Family Ties: School Assignment With Siblings

Umut Dur* Thayer Morrill* William Phan*

October 10, 2020

Abstract

We introduce a generalization of the school choice problem motivated by the following observations: students are assigned to grades within schools, many students have siblings who are applying as well, and school districts commonly guarantee that siblings will attend the same school. This last condition disqualifies the standard approach of considering grades independently as it may separate siblings. We argue that the central criterion in school choice—elimination of justified envy—is now inadequate as it does not consider siblings. We propose a new solution concept, *suitability*, that addresses this concern, and we introduce a new family of *strategy-proof* mechanisms that satisfy it. Using data from the Wake County magnet school assignment, we demonstrate the impact on families of our proposed mechanism versus the "naive" assignment where sibling constraints are not taken into account.

Keywords: School Choice, Matching Theory, Matching with Contracts
JEL Classification: C78, D47, D63, I20

1 Introduction

School assignment has become one of the most well-studied topics in market design. In this expansive literature, every paper that we are aware of considers the problem

*Department of Economics, North Carolina State University. Corresponding author: wphan@ncsu.edu. We thank seminar participants at Frontiers in Market Design (Lund University), Social Choice and Welfare 2018, University of North Carolina Chapel Hill, Ottawa Microeconomic Theory Workshop 2018, INFORMS 2018, Economic Design and Algorithms 2019, and 2019 Conference on Economic Design for comments.

of assigning students to schools. In practice, however, many children have siblings involved in the same assignment procedure, and school boards may *guarantee* that siblings will attend the same school. This motivates a new market design problem: assigning families to schools.

Siblings are assigned concurrently when a family moves to a new city or when a family decides to participate in an auxiliary program, such as attending a magnet school. An assignment procedure that considers grades independently may separate a large number of siblings—violating the guarantee. These complications are likely for *any* school district that has a policy of keeping siblings together.[1]

We generalize the standard school choice problem of Abdulkadiroğlu and Sönmez (2003) by specifying which students are siblings and splitting schools into grades. Each student reports a ranking over schools, but if siblings want the same-school guarantee, they are required to report the same preference ranking.[2] An assignment is feasible if grade capacities and sibling guarantees are respected. For technical reasons, we assume that no child has a twin or more than one sibling. These assumptions are not without loss of generality, and in Appendix A, by means of examples, we illustrate the limitations of them.

The central criterion in the school choice literature is respecting a student's priority.[3] We say that student i has justified envy at, or "blocks", an assignment if i prefers a school s to her own assignment, and i has a higher priority than a student assigned to s. An assignment with justified envy is typically interpreted as being unfair.[4]

This definition is no longer adequate when there are siblings. We discuss two reasons why. First, in order for siblings to attend a school, both must be admitted; therefore, a rejected student might have higher priority than an admitted student if the rejected student has a sibling that did not have high enough priority. A relevant notion of justified envy must somehow take into account all of i's siblings; the standard definition does not. Second, suppose i has justified envy of j, and that i does not

[1]In North Carolina alone, school districts that offer this guarantee include Charlotte-Mecklenberg, Durham, Wake County, Union County, and Winston-Salem. Such a guarantee is offered by school districts in other states including Maryland (Montgomery County) and Oregon (Portland).

[2]For example, in Wake County Public Schools, siblings may opt to be treated as individual students and submit different rankings of schools, but a family that chooses to do this forgoes their guarantee to be assigned to the same school. We do not model the strategic choice of whether siblings should be considered as individuals or as a family.

[3]Many school districts, including Boston, New York, and Chicago, have adopted an assignment procedure that respects students' priorities.

[4]See also the student placement problem of Balinski and Sönmez (1999).

have siblings, but j does. In this case, removing j would also remove all of j's siblings (even if the latter have high priority). We thus argue that one student should not be able to cause more than one other student to be removed from a school.

We generalize what constitutes a block to resolve these issues. Intuitively, under no justified envy, a student ranked x^{th} can block the assignment of the y^{th} ranked student when $x < y$ and both are in the same grade. We extend this to allow the x^{th} and w^{th} ranked students to block the y^{th} and z^{th} ranked students, so long as $x < y$ and $w < z$, and they are in the same respective grades. Specifically, we allow a group of students J to block an assignment if there is a group of students K such that the students of J, one by one, have justified envy of the students of K.[5] Both J and K must be "closed under siblings" in the sense that if J (K) contains one sibling in a family, then it must contain all siblings. Otherwise, replacing K with J would possibly separate a family and the resulting assignment would not be feasible.

We define an assignment as *suitable* if there is no such blocking coalition. We make several observations: When there are no siblings, *suitability* and elimination of justified envy are equivalent. A coalition can be a mixture of siblings, only children, and students from more than one grade. Thus, our blocking coalitions can be quite complex and allow for far more general combinatorial patterns of blocking than the simple blocking considered under justified envy.

Our main result shows that a *suitable* assignment always exists (Theorem 2). This is surprising given the negative results in the closely related matching with couples problem.[6] Our proof of existence is constructive. We introduce a new family of mechanisms called Sequential Deferred Acceptance (SDA) whose members always select a *suitable* assignment. Furthermore, we show that each mechanism is *strategy-proof* (Theorem 2).

Related Literature

We contribute to the literature on school choice pioneered by Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003).[7] To the best of our knowledge, we are

[5]Our formal definition allows J to have more students than K if the school in question is not at full capacity.

[6]We give a more detailed discussion of similarities and differences between our problem and the matching with couples problem in the Related Literature.

[7]Some key contributions in school choice are: Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) and Abdulkadiroğlu, Pathak, and Roth (2005) examine school choice programs in Boston and NYC;

the first to explicitly model siblings, grades, and the ensuing institutional constraint that siblings must be assigned to the same school.[8]

Our paper is related to the matching with couples literature, which has focused primarily on assigning doctors to hospitals. In these papers, couples are allowed to apply together and submit rankings over *pairs* of (possibly different) hospitals. Roth (1984) showed that when couples are present, there may be no *stable* assignment. Klaus and Klijn (2005) provide maximal domain results (in terms of allowable preferences for the couples) to ensure the existence of a *stable* assignment. Some papers also consider the case (as in ours) where each couple must always be assigned to the same hospital (see Dutta and Masso, 1997), but have different assumptions about hospital preferences.[9] The key theoretical difference between our paper and the matching with couples literature is that the complementarities between siblings is more structured than those between couples.[10]

Roth and Peranson (1999) provide a detailed overview of the algorithms implemented in the U.S. National Resident Matching Program (NRMP), wherein couples may apply to hospitals together. Kojima, Pathak, and Roth (2013) and Ashlagi, Braverman, and Hassidim (2014) similarly study the matching market for psychologists. Empirically, the couples problem is significantly different from the siblings problem. Table 1 collates key average statistics from the NRMP, the market for clinical psychologists, and magnet school admission in Wake County Public School System (WCPSS).[11] Two differences stand out: First, couples make up a substantially smaller proportion of the total applicants than sibling pairs. Second, there are a large

Erdil and Ergin (2008) and Abdulkadiroğlu, Pathak, and Roth (2009) study issues regarding priority classes and the breaking of ties; Kesten (2010) proposes a new mechanism for improving on the efficiency of the student optimal *stable* match; and Echenique and Yenmez (2015), Kamada and Kojima (2012), and Kamada and Kojima (2015) study the consequences of distributional constraints and schools' preferences for diversity.

[8]Several recent papers also study extensions where students may apply as groups or are assigned sequences of seats over time. Dur and Wiseman (2018) consider a problem where students may prefer to attend the same school as their neighbors, as opposed to their siblings. Kennes, Monte, and Tumennasan (2014) study the Danish daycare assignment system where each child is assigned to a school in each period, so that each student must express preferences over sequences of schools. Kurino (2014) studies a related dynamic problem where instead of each school having a priority, each agent's current assignment is treated as an endowment.

[9]For example, our school's choice functions do not satisfy Dutta and Masso's weaker form of substitutability (group substitutability).

[10]Section 3.1 provides a formal discussion.

[11]Statistics for columns 1 and 2 are from Roth and Peranson (1999) Table 1, and Kojima, Pathak, and Roth (2013) Table 1, respectively.

Table 1: Matching With Couples vs. School Assignment With Siblings

	NRMP (1993-1996)	Psychology (1999-2007)	WCPSS (2018)
Couples / Siblings as % of Applicants	4.1%	1.3%	19.5%
Number of Hospitals / Schools	3,755	1,094	38
Average Capacity	6.1	2.5	30

number of hospitals each with low capacity, while there are a small number of magnet programs each with high capacity. Roth and Peranson (1999) find that despite the negative theoretical results, a *stable* assignment exists in each year they observe (for the NRMP). Kojima, Pathak, and Roth (2013) show that when there are few couples and preference lists are short (relative to the whole market size), the probability of a *stable* assignment approaches one as the market grows. Using several years of data from the market for clinical psychologists, they also find a *stable* assignment in each year. Ashlagi, Braverman, and Hassidim (2014) show similar results, and introduce the concept of "influence trees"—rejection chains likely to occur because of a couple. In extreme contrast, for each school year in which we have run the WCPSS school assignment (2015-2018), there has never been an assignment without justified envy that keeps siblings together.

Our paper also contributes to the matching with contracts literature. In Appendix B, we model our problem by using the many-to-many matching with contracts framework, and our results have novel implications.

This paper proceeds as follows. In Section 2, after providing information on our motivating application, we present our formal model. In Section 3, we introduce a new respecting priorities criterion, namely *suitability*, and discuss its properties. Next, in Section 4 we define a class of *suitable* and *strategy-proof* mechanisms. Section 5 compares the performances of this class of mechanisms with a naive implementation of the DA mechanism by using school choice data from the WCPSS. The last section concludes.

2 Motivation and Model

2.1 Motivating Application: Design of WCPSS Assignment Procedure

To motivate our theoretical model, we briefly discuss the design and administration of the assignment procedure for the WCPSS magnet program.[12] This was the impetus for our research, as WCPSS requires that siblings be assigned to the same magnet school.[13]

In the 2017-2018 academic year, WCPSS was the 15th largest school system in the United States. The district is comprised of 183 schools (grades K-12) with over 38 magnet schools; seats at the latter are assigned to students via a school choice program. The interested reader may refer to Dur, Hammond, and Morrill (2018), and Dur, Hammond, and Kesten (2018) for extensive discussions of both the assignment procedure and the demographics of Wake County. Here we highlight several details that are critical to our model.

First, all grades at all schools are assigned simultaneously. The vast majority of assignments are for students at the entry-grade year (kindergarten, grade 6, and grade 9). For an entry-grade year, all seats are made through the assignment, but for a non-entry-grade year, most of the seats are taken by students who attended the school the previous year. Therefore, there is a heterogeneity in the capacity (used for the assignment) at each grade, even within the same school.

If a student has an elder sibling already attending a school, then the younger sibling is automatically given highest priority at this school. However, when two siblings apply simultaneously, this is not possible, as the eldest child's assignment has yet to be determined. It is a common occurrence. This happens, for example, when a family has just moved to Wake County or when a family becomes interested in the magnet program after the older child has already begun attending a non-magnet

[12]Since 2015, the authors have been part of the team that designed and implemented the assignment procedure for the WCPSS magnet program.

[13]The following is stated in the Wake County Board of Education Policy Manual. "The highest priority in any of the application processes is for entering grade siblings to attend the same school as an older sibling, so long as the siblings live at the same address... This means that if you apply for more than one sibling to attend a school, the application process will not select one sibling without the other. If there are not available seats for each sibling, the program will select none of the siblings." Retrieved June 2018 at https://www.wcpss.net/Page/33755. Also see the Wake County Board of Education Policy Manual, Policy 6200 and 6200 R & P "Student Assignment."

school.

Each student is given a certain number of points in order to determine her priority.[14] By construction, siblings are each given the same number of priority points. However, for siblings applying to different grades, both the capacities of the grades and the cohorts of students the siblings are competing against differ. Therefore, two siblings do not typically occupy the same place in the ordering of students by priority. Specifically, it is important to note that it is not uncommon for one sibling to be ranked relatively high while another is ranked relatively low.

2.2 Formal Model

We consider the problem of assigning students to schools. As in the standard model, there is a finite set of **students**, I, and a finite set of **schools**, S. Unlike the standard model, the students are part of a family and each school has multiple grades. Let $G = \{1, ..., n\}$ be the finite set of **grades** and $\gamma : I \to G$ be the grade function such that student i applies to grade $\gamma(i)$. Let $\gamma(J) = \bigcup_{i\in J} \gamma(i)$.

For each grade $g \in G$, let I^g denote the set of students applying for grade g, i.e., $I^g = \{i \in I : \gamma(i) = g\}$. Let $\mathcal{F}$ be a partition of students into **families**. For a student i in family $f \in \mathcal{F}$, we will refer to f as i's family. If i and j are in the same family, we refer to them as **siblings**. If i is the only member of i's family, then we call i an **only child**. To avoid technical complications, we restrict families to either one student or two, and we do not allow there to be twins (i.e., if i and j are siblings, then they are applying for different grades). This is not without loss of generality, and we explain in Appendix A complications that these assumptions avoid.[15] Each student $i \in I$ has a **strict ranking** P_i over the set of schools and being unassigned (denoted $\emptyset$). Let $\mathcal{P}$ be the set of strict rankings over $S \cup \{\emptyset\}$. For each $P_i \in \mathcal{P}$, let R_i be the weak ranking associated with P_i.[16] We require that if i and j are siblings,

[14] Typically, the highest number of points are given to a student applying to an entry-grade year with an older sibling already attending the school; the second-highest are given to students who reside in an area designated as high-performing; the third-highest are given to students whose base-school assignment is designated as overcrowded; and the fourth-highest are given to a non-entry-grade student who has an older sibling already attending the school.

[15] In the magnet assignment for Wake County Public Schools, these types comprise a relatively small subset of the applicants. For example, in the 2016-2018 school years, only 3.4% of applicants have a twin and only 2.4% of applicants have more than one sibling applying to a different grade at the same school.

[16] For each s, $s' \in S$, $s\, R_i\, s'$ if and only if $s\, P_i\, s'$ or $s = s'$.

then $P_i = P_j$ (and therefore it is unambiguous to refer to the preferences of a family). This requirement is based on the restriction imposed by WCPSS.[17] A family also has the option for their children to apply separately as non-siblings; in this case, they may express different preferences and forgo their sibling same-school guarantee. We then treat them as two separate families. With slight abuse of notation, we represent the preference of a family f with P_f where P_f is the same as the preference of each member of family f.

Each school $s \in S$ has a **capacity** vector $q_s = (q_s^g)_{g \in G} \in \mathbb{N}^{|G|}$ where q_s^g denotes school s's capacity for grade $g \in G$. In addition, each school $s \in S$ has a vector of **priority rankings** for each grade denoted by $\succ_s = (\succ_s^g)$ where $\succ_s^g$ is a strict ranking of I^g, the students applying for grade $g \in G$.[18] For each $s \in S$, and each $g \in G$, let $\succeq_s^g$ be the weak ranking associated with $\succ_s^g$.

A subset of students $J \subseteq I$ is **closed under siblings** if for each family $f \in \mathcal{F}$, either $f \subseteq J$ or $f \cap J = \emptyset$.[19] In words, if J contains a student, then it also contains that student's sibling (if any). An **assignment** μ is a function $\mu : I \to S \cup \{\emptyset\}$. We refer to the assignment of student i, the students assigned to a school s, and the students assigned to grade g at s as μ_i, μ_s, and μ_s^g, respectively. Mathematically, $\mu_s = \{i \in I : \mu_i = s\}$ and $\mu_s^g = \mu_s \cap I^g$. An assignment μ is **feasible** if for each school $s \in S$, 1) μ_s is closed under siblings, and 2) for each grade $g \in G$, $|\mu_s^g| \leq q_s^g$. In words, all siblings are assigned to the same school, possibly $\emptyset$, and no school is assigned more students in a grade than it has capacity for. We restrict our attention to feasible assignments and for expositional convenience will typically refer to them simply as assignments. When the context is clear, let $\mathcal{A}$ be the set of all possible feasible assignments for the problem at hand. For each assignment $\mu \in \mathcal{A}$, each $s \in S$, and each $J \subseteq I$ with $J \cap \mu_s = \emptyset$, school s has **available seats for $\boldsymbol{J}$ at $\boldsymbol{\mu}$** if for each $g \in \gamma(J)$, $|\{j \in J : \gamma(j) = g\}| \leq q_s^g - |\mu_s^g|$.

A problem is a tuple $(I, S, G, q, \succ, P)$, and a **mechanism** (or rule) φ recommends for each problem an assignment (that is feasible for that problem). We denote the assignment selected by mechanism φ under problem $(I, S, G, q, \succ, P)$ with

[17] This requirement is also present in, for example, Montgomery County Public Schools (Maryland) and Portland Public Schools (Oregon).

[18] Note that this is not equivalent to having a single overall capacity for each school and regarding each family with two siblings as a single student with "size" two. At any school, each grade has both a different capacity and a different priority ranking.

[19] Note that f can either be a set of two siblings or a single student.

$\varphi(I, S, G, q, \succ, P)$ and the assignment of student i with $\varphi_i(I, S, G, q, \succ, P)$.

We say student $i \in I$ (weakly) prefers assignment μ to assignment ν if $\mu_i \; P_i \; \nu_i$ ($\mu_i \; R_i \; \nu_i$). Assignment μ **Pareto-dominates** assignment ν if each $i \in I$ weakly prefers μ to ν, and some $j \in I$ prefers μ to ν. Assignment μ is **Pareto-efficient** if it is not Pareto-dominated by any other assignment. Recall that we restrict two siblings to report the same preference; thus, a manipulation by a family of two is necessarily comprised of changing both students' preferences. The next property states that no single student or pair of siblings is better off when reporting false preferences. A mechanism is **strategy-proof** if for each problem $(I, S, G, q, \succ, P)$, each $f \in \mathcal{F}$, each $P'_f = (P'_i)_{i \in f} \in \mathcal{P}^f$ (with $P'_i = P'_j$ for each $i, j \in f$), and each $i \in f$, $\varphi_i(I, S, G, q, \succ, P) \, R_i \, \varphi_i(I, S, G, q, \succ, P'_f, P_{-f})$. If f is comprised of one student, then the definition is standard. Note that since siblings have the same preference, and are never separated, there is no need to define preferences over arbitrary pairs of schools.

3 A New Criterion for Respecting Priorities

In this section, we define which coalitions are able to block an assignment. Through several illustrative examples, we highlight the issues arising from sibling guarantees. We interpret a blocking coalition as an objection by a parent (or parents) to an assignment that the school board would concur with. Without siblings and grades, student i and school s would form a blocking pair to an assignment if i prefers s to her assignment and i has a higher priority at s than one of the students j assigned to s. We can define an analogous concept in our model. To be consistent with the literature (and to differentiate from how we define *stability*) we say student $i \in I^g$ (i.e. a g^{th} grader) has **justified envy** at assignment μ if there exists a school s and a student $j \in \mu^g_s$ such that $s \, P_i \, \mu_i$ and $i \succ^g_s j$. We emphasize that a student can only have justified envy of another student who is in the same grade. The presence of siblings means that justified envy is not sufficient to constitute a blocking pair in our more general problem. Consider the following example.

Example 1 *Students i_1 and i_2 are siblings, while j and k are only children. School s has two grades, and each grade has one available seat. Each student applies to school s at their respective grade, and priorities for each grade (indicating each student's grade) are shown below.*

$\succ_s^1$	$\succ_s^2$
i_1	k
j	i_2

Consider the assignment where j and k are assigned to s. Student i_1 has justified envy of j. However, it is not feasible to assign i_1 to s unless we also assign her sibling i_2 to s. Note that i_2 does not have sufficiently high priority at s to warrant admission over k—so assigning i_1 and i_2 to s generates justified envy from k.

It is not sufficient for one sibling to be admitted to the school, as this would result in an infeasible assignment. As a result, it is not enough for one sibling to have justified envy. All siblings must have justified envy.

The presence of siblings also restricts the ability of only children to block an assignment. Consider the following example.

Example 2 *Students i_1 and i_2 are siblings, while j is an only child. School s has two grades, and each grade has one available seat. Each student applies to school s at their respective grade, and priorities for each grade are shown below.*

$\succ_s^1$	$\succ_s^2$
i_1	j
	i_2

There are three feasible assignments:

	Grade	
	1	2
Assignment 1	i_1	i_2
Assignment 2	$\emptyset$	j
Assignment 3	$\emptyset$	$\emptyset$

The only feasible assignment in which all seats are filled is the first assignment: i_1 and i_2 are assigned to s.

In Example 2, if we assign i_1 and i_2 to school s, then student j has justified envy. However, since the student she envies has a sibling, honoring j's objection would result in more than one student being removed from the school. As capacity utilization is of central importance in school assignment, we view it as undesirable to have a student's objection result in more than one student being unassigned. Therefore, we do not allow a set of students of size n to block the assignments of more than n students.

These two observations motivate our definition of blocking coalitions. It intuitively extends justified envy in two ways: a block must consider all siblings, and students one by one have justified envy. We also allow for students to block empty seats.

Definition 1 *A set of students $J = \{j_1, \ldots, j_n\}$* ***block*** *an assignment μ if there exists a set of students $K = \{k_1, \ldots, k_m\}$ (possibly empty) and school s such that:*

1. *Both J and K are closed under siblings and $|J| \geq |K|$.*

2. *For each $x \in \{1, \ldots, m\}$, $\mu_{k_x} = s$ and j_x has justified envy at μ of k_x.*[20]

3. *If $K \neq \emptyset$, then s has available seats for $j_{m+1}, \ldots, j_n$ at μ. If $K = \emptyset$, then s has available seats for J at μ.*

An assignment is ***suitable*** *if it is not blocked by any set of students. A mechanism is* suitable *if for each problem it selects an assignment that is suitable for that problem.*

In Definition 1, the first condition addresses our concerns from Examples 1 and 2. Requiring J and K to be closed under siblings and $|J| \geq |K|$ means that 1) for each student $k \in K$, there is a different student in J that has justified envy over k, 2) we do not remove more students than we are assigning, and 3) honoring the priorities of students in J results in a feasible assignment.

Note that this definition is a generalization of justified envy. If there are no siblings, conditions 1 and 3 hold trivially for any instance of justified envy. Moreover, an only child with justified envy of another only child is still sufficient to block an assignment. Thus, with no siblings, the set of *suitable* assignments coincides with set of assignments with no justified envy and no wasted seats. Each grade is independent, as the assignment in one grade does not affect possible blocking in another. Alternatively, if there is only one grade and we allow siblings (or twins), then *suitability* and no justified envy still differ.

The following example illustrates several new types of coalitions admitted under Definition 1.

[20] Under our definition, a student can only have justified envy of a student in her same grade. Therefore, this condition implies that student j_x is in the same grade as student k_x.

Example 3 *We provide three new types of blocking scenarios. Each grade of school s has one available seat. Each student applies to school s at their respective grade, and priorities for each grade are shown below.*

$\succ_s^1$	$\succ_s^2$
i_1	i_2
j_1	j_2

$\succ_s^1$	$\succ_s^2$
m	n
i_1	i_2

$\succ_s^1$	$\succ_s^2$	$\succ_s^3$	$\succ_s^4$	$\succ_s^5$
ℓ	i_1	i_2	k_1	k_2
m_1	m_2	j_1	j_2	n

In the left priority profile, students i_1 and i_2 would be able to block the assignment of j_1 and j_2 to s. Similarly, in the middle profile, m and n would be able to block the assignment of i_1 and i_2 to s. Our notion also allows for interesting combinations of individual and sibling pair students to form a blocking coalition. In the right profile, the combination of single and sibling pair students in the first row would be able to block the assignment of single and sibling pair students in the second row.

If there exists an assignment μ that has no justified envy and no seat is wasted, then μ is also *suitable*. No individual agent has justified envy, so it is not possible for any coalition to have one by one justified envy. A *suitable* assignment, however, may have an agent with justified envy. In Example 2, Assignment 1 is *suitable* but j has justified envy.

Despite our property being an intuitive generalization of justified envy, several interesting properties do not carry over. In the standard school choice problem, there is a unique assignment that has no justified envy and Pareto-dominates any other assignment that has no justified envy. This is no longer true for *suitable* assignments, as the following example shows.

Example 4 *Students i_1 and i_2 are siblings, as are students j_1 and j_2. School s has two grades, priorities for students are listed under each grade, and each grade has one available seat. Each student finds s acceptable.*

$\succ_s^1$	$\succ_s^2$
i_1	j_2
j_1	i_2

There are two suitable *assignments: assigning either i_1 and i_2 to s or j_1 and j_2 to s. There is no Pareto-ranking of these two assignments.*

When schools have rankings over sets of students that are responsive to priorities, the set of unassigned students at each *stable* assignment is the same.[21] Although we have not defined *stability* yet, it is equivalent to having no justified envy when schools have such rankings. In contrast, in the example above, we observe two *suitable* assignments where the set of unassigned students is different. Similarly, our solution concept remains distinct from others that have been proposed in the more general many-to-many matching model.[22]

Any mechanism that selects a non-wasteful assignment without justified envy satisfies an "unassigned invariance" property: adding a new student to the problem does not cause some previously unassigned student to now be assigned. A mechanism that selects from the *suitable* correspondence does not always satisfy this property.

3.1 Choice Among Sets of Students

The primary challenge in our problem is that although we know how a school ranks students within a grade, we do not know how the school would choose among applicants across grades. In practice, it is essentially left to the designer to extend priorities over individual students to selections over sets of students. Here, we consider what choice functions are consistent with priorities, capacities, and the institutional constraint that siblings are assigned to the same school.

Formally, let $\mathscr{P}(I)$ denote the powerset of I. A **choice function** for school s is a function $C_s : \mathscr{P}(I) \to \mathscr{P}(I)$ such that for every $J \subseteq I$, $C_s(J) \subseteq J$. There are two additional requirements for a choice function to be valid: for any subset of students $J \subseteq I$ 1) for every grade g, $|I^g \cap C_s(J)| \leq q_s^g$ and 2) $C_s(J)$ is closed under siblings

[21] This is referred to as the "Rural Hospital Theorem" of Roth (1986). Also see Kojima (2011), Klijn and Yazici (2014), and Martínez, Massó, Neme, and Oviedo (2000).

[22] The most common solution concept is the *setwise-stable* set (Roth 1984 and Sotomayor 1999): the set of *individually rational* matchings that cannot be blocked by a coalition that forms new links among its members, but may preserve its links to members outside of the coalition. *Setwise-stability* along with several other solution concepts such as the *individually rational core* (Sotomayor, 1999), the *pairwise-stable* set, and the *fixed-point* set are discussed in Echenique and Oviedo (2006). Konishi and Ünver (2006) consider a "farsighted" stability notion they call *credibly group-stable*. Similar notions have been studied in the matching with contracts framework by, among others, Klaus and Walzl (2009) and Hatfield and Kominers (2017). Our concept of *suitability* is in the spirit of *setwise stability*, but as *setwise stability* does not consider siblings, the two have no direct relationship. Echenique and Oviedo (2006) show that under some substitution conditions for firms and workers the *setwise-stable* set equals the *pairwise-stable* set; however, when there are siblings, these substitution conditions do not hold. Finally, *suitability* is not implied by *pairwise stability*.

in J.[23] In words, the first condition says that s does not choose more students for grade g than the grade has capacity for. The second condition says that a school must choose either all siblings or none in J. In general, for any $J \subseteq I$, we define K to be a **valid set** in J for school s if $K \subseteq J$, $|I^g \cap K| \leq q_s^g$ for all $g \in G$, and K is closed under siblings in J.

In the literature, a choice function is defined to be **responsive** to a priority order and capacity if it chooses the highest ranked students up to the school's capacity. Responsive choice functions are not consistent in general with the requirement that siblings must be assigned to the same school. This can easily be seen in Example 1.

Responsiveness captures the notion that even when there is some ambiguity regarding preferences, some comparisons are unambiguous. If we want to choose two out of four students, then it is ambiguous whether having the top and last ranked student is better or worse than having the second and third ranked students; however, it is unambiguous that having the first and second ranked students is the best possible outcome. Here, we highlight a second comparison that is unambiguous. It is clear that having your first and third favorite student is better than having your second and fourth favorite student. The assignment has improved in each position.

Definition 2 *Given two sets of students $J, K \subseteq I$, and a priority order $\succ_s = (\succ_s^g)_{g \in G}$, J **rank-dominates** K at school s if there exists an ordering of $J = \{j_1, \ldots, j_n\}$ and an ordering of $K = \{k_1, \ldots, k_m\}$ such that $n \geq m$ and for every $x \leq m$, $j_x \succeq_s^{\gamma(j_x)} k_x$ and for some $x' \leq m$, $j_{x'} \succ_s^{\gamma(j_{x'})} k_{x'}$.*

A school should never choose a rank-dominated set of students—this motivates the following definition.

Definition 3 *Given a priority ordering $\succ_s = (\succ_s^g)_{g \in G}$, a choice function C_s for school s **conforms to $\succ_s$** if for each subset of students $I' \subseteq I$, there does not exist a set of students $J \subset I'$ such that 1) J is a valid set in I' for s and 2) J rank-dominates $C_s(I')$ at s.*

Without siblings, a choice function conforming to a priority is equivalent to a choice function responding to a priority. In particular, when there are no siblings

[23] As a reminder, for each subset of students $J \subseteq I$, a subset $K \subseteq J$ is closed under siblings in $\boldsymbol{J}$ if for each $f \in \mathcal{F}$, either $f \cap J \subseteq K$ or $(f \cap J) \cap K = \emptyset$. In words, if K contains a student, then it also contains that student's sibling if they appear in J.

choosing the q_s^g-highest students for each grade $g \in G$ is both feasible and rank-dominates any other set; therefore, it must be chosen by a choice function that conforms to the ranking. This is also chosen by a responsive choice function, and so the two definitions are equivalent. In the rest of our analysis, we focus on the choice functions that conform to each school's respective priority ordering.

A choice function is **substitutable** if for each $I'' \subset I' \subseteq I$, $i \notin C_s(I'')$ implies $i \notin C_s(I')$. We show that a conforming choice function cannot be substitutable. More precisely, we call a **choice process** for school s a function which, given a set of grades, and a capacity and priority for each grade, outputs a choice function for s. We call a choice process conforming (substitutable) if the output of the choice process is always a conforming (substitutable) choice function.

Theorem 1 *If a choice process is conforming, then it is not substitutable.*

Proof. We prove by means of example. Let $G = \{1, 2\}$, $S = \{s\}$, and $I = \{i_1, i_2, j, k, l, m\}$ where i_1 and i_2 are siblings. Students $\{i_2, j, k\}$ are second graders, and $\{i_1, l, m\}$ are first graders. School s has grade-capacities $q_s^1 = 1$ and $q_s^2 = 2$. Consider the following priorities for s:

$\succ_s^1$	$\succ_s^2$
m	j
i_1	k
l	i_2

We describe several situations where it is unambiguous what must be selected by any conforming choice function (as it is the only undominated alternative). Let C_s be any conforming choice function for the problem.

We first consider the students in $I_1 = \{j, i_1, i_2, \ell\}$. The set of students $\{i_1, i_2, j\}$ rank-dominates any valid set in I_1 for s. Hence, any conforming choice function C_s selects $\{i_1, i_2, j\}$ when I_1 is considered, i.e. $C_s(I_1) = \{i_1, i_2, j\}$.

Second, we consider the students in $I_2 = I_1 \cup \{k\}$. There are two valid sets in I_2 for s that are not rank-dominated by any other valid sets in I_2: $\{j, k, \ell\}$ and $\{i_1, i_2, j\}$. If $C_s(I_2) = \{j, k, \ell\}$, then since $\ell \in I_1 \subseteq I_2$, $\ell \notin C_s(I_1)$, and $\ell \in C_s(I_2)$, C_s is not substitutable. Then, any conforming and substitutable choice function C_s selects $\{i_1, i_2, j\}$ when I_2 is considered, i.e. $C_s(I_2) = \{i_1, i_2, j\}$.

Lastly, we consider all students $I = I_2 \cup \{m\}$. The set of students $\{j, k, m\}$ rank-dominates any valid set in I for s. Hence, any conforming choice function C_s selects $\{j, k, m\}$ when I is considered, i.e. $C_s(I) = \{j, k, m\}$. Since $k \in I_2 \subseteq I$, $k \notin C_s(I_2)$, and $k \in C_s(I)$, C_s is not substitutable. ■

The violations of substitutability are different than what we might initially have expected. A school must either accept both siblings or accept neither, so the most obvious type of complements are this type of "left shoe/right shoe" complements. Although this type of complement is clear, this is not what creates a violation in Theorem 1. These complements are "hidden" in the sense of Hatfield and Kominers (2016). Since each school either accepts both students or rejects both, and since siblings submit the same ranking, a school will never receive the application of just one sibling. Rather, what is present here is what we refer to as "the enemy of my enemy is my friend" complements. Suppose a student i, who has no siblings, is rejected in favor of j, who has an older sibling. If the school receives additional applications for the higher grade, this can cause j's older sibling to be rejected, which in turn causes j to be rejected. This sequence of rejections can result in i being accepted as there is now a "free" seat at the lower grade. In this sense, even an only child can be complements with other only children, as their application can help only children at other grades be accepted.

4 Existence Via A New Family of Mechanisms

Our main result demonstrates that a *suitable* assignment always exists. We introduce a new family of *strategy-proof* mechanisms wherein each member selects a *suitable* assignment for any problem.

First, we will explain the intuition behind each mechanism in the family. At a high level, we run Deferred Acceptance (DA) one grade at a time while taking into account sibling feasibility. Our algorithm can be applied to any ordering of the grades, but for expositional ease, suppose we are assigning students to elementary schools and we have sequenced the grades in decreasing order by age (i.e. fifth grade first and kindergarten last). We first run DA on the fifth grade. Specifically, we have each fifth grader propose to her favorite school. Fix a school s. Let X be the school's fifth-grade applicants, recall that q_s^g denotes s's capacity for grade g, and suppose $|X| > q_s^5$. Before tentatively accepting a set of applicants, the school must first check if it is feasible to accept their younger siblings. For example, consider the fourth grade, and let $Y \subseteq X$ be the set of applicants who have a younger sibling entering the fourth grade. If $|Y| > q_s^4$, then the school cannot accept the set Y without violating its capacity for the fourth grade. At most q_s^4 of these students can be accepted. In other words, we are certain that the lowest $(|Y| - q_s^4)$ ranked students should be rejected. We repeat this pruning of the applicants for each lower grade: if there are $m > q_s^g$ applicants for the fifth grade with a younger sibling entering grade g, then we reject the $(m - q_s^g)$-lowest ranked from these applicants. At the conclusion of this process, it is possible to accept any subset of size q_s^5 of the remaining applicants at grade 5 (or all remaining applicants if there are less than q_s^5), as we are sure there is sufficient capacity at the lower grades to also accept each student's younger sibling.

Now we can proceed as usual: the school tentatively accepts the q_s^5-highest ranked remaining applicants for the fifth grade and permanently rejects all other applicants. The rejected students apply to their next-favorite school, and each school chooses among its applicants (both the new applicants and the applicants they tentatively accepted in a previous round) in a manner analogous to the above. When there are no new applications, the process stops and each school makes its tentative acceptances permanent. When the school accepts a student, it also accepts her younger sibling (if she has one). In this case, we assign the younger sibling to the school, remove her from consideration, and reduce the capacity of the appropriate lower grade by one. Note that if a student is unassigned, then we also remove her younger siblings from consideration at lower grades. This concludes the assignment of students to the fifth grade. The algorithm iterates this process successively for each lower grade.

The intuition for why this algorithm produces a *suitable* assignment is similar to that of the standard DA—when a school considers applicants, it is not certain which students it wishes to accept but it is certain which applicants it wishes to reject. Therefore, a school makes rejections permanent and acceptances tentative. When a school never regrets a rejection, the resulting assignment is guaranteed to be *stable*: the only students who desire the school have been rejected and the school does not regret rejecting these students. The same logic applies to our algorithm and *suitability*. A school never regrets rejecting a student since the student either did not have a high enough priority in the grade she is applying to or else too many higher-ranked applicants also had a sibling applying for the same grade. In either case, the school does not regret rejecting the student, and therefore the resulting assignment is *suitable*.[24]

We now provide a formal definition of the mechanism. Let $\triangleright$ be a strict precedence order over G; for each $g, g' \in G$, $g \triangleright g'$ means the assignment to grade g is performed before grade g'. Label grades so that $g_1 \triangleright g_2 \triangleright ... \triangleright g_{|G|}$. Let $I_1 = I$.

For any problem $(I, S, G, q, \succ, P)$ and any precedence order $\triangleright$, the **Sequential Deferred Acceptance w.r.t.** $\triangleright$ ($\boldsymbol{SDA^{\triangleright}}$) selects its outcome through the following procedure:

Step 1: Deferred Acceptance with Types for Grade $\boldsymbol{g_1}$

Step 1.0: If student $i \in I^{g_1}$ has a sibling in I^g, then she is a type g student. If student i does not have any sibling, then she is a type g_1 student.

Step 1.1: Each student $i \in I^{g_1}$ applies to her favorite choice (possibly being unassigned) under P_i. Given a school $s \in S$ and grade $g \in G$, let A_s and A_s^g be the set of all applicants and type g applicants, respectively. We iteratively determine which students are rejected. Let B_s denote the students rejected by s and initialize $B_s = \emptyset$. For each grade $g \neq g_1$, and each student $i \in A_s^g$, if

$$|\{j | j \in A_s^g \text{ and } j \succ_s^{g_1} i\}| \geq q_s^g,$$

then $i \in B_s$. In words, we consider the type g applicants, i.e. those with a sibling in grade g. Since the school has only a capacity of q_s^g for grade g, we reject a student i if there are at least q_s^g type g applicants who have higher priority than i. Note that this priority is determined by the grade the students are applying for, g_1. Let $A'_s := A_s \setminus B_s$ (the applicants that have not yet been rejected). For each student $i \in A'_s$, if

$$|\{j | j \in A'_s \text{ and } j \succ_s^{g_1} i\}| \geq q_s^{g_1},$$

[24] This also explains why we have restricted students to have at most one sibling. If not, then it is possible that a school could "regret" rejecting a student: Suppose student i has a younger sibling in the fourth grade and in the second grade. Furthermore, suppose i causes student j to be rejected because j has a younger sibling in the fourth grade, and the school has received too many applicants from students with siblings in the fourth grade. In a later round, there are sufficiently many higher-ranked students with siblings in the second grade that i is rejected. If this happens, then the school regrets rejecting j, as there is now space to accommodate j and their sibling. This regret also causes the assignment to violate *suitability*.

then $i \in B_s$. In words, among the remaining applicants, school s rejects a student if there are more remaining, higher-ranked applicants than it has capacity for at grade g_1.

For each $m > 1$:

Step $1.m$**:** Each student $i \in I^{g_1}$ applies to her favorite choice (possibly being unassigned) under P_i that has not rejected her in Steps 1.1 to $1.(m-1)$. Given a school $s \in S$ and grade $g \in G$, let A_s and A_s^g be the set of all applicants and type g applicants, respectively. For each grade $g \neq g_1$ and each student $i \in A_s^g$, if

$$|\{j | j \in A_s^g \text{ and } j \succ_s^{g_1} i\}| \geq q_s^g,$$

then $i \in B_s$. Let $A_s' := A_s \setminus B_s$. For each student $i \in A_s'$, if

$$|\{j | j \in A_s' \text{ and } j \succ_s^{g_1} i\}| \geq q_s^{g_1},$$

then $i \in B_s$.

Step 1 terminates when there are no more rejections. Each student $i \in I_1 \cap I^{g_1}$ and her sibling, if any, are assigned to the school, possibly $\emptyset$, tentatively holding i when Step 1 terminates. Each assigned student is removed, and we denote the remaining students with I_2. We update the number of remaining seats q_s^g in each school s and grade g.

Step $k > 1$**: Deferred Acceptance with Types for Grade** $\boldsymbol{g_k}$

Step $k.0$**:** If student $i \in I_k \cap I^{g_k}$ has a sibling in I^g, then she is a type g student. If student i does not have any siblings, then she is a type g_k student.

For each $m \geq 1$:

Step $k.m$**:** Each student $i \in I_k \cap I^{g_k}$ applies to her favorite choice (possibly being unassigned) under P_i that has not rejected her in Steps $k.1$ to $k.(m-1)$ (if $m = 1$, then no school has rejected her). Given a school $s \in S$ and grade $g \in G \setminus \{g_1, \ldots, g_k\}$, let A_s and A_s^g be the set of all applicants and type g

applicants, respectively. We initialize the set of rejected students at s to be empty, $B_s = \emptyset$. For each grade $g \in G \setminus \{g_1, \ldots, g_k\}$ and each student $i \in A_s^g$, if

$$|\{j | j \in A_s^g \text{ and } j \succ_s^{g_k} i\}| \geq q_s^g,$$

then $i \in B_s$. Let $A'_s := A_s \setminus B_s$. For each student $i \in A'_s$, if

$$|\{j | j \in A'_s \text{ and } j \succ_s^{g_k} i\}| \geq q_s^{g_k},$$

then $i \in B_s$.

Step k terminates when there are no more rejections. Each student $i \in I_k \cap I^{g_k}$ and her sibling, if any, are assigned to the school, possibly $\emptyset$, tentatively holding i when Step k terminates. Each assigned student is removed and we denote the remaining students by I_{k+1}. We update the number of remaining seats q_s^g in each school s and grade g.

The algorithm terminates after we run DA for all grades, i.e. after Step $|G|$.

Notice that each order $\triangleright$ of the grades defines a different mechanism. We illustrate the dynamics of $SDA^{\triangleright}$ in the following example.

Example 5 *Let $I = \{i_1, i_2, j_1, j_2, k_1, k_2, \ell_1, \ell_2, m, n, o\}$, $S = \{s_1, s_2, s_3\}$, $G = \{1, 2, 3\}$, and $1 \triangleright 2 \triangleright 3$. For each $x \in \{i, j, k, \ell\}$, students x_1 and x_2 are siblings. Let $I^1 = \{i_1, j_1, k_1, m\}$, $I^2 = \{i_2, j_2, \ell_1, n\}$, $I^3 = \{k_2, \ell_2, o\}$, $q_{s_1} = (2, 1, 1)$, $q_{s_2} = (1, 2, 1)$, and $q_{s_3} = (1, 1, 1)$. Let preferences and priorities be as below:*[25]

P_i	P_j	P_k	P_ℓ	P_m	P_n	P_o
s_1	s_1	s_2	s_3	s_1	s_1	s_3
s_2	s_2	s_3	s_2	s_3	s_3	s_2
s_3	s_3	s_1	s_1	s_2	s_2	s_1

	s_1			s_2			s_3	
$\succ_{s_1}^1$	$\succ_{s_1}^2$	$\succ_{s_1}^3$	$\succ_{s_2}^1$	$\succ_{s_2}^2$	$\succ_{s_2}^3$	$\succ_{s_3}^1$	$\succ_{s_3}^2$	$\succ_{s_3}^3$
i_1	i_2	k_2	j_1	ℓ_1	k_2	k_1	n	k_2
j_1	j_2	o	i_1	i_2	o	m	ℓ_1	o
m	ℓ_1	ℓ_2	k_1	j_2	ℓ_2	j_1	j_2	ℓ_2
k_1	n		m	n		i_1	i_2	

[25] Abusing notation, we write the preferences $P_{i_1} = P_{i_2}$ as P_i, and similarly for the other siblings.

$SDA^{\triangleright}$ finds its outcome as follows:

Step 1.0: *Students i_1 and j_1 are type 2, student k_1 is type 3, and student m is type 1.*

Step 1.1: *Students i_1, j_1, and m apply to s_1, and student k_1 applies to s_2. School s_1 first considers type 2 applicants, $A^2_{s_1} = \{i_1, j_1\}$, and rejects j_1, since $q^2_{s_1} = 1$ and $i_1 \succ^1_{s_1} j_1$. Then, two remaining applicants, i_1 and m, are tentatively accepted by s_1. Since k_1 is the only applicant for s_2, and it has an available seat at grade 3, s_2 tentatively accepts k_1.*

Step 1.2: *Students i_1 and m apply to s_1, and students k_1 and j_1 apply to s_2. Only student k_1 is rejected from s_2, and all the other students are tentatively accepted.*

Step 1.3: *Students i_1 and m apply to s_1, student j_1 applies to s_2, and student k_1 applies to s_3. Step 1 terminates since no student is rejected. Students i_1, i_2, and m are assigned to s_1, students j_1 and j_2 are assigned to s_2, and students k_1 and k_2 are assigned to s_3. The updated quotas are: $q_{s_1} = (0, 0, 1)$, $q_{s_2} = (0, 1, 1)$ and $q_{s_3} = (0, 1, 0)$.*

Step 2.0: *Student ℓ_1 is type 3, and student n is type 2. All other students in I^2 were assigned in Step 1.*

Step 2.1: *Students ℓ_1 and n apply to s_3 and s_1, respectively. Student ℓ_1 is rejected because there is no remaining seat for her sibling for grade 3, and student n is rejected because all seats of s_1 for grade 2 were allocated in Step 1.*

Step 2.2: *Students ℓ_1 and n apply to s_2 and s_3, respectively. Step 2 terminates since no student is rejected. Student ℓ_1 and ℓ_2 are assigned to s_2 and student n is assigned to s_3. The updated quotas are: $q_{s_1} = (0, 0, 1)$, $q_{s_2} = (0, 0, 0)$, and $q_{s_3} = (0, 0, 0)$.*

Step 3.0: *Student o is type 3. All other students in I^3 were assigned in Steps 1 and 2.*

Step 3.1: *Student o applies to s_3 and is rejected by that school since all seats in s_3 for grade 3 were allocated in Step 1.*

Step 3.2: *Student o applies to s_2 and is rejected by that school since all seats in s_2 for grade 3 were allocated in Step 2.*

Step 3.1: *Student o applies to s_1 and is tentatively accepted. Step 3 terminates since no student is rejected. Student o is assigned to s_1.*

Now we are ready to present the properties of $SDA^{\triangleright}$.

Theorem 2 *For each precedence order $\triangleright$, the $SDA^{\triangleright}$ is* suitable *and* strategy-proof.

Proof. Consider an arbitrary problem $(I, S, G, q, \succ, P)$. Let $\triangleright$ be such that $g_\ell \triangleright g_{\ell+1}$ for each $\ell \in \{1, ..., |G|-1\}$, and $\mu = SDA^{\triangleright}(I, S, G, q, \succ, P)$.

Suitability: We prove by induction and show that μ cannot be blocked by a valid set of students. We first consider the set of students who are assigned in Step 1 of the $SDA^{\triangleright}$ algorithm, i.e. the students in I^{g_1} and their siblings from other grades.

Suppose that there exists student $i \in I^{g_1}$, and school $s \in S$ such that $s \; P_i \; \mu_i$, and either 1) there is an unfilled seat at s at g_1, or 2) $i \succ_s^{g_1} j$ for some $j \in I^{g_1} \cap \mu_s$. If there is an unfilled seat and i has no sibling, then $|\mu_s \cap I^{g_1}| < q_s^{g_1}$. By definition, s tentatively accepts weakly more students from grade g_1 at each step. Thus, at each Step 1.m, if i applies to s, then i is not rejected—contradicting $\mu_i \neq s$. If s does not have an unfilled seat at g_1, i has no sibling, and there is $j \in I^{g_1} \cap \mu_s$ with $i \succ_s^{g_1} j$, then rejection of i implies that there are $q_s^{g_1}$ students with higher priority than i at s, and accepting j contradicts the definition of $SDA^{\triangleright}$.

For each other case, since i was rejected, she must have a sibling applying for some grade $g' \neq g_1$. Moreover, by definition, all available seats of s at grade g' are assigned to students who have a sibling also assigned to s and have higher priority than i under $\succ_s^{g_1}$. Therefore, μ cannot be blocked by a set of students that includes those students assigned in Step 1. Note that the set of students assigned in Step 1 includes all students in I^{g_1} and their siblings.

Suppose μ cannot be blocked by a set of students that includes those students assigned in the first k steps of $SDA^{\triangleright}$. We consider the students assigned in Step $k+1$. By our inductive hypothesis and Definition 1, it suffices to consider the subproblem with the updated quotas that we have in Step $k+1$ of the $SDA^{\triangleright}$ algorithm. Suppose that there exists a remaining student $i \in I^{g_{k+1}}$, and a school $s \in S$ such that $s \; P_i \; \mu_i$ and either 1) there is an unfilled seat at s at g_{k+1}, or 2) $i \succ_s^{g_{k+1}} j$ for some $j \in I^{g_{k+1}} \cap \mu_s$.

If i has no sibling, then the reasoning explained above holds. Additionally, in the case that s does not have an unfilled seat, and there is $j \in I^{g_{k+1}} \cap \mu_s$ with $i \succ_s^{g_{k+1}} j$, it is possible that j has been assigned in some Step $k' < k+1$ and j has a sibling in $I^{g_{k'}}$. By the induction hypothesis, no student in $I^1, \ldots, I^k$ is part of a valid set of students that blocks μ. Thus, since any blocking set of students must include a student that has justified envy of the sibling of j in $I^{g_{k'}}$, μ is not blocked.

Suppose that i has a sibling applying for some grade $g_{k+1} \triangleright g'$. If there is an unfilled seat at s at g_{k+1}, then since i was rejected, by definition, all available seats of s at grade g' are filled with students whose siblings have higher priority than i under $\succ_s^{g_{k+1}}$ (and they are also assigned to s). If seats at s at grade g_{k+1} are filled, then by definition, either all available seats of s at grade g' are filled with students whose siblings have higher priority than i under $\succ_s^{g_{k+1}}$ and they are also assigned to s or, j has been assigned in some Step $k' < k+1$ and j has a sibling in $I^{g_{k'}}$. For the former case, as explained above μ cannot be blocked by a valid set of students including i. For the latter case, by the induction hypothesis and as explained above, μ is not blocked.

Therefore, μ cannot be blocked by a group of students $\bar{I}$ that includes the students assigned in Step $k+1$.

Since the algorithm for $SDA^{\triangleright}$ ends after a finite number of steps ($|G|$), we are done.

Strategy-proofness: Consider Step 1 of the algorithm for $SDA^{\triangleright}$. We will show that students in $I_1 \cap I^{g_1}$ cannot manipulate the mechanism. Let $t : I_1 \cap I^{g_1} \to G$ identify for each agent a type as follows: For each $i \in I_1 \cap I^{g_1}$, let $t(i)$ be g_1 if i has no siblings, and g_k if i has a sibling in grade $g_k \in G$. For each $s \in S$, and each $g \in G$, let $\hat{q}_s^g = \min\{q_s^{g_1}, q_s^g\}$. Then, the tuple $(I_1 \cap I^{g_1}, S, (q_s^{g_1})_{s\in S}, (\succ_s^{g_1})_{s\in S}, P_{I_1\cap I^{g_1}}, t, (\hat{q}_s^g)_{g\in G, s\in S})$ forms a school choice problem with type-specific quotas as in Abdulkadiroğlu and Sönmez (2003) (what they refer to as controlled choice with flexible constraints). The assignment at the end of Step 1 of the $SDA^{\triangleright}$ is then the same as the outcome of their modification of the DA mechanism for $I_1 \cap I^{g_1}$. Notice that no agent in $I_1 \cap I^{g_1}$ is involved in the algorithm again. By Proposition 5 of Abdulkadiroğlu and Sönmez (2003), no agent in $I_1 \cap I^{g_1}$ can manipulate their mechanism, and the same follows for the $SDA^{\triangleright}$. Let I_2 be the remaining students after Step 1 is implied. No agent in I_2 can affect the assignment given by Step 1.

We repeat this procedure for Step 2; the remaining steps are similar. Let $t : I_2 \cap I^{g_2} \to G$ identify for each agent a type as follows: For each $i \in I_2 \cap I^{g_2}$, let $t(i)$ be g_2 if i has no siblings, and g_k if i has a sibling in grade $g_k \in G \backslash \{g_1\}$. For each grade $g \in G \backslash \{g_1\}$, let X_s^g be the number of students assigned to grade g at school s in Step 1 of the $SDA^{\triangleright}$. For each $s \in S$, and each $g \in G \backslash \{g_1\}$, let $\hat{q}_s^g = \min\{q_s^{g_2} - X_s^{g_2}, q_s^g - X_s^g\}$. Then, the tuple $(I_2 \cap I^{g_2}, S, (q_s^{g_2} - X_s^{g_2})_{s\in S}, (\succ_s^{g_2})_{s\in S}, P_{I_2\cap I^{g_2}}, t, (\hat{q}_s^g)_{g\in G, s\in S})$ forms a school choice problem with type-specific quotas. By repeating the same reasoning for grade g_2 and all subsequent, we can show that no student can manipulate the $SDA^{\triangleright}$.

■

Although $SDA^{\triangleright}$ is *suitable* and *strategy-proof*, it is not *Pareto-efficient*. This is not surprising, as in the special case of the standard school choice problem, the DA mechanism is not *Pareto-efficient* and is equivalent to the $SDA^{\triangleright}$.[26]

We may ask whether there exists a *strategy-proof* and *suitable* mechanism whose assignment is not Pareto-dominated by any other *suitable* assignment in each problem. Indeed, in the standard problem (without siblings), the DA is the only such mechanism if we consider no justified envy (Gale and Shapley, 1962; Dubins and Freedman, 1981; Roth, 1982). Unfortunately, our next proposition demonstrates that for our environment and *suitability*, no such mechanism exists.

Proposition 1 *There is no* strategy-proof *and* suitable *mechanism that selects an assignment Pareto-undominated by any other* suitable *assignment in any problem.*

Proof. Suppose by contradiction that there is such a mechanism; call it φ. Let $I = \{i_1, i_2, j_1, j_2, k_1, k_2, \ell_1, \ell_2\}$, $S = \{s_1, s_2, s_3\}$, $G = \{1, 2\}$, and for each $x \in \{i, j, k\}$, x_1 and x_2 are siblings in grades 1 and 2, respectively. At each grade, each school has a capacity of one. Let preferences (of families) and priorities be as below:

P_i	P_j	P_k	P_ℓ
s_2	s_1	s_1	s_3
s_1	s_2	$\emptyset$	s_1

s_1		s_2		s_3	
$\succ^1_{s_1}$	$\succ^2_{s_1}$	$\succ^1_{s_2}$	$\succ^2_{s_2}$	$\succ^1_{s_3}$	$\succ^2_{s_3}$
ℓ_1	ℓ_2	j_1	j_2	k_1	k_2
i_1	i_2	i_1	i_2	ℓ_1	ℓ_2
k_1	j_2				
j_1	k_2				

The only *suitable* assignment that is not Pareto-dominated by some other *suitable* assignment assigns i_1 and i_2 to s_2, j_1 and j_2 to s_1, k_1 and k_2 to $\emptyset$, and ℓ_1 and ℓ_2 to s_3.

Consider a second problem that is the same except that family j reports $P'_j = P'_{j_1} = P'_{j_2}$ where $s_1 \, P'_j \, \emptyset \, P'_j \, s_2 \, P'_j \, s_3$. By *strategy-proofness*, j_1 and j_2 receives the same assignment; by *suitability*, each other agent receives the same assignment.

Finally, consider a third problem that is the same as the second except that family k reports $P'_k = P'_{k_1} = P'_{k_2}$ where $s_1 \, P'_k \, s_3 \, P'_k \emptyset$. By *strategy-proofness*, φ does not assign

[26] Also, in the standard school choice problem, there is no mechanism that has no justified envy and is *Pareto-efficient* as pointed out by Balinski and Sönmez (1999).

k_1 and k_2 to s_1. By *suitability*, φ assigns s_3 to k_1 and k_2. By feasibility and *suitability*, φ assigns s_1 to ℓ_1 and ℓ_2, and s_2 to i_1 and i_2. From this assignment, notice that k and ℓ are better off swapping schools. If they do, this forms a *suitable* and Pareto-dominating assignment—contradicting the assumptions on φ. ■

Finally, one may ask whether or not this family of mechanisms favors siblings or only-children when compared to the grade-by-grade DA (where for each grade, we run the DA separately without taking sibling guarantees into account). In general, this comparison is ambiguous. Siblings benefit from the $SDA^{\triangleright}$ assignment in the case where the older sibling has a high enough priority to be accepted by a school (under the grade-by-grade DA), but the younger sibling's priority is low enough to cause her to be rejected. Under $SDA^{\triangleright}$, both siblings are accepted. Note that while the only-children at the grade of the younger sibling were disadvantaged there, they may benefit in the reverse scenario. If an older sibling has relatively low priority and the younger sibling has relatively high priority, then under $SDA^{\triangleright}$ they are both rejected. This rejection thus benefits only-children at the lower grade.

5 Comparison of the Naive and $SDA^{\triangleright}$ Assignments In Wake County

Using data from the WCPSS magnet program assignment for the 2018-2019 school year, we present a comparison of the assignment outcomes of the grade-by-grade DA (the "naive" assignment) and the $SDA^{\triangleright}$ mechanisms (Table 2). A total of 6,994 students applied for seats across grades K-12. For each level of schooling (elementary, middle, and high school), the grade precedence order $\triangleright$ starts with the highest grade and continues with the next highest grade. We augment the $SDA^{\triangleright}$ to accommodate for twins and family group sizes of more than two.[27,28]

The key finding is that the naive assignment separates siblings at a rate 19 times that of the $SDA^{\triangleright}$. Furthermore, our new assignment also demonstrates that we can

[27]Specifically, for twins we treat each as a single student in the algorithm. Any separation of twins are recorded as a sibling mismatch. For each student with more than one sibling, when we assign the oldest sibling to a school, we also assign her younger siblings to the same school.

[28]In the applications of both mechanisms, we take specific requirements of WCPSS into account, e.g. reserving a portion of seats for students whose parents are not college graduates and assigning 10% of seats via lottery.

Table 2: WCPSS 2018-2019 Comparison of Naive and $SDA^{\triangleright}$ Assignments

Grade	**Total Applicants**	**Applicants with Siblings**	**Assigned w/ Sibling**		**Mismatched Siblings**		**Justified Envy**	
			Naive	$SDA^{\triangleright}$	Naive	$SDA^{\triangleright}$	Naive	$SDA^{\triangleright}$
K	1,342	374	247	250	42	0	0	0
1	308	107	4	7	7	0	0	2
2	260	88	5	7	13	0	0	1
3	269	100	5	6	24	0	0	1
4	234	90	4	4	13	0	0	0
5	154	65	1	1	4	0	0	0
$\boldsymbol{K-5}$	**2,567**	**824**	**266**	**275**	**103**	**0**	**0**	**4**
6	1,785	204	156	163	17	7	0	7
7	164	28	10	11	1	0	0	0
8	121	29	16	19	6	0	0	0
$\boldsymbol{6-8}$	**2,070**	**261**	**182**	**193**	**24**	**7**	**0**	**7**
9	1,871	215	159	174	22	2	0	3
10	336	36	6	10	15	0	0	4
11	115	15	4	5	4	0	0	0
12	35	10	4	7	3	0	0	0
$\boldsymbol{9-12}$	**2,357**	**276**	**173**	**196**	**44**	**2**	**0**	**7**
$\boldsymbol{K-12}$	**6,994**	**1,361**	**621**	**664**	**171**	**9**	**0**	**18**

implement the institutional constraint of keeping siblings together with very few students causing priority violations—only 18 such instances occurred. In Wake County, this may be because siblings have the same priorities; so "criss-crossed" priorities such as Example 4 are not causing violations. Nonetheless, more complicated situations with analogous tensions still appear. In general, other school districts' siblings may have different priorities based on characteristics such as exam scores, grades, etc.

The naive assignment does not meet the WCPSS policy of assigning all siblings to the same school. This is not surprising, as the mechanism does not take into account the existence of siblings at all. In total, over all grades K-12, 171 students are separated from their siblings, or 12.6% of the total number of applicants with siblings (1,361). This percentage is similar for all sub-levels of schooling: elementary (12.5%), middle (9.2%), and high school (15.9%). Much of the mismatch comes from students with siblings in entry grades. The total number of siblings who are all assigned to the same school (as opposed to all being unassigned) is also greater in the $SDA^{\triangleright}$. Over all grades K-12, there are 7% more students assigned under $SDA^{\triangleright}$ than in the naive assignment (621 versus 664 of the total number of siblings).

We now examine the extent to which the constraint of keeping siblings together causes instances of justified envy. We count the number students j for which there is another student i who has justified envy of j at the $SDA^{\triangleright}$ assignment. For the entire school district, there were only 18 such instances. Violation of priorities is mainly caused by the expected situation: a younger sibling follows an older sibling, but individually has a lower priority than some other students who would rather attend. Of all instances, six are caused by students who are part of a sibling triple, and the remaining are caused by students who are part of a sibling pair.

Finally, the reader will note in Table 2 that there are mismatched siblings under $SDA^{\triangleright}$. All mismatched siblings under $SDA^{\triangleright}$ are due to twins. Our algorithm was defined under the assumption that there are no twins, which of course does not hold in practice.[29] In practice, it would be straightforward to ensure that twins are assigned to the same school. In our implementation, we treated twins as individual students. We used this augmentation for two reasons. One, it is simple and transparent. Two, the resulting mismatch may be seen as a conservative estimate regarding the efficacy of the $SDA^{\triangleright}$, as we could make further changes to ensure that twins always attend the same school and thereby decrease the number of mismatched siblings.

6 Conclusion

We demonstrate that the seemingly trivial institutional constraint of keeping siblings together actually forces a careful consideration of grades and sibling structure. From a theoretical standpoint, sibling constraints i) render the previous requirement of no justified envy inadequate, ii) introduce interesting complementarities in terms of school selections across grades, and iii) require the development of new solution concepts and mechanisms. There also remain open questions regarding more general sibling structures (e.g. twins, more than two siblings, etc.).

Most school districts have various types of sibling constraints. We argue that for many, adopting the $SDA^{\triangleright}$ mechanism (or one similar to) is an appropriate, systematic, and fair alternative to heuristic measures used to "patch" the naive assignment

[29] In 2018, the percentage of students with one sibling, students with a twin and students with two or more siblings was 17%, 4%, and 2.6%, respectively.

(to meet the constraints).

References

[1] Atila Abdulkadiroğlu, Parag A Pathak, and Alvin E Roth. The new york city high school match. *American Economic Review*, 95(2):364–367, 2005.

[2] Atila Abdulkadiroğlu, Parag A Pathak, and Alvin E Roth. Strategy-proofness versus efficiency in matching with indifferences: Redesigning the nyc high school match. *American Economic Review*, 99(5):1954–78, 2009.

[3] Atila Abdulkadiroğlu, Parag A. Pathak, Alvin E. Roth, and Tayfun Sönmez. The boston public school match. *American Economic Review*, 95(2):368–371, May 2005.

[4] Atila Abdulkadiroğlu and Tayfun Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

[5] Azar Abizada. Stability and incentives for college admissions with budget constraints. *Theoretical Economics*, 11(2):735–756, 2016.

[6] Azar Abizada and Umut Dur. College admissions with complementarities. 2018.

[7] Itai Ashlagi, Mark Braverman, and Avinatan Hassidim. Stability in large matching markets with complementarities. *Operations Research*, 62(4):713–732, 2014.

[8] Orhan Aygün and Tayfun Sönmez. Matching with contracts: Comment. *American Economic Review*, 103(5):2050–51, 2013.

[9] Michel Balinski and Tayfun Sönmez. A tale of two mechanisms: student placement. *Journal of Economic Theory*, 84(1):73–94, 1999.

[10] Lester E Dubins and David A Freedman. Machiavelli and the gale-shapley algorithm. *The American Mathematical Monthly*, 88(7):485–494, 1981.

[11] Umut Dur, Robert G Hammond, and Onur Kesten. Sequential school choice: Theory and evidence from the field and lab. 2018.

[12] Umut Dur, Robert G Hammond, and Thayer Morrill. Identifying the harm of manipulable school-choice mechanisms. *American Economic Journal: Economic Policy*, 10(1):187–213, 2018.

[13] Umut Dur and Thomas Wiseman. School choice with neighbors. *Journal of Mathematical Economics*, 83:101–109, 2019.

[14] Bhaskar Dutta and Jordi Massó. Stability of matchings when individuals have preferences over colleagues. *Journal of Economic Theory*, 75(2):464–475, 1997.

[15] Federico Echenique and Jorge Ovideo. A theory of stability in many-to-many matching markets. *Theoretical Economics*, 1:233–273, 2006.

[16] Federico Echenique and M Bumin Yenmez. How to control controlled school choice. *American Economic Review*, 105(8):2679–94, 2015.

[17] Aytek Erdil and Haluk Ergin. What's the matter with tie-breaking? improving efficiency in school choice. *American Economic Review*, 98(3):669–89, 2008.

[18] David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

[19] John William Hatfield and Fuhito Kojima. Substitutes and stability for matching with contracts. *Journal of Economic Theory*, 145(5):1704–1723, 2010.

[20] John William Hatfield and Scott Kominers. Hidden substitutes. 2016.

[21] John William Hatfield and Scott Duke Kominers. Contract design and stability in many-to-many matching. *Games and Economic Behavior*, 101:78–97, 2017.

[22] John William Hatfield, Scott Duke Kominers, and Alexander Westkamp. Stability, strategy-proofness, and cumulative offer mechanisms. *Review of Economic Studies*, 2020. Forthcoming.

[23] Yuichiro Kamada and Fuhito Kojima. Stability and strategy-proofness for matching with constraints: A problem in the japanese medical match and its solution. *American Economic Review*, 102(3):366–70, May 2012.

[24] Yuichiro Kamada and Fuhito Kojima. Efficient matching under distributional constraints: Theory and applications. *American Economic Review*, 105(1):67–99, 2015.

[25] John Kennes, Daniel Monte, and Norovsambuu Tumennasan. The day care assignment: A dynamic matching problem. *American Economic Journal: Microeconomics*, 6(4):362–406, 2014.

[26] Onur Kesten. School choice with consent. *The Quarterly Journal of Economics*, 125(3):1297–1348, 2010.

[27] Bettina Klaus and Flip Klijn. Stable matchings and preferences of couples. *Journal of Economic Theory*, 121(1):75–106, 2005.

[28] Flip Klijn and Ayşe Yazıcı. A many-to-many "rural hospital theorem". *Journal of Mathematical Economics*, 54:63–73, 2014.

[29] Fuhito Kojima. The "rural hospital theorem" revisited. *International Journal of Economic Theory*, 8(1):67–76, 2012.

[30] Fuhito Kojima, Parag A Pathak, and Alvin E Roth. Matching with couples: Stability and incentives in large markets. *The Quarterly Journal of Economics*, 128(4):1585–1632, 2013.

[31] Scott Duke Kominers and Tayfun Sönmez. Matching with slot-specific priorities: Theory. *Theoretical Economics*, 11(2):683–710, 2016.

[32] Morimitsu Kurino. House allocation with overlapping generations. *American Economic Journal: Microeconomics*, 6(1):258–89, 2014.

[33] Ruth Martínez, Jordi Massó, Alejandro Neme, and Jorge Oviedo. Single agents and the set of many-to-one stable matchings. *Journal of Economic Theory*, 91(1):91–105, 2000.

[34] Alvin E Roth. The economics of matching: Stability and incentives. *Mathematics of Operations Research*, 7(4):617–628, 1982.

[35] Alvin E Roth. Stability and polarization of interests in job matching. *Econometrica: Journal of the Econometric Society*, pages 47–57, 1984.

[36] Alvin E Roth. On the allocation of residents to rural hospitals: a general property of two-sided matching markets. *Econometrica*, 54(2):425–427, 1986.

[37] Alvin E Roth and Elliott Peranson. The redesign of the matching market for american physicians: Some engineering aspects of economic design. *American Economic Review*, 89(4):748–780, 1999.

[38] Tayfun Sönmez and Tobias B Switzer. Matching with (branch-of-choice) contracts at the united states military academy. *Econometrica*, 81(2):451–488, 2013.

[39] M Bumin Yenmez. A college admissions clearinghouse. *Journal of Economic Theory*, 176:859–885, 2018.

Appendix A: Extensions

We discuss two extensions to the model where we allow for 1) twins, and 2) more than two siblings in each family.

Suppose that we allow for twins and require they cannot be separated. The following example shows that there may not exist a *suitable* assignment.

Example 6 *Let* $I = \{i_1, i_2, j, k, \ell\}$, $S = \{s_1, s_2, s_3\}$, $G = \{1\}$, $q^1_{s_1} = 2$, *and* $q^1_{s_2} = q^1_{s_3} = 1$. *Here,* i_1 *and* i_2 *are twins. Let preferences and priorities be as below:*

P_i	P_j	P_k	P_ℓ
s_1	s_2	s_1	s_3
$\emptyset$	s_1	s_2	s_1
	s_3	$\emptyset$	$\emptyset$

$\succ^1_{s_1}$	$\succ^1_{s_2}$	$\succ^1_{s_3}$
j	k	j
ℓ	j	ℓ
i_1		
i_2		
k		

Let μ *be a* suitable *assignment. If* k *is assigned to either* s_3 *or* $\emptyset$*, then* k *blocks* μ*. So in any* suitable *assignment,* k *is assigned to either* s_1 *or* s_2*.*

Case 1: k *is assigned to* s_1 *in* μ*. Then,* j *is assigned to* s_2*, otherwise* j *blocks at* s_2*. By feasibility,* i_1 *and* i_2 *cannot be separated, so they are unassigned at* μ*. Then,* ℓ *is assigned to* s_3*, otherwise* ℓ *blocks at* s_3*. Notice that* i_1 *and* i_2 *now block* k *at* s_1*, contradicting the* suitability *of* μ*.*

Case 2.1: k *is assigned to* s_2 *in* μ*, and* j *is assigned to* s_1*. Then,* ℓ *is assigned to* s_3*, otherwise* ℓ *blocks at* s_3*. By feasibility,* i_1 *and* i_2 *are unassigned. Notice that* k *blocks at* s_1*, contradicting the* suitability *of* μ*.*

Case 2.2: k *is assigned to* s_2 *in* μ*, and* ℓ *is assigned to* s_1*. By feasibility,* i_1 *and* i_2 *are unassigned. Then,* j *is assigned to* s_1*, otherwise* j *blocks at* s_1*. Notice that* ℓ *blocks at* s_3*, contradicting the* suitability *of* μ*.*

Case 2.3: k *is assigned to* s_2 *in* μ*, and* i_1 *and* i_2 *are assigned to* s_1*. Then,* j *is assigned to* s_3*, otherwise* j *blocks at* s_3*. This leaves* ℓ *unassigned. Notice that* j *and* ℓ *block* i_1 *and* i_2 *at* s_1*, contradicting the* suitability *of* μ*.*

Similarly, suppose that we allow for families of size larger than two and require that they cannot be separated. The following example shows that the natural extension of our algorithm that accounts for three siblings may not result in a *suitable* assignment.

Example 7 *Let* $I = \{i_1, i_2, j_1, j_2, j_3, k_1, k_3\}$*,* $S = \{s_1, s_2\}$*,* $G = \{1, 2, 3\}$*,* $q_{s_1} = (2, 1, 1)$*, and* $q_{s_2} = (1, 1, 1)$*. For each* $h_x \in I$*, let* $\gamma(h_x) = x$*. Let preferences and priorities be as below:*

P_i	P_j	P_k	$\succ^1_{s_1}$	$\succ^2_{s_1}$	$\succ^3_{s_1}$	$\succ^1_{s_2}$	$\succ^2_{s_2}$	$\succ^3_{s_2}$
s_2	s_1	s_1	i_1	i_2	j_3	j_1	j_2	j_3
s_1		s_2	j_1	j_2	k_3	k_1	i_2	k_3
			k_1			i_1		

Let $\triangleright$ *be such that* $1 \triangleright 2 \triangleright 3$*. We follow the steps for the* $SDA^{\triangleright}$*, and extend the algorithm naturally when the three siblings appear. We skip Step 1.0.*

Step 1.1: j_1 *and* k_1 *apply to* s_1*, and* i_1 *applies to* s_2*. Since it is not feasible to assign* $\{j_3, k_3\}$ *to grade 3 at* s_1*, school* s_1 *rejects the lowest priority student in* $\{j_1, k_1\}$ *according to* $\succ^1_{s_1}$*, which is* k_1*. Schools* s_1 *and* s_2 *tentatively accept* j_1 *and* i_1*, respectively.*

Step 1.2: j_1 *applies to* s_1*, and* i_1 *and* k_1 *apply to* s_2*. Since it is not feasible to assign* $\{i_1, k_1\}$ *to grade 1 at* s_2*, school* s_2 *rejects the lowest priority student in* $\{i_1, k_1\}$ *according to* $\succ^1_{s_2}$*, which is* i_1*. Schools* s_1 *and* s_2 *tentatively accept* j_1 *and* k_1*, respectively.*

Step 1.3: i_1 *and* j_1 *apply to* s_1*, and* k_1 *applies to* s_2*. Since it is not feasible to assign* $\{i_2, j_2\}$ *to grade 2 at* s_1*,* j_1 *is rejected. Schools* s_1 *and* s_2 *tentatively accept* i_1 *and* k_1*, respectively.*

The final assignment is $\mu_{i_1} = \mu_{i_2} = s_1$*,* $\mu_{k_1} = \mu_{k_3} = s_2$*, and* $\mu_{j_1} = \mu_{j_2} = \mu_{j_3} = \emptyset$*. Note that we maintain the "sequential" nature of* $SDA^{\triangleright}$ *by only using* $\succ^1_{s_1}$ *and* $\succ^1_{s_2}$ *at Step 1. The only extra comparison required is at Step 1.1 and Step 1.3, which have "reasonable" rejections in the spirit of* $SDA^{\triangleright}$*. Since it is feasible for* k_1 *and* k_3 *to attend* s_1*, this assignment is blocked, and thus not* suitable*.*

Appendix B: Modelling as Matching with Contracts

In this section, we show that there is a natural way to model school assignment with siblings as a "many-to-many" matching with contracts.[30] We may think of each

[30] The matching with contracts problem under a many-to-one framework was first introduced by Hatfield and Milgrom (2005). Hatfield and Kominers (2017) formulate the problem under a many-to-many framework. Yenmez (2018) studies the college admissions problem with early decisions, and is the only other paper that we are aware of that also examines a practical application of many-to-many matching with contracts.

seat at a school as a "contract" between a parent and the school. The terms of the contract specify the student and the grade level; therefore, a family may have multiple contracts with the same school.

We briefly discuss the many-to-one matching with contracts literature, as we will use results therein to prove the main theorem in this section. Hatfield and Milgrom (2005) showed that substitutability of the firms' choice functions is sufficient to guarantee the existence of a *stable* assignment.[31] As such, the property is a central point of discussion in this literature.[32][33] If in addition the Law of Aggregate Demand is satisfied, then the mechanism defined by the Cumulative Offer Process is *strategy-proof* selects a *stable* assignment.

For the more general many-to-many framework, Hatfield and Kominers (2017) show that the domain of substitutable choice functions is a maximal one that guarantees the existence of *stable* assignments.

We contribute to this literature by identifying a real-life mechanism design application where the choice functions of *both* sides violate substitutability, and yet *stable* assignments still exist. Furthermore, we will show that the schools' choice functions do not satisfy the Law of Aggregate Demand, yet there is still a *strategy-proof* mechanism that selects a *stable* assignment.

Let $(I, S, G, q, \succ, P)$ be a problem as before with minor differences. In particular, under this model I is the set of parents and P is the rankings of parents over schools. We allow a school to prioritize parents differently for different grades, e.g. $i \succ_s^g j \succ_s^{g'} i$. We denote the subset of grades parent i is applying to by $\gamma(i)$. For each grade $g \in G$, let I^g denote the set of parents with a child in grade g, i.e., $I^g = \{i \in I : g \in \gamma(i)\}$. Consistent with Section 2, we assume $|\gamma(i)| \leq 2$ for all $i \in I$. Let $X \subseteq I \times S \times G$ be

[31] Aygün and Sönmez (2013) point out that, in addition to substitutability, choice functions must satisfy *irrelvance of removed contracts* for existence.

[32] Hatfield and Kojima (2010) show the extent to which substitutability can be relaxed while maintaining existence and define two progressively weaker notions—unilateral substitutability and bilateral substitutability. Both guarantee the existence of a *stable* assignment, but for the latter, optimality for any side of agents as well as a doctor-*stategy-proof* mechanism is lost. Hatfield et. al (2020) also provide new interesting properties—*observable substitutability*, *observable size monotonicity*, and *nonmanipulability via contractual terms*—that characterize when a cumulative offer mechanism is *strategy-proof* and *stable*.

[33] Abizada (2016) and Abizada and Dur (2018) study the college admissions problem with stipend offers. Despite the presence of complementarities causing the failure of bilateral substitutability, they show the existence of a *pairwise stable* and *strategy-proof* mechanism. In *pairwise stability*, blocking coalitions are restricted to be a single student and school pair.

the finite set of possible contracts such that

$$X = \bigcup_{i \in I} \bigcup_{s \in S} \bigcup_{g \in \gamma(i)} (i, s, g).$$

Let $\mathrm{i}(x)$, $\mathrm{s}(x)$, and $\mathrm{g}(x)$ denote the parent, school, and grade related to contract x, respectively. Similarly, for any subset of contracts $Y \subseteq X$, $\mathrm{i}(Y)$, $\mathrm{s}(Y)$, and $\mathrm{g}(Y)$ denote the set of parents, set of schools and set of grades related to contracts in Y, respectively. Given a subset of contracts $Y \subseteq X$, let Y_i, Y_s, and Y^g denote the contracts related to parent i, school s, and grade g, respectively.

An **assignment** $Y \subset X$ is a set of contracts such that each parent $i \in I$ appears in at most $|\gamma(i)|$ contracts and each (s, g) pair appears in at most q_s^g contracts. We refer to Y_i as i's assignment. A mechanism φ recommends for each possible problem an assignment.

For each $a \in I \cup S$, choice of agent a, denoted by $C_a(\cdot)$, is a function such that for each $Y \subseteq X$, $C_a(Y) \subseteq Y_a$. Here, $C_a(Y) = \emptyset$ means that a rejects all contracts in Y.

For any set of contracts, the chosen subset for each parent is determined as follows: Parent i first considers only schools that give each of their children a seat, then, from these schools, chooses their most preferred one. Formally, given a subset of contracts $Y \subseteq X$, let $\mathrm{s}_i(Y)$ be the set of schools such that $(i, s, g) \in Y$ for all $g \in \gamma(i)$ and $s \, P_i \, \emptyset$. Each parent $i \in I$ has the following choice function:

$$C_i(Y) = \begin{cases} \{\bigcup_{g \in \gamma(i)} (i, s, g) \subseteq Y : s \, R_i \, s' \text{ for all } s, s' \in \mathrm{s}_i(Y)\} & \text{if } \mathrm{s}_i(Y) \neq \emptyset \\ \emptyset & \text{if } \mathrm{s}_i(Y) = \emptyset. \end{cases}$$

Let $Re_i(Y) = Y \setminus C_i(Y)$ be the set of rejected contracts by parent i from Y.

We now turn to schools' choice functions. Given a set of contracts (possibly spanning multiple grades) and its capacities and priorities, a school selects a subset in exactly the same way each school tentatively accepts students in each step of the $SDA^{\triangleright}$ mechanism's algorithm. That is, it proceeds iteratively grade-by-grade and narrows down the pool of contracts/students in a two-step process. First, for each subgroup of students with a sibling in a particular downstream grade, select the highest priority students; this guarantees the feasibility of any selection in the second step. Second, out of the remaining students (who may have siblings across various grades), select the highest priority.

More formally, for each school s, we define the **sequential choice function of s**. Each school s considers grades sequentially according to a precedence order $\triangleright$ where $g \triangleright g'$ means grade g will be processed before grade g'. We denote this choice function with $C_s^{\triangleright}$ and for any given set of contracts Y, the chosen set is calculated as follows:

Step 0: Let $Y^1 = Y_s$. Let $g_k \triangleright g_{k+1}$ for all $k \in \{1, 2, ..., |G| - 1\}$. Let $Re_s^{\triangleright}(Y) = C_s^{\triangleright}(Y) = \emptyset$. Let $\bar{q}_s^{g_k} = q_s^{g_k}$ for all $k \in \{1, 2, ..., |G|\}$.

Step 1: Grade g_1 Selection

(Type Determination) For each $k > 1$, let $a_k = \mathrm{i}(Y^1) \cap \mathrm{i}(Y^{g_1}) \cap \mathrm{i}(Y^{g_k})$.

(Downstream Feasibility) For each $k > 1$, if $|a_k| > \bar{q}_s^{g_k}$, then we add $\bigcup_{g \in \gamma(i)} (i, s, g) \cap Y^1$ to $Re_s^{\triangleright}(Y)$ such that $|\{j \in a_k | j \succ_s^{g_1} i\}| \geq \bar{q}_s^{g_k}$.

(Selection from Remaining) Let $\bar{Y}^1 = (Y^1 \cap Y^{g_1}) \setminus Re_s^{\triangleright}(Y)$. Let $C_s^{g_1}(Y) = \bar{Y} \subseteq \bar{Y}^1$ such that $|\bar{Y}| = \min\left\{\bar{Y}^1, \bar{q}_s^{g_1}\right\}$, and for each $(\bar{i}, i') \in \left(\mathrm{i}(\bar{Y}) \cap I^{g_1}\right) \times \left(\mathrm{i}(\bar{Y}^1) \cap I^{g_1} \setminus \mathrm{i}(\bar{Y})\right)$, $\bar{i} \succ_s^{g_1} i'$. Let $Re_s^{g_1}(Y) = (Y^1 \cap Y^{g_1}) \setminus C_s^{g_1}(Y)$. Add $C_s^{g_1}(Y) \cup (\bigcup_{i \in \mathrm{i}(C_s^{g_1}(Y))} (Y_i \cap Y_s))$ to $C_s^{\triangleright}(Y)$ and add $Re_s^{g_1}(Y) \cup (\bigcup_{i \in \mathrm{i}(Re_s^{g_1}(Y))} (Y_i \cap Y_s))$ to $Re_s^{\triangleright}(Y)$.

(Update Remaining Students and Capacities) Let $Y^2 = Y^1 \setminus (C_s^{\triangleright}(Y) \cup Re_s^{\triangleright}(Y))$ and $\bar{q}_s^{g_k} = q_s^{g_k} - |C_s^{\triangleright}(Y) \cap Y^{g_k}|$ for all $k > 1$.

In general:

Step $\bar{k}$: Grade $g_{\bar{k}}$ Selection

(Type Determination) For each $k > \bar{k}$, let $a_k = \mathrm{i}(Y^{\bar{k}}) \cap \mathrm{i}(Y^{g_{\bar{k}}}) \cap \mathrm{i}(Y^{g_k})$.

(Downstream Feasibility) For each $k > \bar{k}$, if $|a_k| > \bar{q}_s^k$, then we add $\bigcup_{g \in \gamma(i)} (i, s, g) \cap Y^{\bar{k}}$ to $Re_s^{\triangleright}(Y)$ such that $|\{j \in a_k | j \succ_s i\}| \geq \bar{q}_s^{g_k}$.

(Selection from Remaining) Let $\bar{Y}^{\bar{k}} = \left(Y^{\bar{k}} \cap Y^{g_{\bar{k}}}\right) \setminus Re_s^{\triangleright}(Y)$. Let $C_s^{g_{\bar{k}}}(Y)$ be the $\bar{Y} \subseteq \bar{Y}^{\bar{k}}$ such that $|\bar{Y}| = \min\left\{\bar{Y}^{\bar{k}}, \bar{q}_s^{g_{\bar{k}}}\right\}$, and for each $(\bar{i}, i') \in \left(\mathrm{i}(\bar{Y}) \cap I^{g_{\bar{k}}}\right) \times \left(I^{g_{\bar{k}}} \setminus \mathrm{i}(\bar{Y})\right)$, $\bar{i} \succ_s^{g_{\bar{k}}} i'$. Let $Re_s^{g_{\bar{k}}}(Y) = (Y^{\bar{k}} \cap Y^{g_{\bar{k}}}) \setminus C_s^{g_{\bar{k}}}(Y)$. Add $C_s^{g_{\bar{k}}}(Y) \cup (\bigcup_{i \in \mathrm{i}(C_s^{g_{\bar{k}}}(Y))} (Y_i \cap Y_s))$ to $C_s^{\triangleright}(Y)$ and add $Re_s^{g_{\bar{k}}}(Y) \cup (\bigcup_{i \in \mathrm{i}(Re_s^{g_{\bar{k}}}(Y))} (Y_i \cap Y_s))$ to $Re_s^{\triangleright}(Y)$.

(Update Remaining Students and Capacities) Let $Y^{\bar{k}+1} = Y^{\bar{k}} \setminus (C_s^{\triangleright}(Y) \cup Re_s^{\triangleright}(Y))$ and $\bar{q}_s^{g_k} = q_s^{g_k} - |C_s^{\triangleright}(Y) \cap Y^{g_k}|$ for all $k > \bar{k}$.

The process concludes after Step $|G|$.

Next, we turn to properties of assignments. An assignment $Y \subset X$ is **stable** if:

- (Individually Rational) for all $a \in I \cup S$, $C_a(Y) = Y_a$, and
- (Unblocked) there does not exist $Z \subset X$ such that $Z \neq \emptyset$, $Z \cap Y = \emptyset$, $|s(Z)| = 1$, and for all $a \in \mathrm{i}(Z) \cup \mathrm{s}(Z)$, $Z_a \subseteq C_a(Y \cup Z)$.

If there is such a $Z \subset X$ satisfying the above, then we say that Z **blocks** Y (at this problem).

We define two properties of choice functions that are crucial to the existence of *stable* and *strategy-proof* mechanisms in the matching with contracts literature. The first states that a contract y that is rejected from a menu of available contracts is still rejected if another contract y' is added to the menu. A choice function C_a satisfies **substitutability** if for all $Y \subset X$, and $y, y' \in X \backslash Y$, $y \notin C_a(Y \cup \{y\})$ implies $y \notin C_a(Y \cup \{y, y'\})$. The second states that the number of contracts chosen (weakly) increases as the menu size grows. A choice function C_a satisfies the **Law of Aggregate Demand** (LAD) if for all $Y \subseteq Y' \subseteq X$ we have $|C_a(Y)| \leq |C_a(Y')|$.

Proposition 2 *For each $i \in I$, C_i satisfies LAD but not substitutability.*

Proof. We start with LAD. Consider any subset of contracts $Y \subseteq X$. By definition, if $C_i(Y) \neq \emptyset$, then $\mathrm{s}_i(Y) \neq \emptyset$ and $|C_i(Y)| = |\gamma(i)|$. Hence, for any $Y \subset Y'$, $\mathrm{s}_i(Y') \neq \emptyset$ and $|C_i(Y')| = |\gamma(i)|$. That is, the parents' choice functions satisfy LAD.

Next, we show that C_i is not substitutable via example. Let i be a parent with $s \, P_i \, s'$ and $\gamma(i) = \{1, 2\}$. Let $Y = \{(i, s, 1), (i, s', 1), (i, s', 2)\}$ and $Y' = Y \cup \{(i, s, 2)\}$. Then, $C_i(Y) = \{(i, s', 1), (i, s', 2)\}$ and $C_i(Y') = \{(i, s, 1), (i, s, 2)\}$. Hence, parent i's choice function is not substitutable. ■

Proposition 3 *If $C_s^{\triangleright}$ is the sequential choice function of s, then $C_s^{\triangleright}$ satisfies neither substitutability nor LAD.*

Proof. We prove by example. Let $G = \{1, 2\}$. Let $I^1 = \{i, k\}$ and $I^2 = \{i, j\}$. Let $k \succ_s^1 i$ and $i \succ_s^2 j$. For each $g \in \{1, 2\}$ $q_s^g = 1$. Let $1 \triangleright 2$. Let $Y = \{(i, s, 1), (i, s, 2)\}$, $Y' = Y \cup \{(k, s, 1)\}$, $Y'' = Y \cup \{(j, s, 2)\}$, and $Y''' = Y'' \cup \{(k, s, 1)\}$. Then, $C_s^{\triangleright}(Y) =$

$\{(i,s,1),(i,s,2)\}$, $C_s^{\triangleright}(Y') = \{(k,s,1)\}$, $C_s^{\triangleright}(Y'') = \{(i,s,1),(i,s,2)\}$, and $C_s^{\triangleright}(Y''') = \{(k,s,1),(j,s,2)\}$. Since $Y \subset Y'$ and $|C_s^{\triangleright}(Y)| > |C_s^{\triangleright}(Y')|$, $C_s^{\triangleright}$ does not satisfy LAD. Since $Y'' \subset Y'''$, $(j,s,2) \notin C_s^{\triangleright}(Y'')$ and $(j,s,2) \in C_s^{\triangleright}(Y''')$, $C_s^{\triangleright}$ is not substitutable. ■

We consider a problem where $\triangleright$ is a precedence order, and each school s has a sequential choice function $C_s^{\triangleright}$. Note that $\triangleright$ is common across all schools. Despite the fact that no school's choice function is substitutable or satisfies LAD, we will show that there is a *stable* and *strategy-proof* mechanism. This mechanism is analagous to the $SDA^{\triangleright}$ but is written in the language of contracts; it operates by iteratively running the Cumulative Offer Process one grade at a time. The key point is that although no school has a choice function satisfying substitutability or LAD *across* grades, each school's effective choice function when restricted to a *specific* grade is substitutable and satisfies LAD.

For each $s \in S$, each $g \in G$, and each grade-specific capacity vector $\hat{q} = (\hat{q}^{g'})_{g' \in G}$, let a **component choice function for s w.r.t. g and $\hat{q}$** be a mapping $D_s^{g,\hat{q}} : \mathscr{P}(I^g) \to \mathscr{P}(I^g)$ that selects applicants from $\bar{I} \subseteq I^g$ as follows:

Step 1 For each $g' \in G \backslash \{g\}$, if the number of applicants from $\bar{I} \cap I^{g'}$ is more than $\hat{q}^{g'}$, then only the best $\hat{q}^{g'}$ applicants in $\bar{I} \cap I^{g'}$ according to $\succ_s^g$ are tentatively kept and the rest are rejected.

Step 2 Among the unrejected ones in $\bar{I}$ in Step 1, accept the best $\hat{q}^g$ applicants according to $\succ_s^g$.

Proposition 4 *For each $s \in S$, each $g \in G$, and any capacity vector $\hat{q} \in \mathbb{N}^{|G|}$, $D_s^{g,\hat{q}}$ satisfies both substitutability and LAD.*

Proof. We start with LAD. Let $\bar{I} \subset \bar{J} \subseteq I^g$. We compare $D_s^{g,\hat{q}}(\bar{I})$ and $D_s^{g,\hat{q}}(\bar{J})$. We consider the parents in $\bar{I}$ and $\bar{J}$ who are not rejected in Step 1 of $D_s^{g,\hat{q}}$. By definition, the number of parents who are not rejected in $\bar{J} \cap I^{g'}$ is weakly more than $\bar{I} \cap I^{g'}$ for all $g' \in G \setminus \{g\}$. Hence, the number of parents considered in Step 2 is weakly more when we consider $\bar{J}$ compared to $\bar{I}$. Then, $|D_s^{g,\hat{q}}(\bar{I})| \leq |D_s^{g,\hat{q}}(\bar{J})|$.

Next, we show substitutability. Consider any subset of students $\bar{I} \subset I^g$. Let $i \notin D_s^{g,\hat{q}}(\bar{I})$. Then, when $D_s^{g,\hat{q}}(\bar{I})$ is considered, i is rejected in either Step 1 or Step 2. Suppose i is rejected in Step 1. Then, we consider $D_s^{s,\hat{q}}(\bar{I} \cup \{j\})$ where $j \notin \bar{I}$. By definition, there are at least $\hat{q}^{g'}$ parents in $(\bar{I} \cup \{j\}) \cap I^{g'}$ with higher priority than

$i \in I^{g'}$ according to $\succ_s^g$. Hence, $i \notin D_s^{s,\hat{q}}(\bar{I} \cup \{j\})$. Now, suppose i is rejected in Step 2. Then, all parents in $D_s^{s,\hat{q}}(\bar{I})$ have higher priority than i. By LAD and the definition, there will be at least $\hat{q}^g$ parents in Step 2 with higher priority than i when we consider $\bar{I} \cup \{j\}$. Hence, $i \notin D_s^{s,\hat{q}}(\bar{I} \cup \{j\})$. ■

For each precedence order $\rhd$, we will now define the **Sequential Cumulative Offer Mechanism w.r.t. $\rhd$, $SCO^{\rhd}$**.[34] We will determine an assignment as a sequence of many-to-one matching with contracts problems where we process grades in order $\rhd$.

Consider the subproblem at grade g_1—parents I^{g_1}, schools S, preference profile $P^{g_1} = (P_i)_{i \in I^{g_1}}$, capacity profile $q^{g_1} = (q_s^{g_1})_{s \in S}$, and priority profile $\succ^{g_1} = (\succ_s^{g_1})_{s \in S}$. Treating this as a many-to-one matching problem, run the Cumulative Offer Process where each school s uses the choice function $D_s^{g_1,q}$. Let μ^{g_1} be the resulting outcome, that is $\mu^{g_1} : I^{g_1} \to S \cup \emptyset$ —the interpretation being that for each i, if $\mu^{g_1}(i) \neq \emptyset$, then i is assigned at each of the grades $\gamma(i)$ at school $\mu^{g_1}(i)$. Then, the contracts including i and $\mu^{g_1}(i)$ are selected as long as $\mu^{g_1}(i) \in S$.

We revise down capacities for each other grade. For each school $s \in S$, and grade $g' \in G \backslash g_1$, let the capacity now be $\hat{q}_s^{g'} = q_s^{g'} - |\{i \in I^{g_1} : \ \mu^{g_1}(i) = s \ \textit{and} \ g' \in \gamma(i)\}|$.

Consider the subproblem at grade g_2—$I^{g_2} \backslash I^{g_1}$, schools S, preference profile $P^{g_2} = (P_i)_{i \in I^{g_2} \backslash I^{g_1}}$, capacity profile $\hat{q}^{g_2} = (\hat{q}_s^{g_2})_{s \in S}$, and priority profile $\succ^{g_2} = (\succ_s^{g_2})_{s \in S}$. Treating this as a many-to-one matching problem, run the Cumulative Offer Process where each school s uses the choice function $D_s^{g_2,\hat{q}}$. Let μ^{g_2} be the resulting outcome.

Repeat this procedure for the rest of the grades to arrive at $(\mu^g)_{g \in G}$. Let $\hat{\gamma}(i)$ be the $\rhd$-earliest grade in $\gamma(i)$. Finally, let $SCO^{\rhd}$ for this problem select

$$\bigcup_{\substack{i \in I: \\ \mu^{\hat{\gamma}(i)}(i) \neq \emptyset}} \ \bigcup_{g \in \gamma(i)} (i, \mu^{\hat{\gamma}(i)}(i), g).$$

Theorem 3 *Consider an arbitrary precedence order $\rhd$. If each school s has the sequential choice function $C_s^{\rhd}$, then the $SCO^{\rhd}$ mechanism is* stable *and* strategy-proof.

[34] This mechanism is the analogue of the $SDA^{\rhd}$ mechanism for the matching with contracts framework. We refer to it as a Cumulative Offer mechanism instead of the DA mechanism in order to follow the naming conventions in this literature.

Proof. Let $(I, S, G, q, \succ, P)$ be a problem, and $g_1 \triangleright g_2 \triangleright \cdots \triangleright g_n$. Let μ be the assignment recommended by $SCO^{\triangleright}$ for this problem.

Stability: By definition, μ is *individually rational.* Suppose by contradiction that there is $Z \subseteq X$ such that Z blocks μ. First, observe that in the many-to-one subproblem consisting of I^{g_1}, by the substitutability of each school's choice function $D_s^{g_1,q}$ and Hatfield and Milgrom (2005), the assignment μ^{g_1} is not blocked by any contract set that involves parents only in I^{g_1}. Let the $\triangleright$-earliest grade in $\mathrm{g}(Z)$ be g_1. Since for each $s \in S$, $C_s^{\triangleright}$ chooses grade 1 contracts (and siblings) first, $Z' = Z^{g_1} \cup \{(i, s, g) \in Z : i \in \mathrm{i}(Z^{g_1})\}$ is chosen in Step 1 for $C_s^{\triangleright}$. This implies that in the many-to-one subproblem at grade 1, $\mathrm{i}(Z')$ blocks μ^{g_1}—a contradiction. So the $\triangleright$-earliest grade in $g(Z)$ is not g_1.

Let the $\triangleright$-earliest grade in $g(Z)$ be g_2. We can repeat the reasoning above: Since for each $s \in S$, $C_s^{\triangleright}$ chooses grade 2 contracts (and siblings) before any others in Z, $Z' = Z^{g_2} \cup \{(i, s, g) \in Z : i \in \mathrm{i}(Z^{g_2})\}$ is chosen in Step 2 for $C_s^{\triangleright}$. This implies that in the many-to-one subproblem at grade 2, $\mathrm{i}(Z')$ blocks μ^{g_2}—a contradiction to the fact that μ^{g_2} is not blocked by any contract set that involves parents only in $I^{g_2} \setminus I^{g_1}$.

By repetition of this reasoning and the finiteness of G, we conclude that there is no such Z.

Strategy-proofness: Observe that each parent i is only involved in the construction of μ at her earliest grade in $\gamma(i)$, say g. At the processing of grade g, i's report does not affect the assignment of grades processed earlier—i can only affect μ^g. By substitutability and LAD of each school's choice function $D_s^{g,q}$ and Hatfield and Milgrom (2005), the Cumulative Offer Process mechanism is *strategy-proof* and i cannot benefit by misreporting. ■

This result is surprising, since neither the parents' nor the schools' choice functions satisfy substitutability. This is in stark contrast to Hatfield and Kominers (2017), where they show that the substitutable domain is a maximal domain to guarantee the existence of a *stable* assignment. The first key difference is that the grade and sibling structure *forces* schools' preferences to have such complementarities; hence, each schools' sequential choice function falls entirely outside the substitutable domain. The second key difference is that in our environment, while priorities vary, the grade and sibling structure is uniform across schools, and thus the exact same types of resulting complementarities appear in each school's choices. This uniform structure is crucial for our result.