SUSTAINABLE DEBT

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Abstract

We show that debt is sustainable at a competitive equilibrium based solely on the reputation for repayment; that is, even without collateral or legal sanctions available to creditors. In an incomplete asset market, when the rate of interest falls recurrently below the rate of growth of the economy, self-insurance is more costly than borrowing, and repayments on loans are enforced by the implicit threat of loss of the risk-sharing advantages of debt contracts. Private debt credibly circulates as a form of inside money, and it is not valued as a speculative bubble. Competitive equilibria with self-enforcing debt exist under a suitable hypothesis of gains from trade.

Keywords: Rate of interest, self-enforcing debt, Ponzi games, incomplete markets, competitive equilibrium, gains from trade.

JEL Classification: D52, F34, H63.

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1. INTRODUCTION

Conventional wisdom asserts that debt is unsustainable when not secured by collateral or by sanctions that creditors can exercise against debtors upon default. Instead, we argue that, under certain conditions, creditors can rely on the self-interest of debtors in maintaining a reputation for repayment. We identify the implicit enforcement mechanism and show the existence of a competitive equilibrium with unsecured debt. In general, debt is not valued as a speculative bubble: it is priced as the fundamental value of future repayments.

When creditors have no legal rights whatsoever, debtors are able to borrow only if they can maintain a reputation for repayment, as pointed out by Eaton and Gersovitz [16]. Debt is a valuable insurance device and repayments are implicitly enforced by the threat of losing future borrowing privileges. In a celebrated paper, Bulow and Rogoff [12] provided an influential critique of the reputational theory of unsecured debt. The loss of reputation cannot prevent debtors from continuing to save in financial markets after default. Thus, upon default, they maintain access to cash-in-advance insurance contracts involving upfront payments without incurring further debt. As such contracts are available, borrowers that have reached a large debt exposure prefer to declare bankruptcy and to divert saved repayments to acquire cash-in-advance contracts. Creditors anticipate debtors' incentives to default and provide no loans at all. Debt is therefore unsustainable when not secured by collateral or by sanctions against debtors upon default.

The critique of Bulow and Rogoff [12] rests upon a pervasive and robust arbitrage argument. Contrary to a misguided intuition, its logic extends to an incomplete asset market, irrespective of the size of market diversification. The absence of certain cash-in-advance insurance contracts after default, by itself, provides no stronger incentives to debt repayment, because an equally limited insurance opportunity affects the borrower before default. In fact, Auclert and Rognlie [7] and Aguiar and Amador [2] established that debt is unsustainable when only a single bond is traded under a constant interest rate. More generally, with time-varying interest rates and growth, in a diversified incomplete asset market, we proved in [11] that default is unavoidable when the long-term interest rate unambiguously exceeds growth. If the long-term interest rate unambiguously falls below growth, instead, Ponzi games are feasible and, as repayments are unnecessary, default becomes unprofitable.[1] Under a complete asset market, *tertium non datur*. However, these two regimes are not exhaustive when the asset market is incomplete.

The long-term comparison between the interest rate and growth rate under incomplete markets may be ambiguous: the interest rate might be oscillating persistently around growth, exceeding the growth rate in some periods and falling below in other periods. This pattern is of a certain empirical relevance because it is consistent with historically observed safe interest rates in developed countries, as documented by Blanchard [9] and Jordà et al. [19]. Here, we establish that this ambiguous relation between interest rates and growth provides incentives to debt repayment while at the same time ruling out Ponzi games. Furthermore, we show that it arises naturally at a competitive equilibrium under incomplete markets when default only prevents future borrowing. Contrary to the claim of Bulow and Rogoff [12], debt can be sustained by the reputation for repayment alone.

How can debt be self-enforcing? After default, no further debt can be issued and the borrower will have to rely on self-insurance. In Bulow and Rogoff [12]'s arbitrage argument, the accumulation of saved repayments to creditors will provide the same insurance against averse events as debt before default and, in fact, to a larger extent, as interest will accrue on savings. Hence, default will entail no cost in terms of future consumption opportunities. This arbitrage ceases when the long-term interest rate might remain below growth with some probability, even while exceeding growth on average. The cost of self-insurance grows prohibitively high, because low interest rates will deplete asset reserves over time. Instead, when borrowing is permitted, insurance obtains by issuing debt at low interest rates. Hence, debt is a superior instrument, and repayments are implicitly enforced by the threat of losing borrowing privileges.

We provide an argument for the existence of a competitive equilibrium when permanent exclusion from future borrowing is the only punishment for default. For this purpose we develop a novel strategy of proof. We first perturb the economy by introducing a legal

[1] In a working paper version, Bulow and Rogoff [12] mention and dismiss the possibility of debt sustained by Ponzi-type reputational contracts. In fact, to rule out Ponzi games, they assume a finite present value of the borrower's wealth, so implying high interest rates relative to growth. As Bulow and Rogoff [12], we remain within a framework in which Ponzi games are infeasible.

sanction: upon default, a small fraction of the endowment is confiscated. This is sufficient to enforce repayment of any debt not exceeding the present value of confiscated resources. As a result, borrowing and lending occur in the perturbed economy and, at a competitive equilibrium, a claim into each debtor's entire future income is finite. We then progressively remove the auxiliary sanction and consider the limit with no confiscation. This is a competitive equilibrium of the original economy and trade occurs under a suitable *gains from trade* hypothesis: the implicit value of a claim into each debtor's entire income is (robustly) infinite at autarky. Indeed, as this claim has a finite value in the perturbed economy, autarky cannot be the limit as the perturbation is removed.

We also establish that, at a competitive equilibrium, Ponzi games are infeasible and debt is not valued as a speculative bubble. In a speculative regime, borrowers are allowed to exactly roll over their debts, period by period, without repayments. If the long-term interest rate exceeds growth along a path, debt would explode in a roll-over regime. On the other hand, it would vanish over time if the long-term interest rate falls below growth along some other path. Both situations are inconsistent with a competitive equilibrium in which trade persists indefinitely. As a result, a necessary condition for debt roll-over at equilibrium is that the long-term interest rate be unambiguously equal to the rate of growth. This is a fragile property, however, because interest rates need to vary to clear bond markets over time and across contingencies. Hence, Ponzi games cannot occur.

For the purpose of our analysis, we develop a novel dominant root (Perron–Frobenius) approach to time-varying interest rate and growth under uncertainty.[2] A straight comparison between the average interest rate and the growth rate is in general unsatisfactory, and our more sophisticated machinery is necessary. Dominant roots estimate bounds for the long-term interest rate, relative to growth, and govern the long-run tendencies of the debt-to-income ratio. The method was introduced in Bloise et al. [11] for simple Markov frameworks. As competitive equilibrium rarely obeys the Markov property under incomplete markets, we provide an extension in this paper.

[2] For a complete asset market, a dominant root theory is provided by Alvarez and Jermann [5] and Hansen and Scheinkman [17]. Their purpose is to derive a lower bound for the volatility of the permanent component of asset pricing kernels. Our purpose, instead, is to estimate the discrepancy between the long-term interest rate and the growth rate when the asset market is incomplete.

Hellwig and Lorenzoni [18] also showed that debt is sustainable in economies with a complete asset market when the rate of interest is determined endogenously to clear markets. Their insight and contribution rely on an equivalence between speculative bubbles and self-enforcing debt, a variation of the bubble equivalence theorem in Kocherlakota [22]. At a competitive equilibrium with a speculative bubble, an asset with no intrinsic value allows for intertemporal consumption smoothing, as money in Bewley [8]. At a competitive equilibrium with self-enforcing debt, instead, each individual issues private debt and this is valued in the market as a speculative bubble. In other words, the privileges of issuing the speculative bubble are assigned to individuals, as opposed to being embodied in an asset in positive net supply. In fact, incentives to default disappear because individuals are allowed to run limited Ponzi games: they can exactly roll over a certain amount of debt period by period and, as a consequence, no effective repayment is enforced. The amount of debt that a borrower can credibly issue depends on an unspecified process of creditors' expectations coordination: *all* debt limits that allow for exact debt roll-over are self-enforcing and so Hellwig and Lorenzoni [18]'s competitive equilibrium is *indeterminate*. In our framework with uninsurable risk, instead, self-enforcing debt balances at the margin benefits of and costs from default and so it is intimately related to fundamentals.

Bulow and Rogoff [12]'s objection to the reputation argument for repayment posed a powerful challenge to the notion that the threat of exclusion from credit markets, by itself, supports sovereign borrowing. The literature evolved in three distinct directions, and Aguiar and Amador [1] and Panizza, Sturzenegger, and Zettelmeyer [27] provide comprehensive reviews. In a first line of research, as in Bulow and Rogoff [13], debt repayment is sustained by direct punishments, interpreted as the outcome of interferences with the debtor's transactions upon default. A second line of research, as in Kletzer and Wright [21], develops the idea that sovereigns repay because they are worried about the repercussions of default, for instance, for the credit market. In a third line of research, incentives to repay sovereign debt arise from possible broader adverse effects on a borrower's reputation, as in Cole and Kehoe [15]. All previous studies take their cue from the critique of Bulow and Rogoff [12] and explore alternative, and more effective, mechanisms for debt enforcement. We, instead, rely solely on the debtor's reputation for repayment and show a failure of Bulow and Rogoff [12]'s claim with residual uninsurable risks.

We add a short comment on admitting default at equilibrium, as in Eaton and Gersovitz [16] and, more recently, in Aguiar and Gopinath [3] and Arellano [6].[3] For our purposes, it is worth noticing that the prospect of defaulting at some future contingency enhances debt sustainability, because it increases the value of market participation for the borrower. Thus, under conditions in which debt is sustainable when default is not allowed, so will it be when default can occur and the price of the bond reflects the risk of default. In the latter case, debt is sustainable in the sense that lenders are willing to supply credit, though anticipating future default with some probability.

We organize the paper as follows. We begin with the presentation of a simple example in §2 to clarify the debt enforcement mechanism. In §3 we describe the economy and define a competitive equilibrium with self-enforcing debt. As debt sustainability depends on the long-term rate of interest, relative to growth, in §4 we develop our dominant root approach. In §5 we establish the existence of an equilibrium with trade. In §6 we employ the dominant root method to identify a necessary condition for debt roll-over and show that this condition can only be satisfied in singular situations under incomplete markets. Main proofs are collected in Appendix A. Appendix B extends the partial equilibrium example of §2 to a general Markov framework and illustrates the implications of default for debt sustainability. Appendix C provides a self-contained presentation of the dominant root method for simple Markov pricing kernels. Appendix D proves existence of an equilibrium in the perturbed economy where, upon default, individuals lose a constant fraction of their endowment.

2. A MOTIVATING EXAMPLE

How can debt be sustainable when it is not secured by collateral or legal sanctions? A simple example provides the underlying intuition and illustrates the enforcing mechanism. When the rate of interest is recurrently below the rate of growth, self-insurance may be too

[3]These are partial equilibrium frameworks, whereas the issue is more controversial in general equilibrium.

costly, and debt may provide insurance services more efficiently than other instruments. Thus, debt may be implicitly secured by the threat of diminished insurance opportunities upon default, contrary to the claim of Bulow and Rogoff [12]. In the example, these conditions are simply assumed to hold. As our general analysis clarifies, however, they naturally emerge at a competitive equilibrium under incomplete markets.

In each period, there are two states of nature, $S = \{l, h\}$, occurring with equal probability. A risk-free (discount) bond is the only security, and its price is either $q_h > 1$ or $q_l < 1$. An individual can trade the risk-free bond over time, issuing debt when needed, under no commitment for repayment. As in Bulow and Rogoff [12], denial of future credit is the only punishment for default.

Preferences on consumption streams are given by a conventional discounted expected utility. That is,

$$U\left((c_t)_{t=0}^{\infty}\right) = \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t u\left(c_t\right),$$

where $u : \mathbb{R}^+ \to \mathbb{R}$ is a smooth, strictly concave, strictly increasing utility function satisfying $\lim_{c \to 0} u'(c) = \infty$, and δ in $(0, 1)$ is the discount factor. The endowment is constant, $e > 0$. We assume that the discount factor δ lies in $(0, q_l)$. This ensures that the individual is sufficiently impatient and will never save after default. Autarky is, thus, the reservation utility.

We consider a simple consumption plan in which outstanding debt remains constant over time. This is not a roll-over regime, because repayments occur and, though the rate of interest is recurrently positive along some path, debt is not exploding. The outstanding stock of debt is $d > 0$, while consumption is, depending on the current price of the bond,

$$c_s = e + (q_s - 1)\, d > 0.$$

The feature of this plan is that when the rate of interest is positive ($q_l < 1$), some resources are devoted to debt service; when the rate of interest is negative ($q_h > 1$), debt can be refinanced at no cost, and excess resources are diverted to consumption. The stock of debt remains unaltered over time. Is default profitable under these conditions?

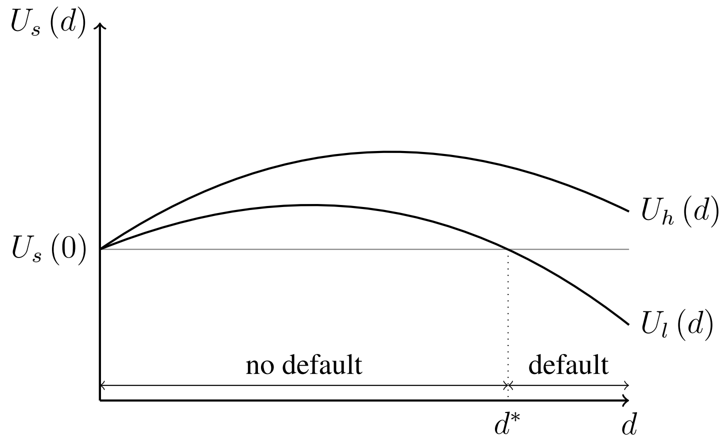


FIGURE 1. No-default incentives

The expected discounted utility, conditional on no default, is

$$U_s(d) = u(c_s) + \left(\frac{\delta}{1-\delta}\right)\left(\frac{u(c_h)+u(c_l)}{2}\right).$$

As saving is never optimal, autarky is the expected discounted utility upon default. Thus, some level $d > 0$ of debt is sustainable whenever

$$U_s(d) \geq U_s(0).$$

This, in turn, amounts to verifying that $U_l'(0) > 0$, which occurs if and only if

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{q_h-1}{2}\right) > (1-q_l) + \left(\frac{\delta}{1-\delta}\right)\left(\frac{1-q_l}{2}\right).$$

The right-hand side is the value of saved repayments, whereas the left-hand side is the value of the excess consumption afforded by refinancing debt at a negative rate of interest. The condition asserts that the marginal cost of default exceeds the marginal benefit from default, and it is certainly satisfied for an open set of $q_h > 1$ and $q_l < 1$. Furthermore, by decreasing marginal utility, the net gain from debt repayment decreases with debt exposure, and default becomes profitable eventually: the maximum level of sustainable debt, $d^* > 0$, is uniquely identified. This property is illustrated in Figure 1.

We can also exhibit local conditions under which the plan is in fact optimal for the borrower when $d^* > 0$ is the debt limit. This requires us to verify the first-order condition

$$q_s > \frac{\delta}{2} \frac{u'(c_h) + u'(c_l)}{u'(c_s)}.$$

This restriction indeed ensures that a reduction of outstanding debt would decrease utility, whereas welfare-increasing additional borrowing is precluded by the binding debt limit. The first-order condition is certainly satisfied for bond prices in a neighborhood of those for which net gains from debt repayment vanish, that is, such that

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{q_h^0 - 1}{2}\right) = \left(1 - q_l^0\right) + \left(\frac{\delta}{1-\delta}\right)\left(\frac{1 - q_l^0}{2}\right).$$

Indeed, by continuity, $d^* > 0$ is sufficiently small and, hence, the expected marginal rate of substitution remains below the price of the bond, as it does in the limit when $d^* = 0$.

Debt is sustainable in this example because the cost of partial repayment when the interest rate is positive is more than compensated by the future prospect of issuing debt at a negative interest rate. This incentive to repayment, though, does not require unduly stringent conditions on average interest rate. In fact, as illustrated in Figure 2, it might well be that the interest rate is positive on average, that is,

$$\frac{1}{2}\frac{1}{q_h} + \frac{1}{2}\frac{1}{q_l} > 1.$$

A positive interest rate on average implies that if debt were rolled over period by period without any repayment, it would explode in expectation. However, even when the interest rate is negative on average, debt would exceed any given threshold with positive probability in a roll-over regime: the interest rate might remain positive with some probability for an arbitrarily long phase and debt accumulation would be explosive along such a path. In fact, the distinguishing feature of our example is that the natural debt limit (the least valuation of the borrower's endowment) is finite. A finite natural debt limit provides an upper bound on debt repayment capacity, as debt would be growing unboundedly when exceeding this

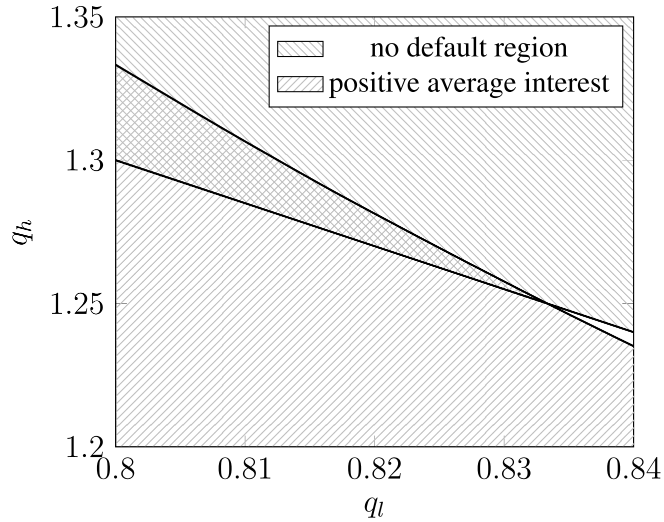


FIGURE 2. Average interest rate ($\delta = 0.8$)

natural limit. In the example,

$$d \leq \left(\frac{1}{1-q_l}\right) e = \sum_{t=0}^{\infty} q_l^t e,$$

so that outstanding debt is in fact bounded by the least valuation of the endowment.

Comparatively, at Hellwig and Lorenzoni [18]'s competitive equilibrium with self-enforcing debt, borrowers are allowed to run limited Ponzi games, that is, to roll over a certain amount of debt without repayments indefinitely.[4] Debt roll-over does not imply explosive paths of debt accumulation because the (long-term) interest rate is unambiguously zero at their equilibrium. This situation occurs in our example exactly when

$$q_l = 1 = q_h.$$

Under Hellwig and Lorenzoni [18]'s circumstances, both the marginal cost of and the marginal benefit from default vanish, and any arbitrary level of debt becomes sustainable (in terms of Figure 1, $U_s(d) = U_s(0)$ for any level of debt $d > 0$). The amount of debt sustainable through Ponzi games is indeterminate and, so, unrelated to fundamentals.

[4]It might seem that individuals *do not* roll over debt in Hellwig and Lorenzoni [18]'s example. This is a deceptive appearance. Rational individuals always exploit the opportunity of Ponzi games if permitted by constraints. In fact, individuals *do* run Ponzi games in their example, but along with an asset accumulation plan, so that their net position does not appear as a Ponzi game.

Why is Bulow and Rogoff [12]'s arbitrage argument failing under incomplete markets? By defaulting, the borrower saves on debt repayments at the cost of no further debt in the future. Under complete markets, saved repayments can be used to pay upfront for the same consumption pattern as without default and, as a result, denial of future credit entails no effective cost. This arbitrage is precluded in the example because markets are incomplete. Indeed, the rate of interest may remain negative for a long phase. Before default, the borrower benefits from refinancing outstanding debt at the negative interest rate. After default, the upfront value of an equivalent positive net consumption is arbitrarily large, because a negative rate of interest accrues on savings. Thus, default entails a large cost, whereas the gain from saved repayments may be relatively small.

To clarify this point, we carry out the natural counter-factual experiment: we introduce elementary Arrow securities so as to complete markets while preserving prices of the risk-free bond. In particular, we consider elementary Arrow securities with prices π in $\mathbb{R}^{S\times S}$ satisfying

$$q_s = \pi_{ss} + \pi_{s\hat{s}}.$$

Furthermore, among the several market values of Arrow securities consistent with the given bond prices, we choose those ensuring a finite present value of the borrower's future endowment, as required by Bulow and Rogoff [12] to rule out Ponzi-type debt contracts (see also Martins-da-Rocha and Vailakis [25]). Notice that the Perron–Frobenius Theorem asserts that there exists a dominant root $\lambda > 0$ such that, for some positive vector b in $\mathbb{R}^S$,

$$\lambda \begin{pmatrix} b_h \\ b_l \end{pmatrix} = \begin{pmatrix} \pi_{hh} & \pi_{hl} \\ \pi_{lh} & \pi_{ll} \end{pmatrix} \begin{pmatrix} b_h \\ b_l \end{pmatrix}.$$

Thus, we select prices of elementary Arrow securities to guarantee that $\lambda < 1$, and this condition is equivalent to a finite valuation of the future endowment (see Appendix C).[5] We then construct the replication policy revealing an incentive to default.

[5] In fact, markets can be completed so as to obtain any value of the dominant root λ in the open interval $(q_l, q_h) \subset \mathbb{R}$.

Before default, the budget constraint imposes

$$-q_s d + (c_s - e) = -d.$$

We modify the debt position by adding a portfolio of Arrow securities in proportions given by the dominant eigenvector and, as long as $\lambda < 1$, we obtain

$$\pi_{ss}\left(b_s - d\right) + \pi_{s\hat{s}}\left(b_{\hat{s}} - d\right) + \left(c_s - e\right) < \left(b_s - d\right).$$

Furthermore, as the eigenvector is given up to a factor of proportionality, we can assume that the portfolio involves no debt in the state in which default is less profitable, $b_h - d > 0$, and no holding of the security in the state in which default is more profitable, $b_l - d = 0$. This reveals that defaulting in state l in S, and adhering to the modified plan without borrowing, permits a further increase in future consumption. Thus, default is unavoidable.

We conclude this long digression by showing that the situation described in our example with incomplete markets is perfectly consistent with a competitive equilibrium. To this purpose, we construct a creditor willing to provide a constant loan to the debtor at the given market prices. This simply requires us to satisfy the system of equations

$$\begin{aligned} q_h &= \delta^* \left(\mu^*_{hh} \frac{u'\left(c^*_h\right)}{u'\left(c^*_h\right)} + \mu^*_{hl} \frac{u'\left(c^*_l\right)}{u'\left(c^*_h\right)} \right), \\ q_l &= \delta^* \left(\mu^*_{lh} \frac{u'\left(c^*_h\right)}{u'\left(c^*_l\right)} + \mu^*_{ll} \frac{u'\left(c^*_l\right)}{u'\left(c^*_l\right)} \right), \end{aligned}$$

where (c^*_l, c^*_h) are the creditor's consumption levels, δ^* in $(0, 1)$ is the discount factor, and μ^* denotes the transition (subjective) probabilities. These equations establish that the bond price is equal to the expected marginal rate of substitution of the creditor, as required by first-order conditions for optimality. Furthermore, the endowment process for the creditor is assumed to satisfy the corresponding budget constraint,

$$c^*_s = e^*_s - \left(q_s - 1\right) d.$$

The bond market clears because the creditor finds it optimal to take the opposite side in the transaction: the positive return on the investment when interest rate is positive compensates for its poor performance when the interest rate happens to be negative. As a matter of fact, we can also suppose that the creditor has full commitment power. Under complete markets, instead, a borrower will not be granted any loan by a creditor with full commitment ability. This marks another crucial difference with respect to Hellwig and Lorenzoni [18]'s Ponzi-type self-enforcing debt.

To verify the robustness of our construction, we extend the example developed in this section to a partial equilibrium Markov framework in Appendix B. Debt is sustainable as long as the advantage of issuing debt at a low interest rate along a path more than compensates for the cost of repayments to creditors when the interest rate is high. We also show that the occurrence of default enhances debt sustainability when insolvency is anticipated by creditors, as in Eaton and Gersovitz [16]. In fact, the prospect of defaulting in the future, contingently on some averse event, increases the value of maintaining an intact borrowing ability and so reduces current incentives to default. In the remaining part of the paper, we study the competitive equilibrium of an economy in which debt can only be sustained by reputation for repayment.

3. THE ECONOMY

3.1. **Fundamentals.** We assume that the economy extends over an infinite horizon, $\mathbb{T} = \{0, 1, 2, \ldots, t, \ldots\}$, subject to uncertainty generated by a Markov process on the finite state space S with irreducible transition $P : S \to \Delta(S)$. When the process is initiated from a given Markov state s_0 in S, it generates a probability space $(\Omega, \mathcal{F}, \mu)$ and a filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ of finite partitions of Ω corresponding to partial histories of Markov states. To be parsimonious on notation, we describe all variables as stochastic processes, under the implicit almost-surely qualification. In particular, we let L be the space of all processes $f : \mathbb{T} \times \Omega \to \mathbb{R}$ adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$, and we let L_t be the space of $\mathcal{F}_t$-measurable random variables $f_t : \Omega \to \mathbb{R}$. A process f in L is positive if $f_t \geq 0$ for every t in $\mathbb{T}$. We use L^+ to denote the positive cone of L.

There is a finite set, I, of individuals. For each individual, the *consumption space* C^i is L^+, and the *endowment* is e^i in C^i. To simplify, we impose restrictive assumptions on preferences, though this is unnecessary for most of our analysis. Every individual is characterized by a canonical expected discounted utility. Preferences over the consumption space C^i are induced by

$$U_t^i\left(c^i\right) = \mathbb{E}_t \sum_{s \in \mathbb{T}} \delta^s u^i\left(c_{t+s}^i\right),$$

where δ in $(0,1)$ is the common discount factor.

Assumption 1 (Endowment)**.** The endowment e^i in C^i is uniformly positive and uniformly bounded with respect to the aggregate, that is, for some sufficiently large $\epsilon_u > 0$ and some sufficiently small $\epsilon_l > 0$,

$$\epsilon_l e_t \leq e_t^i \leq \epsilon_u e_t,$$

where the strictly positive process e in L^+ is the aggregate endowment.

Assumption 2 (Utility)**.** Per-period utility $u^i : \mathbb{R}^{++} \to \mathbb{R}$ is smooth, smoothly strictly increasing and smoothly strictly concave. Furthermore, it is bounded from above and satisfies the strong Inada condition

$$\lim_{c^i \to 0} u^i\left(c^i\right) = -\infty.$$

A consumption plan c^i in C^i is *individually rational* if

$$U_t^i\left(c^i\right) \geq U_t^i\left(e^i\right).$$

Notice that individual rationality is imposed at all contingencies and not only *ex ante*. An allocation c in C specifies a consumption plan c^i in C^i for every individual i in I. It is *feasible* if

$$\sum_{i \in I} c_t^i \leq \sum_{i \in I} e_t^i.$$

The space of individually rational and feasible allocations is denoted by $C(e)$. A simple lemma clarifies that individually rational consumption will be uniformly positive due to the strong Inada condition.

Lemma 3.1 (Lower bound on consumption)**.** *When the aggregate endowment e in L^+ is uniformly positive (that is, $e_t \geq \epsilon > 0$ for every t in $\mathbb{T}$), every individually rational consumption plan c^i in C^i is also uniformly positive.*

3.2. **Competitive markets.** A safe, or riskless, bond is sequentially traded in a competitive market. The bond entitles the holder to receive one unit of consumption uncontingently in the following period, whereas a short sale entails an obligation to deliver. The time-varying price of the bond is q in Q, the space of strictly positive adapted processes in L. Holdings of the bond are denoted by z^i in Z^i, the space of adapted processes in L. A purchase corresponds to $z_t^i > 0$, whereas the individual is issuing, or selling short, the bond if $z_t^i < 0$.

At every contingency, each individual is subject to a budget constraint,

$$q_t z_t^i + c_t^i \leq e_t^i + v_t^i, \tag{3.1}$$

where wealth v^i in V^i (the space of adapted processes in L) evolves according to

$$v_{t+1}^i = z_t^i.$$

As in Zhang [29], an additional solvency constraint requires that

$$-g_{t+1}^i \leq z_t^i, \tag{3.2}$$

where g^i in G^i is the adapted process of debt limits restricting the issuance of the safe bond. Mandatory saving is ruled out, so we assume that the debt limit is always positive (that is, the process belongs to L^+).

At every contingency, an individual maximizes expected discounted utility subject to budget and solvency constraints. Conditional on no default, the indirect utility is denoted by $J_t^i\left(v_t^i, g^i\right)$ in L_t. It depends on the wealth v_t^i in L_t inherited from the past, and on the entire future adapted process for debt limits g^i in G^i, as well as on the process of bond prices q in Q. We shall now determine debt limits so as to ensure that, under limited commitment, no default occurs at equilibrium.

3.3. **Not-too-tight debt limits.** In line with Bulow and Rogoff [12] and Hellwig and Lorenzoni [18], default entails the loss of access to future borrowing opportunities. In other terms, upon default, the risk-free bond cannot be sold short anymore, though saving is unrestricted. Debt limits are set so that no debtor has an incentive to default and no lender can profit from extending credit beyond a borrower's debt limit.

Formally, as in Alvarez and Jermann [4] and Hellwig and Lorenzoni [18], debt limits that are *not too tight* allow for the maximum amount of credit that is compatible with repayment at all contingencies. This requires that, for every individual i in I,

$$J_t^i\left(-g_t^i, g^i\right) = J_t^i\left(0, 0\right). \tag{3.3}$$

The left hand-side is the value of market participation, beginning with the maximum sustainable debt, whereas the right hand-side is the value of default. Indeed, upon default, debt is cleared ($\hat{v}_t^i = 0$) and no borrowing is permitted in the future ($\hat{g}^i = 0$). Debt limits are not too tight if the individual is indifferent between repayment and default.

3.4. **Competitive equilibrium.** Given an initial wealth distribution v_0 in V_0, a *competitive equilibrium with self-enforcing debt* consists of an allocation c in C, a price process q in Q, bond holdings z in Z and debt limits g in G such that the following conditions are satisfied.

(a) For every individual i in I, given initial wealth v_0^i in V_0^i, the plan (c^i, z^i) in $C^i \times Z^i$ is optimal subject to budget constraints (3.1) and solvency constraints (3.2) at debt limits g^i in G^i.

(b) Commodity and financial markets clear, that is,

$$\sum_{i\in I} c_t^i = \sum_{i\in I} e_t^i, \text{ and } \sum_{i\in I} z_t^i = 0.$$

(c) For every individual i in I, debt limits g^i in G^i satisfy the not-too-tight condition (3.3).

This concept of equilibrium follows exactly Alvarez and Jermann [4] except that the asset market is incomplete and the default punishment is the denial of future credit, instead of complete autarky. When the asset market is complete, it coincides with the equilibrium with self-enforcing debt studied by Hellwig and Lorenzoni [18].

4. Dominant root

Whether debt is sustainable or not depends on the long-term pattern of interest rate relative to economic growth. In order to estimate the discrepancy between interest rate and growth under uncertainty, we introduce a dominant root approach. These dominant roots will govern the long-term tendencies of the debt-to-income ratio and hence the pace of debt accumulation. In a steady state, the debt-to-income ratio is explosive when the interest rate exceeds growth, whereas Ponzi schemes are feasible when the interest rate falls below growth. Under uncertainty, and with a time-varying interest rate, discerning these situations requires a more sophisticated method. The interest rate might be exceeding growth for long phases while persistently falling below growth along some other paths. Dominant roots extrapolate tendencies out of these conflicting forces.

Our approach requires us to consider the *valuation operator* $\Pi_t : L_{t+1} \to L_t$ given by

$$\Pi_t \left(b_{t+1}\right) = \min_{z_t \in Z_t} q_t z_t$$

subject to

$$b_{t+1} \leq z_t,$$

where it is understood that the inequality is satisfied almost surely. This operator computes the minimum cost to meet some given future contingent obligations by means of the safe bond only. Formally, this valuation defines a monotone sublinear functional.

Dominant roots correspond to the dominant eigenvalues of the valuation operator, as in a sort of extended Perron–Frobenius Theorem.[6] We consider the suitable space $L(e)$ of all adapted processes that are bounded by some expansion of e in L^+, that is,

$$L(e) = \left\{x \in L : |x| \leq \lambda e \text{ for some } \lambda > 0\right\}.$$

[6]More precisely, our analysis under incomplete markets requires a Perron–Frobenius Theorem for monotone sublinear operators. An exhaustive Markov approach is developed in Appendix C. Unfortunately, this is unsatisfactory, as competitive equilibrium with self-enforcing debt will not in general obey any Markov property. A general theory under our weak assumptions is unavailable, and we have to rely on an approximation method.

This space contains all streams of contingent payoffs that do not grow unboundedly relative to the aggregate endowment. The *upper dominant root* $\rho(q)$ in $\mathbb{R}^+$ is the greatest ρ in $\mathbb{R}^+$ such that, for some non-trivial b in $L^+(e)$,

$$\rho b_t \leq \Pi_t(b_{t+1}).$$

Similarly, the *lower dominant root* $\gamma(q)$ in $\mathbb{R}^+$ is the greatest γ in $\mathbb{R}^+$ such that, for some non-trivial b in $L^+(e)$,

$$\gamma b_t \leq -\Pi_t(-b_{t+1}).$$

Notice that, as the valuation operator is monotone sublinear,

$$\gamma(q) \leq \rho(q).$$

Upper and lower dominant roots are well defined, though we cannot in general establish the existence of their associated eigenprocesses.

Interest rate exceeds growth unambiguously in the long term when $\rho(q) < 1$, whereas it falls below growth when $\gamma(q) > 1$. These regimes are mutually exclusive under complete markets, because upper and dominant roots necessarily coincide. When markets are incomplete, instead, the long-term interest rate might be persistently oscillating around growth, exceeding growth along some paths and falling below growth along some other paths. This ambiguous situation occurs when

$$\gamma(q) < 1 < \rho(q). \tag{4.1}$$

In our previous paper [11], we established that debt is unsustainable when $\rho(q) < 1$, so extending Bulow and Rogoff [12] to incomplete markets under time-varying interest rate and growth.[7] On the other side, Ponzi games are feasible when $\gamma(q) > 1$ and large amounts of debt can be accumulated without inducing default. We shall show in this paper that debt is sustainable by reputation, and not as a Ponzi game, precisely under condition (4.1).

[7] We also provided rather convoluted examples of sustainable debt when this condition fails. Their purpose was to show that the domain for the extension of Bulow and Rogoff [12]'s claim to incomplete markets was tight.

Before turning to competitive equilibrium, we argue that the divergence of dominant roots in (4.1) entails no pathological behavior, as it is in fact consistent even with a single representative individual under empirically plausible calibrations. This exercise also reveals that a sophisticated approach is necessary when the growth rate and interest rate are time-varying: a straight comparison between average growth and average interest rate would be unsatisfactory. We finally clarify that it is the lower dominant root, more than the average interest rate relative to growth, that determines the feasibility of Ponzi games.

Consider a conventional stochastic discount factor of the form

$$m_{t,t+1} = \delta \left(\frac{e_t}{e_{t+1}} \right)^{\alpha},$$

where e in L^+ is interpreted as the consumption of a representative individual and α in $\mathbb{R}^{++}$ is the coefficient of constant relative risk aversion. Consumption grows at a constant secular rate g in $\mathbb{R}^+$ with a cyclical component $\hat{e}$ in L^+; that is,

$$e_t = (1+g)^t \, \hat{e}_t.$$

Furthermore, the cyclical component follows an autoregressive process,

$$\hat{e}_{t+1} = \hat{e}_t^{\theta} \zeta_{t+1},$$

where θ lies in $[0,1]$ and ζ in L^+ is identically and independently distributed with compact support $[\zeta_l, \zeta_h] \subset \mathbb{R}^+$. Under these conditions, the implied time-varying price of the safe bond is

$$q_t = \mathbb{E}_t m_{t,t+1} = \delta \, (1+g)^{-\alpha} \, \hat{e}_t^{\alpha(1-\theta)} \mathbb{E} \left(\frac{1}{\zeta} \right)^{\alpha}.$$

We now compute the upper and lower dominant roots for this pricing kernel.

Claim 4.1 (Computation of dominant roots)**.** *Under the maintained assumptions,*

$$\begin{aligned} \rho(q) &= \delta \, (1+g)^{1-\alpha} \, \zeta_h^{\varphi(\theta,\alpha)} \mathbb{E} \left(\frac{1}{\zeta} \right)^{\alpha}, \\ \gamma(q) &= \delta \, (1+g)^{1-\alpha} \, \zeta_l^{\varphi(\theta,\alpha)} \mathbb{E} \left(\frac{1}{\zeta} \right)^{\alpha}, \end{aligned}$$

where $\varphi(\theta,\alpha) = \alpha$ *if* $\theta < 1$ *and* $\varphi(\theta,\alpha) = 1$ *if* $\theta = 1$.

For an empirical assessment, we consider the normal distribution with mean $\mu = 0$ and standard deviation σ in $\mathbb{R}^{++}$. We assume that the innovation is log-normally distributed with truncations at $\zeta_l = \exp(-2\sigma)$ and $\zeta_h = \exp(2\sigma)$. As a result, we obtain that

$$\mathbb{E}\left(\frac{1}{\zeta}\right)^{\alpha} = \exp\left(\frac{1}{2}\alpha^2\sigma^2\right)\frac{\Phi(-\alpha\sigma + 2) - \Phi(-\alpha\sigma - 2)}{\Phi(2) - \Phi(-2)},$$

where $\Phi : \mathbb{R} \to [0, 1]$ is the standard normal (cumulative) distribution. We then set values of the parameters within standard ranges as in many calibration exercises for the US and other developed economies.

We consider a secular growth of consumption equal to $g = 0.02$. Empirical estimates of log-consumption volatility σ for the US economy range from 0.011 to 0.036, with higher values reported when data refer to total consumption expenditure and cover a longer span of calendar time.[8] For the specification of time preference and risk aversion, we mostly rely on Lucas [24], who argues for an upper bound of 2.5 for the coefficient of relative risk-aversion α, with a value of $\alpha = 1$ chosen as a benchmark, and a lower bound of 0.97 for the discount factor δ. Similar values are used in other calibration experiments.

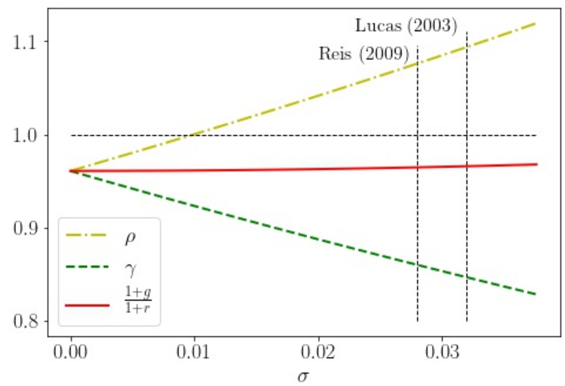
Figure 3 plots the dominant roots against the standard deviation of log-consumption. We also report the consumption volatility for the US economy estimated in several studies. The figure reveals that dominant roots satisfy our divergence property in (4.1) over a range consistent with the time series for US. For comparison, we represent the ratio of secular growth into the unconditional mean interest rate, satisfying

$$\gamma(q) \leq \mathbb{E}\left(\frac{1+g}{1+r}\right) \leq \rho(q).$$

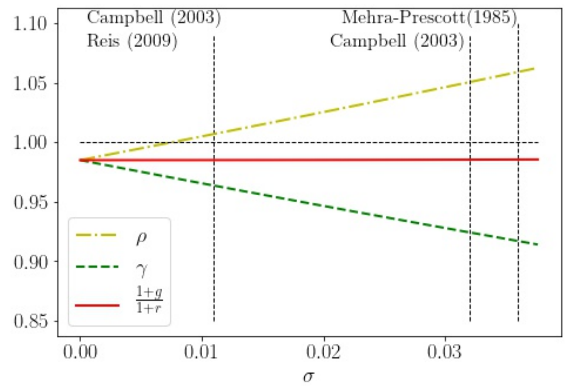
Due to the well-known risk-free rate puzzle (*e.g.*, Campbell [14, Section 3.3]), our calibration over-estimates the average interest rate, predicting a value between 3.5% and 5.5%, while the empirical observation is around $r = 1\%$. Any downward correction of this

[8] Lucas [24] estimates σ directly as the residual variance of detrended log-consumption (*i.e.*, $\theta = 0$). Reis [28] provides estimates of σ for a variety of statistical models, including the least-squares regression for the persistency coefficient ($\theta = 0.92$). Campbell [14] and Mehra and Prescott [26] present estimates of σ assuming that detrended log-consumption is a random walk (*i.e.*, $\theta = 1$). Campbell [14] also documents that consumption volatility is substantially larger in other developed countries and for the US economy over longer time intervals.

erroneous prediction (for instance, setting a discount factor $\delta > 1$ inconsistent with a representative individual or an abnormally high coefficient of relative risk-aversion α) would enlarge the domain of the divergence condition (4.1). Thus, the calibration in Figure 3 is really a worst-case scenario for our plausibility experiment.



$\theta = 0$, $g = 0.02$, $\alpha = 2$, $\delta = 0.985$



$\theta = 1$, $g = 0.02$, $\alpha = 1$, $\delta = 0.985$

FIGURE 3. Dominant roots

To conclude, we verify that dominant roots govern explosive paths of debt accumulation. Consider a situation in which debt is refinanced by issuing new debt period by period forever. This is a *Ponzi game*, or a *debt roll-over* regime, which can be described as a non-trivial process b in L^+ such that

$$\Pi_t\left(-b_{t+1}\right) = -b_t.$$

It turns out that the debt-to-income ratio will be exploding in a Ponzi game if $\gamma\left(q\right) < 1$, and it will not if $\gamma\left(q\right) > 1$. Thus, contrary to a prevailing view, and as illustrated by means of examples in Blanchard and Weil [10], Ponzi games might be infeasible even when the average interest rate is negative (net of growth).

Claim 4.2 (Explosive Ponzi games)**.** *The ratio of debt to endowment in a Ponzi game will be exceeding any given threshold with positive probability if* $\gamma\left(q\right) < 1$, *and it will not if* $\gamma\left(q\right) > 1$.

5. EXISTENCE

We argue that, under a suitable gains from trade hypothesis, a competitive equilibrium with self-enforcing debt exists. Private debt is issued as an insurance device against income fluctuations and it circulates trustworthily as the only store of value in the economy. Default is unprofitable because self-insurance is too costly and individuals prefer to maintain their ability to borrow at low interest rates. As we clarify next in §6, this situation is distinct from a competitive equilibrium in which outside money is valued as a mere speculative bubble, as in Bewley [8], and it is more properly associated with a form of inside money.

We provide a proof of existence when individual endowments evolve according to the irreducible Markov transition $P : S \to \Delta(S)$ on a finite state space S. Thus, the economy cannot grow or decline over time, that is, the aggregate endowment e in L^+ is bounded.[9] In general, at a competitive equilibrium, bond prices are affected by a time-varying wealth distribution and are not measurable with respect to the Markov state space S. Consequently, we cannot impose any Markov restriction on the pricing kernel.

The major difficulty in establishing existence arises from the fact that no trade is always a competitive equilibrium. This resembles the essential property of fiat money: when money is the only store of value and it is not valued in the market, it will not be demanded, because it bears no intrinsic value, and no intertemporal trade will occur; similarly, when all lenders expect that debtors will default, they are not willing to provide credit, and no intertemporal trade will occur. We overcome this obstacle by introducing an approach that, we believe, is novel. Namely, we construct a perturbed economy in which debt is implicitly backed by a share of the private endowment. Trade occurs in this perturbed economy and, as the pledgeable share of the endowment vanishes, debt becomes purely self-enforcing. The dominant root plays an essential role to single out conditions ensuring that trade persists in the limit.

We construct an auxiliary economy in which, upon default, a fraction ϵ in $(0, 1)$ of the endowment is confiscated and no further borrowing is allowed. This is the economy $\mathcal{E}^\epsilon$,

[9] Allowing for growth introduces further technical complexity without adding any substantial insight. Thus, we prefer to avoid unnecessary complications and to focus on the major conceptual issue: when is debt valued at a competitive equilibrium?

whereas the original economy is denoted by $\mathcal{E}^0$. A competitive equilibrium exists in the perturbed economy $\mathcal{E}^\epsilon$.[10] Debt is still unsecured, because confiscated resources are not diverted to satisfy creditors. However, confiscation makes default unprofitable at any level of debt that can be repaid using a fraction ϵ in $(0, 1)$ of the endowment, that is, not exceeding the least present value of confiscable resources. Indeed, why should a debtor default, and lose a fraction of the endowment, when the debt can be repaid using this fraction? The relevant implication is that, at any equilibrium of the perturbed economy, the *least* present value of the endowment is finite, irrespective of the share of confiscable resources.

Lemma 5.1 (Finitely valued endowment)**.** *In any competitive equilibrium of the perturbed economy $\mathcal{E}^\epsilon$, there is an adapted process f^ϵ in $L^+(e)$ such that*

$$f_t^\epsilon = e_t - \Pi_t^\epsilon \left(-f_{t+1}^\epsilon\right),$$

where e in L^+ is the aggregate endowment.

As confiscable resources vanish, the equilibrium allocation cannot converge uniformly to autarky when gains from trade are available. To identify these situations precisely, we construct an implicit pricing at autarky by setting

$$q_t^0 = \max_{i \in I} \delta \mathbb{E}_t \frac{\nabla u^i \left(e_{t+1}^i\right)}{\nabla u^i \left(e_t^i\right)}.$$

The *gains from trade hypothesis* is that $\gamma\left(q^0\right) > 1$. At a competitive equilibrium of the perturbed economy, instead, $\gamma\left(q^\epsilon\right) < 1$, because the endowment would not be finitely valued otherwise, and a reversal cannot occur under uniform convergence. Hence, autarky cannot be an accumulation point.

The role of the dominant root is to identify directions of efficiency that are achieved at a perturbed equilibrium and preserved in the limit. For simple Markov processes with *strictly positive* transitions, the gains from trade hypothesis requires that $q_t^0 > 1$ uniformly. In such a situation, a hypothetical planner can improve upon autarky by means of a simple scheme

[10] Even for the perturbed economy, we cannot rely on any established theorem in the literature, because of the self-enforcing condition. We present our analysis in Appendix D. To establish existence, we truncate the economy by arbitrarily imposing default in the future and progressively remove this further restriction going to the limit.

of transfers: in every period t in $\mathbb{T}$, a small amount $\eta > 0$ is taken from any individual i in I with marginal rate of substitution equal to q_t^0 and distributed to some other individual; in the following period, the donor is compensated with a uncontingent transfer $\eta > 0$; expected utility increases because the compensation is valued more at the margin, that is, $q_t^0 > 1$. This chain of transfers can be continued indefinitely. In this interpretation, the gains from trade hypothesis guarantees a sort of time irreducibility of the economy: the transfer scheme will never be interrupted, as a potential donor will always be available. Private debt is valued at equilibrium because it allows individuals to exploit these welfare gains. On the contrary, in general, it will not be valued when similar welfare gains are not available.

Lemma 5.2 (Trade in the limit)**.** *Under the gains from trade hypothesis, as ϵ in $(0, 1)$ vanishes, no sequence of competitive equilibrium allocations in the perturbed economy $\mathcal{E}^\epsilon$ can converge to autarky uniformly.*

Unfortunately, this established property is not powerful enough to deliver by itself the existence of an equilibrium with trade in the limit. Indeed, it requires uniform convergence, and, in general, sequences of perturbed equilibria may not converge uniformly to a limit equilibrium. However, we exploit the lack of uniform convergence to autarky to extract a sequence of perturbed equilibria (pointwise) converging to a limit equilibrium with trade. We preliminarily verify that the limit remains away from autarky and bounded. For a given ϵ in $(0, 1)$, we denote by $(c^\epsilon, v^\epsilon, g^\epsilon)$ in $C \times V \times G$ a competitive equilibrium of the perturbed economy $\mathcal{E}^\epsilon$.

Lemma 5.3 (Bounds)**.** *Under the gains from trade hypothesis, given any sequence of competitive equilibria in the perturbed economy $\mathcal{E}^\epsilon$,*

$$\liminf_{\epsilon \to 0} \sup_{t \in \mathbb{T}} \|v_t^\epsilon\|_\infty > 0, \tag{5.1}$$

and

$$\limsup_{\epsilon \to 0} \sup_{t \in \mathbb{T}} \|g_t^\epsilon\|_\infty < \infty. \tag{5.2}$$

The most delicate implication of Lemma 5.2 is that borrowing does not vanish in the limit: this is condition (5.1). Assuming not, market clearing would require a progressive contraction of equilibrium debts and credits and thus a uniform contraction of trades, contradicting Lemma 5.2. To establish that debt limits do not explode, that is, condition (5.2), we observe that an individual would otherwise be able to afford arbitrarily large consumption for a long time, by issuing large amounts of debt, and then secure a reservation utility after default, a situation which is inconsistent with the fact that resources are limited in the economy and this large utility value is not feasible.

We now argue that debt is sustainable at equilibrium. In particular, we show that an equilibrium with trade in the original economy can be approached as the (pointwise) limit of a sequence of equilibria in the perturbed economy. A peculiar complication arises because of the endogenous determination of debt limits that is absent in Bewley [8] and in the previous literature.[11]

Proposition 5.1 (Existence)**.** *Under the gains from trade hypothesis, a non-autarkic equilibrium with self-enforcing debt exists.*

A competitive equilibrium with self-enforcing debt will in general be distinct from a Bewley [8]-type monetary equilibrium of the same economy. Furthermore, and differently from Hellwig and Lorenzoni [18], Ponzi games are infeasible and no speculative bubble occurs at equilibrium with self-enforcing debt under incomplete markets. In fact, debt is valued because of the implied future repayments. The purpose of our remaining analysis is to clarify this distinction.

6. Fragility of Ponzi games

Under complete markets, Hellwig and Lorenzoni [18] prove that self-enforcing debt limits necessarily allow borrowers to exactly roll over existing debt, that is, to exactly refinance

[11]Under complete markets it is unnecessary to explicitly consider the not-too-tight condition (3.3) for debt limits because of the equivalence established by Hellwig and Lorenzoni [18, Theorem 1]. The existence of an equilibrium is suggestively devolved to Bewley [8]. In sequential economies with permanent exclusion from markets upon default (*e.g.*, Alvarez and Jermann [4]), the existence of competitive equilibrium is proved via Welfare Theorems and the method is not available in our economy. Nor can we use the proof in Kehoe and Levine [20, Proposition 6], because default is there precluded by a direct restriction of consumption plans.

outstanding obligations by issuing new claims as in a Ponzi game. In fact, equilibrium allocations with self-enforcing private debt are equivalent to allocations that are sustained by unbacked public debt subject to no borrowing. Repayments are not required, and private debt circulates as a speculative bubble. We show that, under incomplete markets, this debt roll-over property fails, in general, when the rate of interest is time-varying.

We consider a competitive equilibrium with non-vanishing debt.[12] We say that a competitive equilibrium involves persistent *debt roll-over* whenever, for some individual i in I, there is an adapted process $b^i \leq g^i$ in the interior of $L^+(e)$ such that $b_0^i = g_0^i$ and

$$\Pi_t\left(-b_{t+1}^i\right) = -b_t^i.$$

This condition guarantees that, beginning from the initial period, any debt level not exceeding the threshold g_0^i in L_0^+ can be perpetually refinanced by issuing further debt without repayments subject to solvency constraints.[13] Over time, the individual can repay an amount b_t^i in L_t^+ of outstanding debt by issuing additional debt up to levels b_{t+1}^i in L_{t+1}^+. Furthermore, debt roll-over is persistent because the adapted process b^i belongs to the interior of $L^+(e)$ and, hence, the amount of debt that can be refinanced does not vanish along any path relative to the aggregate endowment. Our purpose is to verify under which conditions persistent debt roll-over occurs at a competitive equilibrium.

Proposition 6.1 (Necessary condition)**.** *A competitive equilibrium involves persistent debt roll-over only if*

$$\gamma(q) = 1 = \rho(q). \tag{6.1}$$

The necessity of condition (6.1) reveals that debt roll-over is a fragile property. Indeed, under incomplete markets, the upper and the lower dominant root will in general be distinct when the pricing kernel involves some volatility, as certainly occurs under *aggregate*

[12]When debt is unsecured, expectations of future deterioration of solvency conditions might be self-fulfilling, and trade might vanish in the long run. Debt is sustainable, but it disappears over time, inducing no trade in the limit. We neglect competitive equilibria of this nature and focus on those in which trade, and hence debt, occur persistently.

[13]The initial period is used only for narrative convenience: when debt roll-over occurs from some other period, all our arguments apply to the equilibrium beginning from a future contingency.

uncertainty. We first discuss this fragility informally and then identify general conditions in terms of primitives.

Notice that, when debt roll-over occurs at equilibrium, there is an adapted process b in the interior of $L^{+}(e)$ such that

$$b_t = q_t b_{t+1},$$

where q in L^{+} is the price of the risk-free bond.[14] For a *bounded* economy, this implies that the long-term rate of interest is zero along any path, that is,

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=0}^{n-1} q_{t+k}} = 1. \tag{6.2}$$

Thus, debt roll-over imposes severe restrictions at a competitive equilibrium with aggregate uncertainty: the rate of interest will require downward or upward adjustments during phases of prosperity or recession, and this flexibility is precluded by the necessary condition (6.2).

During phases of prosperity, individuals will have a tendency to accumulate assets for precautionary motives, because recessions are expected in the future. Markets will clear only if these savings are balanced by a corresponding supply of bonds. To provide incentives to borrowing, the rate of interest will need to go through downward adjustments and, under some conditions, will be recurrently negative. More formally, notice that first-order conditions require that

$$q_t \geq \max_{i\in I} \delta \mathbb{E}_t \frac{\nabla u\left(c_{t+1}^i\right)}{\nabla u\left(c_t^i\right)}.$$

Thus, under prudence (that is, when marginal utility is weakly convex),

$$q_t \geq \max_{i\in I} \delta \frac{\nabla u\left(\mathbb{E}_t c_{t+1}^i\right)}{\nabla u\left(c_t^i\right)}.$$

When output declines with positive probability, expected consumption will necessarily decrease for some individual and, when individuals are sufficiently patient,

$$q_t \geq \max_{i\in I} \delta \frac{\nabla u\left(\mathbb{E}_t c_{t+1}^i\right)}{\nabla u\left(c_t^i\right)} > 1.$$

[14] This property is the established condition (A.3) in the proof of Proposition 6.1.

Along a path of persistent prosperity, the rate of interest will be recurrently negative, which contradicts condition (6.2).

Proposition 6.2 (No debt roll-over)**.** *Consider the auxiliary pricing*

$$q_t^0 = \min_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \delta \left(\frac{e_t}{\mathbb{E}_t e_{t+1}} \right)^{\alpha}, \tag{6.3}$$

where $\bar{\alpha} > 0$ and $\underline{\alpha} > 0$ are, respectively, an upper bound and a lower bound for the coefficient of relative risk aversion, and marginal utilities are convex (prudence). If $\rho(q^0) > 1$, then there is no persistent debt roll-over at equilibrium.

Claim 6.1 (A class of economies)**.** *In an economy with Markov endowments, condition $\rho(q^0) > 1$ in Proposition 6.2 is certainly satisfied when, for some state s in S,*

$$\sum_{\hat{s} \in S} e_{\hat{s}} \mu_{s,\hat{s}} < \delta^{\frac{1}{\underline{\alpha}}} e_s,$$

where $\underline{\alpha} > 0$ is the lower bound for the coefficient of relative risk aversion.

7. Conclusion

We have shown that, under incomplete markets, private debt is sustainable by the mere reputation for repayment. The implicit enforcement mechanism relies on a high cost of self-insurance compared with the privilege of issuing debt when the rate of interest is low. Private debt reflects the value of expected future repayments and, differently from Hellwig and Lorenzoni [18], does not circulate as a speculative bubble. We interpret this as a genuine failure of the claim in Bulow and Rogoff [12] that lending must be supported by direct sanctions available to creditors.

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APPENDIX A. PROOFS

Proof of Lemma 3.1. By individual rationality, and using Assumptions 1-2, at every t in $\mathbb{T}$,

$$u^i\left(c_t^i\right)+\left(\frac{\delta}{1-\delta}\right)\sup_{\hat{c}^i\in\mathbb{R}^+}u^i\left(\hat{c}^i\right)\geq U_t^i\left(c^i\right)\geq U_t^i\left(e^i\right)\geq U_t^i\left(\epsilon_l e\right)\geq\left(\frac{1}{1-\delta}\right)u^i\left(\epsilon_l\eta\right),$$

where $\eta>0$ is the uniform lower bound for the aggregate endowment; that is, $e_t\geq\eta$ at every t in $\mathbb{T}$. By the Strong Inada Condition (Assumption 2), this suffices to prove the claim. □

Proof of Claim 4.1. We prove the condition for the upper dominant root as the argument is analogous for the lower dominant root. Preliminarily notice that, when $\theta<1$, detrended consumption $\hat{e}$ in L^+ remains in a bounded interval of $\mathbb{R}^{++}$, since

$$\min\left\{\hat{e}_0,\zeta_l^{\frac{1}{1-\theta}}\right\}\leq\hat{e}_t\leq\max\left\{\hat{e}_0,\zeta_h^{\frac{1}{1-\theta}}\right\}.$$

Indeed, to verify this claim, just observe that

$$\hat{e}_t^{\theta}\zeta_l\leq\hat{e}_{t+1}\leq\hat{e}_t^{\theta}\zeta_h.$$

We thus consider the eigenprocess b in L^+ given by $b_t=(1+g)^t\,\hat{e}_t^{\alpha}$. Crucially, this eigenprocess lies in the interior of $L^+(e)$ because $\hat{e}$ in L^+ remains in a compact interval of $\mathbb{R}^{++}$. By direct computation,

$$\begin{aligned}\Pi_t\left(b_{t+1}\right)&=q_t\left(1+g\right)^{t+1}\hat{e}_t^{\alpha\theta}\zeta_h^{\alpha}\\&=\delta\left(1+g\right)^{-\alpha}\hat{e}_t^{\alpha(1-\theta)}\mathbb{E}\left(\frac{1}{\zeta}\right)^{\alpha}\left(1+g\right)^{t+1}\hat{e}_t^{\alpha\theta}\zeta_h^{\alpha}\\&=\delta\left(1+g\right)^{1-\alpha}\mathbb{E}\left(\frac{\zeta_h}{\zeta}\right)^{\alpha}\left(1+g\right)^{t}\hat{e}_t^{\alpha}\\&=\rho\left(q\right)b_t.\end{aligned}$$

Suppose that there exists $\hat{b}$ in $L^+(e)$ such that

$$\rho\hat{b}_t \leq \Pi_t\left(\hat{b}_{t+1}\right).$$

At no loss of generality, we can assume that $\hat{b} \leq b$ and $\hat{b} \not\leq \lambda b$ for any $\lambda < 1$. Monotonicity yields

$$\rho\hat{b}_t \leq \Pi_t\left(\hat{b}_{t+1}\right) \leq \Pi_t\left(b_{t+1}\right) \leq \rho\left(q\right) b_t,$$

thus implying $\rho \leq \rho(q)$. This proves our claim when $\theta < 1$.

When $\theta = 1$, the eigenprocess b in $L^+(e)$ is given by $b_t = e_t = (1+g)^t \hat{e}_t$. Arguing as in the previous part,

$$\begin{aligned}
\Pi_t\left(b_{t+1}\right) &= q_t\left(1+g\right)^{t+1}\hat{e}_t\zeta_h \\
&= \delta\left(1+g\right)^{-\alpha}\mathbb{E}\left(\frac{1}{\zeta}\right)^{\alpha}\left(1+g\right)^{t+1}\hat{e}_t\zeta_h \\
&= \delta\left(1+g\right)^{1-\alpha}\mathbb{E}\left(\frac{1}{\zeta}\right)^{\alpha}\zeta_h\left(1+g\right)^{t}\hat{e}_t \\
&= \rho\left(q\right) b_t.
\end{aligned}$$

As in the previous part, this suffices to establish the claim. □

Proof of Claim 4.2. Let b in L^+ be the evolution of debt in a roll-over regime. If the claim is false, then b is an element of $L^+(e)$. This immediately delivers $\gamma(q) \geq 1$, a contradiction. This proves the first claim. The other claim is true because any approximate lower dominant eigenprocess b in $L^+(e)$ will allow to implement a bounded Ponzi game. □

Proof of Lemma 5.1. Let $f^{i,\epsilon}$ in L^+ be the maximum debt that can be repaid, beginning from each contingency, out of the share ϵ in $(0,1)$ of the endowment. This is well-defined and satisfies, at every t in $\mathbb{T}$,

$$0 \leq f_t^{i,\epsilon} \leq \epsilon e_t^i - \Pi_t^{\epsilon}\left(-g_{t+1}^{i,\epsilon}\right).$$

The upper bound corresponds to devoting the current share of the endowment to debt repayment and borrowing up to the limit. At every t in $\mathbb{T}$, an individual can always repay back a debt not exceeding $f_t^{i,\epsilon}$ in L_t^+ out of share ϵ in $(0,1)$ of the endowment and, at the same time, implement the optimal plan under no borrowing with no initial wealth, so that

$$J_t^{i,\epsilon}\left(-f_t^{i,\epsilon}, g^{i,\epsilon}\right) \geq J_t^{i,\epsilon}(0,0).$$

We claim that $g_t^{i,\epsilon} \geq f_t^{i,\epsilon}$. Indeed, supposing not, at some contingency,

$$J_t^{i,\epsilon}\left(-g_t^{i,\epsilon}, g^{i,\epsilon}\right) > J_t^{i,\epsilon}\left(-f_t^{i,\epsilon}, g^{i,\epsilon}\right) \geq J_t^{i,\epsilon}(0,0),$$

a contradiction. Hence, the adapted process $f^{i,\epsilon}$ in L^+ satisfies, at every t in $\mathbb{T}$, the recursive condition

$$f_t^{i,\epsilon} = \epsilon e_t^i - \Pi_t^\epsilon\left(-f_{t+1}^{i,\epsilon}\right).$$

This suffices to prove the claim, as e^i lies in the interior of $L^+(e)$. □

Proof of Lemma 5.2. We assume uniform convergence to autarky and argue by contradiction. At no loss of generality, the price of the bond satisfies, at every t in $\mathbb{T}$,

$$q_t^\epsilon = \max_{i \in I} \delta \mathbb{E}_t \frac{\nabla u^i\left(c_{t+1}^{i,\epsilon}\right)}{\nabla u^i\left(c_t^{i,\epsilon}\right)}.$$

By the hypothesis on gains from trade, there exists $\gamma > 1$ such that, for some non-zero b^0 in $L^+(e)$, at every t in $\mathbb{T}$,

$$\gamma b_t^0 \leq -\Pi_t^0\left(-b_{t+1}^0\right).$$

As convergence is uniform, for every sufficiently small ϵ in $(0,1)$,

$$b_t^0 \leq -\frac{1}{\gamma}\Pi_t^0\left(-b_{t+1}^0\right) \leq -\Pi_t^\epsilon\left(-b_{t+1}^0\right),$$

where we use the fact that $q_t^0 \leq \gamma q_t^\epsilon$ in computing the minimum-expenditure portfolio. Let $\lambda > 0$ be the greatest value such that $\lambda b^0 \leq f^\epsilon$ and, at no loss of generality, assume that $\lambda = 1$, where f^ϵ in the interior of $L^+(e)$ is given in Lemma 5.1. Monotonicity implies

$$b_t^0 \ll e_t - \Pi_t^\epsilon\left(-b_{t+1}^0\right) \leq e_t - \Pi_t^\epsilon\left(-f_{t+1}^\epsilon\right) \leq f_t^\epsilon,$$

a contradiction. □

Proof of Lemma 5.3. To establish condition (5.1), we argue by contradiction. Supposing not, $\liminf_{\epsilon \to 0} \eta_\epsilon = 0$, where $\eta_\epsilon = \sup_{t \in \mathbb{T}} \|v_t^\epsilon\|_\infty$. For every individual i in I, the budget constraint imposes, at every t in $\mathbb{T}$,

$$v_t^i = \left(c_t^i - e_t^i\right) + q_t v_{t+1}^i.$$

Furthermore, by first-order conditions,

$$\delta \mathbb{E}_t \frac{\nabla u^i \left(c_{t+1}^i\right)}{\nabla u^i \left(c_t^i\right)} \leq q_t,$$

with the equality when the individual is saving. Thus,

$$v_t^i \leq \left(c_t^i - e_t^i\right) + \delta \mathbb{E}_t \frac{\nabla u^i \left(c_{t+1}^i\right)}{\nabla u^i \left(c_t^i\right)} v_{t+1}^i.$$

Evaluating at a competitive equilibrium of the perturbed economy $\mathcal{E}^\epsilon$, and using the bound on wealth,

$$-\eta_\epsilon \leq \left(c_t^{i,\epsilon} - e_t^i\right) + \delta \mathbb{E}_t \frac{\nabla u^i \left(c_{t+1}^{i,\epsilon}\right)}{\nabla u^i \left(c_t^{i,\epsilon}\right)} \eta_\epsilon.$$

As the economy is bounded, by Lemma 3.1, marginal rates of substitution are uniformly bounded, so that, for some sufficiently large $\kappa > 0$,

$$\left(c_t^{i,\epsilon} - e_t^i\right)^- \leq \eta_\epsilon + \kappa \eta_\epsilon.$$

By feasibility, possibly extracting a subsequence, this implies uniform convergence to autarky, which is ruled out by Lemma 5.2. This shows that condition (5.1) holds true.

We now prove that debt limits remain bounded in the perturbed economy $\mathcal{E}^\epsilon$, so that condition (5.2) holds true. At no loss of generality, we can assume that, at every t in $\mathbb{T}$,

$$q_t^\epsilon = \max_{i \in I} \delta \mathbb{E}_t \frac{\nabla u^i \left(c_{t+1}^{i,\epsilon}\right)}{\nabla u^i \left(c_t^{i,\epsilon}\right)}.$$

Indeed, if not, the price process can be replaced without affecting optimal consumption and bond holding. Marginal rates of substitution are uniformly bounded because consumption is uniformly bounded from below (by Lemma 3.1) and from above (by material feasibility). Hence, there exist adapted processes $\underline{q}$ and $\bar{q}$ in the interior of $L^+ (e)$ such that $\underline{q} \leq q^\epsilon \leq \bar{q}$.

Preliminarily notice that equilibrium wealth is uniformly bounded; that is,

$$\limsup_{\epsilon \to 0} \sup_{t \in \mathbb{T}} \|v_t^\epsilon\|_\infty < \infty.$$

In fact, as prices remain bounded, out of a large enough financial wealth, individual i in I can afford a consumption plan $e^i + \bar{e}^t$, where $\bar{e}^t$ in L^+ is the aggregate endowment truncated at period t in $\mathbb{T}$. By impatience, for every individual i in I,

$$\lim_{t \to \infty} U_0^i \left(e^i + \bar{e}^t\right) > U_0^i \left(\bar{e}\right) \geq U_0^i \left(c^{i,\epsilon}\right).$$

Thus, if equilibrium wealth is unbounded, some individual would be able to afford an unfeasibly large value in utility, a contradiction.

Suppose that, by an appropriate choice of the initial state s in S, there is a sequence of equilibria in the perturbed economy $\mathcal{E}^\epsilon$ such that, for some individual i in I, $\lim_{\epsilon \to 0} g_0^{i,\epsilon} = \infty$. Notice that debt limits satisfy, at every t in $\mathbb{T}$,

$$g_t^{i,\epsilon} \leq e_t^i - \Pi_t^\epsilon \left(-g_{t+1}^{i,\epsilon}\right) \leq e_t^i - \bar{\Pi}_t \left(-g_{t+1}^{i,\epsilon}\right),$$

because otherwise the budget set would be empty. The bound in the extreme right hand-side is computed using the pricing functional $\bar{\Pi}_t : L_{t+1} \to L_t$ corresponding to the upper bound on the price process for the risk-free bond. It follows that debt limits diverge at every t in $\mathbb{T}$. Possibly extracting a subsequence, it can be assumed that the sequence of consumption plans $\left(c^{i,\epsilon}\right)_{\epsilon > 0}$ in C^i converges to a consumption plan c^i in C^i. Let $\hat{c}^i$ in C^i be $c_t^i + e_t$ up to period $\hat{t}$ in $\mathbb{T}$ and $\left(1 - \hat{\epsilon}\right) e_t^i$ at any other following period t in $\mathbb{T}$, where $\hat{\epsilon}$ lies in $(0, 1)$. By impatience, period $\hat{t}$ in $\mathbb{T}$ can be chosen sufficiently large so that $U_0^i \left(\hat{c}^i\right) > U_0^i \left(c^{i,\epsilon}\right)$ for every sufficiently small ϵ in $(0, \hat{\epsilon})$. Let $\hat{v}^{i,\epsilon}$ in L be a financial plan supporting $\hat{c}^i$ in C^i, from bounded initial wealth $v_0^{i,\epsilon}$ in L_0, ignoring solvency constraints; that is, such that, at every t in $\mathbb{T}$,

$$q_t^\epsilon \hat{v}_{t+1}^{i,\epsilon} + \hat{c}_t^i = e_t^i + \hat{v}_t^{i,\epsilon}.$$

When the individual defaults in period $\hat{t} + 1$ in $\mathbb{T}$, she can secure a level of utility at least equal to $U_{\hat{t}+1}^i \left(\left(1 - \hat{\epsilon}\right) e^i\right)$. Thus, we only need to verify that $\hat{v}_{t+1}^{i,\epsilon} \geq -g_{t+1}^{i,\epsilon}$ up to period $\hat{t}$ in

$\mathbb{T}$. This is certainly satisfied as debt limits diverge and the financial plan remains bounded on the finite horizon, thus yielding a contradiction. $\square$

Proof of Proposition 5.1. Given a perturbation ϵ in $(0,1)$, we denote $(c^\epsilon, v^\epsilon, g^\epsilon)$ in $C \times V \times G$ a competitive equilibrium of the perturbed economy $\mathcal{E}^\epsilon$. Notice that we do not fix the initial distribution of wealth, which might be varying with ϵ in $(0,1)$. At no loss of generality, the price of the bond is determined, at every t in $\mathbb{T}$, by

$$q_t^\epsilon = \max_{i \in I} \delta \mathbb{E}_t \frac{\nabla u^i \left(c_{t+1}^{i,\epsilon}\right)}{\nabla u^i \left(c_t^{i,\epsilon}\right)}.$$

We can extract a sequence of equilibrium plans $(c^\epsilon, v^\epsilon, g^\epsilon)_{\epsilon > 0}$ in $C \times V \times G$ (pointwise) converging to plans (c, v, g) in $C \times V \times G$ such that $\|v_t\|_\infty > 0$ at some t in $\mathbb{T}$. Indeed, by property (5.1), the distribution of wealth does not vanish at some state s in S and, as the Markov transition is irreducible, this state can be reached from the initial state s_0 in S in finitely many periods with positive probability. We need to verify that plans are optimal and debt limits are self-enforcing.

We first show that, at every t in $\mathbb{T}$, $\lim_{\epsilon \to 0} J_t^{i,\epsilon}(0,0) = J_t^i(0,0)$ and, just to simplify notation, we assume that $t = 0$. To this purpose, consider the (otherwise identical) program truncated at $\hat{t}$ in $\mathbb{T}$. Notice that, as this is basically a maximizaton program over a finite horizon, by canonical arguments,

$$\lim_{\epsilon \to 0} \left| J_0^{i,\hat{t}}(0,0) - J_0^{i,\epsilon,\hat{t}}(0,0) \right| = 0.$$

Any budget feasible plan in the untruncated program can be replicated in the truncated program over the truncated finite horizon. Thus, as continuation utility is bounded from above by some sufficiently large $\Delta^* > 0$ (because utility is bounded from above) and from below by some sufficiently small $\Delta_* < 0$ (because a fraction of the endowment can be consumed), we obtain, for every sufficiently small ϵ in $(0,1)$,

$$\left| J_0^{i,\epsilon}(0,0) - J_0^{i,\epsilon,\hat{t}}(0,0) \right| \leq \delta^{\hat{t}+1} \left(\Delta^* - \Delta_*\right)$$

and

$$\left| J_0^i(0,0) - J_0^{i,\hat{t}}(0,0) \right| \leq \delta^{\hat{t}+1}(\Delta^* - \Delta_*).$$

By a conventional triangular decomposition, this suffices to prove our claim. Similarly, we establish that, at every t in $\mathbb{T}$, $\lim_{\epsilon \to 0} J_t^{i,\epsilon}\left(-g_t^{i,\epsilon}, g^{i,\epsilon}\right) = J_t^i\left(-g_t^i, g^i\right)$.

Clearly, the plan in the limit satisfies budget and solvency constraints. Supposing that it is not optimal, for all sufficiently small ϵ in $(0,1)$, $J_0^i\left(v_0^i, g^i\right) > J_0^{i,\epsilon}\left(v_0^{i,\epsilon}, g^{i,\epsilon}\right) + \Delta$ for some $\Delta > 0$. As a consequence, for any truncation $\hat{t}$ in $\mathbb{T}$,

$$\begin{aligned} J_0^{i,\hat{t}}\left(v_0^i, g^i\right) + \delta^{\hat{t}+1}\Delta^* &\geq \\ J_0^i\left(v_0^i, g^i\right) &> J_0^{i,\epsilon}\left(v_0^{i,\epsilon}, g^{i,\epsilon}\right) + \Delta \\ &\geq J_0^{i,\epsilon,\hat{t}}\left(v_0^{i,\epsilon}, g^{i,\epsilon}\right) + \delta^{\hat{t}+1}\Delta_* + \Delta, \end{aligned}$$

where the upper bound $\Delta^* > 0$ and the lower bound $\Delta_* < 0$ are given as in the previous step. For a sufficiently large $\hat{t}$ in $\mathbb{T}$, this implies that $J_0^{i,\hat{t}}\left(v_0^i, g^i\right) > J_0^{i,\epsilon,\hat{t}}\left(v_0^{i,\epsilon}, g^{i,\epsilon}\right) + \Delta$ for every sufficiently small ϵ in $(0,1)$, thus delivering a contradiction because the value of the truncated program varies continuously. □

Proof of Proposition 6.1. Consider any individual i in I with persistent debt roll-over and drop the index i in I in order to simplify notation. The roll-over property immediately implies $\gamma(q) \geq 1$ and, hence, we show that $\gamma(q) \leq 1$. Supposing not, there exists $\gamma > 1$ such that, for some non-zero process $\hat{b}$ in $L^+(e)$, at every t in $\mathbb{T}$,

$$\gamma \hat{b}_t \leq -\Pi_t\left(-\hat{b}_{t+1}\right).$$

As a consequence, we can find bond holdings Δz in Z such that, at every t in $\mathbb{T}$,

$$q_t \Delta z_t \leq -\gamma \hat{b}_t \leq -\hat{b}_t \tag{A.1}$$

and

$$-\hat{b}_{t+1} \leq \Delta z_t. \tag{A.2}$$

We now show that this process of bond holdings allows for super-replicating the optimal plan under no borrowing, thus delivering a contradiction.

Define λ in $\mathbb{R}^+$ as the greatest value satisfying $g \geq \lambda \hat{b}$. Because debt limits are in the interior of $L^+(e)$, $\lambda > 0$ and, at no loss of generality, $\lambda = 1$. Thus, $g \geq \hat{b}$ and, at some contingency, $g_t < \gamma \hat{b}_t$, since otherwise $g \geq \gamma \hat{b}$, a contradiction as $\gamma > 1$. At no loss of generality, to simplify notation, assume that $g_0 < \gamma \hat{b}_0$ and, so, $\hat{b}_0 > 0$. We argue that

$$J_0(-g_0, g) > J_0\left(-\gamma \hat{b}_0, \hat{b}\right) \geq J_0(0,0),$$

a contradiction. The first strict inequality is obvious, because $g \geq \hat{b}$ and $-g_0 > -\gamma \hat{b}_0$. For the other inequality, take the plan which is optimal at $(0,0)$ in $L_0 \times L^+$ and replicate it at $\left(-\gamma \hat{b}_0, \hat{b}\right)$ in $L_0 \times L^+$ by translation, that is,

$$z_t \mapsto z_t + \Delta z_t.$$

By conditions (A.1)-(A.2), this is feasible, revealing a contradiction.

Clearly, $\rho(q) \geq \gamma(q) = 1$. It only remains to verify that $\rho(q) \leq 1$. The roll-over component b in the interior of $L^+(e)$ satisfies conditions (A.1)-(A.2) with $\gamma = 1$. If an inequality is strict at some contingency, then the previous replication argument would imply

$$J_0(-g_0, g) > J_0(0,0),$$

a contradiction. This means that, at every t in $\mathbb{T}$,

$$b_t = \Pi_t(b_{t+1}) = -\Pi_t(-b_{t+1}). \tag{A.3}$$

Suppose that, given ρ in $\mathbb{R}^+$, there is a process $\hat{b}$ in $L^+(e)$ such that

$$\rho \hat{b}_t \leq \Pi_t\left(\hat{b}_{t+1}\right).$$

Let λ in $\mathbb{R}^+$ be the maximum value such that $\lambda \hat{b} \leq b$ and, at no loss of generality, assume that $\lambda = 1$. Monotonicity yields, at every t in $\mathbb{T}$,

$$\rho \hat{b}_t \leq \Pi_t\left(\hat{b}_{t+1}\right) \leq \Pi_t(b_{t+1}) \leq b_t,$$

so that $\rho \leq 1$. This proves our claim. $\square$

Proof of Proposition 6.2. We show that $\rho(q) \geq \rho(q^0)$ and, to this purpose, it suffices to verify that, at equilibrium, $q_t \geq q_t^0$. Assuming this is not true, we have

$$q_t^0 = \min_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \delta \left(\frac{e_t}{\mathbb{E}_t e_{t+1}} \right)^{\alpha} > q_t.$$

By first-order conditions, along with convexity of marginal utility,

$$q_t \geq \delta \mathbb{E}_t \frac{\nabla u^i \left(c_{t+1}^i \right)}{\nabla u^i \left(c_t^i \right)} \geq \delta \frac{\nabla u^i \left(\mathbb{E}_t c_{t+1}^i \right)}{\nabla u^i \left(c_t^i \right)}.$$

By approximation of the marginal rate of substitution (see below),

$$q_t \geq \delta \left(\frac{c_t^i}{\mathbb{E}_t c_{t+1}^i} \right)^{\alpha^i},$$

where $\alpha^i = \bar{\alpha}$ if $\mathbb{E}_t c_{t+1}^i \geq c_t^i$ and $\alpha^i = \underline{\alpha}$ if $\mathbb{E}_t c_{t+1}^i \leq c_t^i$. Using condition (6.3), we so obtain, for every individual i in I,

$$\left(\frac{e_t}{\mathbb{E}_t e_{t+1}} \right)^{\alpha^i} > \left(\frac{c_t^i}{\mathbb{E}_t c_{t+1}^i} \right)^{\alpha^i},$$

a contradiction. We only have to explain how to approximate the marginal rate of substitution.

We omit the reference to an individual i in I. Suppose the coefficient of relative risk aversion is bounded by $\bar{\alpha} > 0$ and $\underline{\alpha} > 0$. This implies

$$\frac{\underline{\alpha}}{c} \leq -\frac{u''(c)}{u'(c)} \leq \frac{\bar{\alpha}}{c}.$$

Supposing $\bar{c} \geq \underline{c}$, integration of the left hand-side yields

$$\log \bar{c}^{\underline{\alpha}} - \log \underline{c}^{\underline{\alpha}} \leq \log u'(\underline{c}) - \log u'(\bar{c}),$$

that is,

$$\left(\frac{\bar{c}}{\underline{c}} \right)^{\underline{\alpha}} \leq \frac{u'(\bar{c})}{u'(\underline{c})}.$$

Specularly, integration of the right hand-side yields

$$\log u'(\underline{c}) - \log u'(\bar{c}) \leq \log \bar{c}^{\bar{\alpha}} - \log \underline{c}^{\bar{\alpha}},$$

that is,

$$\left(\frac{\underline{c}}{\bar{c}}\right)^{\bar{\alpha}} \leq \frac{u'(\underline{c})}{u'(\bar{c})}.$$

This completes the proof of our claim. □

APPENDIX B. MARKOV PRICING

We complement our analysis with the examination of incentives to default in a partial equilibrium framework. The pricing kernel is fixed exogenously and obeys a simple Markov transition. We provide conditions under which default is unprofitable, even though debt cannot be rolled over without recurrent repayments. As in our simple example (§2), this reveals a failure of Bulow and Rogoff [12] when some risks are uninsurable. We also evaluate the effects of anticipated default on debt sustainability.

We assume that uncertainty is described by an irreducible Markov transition $P : S \to \Delta(S)$ on a finite state space S. The price of the bond is q_s in $\mathbb{R}^{++}$ in state s in S. Debt can be issued only up to state-contingent limits g in $\mathbb{R}^S$. We let $J_s(v_s, g)$ in $\mathbb{R}^*$ be the indirect utility in state s in S beginning with initial wealth v_s in $\mathbb{R}$ and subject to debt limits g in $\mathbb{R}^S$ over the entire infinite horizon. This indirect utility is recursively determined by

$$J_s(v_s, g) = \sup u(c_s) + \delta \sum_{\hat{s} \in S} \mu_{s,\hat{s}} J_{\hat{s}}(z_s, g)$$

subject to the budget constraint,

$$q_s z_s + c_s \leq e_s + v_s,$$

and to the solvency constraint,

$$-g_{\hat{s}} \leq z_s \text{ if } \mu_{s,\hat{s}} > 0.$$

We say that debt is *sustainable* when default is unprofitable, that is, if there exist debt limits $g \gg 0$ in $\mathbb{R}^S$ such that, for every state s in S,

$$J_s(-g_s, g) \geq J_s(0, 0).$$

This condition requires that, beginning with the maximum debt, the debtor has no incentive to default when the interdiction from future borrowing is the punishment for insolvency.

To maintain our analysis tractable, we assume that the borrower is more impatient than the market, that is,

$$\delta \sum_{\hat{s} \in S} \frac{u'(e_{\hat{s}})}{u'(e_s)} \mu_{s,\hat{s}} < q_s. \tag{B.1}$$

As a result, saving after default will not be optimal and the borrower will consume the autarkic endowment. We then exhibit a simple condition ensuring that the benefits from borrowing exceed the costs due to recurrent repayments to creditors. Hence, debt is sustainable by reputation alone.

Claim B.1 (Sustainable debt). *When the borrower is more impatient than the market, debt is sustainable if, for some positive g in $\mathbb{R}^S$, there exists a strictly positive w in $\mathbb{R}^S$ satisfying*

$$w_s = u'(e_s)(-\Pi_s(-g) - g_s) + \delta \sum_{\hat{s} \in S} \mu_{s,\hat{s}} w_{\hat{s}}, \tag{B.2}$$

where $\Pi : \mathbb{R}^S \to \mathbb{R}^S$ is the valuation operator defined in §4 and in Appendix C.

Proof. Given any small λ in $\mathbb{R}^+$, consider the consumption plan

$$c_s(\lambda) = e_s + \lambda(-\Pi_s(-g) - g_s).$$

This plan corresponds to maintaining a state-contingent amount of debt λg in $\mathbb{R}^S$ over time. By the Contraction Mapping Theorem, there exists a unique fixed point $w(\lambda)$ in $\mathbb{R}^S$ of the operator $F_\lambda : \mathbb{R}^S \to \mathbb{R}^S$ given by

$$(F_\lambda w)_s = \frac{u(c_s(\lambda)) - u(e_s)}{\lambda} + \delta \sum_{\hat{s} \in S} \mu_{s,\hat{s}} w_{\hat{s}},$$

where it is understood that

$$\lim_{\lambda \to 0} \frac{u\left(c_s\left(\lambda\right)\right) - u\left(e_s\right)}{\lambda} = u'\left(e_s\right)\left(-\Pi_s\left(-g\right) - g_s\right).$$

Furthermore, the map $\lambda \mapsto w\left(\lambda\right)$ is continuous. Condition (B.2) then implies that $w\left(\lambda\right) \gg 0$ in $\mathbb{R}^S$ for some sufficiently small λ in $\mathbb{R}^{++}$. For the rest of the proof, we fix any λ in $\mathbb{R}^{++}$ with this property and, at no loss of generality, we assume that $e_s > \max_{\hat{s} \in S} \lambda g_{\hat{s}}$.

Consider the Bellman operator defined by

$$\left(TJ\right)_s\left(v_s, \lambda g\right) = \sup u\left(c_s\right) + \delta \sum_{\hat{s} \in S} \mu_{s,\hat{s}} J_{\hat{s}}\left(z_s, \lambda g\right)$$

subject to

$$q_s z_s + c_s \leq e_s + v_s$$

and

$$-\lambda g_{\hat{s}} \leq z_s \text{ if } \mu_{s,\hat{s}} > 0.$$

This operator $T : \mathcal{J} \to \mathcal{J}$ acts on the space of all bounded maps $J : S \times D \to \mathbb{R}$, where the domain is $D = \left\{v_s \in \mathbb{R} : -\max_{\hat{s} \in S} \lambda g_{\hat{s}} \leq v_s\right\} \subset \mathbb{R}$. Notice that the feasible set is certainly non-empty on this restricted domain. The Contraction Mapping Theorem ensures that this Bellman operator admits a unique value function J in $\mathcal{J}$.

For every state s in S, by construction, there exist bond holdings z_s in $\mathbb{R}$ such that

$$q_s z_s = \lambda \Pi_s\left(-g\right)$$

and

$$-\lambda g_{\hat{s}} \leq z_s \text{ if } \mu_{s,\hat{s}} > 0.$$

Budget feasibility in turn implies

$$J_s\left(-\lambda g_s, \lambda g\right) \geq u\left(c_s\left(\lambda\right)\right) + \sum_{\hat{s} \in S} \mu_{s,\hat{s}} J_{\hat{s}}\left(-\lambda g_{\hat{s}}, \lambda g\right).$$

Moreover, by condition (B.1),

$$J_s\left(0, 0\right) = u\left(e_s\right) + \delta \sum_{\hat{s} \in S} \mu_{s,\hat{s}} J_{\hat{s}}\left(0, 0\right).$$

This delivers $w^0 \geq (F_\lambda w^0)$, where

$$w_s^0 = \frac{J_s\left(-\lambda g_s, \lambda g\right) - J_s\left(0, 0\right)}{\lambda}.$$

As the orbit generated by monotone operator $F_\lambda : \mathbb{R}^S \to \mathbb{R}^S$ is weakly decreasing and converges to the only fixed point, we conclude that $w^0 \geq w\left(\lambda\right) \gg 0$, thus proving our claim. □

In the motivating example (§2), the borrower holds a constant amount of debt over time. This obtains as a particular case setting $g_s = d$ for every state s in S, so that

$$-\Pi_s\left(-g\right) - g_s = \left(q_s - 1\right) d.$$

In Claim B.1, instead, the borrower maintains a contingent amount of debt over time, providing repayments to creditors in some states while possibly raising debt exposure in other states. Condition (B.2) ensures that this plan increases the utility level at all contingencies: at the margin, the benefit of rolling over debt in some states exceeds the cost of repayments to creditors in some other states.

To clarify the role of low interest rates, it is worth remarking that condition (B.2) can only be satisfied, by a strictly positive w in $\mathbb{R}^S$, if $q_s > 1$ for some state s in S, that is, only if interest rate is negative in some state. Furthermore, by irreducibility of the Markov transition, given the prices of the bond in all other states, condition (B.2) will certainly be satisfied when the price of the bond in a single state s in S is sufficiently large. Finally, a solution to (B.2) can be determined as the only fixed point of the contraction $F_0 : \mathbb{R}^S \to \mathbb{R}^S$, where

$$\left(F_0 w\right)_s = u'\left(e_s\right)\left(-\Pi_s\left(-g\right) - g_s\right) + \delta \sum_{\hat{s} \in S} \mu_{s,\hat{s}} w_{\hat{s}}.$$

This provides an efficient operational criterion to ascertain whether debt is sustainable under the given market conditions.

We add a short digression on admitting default as in Eaton and Gersovitz [16] and Arellano [6]. Debt limits g in $\mathbb{R}^S$ are reinterpreted as default thresholds, that is, the borrower will default whenever the debt exceeds these state-contingent thresholds. On their side, creditors anticipate that repayments will occur only when the debt falls below the given threshold. The bond is so priced under risk of default, that is,

$$q_s^*\left(z_s, g\right) = q_s z_s \sum_{\hat{s} \in S} \mu_{s,\hat{s}} \mathbf{1}_{\{z_s \geq -g_{\hat{s}}\}},$$

where $\mathbf{1}_E : S \to \{0, 1\}$ is the indicator function of event $E \subset S$. Consistently, under risk of default, the indirect utility is given by

$$J_s^*\left(v_s, g\right) = \sup u\left(c_s\right) + \delta \sum_{\hat{s} \in S} \mu_{s,\hat{s}} \max\left\{J_{\hat{s}}^*\left(z_s, g\right), J_{\hat{s}}\left(0, 0\right)\right\}$$

subject to the budget constraint

$$q_s^*\left(z_s, g\right) + c_s \leq e_s + v_s.$$

At Eaton and Gersovitz [16]'s equilibrium, thresholds reflect the incentives to default, that is,

$$J_s^*\left(-g_s^*, g^*\right) = J_s\left(0, 0\right),$$

where the right-hand side represents the reservation value for the borrower (*i.e.*, the indirect utility when only saving is permitted).

We show that when debt is sustainable without risk of default, so it is with risk of default. This is a simple implication of the fact that default increases the value of market participation,

$$J_s^*\left(v_s, g\right) \geq J_s^{\mathrm{nd}}\left(v_s, g\right),$$

where $J_s^{\mathrm{nd}}\left(v_s, g\right)$ in $\mathbb{R}^*$ denotes the value when default is not permitted, as in our previous analysis. To rule out Ponzi games, we assume that there is $N \gg 0$ in $\mathbb{R}^S$ such that

$$N_s = e_s + q_s \sum_{\hat{s} \in S} \mu_{s,\hat{s}} N_{\hat{s}}.$$

This can be interpreted as requiring a finite present value of the endowment under risk neutrality.

Claim B.2 (Default)**.** *Under a finite risk-neutral present value of the endowment, when debt limits $g \gg 0$ in $\mathbb{R}^S$ satisfy $J_s^{nd}(-g_s, g) \geq J_s(0,0)$, then there exist equilibrium default thresholds g^* in $\mathbb{R}^S$ such that $g^* \geq g$.*

Proof. We first notice that, by budget feasibility,

$$g_s \leq e_s - \Pi_s(-g) \leq e_s + q_s \sum_{\hat{s} \in S} \mu_{s,\hat{s}} g_{\hat{s}} \leq \lambda^* N_s - (\lambda^* - 1) e_s,$$

where λ^* in $\mathbb{R}^+$ is the infimum over all λ in $\mathbb{R}^+$ satisfying $g \leq \lambda N$. Assuming $\lambda^* > 1$, we obtain $g \leq \lambda^* N - (\lambda^* - 1) e \ll \lambda^* N$, violating the definition of λ^* in $\mathbb{R}^+$. Thus, we conclude that $g \leq N$.

Fix any η in $\mathbb{R}$ such that $\eta < J_s(0,0)$, and consider the truncated operator $T : \mathcal{J} \to \mathcal{J}$ defined as

$$(TJ)_s(v_s, g) = \max\left\{\sup u(c_s) + \delta \sum_{\hat{s} \in S} \mu_{s,\hat{s}} \max\left\{J_{\hat{s}}(z_s, g), J_{\hat{s}}(0,0)\right\}, \eta\right\}$$

subject to the budget constraint

$$q_s^*(z_s, g) + c_s \leq e_s + v_s.$$

The space $\mathcal{J}$ contains all bounded continuous maps $J : S \times \mathbb{R} \times G \to \mathbb{R}$, where G is the positive cone of $\mathbb{R}^S$. The truncation at η in $\mathbb{R}$ avoids to deal with unbounded values and, yet, does not interfere with the determination of equilibrium default thresholds as it falls below the reservation utility. We now argue that the operator maps $\mathcal{J}$ into itself. It then follows that it admits a unique fixed point by the Contraction Mapping Theorem.

We have to show that (TJ) lies in $\mathcal{J}$ and, hence, is continuous. To this purpose, notice that

$$(TJ)_s(v_s + q_s \|g' - g''\|_\infty, g') \geq (TJ)_s(v_s, g'').$$

This is an implication of budget feasibility, because

$$q_s^*(z_s, g') - q_s \|g' - g''\|_\infty \leq q_s^*(z_s, g''),$$

so that

$$q_s^*\left(z_s, g''\right) + c_s \leq e_s + v_s \text{ only if } q_s^*\left(z_s, g'\right) + c_s \leq e_s + v_s + q_s \left\| g' - g'' \right\| .$$

Assuming that $\left(v_s^n, g^n\right)_{n \in \mathbb{N}}$ in $\mathbb{R} \times G$ converges to $\left(v_s, g\right)_{n \in \mathbb{N}}$ in $\mathbb{R} \times G$, we obtain

$$(TJ)_s \left(\max\left\{v_s^n, v_s\right\} + q_s \left\|g^n - g\right\|_\infty, g\right) \geq (TJ)_s \left(v^n, g^n\right)$$

and

$$(TJ)_s \left(v^n, g^n\right) \geq (TJ)_s \left(\min\left\{v_s^n, v_s\right\} - q_s \left\|g^n - g\right\|_\infty, g\right) .$$

So, ultimately, continuity requires us to establish that $(TJ) : S \times \mathbb{R} \times G \to \mathbb{R}$ is continuous in v_s in $\mathbb{R}$ given debt limits g in G. This is our next task.

Upper semicontinuity obtains by an application of the Maximum Theorem, as the feasible correspondence is closed. By monotonicity, we only have to show that

$$\lim_{v_s^n \to v_s^-} (TJ)_s \left(v_s^n, g\right) = (TJ)_s \left(v_s, g\right) .$$

Assuming that $(TJ)_s \left(v_s, g\right) > \eta$ at no loss of generality, by the strong Inada condition, the optimal plan in the limit involves some strictly positive level of consumption c_s in $\mathbb{R}^{++}$. Any slight contraction of initial wealth can be balanced by an equal contraction in current consumption, leaving the holdings of the bond unaltered. We so conclude that

$$(TJ)_s \left(v_s, g\right) \geq (TJ)_s \left(v_s^n, g\right) \geq J_s \left(v_s, g\right) + \left(u\left(c_s - \left(v_s - v_s^n\right)\right) - u\left(c_s\right)\right),$$

which establishes our claim.

We are now able to determine equilibrium default thresholds by means of a monotone adjustment process. Set $g^0 = g$ and recursively, for every n in $\mathbb{Z}^+$,

$$J_s^*\left(-g^{n+1}, g^n\right) = J_s\left(0, 0\right) .$$

The process is well-defined by the Intermediate Value Theorem, because

$$J_s^*\left(0, g^n\right) \geq J_s\left(0, 0\right) \geq J_s^*\left(-N_s, g^n\right) .$$

Furthermore, default thresholds are weakly increasing, $g^{n+1} \geq g^n$, as

$$J_s^*(-g^n, g^n) \geq J_s(0,0).$$

This is initially true because

$$J_s^*(-g_s, g) \geq J_s^{\text{nd}}(-g_s, g) \geq J_s(0,0).$$

Monotonicity then implies that

$$J_s^*\left(-g_s^{n+1}, g^{n+1}\right) \geq J_s^*\left(-g_s^{n+1}, g^n\right) = J_s(0,0)$$

Finally, by the strong Inada condition,

$$g_s^{n+1} \leq e_s + q_s \sum_{\hat{s}\in S} \mu_{s,\hat{s}} g_{\hat{s}}^n \leq N_s,$$

because no consumption would be affordable otherwise. As the sequence of thresholds is weakly increasing, it converges to g^* in $\mathbb{R}^S$ such that $g \leq g^* \leq N$. By joint continuity of $J^* : S \times \mathbb{R} \times G \to \mathbb{R}$, we obtain the these default thresholds are indeed an equilibrium. □

APPENDIX C. DOMINANT ROOT

We provide a self-contained presentation of the dominant root method for simple Markov pricing kernels under incomplete markets. We begin with the study of an abstract operator and relate our findings to the asset pricing kernel. Our analysis integrates and expands Bloise, Polemarchakis, and Vailakis [11, Appendix C].

We consider a continuous operator $\Pi : E \to E$ on some Euclidean space E, endowed with its canonical norm and its canonical ordering. The operator is *monotone*, that is, $v' > v''$ implies $\Pi(v') > \Pi(v'')$. It is also *sublinear*, that is, for every λ in $\mathbb{R}^+$, $\Pi(\lambda v) = \lambda \Pi(v)$, and $\Pi(v' + v'') \leq \Pi(v') + \Pi(v'')$. In addition, for some n in $\mathbb{N}$, it satisfies the property

(C.1) $$v > 0 \text{ implies } \Pi^n(v) \gg 0.$$

As usual, E^{+} is the positive cone of the linear space E. Monotone sublinearity is the property inherited by the pricing kernel, under no arbitrage, when markets are incomplete. Condition (C.1) obtains under irreducible Markov transitions.

Dominant roots are defined as in our analysis in §4. The *upper dominant root* $\rho\left(\Pi\right)$ is given by the greatest ρ in $\mathbb{R}^{+}$ such that, for some non-zero b in E^{+},

$$\rho b \leq \Pi\left(b\right).$$

Analogously, the *lower dominant root* $\gamma\left(\Pi\right)$ is given by the greatest γ in $\mathbb{R}^{+}$ such that, for some non-zero b in E^{+},

$$\gamma b \leq -\Pi\left(-b\right).$$

The upper and the lower dominant roots capture the maximum expansion rate of the operator on the positive and on the negative cone, respectively. A simple argument establishes existence of dominant roots.

Claim C.1 (Dominant roots)**.** *Both* $\rho\left(\Pi\right)$ *and* $\gamma\left(\Pi\right)$ *exist and satisfy*

$$\gamma\left(\Pi\right) \leq \rho\left(\Pi\right).$$

Proof. Let Δ be the unitary simplex in E^{+} and consider the map $F : \Delta \twoheadrightarrow \mathbb{R}^{+}$ defined by

$$F\left(d\right) = \left\{\rho \in \mathbb{R}^{+} : \rho d \leq \Pi\left(d\right)\right\}.$$

This is upper hemicontinuous with compact values. By the Maximum Theorem, the value function $f\left(d\right) = \max_{\rho \in F(d)} \rho$ is upper semicontinuous. Its maximum $\rho\left(\Pi\right) = \max_{d \in \Delta} f\left(d\right)$ is the upper dominant root. A similar argument establishes the existence of the lower dominant root. By sublinearity,

$$-\Pi\left(-b\right) \leq \Pi\left(b\right),$$

which shows that $\gamma\left(\Pi\right) \leq \rho\left(\Pi\right)$. □

We also prove that dominant roots are uniquely identified when eigenvectors exist.

Claim C.2 (Identification)**.** *If there is b in the interior of E^+ such that, for some ρ in $\mathbb{R}^+$,*

$$\rho b = \Pi(b),$$

then $\rho(\Pi) = \rho$. Analogously, if there is b in the interior of E^+ such that, for some γ in $\mathbb{R}^+$,

$$\gamma b = -\Pi(-b),$$

then $\gamma(\Pi) = \gamma$.

Proof. As the other proof is analogous, we only verify the second statement. Consider any non-zero b^* in E^+ such that

$$\gamma(\Pi) b^* \leq -\Pi(-b^*).$$

Let λ in $\mathbb{R}^+$ be the maximum value such that $\lambda b^* \leq b$ and, at no loss of generality, assume that $\lambda = 1$. This is consistent because b lies in the interior of E^+. Monotonicity yields

$$\gamma(\Pi) b^* \leq -\Pi(-b^*) \leq -\Pi(-b) \leq \gamma b,$$

which implies $\gamma(\Pi) \leq \gamma$. As $\gamma \leq \gamma(\Pi)$ by the definition of lower dominant root, the claim is proved. $\square$

We relate dominant roots to the existence of well-defined present values. Fixing a (recursive) claim e in E^+, the upper present value is the solution to recursive equation

$$f = e + \Pi(f). \tag{C.2}$$

Analogously, the lower present value is the solution to recursive equation

$$f = e - \Pi(-f). \tag{C.3}$$

We show that present values are finite if and only if dominant roots are less than unity.

Claim C.3 (Present values)**.** *Given a claim e in the interior of E^+, the upper (lower) present value is finite if and only if $\rho(\Pi) < 1$ ($\gamma(\Pi) < 1$).*

Proof. We show the claim for the lower present value, as the argument is analogous in the other case. Suppose that $\gamma(\Pi) \geq 1$ and that f in the interior of E^+ solves equation (C.3).

Let λ be the greatest value in $\mathbb{R}^+$ such that $\lambda b \leq f$, where $\gamma\left(\Pi\right) b \leq -\Pi\left(-b\right)$ and b is a non-zero element of E^+. Monotone sublinearity implies

$$\lambda b \ll e - \lambda \Pi\left(-b\right) \leq e - \Pi\left(-\lambda b\right) \leq e - \Pi\left(-f\right) \leq f,$$

a contradiction. Now assume that $\gamma\left(\Pi\right) < 1$ and define, beginning with $f^0 = 0$, for every n in $\mathbb{Z}^+$,

$$f^{n+1} = e - \Pi\left(-f^n\right).$$

Clearly, $f^{n+1} \geq f^n$. If this sequence converges, we obtain the lower present value by continuity. Otherwise, it diverges and, by linear homogeneity,

$$\frac{f^n}{\|f^n\|} \leq \frac{f^{n+1}}{\|f^n\|} = \frac{e}{\|f^n\|} - \Pi\left(-\frac{f^n}{\|f^n\|}\right).$$

Possibly extracting a converging subsequence, in the limit, for some non-zero b in E^+,

$$b \leq -\Pi\left(-b\right),$$

which implies $\gamma\left(\Pi\right) \geq 1$, a contradiction. $\square$

We are now in the condition of proving existence of dominant eigenvectors. Notice that we do not show that the lower eigenvector lies in the interior of E^+, as instead required for the identification.

Claim C.4 (Dominant eigenvectors)**.** *There exists b in the interior of E^+ such that*

$$\rho\left(\Pi\right) b = \Pi\left(b\right). \tag{C.4}$$

Furthermore, there exists a non-zero b in E^+ such that

$$\gamma\left(\Pi\right) b = -\Pi\left(-b\right). \tag{C.5}$$

Proof. The existence of an eigenvector satisfying (C.4) is proved in Bloise, Polemarchakis, and Vailakis [11, Proposition C.1] and its strict positivity is implied by condition (C.1). To establish the existence of the lower eigenvector, given any ϵ in $(0, 1)$, consider the perturbed operator

$$\Pi^{\epsilon} = \left(\frac{1-\epsilon}{\gamma\left(\Pi\right)}\right)\Pi.$$

Notice that, by linear homogeneity, $\gamma\left(\Pi^{\epsilon}\right) = 1-\epsilon$. Fix a claim e in the interior of E^{+} and observe that the lower present value f^{ϵ} in the interior of E^{+} exists for the perturbed operator (Claim C.3). Observe that $b^{\epsilon} = f^{\epsilon}/\left\|f^{\epsilon}\right\|$ is in E^{+}, with $\left\|b^{\epsilon}\right\| = 1$, and

$$b^{\epsilon} = \frac{e}{\left\|f^{\epsilon}\right\|} - \left(\frac{1-\epsilon}{\gamma\left(\Pi\right)}\right)\Pi\left(-b^{\epsilon}\right).$$

Going to the limit as ϵ in $(0,1)$ vanishes, possibly extracting a subsequence, we obtain the claim because the lower present value grows unboundedly and, hence,

$$\gamma\left(\Pi\right) b = -\Pi\left(-b\right),$$

thus concluding the proof. □

We apply our general analysis to a Markov pricing kernel under incomplete markets when the safe bond is the only asset. To this purpose, we assume that uncertainty is generated by an irreducible Markov transition $P : S \to \Delta\left(S\right)$, with $\mu_{s,\hat{s}}$ in $\mathbb{R}^{+}$ being the probability of moving from state s in S into state $\hat{s}$ in S. We consider the conventional valuation operator generated by the minimum expenditure program, that is,

$$\Pi_{s}\left(v\right) = \min_{z_s \in \mathbb{R}} q_s z_s \tag{C.6}$$

subject to

$$v_{\hat{s}} \leq z_s \text{ if } \mu_{s,\hat{s}} > 0.$$

The specular operation is given by

$$-\Pi_{s}\left(-v\right) = \max_{z_s \in \mathbb{R}} q_s z_s \tag{C.7}$$

subject to

$$z_s \leq v_{\hat{s}} \text{ if } \mu_{s,\hat{s}} > 0.$$

Under the stated assumptions, operator $\Pi : \mathbb{R}^{S} \to \mathbb{R}^{S}$ is continuous, monotone and sublinear (see LeRoy and Werner [23, Chapter 4]). In particular, condition (C.1) obtains because the Markov transition is irreducible. Given a strictly positive bond price q in $\mathbb{R}^S$, we denote $\rho(q)$ and $\gamma(q)$ the dominant roots of the implied pricing operator $\Pi : \mathbb{R}^S \to \mathbb{R}^S$.

We explicitly compute dominant roots for *strictly positive* Markov transitions. In this case, dominant roots correspond to the minimum and the maximum interest rate. The hypothesis of strictly positive Markov transitions, however, is rather restrictive for empirical applications. In addition, at a competitive equilibrium, this property only fortuitously obtains for the pricing kernel, even when satisfied by fundamentals.

Claim C.5 (Strictly positive transitions)**.** *When the Markov transition is strictly positive (that is, $\mu_{s,\hat{s}} > 0$ for all $(s, \hat{s})$ in $S \times S$),*

$$\rho(q) = \max_{s \in S} q_s \text{ and } \gamma(q) = \min_{s \in S} q_s.$$

Proof. Let $b \gg 0$ in $\mathbb{R}^S$ be given by $b_s = q_s$ at every s in S. To satisfy the constraint in (C.6), it is necessary to hold at least a quantity $\max_{\hat{s} \in S} q_{\hat{s}}$ of the risk-free bond (with unitary payoff). Thus,

$$\Pi_s(b) = \left(\max_{\hat{s} \in S} q_{\hat{s}}\right) q_s = \left(\max_{\hat{s} \in S} q_{\hat{s}}\right) b_s = \rho(q) b_s.$$

Similarly, to satisfy the reverse constraint in (C.7), it is necessary to hold no more than quantity $\min_{\hat{s} \in S} q_{\hat{s}}$ of the risk-free bond. Thus,

$$-\Pi_s(-b) = \left(\min_{\hat{s} \in S} q_{\hat{s}}\right) q_s = \left(\min_{\hat{s} \in S} q_{\hat{s}}\right) b_s = \gamma(q) b_s.$$

It may well be true that the upper dominant root is larger than unity, $\rho(q) > 1$, because the rate of interest is negative, $q_s > 1$, in some state s in S. □

APPENDIX D. PERTURBED EQUILIBRIUM

D.1. **Preliminaries.** We here prove the existence of an equilibrium in the perturbed economy $\mathcal{E}^\epsilon$ for a given ϵ in $(0, 1)$. As some parts of the proof are rather involved, we only sketch conventional steps and expand those that require more innovative arguments. In order to simplify notation, we drop any explicit reference to the given ϵ in $(0, 1)$. To establish existence, we artificially force no borrowing out of a finite horizon and progressively relax this additional constraint by taking the limit.

Using Lemma 3.1, by material feasibility and individually rationality, consumption plans are bounded from above by $\bar{\epsilon} > 0$ and from below by $\underline{\epsilon} > 0$. We fix a lower bound $\underline{q}$ and a upper bound $\bar{q}$ in L^{+} on prices such that

$$0 < \underline{q}_t < \min_{i \in I} \delta \frac{\nabla u^i \left(\bar{\epsilon}\right)}{\nabla u^i \left(\underline{\epsilon}\right)} \leq \max_{i \in I} \delta \frac{\nabla u^i \left(\underline{\epsilon}\right)}{\nabla u^i \left(\bar{\epsilon}\right)} < \bar{q}_t. \tag{D.1}$$

Both upper bound and lower bound are taken as constant processes. The auctioneer will vary prices in the truncated interval $Q = \left[\underline{q}, \bar{q}\right] \subset L^{+}$.

We assume that, for every individual i in I, $v_0^i = 0$. We truncate the economy at some s in $\mathbb{T}$ and assume that a fraction ϵ in $(0, 1)$ of the endowment is confiscated and that no borrowing is allowed after this period. On the finite horizon $\mathbb{T}_s = \{0, 1, 2, \ldots, s\}$, instead, borrowing is permitted. We shall then take the limit over truncations in the next step of the proof.

D.2. **Optimal plans.** Given a price q in the interval Q, beginning from every contingency in period t in $\mathbb{T}$, we compute the indirect utility $\bar{J}_t^i(q)$ subject to no borrowing, and no initial wealth, when a fraction ϵ in $(0, 1)$ of the endowment is expropriated. This indirect utility varies continuously with respect to prices q in the interval Q.

For fixed s in $\mathbb{T}$, we also consider a truncated program where borrowing is allowed, subject to participation, only on the finite horizon $\mathbb{T}_s = \{0, \ldots, s\}$. In this truncated program, the endowment $e^{i,s}$ in L^{+} coincides with e^i in L^{+} up to period s in $\mathbb{T}$ and with the unconfiscated fraction $(1 - \epsilon) e^i$ in L^{+} after period s in $\mathbb{T}$. At every t in $\mathbb{T}$, the individual is subject to participation constraint

$$U_t^i \left(c^i\right) \geq \bar{J}_t^i(q).$$

Furthermore, the holding of the bond is restricted, at every t in $(\mathbb{T}/\mathbb{T}_s)$, by the no borrowing constraint $v_t^i \geq 0$. Thanks to the truncation, conventional arguments show that the optimal plan varies continuously with prices, because the participation constraint is effective only over the finite horizon $\mathbb{T}_s$.

D.3. **The adjustment process.** On the domain $Q = \left[\underline{q}, \bar{q}\right]$, we construct a correspondence $F : Q \twoheadrightarrow Q$ by means of the rule

$$F_t\left(q\right) = \mathrm{argmax}_{\tilde{q} \in Q}\, \tilde{q}_t \sum_{i \in I} z_t^i\left(q\right).$$

This correspondence is upper hemicontinuous with convex values on a compact domain and, thus, it admits a fixed point. We next argue by induction and prove that, at a fixed point, $\sum_{i \in I} v_t^i \leq 0$ for every t in $\mathbb{T}$.

Suppose that $\sum_{i \in I} v_t^i \leq 0$ and $\sum_{i \in I} z_t^i > 0$. This implies $q_t = \bar{q}_t$. For some individual i in I such that $z_t^i > 0$, as participation constraint is not binding in the following period when wealth is positive, first-order conditions imply

$$\bar{q}_t \leq \delta \mathbb{E}_t \frac{\nabla u^i\left(c_{t+1}^i\right)}{\nabla u^i\left(c_t^i\right)}.$$

Because $c_t^i \leq \bar{\epsilon}$ by material feasibility (indeed, as $\sum_{i \in I} v_t^i \leq 0$ and $\sum_{i \in I} z_t^i > 0$, material feasibility follows by adding up budget constraints) and $c_{t+1}^i \geq \underline{\epsilon}$ by individual rationality, this violates condition (D.1). Hence, $\sum_{i \in I} z_t^i \leq 0$ and, thus, $\sum_{i \in I} v_{t+1}^i \leq 0$.

No borrowing after period s in $\mathbb{T}$ implies that $\sum_{i \in I} v_t^i = 0$ for all t in $\left(\mathbb{T}/\mathbb{T}_s\right)$. To complete the proof, we proceed by backward induction. Supposing that $\sum_{i \in I} z_{t+1}^i = 0$, we obtain that $c_{t+1}^i \leq \bar{\epsilon}$ by material feasibility (because $\sum_{i \in I} v_{t+1}^i \leq 0$). Furthermore, assuming that $\sum_{i \in I} z_t^i < 0$, then $q_t = \underline{q}_t$. By first-oder conditions, along with material feasibility and individual rationality,

$$\underline{q}_t \geq \delta \mathbb{E}_t \frac{\nabla u^i\left(c_{t+1}^i\right)}{\nabla u^i\left(c_t^i\right)} \geq \delta \mathbb{E}_t \frac{\nabla u^i\left(\bar{\epsilon}\right)}{\nabla u^i\left(\underline{\epsilon}\right)}.$$

This contradicts the lower bound given by (D.1).

D.4. **Relaxing the truncation.** We now take the limit by relaxing the truncation s in $\mathbb{T}$. Previous steps show the existence of a truncated equilibrium prices q^s in Q, with an associated optimal consumption plan $c^{i,s}$ in L^+ for every individual i in I. For fixed s in $\mathbb{T}$, given any contingency in period t in $\mathbb{T}$, we compute the indirect utility $J_t^{i,s}\left(v_t^i\right)$ subject to budget constraints, participation constraints and no borrowing after period s in $\mathbb{T}$ when initial wealth is v_t^i in L_t. By convention, the value is negative infinity when constraints cannot be satisfied. For every t in $\mathbb{T}$, we determine $g_t^{i,s}$ in L_t^+ as

$$J_t^{i,s}\left(-g_t^{i,s}\right) = \bar{J}_t^i, \tag{D.2}$$

where the right hand-side is the indirect utility subject to no borrowing, and no initial wealth, when the fraction ϵ in $(0,1)$ of the endowment is confiscated. A solution exists by continuity, as the participation constraint cannot be satisfied when the initial debt is too large and no borrowing is permitted eventually. Also notice that $g_t^{i,s} = 0$ for every t in $(\mathbb{T}/\mathbb{T}_s)$. The plan remains optimal when participation constraints are substituted by solvency constraints of the form $v_t^i \geq -g_t^{i,s}$. Thus, for the last steps, we only maintain not-too-tight solvency constraints (*i.e.*, satisfying condition (D.2)) and consider the limit with respect to s in $\mathbb{T}$.

Debt limits remain bounded. If not, it would be budget feasible to borrow for a large finite horizon, consuming large amount of resources, and then to revert to the plan ensuring reservation utility (see the last part of the proof of Lemma 5.3 for a similar argument). Hence, possibly extracting a subsequence, consumption plans, financial plans and debt limits converge.

D.5. **Limit.** As budget feasibility is satisfied in the limit, we argue by contradiction to show that the limit plan c^i in L^+ is optimal subject to budget and solvency constraints. Supposing not, there exists an alternative budget feasible plan $\hat{c}^i$ in L^+, with an associated trading plan $\hat{z}^i$ in Z^i, yielding higher utility. By slightly contracting initial consumption and spreading this value over time by saving a fraction and freely disposing of the rest over time, we can assume that budget and solvency constraints are never binding. By discounting, for some sufficiently large $\hat{t}$ in $\mathbb{T}$, we have

$$U_0^i\left(\hat{c}^i\right) + \delta^{\hat{t}+1}\mathbb{E}_0\left(U_{\hat{t}+1}^i\left(\left(1-\epsilon\right)e^i\right) - U_{\hat{t}+1}^i\left(\hat{c}^i\right)\right) > U_0^i\left(c^i\right),$$

where c^i in L^+ is the dominated plan in the limit. For any sufficiently large s in $\mathbb{T}$, the consumption plan $\hat{c}^i$ in L^+ and the financial plan $\hat{v}^i$ in L satisfy budget and solvency constraint at every t in $\mathbb{T}_{\hat{t}} = \left\{0, \ldots, \hat{t}\right\}$, because budget and solvency constraints are not binding.

Furthermore, $\hat{v}^i_{\hat{t}+1}$ in $L_{\hat{t}+1}$ satisfies $\hat{v}^i_{\hat{t}+1} \geq -g^{i,s}_{\hat{t}+1}$. Hence, individual i in I can implement this given plan on $\mathbb{T}_{\hat{t}}$ and the optimal plan starting from wealth $\hat{v}^i_{\hat{t}+1}$ in $L_{\hat{t}+1}$ on $(\mathbb{T}/\mathbb{T}_{\hat{t}})$, so as to secure the utility value given by

$$\begin{aligned}
U^i_0\left(\hat{c}^i\right) + \delta^{\hat{t}+1}\mathbb{E}_0\left(J^{i,s}_{\hat{t}+1}\left(\hat{v}^i_{\hat{t}+1}\right) - U^i_{\hat{t}+1}\left(\hat{c}^i\right)\right) &\geq \\
U^i_0\left(\hat{c}^i\right) + \delta^{\hat{t}+1}\mathbb{E}_0\left(J^{i,s}_{\hat{t}+1}\left(-g^{i,s}_{\hat{t}+1}\right) - U^i_{\hat{t}+1}\left(\hat{c}^i\right)\right) &\geq \\
U^i_0\left(\hat{c}^i\right) + \delta^{\hat{t}+1}\mathbb{E}_0\left(U^i_{\hat{t}+1}\left((1-\epsilon)\, e^i\right) - U^i_{\hat{t}+1}\left(\hat{c}^i\right)\right) &> U^i_0\left(c^i\right),
\end{aligned}$$

where we use the fact that the unconfiscated part of the endowment can be consumed. This shows that, for all sufficiently large s in $\mathbb{T}$, a utility greater than $U^{i,s}_0\left(c^{i,s}\right)$ is budget-affordable, a contradiction.