Asymptotic synthesis of contingent claims with controlled risk in a sequence of discrete-time markets

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Abstract

We examine the connection between discrete-time models of financial markets and the celebrated Black–Scholes–Merton (BSM) continuous-time model in which "markets are complete." Suppose that (a) the probability law of a sequence of discrete-time models converges to the law of the BSM model and (b) the largest possible one-period step in the discrete-time models converges to zero. We prove that, under these assumptions, every bounded and continuous contingent claim can be asymptotically synthesized, controlling for the risks taken in a manner that implies, for instance, that an expected-utility-maximizing consumer can asymptotically obtain as much utility in the (possibly incomplete) discrete-time economies as she can at the continuous-time limit. Hence, in economically significant ways, many discrete-time models with frequent trading resemble the complete-markets model of BSM.

1 Introduction

Arrow [2] shows how a relatively small number of long-lived financial securities, combined with spot markets in all contingencies, *may* allow consumers to attain any contingent consumption bundle that they desire; that is, provide complete markets. This idea is most starkly illustrated by the celebrated model of a securities market originally studied by Black and Scholes [4] and Merton [18],

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in which two securities provide complete markets in the (rough) sense that every well-behaved contingent claim based on the history of stock price can be *synthesized* by continuous trading in the stock and a riskless bond (Harrison and Pliska [10, 11]). Sharpe [19] and Cox, Ross, and Rubinstein [5] show a similar result for discrete-time economies in which the stock price, over each time interval, can move (only) to one of two possible values. But if, in discrete-time models, the stock can move to more than two values over each time interval, markets are "incomplete;" in particular, the arbitrage bounds on the prices of many contingent claims remain wide as we look at a sequence of economies where, along the sequence, trading opportunities are increasingly frequent.

These arbitrage bounds are based on the principle that an investor must be capable of synthesizing a claim that lies (weakly) above or below the given contingent claim *with probability one*, sometimes called *super-* and *sub-hedging*. But for a consumer who desires a specific contingent claim x, those arbitrage bounds are not necessarily relevant; while the consumer may be unable to synthesize x precisely, she may be able to synthesize a claim that is appropriately "close" to x, where "close" means, in terms of her preferences. If, when trading opportunities are frequent, she can get close to her ideal x, then, even if markets are not complete, the inefficiency of the market allocation that results may be minor (Kreps, [15], Section 6.7).

Duffie and Protter [8] provide an approximation result of this character: For a sequence of discrete-time models in which trading happens on an increasingly finer grid of times and for which the price processes converge to the Black–Scholes–Merton (BSM) model, any contingent claim x can be "asymptotically synthesized" in the sense that, for any $\epsilon > 0$, a claim x^n can be synthesized in the nth discrete-time model that is within ϵ of the target claim x with probability greater than $1 - \epsilon$.[1] But unless we have some control over what happens on the "exceptional" set of sample paths for which the synthesized claim and the target claim have widely different values, this is inadequate as economics. This is perhaps most obvious when one considers that, with a classic doubling strategy applied in a discrete-time model, one can produce from an initial investment

[1] This is a rough paraphrase of the Duffie–Protter result, paraphrased to fit in the framework we use here. Duffie and Protter show convergence in distribution of the approximating claims. For a precise statement and proof of the result as paraphrased here, see Kreps [16], Proposition 5.1.

of \$0 a claim that is \$1 with probability as close to 1 as is desired, as long as there are enough trading opportunities. (Of course, the number of trading opportunities required increases with how close to probability 1 one wishes to come.) The problem is that, for the classic doubling strategy, the constructed portfolio will have a massively negative value on the small probability event on which the portfolio does not equal \$1.

In this paper, we prove a result in which the value of synthesized claims x^n are controlled almost surely: We show that every bounded and continuous contingent claim x can be asymptotically synthesized in the sense above and where, moreover, with probability 1, the value of the synthesized claim lies almost surely within the bounds of x itself. Whether this notion of closeness is "good enough" for the preferences of consumers depends, of course, on the nature of those preferences; we present a class of consumer preferences for which this notion of closeness is in fact good enough.[2]

The paper proceeds as follows. Section 2 provides the formal setting for our analysis, our basic definitions of "asymptotic synthesis" with different levels of control on how far the synthesized claims can be from the target claim, our main result,[3] and a corollary that shows how the main result can be applied.

While the theorem of Section 2 provides a positive result—the ability to asymptotically synthesize contingent claims with (appropriately defined) controlled risk—this result does not rule out the possibility that a contingent claim can be asymptotically synthesized with this level of risk for other levels of initial investment. To make the story satisfactory, this must be precluded; in Section 3, we enlist the theory of asymptotic arbitrage (Kabanov and Kramkov [13]; Klein and Schachermayer [14]) to provide conditions under which "the law of one price" holds. Section 4 gives a class of models for which everything works out well; Section 5 shows by example that, in our framework, the even stronger asymptotic synthesis with *vanishing risk* is too much to hope for. Section 6 provides the proof of the main result. Extensions to our main result are briefly

[2]This in turn shows that a consumer in the discrete-time market for large n can do nearly as well as she can in the BSM economy in terms of her preferences. In the limit, she can do no worse. But perhaps she can do even better. In a companion paper [17], we give conditions under which an expected-utility-of-consumption maximizer cannot do better, and we provide examples in which, *in* the limit, a consumer of this sort can attain asymptotically infinite expected utility, although *at* the continuous-time limit, her maximal expected utility is finite.

[3]This result, called Theorem 1 in this paper, is *reported* and then employed in Kreps [16]. However, this paper takes precedence in providing the formal statement and its proof.

discussed in in Section 7.

To the best of our knowledge, this paper is the original treatment of asymptotic synthesis with *economically appropriate* controls on the level of risk undertaken. It is clear that having no controls on what happens on the exceptional set on which x^n differs signficantly from x is inadequate; as noted, no controls allows for the asymptotic synthesis of arbitrage opportunities. We shall show that x-control is the best one can do, within a class of such controls, while a stronger form of control—so-called vanishing risk—is not possible. But this begs the question, can precise meaning be given to the otherwise imprecise "economically appropriate?" Is x-controlled risk good enough in terms of the economics of these models? Answering this question, we assert, requires nailing down the implications for consumer behavior—showing whether and when the utility a consumer can attain *in* the limit of these discrete-time models is just as good as what she can attain *at* the limit of the continuous-time, BSM model—a program that is futher advanced in the companion paper, Kreps and Schachermayer [17].

2 General formulation, definitions, the main result, and an application

We work in the space $\Omega = C_0[0,1]$, the space of all continuous functions ω from $[0,1]$ to R whose value at 0 is 0. We let ω denote a typical element of Ω, with $\omega(t)$ the value of ω at date t. Endow Ω with the sup norm topology, and let $\{F_t; 0 \leq t \leq 1\}$ be the standard augmented filtration, so that $\{F_t; 0 \leq t \leq 1\}$ satisfies the "usual conditions" of right continuity and saturatedness relative to Wiener measure.

Let P be Wiener measure on Ω, so that ω under P is a standard Brownian motion. Expectation with respect to P is denoted by $\mathbf{E}[\cdot]$

The simple Black-Scholes-Merton (BSM) model of one-risky-asset financial market concerns two assets that trade one against the other over the continuous interval $[0,1]$. The bond is the numeraire, whose price (relative to itself) is therefore identically 1. The second security, called the stock, has price $S(t,\omega) = e^{\omega(t)}$ at time t in state ω; that is, under P, the stock price has the law of geometric Brownian motion.

We know that there is a unique probability measure on Ω, denoted P^*, that is equivalent to P and, under which, $S(t)$ is a martingale (Harrison and Kreps [9]). Expectation with respect to P^* is denoted by $\mathbf{E}^*[\cdot]$.

Contingent claims are F_1-measurable functions $x : \Omega \to R$. We let X denote the space of bounded and continuous contingent claims.

The well-known "complete markets" result for the BSM model says that, for every $x \in X$, x can be written

$$x = \mathbf{E}^*[x] + \int_0^1 \alpha dS,$$

for a predictable and S-integrable integrand α (Harrison and Pliska [10, 11]).[4] The interpretation is that a consumer–investor, living in the BSM economy, can *synthesize* the claim x by a *trading strategy* $\big(\mathbf{E}^*[x], \alpha\big)$ that calls for an initial investment $\mathbf{E}^*[x]$, where $\alpha(t, \omega)$ represents the number of shares of stock held at time t in state ω, where initial bond holdings are $\mathbf{E}^*[x] - \alpha(0)$, and subsequent bond holdings are adjusted continuously so that any purchases of stock (after time 0) are financed by the sale of bonds (borrowing) and the proceeds of any sale of stock are used to purchase bonds. Here, the stochastic integral $\int_0^t \alpha(u) dS(u)$ represents the financial-gains from this strategy up to time t.

Now suppose that for $n = 1, 2, \ldots$, we have different probability measures P^n defined on Ω, with the following structure: For each n, the support of P^n consists of piecewise linear functions that, in particular, are linear on all intervals of the form $[k/n, (k+1)/n]$, for $k = 0, \ldots, n-1$. The interpretation is that P^n represents a probability distribution on paths of the log of the stock price in an nth discrete-time economy, in which trading between the stock and bond is possible *only* at times $t = k/n$ for $k = 0, \ldots, n-1$. At time 1, the stock and bond liquidate in state ω at "prices" 1 and $e^{\omega(t)}$.[5]

Consumers in the nth discrete-time economy can implement (state-dependent) self-financing trading strategies $\big(V(0), \{\alpha^n(k/n), k = 0, \ldots, n-1\}\big)$, where the interpretation is that $V(0)$ is the value of the consumer's initial portfolio, $\alpha^n(k/n, \omega)$ is the number of shares of stock held by the consumer *after* she

[4]Harrison and Pliska show that many more claims than those that are bounded and continuous can be synthesized in this sense but, for our purposes, restricting to such claims will be adequate.

[5]The piecewise linearity of ω under the various P^n is a convenient way to have $C_0[0,1]$ be a common state space; in the nth economy, $\omega(t)$ for t not of the form k/n has no economic meaning or consequence. An alternative construction would have $\omega(t)$ piecewise constant over intervals $[k/n, (k+1)/n)$, in which case we would work in the Skorohod space $D[0,1]$.

has traded at time k/n, and, after time 0, bond holdings are adjusted that any adjustments in stock holdings at times k/n are financed with bond purchases/sales. Hence, the consumer at time 0 purchases $V(0) - \alpha(0)$ bonds in addition to $\alpha(0)$ shares of stock. We require that $\alpha^n(k/n)$ is $F_{k/n}$ measurable; in the nth economy, the consumer only knows at time k/n the evolution of the stock price up to and including that date. In the usual fashion, if $V(k/n, \omega)$ is the value of the portfolio formed by this trading strategy at time k/n in state ω, then for all $k = 1, \ldots, n$,

$$V(k/n, \omega) = V(0) + \sum_{j=1}^{k-1} \alpha^n(j, \omega) \times \Big[S\big((j+1)/n, \omega\big) - S\big(j/n, \omega\big) \Big].$$

Please note that for a given n and trading strategy $\big(V(0), \alpha^n\big)$, this defines the corresponding value process $V(k/n, \omega)$ for *all* $\omega \in \Omega$ (and not only for ω in the support of P^n), although we (and our consumer) are interested in this only for those ω that are in the support of P^n.

We maintain throughout the assumption that, for each n, P^n specifies a *viable* model of an economic equilibrium in the usual sense: It is impossible to find in the nth discrete-time model a trading strategy $\big(V(0), \alpha^n\big)$ with $V(0) = 0$, $V(1) \geq 0$ P^n-a.s., and $V(1) > 0$ with P^n-positive probability. This is true if and only if there exists a probability measure P^{*n} that is equivalent to P^n, under which $\{(e^{\omega(k/n)}, F_{k/n}); k = 0, \ldots, n\}$ is a martingale (Dalang, Morton, Willinger [6]). Such a P^{*n} is called an *equivalent martingale measure* (emm) for the nth discrete-time model. Of course, in general there will be more than one emm P^{*n}. However, with respect to any emm P^{*n}, a standard argument shows that $\big(V(k/n), F_{k/n}\big)$ is a martingale with respect to P^{*n}. In particular, the expectation of $V(1)$ under every emm P^{*n} is $V(0)$.

Let $X^n := \big\{x \in X : x(\omega) = V(1, \omega)$ for some trading strategy $\big(V(0), \alpha^n\big)$ for the nth discrete-time economy$\big\}$. We refer to X^n as the space of *synthesizable claims in the nth discrete-time economy.*

Definitions.

a. *The contingent claim $x \in X$ can be* ***asymptotically synthesized with bounded risk*** *if there exists a sequence $\{x^n\}$, where each $x^n \in X^n$, such that:*

(i) *for some finite real number* B*,* $P^n\big(\{\omega : |x^n(\omega)| < B\}\big) = 1$ *for all* n*, and*

(ii) *for every* $\epsilon > 0$*, there exists* N_ϵ *such that, for all* $n > N_\epsilon$*,*

$$P^n\big(\{\omega : |x^n(\omega) - x(\omega)| > \epsilon\}\big) < \epsilon.$$

b. *The claim* x *can be* ***asymptotically synthesized with*** x***-controlled risk*** *if, in the previous definition, we can replace condition (i) with* $P^n\big(\{\omega : \underline{x} \leq x^n(\omega) \leq \overline{x}\}\big) = 1$ *for all* n*, where* $\underline{x} = \inf_\omega x(\omega)$ *and* $\overline{x} = \sup_\omega x(\omega)$*.*

Because claims $x \in X$ are bounded, if x is asymptotically synthesized with x-controlled risk, then it is asymptotically synthesized with bounded risk. While bounded risk is a common concept in the no-arbitrage literature, the notion of x-controlled risk is novel and tailor-made for our present purposes.

If our objective was solely to rule out classic doubling strategies, bounded risk suffices. But we are after more: The more control one has over how far a synthesized claim x^n can be from a target claim, the more likely it is that a consumer will regard the synthesized claim as being close to the target claim in terms of her preferences. Or, to put it differently, stricter "control" widens the class of preferences for which a consumer will regard what she can accomplish in the nth discrete-time economy for large n (in terms of her utility) as close to what she can attain in the limit economy.

In this regard, note that both "bounded risk" and "x-controlled risk" involve probability 1 control on the range of values of the synthesized claims x^n. And, it is apparent, within the class of such controls—that is, probability 1 control on the range of values of the synthesized claims—x-control is the best one could hope for while still maintaining condition $a(ii)$: If the x^n were restricted (with probability 1) to any closed interval of values strictly smaller than $[\underline{x}, \overline{x}]$, then $a(ii)$ could not hold. This is not to say that stronger controls outside of this class are impossible; see the discussion of *vanishing risk* in Section 5.

Our main result is that, under appropriate conditions, x-controlled risk can be attained:

Theorem 1. *Suppose that*

a. $P^n \Rightarrow P$, *and*

b. *for some sequence* $\{\delta_n; n = 1, \ldots, \}$ *of positive numbers tending to zero,*

$$P^n\big(\{\omega : \sup_{0 \le k < n} |\omega(k/n) - \omega((k+1)/n)| \le \delta_n\}\big) = 1. \tag{2.1}$$

Then every (continuous and bounded) $x \in X$ can be asymptotically synthesized with x-controlled risk. Moreover, fixing the claim x, the sequence of claims $\{x^n\}$ that asymptotically synthesize x can be chosen where, for $\big(V^n(0), \alpha^n\big)$ the trading strategy that gives x^n, $V^n(0) \equiv \boldsymbol{E}^*[x]$, the BSM price of the claim x.

The proof of Theorem 1 is left for Section 6.

The two assumptions in Theorem 1 have the following explanation. That $P^n \Rightarrow P$ is saying that, in a somewhat coarse sense, the discrete-time economies asymptotically resemble the BSM economy or, put the other way around, the BSM economy is, in terms of its viewed-from-afar features, an idealization of the nth discrete-time economy for large n. The second assumption is the key to x-controlled risk. Because, in the discrete-time economies, a consumer-investor cannot instantaneously intercede in the face of an "unusual" event, it is necessary that the damage done to her portfolio by the time she can react can be contained. This is especially true when the value of her portfolio is very close to either $\underline{x}$ or $\overline{x}$. In the BSM model, with continuous-time trading, she can intervene instantaneously. Assumption b gives us sufficient control for large n. In a sense, while Assumption a says that the nth discrete-time economy for large n is similar to the BSM economy when viewed on a "macroscopic" scale, Assumption b is the required similarity in terms of the important "microscopic" features.[6]

Why not settle for bounded risk? Consider, for instance, a consumer who is an expected-utility maximizer, with a utility function u that is continuous, strictly increasing, and concave on the domain $[0, \infty)$ or $(0, \infty)$. If her target

[6] As an anonymous referee kindly pointed out, Assumption b can be weakened. For example, suppose that there is a sequence of k/n-valued stopping times τ^n, where $k \in \{0, 1, \ldots, n\}$, such that $\lim_{n\to\infty} P^n(\{\tau^n = 1\}) = 1$, and such that, for each $\delta > 0$, $P^n(\{\omega : \sup_{0 \le k < \tau^n(\omega)} |\omega(k/n) - \omega((k+1)/n)| \le \delta\}) = 1$ for all large enough n. Under this weaker assumption, the assertion of the theorem is still true. And if, for a sequence $\{\tau^n\}$ as above and $M > 0$, $P^n(\{\omega : \sup_{0 \le k < \tau^n(\omega)} |e^{\omega(k/n)} - e^{\omega((k+1)/n)}| \le M\}) = 1$, for each n, then the conclusion of the theorem still holds, but with "bounded risk" in place of "x-controlled risk."

claim has $\underline{x} = 0$, then asymptotic synthesis with bounded risk is inadequate; we don't rule out the possibility that, for the sequence of synthesizable claims $\{x^n\}$ that approach the target claim x, $x^n < 0$ with positive probability for every n. And if $x^n < 0$ with positive probability, then the consumer cannot think that x^n is "close" to x; she cannot even compute the expected utility she gains from x^n.

With x-controlled risk possible, however, we have the following result: *Suppose that a consumer is an expected-utility maximizer with a continuously differentiable, strictly increasing, strictly concave utility function U defined on time-1 consumption, whose domain of definition is either $[0, \infty)$ or $(0, \infty)$ where, in the latter case, the condition $\lim_{r \to 0} U'(r) = \infty$ is imposed. Suppose that, in the continuous-time BSM economy, this consumer's problem of maximizing expected utility given wealth level $w > 0$ has a finite solution. Let u^* be the level of her optimal expected utility in the BSM economy, and let u^n be the supremum expected utility that she can attain in the nth discrete-time economy. Then $\liminf_n u^n \geq u^*$.*

We will not provide details of the proof here, see Kreps and Schachermayer [17] and Kreps [16]. But here is a fast sketch: The assumption that the consumer has a solution in the limit economy implies from the first-order, complementary-slackness conditions that (a version of) her solution is a contingent claim $\hat{x}$ that is a continuous and increasing function of $\omega(1)$ alone. It is also unbounded above and, if $U'(0) = \infty$, approaches but doesn't attain 0 as $\omega(1)$ approaches $-\infty$. For any $\epsilon > 0$, we can find a continuous claim $\hat{x}^\epsilon$ that gives expected utility within ϵ of u^* and that is bounded above and strictly bounded away from 0. Applying Theorem 1 and using the continuity of U, for large enough n, a claim x^n can be synthesized in the nth discrete-time economy for all large-enough n that provides expected utility within ϵ of the expected utility provided by $\hat{x}^\epsilon$, and therefore no less than $u^* - 2\epsilon$. That proves the assertion.

This begs the question, if Theorem 1 implies that $\liminf_n u^n \geq u^*$, can we conclude that $\lim_n u^n = u^*$? The answer, in general, is no. There are two problems: First, the sequence of discrete-time models may admit asymptotic arbitrage, even if the limit model is fully viable. On this point, see the next section. And, even if the sequence of models does not admit asymptotic arbitrage, it is possible that the utility function U is such that $u^* < \infty$, while

$\lim_n u^n = \infty$. See Kreps and Schachermayer [17] for a full exposition of this topic.

3 The law of one price and asymptotic arbitrage

Theorem 1 says that every bounded and continuous contingent claim x can be asymptotically synthesized with x-controlled risk for an initial investment of $\mathbf{E}^*[x]$, the price of x in the "limiting" BSM economy. However, it *does not* say that x cannot be asymptotically synthesized (with, say, bounded risk) for a smaller (or, for that matter, larger) initial investment. In the spirit of the law of one price, it is clearly desirable to know that each claim x can only be asymptotically synthesized for initial investments that approach $\mathbf{E}^*[x]$. This points us in the direction of the issue of *asymptotic arbitrage* (Kabanov and Kramkov [13], Klein and Schachermayer [14]). The following definition adapts the more technical definition of Klein and Schachermayer to the current context:

Definition. *The sequence $\{P^n\}$ admits an* ***asymptotic arbitrage***[7] *if, for some $B > 0$, there exists, for every $\epsilon > 0$, an n and a trading strategy $(V^n(0), \alpha^n))$ for n, with associated portfolio-value process $V^n(t)$, such that*

a. $V^n(0) = 0$,

b. $P^n\big(\{V^n(k/n) \geq -B, k = 1, \ldots, n\}\big) = 1$, *and*

c. $P^n\big(\{V^n(1) \geq 1\}\big) \geq 1 - \epsilon$.

Theorem 2. *Under the conditions of Theorem 1, if the sequence $\{P^n\}$ does not admit asymptotic arbitrage, and if the (bounded and continuous) claim $x \in X$ is asymptotically synthesized with bounded risk by a sequence of trading strategies $\{(V^n(0), \alpha^n)\}$, then $\lim_{n\to\infty} V^n(0) = \boldsymbol{E}^*[x]$.*

Proof (admitting Theorem 1, which is yet to be proved). Suppose that for some given x this failed to be true. Then, by looking along a subsequence as necessary, we can find a sequence $\{V^n(0), \alpha^n\}$ as above for which the limit of $V^n(0)$ exists (possibly ∞ or $-\infty$) and is $\neq \mathbf{E}^*[x]$. Suppose the limit is strictly greater than

[7] In Klein and Schachermayer, following Kabanov and Kramkov, this is called an asymptotic arbitrage of the second kind.

$\mathbf{E}^*[x]$ and is finite. Using Theorem 1, produce a sequence of trading strategies $\{\hat{V}^n(0), \hat{\alpha}^n\}$ which asymptotically synthesizes x with $\hat{V}^n(0) \equiv \mathbf{E}^*[x]$ and with x-controlled (therefore bounded) risk. And, for each n, form the trading strategy $(0, \gamma^n)$ where γ^n is $(\mathbf{E}^*[x], \hat{\alpha}^n) - (V^n(0), \alpha^n) - \mathbf{E}^*[x] + V^n(0)$, where this is short hand for implementing $(\mathbf{E}^*[x], \hat{\alpha}^n)$, plus the negative of $(V^n(0), \alpha^n)$, and holding in addition $\mathbf{E}^*[x] - V^n(0)$ in bonds. It is clear that the portion of this trading strategy that is $(\mathbf{E}^*[x], \hat{\alpha}^n) - (V^n(0), \alpha^n)$ will asymptotically synthesize the 0 contingent claim, while the portion $\mathbf{E}^*[x] - V^n(0)$ approaches a strictly positive amount of time-1 consumption. Scaling this strategy as necessary so that the final consumption level is at least 1, we have an asymptotic arbitrage, a contradiction.

The case where $\lim_n V^n(0)$ is finite and less than $\mathbf{E}^*[x]$ is handled symmetrically.

This leaves cases where $\lim_n V^n(0) = \pm\infty$. Suppose the limit is $+\infty$. Then scale γ^n as described above by $1/(V^n(0) - 1)$ (once $V^n(0)$ exceeds 1), and the result follows. The case $\lim_n V^n(0) = -\infty$ is handled similarly. ■

It is worth observing that conditions a and b in Theorem 1 do not rule out the possibility of asymptotic arbitrage. An example where conditions a and b are satisfied and yet asymptotic arbitrage is possible is due (independently) to K. Pötzelberger and Th. Schlumprecht; for details, see Hubalek and Schachermayer [12] or Kreps [16], Chapter 7.

4 The canonical example

The canonical example of a sequence $\{P^n\}$ that satisfies the conditions of Theorem 1 is based on the following construction. Fix a real-valued random variable ζ with expected value 0, variance 1, and bounded support. Then, for $n = 1, 2, \ldots$, let $\{(\omega(k/n) - \omega((k-1)/n)); k = 1, 2, \ldots, n\}$ be i.i.d. under P^n, with the law of each of these being the law of $\zeta/\sqrt{n}$.

Donsker's Theorem applies and tells us that $P^n \Rightarrow P$, the Wiener measure on $C_0[0, 1]$. As for condition b in Theorem 1, because the support of ζ is bounded, the condition is clearly met. Moreover:

Theorem 3. *Any sequence $\{P^n\}$ created in this fashion does not admit asymptotic arbitrage.*

Proof. We show that there are equivalent martingale measures P^{*n} for the P^n such that the sequences $\{P^n; n = 1, 2, \ldots\}$ and $\{P^{*n}; n = 1, 2, \ldots\}$ are mutually contiguous, hence by Klein and Schachermayer [14], there can be no asymptotic arbitrage.

The emms P^{*n} are given by the Esscher transforms of P^n, a discrete version of Girsanov's Theorem. For each $n = 1, 2, \ldots$, there are unique constants c_n and d_n such that

$$Z^n(\omega) := e^{-c_n\omega(1)-d_n} = \exp\left\{\sum_{k=1}^{n}\left[-c_n\left(\omega\left(\frac{k+1}{n}\right)-\omega\left(\frac{k}{n}\right)\right)-\frac{d_n}{n}\right]\right\}$$

defines the density of a martingale measure P^{*n} for $\{S(t,\omega) = e^{\omega(t)}; t \in [0,1]\}$ and that is equivalent to P^n; that is $dP^{*n}/dP^n = Z^n$. It is straightforward to check that the assumptions on the unscaled increment ζ—namely that $\mathbf{E}[\zeta] = 0$, $\mathbf{Var}[\zeta] = 1$, and ζ has bounded support—imply that $c_n \to 1/2$ and $d_n \to 1/8$, where $Z := e^{-\omega(1)/2-1/8}$ is the Radon-Nikodym derivative dP^*/dP of P^* to P (Wiener measure) by Girsanov's formula.[8] Because $\|Z^n\|_{L^2(P)}$ and $\|(Z^n)^{-1}\|_{L^2(P^*)}$ are both uniformly bounded in n, mutual contiguity of the sequences $\{P^n; n = 1, 2, \ldots\}$ and $\{P^{*n}; n = 1, 2, \ldots\}$ follows, which implies that the sequence $\{P^n\}$ does not admit asymptotic arbitrage (Klein and Schachermayer [14]). ■

5 Vanishing risk?

As we have already noted, among controls that take the form of a probability 1 bound on the range of the x^n, x-controlled risk is as good as it gets. But stronger measures of control can be conceived:

Definition. *The claim x can be* ***asymptotically synthesized with vanishing risk*** *if there exists a sequence $\{x^n\}$ where each $x^n \in X^n$ and such that, for every $\epsilon > 0$, there exists N_ϵ such that, for all $n \geq N_\epsilon$,*

$$P^n\big(\{\omega; |x^n(\omega) - x(\omega)| > \epsilon\}\big) = 0.$$

If x can be asymptotically synthesized with vanishing risk, it can be asymptotically synthesized with x-controlled risk: If $\{x^n\}$ synthesizes x with vanishing

[8] The required calculations are provided in Kreps, [16], Chapter 5.

risk (and if $\overline{x} > \underline{x}$), replace x^n with $\epsilon_n(\overline{x}+\underline{x})/2+(1-\epsilon_n)x^n$ for suitably chosen $\epsilon_n \searrow 0$. (The case $\overline{x} = \underline{x}$ is trivial.)

The stronger notion of asymptotic synthesis with vanishing risk is, however, not feasible, even for examples in which the P^n are created in the fashion of Section 4, for ζ with bounded support. Consider, for instance, ζ with the following distribution:

$$\zeta = \begin{cases} 1.5, & \text{with probability } 2/9, \\ 0, & \text{with probability } 5/9, \text{ and} \\ -1.5, & \text{with probability } 2/9. \end{cases}$$

Note that ζ has mean 0 and variance 1, so discrete-time models created from ζ in the fashion of Section 4 will have laws that converge to the law of BSM; Theorems 1, 2, and 3 all apply.

Imagine trying to synthesize a European put option with strike price 1 on the stock, $x(\omega) = \big(1 - S(1,\omega)\big)^+$. Asymptotically synthesizing x with vanishing risk implies doing so with bounded risk, so by Theorem 2, the initial investment for doing so in the nth model must converge to $\mathbf{E}^*[x] \approx 0.38239 > 0$. But, for any n, in the nth discrete-time economy, there is positive probability of the path ω^o for which $S(t)$ never moves from 1. Along this path, the stock produces neither capital gains nor capital losses, and so every portfolio strategy θ gives $V_\theta(t,\omega^o) = V_\theta(0)$. Synthesis with vanishing risk would require that, for every path ω with positive probability, and in particular, for ω^o, $V_\theta(1,\omega)$ is close to $x(\omega)$. But $x(\omega^o) = (1 - S(1,\omega^o))^+ = 0$. Since $V_\theta(1,\omega^o) = V_\theta(0)$, we can't have both $V_\theta(1,\omega^o)$ close to 0 and $V_\theta(0)$ close to 0.38239.

6 Proof of Theorem 1.

6.1 Preliminaries

Throughout, $\int_0^t \alpha\, dS(u)$ will mean the stochastic integral of α with respect to the process S over the interval from 0 to t, under the usual conditions (α is predictable and S-integrable). Note in particular that if α is constant on intervals of the form $[k/n,(k+1)/n)$, then $\int_0^t \alpha dS(u)$ is just the forward Itô sum

$$\int_0^t \alpha dS(u) = \sum_{j=0}^{k-1} \alpha\left(\frac{j}{n}\right)\left[S\left(\frac{j+1}{n}\right) - S\left(\frac{j}{n}\right)\right] + \alpha\left(\frac{k}{n}\right)\left[S(t) - S\left(\frac{k}{n}\right)\right], \quad (6.1)$$

where k is such that $k/n \leq t \leq (k+1)/n$.

In the standard (Strasbourg) way of doing stochastic integration, simple integrands are meant to be continuous from the left and having right limits (or càglàd), so α would be constant on $(k/n, (k+1)/n]$. Done this way, the interpretation of $\alpha(t)$ in this context would be that it is the portfolio holding at time t prior to any trading. For integrands α that are a.s. continuous, it does not matter, but our interpretation is that, for a trading strategy $(V(0), \alpha)$ that is piecewise constant, $\alpha(t, \omega)$ is the portfolio holding after time t trading is done, and so for such trading strategies, the formula (6.1) for the forward Itô sum is correct.

Theorem 1 is stated for contingent claims x that are bounded and continuous. It is without loss of generality – and saves on notation – to assume as well that $\mathbf{E}^*[x] = 0$: Suppose x is a bounded and continuous contingent claim. Then so is $x' := x - \mathbf{E}^*[x]$. Of course $\mathbf{E}^*[x'] = 0$. And if we can asymptotically replicate x' with bounded risk (in the sense of Theorem 1), then it is clear that we can do so for x as well: In addition to whatever sequence of trading strategies are employed to asymptotically replicate x' with bounded risk, add the purchase of a side portfolio of $\mathbf{E}^*[x]$ bonds, a side portfolio the composition of which never changes.

It is perhaps worth adding here that, as we assume that $\mathbf{E}^*[x] = 0$, our construction of trading strategies that asymptotically synthesize x to follow works entirely with zero-net initial investment strategies. Hence that part of Theorem 1 that states that, for a given x, we are asymptotically synthesizing x with strategies with an initial net investment of $\mathbf{E}^*[x]$ follows immediately from the argument just provided.

As a final preliminary, we have the following:

Lemma 1. *Let x be a bounded and continuous function on $\big(C_0[0,1], \|\cdot\|_\infty\big)$. Then, for each t, there is a bounded and continuous version of $x(t) = \boldsymbol{E}^*[x|F_t]$ (defined in the proof to follow). This version of $\boldsymbol{E}^*[x|F_t]$ is uniquely determined for all continuous trajectories $\omega \in C_0[0,1]$. And if x is Lipschitz continuous with Lipschitz constant Λ, then (for each t) $x(t)$ is Lipschitz continuous with at most the same Lipschitz constant.*[9]

[9] It is also true that this version $x_t(\omega)$ is continuous in t for each ω, but we do not need this. We are very grateful to Rama Cont, who showed us how to prove this.

Proof. Let Ψ be a second copy of $C_0[0,1]$ with generic element ψ. Let $P^*\otimes Q^*$ be the product measure on $\Omega\times\Psi$, such that $(\omega,\psi)\in\Omega\times\Psi$ is two-dimensional Brownian motion with drift $-1/2$ in each coordinate and such that $P^*\otimes Q^*\big(\{(\omega,\psi) : \omega(0)=\psi(0)=0\}\big)=1$. That is, $\{\psi(t); 0\le t\le 1\}$ under Q^* is a Brownian motion with drift $-1/2$ independent of and identically distributed as $\{\omega(t); 0\le t\le 1\}$ under P^*. For the balance of this proof, write $\mathbf{E}^*$ as $\mathbf{E}^{P^*}$ to distinguish from $\mathbf{E}^{Q^*}$.

Define the concatenation at $t\in[0,1]$ of two paths ω and ψ, denoted $\omega\oplus_t\psi$, as follows:

$$(\omega\oplus_t\psi)(u) := \omega(u)\,\mathbf{1}_{[0,t)}(u) + \big(\omega(t)+\psi(u)-\psi(t)\big)\,\mathbf{1}_{[t,1]}(u).$$

It is clear from the independence properties of Brownian motion that, fixing a path ω up to time t, the law that governs ω over $[t,1]$ is the same as the law that governs $\omega\oplus_t\psi$. Hence $\mathbf{E}^{P^*}[x|F_t](\omega)=\mathbf{E}^{Q^*}[x(\omega\oplus_t\psi)]$. That is, if we *define* $x(t,\omega)$ pointwise by

$$x(t,\omega) := \mathbf{E}^{Q^*}[x(\omega\oplus_t\psi)],$$

then $x(t,\cdot)$ is a version of $\mathbf{E}^{P^*}[x|F_t]$. Fix this specific version of $\mathbf{E}^{P^*}[x|F_t]$.

Suppose x is bounded and continuous and that $\{\omega_n\}$ is a sequence in $C_0[0,1]$ with limit ω. Then

$$\begin{aligned}\lim_n x(t,\omega_n) &= \lim_n \mathbf{E}^{Q^*}\big[x(\omega_n\oplus_t\psi)\big]\\ &= \mathbf{E}^{Q^*}\big[\lim_n x(\omega_n\oplus_t\psi)\big] = \mathbf{E}^{Q^*}\big[x(\omega\oplus_t\psi)\big] = x(\omega,t),\end{aligned}$$

where the key step is taking the limit instead the integral, a simple application of bounded convergence and the continuity of x.

To show that this version is the unique continuous version: Suppose $x'(t,\omega)$ is another continuous version of $\mathbf{E}^{P^*}[x|F_t]$. For each ω and $\ell=1,2,\ldots$, because P^* has full support on $C_0[0,1]$, there must be within the $1/\ell$ neighborhood of ω a path ω_ℓ such that $x'(t,\omega_\ell)=x(t,\omega_\ell)$. But then $x'(t,\omega)=\lim_{\ell\to\infty}x'(t,\omega_\ell)=\lim_{\ell\to\infty}x(t,\omega_\ell)=x(t,\omega)$, where the two outside equalities follow from the continuity of $x(t,\cdot)$ and the supposed continuity of $x'(t,\omega)$.

To complete the proof of the lemma, we must show that if x is Liptschitz continuous with Lipschitz constant Λ, then so is $x(t)$. Write

$$
\begin{aligned}
\bigl|x(t,\omega) - x(t,\omega')\bigr| &= \bigl|\mathbf{E}^{Q^*}\bigl[x(\omega \oplus_t \psi)\bigr] - \mathbf{E}^{Q^*}\bigl[x(\omega' \oplus_t \psi)|\bigr]\bigr| \\
&\le \mathbf{E}^{Q^*}\bigl[\bigl|x(\omega \oplus_t \psi) - x(\omega' \oplus_t \psi)\bigr|\bigr] \\
&\le \mathbf{E}^{Q^*}\bigl[\Lambda\bigl\|\bigl((\omega(u)\mathbf{1}_{[0,t)}(u) + \bigl(\omega(t) + \psi(u) - \psi(t)\bigr)\mathbf{1}_{[t,1]}\bigr) \\
&\quad - \bigl((\omega'(u)\mathbf{1}_{[0,t)}(u) + (\omega'(t) + \psi(u) - \psi(t))\mathbf{1}_{[t,1]}(u)\bigr)\bigr\|_\infty\bigr] \\
&\text{(by the presumed Lipschitz continuity of } x) \\
&= \mathbf{E}^{Q^*}\bigl[\Lambda\bigl\|(\omega(u) - \omega'(u))\mathbf{1}_{[0,t)}(u)\bigr\|_\infty\bigr] \\
&\text{(because, path by path, the continuation portion } \psi \text{ cancels out)} \\
&= \Lambda\,\bigl\|(\omega - \omega')\mathbf{1}_{[0,t)}\bigr\|_\infty \\
&\text{(the integrand is constant with respect to } Q^*) \\
&\le \Lambda\|\omega - \omega'\|_\infty.
\end{aligned}
$$

■

Although it is probably obvious, observe that $x(1,\omega) = x(\omega)$.

6.2 A sketch of the proof of Theorem 1

Because the reader may get lost in the details of the proof, here is an overview:

We begin by assuming that the claim x to be asymptotically synthesized is not only continuous and bounded, but also Lipschitz continuous as a function on $\bigl(C_0[0,1], \|\cdot\|_\infty\bigr)$. And—in many steps—we prove the following more technical and precise result:

Proposition 1. *Suppose that x is bounded and Lipschitz continuous. Denote by $x(t)$ the Lipschitz-continuous version of $\boldsymbol{E}^*[x|F_t]$ provided by Lemma 1. (Whenever we write $\boldsymbol{E}^*[x|F_t]$, we mean this version.) Then for every $\epsilon > 0$, there exists N such that for all $n > N$, there is a predictable integrand α^n that is constant on the intervals $\bigl[k/n, (k+1)/n\bigr)$ and a stopping time τ^n, taking values in $\bigl\{k/n; k = 1, \ldots, n\bigr\} \cup \bigl\{\infty\bigr\}$, such that*

$$P^n\bigl(\bigl\{\omega : \tau^n(\omega) = \infty\bigr\}\bigr) > 1 - \epsilon, \quad \textit{and} \tag{6.2a}$$

$$P^n\left(\left\{\omega : \left|\int_0^{\tau^n \wedge 1} \alpha^n dS(u) - x(\tau^n \wedge 1)\right| < \epsilon\right\}\right) = 1, \tag{6.2b}$$

where $\tau^n \wedge 1$ means $\min\{\tau^n, 1\}$.

(This means that we have vanishing risk in synthesizing $x(\tau^n \wedge 1)$. But, of course, this becomes (only) x-controlled risk in the synthesis of $x(1)$.)

We know from the theory of the BSM model that, for the fixed x, there is a predictable integrand α such that $\int_0^1 \alpha dS = x$ holds true P-almost surely. (Recall that we assume that $\mathbf{E}^*[x] = 0$.) Moreover, if we write $x(t,\omega) = \mathbf{E}^*[x|F_t]$ for the specific version of $\mathbf{E}^*[x|F_t]$ provided in Step 2, then $x(t)$ is a version of $\big[\int_0^t \alpha(u)dS(u)\big](\omega)$.

Our first step in proving the proposition is then to find an integer M and a predictable continuous-time process $\big(\alpha^M(t); 0 \le t \le 1\big)$ that is constant on each interval $\big[j/M, (j+1)/M\big)$ and such that

$$\int_0^1 \alpha^M dS \approx x = \int_0^1 \alpha dS = x(1), \quad \text{under the probability } P$$

where the symbol $\approx$ has to be made precise. Note in this regard that, because α^M is constant on intervals $\big[j/M, (j+1)/M\big)$, the value of $\big[\int_0^1 \alpha^M dS\big](\omega)$ can be defined path by path for all $\omega \in C[0,1]$ as the Itô sum,

$$\left[\int_0^1 \alpha^M dS\right](\omega) := \sum_{j=0}^{M-1} \alpha^M\left(\frac{j}{M}, \omega\right) \times \left[S\left(\frac{j+1}{M}, \omega\right) - S\left(\frac{j}{M}, \omega\right)\right],$$

where $S(j/M, \omega) = e^{\omega(j/M)}$. *And*, we can replace $\int_0^1 \alpha dS$ with $x = x(1)$. Using Doob's Maximal Inequality, we can extend this to show that

$$\int_0^t \alpha^M dS(u) \approx x(t) = \int_0^t \alpha dS(u) = x(t), \quad \text{uniformly in } t, \text{ under the probability } P.$$

We then pass to a finer mesh $\big\{k/n; k = 0, \ldots, n\big\}$ which splits each of the intervals from j/M to $(j+1)/M$ into ℓ pieces; that is, $n = \ell M$. If ℓ is large enough ($\ell > L$, for L to be determined), then the estimates in the first part of the proof show that

$$\int_0^{k/n} \alpha^M dS(u) \approx x(k/n), \quad \text{uniformly in } k = 1, \ldots, n,$$

under the probability P^n.

Finally, we stop the process α^M at the first time k/n where either the integral or the stock price is not behaving in a suitably desirable fashion. Because the support of ζ is bounded, for large enough L, stopping allows us to control the damage that can occur over the just-before-stopping interval, $(k - 1/n, k/n]$, which gives us (6.2b). And we show that, for sufficiently large n, the probability (under P^n) that we must intercede in this fashion goes to zero, which is (6.2a).

This will finish the proof of Proposition 1. It should be evident (but we'll give some details) that this proves Theorem 1 for Lipschitz-continuous and bounded contingent claims x. To complete the proof of the theorem, we show that if Theorem 1 holds for Lipschitz-continuous and bounded contingent claims x, it holds for continuous and bounded claims.

6.3 Proof of Proposition 1

Throughout, Λ denotes the Lipschitz constant for the contingent claim x.

Step 1. *For $\epsilon > 0$, there is an integer M and a predictable integrand $\alpha^M = \big(\alpha^M(t); 0 \leq t \leq 1\big)$ that is uniformly bounded and constant on all intervals of the form $\big[j/M, (j+1)/M\big)$ and, for each $t \in \big\{j/M; j = 0, \dots, M\big\}$, Lipschitz in the variable $\omega \in C_0[0,1]$, with the following property:*

$$P\left(\left\{\omega : \sup_{0\leq t\leq 1}\left|\int_0^t \alpha^M dS(u) - x(t)\right| < \epsilon\right\}\right) > 1 - \epsilon/2, \tag{6.3}$$

where $x(t)$ is the continuous version of $\boldsymbol{E}^*[x|F_t]$ given by Lemma 1.

Proof of Step 1. It is convenient to work under the equivalent martingale measure P^* of P. We therefore have that $dS(t) = S(t)dW^*(t)$, where $W^*(t) = W(t) + t/2$ is a P^* Brownian motion. Noting that $S(t)$ has quadratic variation $d\langle S\rangle(t) = S(t)^2 dt$, we obtain the following version of Itô's isometry. Denote by R^* the measure on $[0,1] \times C_0[0,1]$ with density

$$\frac{dR^*}{d\big(\lambda \otimes P^*\big)}(t,\omega) = S(t,\omega)^2,$$

where $\lambda \otimes P^*$ denotes the product of Lebesgue measure λ on $[0,1]$ and P^*. We then have the Itô isometry

$$\big\|\beta(t,\omega)\big\|_{L^2([0,1]\times C_0[0,1], R^*)} = \left\|\int_0^1 \beta dS(t)\right\|_{L^2(P^*)}, \tag{6.4}$$

for every predictable process β for which the left-hand side is finite.

Let $\mathcal{P}$ denote the predictable sigma-algebra on $[0,1] \times C[0,1]$, generated by the filtration $\big\{F_t; 0 \leq t \leq 1\big\}$. That is, $\mathcal{P}$ is the sigma-algebra generated by the stochastic intervals $(\tau, 1]$, where τ runs through the stopping times pertaining to the filtration $\{F_t\}$. Note that, in the present case of the filtration of a Brownian motion, the predictable sigma-algebra coincides with the optional sigma-algebra which, by definition, is generated by the stochastic intervals of the form $[\tau, 1]$, where τ runs through the stopping times with respect to the filtration $\{F_t\}$. This is so because, in this case, every stopping time τ is predictable; i.e., $\tau = \lim_{\ell\to\infty} \tau_\ell$, where $\{\tau_\ell; \ell = 1, 2, \ldots\}$ is a sequence of strictly increasing stopping times. Let $\mathcal{G}^M$ denote the sigma-algebra generated by stochastic intervals of the form $[\tau^M, 1]$ for stopping times τ^M taking values in $\{j/M; j = 0, \ldots, M\}$; we have that $\bigcup_M \mathcal{G}^M$ generates $\mathcal{P}$.

Define by $\overline{\alpha}^M$ the conditional expectation of α with respect to $\mathcal{G}^M$; that is

$$\overline{\alpha}^M = \mathbf{E}^{R^*}[\alpha|\mathcal{G}^M]. \tag{6.5}$$

In fact, a little care is needed here as R^* is not normalized to have mass 1. Hence (6.5) must be interpreted as the conditional expectation with respect to the re-normalized probability measure

$$\frac{R^*}{R^*\big([0,1] \times C_0[0,1]\big)}\ .$$

In any case, as $\alpha \in L^2(R^*)$, the sequence $\big(\overline{\alpha}^M; M = 1, 2, \ldots\big)$ converges to α in the norm of $L^2(R^*)$. Indeed, the sigma-algebras $\big(\mathcal{G}^M; M = 1, 2, \ldots\big)$ generate the sigma-algebra $\mathcal{P}$ and $\|\alpha\|_{L^2(R^*)}$ is finite as $x = \int_0^1 \alpha dS(t)$ is bounded. Hence, by Itô's isometry (6.5), the sequence of random variables $\overline{x}^M = \int_0^1 \overline{\alpha}^M dS(t)$ converges in the norm of $L^2(P^*)$ to $x = \int_0^1 \alpha dS(t)$.

We still have to pass from $\overline{\alpha}^M$ to an F^M-adapted process $\alpha^M = \alpha^M(t, \omega)$ that is uniformly bounded and Lipschitz continuous in ω for each $t = j/M$. To do so, it suffices to approximate each of the finitely many $F^M(j/M)$-measurable random variables $\overline{\alpha}^M(j/M) \in L^2(P^*)$ by an $F^M(j/M)$-measurable and bounded Lipschitz function $\alpha^M(j/M)$ on $\big(C_0[0,1], \|\cdot\|_\infty\big)$with respect to the norm of $L^2(P^*)$. Hence $\|\alpha^M - \overline{\alpha}^M\|_{L^2([0,1]\times C_0[0,1], R^*)}$ can be made arbitrarily small, so that by Itô's isometry, $\|\int_0^1 (\overline{\alpha}^M - \alpha^M) dS\|_{L^2(P^*)}$ also becomes small. The process that results by keeping these values during the respective intervals $\big(j/M, (j+1)/M\big]$, denoted by α^M, does what we want. Indeed, we can make the error $x - \overline{x}^M$ and, therefore, also the error $\int_0^1 (\alpha - \alpha^M) dS$ arbitrarily small with respect to the norm of $L^2(P^*)$. Finally, we apply the L^2-version of Doob's maximal inequality[10] to not only make $\big\|\int_0^1 (\alpha - \alpha^M) dS\big\|_{L^2(P^*)}$ small, but also

[10]Doob [7]; see also Acciaio et al. [1]).

$\left\| \sup_{0\le t\le 1} \int_0^t (\alpha - \alpha^M) dS(u) \right\|_{L^2(P^*)}$. Using the fact that P and P^* are equivalent, we obtain inequality (6.3).

Step 2. *Because the integrand α^M is Lipschitz in $\omega \in C_0[0,1]$ and changes value only finitely many times, we know that the function x^M defined pathwise by*

$$x^M(t)(\omega) := \left[\int_0^t \alpha^M dS(u) \right](u) = \alpha^m \left[S(t,\omega) - S\left(\frac{j}{m},\omega\right) \right] + \sum_{i=0}^{j} \alpha^M \left(\frac{i}{m},\omega\right) \cdot \left[S\left(\frac{i+1}{m},\omega\right) - S\left(\frac{i}{m},\omega\right) \right],$$

where j is such that $j/M \le t < (j+1)/M$, is also Lipschitz on bounded subsets of $C[0,1]$ under the sup norm, uniformly in $t \in [0,1]$.

Proof of Step 2. Consider a bounded set $\mathcal{B}$ in $C[0,1]$, so that there is a constant $C \ge 0$ such that $S(t)(\omega) = e^{\omega(t)} \le C$ for every $\omega \in \mathcal{B}$. We know that there is a uniform Lipschitz constant $\mathcal{L}$ for the functions $\{\alpha^M((j-1)/M); j = 1, \dots, M\}$ on $(C_0[0,1], \|\cdot\|_\infty)$as well as a uniform bound on$\{\|\alpha^M((j-1)/M)\|_\infty; j = 1, \dots, M\}$, which we may assume is the same $C > 0$. We must show that there is a constant $D > 0$ such that

$$\left| x^M(t)(\omega) - x^M(t)(\omega') \right| \le D \left\| \omega - \omega' \right\|, \tag{6.6}$$

for all $\omega, \omega' \in \mathcal{B}$ and $t \in [0,1]$. Clearly, it will suffice to show that there is a constant, denoted by K, such that

$$\left| \int_{(j-1)/M}^t \alpha^M dS(u)(\omega) - \int_{(j-1)/M}^t \alpha^M dS(u)(\omega') \right| \le K \left\| \omega - \omega' \right\|, \tag{6.7}$$

for all $\omega, \omega' \in \mathcal{B}, j = 1, \dots M$, and $t \in [(j-1)/M, j/M]$. For then (6.7) will imply (6.6), where we take $D = KM$.

And to show (6.7), note that, for $d = \left\| \omega - \omega' \right\|_\infty$, we have the estimates

$$\left| \alpha^M \left(\frac{j-1}{M}\right)(\omega) - \alpha^M \left(\frac{j-1}{M}\right)(\omega') \right| \le \mathcal{L} d,$$

$$\left| S(t)(\omega) - S(t)(\omega') \right| \le 2Cd, \quad \text{and} \quad \left| S\left(\frac{j-1}{M}\right)(\omega) - S\left(\frac{j-1}{M}\right)(\omega') \right| \le 2Cd.$$

Putting these together, we have

$$\left| \int_{(j-1)/M}^{t} \alpha^M dS(u)(\omega) - \int_{(j-1)/M}^{t} \alpha^M dS(u)(\omega') \right|$$

$$= \left| \alpha^M \left(\frac{j-1}{M} \right)(\omega) \times \left[S(t)(\omega) - S\left(\frac{j-1}{M} \right)(\omega) \right] \right.$$

$$\left. - \alpha^M \left(\frac{j-1}{M} \right)(\omega) \times \left[S(t)(\omega) - S\left(\frac{j-1}{M} \right)(\omega) \right] \right|$$

$$\leq 2C(\mathcal{L} + C)d,$$

where we have used the inequality $|ab - a'b'| \leq 2C(\mathcal{L} + C)d$, provided that $|a - a'| \leq \mathcal{L}d$, $|b - b'| \leq Cd$, and $\max\{|a|, |a'|, |b|, |b'|\} \leq 2C$.

This finishes the proof of Step 2.

Step 3. Because $P^n \Rightarrow P$, the family of distributions $\{P^n; n = 1, \ldots\} \cup \{P\}$ is tight, and so for given $\epsilon > 0$ there is a bound B such that the event

$$\mathcal{D}_B := \{\omega : |\omega(t)| < B \text{ for all } t \in [0, 1]\}$$

has $P^n(\mathcal{D}_B) \geq 1 - \epsilon/2$ for all n and $P(\mathcal{D}_B) \geq 1 - \epsilon/2$.

Step 4. We know from (6.3) that the Borel set

$$\mathcal{D}^{M,\epsilon} := \left\{ \omega : \sup_{0 \leq t \leq 1} \left| x(t)(\omega) - x^M(t)(\omega) \right| < \epsilon \right\}$$

has $P(\mathcal{D}^{M,\epsilon}) > 1 - \epsilon/2$. Recall the uniform Lipschitz continuity of the functions $x(t)$ established in Lemma 1 and the uniform Liptschitz continuity of the function $x^M(t)$ shown to hold true on bounded subsets of $C[0, 1]$ shown in Step 3. We therefore obtain that the set $\mathcal{D}^{M,\epsilon} \cap \mathcal{D}_B$ is *open* in $(C_0[0, 1], \| \cdot \|_\infty)$. Because $P(\mathcal{D}^{M,\epsilon} \cap \mathcal{D}_B) > 1 - \epsilon$, by applying the Portmanteau Theorem for functional weak convergence (Billingsley [3], Theorem 2.1), we conclude that, for large enough n,

$$P^n(\mathcal{D}^{M,\epsilon} \cap \mathcal{D}_B) > 1 - \epsilon. \tag{6.8}$$

Therefore, for large enough L and all $n = \ell M$ for $\ell > L$, (6.8) is true.[11]

[11] We look at integer multiples $n = \ell M$ of M so that the time grid $\{0, 1/n, 2/n, \ldots, 1\}$ contains j/M for integer j.

Step 5. Define stopping times τ_1^n, τ_2^n, and τ^n for each $\omega \in C[0,1]$ by

$$\tau_1^n(\omega) := \inf\left\{t = k/n : \left|x(k/n)(\omega) - x^M(k/n)(\omega)\right| \geq \epsilon\right\},$$
$$\tau_2^n(\omega) := \inf\left\{t = k/n : \ln\big(S(t,\omega)\big) \notin \big[-B, B\big]\right\}, \quad \text{and}$$
$$\tau^n(\omega) = \min\big\{\tau_1^n(\omega), \tau_2^n(\omega)\big\},$$

where, by convention, $\tau_1^n(\omega) = \infty$ if $\left|x(k/n)(\omega) - x^M(k/n)(\omega)\right| < \epsilon$ for $k = 0, \ldots, n$, and $\tau_2^n(\omega) = \infty$ if $\ln\big(S(k/n,\omega)\big) \in \big[-B, B\big]$ for $k = 0, \ldots, n$. These stopping times are all well defined for all $\omega \in C[0,1]$.

In particular, an investor using the portfolio strategy α^M in the nth discrete-time economy can stop according to these stopping rules: Clearly, she will know the current stock price, so implementing τ_2 is trivial. As for implementing τ_1, $x^M(k/n)(\omega)$ is just the value of her portfolio at time k/n, for the path ω she has observed (where we fill in between times k/n and $(k+1)/n$ with linear interpolation), while $x(k/n)(\omega)$ is calculated from the path ω up to time k/n as in Lemma 1.

Step 6. We now verify (6.2a). Note that for any path $\omega \in \mathcal{D}_B \cap \mathcal{D}^{M,\epsilon}$, $|\omega(t)| < B$ for all t, hence for all t of the form k/n. And for $\omega \in \mathcal{D}_B \cap \mathcal{D}^{M,\epsilon}$, $\left|x(t)(\omega) - x^M(t)(\omega)\right| < \epsilon$ for all t, hence for all t of the form k/n. This implies that for all $\omega \in \mathcal{D}_B \cap \mathcal{D}^{M,\epsilon}$, $\tau_1^n = \tau_2^n = \tau^n = \infty$. Hence, for large enough L (which gives large enough n),

$$P^n\big(\{\omega : \tau^n(\omega) = \infty\}\big) \geq P^n\big(\mathcal{D}_B \cap \mathcal{D}^{M,\epsilon}\big) > 1 - \epsilon,$$

which is (6.2a).

Step 7. It remains to show that (6.2b) holds. Please recall that, by assumption, there is a sequence $\{\delta_n\}$ of positive numbers such that $\delta_n \to 0$ and

$$P^n\big(\{\omega : |\omega(k/n) - \omega((k-1)/n)| > \delta_n\}\big) = 0.$$

If either $\tau_1^n(\omega) = \infty$ or if $\tau_2^n(\omega) < \tau_1^n(\omega) < \infty$, then at time $\tau^n \wedge 1$, we know that $\left|x(\tau^n \wedge 1)(\omega) - x^M(\tau^n \wedge 1)(\omega)\right| < \epsilon$. Hence all such ω belong to the event in (6.2b).

This leaves the case of paths ω such that $\tau_1^n(\omega) \leq \tau_2^n(\omega)$ and $\tau_1^n(\omega) \leq 1$. For such an ω, let $\tau_1^n(\omega) = k/n$, and consider the state of affairs at time $(k-1)/n$.

Since neither stopping time τ_1^n nor τ_2^n has triggered, we know that

$$S((k-1)/n) \le e^B \quad \text{and} \quad \big|x((k-1)/n) - x^M((k-1)/n)\big| < \epsilon.$$

We must show that

$$\big|x(k/n) - x^M(k/n)\big| < 2\epsilon,$$

which we do by showing that, for large enough L,

$$\big|x(k/n) - x((k-1)/n)\big| \le \epsilon/2 \;\text{ and }\; \big|x^M(k/n) - x^M((k-1)/n)\big| \le \epsilon/2.$$

The first of these follows easily from the Lipschitz continuity of x. Fixing the path of ω up to time $(k-1)/n$, consider two possible continuations, ω and ω'. That is, ω and ω' are partial paths up to time k/n that coincide up to time $(k-1)/n$. Then

$$\sup\big\{\big|\omega(i/n) - \omega'(i/n)\big|; i = 0, \dots, k\big\} = \big|\omega(k/n) - \omega'(k/n)\big|,$$

and, since ω and ω' coincide up to time $(k-1)/n$, $\big|\omega(k/n) - \omega'(k/n)\big|$ can be no larger than $2\delta_n$, P^n-a.s. By choosing L to be large enough, this can be made as small as needed so that, taking into account the Lipschitz constant for x, we get the desired bound on $\big|x(k/n) - x((k-1)/n)\big|$.

And, finally, to bound $\big|x^M(k/n) - x^M((k-1)/n)\big|$, write

$$\begin{aligned}
x^M\left(\frac{k}{n}\right) - x^M\left(\frac{k-1}{n}\right) &= \alpha^M\left(\frac{k-1}{n}\right) \times \left[S\left(\frac{k}{n}\right) - S\left(\frac{k-1}{n}\right)\right] \\
&= \alpha^M\left(\frac{k-1}{n}\right) \times S\left(\frac{k-1}{n}\right) \times \left[\frac{S(k/n) - S((k-1)/n)}{S((k-1)/n)}\right] \\
&= \alpha^M\left(\frac{k-1}{n}\right) \times S\left(\frac{k-1}{n}\right) \times \left[\frac{e^{\omega(k/n)}}{e^{\omega((k-1)/n)}} - 1\right] \\
&= \alpha^M\left(\frac{k-1}{n}\right) \times S\left(\frac{k-1}{n}\right) \times \left[e^{\omega(k/n) - \omega((k-1)/n)} - 1\right].
\end{aligned}$$

We are looking at paths ω such that $\tau_1^n \le \tau_2^n$, so we know that $S((k-1)/n) \le e^B$. And we know that α^M is uniformly bounded. And the final term in the product can be made as small as necessary to make the product less than $\epsilon/2$, because $|\omega(k/n) - \omega((k-1)/n)| < \delta_n$, P^n-a.s. Noting that $\epsilon > 0$ is arbitrary here, we have (6.2b).

Step 8. *We therefore have the result for all n that are of the form ℓM, for any $\ell > L$. To finish the proof of Proposition 1, we must show that, enlarging L still further as necessary, the result is true for all $n > (L+1)M$.*

This is accomplished as follows. We have fixed M and L. For every $n > (L+1)M$, and for $j = 0, \ldots, M$, let $k_{j,n}$ be the least integer such that $k_{j,n}/n \geq j/M$. Of course, $k_{j,n} - j/M < 1/n$. Modify the construction given above: writing $\alpha_j^M(\omega)$ for the value of α^M previously applied on the interval $[j/M, (j+1)/M)$ (which is based on the path of stock prices up to time j/M), delay slightly the shift from α_{j-1}^M to α_j^M by (instead) holding $\alpha_j^M(\omega)$ shares of stock over the interval $[k_{j,n}/n, k_{j+1,n}/n)$. The stopping rule τ^n is defined just as before.

This (slight) shift in when the portfolio's composition changes is properly adapted to the information received. And, as $n \to \infty$ for fixed M, the change it causes in the value of the portfolio uniformly tends to zero: The stopping rule puts a bound on the price of the stock and the number of shares of stock held is uniformly bounded, so the "error" introduced by this slight delay vanishes as the amount by which the stock price can move over any single interval of length $1/n$ vanishes (in n, again relying on the bound in stock prices before stopping). And, since M is fixed, there is a fixed number of such "errors" that are introduced. Enlarging L as necessary (holding M fixed), the sum of these M errors become (uniformly) arbitrarily small, completing the proof of Proposition 1.

6.4 Completing the proof of Theorem 1

We have proved Proposition 1 for Lipschitz continuous x. This almost immediately gives Theorem 1 for such x: Given ϵ, find N sufficiently large so that for all $n > N$, there exist α^n and τ^n that satisfy (6.1) and (6.2). Re-interpret α^n and τ^n as a (0 initial investment) trading strategy $\hat{\alpha}^n$ for the nth discrete-time economy where

$$\hat{\alpha}^n(t,\omega) = \begin{cases} \alpha^n(t,\omega), & \text{if } t < \tau^n(\omega), \text{ and} \\ 0, & \text{if } t \geq \tau^n(\omega). \end{cases}$$

In words, "stopping" according to τ^n is interpreted as converting the value of the portfolio held at that time entirely into bonds and doing no further trading until time 1. On the event $\{\omega : \tau^n(\omega) = \infty\}$, which has probability greater than $1-\epsilon$, the integral $\left[\int_0^1 \hat{\alpha}^n dS(u)\right](\omega) = \left[\int_0^1 \alpha^n dS(u)\right](\omega) = x(1,\omega)$ will be within ϵ of $x(1,\omega) = x(\omega)$. And on the event $\{\omega : \tau^n(\omega) < \infty\}$, $\left[\int_0^1 \hat{\alpha}^n dS(u)\right](\omega) = \left[\int_0^{\tau^n} \hat{\alpha}^n dS(u)\right](\omega)$, which is within ϵ of $x(\tau^n)$, which of course has expected value less or equal to $\|x\|_\infty$.

This almost gives us Theorem 1, except that, for fixed $\epsilon > 0$, the probability-one bound on the synthesized claims are that their values lie in the interval $(\underline{x} - \epsilon, \overline{x} + \epsilon)$. We want the synthesized claims to have values that lie in the interval $(\underline{x}, \overline{x})$. Recall that we assumed, without loss of generality, $\mathbf{E}^*[x] = 0$. Except for the trivial case where $x \equiv 0$, this implies that $\underline{x} < 0$ and $\overline{x} > 0$. For the fixed ϵ, let α be close enough to 1 so that $(1-\alpha)\overline{x} < \epsilon/2$ and $(\alpha-1)\underline{x} < \epsilon/2$. Let $\delta = \min\{(\alpha-1)\underline{x}, (1-\alpha)\overline{x}\}$. Of course, $\delta < \epsilon/2$.

Consider the claim $x' = \alpha x$: (a) if x is Lipschitz continuous, then so is x'; (b) $\mathbf{E}^*[x] = 0$ implies $\mathbf{E}^*[x'] = 0$; and (c) $\underline{x}' = \alpha\underline{x}$ and $\overline{x}' = \alpha\overline{x}$. Employ Proposition 1 for the claim x' and for δ in place of ϵ. This guarantees that there exists N sufficiently large so that, for all $n > N$, a claim x^n can be synthesized in the nth discrete time economy such that, with P^n-probability $1-\delta$ or greater, $|x^n(\omega) - x'(\omega)| < \delta$, and $\underline{x}' - \delta < x^n(\omega) < \overline{x}' + \delta$ with P^n-probability 1. Since $|x'(\omega) - x(\omega)| = |\alpha x(\omega) - x(\omega)| = |(1-\alpha)x(\omega)| < \epsilon/2$ by the choice of α, we know that $|x^n(\omega) - x(\omega)| \leq |x^n(\omega) - x'(\omega)| + |x'(\omega) - x(\omega)| < \delta + \epsilon/2 < \epsilon$ with P^n-probability $1-\delta$ or more, which is certainly $1-\epsilon$ or more under P^n. And if $x^n(\omega) \in (\underline{x}' - \delta, \overline{x}' + \delta)$, since $\underline{x}' - \delta = \alpha\underline{x} - \delta > \underline{x}$, and $\overline{x}' + \delta = \alpha\overline{x} + \delta < \overline{x}$, we know that, with P^n-probability one, $x^n \in (\underline{x}, \overline{x})$. Hence we have the tighter probability-one bounds on the synthesized claims.

As a final step, we need to extend Theorem 1 from Liptschitz-continuous and bounded contingent claims x to continuous and bounded contingent claims. We employ the following Lemma:

Lemma 2. *Fix a bounded and continuous function x on $\big(C_0[0,1], \|\cdot\|_\infty\big)$. For $\Lambda > 0$, define $x^\Lambda(\omega) := \inf\big\{x(\omega') + \Lambda\|\omega - \omega'\|; \omega' \in C_0[0,1]\big\}$ for each $\Lambda > 0$. Then:*

a. *If $\Lambda < \Lambda'$, then $x^\Lambda(\omega) \leq x^{\Lambda'}(\omega) \leq x(\omega)$. And if $\underline{x} = \inf_\omega x(\omega)$, then $x^\Lambda(\omega) \geq \underline{x}$. Hence for all Λ, $\|x^\Lambda\| \leq \|x\|$.*

b. *For all ω, $\lim_{\Lambda\to\infty} x^\Lambda(\omega) = x(\omega)$. Hence, by monotone convergence, $\lim_{\Lambda\to\infty} \boldsymbol{E}^*[x^\Lambda] = \boldsymbol{E}^*[x]$. And for any compact set K in $\big(C_0[0,1], \|\cdot\|_\infty\big)$ and for any $\epsilon > 0$, there is sufficiently large Λ (depending on K and ϵ) such that $x(\omega) - x^\Lambda(\omega) \leq \epsilon$ for all $\omega \in K$.*

c. *x^Λ is Lipschitz continuous with Lipschitz constant at most Λ.*

The proof is straightforward: Part a is obvious. Part b follows from compactness, and c follows from the triangle inequality.

Now fix a bounded and continuous claim x such that $\mathbf{E}^*[x] = 0$ and some $\epsilon > 0$. We have the following (asymptotic) estimates.

i. Since $\mathbf{E}^*[x^\Lambda] \nearrow \mathbf{E}^*[x] = 0$, for all large enough Λ, $|\mathbf{E}^*[x^\Lambda]| \leq \epsilon/3$.

ii. Since $P^n \Rightarrow P$, the tightness of probability measures $\{P^n\}$ allows us to produce a compact subset K of $C[0,1]$ such that $P^n(K) > 1 - \epsilon/3$ for all n.

iii. Apply Lemma 2: For this compact set K and for all large enough Λ, $|x^\Lambda(\omega) - x(\omega)| \leq \epsilon/3$ for all $\omega \in K$.

iv. Since x^Λ is Lipschitz, so is $\hat{x}^\Lambda := x^\Lambda - \mathbf{E}^*[x^\Lambda]$. And $\|\hat{x}^\Lambda\| \leq \|x^\Lambda\| + \big|\mathbf{E}^*[x^\Lambda]\big| \leq \|x\| + \epsilon/3$

v. Theorem 1 for Lipschitz continuous and bounded functions ensures that for all sufficiently large n, we can produce in the nth discrete time economy a contingent claim x^n such that

$$P^n\big(\{|x^n(\omega)| \ \leq \|\hat{x}^\Lambda\| + \epsilon/3\}\big) = 1 \text{ and } P^n\big(\{|x^n(\omega) - \hat{x}^\Lambda(\omega)| \leq \epsilon/3\}\big) \geq 1 - \epsilon/3.$$

Combining these estimates, we have that, for all sufficiently large n:

vi. $P^n\big(\{|x^n(\omega)| \ \leq \|x\| + 2\epsilon/3\}\big) = 1$, and

vii. If we denote by $\mathcal{J}^n$ the set $\big\{|x^n(\omega) - \hat{x}^\Lambda(\omega)| \leq \epsilon/3\big\}$, then on the set $\mathcal{J}^n \cap K$, $\big|x^n(\omega) - x(\omega)\big| \leq \big|x^n(\omega) - \hat{x}^\Lambda(\omega)\big| + \big|\hat{x}^\Lambda(\omega) - x^\Lambda(\omega)\big| + \big|x^\Lambda(\omega) - x(\omega)\big| \leq \epsilon$.

viii. And $P^n(\mathcal{J}^n \cap K) \geq 1 - 2\epsilon/3$.

Repeating the argument from just before Lemma 2 (since vi doesn't quite give x-control), this completes the proof of Theorem 1. ■

7 Extensions

Theorem 1 is easily extended to claims that are unbounded on one side. Suppose x is a continuous claim with $x \geq 0$ (for all ω) and $\mathbf{E}^*[x] < \infty$. Then a simple corollary to Theorem 1 is that, for any $\epsilon > 0$, there exists N such that for all $n > N$, a claim x^n can be synthesized in the nth discrete-time economy for an initial investment of $\mathbf{E}^*[x]$ and such that $P^n\big(\{\omega : x^n(\omega) - x(\omega) > \epsilon\}\big) < \epsilon$ and $P^n\big(\{\omega : x^n(\omega) > -\epsilon\}\big) = 1$. Proof: Define x^B by $x^B(\omega) := \min\{x(\omega), B\}$. Choose B large enough so that $P^n\big(\{\omega : x(\omega) > B\}\big) \leq \epsilon/3$ uniformly in n (recall that $P^n \Rightarrow P$ for the uniformity), and such that $\mathbf{E}^*[x] - \mathbf{E}^*[x^B] < \epsilon/3$. And then apply Theorem 1 to x^B for $\epsilon/3$ in place of ϵ.

(One must be careful in general how to interpret this. Kreps [16] Chapter 4, provides an example in which the P^* are given by a symmetric binomial random walk—hence for each n there is a unique emm—such that, for a nonnegative and continuous (but not Lipschitz continuous) claim x, $\mathbf{E}^*[x]$ is finite but $\mathbf{E}^{*n}[x] \to \infty$.)

An obvious question is the following: Suppose the objective is to asymptotically synthesize the claim x. Suppose that $x(t) = \int_0^t \alpha dS$. Do the portfolio strategies α^M that we employ in the $N = LM$th discrete-time economy approach α; that is, to the portfolio strategies converge in any meaningful sense? Indeed, will it work simply to take $\alpha^M(j/M, \omega) = \alpha(j/M)$, if not in general, then at least in cases where x is suitably well-behaved. In Kreps [16], for instance, something like this is done, seemingly successfully, in simulating the asymptotic construction of a call option for a trinomial random-walk model. (It is "something like this" because, in that simulation, for the Nth discrete-time model, the portfolio strategy is adjusted at *every* time $t = k/N$, while here we adjust only at times $t = j/M$ where $N = LM$ for large L.) Moreover, in the very specific context of expected-utility maximization for both CRRA and CARA utility functions, Kreps [16] shows convergence of the optimal portfolio rules. We conjecture that general results concerning the convergence of the α^M to α are available; this is a topic we hope to pursue in future work.

Our analysis has been conducted entirely in the special context where the limit economy is the classic BSM economy. Extensions to cases in which the (one) risky price process is a well-behaved Itô integral are clearly available. The essential elements of our proof are:

a. The stochastic integral $\int_0^t \alpha dS$ can be approximated to any degree desired by an integral $\int_0^t \alpha^M dS$, for an integrand α^M that is piece-wise constant on intervals of the form $[j/M, (j+1)/M]$, for large enough M. This is

Step 1 of our proof.

b. This can be done with integrands that are uniformly bounded, which is where the assumption that x is Lipschitz enters.

c. If $\{S(t)\}$ is a well-behaved Itô integral, where "well-behaved" implies these two crucial properties can be satisfied, we are in business. Indeed, given these properties can be satisfied, there seems no reason that $\{S(t)\}$ cannot be multi-dimensional, as long as the limit economy provides for complete markets.

Extending beyond this, to cases where $\{S(t)\}$ is a general semi-martingale, is a much greater challenge. By their very nature, price processes with jumps requires machinery that our analysis is able to avoid. But one can imagine a theory in which, even if markets in the continuous-time economy are not complete, any contingent claim that can be created (with continuous trading) in the limit economy can be asymptotically synthesized with controlled risk in "sufficiently similar" discrete-time economies. This, we believe, is a great direction for future research.

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