg_t = \mu_t dt + \sigma_t dW_t, \tag{4}$$

where the drift μ and the diffusion σ are adapted stochastic processes (i.e., μ_t and σ_t are $\mathcal{F}(t)$-measurable for each t, see Shreve, 2004, Footnote 1 on page 97) and are both continuous in t. We assume that $\mu_0 > 0$ and $\sigma_0 > 0$ and that $\sigma_t \geq 0$ for all $t > 0$. We also assume that $E \int_0^t \sigma_s^2 ds$ and $E \int_0^t |\mu_s| ds$ are finite for every $t > 0$. This implies that the integrals $E \int_0^t \sigma_s dW_s$ and $E \int_0^t \mu ds$ are well defined, and the Ito integral $\int_0^t \sigma_s dW_s$ is a martingale; see Shreve (2004, Footnote 2 on page 143).

The process g is interpreted as the additive return of some risky asset. Specifically, let P_t be the price of some risky asset at time t and assume that P_0 is known. Then,

$$g_t = P_t - P_0,$$

is the additive return of the asset at time t. In particular, observe that $g_0 = P_{t=0} - P_0 \equiv 0$. For simplicity we assume that g is bounded from below (i.e., there exists $M_g \in \mathbb{R}$, such that $g_t \geq M_g$ for each $t > 0$). Under these assumptions, for a sufficiently small time t, g_t is a gamble (see Footnote 8 in Appendix B.2); i.e., it has positive expectation and takes negative values with positive probability. Thus, we can apply our definitions of decision functions and indices to g_t for each specific value of $t > 0$.

In our setup, a decision maker has to make a decision at time zero, where he cares only about his wealth at some time t. From the perspective of this decision maker, P_0 is a pure number and g_t is a gamble, just as before.

[6] For ease of exposition we limit the Wiener process to one dimension. All the results remain the same with a multidimensional Wiener process with the corresponding adjustments.

Let Γ denote the set of all continuous-time random processes satisfying these assumptions. Observe that the set Γ is quite general. In particular, it includes returns on assets whose prices are described by geometric Brownian processes (Black & Scholes, 1973; Merton, 1992, Chapters 4 and 5),[7] and variants of arithmetic Brownian processes and of mean-reverting processes that are bounded from below (also known as Ornstein–Uhlenbeck processes; see, e.g., Merton, 1992, Chapter 5; Hull & White, 1987; Meddahi & Renault, 2004), such as Cox *et al.*'s (1985) process for modeling interest rate.

4.2 Adapted Definitions

We define a *local-risk index* at time zero as a function $Q^l : \Gamma \to \mathbb{R}^{++}$ that assigns to each process $g \in \Gamma$ a positive number, which is interpreted as the process's short-term riskiness at $t = 0$. Given $g \in \Gamma$ with initial parameters μ_0 and σ_0, we define three specific local-risk indices (analogous to the corresponding definitions in Section 2.2):

1. The *variance-to-mean local index* $Q^l_{VM}(g)$ is equal to:

$$Q^l_{VM}(g) = \frac{\sigma_0^2}{\mu_0}.$$

2. The *inverse Sharpe local index* $Q^l_{IS}(g)$ is equal to:

$$Q^l_{IS}(g) = \frac{\sigma_0}{\mu_0}.$$

3. The *standard deviation local index* $Q^l_{SD}(g)$ is equal to: $Q^l_{SD}(g) = \sigma_0$.

Given a continuous-time process $g \in \Gamma$, decision function f and agent $(u, w) \in \mathcal{DM}$, let $f_g^{(u,w)}(t) \equiv f((u, w), g_t)$ be the value of the decision function of agent (u, w) with respect to an investment in g as a function of the duration of the investment t.

The following definition, which deals with general real-valued functions, will be useful for defining the concept of consistency of indices in the continuous-time framework.

[7] When one models an asset's price P by a geometric Brownian motion, then P_t (the asset value at time t) obtains only positive values. In this case, the additive return is defined as the difference between the asset's value at time t and its initial value, i.e., $g_t = P_t - P_0$. Obviously, the additive return can obtain both positive and negative values (for any time $t > 0$), which is consistent with our requirement that g_t be a gamble. Specifically, in the case of a geometric Brownian motion, $dg_t \equiv dp_t = p_t \cdot \mu \cdot dt + p_t \cdot \sigma \cdot dW$, which implies that $\mu_t = p_t \cdot \mu$ and $\sigma_t = p_t \cdot \sigma$ as in Equation (4).

Definition 3. Let $f, h : \mathbb{R}^+ \Rightarrow \mathbb{R}$, and assume that $\lim_{t\to 0} \frac{f(t)}{h(t)}$ is well defined. We say that f is *uniformly higher* than h (around zero) and denote it by $f >> h$ if (1) there exists $\bar{t} > 0$, such that $f(t) > h(t)$ for each $t \in \left(0, \bar{t}\right)$ and (2)

$$\lim_{t\to 0} \frac{f(t)}{h(t)} \neq 1.$$

That is, $f >> h$ if $f(t)$ is strictly higher than $h(t)$ for any sufficiently small t, and the relative difference between the two functions does not become negligible (as measured by the ratio $\frac{f(t)}{h(t)}$ not converging to one) in the limit of $t \to 0$.

We say that a local-risk index is consistent with a decision function over continuous returns, if the local-risk index of g' is lower than the index of g iff all risk-averse agents find g_t uniformly more attractive than g'_t with respect to any short-term investment.

Definition 4. Local-risk index $Q^l : \Gamma \to \mathbb{R}^+$ is *consistent* with decision function f over the set of continuous returns Γ if, for each pair of continuous-time processes $g, g' \in \Gamma$ and each agent $(u, w) \in \mathcal{DM}$, we have that

$$Q^l(g) > Q^l(g') \Leftrightarrow f_g^{(u,w)} >> f_{g'}^{(u,w)}.$$

Note that a consistent risk index is unique up to strictly monotone transformations.

We say that a risk-aversion index is consistent with a decision function over continuous-time returns, if the risk-aversion index of agent (u, w) is strictly higher than the index of (u', w') iff agent (u, w) finds all gambles uniformly less attractive than agent (u', w').

Definition 5. Risk-aversion index $\phi : \mathcal{DM} \longrightarrow \mathbb{R}^+$ is *consistent* with decision function f over the set of continuous returns Γ if, for each pair of agents $(u, w), (u', w') \in \mathcal{DM}$ and for each gamble $g \in \Gamma$, we have that

$$\phi(u, w) > \phi(u', w') \Leftrightarrow f_g^{(u,w)} << f_g^{(u',w')}$$

Note that a consistent risk-aversion index is unique up to monotone transformations.

4.3 Main Result

Our main result shows that in each of the decision functions described above, all agents have the same ranking over all short-term continuous returns. Moreover, the rankings are consistent with the three risk indices presented above, and they are the instantaneous versions of the corresponding indices in the case of normally distributed gambles and CARA utilities analyzed in Claim 1. Finally, we adapt to the present setup the classic result that all continuous short-term returns induce the same ranking over all agents, which is consistent with the Arrow–Pratt coefficient of absolute risk aversion (as in the case of normally distributed gambles and CARA utilities). Formally:

Theorem 1. *The following conditions hold over the set of continuous returns* Γ*:*

1. *The standard deviation index* Q^l_{SD} *is consistent with the risk premium function* f_{RP}*.*
2. *The variance-to-mean index* Q^l_{VM} *is consistent with the capital allocation function* f_{CA}*.*
3. *The inverse Sharpe index* Q^l_{IS} *is consistent with the optimal certainty equivalent function* f_{CE}*.*
4. *The Arrow–Pratt coefficient of absolute risk aversion* ρ *is consistent with decision functions* f_{CA}*,* f_{CE}*, and* f_{RP}*.*

Sketch of proof; formal proof is presented in Appendix B.2. The value of an asset with a continuous-time return g after a sufficiently small time t is represented by a gamble g_t for which the magnitudes of all high moments are small relative to the magnitude of the second moment. Assuming random variables of this type allows us to use Taylor expansion to approximate the decision functions, and to obtain the consistent risk indices.

Recall that $\lim_{t\to 0}\frac{\sigma^2[g_t]}{t}=\sigma_0^2$ and $\lim_{t\to 0}\frac{\mu[g_t]}{t}=\mu_0$, which implies for sufficiently small t-s that $\sigma^2\left[g_t\right]\approx t\cdot\sigma_0^2$ and $\mu\left[g_t\right]\approx t\cdot\mu_0$. We begin with a standard approximation of the risk premium function f_{RP} (see, e.g., Eeckhoudt *et al.*, 2005, Chapter 1). Recall that the risk premium was defined implicitly as

$$\mathbb{E}\left[u\left(w+g_t\right)\right]=u\left(w+\mathbb{E}\left[g_t\right]+f_{RP}\right). \tag{5}$$

A second-order Taylor expansion of the left-hand side around $w+\mathbb{E}[g_t]$ yields

$$\begin{aligned}\mathbb{E}\left[u\left(w+g_t\right)\right]\approx&\ \mathbb{E}\left[u\left(w+\mathbb{E}\left[g_t\right]\right)+u'(w+\mathbb{E}[g_t])(g_t-\mathbb{E}[g_t])+\frac{1}{2}u''\left(w+\mathbb{E}\left[g_t\right]\right)\left(g_t-\mathbb{E}[g_t]\right)^2\right]\\ =&\,u\left(w+E\left[g\right]\right)+\frac{1}{2}u''\left(w+E\left[g_t\right]\right)\sigma^2\left[g_t\right].\end{aligned}$$

A first-order Taylor expansion of the right-hand side around $w+E[g]$ yields

$$u\left(w+\mathbb{E}\left[g_t\right]+f_{RP}\right)\approx u\left(w+\mathbb{E}\left[g_t\right]\right)+u'\left(w+\mathbb{E}\left[g_t\right]\right)\cdot f_{RP}.$$

Combining these equations and isolating f_{RP} yields

$$f_{RP}\approx\frac{1}{2}\frac{u''\left(w\right)}{u'\left(w\right)}\sigma^2\left[g_t\right]\approx\frac{1}{2}\frac{u''\left(w\right)}{u'\left(w\right)}t\cdot\sigma_0^2,$$

which implies that $Q^l_{SD}=\sigma_0$ (resp., $\rho{=}-\frac{1}{2}\frac{u''(w)}{u'(w)}$) is a consistent risk index (resp., risk-aversion index) for decision function f_{RP}.

In order to analyze f_{CA}, we define $C\left(\alpha\right)=\left(f_{RP}\left(\alpha\cdot g_t\right)+\mathbb{E}[\alpha\cdot g_t]\right)$ to be the certainty equivalent of investment α in g_t. Substituting the value of f_{RP} calculated above we get

$$C\left(\alpha\right)\approx\alpha^2\frac{1}{2}\frac{u''\left(w\right)}{u'\left(w\right)}\sigma^2\left[g_t\right]+\alpha\cdot\mathbb{E}[g_t].$$

In order to maximize $C\left(\alpha\right)$, we compare the derivative to zero, to get

$$\frac{\partial C\left(\alpha\right)}{\partial\alpha}=0\;\Leftrightarrow\;\alpha^*\cdot\frac{u''\left(w\right)}{u'\left(w\right)}\sigma^2\left[g_t\right]+\mathbb{E}[g_t]=0\;\Leftrightarrow\;f_{CA}=\alpha^*\approx-\frac{\mathbb{E}[g_t]}{\frac{u''}{u'}\sigma^2\left[g_t\right]}\approx-\frac{\mu_0}{\frac{u''}{u'}\sigma_0^2},$$

which implies that the variance-to-mean index Q^l_{VM} (resp., the Arrow–Pratt coefficient ρ) is a consistent risk index (resp., risk-aversion index) for decision function f_{CA}.

Finally, if we calculate $f_{CE}=C\left(\alpha^*\right)=C\left(f_{CA}\right)$ we get

$$\begin{aligned}f_{CE}&\approx\left(\frac{\mathbb{E}[g]}{\frac{u''(w)}{u'(w)}\sigma^2\left[g_t\right]}\right)^2\frac{1}{2}\frac{u''\left(w\right)}{u'\left(w\right)}\sigma^2\left[g_t\right]+\frac{\mathbb{E}[g_t]}{-\frac{u''(w)}{u'(w)}\sigma^2\left[g_t\right]}\mathbb{E}[g_t]\\&=\frac{1}{2}\frac{1}{-\frac{u''(w)}{u'(w)}}\left(\frac{\mathbb{E}[g_t]}{\sigma\left[g_t\right]}\right)^2\approx\frac{1}{2}\frac{1}{-\frac{u''(w)}{u'(w)}}t\cdot\left(\frac{\mu_0}{\sigma_0}\right)^2,\end{aligned}$$

which implies that the inverse Sharp index Q^l_{IS} (the Arrow–Pratt coefficient ρ) is a consistent risk (risk aversion) index for decision function f_{CE}. □

Remark 1 (on why the indices in the continuous-time setup coincide with the indices in the CARA-normal setup). The expressions that approximate the various functions in the continuous-time setup consist of two elements: the coefficient of risk aversion with respect to the initial wealth level, and a function of the first and second moment of the "small" gamble. In the CARA-normal setup of Section 3, the risk-aversion coefficient is constant over all wealth levels, and, thus, it is relevant also to large gambles. In addition, the only moments that matter to an agent with CARA utility who invests in a normally distributed gamble are the first two moments. To see that, recall that for CARA utility u (with coefficient of risk aversion ρ) and normal gamble g,

$$\mathbb{E}[u(w+g)] = \mathbb{E}[1 - e^{-\rho\cdot(w+g)}] = 1 - e^{-\rho\mathbb{E}[w+g]+0.5\rho^2\sigma^2[g]}.$$

Therefore, it seems plausible that the expressions that represent the decision functions in the CARA-normal setup, depend only on the first two moments, and, thus, they coincide with the approximated decision functions that are relevant for short-term investments in assets with continuous returns.

4.4 Weak Consistency for Acceptance/Rejection

The case of the the acceptance/rejection function f_{AR} has been analyzed in Schreiber (2016). As the function f_{AR} has only two feasible values (0 or 1), it cannot admit of consistent risk indices, as in many cases in which one gamble is riskier than another, an agent may choose to reject both gambles (and his value of f_{AR} of both gambles would be zero). Nevertheless, one can define the milder notion of weak consistency, and show that a corollary to Schreiber's (2016) result is that the risk index Q^l_{VM} is weakly consistent with the acceptance/rejection function f_{AR}.

A local-risk index is weakly consistent with a decision function over the set of continuous returns, if each agent chooses a weakly lower value of his investment decision in gamble g_t relative to g'_t for a sufficiently small t if the local risk of g is strictly higher than the local risk of g'. Formally:

Definition 6. Local-risk index $Q^l : \Gamma \to \mathbb{R}^+$ is *weakly consistent* with decision function f over the set Γ if for each agent $(u, w) \in \mathcal{DM}$ and each pair of continuous-time processes $g, g' \in \Gamma$, there exists time $\bar{t}$, such that, for each time $t < \bar{t}$, we have that

$$Q^l(g) > Q^l(g') \Rightarrow f((u,w), g_t) \leq f((u,w), g'_t).$$

Note that weak consistency does not restrict the agents' choices when both gambles have the same local-risk index. As a result, a weakly consistent risk index is unique only up to weakly monotone transformations; i.e., if Q is a weakly consistent local-risk index with decision function f over the set of continuous returns Γ, then risk index Q' is consistent with function f over this domain if there exists a weakly increasing mapping $\theta : Q(G) \rightarrow Q'(G)$, such that $Q'(g) = \theta(Q(g))$ for each $g \in \Gamma$. In particular, a constant index is trivially a weakly consistent local-risk index of any decision function.

We say that a risk-aversion index is weakly consistent with a decision function over continuous-time returns, if for each short-term return, an agent chooses a (weakly) higher value for his investment decision in the asset relative to another agent's decision if the former agent's risk aversion is smaller. Formally:

Definition 7. Risk-aversion index $\phi : \mathcal{DM} \longrightarrow \mathbb{R}^+$ is *weakly consistent* with decision function f over the domain of short-term continuous gambles if for each continuous-time process $g \in \Gamma$ and each pair of agents $(u, w), (u', w') \in \mathcal{DM}$, there exists a time $\bar{t}$, such that, for each time $t < \bar{t}$, we have that

$$\phi(u, w) > \phi(u', w') \Rightarrow f((u, w), g_t) \leq f((u', w'), g_t).$$

The following corollary, which is implied by Schreiber (2016, Theorems 2.2 & 3.3), shows that the standard deviation index Q^l_{VM} and the Arrow–Pratt coefficient of absolute risk aversion ρ are weakly consistent with the acceptance/rejection function f_{AR}.

Corollary 1 (Implied by Schreiber (2016, Theorems 2.2 & 3.3))**.** *The following conditions hold over the domain of continuous short-term decisions:*

1. *The variance-to-mean index Q^l_{VM} is weakly consistent with decision function f_{AR}.*
2. *The Arrow–Pratt coefficient ρ is weakly consistent with decision function f_{AR}.*

4.5 Continuous-Time Processes with Jumps

The set of continuous-time gambles Γ analyzed in this paper does not allow for jumps. In what follows we show that the absence of jumps is necessary for our main result. Specifically, we demonstrate that risk-averse agents rank continuous-time processes with jumps differently, even for short-term investments, which rules out the existence of consistent risk indices. Consider, for example, the acceptance/rejection function f_{AR} (similar conclusions can be drawn for the other decision functions). Hart (2011) observes that there are many pairs of (discrete-time) gambles that are ranked differently by different risk-averse agents (see Hart 2011, Footnote 23). Let $h, \tilde{h}$ be such a pair of gambles.

Consider the following compound Poisson processes $g, \tilde{g}$, where each has an initial value of zero. The value of each process changes only when there is a jump. The jumps arrive randomly with a rate λ. In process g (resp., $\tilde{g}$) the size of each jump is distributed according to h (resp., $\tilde{h}$). Observe that for sufficiently short times the probability of having two jumps is negligible, and the decision whether to accept or to reject a gamble depends only on what may happen after a single jump. This implies that agents who rank the gambles $h, \tilde{h}$ differently, would also rank $g_t, \tilde{g}_t$ differently, for any sufficiently short time t. This rules out the existence of a consistent risk index in this setup.

5 Conclusion

Our main result is that in four central decision problems all risk-averse agents have the same (problem-dependent) ranking over short-term investments in risky assets whose returns evolve continuously, and these rankings are represented by simple well-known indices of risk. The indices obtained are the same as in the classic model of CARA utilities and normally distributed gambles. Each problem relates to a different dimension of risk, and, thus, its ranking is represented by a different risk index. Finally, adapting a classic result to the present setup, we show in all of the decision functions analyzed above, the decisions of agents are consistent with their Arrow–Pratt coefficients of risk aversion.

The proposed indices in our paper are all based on the first two moments. This is a result of the known property of continuous stochastic processes for which higher moments go quickly to zero as the time parameter goes to zero. Hence, multiple indices of risk that do use higher moments might coincide with our indices when they are applied to continuous-time processes and short-term investments. For instance, Schreiber (2016) shows that the index of Aumann & Serrano (2008) and that of Foster & Hart (2009) (which, in general, both depend on all moments of the gamble) coincide with the variance-to-mean index Q_{VM} for continuous processes in the limit of $t \to 0$, and Shorrer (2014) shows that there is a continuum of risk indices (which depend also on higher moments) that are consistent with acceptance/rejection decisions of agents with respect to small discrete gambles. Indeed, under the assumption that returns evolve continuously in time, the only relevant parameters for measuring risk are the first two moments. Our results can be interpreted as characterizing a necessary condition for a plausible risk index, namely, that a plausible risk index (with respect to one of the four decision functions analyzed in the paper) should depend on the first two moments in the same way as presented in our main result. We leave for future research the interesting question of how to choose among the various risk indices that satisfy this necessary condition. One possible direction for analyzing this question is the axiomatic approach applied in Shorrer (2014) to acceptance/rejection decisions.

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A Multiplicative Gambles

In the main text we followed the recent literature of riskiness (initiated by Aumann & Serrano, 2008 and Foster & Hart, 2009) and focused on decision problems with regard to additive gambles in units of dollars. However, in most financial applications, it is common to describe the returns of an asset in relative terms, namely, percentages (see, e.g., Markowitz, 1959 and Merton, 1992), as this is the way in which returns are described in practice in exchange markets. Hence, in this section we show that our results hold also with regard to multiplicative returns.

In some sense, the difference between multiplicative and additive returns is only a matter of presentation: if one invests x dollars in a multiplicative gamble r, one's payoff will be $x(1+r)$ dollars, and this is just the same payoff as if one invests in an additive return of $x \cdot r$ dollars. Nevertheless, we think that presenting the results for multiplicative returns is important for two reasons: first, as argued in Schreiber (2014), each investment might have two different aspects of riskiness, absolute and relative; given two assets, one of them might be riskier in relative terms but less risky in absolute terms. Therefore it is worthwhile to study the difference between multiplicative and additive returns in our setup. As it turns out, this potential difference vanishes when focusing on short-term investments and we derive in the multiplicative setup results analogous to those that we have in the additive setup. Second, in many situations of decision making under risk, the risk-free interest rate should be taken into account. Since the risk-free interest rate is calculated in terms of percentages, it is natural to combine it in decision problems with relative return, as we do here.

A.1 Adaptation to the Model

Let $r_f > 0$ be the risk-free interest rate available for all investors. A *multiplicative risky asset (multiplicative gamble)* r is a real-valued random variable with an expectation that is greater than r_f, and some negative values greater than -1, i.e., $\boldsymbol{E}[r] > r_f$, $\boldsymbol{P}[r < 0] > 0$, and $r \geq -1$. We interpret r as the per-dollar return of the asset. Let $\mathcal{R}$ denote the set of all multiplicative risky assets.

We adapt the definitions of our decision functions to the case of multiplicative gambles.

1. The acceptance function $f^m_{AR} : \mathcal{DM} \times \mathcal{R} \to \{0, 1\}$ is given by

$$f^m_{AR}\left((u, w), r\right) = \begin{cases} 1 & \mathbb{E}\left[u\left(w \cdot (1+r)\right)\right] \geq u\left(w \cdot (1+r_f)\right) \\ 0 & \mathbb{E}\left[u\left(w \cdot (1+r)\right)\right] < u\left(w \cdot (1+r_f)\right), \end{cases}$$

   where we consider a situation in which an agent faces a binary choice between investing his entire wealth in a multiplicative gamble r and investing it in the riskless asset with return r_f.

2. The capital allocation function $f^m_{CA} : \mathcal{DM} \times \mathcal{R} \to \mathbb{R}^+ \cup \{\infty\}$ is given by

$$f^m_{CA}\left((u, w), r\right) = \arg\max_{\alpha \in \mathbb{R}^+} \mathbb{E}\left[u\Big(w \cdot (1+r_f) + \alpha \cdot w \cdot (r - r_f)\Big)\right]; \tag{6}$$

if Equation (6) does not admit of a solution (i.e., $\mathbb{E}\left[u\Big(w\cdot(1+r_f)+\alpha\cdot w\cdot(r-r_f)\Big)\right]$ is increasing for all α-s), then we set $f^m_{CA}\left((u,w),r\right)=\infty$. This function deals with a situation in which an agent decides on the optimal share $\alpha\geq 0$ of his wealth w to invest in the multiplicative gamble r (where $\alpha>1$ can be induced by leverage), where his remaining wealth is invested in the riskless asset.

3. The optimal certainty equivalent function $f^m_{CE}:\mathcal{DM}\times\mathcal{R}\to\mathbb{R}^+\cup\{\infty\}$ is defined implicitly as the unique solution to the equation

$$
\begin{aligned}
u\left(w\cdot(1+f^m_{CE})\right) &= \max_{\alpha\in\mathbb{R}^+}\mathbb{E}\left[u\left(w\cdot(1+r_f)+\alpha\cdot w\cdot(r-r_f)\right)\right]\\
&\equiv \mathbb{E}\left[u\Big(w(1+r_f)+f^m_{CA}\left((u,w),r\right)\cdot w\cdot(r-r_f)\Big)\right]; \qquad (7)
\end{aligned}
$$

if Equation (7) does not admit of a solution (i.e., $\mathbb{E}\left[u\Big(w\cdot(1+r_f)+\alpha\cdot w\cdot(r-r_f)\Big)\right]$ is increasing for all α-s), then we set $f^m_{CE}\left((u,w),r\right)=\infty$. This function describes the rate of a constant return that is equivalent to investing optimally in a multiplicative gamble r, where the remaining wealth is invested in the riskless asset.

4. The risk-premium function $f^m_{RP}:\mathcal{DM}\times\mathcal{R}\to\mathbb{R}^-$ is defined implicitly as the unique solution to the equation

$$\mathbb{E}\left[u\left(w\cdot(1+r)\right)\right]=u\left(w\cdot\left(1+\mathbb{E}\left[r\right]+f^m_{RP}\right)\right),$$

where f^m_{RP} represents the constant (negative) return that makes the agent indifferent between investing all his wealth in the multiplicative gamble r and investing in an asset with a constant return that is equal to the expectation of r plus f^m_{RP}.

Let $R_N\subseteq\mathcal{R}$ be the set of normally distributed multiplicative gambles (defined analogously to the definition of $\mathcal{G}_N$). The Arrow–Pratt coefficient of *relative* risk aversion, denoted by $\varrho:\mathcal{DM}\to\mathbb{R}^{++}$, is defined as follows:

$$\varrho\left(u,w\right)=\frac{-w\cdot u''\left(w\right)}{u'\left(w\right)}.$$

We adapt the three indices of risk in the main text to the multiplicative setup and the existence of a risk-free interest rate. Specifically:

1. The *variance-to-mean index* $Q^m_{VM}(r)$ is equal to

$$Q^m_{VM}(r) = \frac{\sigma^2[r]}{\boldsymbol{E}[r] - r_f}, \qquad \text{where } \sigma^2[r] \equiv E\left[(r - E[r])^2\right].$$

2. The *inverse Sharpe index* $Q^m_{IS}(r)$ is equal to

$$Q^m_{IS}(r) = \frac{\sigma[r]}{\boldsymbol{E}[r] - r_f}.$$

3. The *standard deviation index* $Q^m_{SD}(r)$ is equal to $Q^m_{SD}(r) = \sigma[r]$.

A.2 Adapted Results

The adaptation of Claim 1 and Theorem 1 to multiplicative gambles is as follows. Observe that all the results remain the same, except that the Arrow–Pratt coefficient of relative risk aversion replaces the coefficient of absolute risk aversion.

Claim 2. The following conditions hold over the domain $DM_{CARA} \times R_N$:

1. The standard deviation index Q^m_{SD} is consistent with decision function f^m_{RP}.
2. The variance-to-mean index Q^m_{VM} is consistent with both the capital allocation function f^m_{CA} and the acceptance/rejection function f^m_{AR}.
3. The inverse Sharpe index Q^m_{IS} is consistent with the decision function f^m_{CE}.
4. The Arrow–Pratt coefficient of relative risk aversion ϱ is consistent with all four decision functions: f^m_{AR}, f^m_{CA}, f^m_{CE}, and f^m_{RP}.

The proof of Claim 2 is made analogous to the corresponding proof in the additive case by using the following identities (details are omitted for brevity):

1. $f^m_{AR}((u,w),r) \equiv f_{AR}((u, w(1+r_f)), w(r - r_f))$,
2. $f^m_{CA}((u,w),r) \equiv f_{CA}((u, w(1+r_f)), w(r - r_f))$,
3. $f^m_{CE}((u,w),r) \equiv f_{CE}((u, w(1+r_f)), w(r - r_f))/w$, and
4. $f^m_{RP}((u,w),r) \equiv f_{RP}((u, w(1+r_f)), w(r - r_f))/w$.

Recall that in the continuous-time setup, the decision problems are parameterized by t, which is the investment horizon. Previously, we assumed that a continuous-time random process g represents the additive return of a financial investment. Now the continuous-time random process r represents the excess multiplicative return: $r_t = (P_t - P_0)/P_0$. We assume that the *compound risk-free interest rate* is r_f and hence the riskless return over period t is $r_f(t) = e^{\mu_f \cdot t} - 1$. The adapted definitions of the risk indexes in the multiplicative setup for the local risk indices are as follows:

1. The *variance-to-mean local index* $Q_{VM}^{l,m}(g)$ is equal to

$$Q_{VM}^{l,m}(g) = \frac{\sigma_0^2}{\mu_0 - \mu_f}.$$

2. The *inverse Sharpe local index* $Q_{IS}^{l,m}(g)$ is equal to

$$Q_{IS}^{l,m}(g) = \frac{\sigma_0}{\mu_0 - \mu_f}.$$

3. The *standard deviation local index* $Q_{SD}^{l,m}(g)$ is equal to $Q_{SD}^{l,m}(g) = \sigma_0$.

The analogous result to Theorem 1 is as follows.

Theorem 2. *The following conditions hold over the domain of continuous short-term decisions with respect to multiplicative gambles:*

1. *The standard deviation index $Q_{SD}^{l,m}$ is consistent with the risk premium function f_{RP}^m.*
2. *The variance-to-mean index $Q_{VM}^{l,m}$ is consistent with the capital allocation function f_{CA}^m, and it is weakly consistent with the acceptance/rejection function f_{AR}^m.*
3. *The inverse Sharpe index $Q_{IS}^{l,m}$ is consistent with the decision function f_{CE}^m.*
4. *The Arrow–Pratt coefficient of relative risk aversion ϱ is consistent with decision functions: f_{CA}^m, f_{CE}^m, and f_{RP}^m, and it is weakly consistent with f_{AR}^m.*

The proof of Theorem 2 is made analogous to the corresponding proof in the additive case by using the following identities (details are omitted for brevity):

1. $f_{AR}^m((u,w), r_t) \equiv f_{AR}((u, w(1 + r_f(t))), w(r_t - r_f(t)))$,

2. $f^m_{CA}((u,w),r_t) \equiv f_{CA}((u,w(1+r_f(t))),w(r_t - r_f(t)))$,

3. $f^m_{CE}((u,w),r_t) \equiv f_{CE}((u,w(1+r_f(t))),w(r_t - r_f(t)))/w$, and

4. $f^m_{RP}((u,w),r_t) \equiv f_{RP}((u,w(1+r_f(t))),w(r_t - r_f(t)))/w$.

B Proofs

B.1 Proof of Claim 1

The following well-known fact, which describes the expectation of a log-normal distribution, will be useful in our proofs (the standard proof, which relies on the Laplace transform of the normal distribution, is omitted for brevity; see, e.g., Forbes *et al.*, 2011, page 132).

Fact 1. *If y is normally distributed with expectation μ and standard deviation σ, then* $\mathbb{E}\left[e^y\right] = e^{\mu+0.5\sigma^2}$.

Next, we prove Claim 1. Let g be a normally distributed random variable with expectation μ and standard deviation σ. Let u be a CARA utility with parameter ρ, i.e., $u(x) = 1 - e^{-\rho \cdot x}$. Let w be the arbitrary initial wealth.

1. Q_{SD} and ρ are consistent with f_{RP}. The risk premium x is defined implicitly by

$$\mathbb{E}\left[u(w+g)\right] = \mathbb{E}\left[1 - e^{-\rho(w+g)}\right] = u(w + E[g] + x) = 1 - e^{-\rho(w+\mu+x)}$$

$$\Leftrightarrow 1 - \mathbb{E}\left[e^{-\rho(w+g)}\right] = 1 - e^{-\rho(w+\mu+x)} \Leftrightarrow \mathbb{E}\left[e^{-\rho(w+g)}\right] = e^{-\rho(w+\mu+x)}.$$

By Fact 1

$$\mathbb{E}\left[e^{-\rho(w+g)}\right] = e^{-\rho(w+\mu)+0.5\rho^2\sigma^2},$$

which implies

$$e^{-\rho(w+\mu)+0.5\rho^2\sigma^2} = e^{-\rho(w+\mu+x)} \Leftrightarrow -\rho(w+\mu)+0.5\rho^2\sigma^2 = -\rho(w+\mu+x) \Leftrightarrow x = 0.5\rho\sigma^2.$$

Thus, $f_{RP}\left((u,w),g\right) = 0.5\rho\sigma^2$, which implies that $Q_{SD} = \sigma$ is a consistent risk index (and that ρ is a consistent risk-aversion index with respect to f_{RP}.

2. Q_{VM} and ρ are consistent with f_{AR}. The agent accepts the gamble iff

$$\mathbb{E}\left(1 - e^{-\rho(w+g)}\right) > \mathbb{E}\left(1 - e^{-\rho w}\right)$$

which, by Fact 1 is equivalent to

$$e^{-\rho(w+\mu)+0.5\rho^2\sigma^2} < e^{-\rho w} \Leftrightarrow 0.5\rho < \frac{\mu}{\sigma^2}.$$

Thus, $f_{AR}\left(\left(u,w\right),g\right) = \mathbf{1}_{\left\{0.5\rho<\frac{\mu}{\sigma^2}\right\}}$, which implies that $Q_{VM} = \frac{\sigma^2}{\mu}$ is a consistent risk index (and that ρ is a consistent risk-aversion index) with respect to f_{AR}.

3. Q_{VM} and ρ are consistent with f_{CA}. The capital allocation function is given by

$$f_{CA}\left(\left(u,w\right),g\right) = \arg\max_{\alpha\in\mathbb{R}^+} \mathbb{E}\left[u\Big(w+\alpha g\Big)\right] = \arg\max_{\alpha\in\mathbb{R}^+} \mathbb{E}\left[1-e^{-\rho\left(w+\alpha g\right)}\right].$$

It follows from Fact 1 that the r.h.s. of the above equation is equivalent to

$$\arg\max_{\alpha\in\mathbb{R}^+} \mathbb{E}\left[1-e^{-\rho\left(w+\alpha g\right)}\right] = \arg\max_{\alpha\in\mathbb{R}^+}\left(1-e^{-\rho w-\rho\alpha\mu+0.5\rho^2\alpha^2\sigma}\right)$$

$$= \arg\min{}_{\alpha\in\mathbb{R}^+}\left(-\rho\alpha\mu+0.5\rho^2\alpha^2\sigma\right).$$

The first-order condition is

$$-\mu+\rho\alpha^*\sigma = 0 \Leftrightarrow \alpha^* = \frac{1}{\rho}\frac{\mu}{\sigma^2}.$$

Thus, $f_{CA}\left(\left(u,w\right),g\right) = \frac{1}{\rho}\frac{\mu}{\sigma^2}$, which implies that $Q_{VM} = \frac{\sigma^2}{\mu}$ is a consistent risk index (and that ρ is a consistent risk-aversion index) with respect to f_{CA}.

4. Q_{IS} and ρ are consistent with f_{CE}. The optimal certainty equivalent function is given by

$$1-e^{-\rho(w+f_{CE})} = u\left(w+f_{CE}\right) = \mathbb{E}\left[u\Big(w+f_{CA}\left(\left(u,w\right),g\right)\cdot g\Big)\right]$$

$$= \mathbb{E}\left[u\Big(w+\frac{1}{\rho}\frac{\mu}{\sigma^2}\cdot g\Big)\right] = \mathbb{E}\left[1-e^{-\rho\cdot\left(w+\frac{1}{\rho}\frac{\mu}{\sigma^2}\cdot g\right)}\right] = 1-e^{-\rho w-\frac{\mu^2}{\sigma^2}+0.5\frac{\mu^2}{\sigma^2}},$$

where the last equality uses Fact 1. This implies that

$$1-e^{-\rho(w+f_{CE})} = 1-e^{-\rho w-\frac{\mu^2}{\sigma^2}+0.5\frac{\mu^2}{\sigma^2}} \Leftrightarrow -\rho\left(w+f_{CE}\right) = -\rho w-\frac{\mu^2}{\sigma^2}+0.5\frac{\mu^2}{\sigma^2} \Leftrightarrow f_{CE} = \frac{1}{2\rho}\frac{\mu^2}{\sigma^2}.$$

Thus, $f_{CE}\left(\left(u,w\right),g\right) = \frac{1}{2\rho}\frac{\mu^2}{\sigma^2}$, which implies that $Q_{IS} = \frac{\sigma}{\mu}$ is a consistent risk index (resp., ρ is a consistent risk-aversion index) with respect to f_{CE}.

B.2 Proof of Theorem 1

The following three lemmas will be useful in our proofs. The first lemma is a simple version of Ito's well-known lemma (see, e.g., Shreve, 2004, Equation 4.4.24).

Lemma 1 (Ito's lemma)**.** *Let $s(t)$ be a random process described by $ds_t = \mu_t dt + \sigma_t dW$. Let $f(t,s)$ be a twice-differentiable function; then*

$$df = \left(\mu_t \frac{\partial f}{\partial s} + 0.5\sigma_t^2 \frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial t}\right) dt + \frac{\partial f}{\partial s}\sigma_t dW.$$

The next two lemmas are standard calculus results.

Lemma 2. *Let $F_t(y)$ be a set of real-valued, continuous, and weakly increasing functions, with $0 < t \leq T$ and $y \in \mathbb{R}$. Assume that there exists a continuous and strictly increasing function $F(y)$ such that (1) $\forall y, F(y) = \lim_{t\to 0} F_t(y)$, and (2) $\exists y^*$, s.t. $F(y^*) = 0$. Then, there exists $\bar{t} > 0$ s.t.*

$$\forall t < \bar{t}\ \exists y_t \text{ s.t. } F_t(y_t) = 0, \text{ and } \lim_{t\to 0} y_t = y^*.$$

Proof. Let $\delta > 0$. We have to show that there exists $\bar{t}$ s.t. $\forall t < \bar{t}$ there is a value y_t satisfying $|y_t - y^*| < \delta$ and $F_t(y_t) = 0$. Since $F(y)$ is strictly increasing there exists a positive number C such that $F(y^* - \delta) < -C$ and $F(y^* + \delta) > C$. Condition (1) implies that there exists $\bar{t}$ s.t. $\forall t < \bar{t}$,

$$|F_t(y^* + \delta) - F(y^* + \delta)| < C, \text{ and } |F_t(y^* - \delta) - F(y^* - \delta)| < C.$$

Hence, $F_t(y^* - \delta) < 0$ and $F_t(y^* + \delta) > 0$. Since F_t is continuous, $\exists y_t \in (y^* - \delta, y^* + \delta)$ s.t. $F_t(y_t) = 0$. □

Lemma 3. *Let $F_t(\alpha)$ be a set of twice-differentiable strictly concave functions where $0 < t \leq T$ and $\alpha \in \mathbb{R}$, and let F be a twice-differentiable strictly concave function such that (1) $\forall \alpha,\ F(\alpha) = \lim_{t\to 0} F_t(\alpha)$, and (2) $\exists \alpha^* \in \mathbb{R}$ such that $\alpha^* = \arg\max_{\alpha\in\mathbb{R}} F(\alpha)$. Then, there exists $\bar{t} > 0$ such that*

$$\forall t < \bar{t}, \exists \alpha_t \in \mathbb{R} \text{ s.t. } \alpha_t = \arg\max_{\alpha} F_t(\alpha), \text{ and } \lim_{t\to 0} \alpha_t = \alpha^*.$$

Proof. We have to show that, given $\delta > 0$, there exists $\bar{t} > 0$ such that $\forall t < \bar{t}$, $\exists \alpha_t$, which maximizes $F_t(\alpha)$, and that $|\alpha_t - \alpha^*| < \delta$. Let $\delta_1 = \min\{F(\alpha^*) - F(\alpha^* - \delta), F(\alpha^*) - F(\alpha^* + \delta)\}$. There exists $\bar{t}$ s.t. $\forall t < \bar{t}$,

$$|F_t(\alpha^*) - F(\alpha^*)| < \delta_1/3, \quad |F_t(\alpha^* + \delta) - F(\alpha^* + \delta)| < \delta_1/3, \text{ and}$$

$$|F_t(\alpha^* - \delta) - F(\alpha^* - \delta)| < \delta_1/3.$$

Hence, $\forall t < \bar{t}$,

$$F_t(\alpha^*) > F_t(\alpha^* - \delta) \text{ and } F_t(\alpha^*) > F_t(\alpha^* + \delta).$$

Since for all t, F_t is weakly concave, there exists $\alpha_t \in (\alpha^* - \delta, \alpha^* - \delta)$, which is the argmax of F_t. ☐

Next, we prove the main theorem. Let $g \in \Gamma$ be a continuous-time random process, and let $(u, w) \in \mathcal{DM}$ be a decision maker.

1. Q^l_{SD} and ρ are consistent with f_{RP}. For every $t > 0$, let F_t be defined as follows:

$$F_t(x) = \frac{u\Big(w + \mathbb{E}\left[g_t\right] + x \cdot t\Big) - \mathbb{E}\Big[u\Big(w + g_t\Big)\Big]}{t}.$$

   By definition, if for some value of x, $F_t(x) = 0$, then $x \cdot t = f_{RP}((u, w), g_t)$. To calculate the limit of F_t as t goes to zero, it is simpler to look at F_t as the difference between two functions k_t and h_t, defined by

$$k_t(x) = \frac{u(w + \mathbb{E}\left[g_t\right] + x \cdot t) - u(w)}{t}, \quad \text{and}$$

$$h_t = \frac{\mathbb{E}\left[u\left(w + g_t\right)\right] - u(w)}{t}. \tag{8}$$

   Clearly,

$$F_t(x) = k_t(x) - h_t$$

   for every value of x. The limit of $k_t(x)$ as t goes to zero is simply the derivative with respect to t at w:

$$\lim_{t \to 0} k_t(x) = u'(w) \cdot (\mu_0 + x). \tag{9}$$

By applying Ito's lemma

$$h_t = \frac{\mathbb{E}\left[\int_0^t \left(\mu_q u'_q + \frac{1}{2}\sigma_q^2 u''_q\right) dq\right]}{t} + \frac{\mathbb{E}\left[\int u'_q \sigma_q dW\right]}{t},$$

where $u'_q \equiv du(w_q)/d(w_q)$, $u''_q \equiv du^2(w_q)/d^2(w_q)$, and $w_q = w + g_q$. Since we assumed that g is bounded from below, the concavity and monotonicity of u implies that u'_q is bounded. In addition, we assumed that the σ_t satisfies the square-integrability condition and, therefore, that $E\left[\int_0^t \sigma_q^2 dq\right]$ is finite. These two assumptions imply that $E\left[\int_0^t (u'_q \sigma_q)^2 dq\right]$ is finite and, therefore, that $\int_0^t u'_q \sigma_t dW$ is a martingale; see Shreve (2004, Theorem 4.3.1. on page 134). Hence, h_t can be rewritten as follows:

$$h_t = \frac{\mathbb{E}\left[\int_0^t \left(\mu_q u'_q + \frac{1}{2}\sigma_q^2 u''_q\right) dq\right]}{t}.$$

Since μ_q, σ_q, u'_q and u''_q are all continuous, according to the mean-value theorem for integration, for each realization of g there exists some $x \in (0, t)$ for which

$$\frac{\int_0^t \left(\mu_q u'_q + \frac{1}{2}\sigma_q^2 u''_q\right) dq}{t} = \mu_x u'_x + \frac{1}{2}\sigma_x^2 u''_x.$$

As t goes to zero this expression converges to $\mu_0 u'(w) + \frac{1}{2}\sigma_0^2 u''(w)$. Since for every realization of g it converges to the exact same number, the expectation of this expression also converges to this number. Therefore,

$$\lim_{t \to 0} h_t = \mu_0 u'(w) + \frac{1}{2}\sigma_0^2 u''(w). \tag{10}$$

It follows from Equations (9) and (10) that

$$F(x) \equiv \lim_{t \to 0} F_t(x) = u'(w)x - \frac{1}{2}\sigma_0^2 u''(w).$$

Let x^* be the real number s.t. $F(x^*) = 0$, i.e.,

$$x^* = \frac{1}{2}\frac{u''(w)}{u'(w)}\sigma_0^2.$$

It is easy to see that the two conditions of Lemma 2 are satisfied: for all t, first F_t

is continuous as it is the sum of continuous functions, and second, F_t is a strictly increasing function since u is an increasing function. It follows from the lemma that there exists $\bar{t}$ and x_t such that $F_t(x_t) = 0$ for each $t < \bar{t}$, and

$$\lim_{t \to 0} x_t = x^*,$$

where, by definition, $f_{RP}((u,w), g_t) = x_t t$. Note that since $\frac{u''(w)}{u'(w)}$ is negative, x^* is negative as well, and therefore x^* (and $x^* \cdot t$ for all $t > 0$) is strictly decreasing with $\rho = -\frac{u''(w)}{u'(w)}$ and with $Q^l_{SD} = \sigma_0$.

Next, we would like to show that Q^l_{SD} and ρ are consistent with f_{RP}. We begin by showing that $Q^l_{SD}(g) > Q^l_{SD}(g')$ implies that $(f_{RP})_g^{(u,w)} << (f_{RP})_{g'}^{(u,w)}$ for any $(u,w) \in \mathcal{DM}$. Fix a decision maker (u,w), and let $x^*(g) \equiv x^*((u,w), g)$ (and use a similar notation for $x_t(g)$). Let $g, g' \in \Gamma$ be two processes satisfying $Q^l_{SD}(g) > Q^l_{SD}(g')$. Then $x^*(g) < x^*(g')$, and from the fact that $x_t \to x^*$it follows that there exists $\bar{t} > 0$, such that for each $t \in \left(0, \bar{t}\right)$, $x_t(g) \cdot t < x_t(g') \cdot t$, which implies that $f_{RP}((u,w), g_t) < f_{RP}((u,w), g'_t)$. In addition,

$$lim_{t\to 0} \frac{f_{RP}((u,w), g_t)}{f_{RP}((u,w), g'_t)} = lim_{t\to 0} \frac{x_t(g_t)\, t}{x_t(g'_t)\, t} = \frac{x^*(g)}{x^*(g')} = \frac{\left(Q^l_{SD}(g)\right)^2}{\left(Q^l_{SD}(g')\right)^2} \neq 1,$$

which proves that $(f_{RP})_g^{(u,w)} << (f_{RP})_{g'}^{(u,w)}$. Similarly, we show that $\rho(u', w') > \rho(u'', w'')$ implies that $(f_{RP})_g^{(u',w')} << (f_{RP})_g^{(u'',w'')}$ for any $g \in \Gamma$. Fix a process $g \in \Gamma$, and let $x^*(u,w) \equiv x^*((u,w), g)$ (and use a similar notation for $x_t(u,w)$). Let $(u', w'), (u'', w'') \in \mathcal{DM}$ be two agents satisfying $\rho(u', w') > \rho(u'', w'')$. Then $x^*(u', w') < x^*(u'', w'')$, and from the fact that $x_t \to x^*$ it follows that there exists $\bar{t} > 0$, such that for each $t \in \left(0, \bar{t}\right)$, $x_t(u'', w'') \cdot t < x_t(u', w') \cdot t$, implying that $f_{RP}((u'', w''), g_t) < f_{RP}((u', w'), g_t)$. In addition,

$$lim_{t\to 0} \frac{f_{RP}((u'', w''), g_t)}{f_{RP}((u', w'), g_t)} = lim_{t\to 0} \frac{x_t(u'', w'')\, t}{x_t(u', w')\, t} = \frac{x^*(u'', w'')}{x^*(u', w')} = \frac{\rho(u'', w'')}{\rho(u', w')} \neq 1,$$

which proves that $(f_{RP})_g^{(u',w')} << (f_{RP})_g^{(u'',w'')}$.

For the other direction, given some agent (u,w), if for two processes g and g' there is some $\bar{t}$ s.t. $f_{RP}((u,w), g_t) < f_{RP}((u,w), g'_t)$ for every $0 < t < \bar{t}$, and the ratio $f_{RP}((u,w), g_t) / f_{RP}((u,w), g'_t)$ does not go to 1 when t goes to zero, then

$x_t < x'_t$ for all $t < \bar{t}$, implying that the limits also satisfy $x^* < x'^*$ and, therefore, $\sigma_0 > \sigma'_0$. Similarly, given some process g, if for two agents (u', w') and (u'', w'') there is some $\bar{t}$ s.t. $f_{RP}\left((u', w'), g_t\right) < f_{RP}\left((u'', w''), g_t\right)$ for every $0 < t < \bar{t}$, and the ratio $f_{RP}\left((u, w), g_t\right) / f_{RP}\left((u', w'), g_t\right)$ does not go to 1 when t goes to zero, then $x^* < x'^*$, implying that (u', w') is locally more averse to risk than (u'', w'').

2. Q^l_{VM} and ρ are consistent with f_{CA}. The capital allocation function is defined by

$$f_{CA}\left((u, w), g_t\right) = \arg\max_{\alpha \in \mathbb{R}^+} \mathbb{E}\left[u\Big(w + \alpha \cdot g_t\Big)\right],$$

where $f_{CA}\left((u, w), g_t\right)$ equals infinity if there is no internal solution. For every $t > 0$, let F_t be the function defined as follows:

$$F_t(\alpha) = \frac{\mathbb{E}\left[u\left(w + \alpha g_t\right)\right] - u(w)}{t}. \tag{11}$$

By Ito's lemma,

$$F_t(\alpha) = \frac{\mathbb{E}_0\left[\int_0^t \alpha\mu_q u'_q + \frac{1}{2}\alpha^2\sigma_q^2 u''_q \; dq\right]}{t} + \frac{\mathbb{E}\left[\int_0^t \alpha u'_q \sigma_q dW\right]}{t},$$

where $u'_q \equiv du(w_q)/d(w_q)$, $u''_q \equiv du^2(w_q)/d^2(w_q)$, and $w_q = w + \alpha g_q$. For the same reason as in the case of f_{RP}, the expression on the right-hand side $\mathbb{E}\left[\int_0^t \alpha u'_q \sigma_q dW\right]$ is zero and therefore it can be omitted.

We define $F(\alpha)$ to be the limit of $F_t(\alpha)$ as t goes to zero. For the same reason as in the case of f_{RP} it equals to:

$$F(\alpha) \equiv \lim_{t \to 0} F_t(\alpha) = \alpha\mu_0 u'(w) + \frac{1}{2}\alpha^2\sigma_0^2 u''(w). \tag{12}$$

We denote by α^* the value of α that maximizes $F(\alpha)$:

$$\alpha^* = \arg\max_{\alpha} F(\alpha) = -\frac{u'(w)}{u''(w)}\frac{\mu_0}{\sigma_0^2}. \tag{13}$$

The two conditions of Lemma 3 are satisfied: first, by definition, the limit of F_t is

F. Second, we represent F_t as the sum of two expressions

$$F_t(\alpha) = \frac{\alpha \cdot \mathbb{E}_o\left[\int_0^t \mu_q u'_q dq\right]}{t} + \frac{\alpha^2 \cdot \mathbb{E}_o\left[\frac{1}{2}\int_0^t \sigma_q^2 u''_q \, dq\right]}{t}.$$

Since we assume that u'' is negative, F_t is strictly concave with α and the second condition of the lemma is satisfied.

By the lemma, there exists $\bar{t} > 0$ such that α_t maximizes F_t for all $t < \bar{t}$, and[8]

$$\lim_{t \to 0} \alpha_t = \alpha^*.$$

Note that the limit α^* is strictly decreasing with $\rho = -u''(w)/u'(w)$, and with $Q^l_{VM} = \sigma_0^2/\mu_0$.

Next we would like to show that Q^l_{VM} and ρ are consistent with f_{CA}, where by definition $f_{CA}\left((u,w), g_t\right) = (f_{CA})_g^{(u,w)}(t) = \alpha_t((u,w), g_t)$. For the first direction, we have to show that if for two processes $g, g' \in \Gamma$, $Q^l_{VM}(g) > Q^l_{VM}(g')$, then $(f_{CA})_g^{(u,w)} << (f_{CA})_{g'}^{(u,w)}$. Indeed, $Q^l_{VM}(g) > Q^l_{VM}(g')$ implies that $\alpha^*(g) < \alpha^*(g')$, and from the convergence of α_t it follows that there exists $\bar{t} > 0$, such that for each $t \in \left(0, \bar{t}\right)$, $\alpha_t(g) < \alpha_t(g')$. Since $\alpha^*(g) < \alpha^*(g')$, it follows that $\lim_{t\to 0} \alpha_t(g)/\alpha_t(g') \neq 1$ and therefore that $(f_{CA})_g^{(u,w)} << (f_{CA})_{g'}^{(u,w)}$. Similarly, if for two agents (u', w') and (u'', w''), $\rho(u', w') > \rho(u'', w'')$, then $\alpha^*(u', w') < \alpha^*(u'', w'')$, and from the convergence of α_t it follows that there exists some $\bar{t} > 0$, such that for each $t \in \left(0, \bar{t}\right)$, $\alpha_t(u', w') < \alpha_t(u'', w'')$. Since α^* (the limit of α_t) is positive, it follows that $\lim_{t\to 0} \alpha_t(u', w')/\alpha(u'', w'') \neq 1$ and therefore that $(f_{CA})_g^{(u',w')} << (f_{CA})_g^{(u'',w'')}$.

For the other direction, let $g, g' \in \Gamma$ be two processes for which, for any decision maker (u, w), $(f_{CA})_g^{(u,w)} << (f_{CA})_{g'}^{(u,w)}$. Indeed, the limits of $f_{CA}\left((u,w), g_t\right)$ and $f_{CA}\left((u,w), g'_t\right)$ when t goes to zero satisfy $\alpha^*(g') > \alpha^*(g)$ and, therefore, $Q^l_{VM}(g) > Q^l_{VM}(g')$. Similarly, let (u, w) and (u', w') be two decision makers for which $(f_{CA})_g^{(u',w')} << (f_{CA})_g^{(u'',w'')}$. Indeed, the limits of $f_{CA}\left((u,w), g_t\right)$ and

[8] The analysis implies that g_t is a "gamble" for each $t < \bar{t}$. To see that, note that we have shown that for every process g, and for every strictly concave utility function, there exists $\bar{t}$ such that for every $t < \bar{t}$, the solution of the maximization problem is internal. This implies that for every such t, $\boldsymbol{E}[g_t] > 0$. Otherwise, a risk-averse agent would be better off by choosing $\alpha_t = 0$, contradicting our result here that $\alpha_t > 0$ for a sufficiently short time t. Similarly, the analysis implies that $P(g_t < 0) > 0$ for a sufficiently short time t. Otherwise, for every α_t and $\epsilon > 0$, $(\alpha_{t.} + \epsilon)\, g_t$ would first-order stochastically dominate $\alpha_t g_t$ and therefore any agent would be better off enlarging any given $\alpha_{t.}$, which implies that the solution is not internal, contradicting our result that some finite $\alpha_t > 0$ maximizes F_t. These two properties of g_t imply that g_t is a gamble.

$f_{CA}\left(\left(u,w\right),g_t'\right)$ when t goes to zero satisfy $\alpha^*(u',w') < \alpha^*(u'',w'')$ and, therefore, $\rho\left(u',w'\right) > \rho\left(u'',w''\right)$.

3. Q^l_{IS} and ρ are consistent with f_{CE}. For every $t>0$, let F_t be defined as follows:

$$F_t(z) = \frac{u(w+z\cdot t) - \mathbb{E}\left[u\Big(w+\alpha g_t\Big)\right]}{t}.$$

It is easy to see that if α is the optimal allocation and $F_t(z)=0$ then $z\cdot t = f_{CA}(\left(u,w\right),g_t)$. To calculate the limit of F_t as t goes to zero, it is simpler to look at F_t as the difference between two functions k_t and h_t, defined by

$$k_t(z) = \frac{u(w+zt)-u(w)}{t}, \quad \text{and}$$

$$h_t = \frac{\mathbb{E}\left[u\Big(w+\alpha g_t\Big)\right] - u(w)}{t}. \tag{14}$$

Clearly,

$$F_t(z) = k_t(z) - h_t$$

for every value of z. The limit of $k_t(z)$ as t goes to zero is simply the derivative:

$$\lim_{t\to 0} k_t(z) = u'(w)\cdot z.$$

Using Ito's lemma, and taking the limit (as we did in Equations 11 and 12), we get

$$\lim_{t\to 0} h_t = \alpha\mu_0 u'(w) + \frac{1}{2}\alpha^2\sigma_0^2 u''(w).$$

Recall that according to Equation (13),

$$\alpha^* = -\frac{u'(w)}{u''(w)}\frac{\mu_0}{\sigma_0^2}.$$

Plugging $\alpha=\alpha^*$ into h_t, we get

$$\lim_{t\to 0} h_t = -\frac{\left(u'\left(w\right)\right)^2\mu_0^2}{u''\left(w\right)\sigma^2} + \frac{1}{2}\frac{\left(u'\left(w\right)\right)^2\mu_0^2}{u''\left(w\right)\sigma^2} = -\frac{1}{2}\frac{\left(u'\left(w\right)\right)^2\mu_0^2}{u''\left(w\right)\sigma^2}.$$

We define $F(z)$ to be the limit of $F_t(z)$, where t goes to zero:

$$F(z) \quad \equiv \quad \lim_{t\to 0} F_t(z) = \lim_{t\to 0} k_t(z) - \lim_{t\to 0} h_t(z) = u'(w)z + \frac{1}{2}\frac{(u_0')^2}{u_0''}\Big(\frac{\mu_0}{\sigma_0}\Big)^2.$$

We define z^* to be the value that results in $F(z^*) = 0$:

$$z^* = -\frac{1}{2}\frac{u'(w)}{u''(w)}\Big(\frac{\mu_0}{\sigma_0}\Big)^2.$$

For every t, $F_t(z)$ is continuous and strictly increasing satisfying the conditions of Lemma 2, therefore by the lemma there is $\bar{t}$ such that $F_t(z_t) = 0$ for every $t \in (0, \bar{t})$, implying that $z_t \cdot t$ is the certainty equivalent of the optimal investment in the gamble with horizon t, and that

$$\lim_{t\to 0} z_t = z^*.$$

It is easy to see that z^* (and therefore z^*t for all t) is strictly decreasing with $\rho = -\frac{u''(w)}{u'(w)}$ and with $Q^l_{IS} = \frac{\sigma}{\mu}$.
Next, we would like to show that Q^l_{IS} and ρ are consistent with f_{CE}. We begin by showing that $Q^l_{IS}(g) > Q^l_{IS}(g')$ implies that $(f_{CE})^{(u,w)}_g << (f_{CE})^{(u,w)}_{g'}$ for any $(u, w) \in \mathcal{DM}$. Fix a decision maker (u, w), and let $z^*(g) \equiv z^*((u, w), g)$ (and use a similar notation for $z_t(g)$). Let $g, g' \in \Gamma$ be two processes satisfying $Q^l_{IS}(g) > Q^l_{IS}(g')$. Then $z^*(g) < z^*(g')$, and from the fact that $z_t \to z^*$ it follows that there exists $\bar{t} > 0$, such that for each $t \in \left(0, \bar{t}\right)$, $z_t(g) \cdot t < z_t(g') \cdot t$, which implies that $f_{CE}((u, w), g_t) < f_{CE}((u, w), g'_t)$. In addition,

$$lim_{t\to 0}\frac{f_{CE}((u, w), g_t)}{f_{CE}((u, w), g'_t)} = lim_{t\to 0}\frac{z_t(g_t)\, t}{z_t(g'_t)\, t} = \frac{z^*(g)}{z^*(g')} = \frac{\left(Q^l_{IS}(g)\right)^2}{\left(Q^l_{IS}(g')\right)^2} \neq 1,$$

which proves that $(f_{CE})^{(u,w)}_g << (f_{CE})^{(u,w)}_{g'}$. Similarly, we show that $\rho(u', w') > \rho(u'', w'')$ implies that $(f_{CE})^{(u',w')}_g << (f_{CE})^{(u'',w'')}_g$ for any $g \in \Gamma$. Fix a process $g \in \Gamma$, and let $z^*(u, w) \equiv z^*((u, w), g)$ (and use a similar notation for $z_t(u, w)$). Let $(u', w'), (u'', w'') \in \mathcal{DM}$ be two agents satisfying $\rho(u', w') > \rho(u'', w'')$. Then $z^*(u', w') < z^*(u'', w'')$, and from the fact that $z_t \to z^*$ it follows that there exists $\bar{t} > 0$, such that for each $t \in \left(0, \bar{t}\right)$, $z_t(u'', w'') \cdot t < z_t(u', w') \cdot t$, implying that

$f_{CE}((u'', w''), g_t) < f_{CE}((u', w'), g_t)$. In addition,

$$lim_{t\to 0}\frac{f_{CE}((u'', w''), g_t)}{f_{CE}((u', w'), g_t)} = lim_{t\to 0}\frac{z_t(u'', w'')\,t}{z_t(u', w')\,t} = \frac{z^*(u'', w'')}{z^*(u', w')} = \frac{\rho(u'', w'')}{\rho(u', w')} \neq 1,$$

which proves that $(f_{CE})_g^{(u',w')} << (f_{CE})_g^{(u'',w'')}$.

For the other direction, given some agent (u, w), if for two processes g and g' there is some $\bar{t}$ s.t. $f_{CE}((u, w), g_t) < f_{CE}((u, w), g'_t)$ for every $0 < t < \bar{t}$, and the ratio $f_{CE}((u, w), g_t)/f_{CE}((u, w), g'_t)$ does not go to 1 when t goes to zero, then $x_t < x'_t$ for all $t < \bar{t}$, implying that the limits also satisfy $z^* < z'^*$ and, therefore, $\sigma_0 > \sigma'_0$. Similarly, given some process g, if for two agents (u', w') and (u'', w'') there is some $\bar{t}$ such that $f_{CE}((u', w'), g_t) < f_{CE}((u'', w''), g_t)$ for every $t \in \left(0, \bar{t}\right)$, and the ratio $f_{CE}((u'', w''), g_t)/f_{CE}((u', w'), g_t)$ does not go to 1 when t goes to zero, then $z^* < z'^*$, implying that (u', w') is locally more averse to risk than (u'', w'').