The shape of luck and competition in winner-take-all tournaments*

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Abstract

In winner-take-all tournaments, agents' performance is determined jointly by effort and luck, and the top performer is rewarded. We study the impact of the "shape of luck" – the details of the distribution of performance shocks – on incentives in such settings. We are concerned with the effects of increasing the number of competitors, which can be deterministic or stochastic, on individual and aggregate effort. We show that these effects are determined by the shape of the density and failure (hazard) rate of the distribution of shocks. When shocks have heavy tails, aggregate effort can decrease in the number of competitors.

Keywords: tournament, competition, heavy tails, stochastic number of players, unimodality, log-supermodularity, failure rate

JEL codes: C72, D72, D82

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1 Introduction

> *[...] because the contests that mete out society's biggest prizes are so bitterly competitive, talent and effort alone are rarely enough to ensure victory. In almost every case, a substantial measure of luck is also necessary.*
>
> Robert H. Frank,
> "Success and Luck: Good Fortune and the Myth of Meritocracy"

Luck, or lack thereof, plays a crucial role in people's lives. The success stories we observe in business, academia, sports or the arts can often be traced back to a "lucky moment" or an unlikely sequence of events that defined the future path of success. Notable examples are the stories of Bill Gates and Microsoft, Da Vinci's *Mona Lisa* and actor Bruce Willis. In fact, a lucky break, or sequence of breaks, underlying a success story is not an exception but a rule (Mlodinow, 2009; Frank, 2016).[1]

Luck is especially important in *winner-take-all* (WTA) settings where rewards accrue to the select few. Examples include R&D competition, admission to top universities, job applications for an attractive position or competition for promotion in organizations. Careers in professional sports or the arts are predicated almost entirely on WTA incentives. When many hard-working, equally able people are trying to achieve the same thing, success requires a nontrivial amount of luck. As the number of competitors increases, so does the chance that *someone else* will get a better draw, which should discourage individual effort. Yet, economists typically believe that competition provides incentives in markets, at least on the aggregate, leading to larger output, lower prices and higher efficiency (e.g., Ruffin, 1971). In symmetric auctions, revenue increases in the number of bidders (McAfee and McMillan, 1987*a*). For R&D competition, empirical evidence shows a positive effect of competition on investment in innovation even at the individual firm level (Vives, 2008).

[1] When Bill Gates was growing up, he was one of 50 or so students *in the world* who, by sheer chance, had access to a programming terminal allowing to run code with instant feedback (Frank, 2016). The key contract between IBM and Microsoft, which transformed the latter into a world-dominating software company, was signed due to a series of random events; Microsoft did not even develop the initial version of its famous operating system DOS (Mlodinow, 2009). *Mona Lisa* was not considered an exceptional work of art until it was stolen from Louvre in 1911. The newspaper coverage of the painting's theft and recovery two years later created its global fame (Watts, 2011). Bruce Willis acted for seven years in small roles in New York, his main income coming from bartending. He flew to Los Angeles for personal reasons, went to a few television auditions, and got a role in *Moonlighting* far from being everyone's top choice. The first season flopped, but the second one became a hit, and the rest is history (Mlodinow, 2009).

In this paper, we study the effect of competition on incentives in WTA settings with a significant luck component. To do so, we utilize the classic rank-order tournament model of Lazear and Rosen (1981). Agents' output is given by effort distorted by additive noise, and the agent whose output is the highest wins the tournament and receives a fixed prize. The idiosyncratic noise is synonymous with luck in this model, and different distributions of noise allow for different "shapes of luck."

We consider tournaments where the number of players can be stochastic, following an arbitrary distribution. Indeed, in many situations the number of competitors is unknown to the tournament participants at the time they decide how much to invest in competition. This is the case, for example, in coding contests where an unknown and potentially very large number of coders submit their solutions, such as The Netflix Prize;[2] in hiring tournaments where a job seeker does not know how many others she is up against; or in promotion tournaments where an employee may not know how many of her colleagues the management is considering for a senior position.

The contribution of this paper is to answer a very basic question: How are individual and aggregate efforts in WTA tournaments affected by an increase in the *number* of competitors, in the first-order stochastic dominance (FOSD) sense? We show that the shape of the distribution of noise – specifically, of its density and failure (hazard) rate – is crucial for any prediction about the effect of competition on effort. There is not a single prediction, either for individual or aggregate effort, that cannot be reversed for at least some distribution of noise. Individual and aggregate equilibrium effort can be increasing, decreasing or nonmonotone in the number of players. We systematize and provide new general results for these effects both when the number of players is known and when it is random. The results have many testable implications, as well as far-reaching applications for tournament design.

In order to cleanly delineate the effects of the number of players and the distribution of noise, we focus on a setting with symmetric players; that is, we assume away differences in ability. While these differences undoubtedly play a critical role in success across the society at large, the most intense competition takes place, and the impact of luck is especially pronounced, in stratified sub-tournaments among

[2]The Netflix Prize competition where the task was to improve the Netflix recommendation algorithm for movies ran for three years overall and about 40,000 teams registered at some point. The final stage lasted 30 days and the two best teams tied in terms of the score. One of the teams won because it submitted its solution *twenty minutes* before the rival (The New York Times, 2009).

(roughly) equally able contestants.[3] Then, in the symmetric pure-strategy equilibrium, there are no differences in effort, and the winner is the luckiest player, i.e., the one with the highest realization of noise.

A general intuition for our results is the following. Start with a deterministic number of players $k \geq 2$. A marginal increase in player i's effort in the symmetric equilibrium is pivotal – that is, it makes this player the winner – if the player's noise realization, X_i, is equal to order statistic $X_{(k-1:k-1)}$ – the highest realization of noise among the other $k-1$ players. This means, formally, that the equilibrium effort is determined by the probability density of the difference $X_i - X_{(k-1:k-1)}$ at zero, which is equal to the expectation $\mathbb{E}(f(X_{(k-1:k-1)}))$, where $f(\cdot)$ is the pdf of noise. Order statistic $X_{(k-1:k-1)}$ is FOSD-increasing in k, and the comparative statics for monotone densities then follow immediately. For example, in the case of uniformly distributed noise the number of players does not affect the individual equilibrium effort. More generally, adapting results from Karlin (1968), we show that the *unimodality* of the pdf of noise leads to the individual equilibrium effort being unimodal in the number of players. The result follows from a log-supermodularity condition that the pdf of order statistic $X_{(k-1:k-1)}$ satisfies.[4] We provide a general characterization of the equilibrium comparative statics for unimodal noise distributions, from which all existing results follow as special cases.

For aggregate effort, using arguments similar to the ones in the previous paragraph, we show that when the cost of effort is quadratic, aggregate effort can be written as the expectation $\mathbb{E}(h(X_{(k-1:k)}))$, where $h(\cdot)$ is the failure (hazard) rate of noise, and $X_{(k-1:k)}$ is the second-highest order statistic among k noise realizations. Then, aggregate effort is increasing in the number of players if the noise distribution has an increasing failure rate (IFR), such as the normal, uniform and Gumbel distributions. The comparative statics are reversed for distributions with decreasing failure rates (DFR), such as Pareto. We then generalize these results for cost functions more

[3]For example, each year thousands of top high school graduates compete for admission to elite universities; the presence of unqualified applicants in the mix is largely irrelevant. A similar stratification happens naturally in the job market for academic positions or in competition among papers submitted to top journals. Even if quality varies substantially in the initial pool, the actual competition boils down to a subset where quality is very close and, inevitably, luck comes into play. It is also widely believed that tournaments become inefficient as agents' heterogeneity increases (Lazear and Rosen, 1981). Thus, tournament-based incentives are most likely to emerge in settings with symmetric agents.

[4]See Athey (2002) and related papers for a discussion of the role of log-supermodularity in monotone comparative statics problems under uncertainty.

or less convex than quadratic in the sense of the convex transform order (Shaked and Shanthikumar, 2007).

Turn now to the case when the number of players, k, is stochastic. The equilibrium effort is then determined by an expectation over k of the expectations of functions of order statistics described in the previous two paragraphs. For the equilibrium effort to be unimodal in the number of players as the latter increases in the FOSD sense, the result on the preservation of unimodality has to be applied once more, this time to the distribution of k. Unlike the distributions of order statistics $X_{(k-1:k-1)}$ and $X_{(k-1:k)}$, an arbitrary distribution of k may not satisfy the corresponding log-supermodularity condition; hence, a restriction on the distribution of k has to be imposed. This restriction is rather weak; it is satisfied, for example, by the family of power series distributions which includes the distributions typically used in the literature to model population uncertainty, such as the Poisson, binomial, negative binomial and logarithmic distributions.

The most surprising results are obtained for aggregate effort in the presence of a *heavy tail* in the distribution of noise. Such noise distributions, most notably *power laws* (Gabaix, 2016), are usually characterized by a decreasing or (interior) unimodal failure rate. Our results then imply a reduction in aggregate effort with the number of players, at least in sufficiently large tournaments. A principal whose goal is to maximize aggregate effort or investment, e.g., in a promotion tournament or an R&D race, would benefit from restricting the number of participants. This is a new mechanism that is very different from the ones identified in the literature on optimal exclusion of agents in contests.[5]

Fluctuations following heavy-tailed distributions, including power laws (also known as the Pareto distribution), have been widely identified in economics, finance and other domains. For example, it has been known for a long time that economic variables such as income (Pareto, 1896), city sizes (Auerbach, 1913), firm sizes (Axtell, 2001), stock market movements (Mandelbrot, 1963) and CEO compensation (Roberts, 1956) follow power laws. More recently, power laws have been found to describe demand for books at Amazon (Chevalier and Goolsbee, 2003) and movie ratings in

[5]The existing literature shows that it may be optimal to exclude certain types from contests with heterogeneous agents (see, e.g., Baye, Kovenock and De Vries, 1993; Taylor, 1995; Fullerton and McAfee, 1999; Che and Gale, 2003), or restrict entry in contests with symmetric agents and endogenous participation (Fu, Jiao and Lu, 2015). The present paper shows, instead, that exclusion may be optimal even when agents are symmetric and their number is exogenous.

Netflix (Bimpikis and Markakis, 2016).[6] The nature of innovation as an unlikely breakthrough resulting from a large number of mostly unsuccessful attempts produces heavy tails in the value, quality and financial returns of inventions (Fleming, 2007).

To date, there is virtually no empirical research on the effects of variation in the shape of shocks on behavior in tournaments. The existing studies of tournaments using natural data (e.g., Ehrenberg and Bognanno, 1990; Knoeber and Thurman, 1994; Eriksson, 1999) treat noise as a nuisance and do not attempt to estimate its distribution. Similarly, laboratory experiments typically rely on a specific distribution of noise in their winner determination process – most often, a lottery contest or uniformly distributed additive shocks (for a review, see Dechenaux, Kovenock and Sheremeta, 2015) – and do not explore variation in its shape. We only know of one exception. List et al. (2014) study how effort depends on the number of players in tournaments with varying noise densities. They consider distributions with constant, increasing and decreasing densities and find, consistent with theory, that the comparative statics of individual effort follow similar patterns. However, all three distributions in their study are light-tailed with increasing failure rates, and, consistent with our results, they observe aggregate effort increasing in the number of players in all three cases.

Finally, we study the design of optimal tournaments with stochastic participation. We identify conditions for when the uncertainty about the number of players increases or decreases equilibrium effort. We also explore the optimality of disclosing the realized number of players. Among the ∼40,000 teams registered for The Netflix Prize competition many were not active. Should Netflix have disclosed the number of active participants?

Relation to prior literature Starting with the seminal contributions of Tullock (1980) and Lazear and Rosen (1981), there is by now a large theoretical literature on tournaments using the respective models.[7] An important feature of these models distinguishing them from "perfectly discriminating" contests or all-pay auctions (e.g.,

[6]See Gabaix (2016) for a survey of many identified power laws and their underlying mechanisms. Many patterns outside economics are described by power laws as well, such as the frequency of words in natural languages (Zipf, 1949), the intensity of earthquakes (Christensen et al., 2002) or popularity in social networks (Barabási and Albert, 1999).

[7]See surveys by, e.g., Konrad (2009), Connelly et al. (2014), Corchón and Serena (2018).

Hillman and Riley, 1989; Baye, Kovenock and De Vries, 1996; Siegel, 2009; Moldovanu and Sela, 2001) is the presence of uncertainty, or "noise," in the winner determination process.

Yet, the existing analysis of *general* tournament models is quite scarce even in the case of non-random number of players. For tractability reasons, most of the literature uses either the Tullock CSF (also known as the lottery contest) and its lottery-form generalizations satisfying the axioms of Skaperdas (1996), or the Lazear-Rosen tournament with two players.[8] Relatively little is known about the basic comparative statics of the WTA tournament model in general. While the symmetric equilibrium effort decreases in the number of players in the Tullock contest (see, for example, surveys by Nitzan, 1994; Corchón and Serena, 2018), it is independent of the number of players in a Lazear-Rosen tournament when the distribution of noise is uniform. For general tournaments with a fixed number of players, Gerchak and He (2003) provide an important first step showing that the equilibrium effort is decreasing in the number of players when the noise density is decreasing or unimodal and symmetric, and increasing when the density is increasing (similar results are obtained by Ales, Cho and Körpeoğlu, 2017*a*). Even less is known about the behavior of aggregate effort beyond the Tullock contest and Lazear-Rosen tournament with uniformly distributed noise where it is increasing in the number of players.

There is no study of general rank-order tournaments with a stochastic number of players. The previous literature is restricted to the Tullock contest model (and its lottery-form generalizations), which we generalize. Myerson and Wärneryd (2006) compare aggregate equilibrium effort in the case of an arbitrary distribution of group size with expectation μ with the case when the number of players is equal to μ with certainty. Münster (2006) and Lim and Matros (2009) study the comparative statics of effort when the distribution of contest size is binomial.[9] Fu, Jiao and Lu (2011) study the effect of disclosing the number of participating players on aggregate effort. Boosey, Brookins and Ryvkin (2018) provide results on the effects of disclosure in contests between groups with stochastic sizes. More generally, our paper is related to the literature on games with population uncertainty, including auctions[10] and Poisson

[8]Notable exceptions are the papers analyzing optimal prize structures in tournaments (Nalebuff and Stiglitz, 1983; Green and Stokey, 1983; Krishna and Morgan, 1998; Kalra and Shi, 2001; Akerlof and Holden, 2012; Ales, Cho and Körpeoğlu, 2017*b*; Balafoutas et al., 2017; Drugov and Ryvkin, 2018, 2019*b*). See also a survey of the earlier literature by McLaughlin (1988).

[9]Münster (2006) also explores the effect of risk-aversion in the same setting.

[10]For a theoretical analysis of auctions with a stochastic number of bidders see, e.g., McAfee and

games.[11]

The rest of the paper is organized as follows. Section 2 sets up the model and provides some preliminary steps. Sections 3 and 4 present comparative statics results for individual and aggregate effort, respectively. Applications for tournament design are discussed in Section 5, and Section 6 concludes. More technical results on the existence of equilibrium and comparative statics for multimodal noise distributions are provided in Appendix A. All proofs are contained in Appendix B.

2 The model and preliminaries

2.1 The model setup

We consider a tournament game in which the number of players, K, is a random variable taking nonnegative integer values. Let $p = (p_0, p_1, \dots, p_n)$ denote the probability mass function (pmf) of K, where $p_k = \mathbb{P}(K = k)$ is the probability of having k players in the tournament. We will use $P = (P_0, P_1, \dots, P_n = 1)$ to denote the corresponding cumulative mass function (cmf), $P_k = \sum_{l=0}^{k} p_l$, and $G(z) = \sum_{k=0}^{n} p_k z^k$, $z \in [0, 1]$, to denote the probability generating function (pgf) of K. The maximum possible number of players $n \geq 2$ can be finite or infinite, $\mathbb{P}(K \geq 2) > 0$, and $\mathbb{E}(K^3)$ is finite.

Operationally, it is convenient to think about a set of *potential* participants $\mathcal{N} = \{1, \dots, n\}$ from which a subset $\mathcal{K} \subseteq \mathcal{N}$ is randomly drawn such that $\mathbb{P}(|\mathcal{K}| = k) = p_k$, and subsets of the same cardinality $|\mathcal{K}|$ have the same probability of being drawn. Each player is informed if she is selected, but is not informed about the value of K.

All participating players $i \in \mathcal{K}$ simultaneously and independently choose efforts $e_i \in \mathbb{R}_+$. The cost of effort e_i to player i is $c(e_i)$. Function $c : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing[12] and strictly convex on $[0, \bar{e}]$, and C^2 on $(0, \bar{e}]$, where $\bar{e} \equiv c^{-1}(1) < \infty$. Furthermore, $c(0) = c'(0) = 0$. Efforts e_i are perturbed by random additive shocks X_i to generate the players' output levels $y_i = e_i + X_i$.[13] Shocks X_i

McMillan (1987*b*), Harstad, Kagel and Levin (1990) and Levin and Ozdenoren (2004).

[11]See, e.g., Myerson (1998, 2000); Makris (2008, 2009); De Sinopoli and Pimienta (2009); Mohlin, Östling and Wang (2015); Kahana and Klunover (2015, 2016).

[12]Throughout this paper, unless noted otherwise, "increasing" will mean nondecreasing and "decreasing" will mean nonincreasing. When distinctions are important, "strictly increasing" and "strictly decreasing" will be used.

[13]We assume that shocks are additive. A model with multiplicative shocks can be cast into this form via a logarithmic transformation, see Section 2.4.

are i.i.d. with absolutely continuous cdf $F(\cdot)$ and continuous, bounded, piecewise differentiable, and square-integrable pdf $f(\cdot)$ defined on interval support $\mathcal{X} = [\underline{x}, \overline{x}]$, where the bounds $\underline{x}$ and $\overline{x}$ may be finite or infinite.[14] The winner of the tournament – the player whose output is the highest – receives a prize normalized to one. Ties occur with probability zero. The players are risk-neutral expected payoff maximizers.

2.2 Equilibrium characterization

Let S_i denote a random variable equal to 1 if player $i \in \mathcal{N}$ is selected for participation and zero otherwise, and let $\tilde{K} = (K|S_i = 1)$ denote the random number of players in the tournament from the perspective of a participating player. The pmf of $\tilde{K}$ is given by (see, e.g., Harstad, Kagel and Levin, 1990)

$$\tilde{p}_k = \mathbb{P}(\tilde{K} = k) = \frac{p_k k}{\bar{k}}, \quad k = 1, \ldots, n, \tag{1}$$

where $\bar{k} = \mathbb{E}(K)$. The corresponding cmf will be denoted as $\tilde{P}$.

Equation (1) can be understood as follows (cf. Myerson and Wärneryd, 2006). Suppose n is finite (for an infinite n, a similar argument applies in the limit $n \to \infty$). For a given k, the probability for player i to be selected for participation is $\mathbb{P}(S_i = 1|K = k) = k/n$; thus,

$$\tilde{p}_k = \mathbb{P}(K = k|S_i = 1) = \frac{\mathbb{P}(S_i = 1|K = k)p_k}{\sum_{l=0}^{n} \mathbb{P}(S_i = 1|K = l)p_l} = \frac{\frac{k}{n}p_k}{\sum_{l=0}^{n} \frac{l}{n}p_l},$$

which gives (1).

We focus on a symmetric pure strategy equilibrium in which all participating players choose effort $e^* > 0$. The expected payoff of a participating player $i \in \mathcal{K}$ from some deviation effort e_i is

$$\pi^{(i)}(e_i, e^*) = \sum_{k=1}^{n} \tilde{p}_k \int F(e_i - e^* + x)^{k-1} dF(x) - c(e_i). \tag{2}$$

[14] In this type of models, it is typically assumed that the shocks are zero-mean. While this assumption can be made without loss of generality, it is not necessary because the probability of winning is determined by differences in shocks. Moreover, shocks can be i.i.d. conditional on an additive common component.

Here and below, integration over $\mathcal{X}$ is implied unless noted otherwise. Define b_k as

$$b_k = (k-1)\int F(x)^{k-2} f(x) dF(x). \tag{3}$$

The symmetric first-order condition for payoff maximization, $\pi^{(i)}_{e_i}(e^*, e^*) = 0$, gives

$$c'(e^*) = B_p \equiv \sum_{k=1}^{n} \tilde{p}_k b_k = \frac{1}{\bar{k}} \sum_{k=2}^{n} p_k k b_k. \tag{4}$$

The summation can start with $k = 2$ instead of $k = 1$ because $b_1 = 0$. In other words, only group sizes $k \geq 2$ contribute to the equilibrium effort.

Let e^*_p denote the unique positive solution of (4), assuming it exists. This solution is the candidate for the symmetric pure strategy equilibrium effort. The equilibrium existence can be ensured by assuming function $c(\cdot)$ has a second derivative bounded away from zero and the distribution of noise is sufficiently dispersed. Technical details are provided in Appendix A.1. Since $c'(e^*)$ is strictly increasing in e^*, the comparative statics of equilibrium effort e^*_p with respect to parameters of distribution p are determined entirely by B_p.

To understand the role of coefficients b_k, consider the degenerate case of a tournament with a fixed number of players k. With a slight abuse of notation, let e^*_k denote the corresponding equilibrium effort. Equation (4) then gives

$$c'(e^*_k) = b_k. \tag{5}$$

Thus, b_k is the equilibrium marginal benefit of effort in a tournament with k players, and total marginal benefit B_p is the expectation of $b_{\tilde{K}}$, cf. Eq. (4).

Combining (3) and (4), it will sometimes be convenient to write B_p in the form

$$B_p = \int f(x) d\tilde{G}(F(x)), \tag{6}$$

where $\tilde{G}(z) = \sum_{k=1}^{n} \tilde{p}_k z^{k-1}$ is the pgf of $\tilde{K}$.

2.3 Properties of $\tilde{p}$

We are interested in the effects of changes in distribution p on coefficients B_p, which then monotonically map into the comparative statics for equilibrium effort e_p^*. In particular, we explore how B_p responds to a stochastic increase (in the standard FOSD sense) in the number of players in the tournament. To this end, consider a parameterized family of group size distributions $\{p(\theta)\}_{\theta\in\Theta}$, where $\Theta \subseteq \mathbb{R}$ is an ordered set. Let $P(\theta)$ and $G(z,\theta)$ denote the corresponding cmf and pgf. We assume that $p(\theta)$ is FOSD-ordered in θ so that $P_k(\theta)$ and $G(z,\theta)$ are decreasing in θ. However, the corresponding updated pmf, $\tilde{p}(\theta)$, may not be FOSD-ordered in θ, in general. The following lemma provides a sufficient condition for when FOSD ordering is preserved under participation updating.

Lemma 1 *Suppose $p(\theta)$ satisfies the increasing likelihood ratio (ILR) property: For any $\theta' > \theta$, $p_k(\theta')/p_k(\theta)$ is increasing in k. Then both $p(\theta)$ and $\tilde{p}(\theta)$ are FOSD-ordered.*

Two distributions used most prominently in the literature to model population uncertainty are the Poisson and binomial distributions. Along with the negative binomial and logarithmic distributions, they belong to a family known as *power series distributions* (PSD) that are characterized by pmfs of the form

$$p_k(\theta) = \frac{a_k \theta^k}{A(\theta)}. \tag{7}$$

Here, a_k are nonnegative numbers, $\theta \geq 0$ is a parameter, and $A(\theta) = \sum_{k=0}^{\infty} a_k \theta^k$ (it is assumed that the sum exists) is the normalization function (Johnson, Kemp and Kotz, 2005). The pgf of PSD distributions is $G(z,\theta) = A(\theta z)/A(\theta)$. Lemma 2 lists several properties of the PSD family that will prove useful later. $G_\theta(z,\theta)$ and $P_k'(\theta)$ denote, respectively, the derivatives of the pgf and cmf with respect to θ.

Lemma 2 *Suppose pmf $p(\theta)$ is in the PSD family (7). Then*
(i) the updated pmf $\tilde{p}(\theta)$ is also in the PSD family;
(ii) $p(\theta)$ satisfies the ILR property;
(iii) $P_k'(\theta) \leq 0$ and $G_\theta(z,\theta) \leq 0$;
(iv) $-G_\theta(z,\theta)$ is log-supermodular: For all $\theta' > \theta$, $G_\theta(z,\theta')/G_\theta(z,\theta)$ is increasing in z.

(v) $-P_k'(\theta)$ *is log-supermodular: For all* $\theta' > \theta$, $P_k'(\theta')/P_k'(\theta)$ *is increasing in* k.

Property (i) states that the PSD family is closed under the participation updating (1).[15] From property (ii) and Lemma 1, PSD distributions are FOSD-ordered by θ (implying (iii)) and preserve the ordering under updating. Finally, the log-supermodularity properties (iv) and (v) will play a key role in comparative statics.

2.4 Tullock contests

The equilibrium of the popular contest model of Tullock (1980), with the probability of player i winning the contest given by contest success function (CSF) $e_i^r / \sum_{j\in\mathcal{K}} e_j^r$, can be obtained as a special case of (4).

Consider a tournament with deterministic size $k \geq 2$ and multiplicative noise with support in $\mathbb{R}_+$ such that player i's output is $y_i = e_i X_i$. Letting $\hat{e}_i = \log e_i$ and $\hat{X}_i = \log X_i$, this model is transformed into the additive noise model in which noise has cdf $\hat{F}(x) = F(\exp(x))$ and pdf $\hat{f}(x) = f(\exp(x))\exp(x)$, and the cost of effort is $\hat{c}(\hat{e}) = c(\exp(\hat{e}))$. The first-order condition (5) then becomes $\hat{c}'(\hat{e}) = \hat{b}_k$, where $\hat{b}_k$ is given by (3) with F and f replaced by $\hat{F}$ and $\hat{f}$.

Notice that $\hat{c}'(\hat{e}) = c'(\exp(\hat{e}))\exp(\hat{e}) = c'(e)e$. Thus, if $c(e)$ is the original cost function in the model with multiplicative noise, the corresponding additive noise model will have the transformed cost function of the form

$$c_{\mathrm{m}}(e) = \int_0^e c'(t)t dt. \tag{8}$$

To obtain the equilibrium of the Tullock contest, consider the model with multiplicative noise following the inverse exponential distribution, $F(x) = \exp(-x^{-r})$ (Hirshleifer and Riley, 1992; Jia, 2008; Jia, Skaperdas and Vaidya, 2013). The transformed noise then has the generalized extreme value type I (or Gumbel) distribution with cdf $\hat{F}(x) = \exp[-\exp(-rx)]$ and pdf $\hat{f}(x) = r\exp[-rx - \exp(-rx)]$. Equation

[15] In some cases, the updated distribution is of the same type as the initial distribution. For example, for $K \sim \text{Binomial}(n,q)$ we have $p_k = \binom{n}{k} q^k (1-q)^{n-k}$ (for $k = 0,\dots,n$) and $\tilde{p}_k = \binom{n-1}{k-1} q^{k-1}(1-q)^{n-k}$ (for $k = 1,\dots,n$); that is, $(\tilde{K}-1) \sim \text{Binomial}(n-1,q)$. Similarly, for $K \sim \text{Poisson}(\lambda)$ we have $p_k = \exp(-\lambda)\lambda^k/k!$ (for $k = 0,1,\dots$) and $\tilde{p}_k = \exp(-\lambda)\lambda^{k-1}/(k-1)!$ (for $k = 1,2,\dots$); that is, $(\tilde{K}-1) \sim \text{Poisson}(\lambda)$. It is possible, however, for the updated distribution to be of a different type (albeit still within the PSD family). For example, for $K \sim \text{Logarithmic}(\theta)$, where $\theta \in (0,1)$, we have $p_k = -\theta^k/[k\ln(1-\theta)]$, $\bar{k} = -\theta/[(1-\theta)\ln(1-\theta)]$, and $\tilde{p}_k = (1-\theta)\theta^{k-1}$; that is, $\tilde{K}$ has the geometric distribution with parameter $1-\theta$.

(3) then produces

$$\hat{b}_k = \frac{r(k-1)}{k^2}. \tag{9}$$

Assuming the original contest has a linear cost of effort $c(e) = e$, the transformed cost of effort is quadratic: $c_{\mathrm{m}}(e) = \int_0^e t dt = e^2/2$. First-order condition (5) then gives the well-known equilibrium effort in the Tullock contest, $e_k^* = r(k-1)/k^2$.

For stochastic contest size, Eq. (4) reproduces the model studied by Myerson and Wärneryd (2006) and, when K is binomial, the models of Münster (2006) and Lim and Matros (2009). The approach can also be extended to generalized Tullock contests with CSFs of the form $g(e_i)/\sum_{j=1}^{k} g(e_j)$, where $g(\cdot)$ is a strictly increasing "impact function" (Jia, 2008; Ryvkin and Drugov, 2017).

3 Individual equilibrium effort

In this section, we consider how individual equilibrium effort changes when the number of players in the tournament is stochastically increased. For maximum generality, we will directly assume that $\tilde{K}$ is FOSD-ordered in parameter θ. By Lemma 1, this holds when $p(\theta)$ has the ILR property (e.g., when K belongs to the PSD family, cf. Lemma 2), but the results of this section are still valid even if it does not.

Parameter θ can be continuous, such as q when $K \sim \text{Binomial}(n, q)$ or λ when $K \sim \text{Poisson}(\lambda)$, or a discrete index, such as n when $K \sim \text{Binomial}(n, q)$ or k when $K = k$ deterministically. In the continuous case, we will assume that $p(\theta)$ is differentiable in θ. Subscript θ will be used to denote both the derivative and the first difference.

In Section 3.1, we formulate our main result for unimodal noise densities, Proposition 1, and explain the underlying intuition. In Section 3.2, we revisit the existing results in the literature connecting them to special cases of Proposition 1, and provide some further results.

3.1 Comparative statics for unimodal noise distributions

Proposition 1 *Suppose $\tilde{K}$ is FOSD-increasing in θ and*
(a) $f(x)$ is unimodal;
(b) $-\tilde{G}_\theta(z, \theta)$ is log-supermodular.
Then $e^*_{p(\theta)}$ is unimodal in θ.

Proposition 1 states that the unimodality of the pdf of noise $f(\cdot)$ is "inherited" by the equilibrium effort $e^*_{p(\theta)}$ under some condition on the distribution of the number of players.

For the intuition, consider a tournament of deterministic size $k \geq 2$. Coefficients b_k in (3) can be rewritten as

$$b_k = \int f(x)dF(x)^{k-1} = \int f(x)f_{(k-1:k-1)}(x)dx, \tag{10}$$

where $F(x)^{k-1}$ is the cdf of the $(k-1)$-th order statistic among $k-1$ i.i.d. draws from distribution F, and $f_{(k-1:k-1)}(x) = dF(x)^{k-1}/dx$ is the corresponding pdf. In the symmetric equilibrium, player i wins the tournament if her realization of noise, X_i, exceeds $X_{(k-1:k-1)} = \max_{j \neq i} X_j$ – the largest shock among the other $k-1$ players. A marginal increase in the player's effort is then pivotal when there is a tie between the two shocks, i.e., it is determined by the probability density of $X_i - X_{(k-1:k-1)}$ at zero.

Representation (10) immediately leads to comparative statics results for monotone pdfs $f(x)$ (see Section 3.2). Indeed, $b_k = \mathbb{E}(f(X_{(k-1:k-1)}))$, and $X_{(k-1:k-1)}$ is FOSD-increasing in k. For nonmonotone distributions, however, the effect of an increase in k is ambiguous: both higher and lower values of $f(x)$ acquire higher weights in the expectation. To obtain comparative statics for unimodal distributions, Proposition 1 applies results on the preservation of unimodality under integration. These results are parallel to those of Karlin (1968) on the preservation of single-crossing, since a unimodal function has a single-crossing derivative.

For illustration, consider expectation $\gamma(\theta) = \int_0^1 u(z)dH(z,\theta)$, where $u(z)$ is interior unimodal, continuous and piecewise differentiable, and $H(z,\theta)$ is an absolutely continuous cdf of a random variable FOSD-increasing in θ. Since $u(1)$ is finite for an interior unimodal $u(z)$, integration by parts gives $\gamma(\theta) = u(1) - \int_0^1 u'(z)H(z,\theta)dz$. Then the derivative, or first difference, of $\gamma(\theta)$ is $\gamma_\theta(\theta) = -\int_0^1 u'(z)H_\theta(z,\theta)dz$. Following Karlin (1968), for $u'(z)$ single-crossing $+-$ and $-H_\theta(z,\theta)$ log-supermodular, $\gamma_\theta(\theta)$ is also single-crossing $+-$, and hence $\gamma(\theta)$ is unimodal.[16]

[16] Karlin (1968) refers to log-supermodularity as *total positivity* of order 2, TP_2. More generally, total positivity of order $r \geq 2$, TP_r, is sufficient for a *variation-diminishing property* of integration whereby the number of crossings of an integral cannot exceed the number of crossings of the integrated function as long as the latter does not exceed $r-1$. Therefore, Proposition 1 can be generalized to multimodal densities. For details, see Appendix A.2.

For tournaments with deterministic size k, $\theta = k$ and the role of $H(z, \theta)$ is played by $F(x)^{k-1}$, cf. (10). It is easy to see that $-H_\theta = F(x)^{k-1} - F(x)^k$ is log-supermodular, leading to the following result.

Corollary 1 *In tournaments with deterministic size k, if $f(x)$ is unimodal then e_k^* is unimodal in k.*

When the number of players is stochastic, the result on the preservation of unimodality has to be applied once more, this time to the expectation of a unimodal sequence $B_{p(\theta)} = \sum_{k=1}^n \tilde{p}_k(\theta) b_k$. In this case, the role of $H(z, \theta)$ is played by $\tilde{P}_k(\theta)$, the cmf of $\tilde{K}$. However, due to representation (6) the log-supermodularity condition can be checked for $-\tilde{G}_\theta(z, \theta)$ instead. Proposition 1 then proves that $B_{p(\theta)}$ (and $e^*_{p(\theta)}$) is unimodal in θ.

The log-supermodularity condition in Proposition 1 holds for PSD distributions of tournament size, cf. Lemma 2(iv), producing the following result.

Corollary 2 *If $p(\theta)$ is in the PSD family (7) and $f(x)$ is unimodal, then $e^*_{p(\theta)}$ is unimodal in θ.*

For an example of interior unimodal effort, consider the type I generalized logistic distribution, which has cdf $F(x) = 1/(1+\exp(-x))^a$ with parameter $a > 0$ (Johnson, Kotz and Balakrishnan, 1995). The standard logistic distribution is obtained for $a = 1$.[17] Then, $b_k = a(k-1)/[k(ak+1)]$. Since $b_{k+1} - b_k \propto 1 + a - ak(k-1)$ is decreasing in k, b_k (and hence, e_k^*) is either monotonically decreasing or interior unimodal. In particular, b_k reaches its maximum at $\hat{k}$ if $a = 1/(\hat{k}^2 - \hat{k} - 1)$. When the number of players is stochastic, with a distribution satisfying the log-supermodularity condition (b) of Proposition 1, this unimodality is transferred to the equilibrium effort as the number of players is FOSD-increasing. Figure 1 shows the dependence of the equilibrium effort on k when the tournament size is deterministic (center), and on parameter q when the number of players is stochastic and distributed as Binomial$(10, q)$ (right). The latter distribution is PSD and hence satisfies condition (b).

Figure 3 gives an example of a lognormal distribution which is unimodal. The equilibrium effort is monotonically decreasing in the deterministic case and is interior

[17] The example has to involve an asymmetric unimodal distribution since effort is monotonically decreasing when $f(\cdot)$ is symmetric unimodal, see Section 3.2.

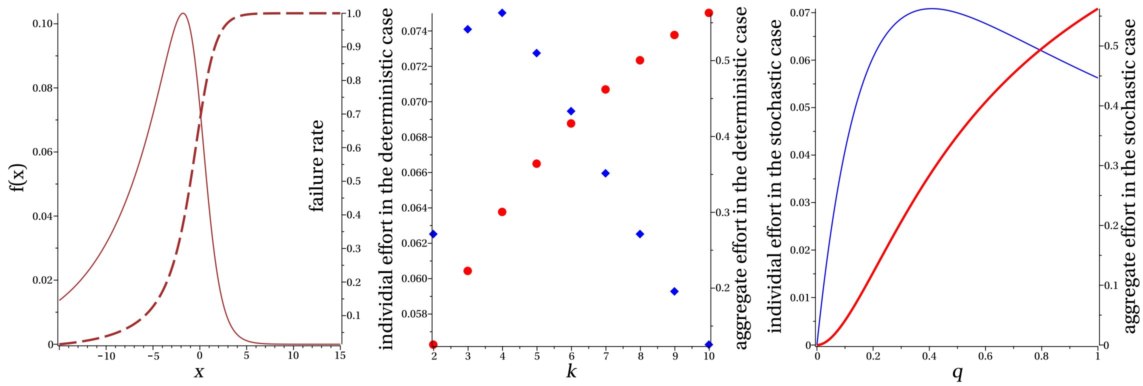

Figure 1: *Left*: The pdf $f(x)$ (thin solid line) and the failure rate $h(x)$ (thick dashed line) of the type I generalized logistic distribution with $a = \frac{1}{6}$. *Center*: Individual (blue diamonds) and aggregate (red circles) equilibrium efforts in the deterministic case as a function of the number of players k for effort cost function $c(e) = e^2/2$. *Right*: Individual (thin blue line) and aggregate (thick red line) equilibrium efforts in the stochastic case, with $K \sim \text{Binomial}(10, q)$ for effort cost function $c(e) = e^2/2$.

unimodal in the stochastic case. These comparative statics are similar to those for the Tullock contest which we discuss in Section 3.2.

Proposition 1 also allows us to characterize the behavior of $e^*_{p(\theta)}$ for *U-shaped* noise distributions such that $-f(x)$ is unimodal.

Corollary 3 *Suppose $\tilde{K}$ is FOSD-increasing in θ and*
(a) $f(x)$ is U-shaped;
(b) $-\tilde{G}_\theta(z, \theta)$ is log-supermodular;
(c) a one-player tournament is not possible, $p_1(\theta) = 0$.
Then $e^*_{p(\theta)}$ is U-shaped in θ.

Corollary 3 is a mirror image of Proposition 1, bar condition (c) that appears for the following reason. When $f(x)$ is U-shaped, sequence b_k (and hence, effort e^*_k in the deterministic case) is also U-shaped but only starting with $k = 2$, since $b_1 = 0$. Therefore, the entire sequence b_k starting with $k = 1$ may be (interior) bimodal and hence, the comparative statics may also be bimodal if $p_1 > 0$. Condition (c) excludes this possibility. Figure 2 shows an example of a U-shaped distribution of noise that produces a U-shaped e^*_k for $k \geq 2$ in the deterministic case, and a U-shaped dependence of $e^*_{p(\theta)}$ on θ for stochastic K distributed as Binomial$(10, q)$ truncated so that $K \geq 2$.

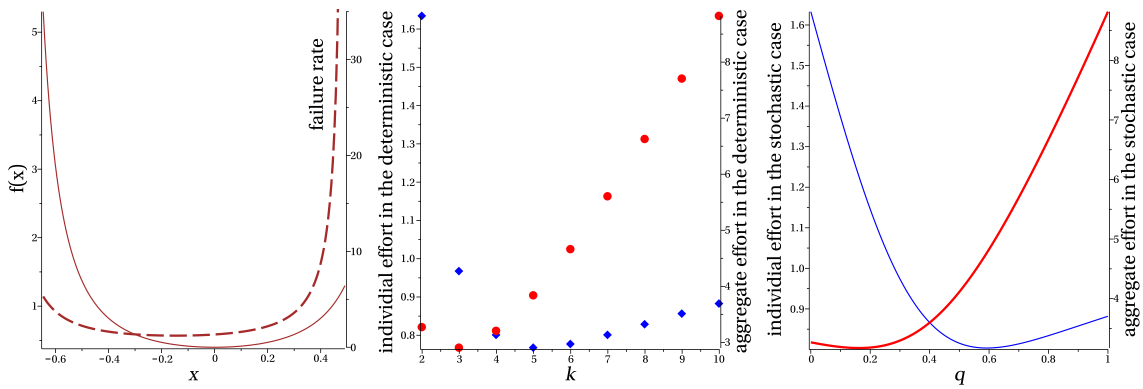

Figure 2: *Left*: The pdf $f(x)$ (thin solid line) and the failure rate $h(x)$ (thick dashed line) of a noise distribution with cdf $F(x) = 0.2\tan(2x) + 0.7$ defined on $[-0.646, 0.491]$. *Center*: Individual (blue diamonds) and aggregate (red circles) equilibrium efforts in the deterministic case as a function of the number of players k for effort cost function $c(e) = e^2/2$. *Right*: Individual (thin blue line) and aggregate (thick red line) equilibrium effort in the stochastic case, with K distributed as truncated Binomial$(10, q)$ with $K \geq 2$, for effort cost function $c(e) = e^2/2$.

3.2 Comparative statics under stronger assumptions on $f(\cdot)$

The literature so far has studied the comparative statics of individual effort in Lazear-Rosen tournaments only in the deterministic case and assuming monotone or symmetric unimodal noise densities. For Tullock contests, the analysis has been done in the stochastic case for K binomial. In this section, we review and comment on these results, and present more general versions.

As discussed below representation (10) for coefficients b_k, the case of monotone densities $f(x)$ is straightforward.

Corollary 4 *Suppose $\tilde{K}$ is FOSD-increasing in θ.*
(i) If $f(x)$ is increasing, then $e^*_{p(\theta)}$ is increasing in θ.
(ii) If $f(x)$ is decreasing and a one-player tournament is not possible ($p_1(\theta) = 0$), then $e^*_{p(\theta)}$ is decreasing in θ.
(iii) $e^*_{p(\theta)}$ is constant in θ for any $p(\theta)$ such that $p_1(\theta) = 0$ if and only if $f(x)$ is constant.

The deterministic version of Corollary 4 (except the "only if" part of (iii)) was proved by Gerchak and He (2003) and List et al. (2014). The stochastic version follows simply from the representation $B_{p(\theta)} = \mathbb{E}(b_{\tilde{K}})$, cf. (4).

Lim and Matros (2009) provide the most comprehensive comparative statics results for Tullock contests with $K \sim \text{Binomial}(n, q)$, showing that individual equilibrium effort is unimodal in q and in n. The extreme value type I distribution underlying the Tullock contest model has a unimodal density. While coefficients b_k are decreasing for $k \geq 2$, cf. Eq. (9), the entire sequence $\{b_k\}_{k=1}^{n}$ including $b_1 = 0$ is interior unimodal. Since $p_1 = nq(1-q)^{n-1} > 0$, the unimodality of e_p^* in q and n follows from Proposition 1.[18] Truncating the binomial distribution so that $p_0 = p_1 = 0$ would produce a decreasing dependence on q.

For the deterministic case, Gerchak and He (2003) show that $e_2^* = e_3^*$ when $f(x)$ is symmetric. If $f(x)$ is also unimodal, then e_k^* is decreasing in k for $k \geq 3$. The following proposition generalizes this result to the stochastic case.

Proposition 2 *Suppose $\tilde{K}$ is FOSD-increasing in θ.*
(i) If $f(x)$ is symmetric and $\operatorname{supp}(K) = \{2, 3\}$, then $e_{p(\theta)}^*$ is constant in θ.
(ii) If $f(x)$ is unimodal and symmetric, and $p_1(\theta) = 0$, then $e_{p(\theta)}^*$ is decreasing in θ.

Moreover, for a U-shaped and symmetric $f(x)$, part (ii) of Proposition 2 implies that $e_{p(\theta)}^*$ is increasing in θ.

Finally, Ryvkin and Drugov (2017) also analyze comparative statics in tournaments with a large deterministic number of players. The representation $b_k = \mathbb{E}(f(X_{(k-1:k-1)}))$ shows that, as k becomes large, the comparative statics are determined by the shape of the upper tail of $f(\cdot)$.

4 Aggregate equilibrium effort

In this section, we explore the effects of a stochastic increase in the number of players on expected aggregate effort,

$$E_{p(\theta)}^* = \bar{k}(\theta) e_{p(\theta)}^* = \bar{k}(\theta) c'^{-1}(B_{p(\theta)}). \tag{11}$$

We will use $E_k^* = kc'^{-1}(b_k)$ to denote aggregate effort for tournaments with deterministic size k. Throughout this section, it can be assumed that the original number

[18]The binomial distribution is in the PSD family with respect to parameter q; hence, for the dependence on q the result follows from Corollary 2. For the dependence on n, note that $\tilde{G}(z, n) = (1 - q + qz)^{n-1}$ and $-\tilde{G}_n(z, n)$ is log-supermodular.

of players, K, is FOSD increasing in θ. We no longer have to deal with $\tilde{K}$ because, unlike individual effort, aggregate effort is determined from an outsider's perspective.

If individual effort in the deterministic case, $e_k^* = c'^{-1}(b_k)$, is constant or increasing in k (i.e., when b_k is constant or increasing), it is obvious that aggregate effort is increasing in θ. However, as discussed in Section 3, for many noise distributions individual effort is decreasing or nonmonotone in the number of players, and hence, the comparative statics of aggregate effort are unclear. For example, in the Tullock contest with linear costs $c(e) = e$ individual effort $e_k^* = r(k-1)/k^2$ is decreasing but aggregate effort $E_k^* = r(k-1)/k$ is increasing in k for $k \geq 2$ (cf. Section 2.4).

Yet, a very crisp characterization of aggregate effort is possible through the behavior of the failure (or hazard) rate of noise, $h(x) = f(x)/[1 - F(x)]$. We say that distribution $f(x)$ is IFR (increasing failure rate) if $h(x)$ is increasing, and DFR (decreasing failure rate) if $h(x)$ is decreasing. Further, for two strictly increasing functions $c_1(\cdot)$ and $c_2(\cdot)$, we say that $c_1(\cdot)$ is *more (less) convex* than $c_2(\cdot)$ if $c_1(c_2^{-1}(\cdot))$ is convex (concave). Our main result for aggregate effort is given by the following proposition.

Proposition 3 *Suppose K is FOSD-increasing in θ.*
(i) If $f(x)$ is IFR and $c(e)$ is more convex than quadratic, then $E^*_{p(\theta)}$ is increasing in θ.
(ii) If $f(x)$ is DFR, $c(e)$ is less convex than quadratic, and there are at least two players ($p_0(\theta) = p_1(\theta) = 0$), then $E^*_{p(\theta)}$ is decreasing in θ.

Let us first explain why the failure rate of noise plays a prominent role in the analysis of aggregate effort. Coefficients b_k, Eq. (3), can be rewritten as $b_k = (k-1)\int F(x)^{k-2}[1-F(x)]h(x)dF(x)$. Note that the density of $X_{(k-1:k)}$, the second highest order statistic among k noise realizations, is $f_{(k-1:k)}(x) = k(k-1)F(x)^{k-2}[1-F(x)]f(x)$. Therefore, b_k becomes

$$b_k = \frac{1}{k}\int f_{(k-1:k)}(x)h(x)dx = \frac{1}{k}\mathbb{E}(h(X_{(k-1:k)})). \tag{12}$$

In equilibrium, winning a tournament with $k \geq 2$ players can be interpreted as both surpassing $X_{(k-1:k-1)}$, the highest shock among the other $k-1$ players, and surpassing $X_{(k-1:k)}$, the second highest shock among all k players. Note that the failure rate can be written as $h(x) = f(x|X \geq x)$, i.e., the pdf of noise at $X = x$

conditional on $X \geq x$. However, only realizations of noise exceeding $X_{(k-1:k)}$ can lead to winning; therefore, $\mathbb{E}(h(X_{(k-1:k)})) = \mathbb{E}(f(X_{(k-1:k)}|X \geq X_{(k-1:k)}))$ gives exactly the relevant conditional expectation. In order to obtain b_k, it needs to be multiplied by $\mathbb{P}(X \geq X_{(k-1:k)}) = 1/k$.

Consider a tournament with deterministic size $k \geq 2$ and quadratic cost function, $c(e) = e^2/2$, producing individual effort $e_k^* = b_k$. Using (12), aggregate effort is then $E_k^* = kb_k = \mathbb{E}(h(X_{(k-1:k)}))$. Since $X_{(k-1:k)}$ is FOSD-increasing in k, the comparative statics in the case of a monotone failure rate $h(x)$ follow immediately.

Most standard distributions fall into one of the monotone failure rate classes. IFR is implied by the log-concavity of pdf $f(x)$, while DFR is implied by the log-convexity of $f(x)$ provided $f(\overline{x}) = 0$. The exponential distribution, with $f(x) = \lambda \exp(-\lambda x)$, has a constant failure rate λ and hence is both IFR and DFR.

For tournaments with a stochastic number of players, still assuming the cost function is quadratic, Eq. (4) gives aggregate effort in the form

$$E_{p(\theta)}^* = \bar{k}(\theta) \sum_{k=1}^{n} \tilde{p}_k(\theta) b_k = \sum_{k=0}^{n} p_k(\theta) k b_k = \sum_{k=0}^{n} p_k(\theta) E_k^*. \tag{13}$$

When noise has a monotone failure rate, E_k^* is monotone for $k \geq 2$, as explained above. Moreover, when noise is IFR, E_k^* is increasing for $k \geq 0$ because $E_0^* = E_1^* = 0$; therefore, $E_{p(\theta)}^*$ is increasing in θ. However, when noise is DFR, E_k^* is no longer monotone for all k, and a restriction $K \geq 2$ has to be imposed to ensure that $E_{p(\theta)}^*$ is decreasing in θ.

Finally, consider the effect of the cost function. For a general cost function, aggregate effort $E_{p(\theta)}^*$ is given by (11). An increase in θ has two effects: A direct effect, due to the increase in $\bar{k}(\theta)$, and an indirect effect, due to the equilibrium change in $B_{p(\theta)}$. The latter effect can go in either direction, but it becomes less important as the cost function becomes "more convex." In other words, if the direct effect of a higher $\bar{k}(\theta)$ dominates the effect of $B_{p(\theta)}$ for some cost function, this is also the case for "more convex" cost functions.

The definition of cost function $c_1(\cdot)$ being more convex than $c_2(\cdot)$ is equivalent to requiring that there exists a strictly increasing, convex function $\eta(\cdot)$ such that $c_1(e) = \eta(c_2(e))$; indeed, defining $t = c_2(e)$, obtain $\eta(t) = c_1(c_2^{-1}(t))$. This partial order is related to the likelihood ratio order of random variables, whereby a random variable X is said to be smaller than random variable Y if the ratio of pdfs $f_Y(x)/f_X(x)$ is increasing in x. An equivalent condition is that $F_Y(F_X^{-1}(z))$ is convex (the convex transform order in Shaked and Shanthikumar, 2007). In our case, it implies that the ratio of marginal costs $c_1'(e)/c_2'(e)$ is increasing in e. The definition of a less convex function is analogous.[19]

The results for Tullock contests with linear costs mentioned at the beginning of this section follow from Proposition 3 because, as shown in Section 2.4, the properties of equilibrium in such a contest are equivalent to those of a tournament with a quadratic cost function and Gumbel distribution of noise, which is IFR. Figure 1 illustrates the case of a logistic distribution which is also IFR.

The exponential distribution has a constant failure rate, leading to the following result.

Corollary 5 *Suppose K is FOSD-increasing in θ; noise is exponentially distributed with pdf $f(x) = \lambda \exp(-\lambda x)$; the cost of effort is quadratic, $c(e) = e^2/2$; and there are at least two players ($p_0(\theta) = p_1(\theta) = 0$). Then $E^*_{p(\theta)} = \lambda$ is constant.*

To understand why aggregate effort can decrease in the number of players for DFR distributions of noise, note that such a distribution has a decreasing pdf which falls faster than its cdf is increasing. Hence, individual effort is decreasing (see Corollary 4(ii)), and so fast that aggregate effort decreases too. For a simple example, consider the F-distribution with (2,2) degrees of freedom, whose pdf and cdf are $f(x) = 1/(1+x)^2$ and $F(x) = x/(1+x)$ defined for $x \geq 0$. This gives $b_k = 2/[k(k+1)]$ and, for the quadratic cost function, aggregate effort $E^*_k = 2/(k+1)$ is strictly decreasing in k.

A result similar to Proposition 1, which shows that individual effort is unimodal for a unimodal pdf $f(x)$ under an additional log-supermodularity condition, can be formulated for aggregate effort when failure rate $h(x)$ is unimodal and the cost of effort is quadratic. Here, $P'_k(\theta)$ denotes the derivative or first difference of the cmf with respect to θ.

Proposition 4 *Suppose K is FOSD-increasing in θ, $c(e) = e^2/2$, and*

(a) $h(x)$ is unimodal;

[19] It follows that a cost function $c(e)$ is more convex than quadratic if $c(\sqrt{t})$ is convex in t or, equivalently, the ratio $c'(e)/e$ is increasing. Thus, a cost function is more convex than quadratic if the marginal cost increases faster than linear. For thrice differentiable functions, this condition implies $c''' \geq 0$, and is equivalent to it provided $c'(0) = 0$. Indeed, the condition that $c'(e)/e$ is increasing is equivalent to $c''(e)e \geq c'(e)$, which implies that $c'(e)$ is convex. Conversely, if $c'(0) = 0$, the convexity of $c'(e)$ implies $c''(e)e \geq c'(e)$. A less convex than quadratic function has $c''' \leq 0$.

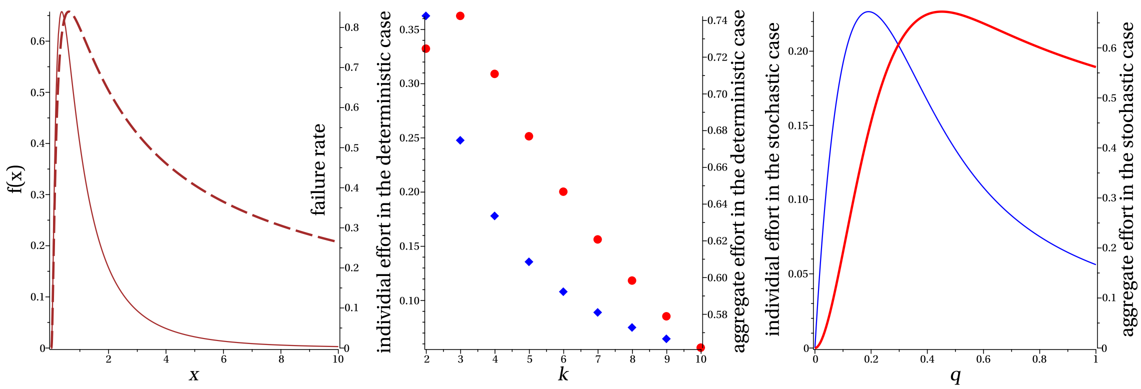

Figure 3: *Left*: The pdf $f(x)$ (thin solid line) and the failure rate $h(x)$ (thick dashed line) of the log-normal distribution with parameters $(0, 1)$. *Center*: Individual (blue diamonds) and aggregate (red circles) equilibrium efforts in the deterministic case as a function of the number of players k for effort cost function $c(e) = e^2/2$. *Right*: Individual (thin blue line) and aggregate (thick red line) equilibrium efforts in the stochastic case, with $K \sim \text{Binomial}(10, q)$ for effort cost function $c(e) = e^2/2$.

(b) $-P_k'(\theta)$ *is log-supermodular.*
Then $E^*_{p(\theta)}$ *is unimodal in* θ*.*

In the deterministic case, $E_k^* = \mathbb{E}(h(X_{(k-1:k)})) = \int h(x)dF_{(k-1:k)}(x)$, and the unimodality of E_k^* follows from the log-supermodularity of $-F_{(k-1:k)}(x)$. Therefore, in the stochastic case, $E^*_{p(\theta)} = \sum_{k=0}^{n} p_k(\theta)E_k^*$ is an expectation of a unimodal sequence. The unimodality of $E^*_{p(\theta)}$ is then ensured by the log-supermodularity condition (b). It is satisfied by PSD distributions, cf. Lemma 2(v), leading to the following result. For an illustration, see Figure 3.

Corollary 6 *If* K *is in the PSD family (7),* $c(e) = e^2/2$*, and* $h(x)$ *is unimodal, then* $E^*_{p(\theta)}$ *is unimodal in* θ*.*

Similar to the case of individual effort (Proposition 1 and Corollary 3), Proposition 4 also implies that $E^*_{p(\theta)}$ is U-shaped for distributions with a U-shaped failure rate (assuming $p_0 = p_1 = 0$). For an illustration, see Figure 2.[20]

[20]More generally, as in the case of individual effort, Proposition 4 can be generalized to multimodal $h(x)$ under higher order total positivity of $-P_k'(\theta)$, cf. footnote 16. The behavior of aggregate effort in large tournaments is determined by the shape of the failure rate in the upper tail.

5 Applications to tournament design

In this section, we investigate optimal design questions for tournaments with stochastic participation. Section 5.1 looks at how the level of uncertainty in the number of players affects aggregate effort. In Section 5.2 the question is whether it is optimal, from an *ex ante* perspective, to disclose the realized number of players. Finally, in Section 5.3 we briefly comment on the objective of maximizing the best performance.

5.1 The effect of uncertainty in the number of players

Is uncertainty in the number of players beneficial or detrimental for aggregate effort? The following proposition provides a general answer for any two distributions of the number of players ranked by second-order stochastic dominance (SOSD).

Proposition 5 *Consider two group size distributions, p and p', with the same mean $\bar{k}$ such that there are at least two players ($p_0 = p_1 = p'_0 = p'_1 = 0$) and p' SOSD p. Then, $E^*_{p'} \geq (\leq) E^*_p$ if $f(x)$ is log-concave (log-convex); moreover, the inequality is strict if $f(x)$ is strictly log-concave (log-convex).*

Proposition 5 says that higher uncertainty about the number of players (in the SOSD sense) reduces expected aggregate effort if noise has a log-concave distribution. For a log-convex noise distribution, the relationship is reversed. The comparison between aggregate efforts $E^*_{p'}$ and E^*_p is equivalent to a comparison between individual efforts $e^*_{p'}$ and e^*_p since the mean group size $\bar{k}$ is the same, which amounts to a comparison between $B_{p'}$ and B_p. Equation (4) can be written as

$$B_p = \frac{1}{\bar{k}}\mathbb{E}(Kb_K | K \geq 2)\mathbb{P}(K \geq 2); \tag{14}$$

that is, B_p is proportional to the expectation of Kb_K conditional on $K \geq 2$, and the assumption that there are always at least two players makes this expectation unconditional. Jensen's inequality can then be applied if kb_k is concave (convex) in k for $k \geq 2$, which is the case when $f(x)$ is log-concave (log-convex).

As a special case, Proposition 5 allows for a comparison of aggregate effort between tournaments with deterministic and stochastic group sizes. It implies that the presence of uncertainty in the number of players, as opposed to a tournament where the number of players is fixed and equal to $\bar{k}$, reduces expected aggregate effort in the Tullock contest (since the Gumbel distribution is log-concave) and increases it for many heavy-tailed distributions such as Pareto (which is log-convex). This is in contrast to the existing studies – restricted to Tullock contests – comparing aggregate effort in contests with deterministic and stochastic participation: Myerson and Wärneryd (2006), Lim and Matros (2009) and Boosey, Brookins and Ryvkin (2018). All three show that uncertainty in the number of players always reduces aggregate effort.[21]

However, Proposition 5 is more general than that and relates the ranking of expected aggregate effort to the SOSD order of group size distributions. It also shows that the presence of a heavy tail in the distribution of noise reverses the prevailing "intuition" that uncertainty in the number of players is detrimental for aggregate effort.

5.2 Optimal disclosure of the number of players

When the number of players K is stochastic, it might be possible for the tournament designer to reveal the realization of K to the players before they choose their efforts. Assuming commitment power, when does the tournament designer prefer to (commit to) disclose K? Lim and Matros (2009) show that in a standard Tullock contest with the binomial distribution of the number of players aggregate effort is independent of disclosure. Fu, Jiao and Lu (2011) generalize this result to contests with CSFs of the form $g(e_i)/\sum_{j=1}^k g(e_j)$. They show that full disclosure (no disclosure) is optimal if $g(e)/g'(e)$ is strictly convex (concave), while the indifference is recovered when $g(e)/g'(e)$ is linear. The following proposition generalizes these results to arbitrary tournaments and arbitrary distributions of the number of players.

Proposition 6 *Suppose b_k is non-constant for $k \geq 1$ in the support of p and $c'(\cdot)$ is nonlinear. Then it is optimal to disclose (not disclose) the number of participants in the tournament if $c''' \leq (\geq) 0$.*

Disclosure creates a mean-preserving variation in the marginal benefit of effort. Indeed, without disclosure the (expected) marginal benefit of effort is $c'(e_p^*) = B_p =$

[21] Proposition 5 is not a direct generalization of Myerson and Wärneryd (2006) and Lim and Matros (2009) because these two papers allow for the possibility of having fewer than two players in the contest, under additional restrictions. A version of Proposition 5 directly generalizing Myerson and Wärneryd (2006) and Lim and Matros (2009) is more nuanced and is available in Ryvkin and Drugov (2017). Boosey, Brookins and Ryvkin (2018) consider contests between groups with stochastic size.

$\mathbb{E}(b_{\tilde{K}})$, cf. (4), whereas with disclosure the realization of K is observed and effort is chosen according to $c'(e_k^*) = b_k$. Such variation then increases (decreases) expected individual effort if the marginal cost function is concave (convex); that is, if $c''' \leq (\geq)0$. For a quadratic cost function (i.e., when $c'(\cdot)$ is linear) disclosure is irrelevant. Note that the nature of coefficients b_k does not affect the optimality of disclosure. The only special case is when b_k is constant in the support of p for $k \geq 1$ (for example, noise is uniformly distributed and $p_1 = 0$), in which case disclosure does not matter.

The results of Fu, Jiao and Lu (2011) are recovered as a special case by introducing effective effort $y = g(e)$, which transforms their CSF into the lottery form and the cost of effort into $c(y) = g^{-1}(y)$. Following Section 2.4, the resulting cost of effort in the corresponding tournament with additive noise is $c_{\mathrm{m}}(y) = \int_0^y c'(t)t dt$, which gives the marginal cost $c'_{\mathrm{m}}(y) = c'(y)y = y/g'(g^{-1}(y)) = g(e)/g'(e)$.

A similar effect of a (mean-preserving) variation in the marginal benefit of effort emerges in static biased contests (see Drugov and Ryvkin, 2017) and dynamic contests where revealing interim information is equivalent to biasing the next stage (see Lizzeri, Meyer and Persico, 1999, 2002; Aoyagi, 2010). Parallel results regarding the role of c''' hold in those settings as well.

5.3 Best performance

Focusing on effort as the designer's objective is standard, and it is most easily justified under the assumption that noise, or luck, is unproductive and only distorts the perception, or measurement, of individual efforts. However, the designer may also be interested in maximizing the best performance. This objective is relevant in settings where noise is productive, such as innovation or design tournaments in which the winner's idea ends up being implemented and the rest are discarded.

For simplicity, consider a tournament with a deterministic number of players $k \geq 2$ and define expected best performance $y_k^{\max} = e_k^* + \mathbb{E}(X_{(k:k)})$. As k increases, $\mathbb{E}(X_{(k:k)})$ – the expectation of the best shock – increases as well, but equilibrium effort e_k^* may decrease, as, for example, in the Tullock contest or when $f(x)$ is decreasing or symmetric unimodal (cf. Corollary 4 and Proposition 2). Due to this trade-off, $y_k^{\max}$ may be nonmonotone in k.

Figure 4 shows individual effort e_k^*, the expectation of the best shock $\mathbb{E}(X_{(k:k)})$ and the resulting expected best performance $y_k^{\max}$ for a tournament with the Gumbel distribution of noise. The equilibrium effort in this tournament is the same as in the Tullock contest (cf. Section 2.4). As seen from the figure, while eventually $y_k^{\max}$ is increasing in k, there is a range of k where it is declining.[22]

A similar trade-off can emerge for a given number of players k when the intensity of noise increases.[23] For an explicit characterization, suppose the distribution of noise has a scale parameter $\sigma > 0$ such that the pdf and cdf are $f_\sigma(x) = (1/\sigma)f_1(x/\sigma)$ and $F_\sigma(x) = F_1(x/\sigma)$, where subscript 1 refers to the "standardized" distribution. Examples include the standard deviation of the normal distribution, $1/\lambda$ for the exponential distribution, and $1/r$ for the Gumbel distribution. The expected best performance is then

$$y_k^{\max}(\sigma) = c'^{-1}\left(\frac{1}{\sigma}b_{1,k}\right) + \sigma\mathbb{E}(X_{1,(k:k)}). \tag{15}$$

The first term is decreasing in σ, while the second term is increasing; therefore, $y_k^{\max}(\sigma)$ can be U-shaped. For illustration, suppose the noise is uniform on $[-\sigma/2, \sigma/2]$, and the cost function is $c(e) = e^2/2$. A sufficient condition for the equilibrium existence is $D_- < 1$, where $D_- = (k-1)/\sigma^2$ (see Proposition 7 in Appendix A.1). The equilibrium effort is $e_k^* = 1/\sigma$, and the expected best performance is

$$y_k^{\max}(\sigma) = \frac{1}{\sigma} + \frac{\sigma(k-1)}{2(k+1)}.$$

The derivative $\partial y_k^{\max}/\partial\sigma$ changes sign at $\sigma_0^2 = 2(k+1)/(k-1)$, and the equilibrium exists for $\sigma^2 > k-1$; therefore, $y_k^{\max}(\sigma)$ is U-shaped in σ for $k = 2, 3, 4$.

[22] An increase in k may interfere with the existence of the pure-strategy equilibrium we are studying (cf. Proposition 7 in Appendix A.1). The conditions of Proposition 7 are very general and hence, rather strong. For the example in Figure 4, we explicitly verified that the payoff function $\pi^{(i)}(e_i, e^*)$, Eq. (2), is maximized at $e_i = e_k^*$ and hence the equilibrium exists for all parameter values used in the figure.

[23] In a large tournament model including the analysis of mixed-strategy equilibria for small noise, Morgan, Tumlinson and Vardy (2018) show that aggregate effort can be nonmonotone in noise intensity due to players dropping out when the winner determination process becomes "too meritocratic." Using a large contest assortative matching model of competition for college admissions, Olszewski and Siegel (2019) show that more noise in contest outcomes can increase students' welfare. In a setting similar to the one in this paper, Drugov and Ryvkin (2019*a*) provide a systematic study of how equilibrium effort is affected by changes in the distribution of noise, and what it means to have "more noise" in a tournament.

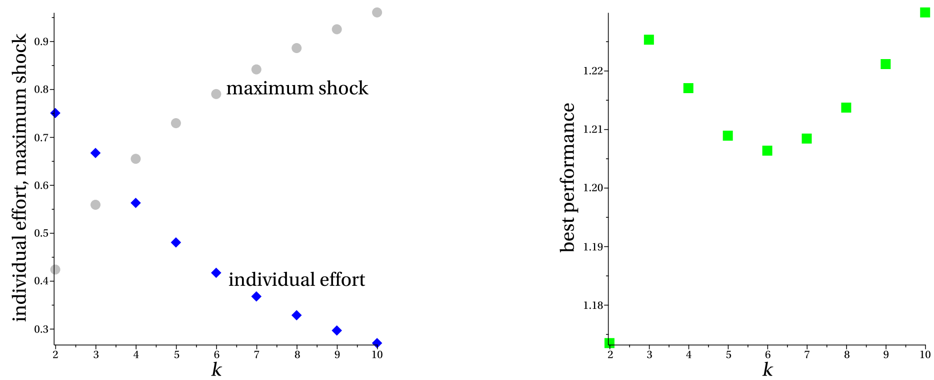

Figure 4: A tournament with the Gumbel distribution of noise with parameter $\frac{1}{3}$ and cost function $c(e) = e^2/2$. *Left*: Individual effort e_k^* and expected maximum shocks $\mathbb{E}(X_{(k:k)})$. *Right*: Expected best performance $y_k^{\max}$.

6 Conclusion

Tournament incentives are ubiquitous. Students applying to universities, researchers competing for grants, R&D firms competing for innovation, job candidates applying for an opening or employees competing for promotion, and numerous other examples, are situations where participants' outcomes are determined jointly by ability, effort and luck. Differences in ability stratify the playing field to some extent, but competition is the most fierce, and luck plays the biggest role, in tournaments among equally able contestants.

It is traditionally believed that competition increases productivity, fosters innovation, and promotes economic growth.[24] However, it is also easy to imagine how competition may discourage effort in winner-take-all environments where luck plays a significant role. Our results demonstrate that there is a nontrivial interplay between the two effects, and the nature of shocks – the "shape of luck" – matters for the willingness to compete.

We show that individual effort reacts to an increase in competition, be it deterministic or stochastic, in a way that essentially follows the shape of the density of noise. As long as the density is unimodal, individual effort is also unimodal in the

[24]There is a strand of literature analyzing the effect of market competition among firms on managerial effort that shows effort may go down due to endogenously adjusted managerial incentives (e.g., Schmidt, 1997); (for an earlier survey, see Vickers, 1995). There, competition has more than one effect (say, an information effect and changing margins), and the focus is on individual effort.

number of players, but it can be increasing, deceasing or nonmonotone when the distribution of noise is skewed. Aggregate effort behaves similarly, but following the shape of the failure rate of noise. Hence, the presence of heavy tails – a decreasing or interior unimodal failure rate – in the distribution of noise can lead to a *reduction* in aggregate effort with competition.

The results of this paper predict diverging effects of competition on aggregate effort (or investment) in tournaments characterized by different types of noise. Given the various contradictory findings and nonmonotonicities in the literature on the effects of competitive pressure on innovation (e.g., Aghion et al., 2005; Vives, 2008), our results provide an independent mechanism through which different reactions to competitive pressure may arise across industries, or even within the same industry across time.

Heavy-tailed fluctuations are common in many areas often associated with tournament incentives, such as sales of creative and innovative products or the financial sector. Our results suggest that restricting competition can be beneficial in these settings.

Our last comment is methodological. The techniques developed in this paper can be extended to many applications of general tournament models, including optimal contract design and dynamic tournaments,[25] giving a new life to the literature that so far has been limited to considering a number of special cases.

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[25]For example, in companion papers Drugov and Ryvkin (2018) and Drugov and Ryvkin (2019*b*), we use them to obtain new general results on the optimal allocation of prizes.

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Appendix A Technical results

A.1 Equilibrium existence

Equilibrium existence and comparative statics are two separate issues, and in the paper we have focused on the latter. In this section, however, we address the equilibrium existence, which so far did not receive an adequate treatment in the literature on Lazear-Rosen tournaments. It is generally understood that a symmetric pure strategy equilibrium exists if shocks X_i are sufficiently dispersed and/or the effort cost function $c(\cdot)$ is sufficiently convex (see, e.g., Nalebuff and Stiglitz, 1983), but general sufficient conditions for equilibrium existence have remained unknown.[26]

For e_p^* to be the unique symmetric equilibrium, it is sufficient to require that (i) Eq. (4) has a solution and (ii) payoff function $\pi^{(i)}(e_i, e_p^*)$, Eq. (2), is strictly concave in e_i.[27] The main difficulty is in the "revenue" part of the payoff function that may not be globally concave because, in general, $F(\cdot)$ is not concave; moreover, even if $F(\cdot)$ is concave, $F(\cdot)^{k-1}$ may not be, for a sufficiently large k. At the same time, $c(\cdot)$ is strictly convex, and hence a version of sufficient conditions can be obtained if the convexity of $c(\cdot)$ is restricted in some way. The simplest approach is to impose a uniform restriction on $c''(\cdot)$ on $[0, \bar{e}]$.[28]

Let $f_m = \sup\{f(x) : x \in \mathcal{X}\}$, $f'_{\max} = \sup\{f'(x) : x \in \mathcal{X}\}$ and $f'_{\min} = \inf\{f'(x) : x \in \mathcal{X}\}$ denote the tight, possibly infinite, bounds of pdf $f(\cdot)$ and its derivative $f'(\cdot)$ on $\mathcal{X}$. We impose the following restrictions on the pdf of noise.

Assumption 1 *(a) $f(\cdot)$ is uniformly bounded; that is, $f_m < \infty$.*
(b) $f'(\cdot)$ is uniformly bounded above or below or both; that is, either $f'_{\max} < \infty$ or $f'_{\min} > -\infty$ or both.

[26] For WTA Tullock contests, equilibrium existence and uniqueness are well-understood, see Szidarovszky and Okuguchi (1997).

[27] Since $\pi^{(i)}(0, e_p^*) \geq 0$, conditions (i) and (ii) automatically imply that the symmetric equilibrium payoff is positive, $\pi^{(i)}(e_p^*, e_p^*) > 0$.

[28] Any effort $e_i > \bar{e}$ is strictly dominated.

Proposition 7 *Suppose Assumption 1 is satisfied and*
(a) There exists a $c_0 > 0$ *such that* $c''(e) \geq c_0$ *for all* $e \in [0, \bar{e}]$.
(b) $c_0 > D \equiv \min\{D_+, D_-\}$, *where*

$$D_+ = \frac{\mathbb{E}[K(K-1)^2]}{\bar{k}} f_m^2 + \frac{\mathbb{E}[K(K-1)]}{\bar{k}} f'_{\max},$$
$$D_- = \frac{\mathbb{E}[K(K-1)]}{\bar{k}} (f_m^2 - f'_{\min}). \tag{16}$$

(c) $\mathbb{E}(K|K \geq 1) c(c'^{-1}(f_m)) < 1$.
Then e_p^* *is the unique equilibrium in the tournament.*

Conditions (a) and (b) in Proposition 7 guarantee the global strict concavity of payoff function (2) in e_i, while condition (c) ensures that Eq. (4) has a solution. The conditions are consistent with the intuition described above. For a given tournament model, they are easier to satisfy as noise becomes more dispersed (leading to a decrease in f_m, $f'_{\max}$ and $|f'_{\min}|$).

More explicitly, suppose noise has a scale parameter $\sigma > 0$ such that its cdf and pdf are, respectively, $F_\sigma(x) = F_1(x/\sigma)$ and $f_\sigma(x) = (1/\sigma) f_1(x/\sigma)$, where $F_1(\cdot)$ and $f_1(\cdot)$ characterized the "standardized" distribution. An increase in σ produces a larger variance, a reduction in the SOSD order and in the *dispersive order* (Lewis and Thompson, 1981).[29] This gives the bounds $f_m(\sigma) = (1/\sigma) f_{1m}$, $f'_{\min}(\sigma) = (1/\sigma^2) f'_{1\min}$ and $f'_{\max}(\sigma) = (1/\sigma^2) f'_{1\max}$, producing $D(\sigma) = (1/\sigma^2) D_1$. Subscript 1 refers to the bounds for the standardized distribution. Condition (b) then takes the form $\sigma^2 c_0 > D_1$, and condition (c) is $\sigma c'(c^{-1}(1/\mathbb{E}(K|K \geq 1))) > f_{1m}$.

Additionally, conditions (b) and (c) are harder to satisfy as the number of players increases. Overall, the conditions of Proposition 7 are rather strong because the global strict concavity of the payoff function is not necessary. An alternative approach can be to impose a weaker restriction on $c(\cdot)$ but restrict attention to particular families of noise distributions. In contrast, for the purposes of this paper we have chosen to formulate conditions with maximum flexibility for the shape of the distribution of noise, at the expense of a rather restrictive positivity of $c''(\cdot)$ and substantial noise dispersion.

[29] Drugov and Ryvkin (2019*a*) show that ranking noise distributions in the dispersive order is necessary and sufficient to rank equilibrium effort in tournaments with arbitrary sizes and prize schedules.

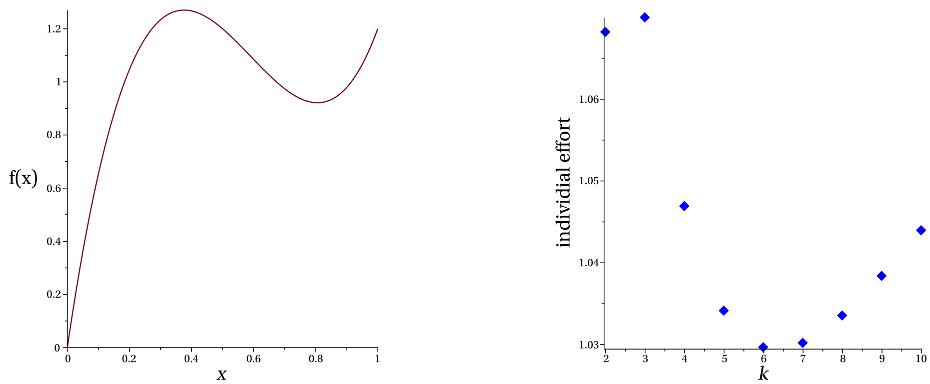

Figure 5: *Left*: $f(x) = 8.8x^3 - 15.6x^2 + 8x$. *Right*: Individual equilibrium effort e_k^* as a function of k for effort cost function $c(e) = e^2/2$.

A quadratic cost function, $c(e) = c_0 e^2/2$, satisfies condition (a). Generally, functions satisfying condition (a) have the form $c(e) = c_0 e^2/2 + \kappa(e)$, where $\kappa : [0, \bar{e}] \to \mathbb{R}_+$ is convex. Note that a function can satisfy the condition even if it is less convex than quadratic. For example, function $c(e) = c_1 e^\xi$ has a positive second derivative bounded below by $c_0 = \xi(\xi - 1)c_1 \bar{e}^{\xi-2}$ when $\xi \in (1, 2]$.

A.2 Comparative statics for multimodal noise densities

More generally, one may ask whether there is any "higher-order" universality in the behavior of $B_{p(\theta)}$ (and $e^*_{p(\theta)}$) for multimodal densities. The answer is yes, to some extent. For deterministic tournament size, Proposition 1 relies on the fact that $f'(x)$ is single-crossing and $z^{k-1}(1-z)$ is log-supermodular in (z, k). Log-supermodularity is also known as total positivity of order 2 (TP_2), a special case of total positivity of order r (TP_r), introduced by Karlin (1968).

Function $v : S_1 \times S_2 \to \mathbb{R}$, with $S_1, S_2 \subseteq \mathbb{R}$, is TP_r if for all $l = 1, \ldots, r$ and all sequences $x_1 < \ldots < x_l$, $y_1 < \ldots < y_l$ ($x_i \in S_1$, $y_j \in S_2$),

$$\det \begin{pmatrix} v(x_1, y_1) & \ldots & v(x_1, y_l) \\ \vdots & & \vdots \\ v(x_l, y_1) & \ldots & v(x_l, y_l) \end{pmatrix} \geq 0.$$

The *variation-diminishing property* of totally positive kernels (Karlin, 1968) states that if function $v(\cdot, \cdot)$ is TP_r and function $\phi : S_2 \to \mathbb{R}$ changes sign $j \leq r - 1$ times on S_2 then function $\tilde{\phi}(x) = \int_{S_2} v(x,y)\phi(y)dy$ changes sign at most j times on S_1. Moreover, if $\tilde{\phi}$ changes sign exactly j times then it follows the same sequence of sign changes as ϕ. It can be shown that $z^{k-1}(1-z)$ is, in fact, TP_∞ (Marshall, Olkin and Arnold, 2011, p. 759); therefore, if $f'(x)$ has any number j of sign changes then $b_{k+1} - b_k$ will have at most j sign changes.

We conclude that if $f(x)$ has j modes, b_k (and e_k^*) will have at most j modes, and if b_k has exactly j modes then the sequence of local minima and maxima of b_k (and e_k^*) will follow the shape of $f(x)$. The case of unimodal (or U-shaped) $f(x)$ is special because $f'(x)$ has at most one sign change, and hence e_k^* is either monotone or interior unimodal (or U-shaped). Figure 5 illustrates a case when b_k and $f(x)$ both have two modes, and e_k^* follows the shape of $f(x)$.

When K is stochastic, Proposition 1 can be generalized to the multimodal case with condition (b) replaced by the requirement that $-\tilde{G}_\theta(z,\theta)$ is TP_r. Similarly, Proposition 4 can be generalized to multimodal failure rates if $-P_k'(\theta)$ is TP_r.

Appendix B Proofs

Proof of Lemma 1 The fact that ILR implies FOSD ordering is well-known (see, e.g., Shaked and Shanthikumar, 2007). To prove that $\tilde{p}(\theta)$ is also FOSD-ordered, it is sufficient to show that $\tilde{P}_k(\theta)$ is decreasing in θ. For some $\theta' > \theta$,

$$\tilde{P}_k(\theta') - \tilde{P}_k(\theta) = \sum_{l=0}^{k}\left[\frac{lp_l(\theta')}{\bar{k}(\theta')} - \frac{lp_l(\theta)}{\bar{k}(\theta)}\right] = \frac{1}{\bar{k}(\theta)\bar{k}(\theta')}\sum_{l=0}^{k} l[p_l(\theta')\bar{k}(\theta) - p_l(\theta)\bar{k}(\theta')]$$
$$= \frac{1}{\bar{k}(\theta)\bar{k}(\theta')}\sum_{l=0}^{k}\sum_{l'=0}^{n} ll'[p_l(\theta')p_{l'}(\theta) - p_l(\theta)p_{l'}(\theta')]$$
$$= \frac{1}{\bar{k}(\theta)\bar{k}(\theta')}\sum_{l=0}^{k}\sum_{l'=k+1}^{n} ll'[p_l(\theta')p_{l'}(\theta) - p_l(\theta)p_{l'}(\theta')] \leq 0.$$

The terms with $l' \leq k$ vanish due to symmetry, and the inequality on the last line follows directly from the ILR property of $p(\theta)$. ■

Proof of Lemma 2 (i) From (1),

$$\tilde{p}_k = \frac{kp_k}{\bar{k}} = \frac{ka_k\theta^k}{\sum_{k=1}^{\infty} ka_k\theta^k} = \frac{\tilde{a}_k\theta^k}{\tilde{A}(\theta)},$$

where $\tilde{a}_k = ka_k$ and $\tilde{A}(\theta) = \sum_{k=1}^{\infty} \tilde{a}_k \theta^k$; that is, $\tilde{p}_k$ also has the PSD form.
(ii) Consider some $\theta' > \theta$. Then

$$\frac{p_k(\theta')}{p_k(\theta)} = \frac{A(\theta)}{A(\theta')} \left(\frac{\theta'}{\theta} \right)^k,$$

which is increasing in k.
(iii) The cmf of a PSD distribution (7) is $P_k(\theta) = \sum_{l=0}^{k} a_l \theta^l / \sum_{l=0}^{\infty} a_l \theta^l$, which gives

$$\begin{aligned} P_k'(\theta) &= \frac{1}{A(\theta)^2} \left[\sum_{l=0}^{k} l a_l \theta^{l-1} \sum_{l'=0}^{\infty} a_{l'} \theta^{l'} - \sum_{l=0}^{k} a_l \theta^l \sum_{l'=0}^{\infty} l' a_{l'} \theta^{l'-1} \right] \\ &= \frac{1}{A(\theta)^2} \sum_{l=0}^{k} \sum_{l'=0}^{\infty} a_l a_{l'} \theta^{l+l'-1} (l - l') = \frac{1}{A(\theta)^2} \sum_{l=0}^{k} \sum_{l'=k+1}^{\infty} a_l a_{l'} \theta^{l+l'-1} (l - l') \leq 0. \end{aligned} \tag{17}$$

The terms with $l' \leq k$ vanish due to symmetry.

Using summation by parts,

$$G(z, \theta) = \sum_{k=0}^{n} p_k(\theta) z^k = P_n(\theta) z^n - \sum_{k=0}^{n-1} P_k(\theta)(z^{k+1} - z^k) = z^n + (1 - z) \sum_{k=0}^{n-1} P_k(\theta) z^k,$$

which gives

$$G_\theta(z, \theta) = (1 - z) \sum_{k=0}^{n-1} P_k'(\theta) z^k, \tag{18}$$

and the result for $G_\theta(z, \theta)$ follows from (17).
(iv) We will prove this part assuming part (v) holds, and then prove part (v) directly. Consider some $\theta' > \theta$. From (18),

$$\rho(z, \theta, \theta') = \frac{G_\theta(z, \theta')}{G_\theta(z, \theta')} = \frac{\sum_{k=0}^{n-1} P_k'(\theta') z^k}{\sum_{k=0}^{n-1} P_k'(\theta) z^k}.$$

This gives, up to a positive multiplier,

$$\begin{aligned} \rho_z(z, \theta, \theta') &\propto \sum_{k=0}^{n-1} k P_k'(\theta') z^{k-1} \sum_{l=0}^{n-1} P_l'(\theta) z^l - \sum_{k=0}^{n-1} P_k'(\theta') z^k \sum_{l=0}^{n-1} l P_l'(\theta) z^{l-1} \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} P_k'(\theta') P_l'(\theta) z^{k+l-1} (k - l). \end{aligned}$$

The terms with $k = l$ are zero. Separating the double sum into two parts, and swapping the indices in the second part, obtain

$$\rho_z(z,\theta,\theta') \propto \sum_{k>l} P_k'(\theta')P_l'(\theta)z^{k+l-1}(k-l) + \sum_{k<l} P_k'(\theta')P_l'(\theta)z^{k+l-1}(k-l)$$
$$= \sum_{k>l}[P_k'(\theta')P_l'(\theta) - P_l'(\theta')P_k'(\theta)]z^{k+l-1}(k-l) \geq 0,$$

where the inequality follows because each term is positive due to part (v).
(v) Consider some $\theta' > \theta$ and let $\beta = \theta'/\theta > 1$. For convenience, introduce the notation $\alpha_{ll'} = a_l a_{l'} \theta^{l+l'-1}(l'-l)$. Using (17),

$$s_k(\theta,\theta') = \frac{P_k'(\theta')}{P_k'(\theta)} = \frac{A(\theta)^2}{A(\theta')^2}\frac{N_k}{D_k},$$

where

$$N_k = \sum_{l=0}^{k} \sum_{l'\geq k+1} \beta^{l+l'-1}\alpha_{ll'}, \quad D_k = \sum_{l=0}^{k} \sum_{l'\geq k+1} \alpha_{ll'}.$$

We need to show that N_k/D_k is increasing in k, or, equivalently, that $N_{k+1}D_k - N_k D_{k+1} \geq 0$. Notice that N_{k+1} can be expressed through N_k as follows:

$$N_{k+1} = N_k - \sum_{l=0}^{k} \beta^{l+k}\alpha_{l,k+1} + \sum_{l'\geq k+2} \beta^{l'+k}\alpha_{k+1,l'}.$$

Similarly,

$$D_{k+1} = D_k - \sum_{l=0}^{k} \alpha_{l,k+1} + \sum_{l'\geq k+2} \alpha_{k+1,l'}.$$

Therefore,

$$N_{k+1}D_k - N_k D_{k+1} = \left(N_k - \sum_{l=0}^{k} \beta^{l+k}\alpha_{l,k+1} + \sum_{l'\geq k+2} \beta^{l'+k}\alpha_{k+1,l'}\right) D_k$$
$$- N_k\left(D_k - \sum_{l=0}^{k} \alpha_{l,k+1} + \sum_{l'\geq k+2} \alpha_{k+1,l'}\right)$$
$$= \sum_{l=0}^{k} \alpha_{l,k+1}(N_k - \beta^{l+k}D_k) + \sum_{l'\geq k+2} \alpha_{k+1,l'}(\beta^{l'+k}D_k - N_k).$$

It can be shown that each of the two terms in the last line is nonnegative. We demonstrate it explicitly for the first term; for the second term, the derivation is similar.

$$\sum_{l=0}^{k}\alpha_{l,k+1}(N_k-\beta^{l+k}D_k)=\sum_{l=0}^{k}\sum_{j=0}^{k}\sum_{l'\geq k+1}\left(\beta^{j+l'-1}\alpha_{jl'}\alpha_{l,k+1}-\beta^{l+k}\alpha_{jl'}\alpha_{l,k+1}\right)$$
$$=\sum_{l=0}^{k}\sum_{j=0}^{k}\sum_{l'\geq k+1}\left(\beta^{l+l'-1}\alpha_{ll'}\alpha_{j,k+1}-\beta^{l+k}\alpha_{jl'}\alpha_{l,k+1}\right)$$
$$\geq\sum_{l=0}^{k}\sum_{j=0}^{k}\sum_{l'\geq k+1}\beta^{l+k}\left(\alpha_{ll'}\alpha_{j,k+1}-\alpha_{jl'}\alpha_{l,k+1}\right)$$
$$=\sum_{l=0}^{k}\sum_{j=0}^{k}\sum_{l'\geq k+1}\beta^{l+k}a_la_{l'}a_ja_{k+1}\theta^{l+l'-1+j+k}\left[(l'-l)(k+1-j)-(l'-j)(k+1-l)\right]$$
$$=\sum_{l=0}^{k}\sum_{j=0}^{k}\sum_{l'\geq k+1}\beta^{l+k}a_la_{l'}a_ja_{k+1}\theta^{l+l'-1+j+k}(l'-k-1)(l-j)$$
$$=\sum_{l'\geq k+1}\beta^{k}a_{l'}a_{k+1}\theta^{l'-1+k}(l'-k-1)\sum_{l=0}^{k}\sum_{j=0}^{k}\beta^{l}a_la_j\theta^{l+j}(l-j).$$

The sum over l and j can be rewritten as

$$\sum_{l=0}^{k}\sum_{j=0}^{k}\beta^{l}a_la_j\theta^{l+j}(l-j)=\sum_{l>j}\beta^{l}a_la_j\theta^{l+j}(l-j)+\sum_{l<j}\beta^{l}a_la_j\theta^{l+j}(l-j)$$
$$=\sum_{l>j}(\beta^{l}-\beta^{j})a_la_j\theta^{l+j}(l-j)\geq 0.$$

The inequality follows because $\beta>1$. ■

Proof of Proposition 1 When $f(x)$ is monotone, the result is trivial. Suppose $f(x)$ is interior unimodal. Integrating representation (6) by parts, obtain

$$B_{p(\theta)}=f(\overline{x})\tilde{G}(1,\theta)-f(\underline{x})\tilde{G}(0,\theta)-\int f'(x)\tilde{G}(F(x),\theta)dx.$$

From the definition of pgf $\tilde{G}$, we have $\tilde{G}(1,\theta)=1$. The derivative, or first difference, of $B_{p(\theta)}$ with respect to θ, therefore, is

$$B'_{p(\theta)} = -f(\underline{x})\tilde{G}_\theta(0,\theta) - \int f'(x)\tilde{G}_\theta(F(x),\theta)dx.$$

In order to show that $B_{p(\theta)}$ is unimodal, it is sufficient to show that $B'_{p(\theta)}$ is single-crossing $+-$; that is, if $B'_{p(\theta)} < 0$ for some θ then $B'_{p(\theta')} \leq 0$ for all $\theta' > \theta$.

Suppose $B'_{p(\theta)} < 0$ and consider some $\theta' > \theta$. Function $f'(x)$ is single-crossing $+-$. Let $\hat{x} \in \mathrm{int}(\mathcal{X})$ denote a mode of $f(x)$ such that $f'(x) \geq (\leq) 0$ for $x \leq (\geq)\hat{x}$. Furthermore, let $\tilde{\rho}(z,\theta,\theta') = \tilde{G}_\theta(z,\theta')/\tilde{G}_\theta(z,\theta)$. Splitting the integral, rewrite $B'_{p(\theta')}$ as

$$\begin{aligned}
B'_{p(\theta')} &= -f(\underline{x})\tilde{G}_\theta(0,\theta') - \int_{\underline{x}}^{\hat{x}} f'(x)\tilde{G}_\theta(F(x),\theta')dx - \int_{\hat{x}}^{\overline{x}} f'(x)\tilde{G}_\theta(F(x),\theta')dx \\
&= -f(\underline{x})\tilde{\rho}(0,\theta,\theta')\tilde{G}_\theta(0,\theta) - \int_{\underline{x}}^{\hat{x}} f'(x)\tilde{\rho}(F(x),\theta,\theta')\tilde{G}_\theta(F(x),\theta)dx \\
&- \int_{\hat{x}}^{\overline{x}} f'(x)\tilde{\rho}(F(x),\theta,\theta')\tilde{G}_\theta(F(x),\theta)dx \\
&\leq -f(\underline{x})\tilde{\rho}(F(\hat{x}),\theta,\theta')\tilde{G}_\theta(0,\theta) - \tilde{\rho}(F(\hat{x}),\theta,\theta')\int_{\underline{x}}^{\hat{x}} f'(x)\tilde{G}_\theta(F(x),\theta)dx \\
&- \tilde{\rho}(F(\hat{x}),\theta,\theta')\int_{\hat{x}}^{\overline{x}} f'(x)\tilde{G}_\theta(F(x),\theta)dx \\
&= \tilde{\rho}(F(\hat{x}),\theta,\theta')\left[-f(\underline{x})\tilde{G}_\theta(0,\theta) - \int f'(x)\tilde{G}_\theta(F(x),\theta)dx\right] = \tilde{\rho}(F(\hat{x}),\theta,\theta')B'_{p(\theta)} \leq 0.
\end{aligned}$$

The first inequality follows because $\tilde{\rho}(z,\theta,\theta')$ is increasing in z, due to the log-supermodularity condition. ■

Proof of the "only if" part of Corollary 4(iii) It is sufficient to show that if b_k is independent of k for $k \geq 2$ then $f(x)$ is constant. Let $\Delta^l b_k$ denote the l-th difference of b_k, defined recursively as $\Delta^{l+1}b_k = \Delta^l b_{k+1} - \Delta^l b_k$, with $\Delta^1 b_k = b_{k+1} - b_k$. The fact that b_k is constant implies that $\Delta^l b_k = 0$ for all $k \geq 2$, $l \geq 1$. Note that $\Delta^l z^{k-1} = (-1)^l z^{k-1}(1-z)^l$ for $z \in [0,1]$; therefore,

$$\Delta^l b_k = (-1)^l \int f(x)d[F(x)^{k-1}(1-F(x))^l] = 0; \quad k \geq 2,\, l \geq 1. \tag{19}$$

Fix some $k, l \geq 1$. Then

$$\sum_{j=k}^{k+l}(-1)^{k+l-j}\binom{k+l}{j}\Delta^{k+l-j}b_{j+1}$$
$$= \sum_{j=k}^{k+l}\binom{k+l}{j}\int f(x)d[F(x)^j(1-F(x))^{k+l-j}] = \int f(x)dF_{(k:k+l)}(x),$$

where $F_{(k:k+l)}(x)$ is the cdf of order statistic $X_{(k:k+l)}$. In the sum in the first line of the equation above, all terms except the one with $j = k+l$ are zero due to (19); therefore,

$$A_{kl} = \int f(x)dF_{(k:k+l)}(x) = b_{k+l+1} = b_2; \quad k, l \geq 1.$$

Suppose $f(\tilde{x}_1) \neq f(\tilde{x}_2)$ for some points $\tilde{x}_1, \tilde{x}_2 \in \mathcal{X}$. Then, by continuity of $f(\cdot)$, there exist points $x_1, x_2 \in \text{int}(\mathcal{X})$ such that $f(x_1) \neq f(x_2)$, $f(x_1) > 0$, $f(x_2) > 0$, and $F(x_1), F(x_2)$ are rational numbers. First, let $k, l \to \infty$ so that $k/(k+l) = F(x_1)$. Then $\sqrt{k+l}(X_{(k:k+l)} - x_1)$ converges in distribution to $N\left(0, F(x_1)(1-F(x_1))/f(x_1)^2\right)$, which implies A_{kl} converges to $f(x_1)$. Second, let $k, l \to \infty$ so that $k/(k+l) = F(x_2)$. Then, similarly, A_{kl} converges to $f(x_2)$. However, A_{kl} is a constant independent of k, l – a contradiction. ■

Proof of Proposition 2 Using (3), define

$$\Delta b_{k+3} = b_{k+3} - b_{k+2} = \int \left[(k+2)F(x)^{k+1} - (k+1)F(x)^k\right] f(x)dF(x), \quad k \geq 0.$$

Integrating by parts, obtain

$$\Delta b_{k+3} = \int f(x)d(F(x)^{k+2} - F(x)^{k+1}) = \int F(x)^{k+1}(1-F(x))f'(x)dx. \tag{20}$$

The symmetry of $f(x)$ around its mean μ implies $f(x) = f(2\mu - x)$ and $F(x) = 1 - F(2\mu - x)$ for all $x \in \mathcal{X}$.
(i) From (3), $b_2 = \int f(x)dF(x)$ and

$$b_3 = 2\int F(x)f(x)dF(x) = 2\int F(2\mu - x)f(2\mu - x)dF(2\mu - x)$$
$$= 2\int(1-F(x))f(x)dF(x) = 2b_2 - b_3,$$

which implies $b_2 = b_3$.
(ii) From (20), via the change of variable $x \to 2\mu - x$, due to $f'(2\mu - x) = -f'(x)$,

$$\begin{aligned}
\Delta b_{k+3} &= \int_{\underline{x}}^{\mu} F(x)^{k+1}(1-F(x))f'(x)dx + \int_{\mu}^{\overline{x}} F(x)^{k+1}(1-F(x))f'(x)dx \\
&= \int_{\underline{x}}^{\mu} F(x)^{k+1}(1-F(x))f'(x)dx - \int_{\underline{x}}^{\mu} (1-F(x))^{k+1}F(x)f'(x)dx \\
&= -\int_{\underline{x}}^{\mu} F(x)(1-F(x))\left[(1-F(x))^k - F(x)^k\right] f'(x)dx \le 0,
\end{aligned}$$

with equality only for $k = 0$ or when $f(x)$ is constant. Thus, b_k is decreasing for $k \ge 2$ and the result follows from (4). ■

Proof of Proposition 3 We will prove part (i); the derivation for part (ii) is similar. (i) Suppose $f(x)$ is IFR and $c(e)$ is more convex than quadratic. From representation (12), kb_k is increasing in k, which implies $B(\theta) = \sum_{k=0}^{n} p(\theta)kb_k$ is increasing in θ. From (4), $B_{p(\theta)} = B(\theta)/\bar{k}(\theta)$.

By definition, $E^*_{p(\theta)} = \bar{k}(\theta)e^*_{p(\theta)}$. Differentiating with respect to θ, obtain

$$(E^*_{p(\theta)})' = \bar{k}'(\theta)e^*_{p(\theta)} + \bar{k}(\theta)(e^*_{p(\theta)})'.$$

Differentiating the first-order condition $c'(e^*_{p(\theta)}) = B_{p(\theta)}$ with respect to θ, obtain $(e^*_{p(\theta)})' = B'_{p(\theta)}/c''(e^*_{p(\theta)})$, where $B'_{p(\theta)} = B'(\theta)/\bar{k}(\theta) - B(\theta)\bar{k}'(\theta)/\bar{k}(\theta)^2$. This gives

$$\begin{aligned}
(E^*_{p(\theta)})' &= \bar{k}'(\theta)e^*_{p(\theta)} + \frac{\bar{k}(\theta)}{c''(e^*_{p(\theta)})}\left[\frac{B'(\theta)}{\bar{k}(\theta)} - \frac{B(\theta)\bar{k}'(\theta)}{\bar{k}(\theta)^2}\right] \\
&= \frac{B'(\theta)}{c''(e^*_{p(\theta)})} + \bar{k}'(\theta)\left[e^*_{p(\theta)} - \frac{c'(e^*_{p(\theta)})}{c''(e^*_{p(\theta)})}\right].
\end{aligned}$$

It is sufficient to show that the expression in square brackets is positive. Since $c(\sqrt{t})$ is convex in t, we have

$$\frac{d^2}{dt^2}c(\sqrt{t}) = \frac{d}{dt}\left[\frac{c'(\sqrt{t})}{2\sqrt{t}}\right] = \frac{c''(\sqrt{t})\sqrt{t} - c'(\sqrt{t})}{4t^{3/2}} \ge 0;$$

therefore, $c''(e)e \ge c'(e)$. ■

Proof of Proposition 4 We first show that E^*_k is unimodal. Following the same argument as in the proof of Proposition 1, it is sufficient to show that $-F_{(k-1:k)}(x) = kF(x)^{k-1} - (k-1)F(x)^k$ is log-supermodular. Consider some $l > k \geq 2$, and let $z = F(x)$. We will show that the ratio $[lz^{l-1} - (l-1)z^l]/[kz^{k-1} - (k-1)z^k]$ is increasing in z. The derivative of the ratio with respect to z is

$$\begin{aligned}
&\frac{1}{[kz^{k-1} - (k-1)z^k]^2}\big[(l(l-1)z^{l-2} - l(l-1)z^{l-1})(kz^{k-1} - (k-1)z^k) \\
&-(k(k-1)z^{k-2} - k(k-1)z^{k-1})(lz^{l-1} - (l-1)z^k)\big] \\
&= \frac{z^{k+l-3}(1-z)}{[kz^{k-1} - (k-1)z^k]^2}\left[lk(l-1) - l(l-1)(k-1)z - lk(k-1) + k(l-1)(k-1)z\right] \\
&= \frac{z^{k+l-3}(1-z)(l-k)[lk - (l-1)(k-1)z]}{[kz^{k-1} - (k-1)z^k]^2} \geq 0.
\end{aligned}$$

Thus, E_k^* is unimodal for $k \geq 2$; therefore, it is also unimodal for all $k \geq 0$ because $E_0^* = E_1^* = 0$. Using summation by parts, we can write

$$E^*_{p(\theta)} = \sum_{k=0}^{n} p_k(\theta) E_k^* = P_n(\theta) E_n^* - \sum_{k=0}^{n-1} P_k(\theta) \Delta E_k^*,$$

where $\Delta E_k^* = E_{k+1}^* - E_k^*$ – the first difference of E_k^* – is single-crossing $+-$. Note that $P_n(\theta) = 1$ is independent of θ; hence, the derivative, or first difference, of $E^*_{p(\theta)}$ with respect to θ is

$$E^{*\prime}_{p(\theta)} = -\sum_{k=0}^{n-1} P_k'(\theta) \Delta E_k^*,$$

where $P_k'(\theta)$ is the derivative, or the first difference, of the cmf.

We will now show that $E^{*\prime}_{p(\theta)}$ is single-crossing $+-$. Let $\hat{k}$ denote a mode of E_k^* such that $\Delta E_k^* \geq (\leq) 0$ for $k \leq (\geq) \hat{k}$. Suppose $E^{*\prime}_{p(\theta)} < 0$ and consider some $\theta' > \theta$. Let $s_k(\theta, \theta') = P_k'(\theta')/P_k'(\theta)$. Separating the sum, obtain

$$\begin{aligned}
E^{*\prime}_{p(\theta')} &= -\sum_{k \leq \hat{k}} P_k'(\theta') \Delta E_k^* - \sum_{k > \hat{k}} P_k'(\theta') \Delta E_k^* \\
&= -\sum_{k \leq \hat{k}} s_k(\theta, \theta') P_k'(\theta) \Delta E_k^* - \sum_{k > \hat{k}} s_k(\theta, \theta') P_k'(\theta) \Delta E_k^* \\
&\leq -s_{\hat{k}}(\theta, \theta') \sum_{k \leq \hat{k}} P_k'(\theta) \Delta E_k^* - s_{\hat{k}}(\theta, \theta') \sum_{k > \hat{k}} P_k'(\theta) \Delta E_k^* = s_{\hat{k}}(\theta, \theta') E^{*\prime}_{p(\theta)} \leq 0.
\end{aligned}$$

The first inequality follows from condition (b) whereby $s_k(\theta, \theta')$ is increasing in k. Thus, $E^{*\prime}_{p(\theta)}$ is single-crossing $+-$, and hence $E^*_{p(\theta)}$ is unimodal. ■

Proof of Proposition 5 First, we prove that if $f(x)$ is log-concave (log-convex) then kb_k is concave (convex) in k for $k \geq 2$. Integrating (3) by parts, obtain

$$kb_k = k(k-1)\int F(x)^{k-2}f(x)dF(x) = k\left[f(x)F(x)^{k-1}|_{\underline{x}}^{\overline{x}} - \int F(x)^{k-1}f'(x)dx\right]$$
$$= kf(\overline{x}) - k\int F(x)^{k-1}f'(x)dx.$$

This gives the second difference

$$\begin{aligned}\Delta^2(kb_k) &= (k+2)b_{k+2} - 2(k+1)b_{k+1} + kb_k \\ &= \int f'(x)F(x)^{k-1}[-(k+2)F(x)^2 + 2(k+1)F(x) - k]dx \\ &= \int f'(x)F(x)^{k-1}(1-F(x))[(k+2)F(x) - k]dx. \qquad (21)\end{aligned}$$

Suppose $f(x)$ is log-concave (when $f(x)$ is log-convex, the argument is similar), then $f'(x)/f(x)$ is decreasing. Let $\hat{x} \in \mathcal{X}$ denote a point such that $F(\hat{x}) = k/(k+2)$. Splitting the integral, obtain

$$\begin{aligned}\Delta^2(kb_k) &= \int_{\underline{x}}^{\hat{x}} \frac{f'(x)}{f(x)}F(x)^{k-1}(1-F(x))[(k+2)F(x)-k]dF(x) \\ &+ \int_{\hat{x}}^{\overline{x}} \frac{f'(x)}{f(x)}F(x)^{k-1}(1-F(x))[(k+2)F(x)-k]dF(x) \\ &\leq \frac{f'(\hat{x})}{f(\hat{x})}\int_{\underline{x}}^{\hat{x}} F(x)^{k-1}(1-F(x))[(k+2)F(x)-k]dF(x) \\ &+ \frac{f'(\hat{x})}{f(\hat{x})}\int_{\hat{x}}^{\overline{x}} F(x)^{k-1}(1-F(x))[(k+2)F(x)-k]dF(x) \\ &= \frac{f'(\hat{x})}{f(\hat{x})}\int F(x)^{k-1}(1-F(x))[(k+2)F(x)-k]dF(x).\end{aligned}$$

The inequality follows because $f'(x)/f(x)$ is decreasing and the integrand is negative (positive) for $x \leq (\geq)\hat{x}$. Moreover, the inequality is strict when $f'(x)/f(x)$ is strictly decreasing. The last integral is

$$\int_0^1 z^{k-1}[-(k+2)z^2 + 2(k+1)z - k]dz = -(k+2)\int_0^1 z^{k+1}dz + 2(k+1)\int_0^1 z^k dz - k\int_0^1 z^{k-1}dz = -1 + 2 - 1 = 0.$$

Thus, $\Delta^2(kb_k) \le 0$, i.e., kb_k is concave.

Second, we compare $E_p^* = \bar{k}e_p^*$ to $E_{p'}^* = \bar{k}e_{p'}^*$. This is equivalent to comparing e_p^* and $e_{p'}^*$, i.e., it is sufficient to compare B_p to $B_{p'}$. Under the assumption $p_0 = p_1 = 0$, Eq. (4) can be written as $B_p = (1/\bar{k})\mathbb{E}_p(Kb_K)$. Therefore, we are comparing $\mathbb{E}_p(Kb_K)$ to $\mathbb{E}_{p'}(Kb_K)$. The result then follows from the definition of second-order stochastic dominance. ■

Proof of Proposition 6 Without disclosure, the expected aggregate effort in the tournament is $E_p^* = \bar{k}e_p^* = \bar{k}c'^{-1}(B_p)$, where, from (4), $B_p = \mathbb{E}(b_{\tilde{K}})$. With disclosure, the expected aggregate effort is $\mathbb{E}(Kc'^{-1}(b_K))$, which can be rewritten as

$$\mathbb{E}(Kc'^{-1}(b_K)) = \sum_{k=1}^{n} p_k k c'^{-1}(b_k) = \bar{k}\sum_{k=1}^{n} \tilde{p}_k c'^{-1}(b_k) = \bar{k}\mathbb{E}(c'^{-1}(b_{\tilde{K}})).$$

Thus, comparing E_p^* and $\mathbb{E}(Kc'^{-1}(b_K))$ is equivalent to comparing $c'^{-1}(\mathbb{E}(b_{\tilde{K}}))$ and $\mathbb{E}(c'^{-1}(b_{\tilde{K}}))$.

It follows that when b_k is not constant in the support of $\tilde{K}$, and c'^{-1} is concave (convex) and nonlinear for at least some distinct values of b_k, disclosure is not optimal (optimal). The concavity (convexity) of c'^{-1} is equivalent to the convexity (concavity) of c', i.e., to the condition $c''' \ge (\le) 0$. ■

Proof of Proposition 7 We start by showing that condition (c) guarantees the existence of a unique e_p^* solving (4). Recall that $c'(\cdot)$ is strictly increasing and $c'(0) = 0$. It is, therefore, sufficient to show that $c'(\bar{e}) > B_p$. Representation (10) gives $b_k \le f_m$; therefore, $B_p = \mathbb{E}(b_{\tilde{K}}) \le f_m$. Condition (c) implies $c'^{-1}(f_m) < c^{-1}(1/\mathbb{E}(K|K \ge 1)) \le c^{-1}(1)$; therefore, $f_m < c'(c^{-1}(1)) = c'(\bar{e})$, which produces the desired result.

Next, we use conditions (a) and (b) to show that payoff function (2) is strictly concave in e_i. Let $R(e) = \sum_{k=1}^{n} \tilde{p}_k \int F(e - e^* + x)^{k-1} dF(x)$ and suppose $c''(e) \ge c_0 > 0$ on $[0, \bar{e}]$. We need to show that $R''(e) < c_0$. For convenience, let $\Delta e = e - e^*$.

Differentiating $R(e)$ once, obtain

$$R'(e) = \sum_{k=1}^{n} \tilde{p}_k (k-1) \int_{\underline{x}}^{\overline{x}} F(\Delta e + x)^{k-2} f(\Delta e + x) dF(x). \tag{22}$$

We need to evaluate the second derivative $R''(e)$. Note that the integrand in (22) is nonzero only for $x \in [\max\{\underline{x}, \underline{x} - \Delta e\}, \min\{\overline{x}, \overline{x} - \Delta e\}]$, and is continuous and piecewise differentiable in this interval under our assumptions; however, the integrand may be discontinuous on $\mathcal{X}$. We, therefore, consider the cases when $\Delta e \geq 0$ and $\Delta e < 0$ separately.

(i) Suppose that $\Delta e \geq 0$. Then the interval of integration in (22) is $[\underline{x}, \overline{x} - \Delta e]$ and

$$\begin{aligned}
R''(e) &= \sum_{k=1}^{n} \tilde{p}_k (k-1) \Big[(k-2) \int F(\Delta e + x)^{k-3} f(\Delta e + x)^2 dF(x) \\
&+ \int F(\Delta e + x)^{k-2} f'(\Delta e + x) dF(x) - f(\overline{x}) f(\overline{x} - \Delta e) \Big] \\
&\leq \sum_{k=1}^{n} \tilde{p}_k (k-1) [(k-2) f_m^2 + f'_{\max}] \leq \frac{\mathbb{E}(K^3)}{\bar{k}} f_m^2 + \frac{\mathbb{E}(K^2)}{\bar{k}} f'_{\max}.
\end{aligned}$$

(ii) Suppose that $\Delta e < 0$. Then the interval of integration in (22) is $[\underline{x} - \Delta e, \overline{x}]$ and

$$\begin{aligned}
R''(e) &= \sum_{k=1}^{n} \tilde{p}_k (k-1) \Big[(k-2) \int_U F(\Delta e + x)^{k-3} f(\Delta e + x)^2 dF(x) \\
&+ \int_U F(\Delta e + x)^{k-2} f'(\Delta e + x) dF(x) + F(\underline{x})^{k-2} f(\underline{x}) f(\underline{x} - \Delta e) \Big] \\
&\leq \sum_{k=1}^{n} \tilde{p}_k (k-1) [(k-1) f_m^2 + f'_{\max}] = \frac{\mathbb{E}[K(K-1)^2]}{\bar{k}} f_m^2 + \frac{\mathbb{E}[K(K-1)]}{\bar{k}} f'_{\max}.
\end{aligned}$$

Thus, D_+ given by (16) is a bound such that $c_0 > D_+$ ensures $R''(e) - c_0 < 0$.

An alternative bound on $R''(e)$ can be obtained by transforming (22) via a change of variable $x + \Delta e \to x$ into the form

$$R'(e) = \sum_{k=1}^{n} \tilde{p}_k (k-1) \int_{\underline{x}}^{\overline{x}} F(x)^{k-2} f(x - \Delta e) dF(x). \tag{23}$$

In this case the integrand is nonzero, continuous and piecewise differentiable for $x \in [\max\{\underline{x}, \underline{x} + \Delta e\}, \min\{\overline{x}, \overline{x} + \Delta e\}]$. We consider the same two cases as above.

(i) For $\Delta e \geq 0$, the interval of integration in (23) is $[\underline{x} + \Delta e, \overline{x}]$ and

$$R''(e) = \sum_{k=1}^{n} \tilde{p}_k(k-1) \left[-\int F(x)^{k-2} f'(x - \Delta e) dF(x) - F(\underline{x} + \Delta e)^{k-2} f(\underline{x}) f(\underline{x} + \Delta e) \right]$$
$$\leq -\sum_{k=1}^{n} \tilde{p}_k(k-1) f'_{\min} = -\frac{\mathbb{E}[K(K-1)]}{\bar{k}} f'_{\min}.$$

(ii) For $\Delta e < 0$, the interval of integration in (23) is $[\underline{x}, \overline{x} + \Delta e]$ and

$$R''(e) = \sum_{k=1}^{n} \tilde{p}_k(k-1) \left[-\int F(x)^{k-2} f'(x - \Delta e) dF(x) + F(\overline{x} + \Delta e)^{k-2} f(\overline{x}) f(\overline{x} + \Delta e) \right]$$
$$\leq \sum_{k=1}^{n} \tilde{p}_k(k-1)(f_m^2 - f'_{\min}) = \frac{\mathbb{E}[K(K-1)]}{\bar{k}} (f_m^2 - f'_{\min}).$$

This produces bound D_- in (16) such that $c_0 > D_-$ implies $R''(e) - c_0 < 0$. Since both bounds are valid, condition $c_0 > \min\{D_+, D_-\}$ is sufficient.

Finally, we check the participation constraint. From (2) and (1), the equilibrium payoff is

$$\pi^{(i)}(e_p^*, e_p^*) = \sum_{k=1}^{n} \tilde{p}_k \int F(x)^{k-1} dF(x) - c(e_p^*) = \frac{1}{\bar{k}} \sum_{k=1}^{n} p_k - c(e_p^*) = \frac{1}{\mathbb{E}(K|K \geq 1)} - c(e_p^*),$$

where $\mathbb{E}(K|K \geq 1) = \sum_{k=1}^{n} k p_k / \sum_{k=1}^{n} p_k$ is the expected number of players conditional on there being at least one player in the tournament. Using the bound $B_p \leq f_m$ derived above and condition (c), obtain

$$\mathbb{E}(K|K \geq 1) c(e_p^*) = \mathbb{E}(K|K \geq 1) c(c'^{-1}(B_p)) \leq \mathbb{E}(K|K \geq 1) c(c'^{-1}(f_m)) < 1,$$

which gives $\pi^{(i)}(e_p^*, e_p^*) > 0$. ∎