Dynamically Stable Matching*

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Abstract

I introduce a stability notion, *dynamic stability*, for two-sided dynamic matching markets where (i) matching opportunities arrive over time, (ii) matching is one-to-one, and (iii) matching is irreversible. The definition addresses two conceptual issues. First, since not all agents are available to match at the same time, one must establish which agents are allowed to form blocking pairs. Second, dynamic matching markets exhibit a form of externality that is not present in static markets: an agent's payoff from remaining unmatched cannot be defined independently of other contemporaneous agents' outcomes. Dynamically stable matchings always exist. Dynamic stability is a necessary condition to ensure timely participation in the economy by ensuring that agents do not strategically delay the time at which they are available to match.

KEYWORDS: *dynamic stability, dynamic matching, stable matching, non-transferable utility, externalities, credibility, market design, dynamic arrivals, aftermarkets, sequential assignment*
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1 INTRODUCTION

I formulate a stability notion, denoted *dynamic stability*, for two-sided dynamic matching markets where (i) matching opportunities arrive over time, (ii) matching is one-to-one, and (iii) matching is irreversible. Stability notions provide an analyst with a set of predictions for the self-enforcing outcomes of decentralized matching markets that depend only on the primitive payoff structure. While stability notions are extensively used in the study of static matching markets, they have not been systematically studied for dynamic matching markets, even though the latter are ubiquitous and cover many important applications, such as labor markets and child adoption.

Defining stability in a dynamic matching market brings forth two new challenges that arise when taking into account agents' intertemporal incentives. First, since not all agents are available to match at the same time, it is natural to ask which pairs of agents can object to a proposed matching. Dynamic stability assumes that only agents who are available to match at the same time can form a blocking pair. Second, whether an agent finds their matching partner acceptable depends on what their value of remaining unmatched is. In turn, this value depends on what matching the agent *conjectures* would ensue upon their decision to remain unmatched. Given a conjectured continuation matching, one could define an agent's acceptable partners to be those who are preferred to the continuation matching. This, together with a specification of the set of blocking pairs, is enough to determine whether a matching is stable in the dynamic economy: it should have no blocking pairs and agents should always be matched to acceptable partners.

The missing step is then to determine what matching the agent conjectures would result following their decision to remain unmatched. The first difficulty is that the set of agents available to match from tomorrow onward depends on both the arrivals into the economy and who remains unmatched from previous periods. In other words, today's matching together with tomorrow's arrivals define the set of feasible continuation matchings. When contemplating remaining unmatched, the agent then needs to conjecture both who else remains unmatched today and tomorrow's continuation matching. Thus, as in the literature on the core with externalities (see, for instance, Shapley and Shubik, 1969; Rosenthal, 1971; Richter, 1974; Sasaki and Toda, 1996; Pycia and Yenmez, 2017; Rostek and Yoder, 2017), an agent's payoff from remaining unmatched cannot be defined independently of other contemporaneous agents' matching outcomes. This externality sets apart dynamic matching markets from their static counterparts.

Given a conjecture about who else remains unmatched today, not all continuation matchings are equally *reasonable*. Indeed, the agent should correctly anticipate that the continuation matching should be itself self-enforcing. Thus, for a given conjecture about today's matching outcome, the agent rules out those continuation matchings that are not self-enforcing. This is still not enough to pin down a unique continuation matching and, thus, the value of remaining unmatched. For a given conjecture about who else remains unmatched today, there can be many self-enforcing matchings. Moreover, there can be many conjectures about who else remains unmatched today. Thus, the last step in determining whether the agent finds their matching partner acceptable is an assumption on how the agent selects among the reasonable conjectures. Following Sasaki and Toda (1996), I assume that the agent prefers their matching partner to remaining unmatched if the agent prefers their matching partner to one of the conjectured continuation matchings. Unlike Sasaki and Toda (1996), the agent does not entertain all continuation matchings but only those that are self-enforcing in the continuation economy.

Dynamic stability (Definition 6) is a recursive definition that builds on the elements previously described. A matching for the dynamic economy is dynamically stable if (i) there is no pair of agents who are available to match at the same time who prefer to match together and (ii) there is no agent who is matched to someone who is unacceptable. Similar to the static notion of stability, dynamic stability is defined by the absence of pairwise blocks and the requirement that each agent is matched to an acceptable partner. In contrast to static notions of stability, the set of acceptable partners today is defined using the set of dynamically stable matchings from tomorrow onward. Dynamically stable matchings always exist in any finite horizon economy (Theorem 1); I discuss their properties in Section 4. As I explain in Section 4, the proof of Theorem 1 builds on the insights in Sasaki and Toda (1996) that an agent's most pessimistic conjecture can be used to define an artificial economy without externalities in which (static) stable matchings are known to exist.

Dynamic matching markets pose new challenges for market design, two of which are analyzed in Section 5. First, in a dynamic matching market, agents choose whether and *when* to participate. Proposition 2 shows that dynamic stability is a necessary condition for *timely* participation in the market: whenever a matching fails to be dynamically stable, market participants have an incentive to delay the time at which they are available to match.[1] This echoes the observation in static matching markets that stability is a necessary condition for participation (Roth, 1984). Second, in applications such as school choice, college admissions, and teacher assignment, assign-

[1]The mechanism design literature on revenue management has studied the problem of strategic participation (see, for instance, Gershkov et al., 2015; Garrett, 2016; Bergemann and Strack, 2019).

ments are performed sequentially because not all agents are available to match at the same time (e.g., Westkamp, 2013; Andersson et al., 2018; Dur and Kesten, 2019). The centralized markets that perform these assignments operate via *spot* mechanisms, that output a matching as a function of the current set of available agents and their reported preferences, but do not condition on future matching possibilities. Unsurprisingly, spot mechanisms do not necessarily induce dynamically stable matchings (Example 4). This raises the question of what outcomes may arise when employing spot mechanisms to perform assignments in a dynamic economy. Theorem 2 shows that only dynamically stable matchings can arise as outcomes of subgame perfect Nash equilibrium of the non-cooperative game induced by a sequence of spot mechanisms. Thus, agents' forward-looking behavior is enough to overcome the mechanisms' inability to condition on all parameters of the economy. However, achieving dynamically stable matchings may be at odds with truthful behavior: even if the spot mechanism is strategy-proof for a static economy, in the dynamic economy agents have an incentive to truncate their preferences above and beyond what they would do in a static economy (Roth and Vande Vate, 1991) to ensure that their assignment reflects what they could have instead obtained by waiting to be matched.

Related literature Outside of the literature on the core with externalities, this paper relates to five other strands of literature. The first strand is the literature on market design, which studies dynamic matching markets such as those in this paper, but from the point of view of optimality instead of stability (Ünver, 2010; Anderson et al., 2015; Leshno, 2017; Schummer, 2015; Bloch and Cantala, 2017; Ashlagi et al., 2018; Thakral, 2019; Akbarpour et al., 2020; Arnosti and Shi, 2020; Baccara et al., 2020). An exception is Altinok (2019) who studies stability in dynamic many-to-one matching markets. The present study of stability is important because stability is considered a key property for the success of algorithms (Roth, 1991) and because it highlights the potential issues in applying the static notions of stability to dynamic environments.

The second strand is the literature on matching with frictions, which studies dynamic matching markets, such as those in this paper, in a non-cooperative framework (see Burdett and Coles (1997); Eeckhout (1999); Adachi (2003); Lauermann and Nöldeke (2014) for non-transferable utility, and Shimer and Smith (2000) for transferable utility). As in this strand of the literature, an agent's value of remaining unmatched (their continuation value) is determined endogenously by the remaining agents in the market and the future matching opportunities.

The third strand studies stability notions for markets in which matching opportunities are fixed and pairings can be revised over time (e.g., Damiano and Lam, 2005; Kurino, 2009; Kadam and Kotowski, 2018; Liu, 2018; Kotowski, 2019). The contribution relative to this strand is to provide stability notions for markets in which matching opportunities arrive over time and matching is irreversible. As discussed in the introduction, that matching opportunities arrive over time and that matching is irreversible introduces a primitive externality that is absent from these papers and must be addressed when defining what stability means. In particular, while this paper shares with Liu (2018) and Kotowski (2019) the perfection requirement (see also Doval, 2015) and with Kotowski (2019) the use of the approach pioneered by Sasaki and Toda (1996), the motivation for using this approach is different. Since in my paper the set of *feasible* continuation matchings cannot be defined independently of other contemporaneous agents' matching outcomes, the externality is an intrinsic feature of the environment that must be addressed by the stability notion. Instead, in Kotowski (2019), the set of feasible continuation matchings does not depend on other contemporaneous agents' matching outcomes, but the set of *equilibrium* continuation matchings may because of the perfection requirement in the stability notion, together with the assumption of non-separable payoffs.

The fourth strand studies sequential assignment problems (e.g., Westkamp, 2013; Dogan and Yenmez, 2018; Andersson et al., 2018; Dur and Kesten, 2019; Haeringer and Iehlé, 2019; Mai and Vazirani, 2019; Feigenbaum et al., 2020). Of these, only Westkamp (2013), Andersson et al. (2018), Dur and Kesten (2019), and Mai and Vazirani (2019) study models in which not all agents are available to be matched at the same time. However, their focus is on the properties of the matching implemented by the mechanism from the point of view of stability in a static market. Theorem 2 echoes observations in Westkamp (2013), Andersson et al. (2018), and Dur and Kesten (2019) that sequential assignment may be at odds with stability or truthful behavior. Theorem 2 complements these results by identifying dynamic stability as the solution concept for sequential assignment problems. Since stability notions are oftentimes used for preference identification, Theorem 2 can inform the empirical study of sequential assignment problems as in Narita (2018) and Neilson et al. (2020).

The fifth strand is the literature on the farsighted stable set (e.g., Harsanyi, 1974; Chwe, 1994; Mauleon et al., 2011; Ray and Vohra, 2015), which is used to model externalities in coalition formation games (e.g., Acemoglu et al., 2012 apply farsightedness to a dynamic noncooperative coalitional game). As in this literature, agents in my model understand the terminal consequences of their moves. While farsighted stability focuses on the credibility of coalitional blocks, dynamic stability focuses on the credibility of the continuation matchings used to dissuade agents from blocking. Relatedly, Ray and Vohra (1997) provide a recursive definition of binding agreements in a static model in which a blocking coalition anticipates, amongst other things, that the agents outside the coalition form a binding agreement among themselves.

Organization The rest of the paper is organized as follows. Section 2 describes the model and Section 3 defines dynamic stability. Section 4 shows that dynamically stable matchings exist and discusses their properties. Section 5 studies participation and incentives in dynamic matching markets. All proofs are in the appendices.

2 Model

The economy lasts for $T < \infty$ periods. There are two sides, A and B. Agents on side A are labeled $a \in \mathscr{A}$, while agents on side B are labeled $b \in \mathscr{B}$, where $\mathscr{A}$, $\mathscr{B}$ are finite sets.

An economy of length T is defined by two sequences $(A_1, \ldots, A_T)$ and $(B_1, \ldots, B_T)$ of subsets of $\mathscr{A}$ and $\mathscr{B}$, respectively, that satisfy that $A_s \cap A_r = \varnothing$ and $B_s \cap B_r = \varnothing$, whenever $s \neq r$. I denote it by $E^T = (A_1, B_1, \ldots, A_T, B_T)$. For any $t \leq T$, let $\overline{A}_t = \cup_{s=1}^t A_s$ denote the implied arrivals on side A through period t; similarly, let $\overline{B}_t = \cup_{s=1}^t B_s$ denote the implied arrivals on side B through period t.

Definitions 1 and 2 define the set of feasible allocations for E^T:

Definition 1. A period-t matching for economy E^T is a mapping

$$m_t : \overline{A}_t \cup \overline{B}_t \mapsto \overline{A}_t \cup \overline{B}_t$$

such that

1. For all $a \in \overline{A}_t$, $m_t(a) \in \{a\} \cup \overline{B}_t$,
2. For all $b \in \overline{B}_t$, $m_t(b) \in \overline{A}_t \cup \{b\}$,
3. For all $k \in \overline{A}_t \cup \overline{B}_t$, $m_t(m_t(k)) = k$.

Definition 2. A matching m for economy E^T is a tuple $(m_1, \ldots, m_T)$ such that

1. For all $t \in \{1, \ldots, T\}$, m_t is a period-t matching,
2. For all $t \in \{1, \ldots, T\}$, for all $a \in \overline{A}_t$, if $m_t(a) \neq a$, then $m_s(a) = m_t(a)$ for all $s \geq t$.

Part 2 of Definition 2 incorporates the idea that matchings in the economy are irreversible. Let $\mathcal{M}_T$ denote the set of matchings for economy E^T. Given a matching m and a period t, let $\mathcal{M}_T(m^{t-1})$ denote the set of matchings in $\mathcal{M}_T$ that coincide with m through period $t-1$.

Fix a matching m, and suppose that agents have matched according to m through period $t-1$. Then, the set of agents who can match in period t is determined by the unmatched agents in period $t-1$ and the new arrivals in period t, (A_t, B_t). Formally,

$$\mathcal{A}(m^{t-1}) = \{a \in \overline{A}_{t-1} : m_{t-1}(a) = a\} \cup A_t,$$
$$\mathcal{B}(m^{t-1}) = \{b \in \overline{B}_{t-1} : m_{t-1}(b) = b\} \cup B_t,$$

where m^{t-1} denotes the tuple $(m_1, \ldots, m_{t-1})$, with $m^0 = \{\varnothing\}$. A matching m also defines a continuation economy of length $T-t$, denoted by $E_{t+1}^T(m^t)$, with side-A arrivals given by $(\mathcal{A}(m^t), \ldots, A_T)$ and side-B arrivals given by $(\mathcal{B}(m^t), \ldots, B_T)$.

I close the model by defining agents' preferences. Each $a \in \mathscr{A}$ defines a discount factor $\delta_a \in [0,1]$ and a Bernoulli utility, $u(a, \cdot) : \mathscr{B} \cup \{a\} \mapsto \mathbb{R}$. Similarly, each $b \in \mathscr{B}$ defines a discount factor $\delta_b \in [0,1]$ and a Bernoulli utility, $v(\cdot, b) : \mathscr{A} \cup \{b\} \mapsto \mathbb{R}$. I assume that for all $a \in \mathscr{A}$ and all $b \in \mathscr{B}$, $u(a,a) = v(b,b) = 0$.

Fix a matching m and a period t. For every $a \in \mathcal{A}(m^{t-1})$, let $t_m(a)$ denote the first date at which a is matched under m. That is, $t_m(a)$ is the smallest index s such that $t \leq s$ and $m_s(a) \neq a$; otherwise, let $t_m(a) = T$. Then, let

$$U_t(a, m) = \delta_a^{t_m(a)-t} u(a, m_T(a)),$$

denote a's payoff from matching m at date t. Similarly, for $b \in \mathcal{B}(m^{t-1})$, let

$$V_t(b, m) = \delta_b^{t_m(b)-t} v(m_T(b), b),$$

denote b's payoff from matching m at date t.

For future reference, I record two properties that matchings m may satisfy: individual rationality (Definition 3) and stability for static markets (Definition 4).

Definition 3. The matching m for economy E^T is *individually rational* if for all $a \in \overline{A}_T$, $u(a, m_T(a)) \geq 0$ and for all $b \in \overline{B}_T$, $v(m_T(b), b) \geq 0$.

That is, matching m is individually rational if each agent prefers their matching partner to remaining unmatched through period T.

Definition 4. [Gale and Shapley, 1962] Suppose $T = 1$. The matching m for economy $E^1 = (A_1, B_1)$ is *stable* if the following hold:

(S_1) For all $a \in A_1$, $U_1(a, m) \geq 0$,

(S_2) For all $b \in B_1$, $V_1(b, m) \geq 0$,

(S_3) There is no pair $(a,b) \in A_1 \times B_1$ such that $u(a,b) > U_1(a,m)$ and $v(a,b) > V_1(b,m)$.

Let $\mathcal{S}(E^1)$ denote the set of stable matchings for E^1.

In a one-period economy, matching m is stable if each agent is matched to an acceptable matching partner, that is, a partner who is preferred to remaining unmatched, and there is no pair of agents who prefer each other to the partners assigned by m. Importantly, the value of remaining unmatched can be determined using the model primitives.

Throughout, I use the following example to illustrate the concepts in the paper:

Example 1. The economy lasts for two periods, that is, T=2. In $t = 1$, Jordan, LeBron, and Shaquille arrive on side A, while Bulls and Heat arrive on side B. In $t = 2$, there are no arrivals on side A, and Lakers and Cavalliers arrive on side B. That is,

$$A_1 = \{\text{Jordan}, \text{LeBron}, \text{Shaquille}\}, B_1 = \{\text{Bulls}, \text{Heat}\}, A_2 = \{\varnothing\}, B_2 = \{\text{Lakers}, \text{Cavalliers}\}.$$

Below, I list the agents' preferences. If (Lakers, 1) (resp., (Cavalliers, 1)) appears before (Heat, 0) in an agent's ranking, then they prefer to wait 1 period to match with Lakers (resp., Cavalliers) over matching immediately with Heat. That is, the 0s and 1s are the exponents of the discount factors, and the list provides the ranking of the discounted utilities, $\{\delta_\cdot^{t-1} u(\cdot, b) : a \in A_1\}$. For side B, the list represents the ranking of utilities, $\{v(a,\cdot) : b \in B_1 \cup B_2\}$.

Jordan :	(Lakers, 1)	(Bulls, 0)	(Cavalliers , 0)	
LeBron :	(Lakers, 1)	(Cavalliers , 0)	(Heat, 0)	(Cavalliers, 1)
Shaquille :	(Heat, 0)	(Lakers, 1)		

Bulls :	Jordan		
Heat :	LeBron	Shaquille	
Lakers :	Shaquille	Jordan	LeBron
Cavalliers :	LeBron	Jordan	

Consider the following three matchings (in what follows, a horizontal line separates matchings that occur in different periods):

$$m^L = \left(\begin{array}{ccc} \text{Jordan} & \_ & \text{Bulls} \\ \text{Shaquille} & \_ & \text{Heat} \\ \hline \text{LeBron} & \_ & \text{Lakers} \\ \varnothing & \_ & \text{Cavalliers} \end{array}\right) \quad m^C = \left(\begin{array}{ccc} \text{Jordan} & \_ & \text{Bulls} \\ \hline \text{LeBron} & \_ & \text{Heat} \\ \text{Shaquille} & \_ & \text{Lakers} \\ \varnothing & \_ & \text{Cavalliers} \end{array}\right) \quad m^R = \left(\begin{array}{ccc} \text{Shaquille} & \_ & \text{Heat} \\ \hline \text{Jordan} & \_ & \text{Lakers} \\ \text{LeBron} & \_ & \text{Cavalliers} \\ \varnothing & \_ & \text{Bulls} \end{array}\right)$$

Figure 1: Three matchings for the economy in Example 1.

Figure 1 illustrates for each matching only the pairings that happen within each period. For instance, note that m_1^L specifies that Jordan matches with Bulls and Shaquille matches with Heat, but also that LeBron is single. That is, $m_1^L(\text{LeBron}) = \text{LeBron}$. Similarly, m_2^L specifies that LeBron matches with Lakers and also records the period-1 matching, $m_2^L(\text{Jordan}) = \text{Bulls}$, $m_2^L(\text{Shaquille}) = \text{Heat}$.

The three matchings in Figure 1 satisfy Definition 3: each agent's matching partner is preferred to remaining single though period 2. However, only the period-2 matchings m_2^L and m_2^R satisfy Definition 4. To see this, consider m^C. If agents match according to m_1^C in $t = 1$, this induces a one-period economy in $t = 2$, with agents on side A, $\mathcal{A}(m_1^C) = \{\text{LeBron}, \text{Shaquille}\}$, and agents on side B, $\mathcal{B}(m_1^C) = \{\text{Heat}, \text{Lakers}, \text{Cavalliers}\}$. Note that m_2^C does not satisfy Definition 4 for this one-period economy: LeBron and Cavalliers form a block. Instead, it is immediate to verify that m_2^L and m_2^R do satisfy Definition 4 for the one-period economies $E_2^2(m_1^L)$ and $E_2^2(m_1^R)$, respectively.

Remark 1 highlights three assumptions that simplify notation, but are otherwise unnecessary for the results:

Remark 1. First, while the model presumes that agents can perfectly foresee when each agent arrives in the economy, all of the results extend to the case in which arrivals are *stochastic*. In this case, an economy of length T is defined as a distribution G_T over sequences E^T. Second, I assume that no two agents with the same characteristic arrive within or across periods. Both of these extensions can be found in Doval (2021). Third, while the model is written in terms of time-discounted preferences, all that matters for the results is that time preferences are dynamically consistent.

3 Dynamic Stability

Section 3 defines dynamic stability (Definition 6). Similar to the static notion of stability, dynamic stability is defined by the absence of pairwise blocks and the requirement that each agent is matched to an acceptable partner; that is, someone that is preferred to remaining unmatched. Unlike the static notion of stability, the value of remaining unmatched is determined endogenously (see Equation 1 below). In what follows, I first introduce the definition, using Example 1 to illustrate its components. I then discuss in detail the different elements in the definition.

Given a matching m, the goal is to determine whether the agents will follow the prescription of m in every period $t \in \{1, \dots, T\}$. Fix a period t and suppose that the agents have matched according to m through period $t-1$. Thus, the set of agents who can match in period t is $\mathcal{A}(m^{t-1}) \cup \mathcal{B}(m^{t-1})$.

There are two reasons the agents who can match in period t, $\mathcal{A}(m^{t-1}) \cup \mathcal{B}(m^{t-1})$, may prefer not to follow the prescription of m in period t. First, there could be a pair of agents who prefer to match together over matching according to m. Second, there could be a single agent who prefers not to match according to m. The latter could be either because m dictates that the agent matches in period t, but the agent could do better by remaining unmatched, or because m dictates that the agent is unmatched in period t, but the continuation matching, $(m_s)_{s=t+1}^T$, is not self-enforcing. In either case, I say that the agent prefers to be unavailable to match in period t. Note that an agent who is unavailable to match in period t automatically remains unmatched in period t, but an agent who is available to match in period t could remain unmatched at the end of period t.[2]

An agent can determine whether they prefer to match according to m in period t or with a contemporaneous partner on the other side by comparing their payoff under m with their payoff from matching with the new partner. Instead, the matching m, together with the model primitives do not provide enough information to determine whether an agent should be available to match in period t. To see this, recall the three matchings in Figure 1. Note that m^C and m^R can be ruled out as predictions for the economy in Example 1 using the information contained in the matching alone. First, as argued in Section 2, m_2^C is not stable in $t=2$. Thus, even if agents match according to m_1^C in $t=1$, m_2^C cannot be enforced in $t=2$. Second, m^R can also be ruled out as a prediction for the economy: LeBron and Heat prefer each other to their outcome under m^R. Thus, they would match together in $t=1$ if they anticipate that m^R is the suggested outcome for the economy.

Instead, m^L cannot be ruled out using the information contained in the matching alone. In this matching, there is only one pair who prefers to match together over their outcome under m^L: Jordan and Lakers. However, under m^L, Jordan and Lakers never meet: Jordan is supposed to match with Bulls and exit before Lakers arrives. Thus, for Jordan to match with Lakers, Jordan must prefer to wait for Lakers to arrive over matching with Bulls. Deciding whether waiting for Lakers is preferred to matching with Bulls in $t=1$ depends on what Jordan expects the matching would be should he remain unmatched in $t=1$. However, m^L says nothing about what would happen if Jordan chooses to be unavailable to match in $t=1$. Hence, he cannot decide whether he prefers matching with Bulls to remaining unmatched in $t=1$.

In the dynamic economy, together with the matching m one should at least prescribe for each period t and for each agent who can match in period t, the matching

[2]As I explain after stating Definition 6, it is enough to consider in each period t, blocks by agents who are matched under m in period t. Considering both types of blocks allows me to introduce a property of dynamically stable matchings that is key in the proof of Theorem 1 (see Remark 2).

that would ensue if the agent chose to be unavailable to match in period t. This is the role of what I dub the agent's *conjectures*, to which I turn next.

Consider an agent k who can match in period t under matching m and, instead, contemplates being unavailable to match in period t. A conjecture for agent k is a matching $\overline{m}$ such that $\overline{m}_t(k) = k$. Since k cannot alter the matchings through period $t-1$, $\overline{m}$ must be an element of $\mathcal{M}_T(m^{t-1})$. Note that the conjecture defines both who matches in period t, $\overline{m}_t$, and the continuation matching $(\overline{m}_s)_{s=t+1}^T$. The reason is that to determine the set of feasible matchings from period $t+1$ onward when k is not available to match in period t, one needs to determine who else remains unmatched in period t. After all, the unmatched agents in period t *together* with the new arrivals define the set of agents who can match from period $t+1$ onward.

Having determined the set of *feasible* conjectures for k, I identify in two steps the matchings that k can exclude from consideration.

First, note that each $(m^{t-1}, \overline{m}_t)$ determines a continuation economy $E_{t+1}^T(m^{t-1}, \overline{m}_t)$. Because the remaining unmatched agents and the new entrants can (and will) object to any matching that is not self-enforcing, k should exclude from consideration any such matching. Recall that dynamic stability is a recursive definition and let $\mathcal{D}_t$ denote the correspondence that maps economies of length t, E^t, to the set of dynamically stable matchings for E^t. Then, k can exclude from consideration any continuation matching $(\overline{m}_s)_{s=t+1}^T$ that is not an element of $\mathcal{D}_{T-t}(E_{t+1}^T(m^{t-1}, \overline{m}_t))$.[3]

Second, one needs to determine the period-t matchings (if any) that k can exclude from consideration. The minimal restriction on the set of period-t matchings that k should entertain is that the matching formed by the agents who match in period t forms a (static) stable matching. Thus, k never conjectures that the agents who exit the economy in period t could have found a better matching among themselves. Formally,

Definition 5. Fix a matching, m. The period-t matching, m_t, is *stable among those who match in period t* if $m_t \in \mathcal{S}(A_t^m, B_t^m)$ where $A_t^m = \{a \in \mathcal{A}(m^{t-1}) : m_t(a) \neq a\}$, and similarly, $B_t^m = \{b \in \mathcal{B}(m^{t-1}) : m_t(b) \neq b\}$.

Given these restrictions, one can define the set of matchings that k conjectures may ensue if k decides to be unavailable to match in period t. I denote this set by

[3]There is a slight abuse of notation. Whereas $\mathcal{D}_{T-t}\big(E_{t+1}^T(m^{t-1}, \overline{m}_t)\big)$ defines a matching only for the agents who can match from $t+1$ onward, $\overline{m}$ also specifies the outcome for those who have matched through period t. Thus, this should be read as "$(\overline{m}_s)_{s=t+1}^T$ coincides with an element of $\mathcal{D}_{T-t}$ for the agents who are *yet* to be matched at the end of period t."

$M_{\mathcal{D}}(k, m^{t-1})$. Formally,

$$M_{\mathcal{D}}(k, m^{t-1}) = \left\{ \begin{array}{ll} \overline{m} \in \mathcal{M}_T(m^{t-1}): & \text{(i) } \overline{m}_t(k) = k, \\ & \text{(ii) } (\overline{m}_s)_{s=t+1}^T \in \mathcal{D}_{T-t}\left(E_{t+1}^T(\overline{m}^t)\right), \\ & \text{(iii) } \overline{m}_t \text{ satisfies Definition 5} \end{array} \right\}. \tag{1}$$

Example 1 (continued)**.** I now illustrate the set $M_{\mathcal{D}}(k, m^{t-1})$ for the case in which k = Jordan and $m = m^L$. Consider the following four matchings:

$$\overline{m}_{J1} = \begin{pmatrix} \text{Shaquille} & \_ & \text{Heat} \\ \hline \text{LeBron} & \_ & \text{Cavalliers} \\ \text{Jordan} & \_ & \text{Lakers} \\ \varnothing & \_ & \text{Bulls} \end{pmatrix} \quad \overline{m}_{J2} = \begin{pmatrix} \text{LeBron} & \_ & \text{Heat} \\ \hline \text{Shaquille} & \_ & \text{Lakers} \\ \text{Jordan} & \_ & \text{Bulls} \\ \varnothing & \_ & \text{Cavalliers} \end{pmatrix}$$

$$\overline{m}_{J3} = \begin{pmatrix} \text{Shaquille} & \_ & \text{Heat} \\ \hline \text{LeBron} & \_ & \text{Lakers} \\ \text{Jordan} & \_ & \text{Bulls} \\ \varnothing & \_ & \text{Cavalliers} \end{pmatrix} \quad \overline{m}_{J4} = \begin{pmatrix} \text{Shaquille} & \_ & \text{Heat} \\ \text{LeBron} & \_ & \text{Bulls} \\ \hline \text{Jordan} & \_ & \text{Lakers} \\ \varnothing & \_ & \text{Cavalliers} \end{pmatrix}$$

Figure 2: Four conjectures for Jordan, only $\overline{m}_{J1}$ and $\overline{m}_{J2}$ are valid.

In Figure 2, only $\overline{m}_{J1}, \overline{m}_{J2}$ are valid conjectures for Jordan. As will be clear after the statement of Definition 6, dynamic stability reduces to the static notion of stability in one-period economies. Since $\overline{m}_{J3}$ does not prescribe a stable matching in $t = 2$ (Jordan and Lakers are a block), then it fails to satisfy condition (ii) in Equation 1. Instead, $\overline{m}_{J4}$ is not a valid conjecture for Jordan because the period-1 matching does not satisfy condition (iii) in Equation 1. Under $\overline{m}_{J4}$, LeBron and Heat are supposed to match in $t = 1$, but prefer to match with each other over matching with their matching partners in $\overline{m}_{J4}$, so that the period-1 matching does not satisfy Definition 5.

Definition 6 defines dynamic stability. Together with prescribing how to use the conjectures to define an agent's value of remaining unmatched, it also prescribes the pairs of agents that can object to a given matching:

Definition 6. Given the correspondences $(\mathcal{D}_t)_{t \le T-1}$, matching m is *dynamically stable* for E^T if for all $t \in \{1, \ldots, T\}$ the following hold:

(D_1) For all $a \in \mathcal{A}(m^{t-1})$, there exists $\overline{m} \in M_{\mathcal{D}}(a, m^{t-1})$ such that $U_t(a, m) \ge U_t(a, \overline{m})$,

(D_2) For all $b \in \mathcal{B}(m^{t-1})$, there exists $\overline{m} \in M_{\mathcal{D}}(b, m^{t-1})$ such that $V_t(b, m) \ge V_t(b, \overline{m})$,

(D_3) There is no pair $(a, b) \in \mathcal{A}(m^{t-1}) \cup \mathcal{B}(m^{t-1})$ such that $u(a, b) > U_t(a, m)$ and $v(a, b) > V_t(b, m)$.

Let $\mathcal{D}_T(E^T)$ denote the set of dynamically stable matchings for E^T.

In words, a matching m is dynamically stable if the following hold. First, for each period t and each agent k who can match in period t under m, there exists a matching $\overline{m}_k \in M_{\mathcal{D}}(k, m^{t-1})$, such that k's payoff under m is at least the payoff that k would obtain by remaining unmatched in period t when the matching is $\overline{m}_k$. Second, there is no pair of agents who can match in the same period and prefer matching together to matching according to m. Since $\mathcal{A}(m^{t-1}) \cup \mathcal{B}(m^{t-1})$ may contain agents who arrive before period t, condition (D_3) does not rule out *all* blocks involving agents who arrive in different periods. Instead, it requires that both agents are present when they block, since only then they can evaluate whether they want to carry out the block.

When $T = 1$, the definition of dynamic stability coincides with the static notion of stability. Indeed, fix a one-period economy $E^1 = (A_1, B_1)$ and a matching m for E^1. Fix a in A_1 and note that all matchings $\overline{m}$ in the set $M_{\mathcal{D}}(a, m^0)$ satisfy that $\overline{m}_1(a) = a$ (part (ii) in the definition of $M_{\mathcal{D}}$ is vacuous). Thus, condition (D_1) simply states that a prefers m to remaining single. Condition (D_3) in Definition 6 states that the matching m has no pairwise blocks. Thus, when $T = 1$, Definition 6 reduces to Definition 4, so that the correspondence $\mathcal{D}_1$ coincides with $\mathcal{S}$.

Whereas the comparison of Definitions 4 and 6 suggests that in the dynamic economy, the value of waiting to be matched replaces the value of being single, this analogy is potentially incomplete. In a static economy, an agent who is single under a stable matching receives *exactly* the value they can guarantee by being single. Instead, Definition 6 suggests that for an agent who is unmatched in period t, there may be a gap between the utility of matching m and the utility of the conjectured matching $\overline{m}$, which represents the value that the agent can guarantee by choosing to be unavailable to match. This gap is only superficial since dynamically stable matchings satisfy the following property, which I record for future reference:

Remark 2. Let m be a dynamically stable matching for E^T. For all $t \geq 1$, for all $k \in \mathcal{A}(m^{t-1}) \cup \mathcal{B}(m^{t-1})$ such that $m_t(k) = k$, then $m \in M_{\mathcal{D}}(k, m^{t-1})$.

Thus, for those agents who are unmatched under m in a given period, the conjectured matching, $\overline{m}$, can be picked to exactly coincide with the matching m. That is, if m is dynamically stable, an agent who can match in period t, but remains unmatched in period t under m, is indifferent between being available to match in period t and being unavailable to match in period t. It follows that if a matching m satisfies condition (D_3) and for each period t, conditions (D_1) and (D_2) for agents k such that $m_t(k) \neq k$, then m is dynamically stable.

I now illustrate Definition 6 for the case in which $T = 2$ using Example 1:

Example 1 (continued)**.** There are two dynamically stable matchings:

$$m^L = \left(\begin{array}{lcl} \text{Jordan} & \_\_ & \text{Bulls} \\ \text{Shaquille} & \_\_ & \text{Heat} \\ \hline \text{LeBron} & \_\_ & \text{Lakers} \\ \varnothing & \_\_ & \text{Cavalliers} \end{array}\right) \qquad \overline{m}^L = \left(\begin{array}{lcl} \text{Jordan} & \_\_ & \text{Bulls} \\ \text{LeBron} & \_\_ & \text{Heat} \\ \hline \text{Shaquille} & \_\_ & \text{Lakers} \\ \varnothing & \_\_ & \text{Cavalliers} \end{array}\right)$$

Figure 3: Dynamically stable matchings in Example 1.

To check that m^L is dynamically stable, it only remains to verify that Jordan cannot improve on his matching outcome by waiting to be matched. Since $\overline{m}_{J2}$ in Figure 2 is a reasonable conjecture for Jordan, it follows that by being unavailable to match in $t = 1$, he can at most guarantee his payoff from matching with Bulls in $t = 2$. Therefore, Jordan prefers to match with Bulls in $t = 1$ rather than to wait for Lakers to arrive. Note how the requirement that Jordan's conjectured matching induces a stable matching in period 2 "prevents" Jordan and Lakers from blocking m^L. Under the conjectured matching $\overline{m}_{J2}$, Lakers matches with Shaquille, who is preferred to Jordan. Indeed, this is the only stable matching in $t = 2$. Thus, when Jordan and Lakers cannot agree in advance that they will match together in $t = 2$, there are instances in which Lakers is not willing to match with Jordan, once Jordan waits for Lakers to arrive. Anticipating this, Jordan prefers to match with Bulls in $t = 1$.

The matching $\overline{m}^L$ is also dynamically stable. Indeed, the matching $\overline{m}_{J1}$ in Figure 2 is a valid conjecture for LeBron when he considers being unavailable to match in $t = 1$. Since LeBron prefers to match with Heat in $t = 1$ over matching with Cavalliers in $t = 2$, LeBron cannot object to $\overline{m}^L$ in $t = 1$.

Discussion: Having introduced the definition of dynamic stability, I now discuss the role of its different components, starting with the set of conjectures, $M_{\mathcal{D}}(k, m^{t-1})$.

Two assumptions about the timing of an agent's decision to match in period t according to matching m underlie the definition of the set of conjectures, $M_{\mathcal{D}}(k, m^{t-1})$.

First, this decision is made before agents are matched according to the prescribed matching in period t. To see this, note that k does not necessarily assume that the remaining agents, except perhaps for k's matching partner, match according to m_t. As an illustration, consider Example 1. When considering whether Jordan could block m^L, both $\overline{m}_{J1}$ and $\overline{m}_{J2}$ in Figure 2 are valid conjectures under dynamic stability. However, only $\overline{m}_{J1}$ respects that Shaquille's assignment is the same as in m^L. It is well-known in static matching models with externalities that if agents make decisions to block only after agents are matched according to m_t, stable matchings may not exist (Chowdhury, 2004). Doval (2015) shows that this observation extends to dynamic matching markets.

Second, all agents who can match in period t under m, that is, the remaining unmatched agents together with the new arrivals, decide *simultaneously* whether to be available to match in period t (and also whether to form blocking pairs). That is, when agent k contemplates being unavailable to match in period t, two things are true: (i) agent k is not yet unavailable to match, and (ii) no one else has yet made this decision.

To understand the role of the second assumption in Equation 1, note the following. The requirement that a conjectured matching $\overline{m}$ satisfies Definition 5 in period t and that it induces a dynamically stable matching from period $t+1$ onward says nothing about (a) the existence of blocking pairs in period t involving at least one agent who remains unmatched in period t, or (b) whether agents who match in period t under $\overline{m}$ prefer to remain unmatched. Both decisions depend on the agents other than k understanding what the *new* continuation matching is, i.e., $(\overline{m}_s)_{s=t+1}^T$, and making decisions optimally relative to this. The optimality of these decisions depends on the matching that they conjecture would ensue if they did not follow the prescription of $\overline{m}$. This, in turn, requires knowing the set of agents who remain available to match in period t, which presumably does not include k. In this case, it is *as if* the agents other than k learn that k is unavailable to match in period t *before* making the same decision themselves. This contradicts the assumption that these decisions are made simultaneously.

Since agents make their decisions to match simultaneously, the set of conjectures *does not* describe how k expects the remaining agents (and the continuation economy) to *react* to k's decision to be unavailable to match in period t. Instead, it describes that even if k chooses to object to m in period t, *some* matching, $\overline{m}_t$, will happen in period t. Since dynamic stability describes the set of predictions in the continuation economy, it is possible to pin down the properties of the matching that would ensue in the continuation economy. Regardless of the period-t matching $\overline{m}_t$, the continuation matching will be dynamically stable for $E_{t+1}^T(m^{t-1}, \overline{m}_t)$. However, it is not immediate to prescribe exactly what the period-t matching $\overline{m}_t$ should be. On the one hand, agents other than k could also decide to be unavailable to match or form blocking pairs. On the other hand, even if k is the only one to find fault with the proposed matching, the remaining period-t agents evaluate their decisions to be available to match and/or form blocking pairs using m as the prescribed outcome and under the assumption that all agents who can match in period t, including k, are indeed available to match in period t. It is natural to assume that $\overline{m}_t$ at the very least satisfies Definition 5: when making their decisions to match in period t, no agent should exit with a partner worse than remaining single forever, and no two agents who are exiting should have preferred to exit with each other.

The previous discussion offers two takeaways. First, the discussion highlights the externality that sets aside dynamic matching markets from their static counterparts. Although one should describe the final outcome when there is a block in a static matching market, doing so is not needed to define the payoffs for the blocking agents. Instead, the dynamic economy forces the analyst to explicitly describe the outcome in period t, even if there is a block. Otherwise, it is not even possible to define the set of feasible continuation matchings. Second, a potential avenue for further research is to consider refinements of dynamic stability by strengthening Definition 5. For instance, one could require that the conjectured period-t matching $\overline{m}_t$ satisfies that no agent exits with a partner worse than the value of remaining unmatched in period t, as described by the set $M_{\mathcal{D}}(k, m^{t-1})$. For instance, if $\overline{m}_t$ specifies that $\overline{m}_t(a) \neq a$, then a prefers $\overline{m}_t(a)$ to the worst element of $M_{\mathcal{D}}(a,\cdot)$. This would be consistent with the idea that agents in period t make their matching decisions using their set of conjectures. The appropriate strengthening of Definition 5 may depend on the application at hand. The analysis in Section 5.2 identifies dynamic stability as the solution concept for sequential assignment problems, highlighting how the set of conjectures and, in particular, Definition 5 arise in a particular application.

Together with the set of conjectures, two other elements of Definition 6 are worth noting.

First, Definition 6 implies that in order to prevent an agent from blocking matching m, it suffices to find a conjectured matching that is worse than m. As I explain after the statement of Theorem 1, the ability to *select* conjectures to dissuade agents from objecting to a matching is important for the existence result. Alternatively, one could require that in order to prevent an agent from blocking matching m, *all* conjectured matchings must be worse than m. Under this alternative definition, there are economies for which no matching is self-enforcing. To illustrate, consider Example 1. Under the alternative definition, Jordan blocks m^L since $\overline{m}_{J1}$ is a valid conjecture for Jordan, and Jordan prefers $\overline{m}_{J1}$ to m^L Similarly, under the alternative definition, LeBron blocks $\overline{m}^L$: a valid conjecture for LeBron involves Jordan and Bulls matching in $t = 1$, in which case, the only stable matching in $t = 2$ matches LeBron with Lakers. Thus, in the economy in Example 1, there is no dynamically stable matching under the alternative definition of blocking.

The existing literature offers no general prescription between the two notions of blocking previously described (see Ray and Vohra (1997) for a similar discussion). However, considering the alternative notion of blocking might require changing other assumptions in the model, in particular, the timing of decisions in each period. To see this, consider an agent k who chooses to be unavailable to match in period t. A typical argument in favor of the alternative notion of blocking uses *forward induction*: when the agents in period $t+1$ observe that k deviated from the prescribed matching m, they should infer that this only makes sense if there is a continuation matching that k preferred to matching m. However, to select k's most preferred conjecture, the forward induction reasoning should apply to both the period-t matching *and* the continuation matching. That is, the remaining period-t agents should be able to *react* in period t to k's decision to block. This is inconsistent with the assumption that agents in period t make decisions simultaneously.

Second, Definition 6 does not require that agents hold "common beliefs" about the matching that would ensue when they decide to be unavailable to match in period t. That is, suppose two agents k and k' anticipate that the matchings $\overline{m}_k$ and $\overline{m}_{k'}$ would ensue when they choose to be unavailable to match in period t. Then, $\overline{m}_k$ and $\overline{m}_{k'}$ need not coincide for the agents other than k and k' in period t. Furthermore, even if $\overline{m}_k$ and $\overline{m}_{k'}$ coincide in period t, they need not coincide in the continuation economy. This property is shared with the model of Sasaki and Toda (1996) and with the models of Ambrus (2006) and Liu et al. (2014), where the solution concepts are akin to rationalizability. Note, however, that this lack of commonality only applies to those agents who match in period t and for whom remaining unmatched in period t is an *off path* event. Indeed, the property in Remark 2 implies that in a dynamically stable matching m, the agents who remain unmatched in period t under m all share the same conjecture: even if they chose to object to m in period t, they all agree that m would still be the outcome from period t onward.

4 Properties

Theorem 1 presents the main result of the paper: the set of dynamically stable matchings is non-empty.

Theorem 1. *For all $T \in \mathbb{N}$, the correspondence $\mathcal{D}_T$ is non-empty valued.*

See Appendix A for the proof, which shows how to find a matching (labeled $m^\star$ in the proof) that is dynamically stable. Because of the recursive nature of dynamic stability, the proof needs to simultaneously determine the conjectures, $M_{\mathcal{D}}(k, {m^\star}^{t-1})$, and the matching, $m^\star$, that is dynamically stable given these conjectures.

To determine the conjectures, the proof proceeds by induction on $T \geq 1$: to show that dynamic stability is well defined for T, one must show that it is well defined for $T' < T$. This is the step that uses the assumption that $T < \infty$. As argued in Section 3, dynamic stability coincides with the static notion of stability when $T = 1$. It follows from Gale and Shapley (1962) that $\mathcal{D}_1(\cdot)$ is non-empty for all one-period economies.

Given the set of conjectures, I adapt the proof technique in Sasaki and Toda (1996) to find a matching that is dynamically stable. Sasaki and Toda (1996) show how to use a set of conjectures to construct an economy without externalities. Building on their insights, I use the agents' conjectures to construct a "static" economy in period t as follows. To be concrete, consider the case in which $T = 2$. For each agent $k \in A_1 \cup B_1$, I calculate k's continuation value to be the payoff from the worst matching in $M_{\mathcal{D}}(k, \cdot)$. Given these continuation values, I truncate k's preferences so that k is only willing to match with period-1 agents that are preferred to k's continuation value. I choose the period-1 matching, $m_1^\star$, to be a stable matching for the one-period economy with the truncated preferences. This, in turn, determines the set of unmatched agents at the end of $t = 1$. I then choose the period-2 matching, $m_2^\star$, to be a stable matching among the newly arriving agents and the remaining ones from period 1.

By construction, $m^\star$ satisfies conditions (D_1) and (D_2) of Definition 6. In particular, for agents who do not match in $t = 1$ under $m^\star$, I show that $m^\star$ is a valid conjecture (recall Remark 2). It only remains to check that $m^\star$ satisfies condition (D_3) in Definition 6. Suppose there is a pair (a, b) of agents in period 1, who prefer matching together over $m^\star$. By construction of $m^\star$, it must be that at least one of the members of the pair is unmatched in $t = 1$. For concreteness, say it is a. It then follows that a truncated their preferences "too much:" $m^\star$ is worse than the worst matching in $M_{\mathcal{D}}(a, \cdot)$. However, this contradicts that $m^\star$ is an element of $M_{\mathcal{D}}(a, \cdot)$.

Both the ability to select conjectures to dissuade agents from blocking a matching together with the property in Remark 2 are key to show that dynamically stable matchings exist. To see the role of the former, suppose instead that the existence of a conjecture $\overline{m}$ that is preferred to m was enough for an agent to block matching m. Then one could modify the construction in the proof as follows. Instead of truncating each agent's preferences using the worst element in $M_{\mathcal{D}}(\cdot)$, one would truncate each agent's preferences using the best element in that set. The same construction would again lead to a matching that satisfies the property in Remark 2 for those agents who remain unmatched in $t = 1$. However, this is not enough to argue that the unmatched agents would not form a blocking pair with another agent in $t = 1$. After all, the constructed matching need not be the best conjecture for all the unmatched agents.

Similarly, without the property in Remark 2, the ability to select conjectures is not enough for the result in Theorem 1. The ability to promise a blocking agent the worst element in $M_{\mathcal{D}}(\cdot)$ maximally dissuades agents who match in period t from being unavailable to match. However, there may be agents for whom remaining unmatched in period t is the best they can do given this promise. The property in Remark 2 implies that one can effectively deliver this promise simultaneously to all of the agents who remain unmatched in period t when choosing a dynamically stable matching from period $t+1$ onward.[4] The reason is that all agents agree that (i) (static) stability is a minimal requirement of the outcome for the agents who match in period t and (ii) continuation matchings are dynamically stable for the continuation economy.

Algorithms: Even though it is not constructive, the proof of Theorem 1 suggests an algorithm to find dynamically stable matchings for economies of length 2. To see this, fix an economy, $E^2 = (A_1, B_1, A_2, B_2)$. The algorithm consists of three steps:

(a) Conjectures: For each $k \in A_1 \cup B_1$ and for each $C \subseteq (A_1 \cup B_1) \setminus \{k\}$, define the set $M(k, C)$ as follows. Any matching m in $M(k, C)$ is such that (i) $m_1(k') = k'$ if $k' \notin C$, (ii) the restriction of m_1 to C coincides with side-A deferred acceptance on C, and (iii) m_2 is a stable matching for $\mathcal{A}(m_1) \cup \mathcal{B}(m_1)$. Let $M(k)$ denote the union of $M(k, C)$ over the (possibly empty) subsets $C \subseteq (A_1 \cup B_1) \setminus \{k\}$.

(b) Period-1 matching: As in the proof of Theorem 1, for each agent $k \in A_1 \cup B_1$, truncate k's preferences using the set $M(k)$. Let $m_1^\star$ denote the outcome of side-A deferred acceptance on $A_1 \cup B_1$ using the truncated preferences.

(c) Period-2 matching: The period-2 matching, $m_2^\star$, is obtained by running side-A deferred acceptance on $\mathcal{A}(m_1^\star) \cup \mathcal{B}(m_1^\star)$.

Two comments are in order. First, step (a) does not recover the entire set of conjectures, $M_{\mathcal{D}}(k, \cdot)$: In order to do so, one should repeat step (a) for each subset C and each (static) stable matching amongst the agents in C. However, k's payoff from remaining unmatched in period 1 depends on the period-1 matching amongst the agents in C only through the set of unmatched agents that it induces. By the Lone Wolf theorem (McVitie and Wilson, 1970), the latter set is independent of the stable matching one selects for C. Second, by changing side-A deferred acceptance in steps (b) and (c) to other (static) stable matchings, the algorithm retrieves different dynamically stable matchings.

Whether an algorithm to find dynamically stable matchings for economies of length

[4] The lack of commonality in the conjectures could have hindered existence by making it difficult to find one continuation matching that "works" for all agents who remain unmatched in period t.

$T \geq 3$ exists is still an open question. The key difficulty in designing an algorithm for an economy of length T is to find an algorithm that recovers an agent's worst dynamically stable matching for an economy of length $T-1$. After all, step (b) only uses each agent's worst conjectured matching to truncate their preferences. As an example in Section A.1 illustrates, the algorithm described above for $T=2$ does not recover every agent's worst dynamically stable matching. In this example, there are two dynamically stable matchings, labeled m^A and m^B. The algorithm only recovers m^A, whereas there are agents who are worse off under m^B. Therefore, the algorithm for $T=2$ cannot be extended to economies of length $T \geq 3$.

However, when agents do not discount the future, one can find dynamically stable matchings using the algorithms for the static notion of stability. To see this, let $\overline{m}_T$ be a (static) stable matching for $(\overline{A}_T, \overline{B}_T)$, that is, an element of $\mathcal{S}(\overline{A}_T, \overline{B}_T)$. When the agents do not discount the future, it is immediate to verify that the matching m that leaves all agents unmatched until period T and matches them according to $\overline{m}_T$ in period T is dynamically stable. Nevertheless, this construction retrieves some, but not all, dynamically stable matchings. There are two reasons for this. First, the ability to select conjectures can be used to dissuade agents from waiting to be matched and instead, accept matching partners not consistent with the static notion of stability. Second, as Example 2 illustrates, even if agents do not discount the future, the dynamic and static notions of stability allow for different pairwise blocks:

Example 2. Consider the following variant of Example 1. Jordan and Bulls arrive at $t=1$. The remaining agents arrive at $t=2$, except for Cavalliers, who no longer arrives. That is, $A_1 = \{\text{Jordan}\}$, $B_1 = \{\text{Bulls}\}$, $A_2 = \{\text{LeBron}, \text{Shaquille}\}$, and $B_2 = \{\text{Heat}, \text{Lakers}\}$. The agents do not discount the future. Their rankings are as follows:

$$\begin{array}{rccc}
\text{Jordan}: & \text{Lakers} & \text{Bulls} & \\
\text{LeBron}: & \text{Lakers} & \text{Heat} & \\
\text{Shaquille}: & \text{Heat} & \text{Lakers} &
\end{array}
\qquad
\begin{array}{rccc}
\text{Bulls}: & \text{Jordan} & & \\
\text{Heat}: & \text{LeBron} & \text{Shaquille} & \\
\text{Lakers}: & \text{Shaquille} & \text{Jordan} & \text{LeBron}
\end{array}$$

The following two matchings are dynamically stable:

$$m^{\mathcal{S}} = \begin{pmatrix} \varnothing \\ \hline \text{Jordan} \; \_\_ \; \text{Bulls} \\ \text{LeBron} \; \_\_ \; \text{Heat} \\ \text{Shaquille} \; \_\_ \; \text{Lakers} \end{pmatrix} \qquad m^{\mathcal{D}} = \begin{pmatrix} \text{Jordan} \; \_\_ \; \text{Bulls} \\ \hline \text{LeBron} \; \_\_ \; \text{Lakers} \\ \text{Shaquille} \; \_\_ \; \text{Heat} \end{pmatrix}.$$

The matching $m^{\mathcal{S}}$ is the unique matching obtained by the construction described above: no one matches in $t=1$ and a (static) stable matching is chosen for $t=2$.

Instead, matching $m^{\mathcal{D}}$ is dynamically stable, but cannot be obtained as a (static) stable matching when everyone waits to be matched until $t = 2$. To see this, note that if this matching were to happen in $t = 2$, then Jordan and Lakers would form a (static) block. However, under dynamic stability, when Jordan considers remaining unmatched in $t = 1$, there is a unique valid conjecture and it corresponds to matching $m^{\mathcal{S}}$. Under this conjecture, Jordan matches with Bulls, whereas Lakers matches with Shaquille, whom they prefer to Jordan. Thus, when applied to the dynamic environment, the static notion of stability may rule out matchings by allowing for pairwise blocks that are not *credible*: Anticipating that Lakers will not form a block with Jordan once Jordan waits for Lakers to arrive, Jordan prefers not to wait for Lakers in the first place.

Further properties: In static matching markets, the Lone Wolf Theorem (McVitie and Wilson, 1970) and the lattice property are important structural properties. However, the set of dynamically stable matchings inherits neither.

Proposition 1. *There exist economies for which the set $\mathcal{D}_T(E^T)$ does not satisfy the Lone Wolf theorem. Furthermore, the set $\mathcal{D}_T(E^T)$ does not form a lattice.*

Proposition 1 follows from an example in Section A.1. Despite the differences between the dynamic and static notions of stability, there are also important similarities. Similar to stability in static matching markets, dynamic stability of a matching is a necessary condition for voluntary participation. This is the topic of Section 5.

5 Participation, timing, and incentives in dynamic matching markets

In static matching markets, mechanisms that output stable matchings simultaneously address two problems. First, as observed by Roth (1984), stability is a necessary condition for voluntary participation in the mechanism. This identifies mechanisms that output stable matchings for the reported preferences as the only candidates to induce participation. Second, conditional on participating, the mechanism must be such that participants have an incentive to truthfully report their preferences. It is well known that no stable mechanism exists that makes it optimal for both sides of the market to truthfully reveal their preferences. Instead, if the mechanism outputs, say, the matching obtained by side A-proposing deferred acceptance (henceforth, DA), then it is a dominant strategy for agents on side A to truthfully report their preferences. Thus, in settings in which side B is non-strategic, side A-proposing DA is used to simultaneously address the problems of participation and incentives.

This section analyzes the issues of participation and incentives in the context of dynamic matching markets. Proposition 2 in Section 5.1 shows that dynamic stability is a necessary condition for *timely* participation in the matching market. Furthermore, an extension of DA to the dynamic economy is shown to give agents on the proposing side the correct incentives to participate as soon as they arrive. Section 5.2 studies the issues of participation and incentives in the context of sequential assignment problems. The main result in Section 5.2, Theorem 2, shows that only dynamically stable matchings can arise as subgame perfect equilibrium outcomes of the preference revelation game induced by a sequence of stable mechanisms.

5.1 Voluntary and timely participation

Roth and Sotomayor (1992, pp. 22–23) argue that stability is a necessary condition for voluntary participation in a static economy as follows. Suppose a matchmaker can recommend a matching for the economy but cannot compel the agents to accept the matching. Instead, the agents are free to form pairs among themselves or choose to remain unmatched. Then, the agents follow the matchmaker's recommended matching only if it is a stable matching. Proposition 2 shows that dynamic stability plays the same role in the dynamic economy: the agents follow the matchmaker's recommended matching only if it is dynamically stable.

Consider a matchmaker who recommends a matching for the agents in E^T and whose objective is that the agents follow her recommendation. In each period t, agents arrive in the economy according to E^T. Each of the newly arriving agents, together with the remaining unmatched agents from previous periods, announce simultaneously either that they want to follow the matchmaker's recommendation or the name of an agent they wish to match with (including themselves). The matchmaker's recommendation is implemented only if all the agents agree to follow it. Instead, any agents k and k' who announce that they wish to match with each other are matched, while the remaining agents remain unmatched. Agents who match exit. The remaining unmatched agents join the newly arriving agents in period $t+1$.

In the dynamic economy, the matchmaker has to recommend not only a matching for economy E^T (the one that would ensue if all the agents follow her recommendation), but also a continuation matching in the event that some agent chooses not to follow her recommendation. The matchmaker could potentially condition her recommendation for period t on (i) the matching that has ensued through period $t-1$, $\hat{m}^{t-1}$, and (ii) on when and which agents in $\mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1})$ chose not to follow her recommendation. Because it does not affect the conclusion of Proposition 2, I make the simplifying assumption that the matchmaker's recommendation in period t only depends on (i). I denote this recommendation by $\mu_t(\hat{m}^{t-1})$, where $\mu_t(\hat{m}^{t-1})$ is a period-t matching that respects the matchings that have happened through period $t-1$. That is, for all $a \in \overline{A}_{t-1}$, $\hat{m}_{t-1}(a) \neq a$ implies that $\mu_t(\hat{m}^{t-1})(a) = \hat{m}_{t-1}(a)$.

I focus on situations where following the matchmaker's recommendation is a subgame perfect Nash equilibrium (henceforth, SPNE) such that there is no pair of contemporaneous agents who have a joint deviation.[5] This refinement is standard in the literature that studies the non-cooperative implementation of stable matchings (see, for instance, Ma, 1995, Shin and Suh, 1996, and Sönmez, 1997); I refer to this as a *pairwise* SPNE. Subgame perfection implies that the agents should find it optimal to follow the matchmaker's recommendation even after (other) agents in previous periods have chosen not to follow the matchmaker's recommendation. Allowing for joint deviations by pairs of contemporaneous agents allows us to recover the result in Roth and Sotomayor (1992): when $T = 1$, following the matchmaker's recommendation is an equilibrium only if the suggested matching is stable.

Under such an equilibrium, the matchmaker's plan describes both the matching that would ensue if everyone follows her recommendation and the matching that would ensue if some agent chooses not to follow her recommendation (but from then on, everyone chooses to follow her recommendation). This addresses two potential difficulties. First, it identifies the recommendations the agents would find optimal to follow and hence, which matchings the matchmaker can "credibly" recommend. Second, it does away with having to specify the particular protocol by which agents form matchings when they do not follow the matchmaker's recommendations. To see this, note that the matching plan in period t can depend on whether the matching through period $t-1$ coincides with the matchmaker's recommendation. For instance, it could specify that everyone remains unmatched whenever the agents do not follow the matchmaker's recommendation. However, if the agents are free to form matchings among themselves, it is not clear that the agents would follow such a recommendation. In this case, the final outcome would be determined by a combination of the matchmaker's plan together with the agents' decisions to form matchings. This could require making further assumptions about how the agents form matchings when they do not follow the matchmaker's recommendations. Requiring that the agents find it optimal to follow the matchmaker's recommendation on and off the path of play circumvents these difficulties.

To state Proposition 2, I introduce one final piece of notation. For any matching through period t, $\hat{m}^{t-1}$, let $m_\mu(\hat{m}^{t-1})$ denote the matching for $E_t^T(\hat{m}^{t-1})$ that arises

[5] At the cost of more notation, I could assume that in each period the agents announce one at a time either that they want to follow the recommendation or an agent on the other side (or themselves) they want to match with and use SPNE as the solution concept (see Lagunoff, 1994).

when after $\hat{m}^{t-1}$, the remaining agents and the new entrants follow the matchmaker's recommendation. Formally, $m_\mu(\hat{m}^{t-1}) = (\mu_t(\hat{m}^{t-1}), \mu_{t+1}(\hat{m}^{t-1}, \mu_t(\hat{m}^{t-1})), \ldots)$. In particular, $m_\mu \equiv m_\mu(\varnothing) \in \mathcal{M}_T$ denotes the matching that is implemented if everyone follows the matchmaker's recommendation.

Proposition 2. *Suppose that following the matchmaker's recommendation is a pairwise SPNE. Then, for all $t \geq 1$ and $\hat{m}^{t-1}$, $m_\mu(\hat{m}^{t-1})$ is dynamically stable for $E_t^T(\hat{m}^{t-1})$. In particular, m_μ is dynamically stable for E^T.*

Proposition 2 echoes the observations in Roth (1984) and Roth and Sotomayor (1992) that stability is a necessary condition for voluntary participation. It is immediate to see that the ability of the agents to remain single or form blocking pairs implies that the matching $m_\mu(\hat{m}^{t-1})$ must be individually rational (Definition 3) and satisfy condition (D_3). Proposition 2 implies that this is not enough to guarantee that the agents will follow the matchmaker's recommendation: $m_\mu(\hat{m}^{t-1})$ must be dynamically stable. Underlying this observation is that from period $t+1$ onward, the agents follow the matchmaker's recommendation only if it leads to a dynamically stable matching. In turn, this limits the set of matchings that an agent can expect when they consider not following the matchmaker's recommendation in period t.

Similar to the literature on strategic participation in mechanism design (e.g., Gershkov et al., 2015, Garrett, 2016, and Bergemann and Strack, 2019), Proposition 2 highlights that in a dynamic economy voluntary participation involves not only the decision of whether to participate, but also *when*. Indeed, since in the final period the matchmaker's recommendation is followed only if the matching satisfies the static notion of stability, the result in Roth and Sotomayor (1992) implies that any agent who has not yet participated in the matchmaker's plan will participate then. Dynamic stability of the matchmaker's recommended matching guarantees that agents are willing to participate as soon as they arrive.

Remark 3. An alternative description of the game would have the matchmaker provide a recommendation as a function of the set of agents who agree to follow her recommendation. That is, μ_t depends on both the matching through period $t-1$, $\hat{m}^{t-1}$, and the set of agents who follow the matchmaker's recommendation in period t, $\hat{A}_t \cup \hat{B}_t \subseteq \mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1})$. Then, in each period t, the agents who choose to follow the matchmaker's recommendation are matched according to μ_t, whereas the remaining agents either remain unmatched or form matching pairs.

This alternative specification of the game raises the question of what properties the matching should have whenever the set of agents who follow the matchmaker's recommendation differs from the set of agents who follow her recommendation on the path of play. For concreteness, suppose that in equilibrium agents in $\hat{A}_t \cup \hat{B}_t$ are supposed to follow the matchmaker's recommendation. As in Proposition 2, this will identify properties, such as individual rationality, that the matching starting from period t must satisfy when $\hat{A}_t \cup \hat{B}_t$ follows the matchmaker's recommendation. Otherwise, it would not be part of an equilibrium. Consider now an agent $k \in \hat{A}_t \cup \hat{B}_t$ who contemplates not following the matchmaker's plan in period t. Then, agent k anticipates that in period t, the matchmaker's plan for $(\hat{A}_t \cup \hat{B}_t) \setminus \{k\}$ would be implemented. Because this is an off the path event, equilibrium play does not even guarantee that this period-t matching is stable for the agents in $(\hat{A}_t \cup \hat{B}_t) \setminus \{k\}$, let alone individually rational. This, in turn, implies that the matchmaker can threaten k with period-t matchings that the agents in $\hat{A}_t \cup \hat{B}_t \setminus \{k\}$ would not follow on the path of play. The ability to implement off the path of play period-t matchings that fail to even satisfy Definition 5 runs counter to the matchmaker's inability to compel the agents to accept such a matching on the path of play.[6] In turn, the matchmaker's ability to implement these types of matchings off the path of play may limit what the agents themselves can achieve on the path of play. This again runs counter to the matchmaker's inability to compel the agents to accept any given matching.

The game considered in Proposition 2 has the advantage that there is no conflict between what the matchmaker can achieve on and off the path of play. However, the result in Proposition 2 would go through in the alternative specification of the game if one requires that for each period t, each matching $\hat{m}^{t-1}$ through period $t-1$, and each subset of agents who follows the recommendation in period t, $\hat{A}_t \cup \hat{B}_t \subseteq \mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1})$, the matchmaker's plan, $\mu_t(\hat{m}^{t-1}, \hat{A}_t, \hat{B}_t)$, satisfies Definition 5. Indeed, the following holds:[7]

Proposition 2*. *Suppose that for all $t \geq 1$, $\hat{m}^{t-1}$ and $\hat{A}_t \cup \hat{B}_t \subseteq \mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1})$, $\mu_t(\hat{m}^{t-1}, \hat{A}_t, \hat{B}_t)$ satisfies Definition 5. Then, following the matchmaker's recommendation is a pairwise SPNE only if for all $t \geq 1$, $\hat{m}^{t-1}$, $m_\mu(\hat{m}^{t-1})$ is dynamically stable for $E_t^T(\hat{m}^{t-1})$.*

I omit the proof because it follows the same steps as that of Proposition 2.

Proposition 3 and Example 3 below further illustrate that, even if preferences are known, the option to delay the time at which an agent is available to match creates an incentive problem in dynamic matching markets. While in static matching markets

[6] Indeed, as Proposition 2 shows, the matchmaker can only implement individually rational matchings that satisfy condition (D_3), which implies that the period-t matching satisfies Definition 5.

[7] Note that in this case the matching $m_\mu(\hat{m}^{t-1})$ denotes $(\mu_t(\hat{m}^{t-1}, \mathcal{A}(\hat{m}^{t-1}), \mathcal{B}(\hat{m}^{t-1})), \mu_{t+1}(\mu_t(\cdot), \cdot), \ldots)$.

DA provides the agents on the proposing side with the correct incentives to report their preferences, Proposition 3 shows that when $T = 2$, a natural extension of DA to the dynamic economy provides the agents on the proposing side with the correct incentives to participate as soon as they arrive.

To state Proposition 3, let m^{A-DA} denote the matching obtained by running the following dynamic version of DA: the DA algorithm is run with agents in $A_1 \cup A_2$ making proposals (using their intertemporal preferences) to agents in $B_1 \cup B_2$ (who choose between proposals using their intertemporal preferences). The following holds:

Proposition 3. *Let $T = 2$. Then, for all $a \in A_1$, there exists $\overline{m} \in M_{\mathcal{D}}(a,\cdot)$ such that*

$$U_1(a, m^{A-DA}) \geq U_1(a, \overline{m}).$$

Proposition 3 states that in a two-period economy, agents in A_1 cannot improve on the outcome of m^{A-DA} by delaying the time at which they are available to match. To see this, fix an agent $a \in A_1$. Consider the matching $\overline{m}^{A-DA}$ obtained by running the dynamic version of side-A DA in the economy $(A_1 \setminus \{a\}, B_1, A_2 \cup \{a\}, B_2)$. That is, agent a makes proposals *as if* a arrived in $t = 2$, and agents $b \in B_1$ evaluate the payoff from matching with agent a *as if* a arrived in $t = 2$. I show that $\overline{m}^{A-DA}$ is an element of $M_{\mathcal{D}}(a,\cdot)$. If a could improve on m^{A-DA} by waiting to be matched, then a strictly prefers $\overline{m}^{A-DA}$ to m^{A-DA}. The same lemma used to prove that DA is strategy-proof for the proposing side in static matching markets implies that a prefers m^{A-DA} to $\overline{m}^{A-DA}$. Thus, $\overline{m}^{A-DA}$ can be chosen as the matching a expects would arise if a blocked matching m^{A-DA} in $t = 1$.

Proposition 3 provides a way to construct a matching plan to support m^{A-DA} that satisfies the properties in Proposition 2* and such that all agents on side A would follow it in $t = 1$. This follows from three observations. First, m^{A-DA} induces a stable matching in $t = 2$, so that if agents match according to m^{A-DA} in $t = 1$, then it is optimal for the agents in $E_2^2(m_1^{A-DA})$ to follow the matchmaker's recommendation. Second, by construction, no pair of agents in $t = 1$ can improve on m^{A-DA} by matching together. Third, the matchmaker can credibly promise each $a \in A_1$, that $\overline{m}^{A-DA}$ is the outcome if they choose not to follow the recommendation in $t = 1$. When a objects in $t = 1$, the matchmaker implements the period-1 matching $\overline{m}_1^{A-DA}$, which by construction satisfies Definition 5. Since $\overline{m}_2^{A-DA}$ is stable for $(\mathcal{A}(\overline{m}_1^{A-DA}), \mathcal{B}(\overline{m}_1^{A-DA}))$, agents will participate and follow the matchmaker's recommendation in $t = 2$.

However, the matchmaker may not be able to convince the agents in B_1 to participate when the matching is m^{A-DA}. As Example 3 illustrates, the matching obtained by running the dynamic version of DA with side B proposing may be improved on by agents on side A waiting to be matched:

Example 3. Consider the following variant of Example 1. LeBron and Shaquille continue to arrive at $t = 1$, whereas Jordan now arrives at $t = 2$. Arrivals on side B are as before, except that now Bulls no longer arrives. That is, $A_1 = \{\text{LeBron}, \text{Shaquille}\}$, $B_1 = \{\text{Heat}\}$, $A_2 = \{\text{Jordan}\}$ and $B_2 = \{\text{Lakers}, \text{Cavalliers}\}$. Preferences are given by:

$$\begin{array}{lllll} \text{Jordan}: & \text{Cavalliers} & & & \\ \text{LeBron}: & (\text{Lakers}, 1) & (\text{Cavalliers}, 0) & (\text{Heat}, 0) & (\text{Cavalliers}, 1) \\ \text{Shaquille}: & (\text{Heat}, 0) & (\text{Lakers}, 1) & & \end{array} \qquad \begin{array}{lll} \text{Heat}: & \text{LeBron} & \text{Shaquille} \\ \text{Lakers}: & \text{Shaquille} & \text{LeBron} \\ \text{Cavalliers}: & \text{LeBron} & \text{Jordan} \end{array}$$

Figure 4 illustrates the matchings obtained by running the dynamic version of DA with sides B and A proposing, respectively:

$$m^{B-DA} = \left(\begin{array}{lcl} \text{LeBron} & \_\_ & \text{Heat} \\ \hline \text{Shaquille} & \_\_ & \text{Lakers} \\ \text{Jordan} & \_\_ & \text{Cavalliers} \end{array} \right) \qquad m^{A-DA} = \left(\begin{array}{lcl} \text{Shaquille} & \_\_ & \text{Heat} \\ \hline \text{LeBron} & \_\_ & \text{Lakers} \\ \text{Jordan} & \_\_ & \text{Cavalliers} \end{array} \right)$$

Figure 4: Matchings obtained by running the dynamic version of DA.

The matching m^{B-DA} is not dynamically stable: LeBron can guarantee to be matched with Lakers by remaining unmatched in $t = 1$. To see this, note that Shaquille also needs to match in $t = 2$ for LeBron not to match with Lakers in $t = 2$. Hence, LeBron needs to conjecture that *everyone* matches in $t = 2$ when he waits to be matched. However, when this is the case, all stable matchings match LeBron with Lakers. Thus, if the matchmaker suggests m^{B-DA}, LeBron does not follow the matchmaker's recommendation in $t = 1$.

Proposition 3 and Example 3 echo the static matching markets' results that side A-proposing DA is strategy-proof for agents on side A, whereas side B-proposing DA is not. Indeed, as previously explained, the proof of Proposition 3 is intimately related to the strategy-proofness of DA for the proposing side. However, the analogy between the result in Proposition 3 and the strategy-proofness of DA for the proposing side is incomplete. Whereas the latter refers to agents' incentives to truthfully report their preferences *conditional* on participating in the mechanism, the former refers to agents' incentives to timely participate in the mechanism when their preferences are commonly known. In other words, in a dynamic matching market, agents can strategically decide when to participate and what preferences to report. This is the focus of Section 5.2.

5.2 *Sequential assignment problems*

Section 5.2 studies the issues of timely participation and preference manipulation in dynamic matching markets within the context of sequential assignment problems. In sequential assignment, matchings are performed in multiple stages via a sequence of *spot* mechanisms that output a matching as a function of the current set of available agents and their reported preferences, but do not condition on future matching opportunities.

Sequential assignment covers important applications such as school choice and college admissions, in which both students and school seats become available over time. In school choice, admissions to public, charter, and private schools often occur at different times and through different mechanisms, schools update the number of seats available as the beginning of the school year approaches, and new students join the public school system during the summer as families move across district and/or state boundaries (Andersson et al., 2018 provide an excellent description of different school districts and their sequential algorithms). In many districts in the United States, this leads to *aftermarkets* (see Pathak, 2016): Public school districts run their matching algorithms several times to accommodate newly incoming students, newly available seats, and also the timing of private and charter schools' decisions. While in the United States private schools do not participate in the centralized matching procedure, they do in Turkey and some localities in Sweden. For instance, seats in public and private schools are assigned via a two-stage procedure in Turkey. Since 2015, this two-stage procedure operates as follows. In the first stage, students are assigned based on test scores via serial dictatorship to private schools. In the second stage, unmatched students from the first stage are assigned based on test scores via serial dictatorship to public schools.[8] Another example is college admissions in Germany (Westkamp, 2013). In the first stage, students with high grades and/or high wait times are assigned through the Boston mechanism to a subset of the available seats. In the second stage, the remaining students and seats are matched through college-proposing DA.

The use of spot mechanisms presents a well-known problem in dynamic environments: implementing an allocation that has good dynamic properties may require using information beyond that which is available in the current period (Parkes, 2007). Proposition 2 identifies dynamically stable matchings as those that induce the right incentives to participate. As Example 4 illustrates, spot mechanisms do not necessarily implement dynamically stable matchings:

[8] Before 2015, public school seats were assigned first and even if they received a match in the first round, students could participate in the second round, where private school seats were assigned.

Example 4. Consider again the economy in Example 3. I reproduce below the matchings obtained using the dynamic version of DA:

$$m^{B-DA} = \left(\begin{array}{lcl} \text{LeBron} & \_\_ & \text{Heat} \\ \hline \text{Shaquille} & \_\_ & \text{Lakers} \\ \text{Jordan} & \_\_ & \text{Cavalliers} \end{array}\right) \qquad m^{A-DA} = \left(\begin{array}{lcl} \text{Shaquille} & \_\_ & \text{Heat} \\ \hline \text{LeBron} & \_\_ & \text{Lakers} \\ \text{Jordan} & \_\_ & \text{Cavalliers} \end{array}\right)$$

Figure 5: The matchings in Figure 4.

Suppose instead that one runs DA (with either side proposing) among the agents in $t = 1$, and then one runs DA among the remaining unmatched agents and the new arrivals in $t = 2$. In this example, the resulting matching would be m^{B-DA}. Furthermore, in this example, m^{B-DA} would also be the outcome of running the Boston mechanism in $t = 1$, followed by side-B DA in $t = 2$, as in German college admissions. Since m^{B-DA} is not dynamically stable, it follows that neither of these sequences of spot mechanisms can produce a dynamically stable matching for this economy.

Instead, the analysis in Example 3 implies that the matching m^{A-DA} is dynamically stable and can be achieved via the dynamic version of side-A DA. To understand why m^{A-DA} cannot be achieved using either sequence of spot mechanisms in the previous paragraph, note the following. The period-1 matching under m^{A-DA} matches Shaquille with Heat, leaving LeBron unmatched. This poses no challenges to stability in the dynamic economy: LeBron prefers to remain unmatched in $t = 1$ to match with Lakers in $t = 2$. However, this matching is not stable in $t = 1$ relative to the agents' true preferences: Heat prefers LeBron over Shaquille, and LeBron prefers Heat over remaining unmatched.

The results in Section 5.1 imply that if m^{B-DA} is the matching that is to be implemented by the sequence of spot mechanisms, LeBron will not find it optimal to participate in $t = 1$. In sequential assignment problems, LeBron not only chooses when to participate, but also what preferences to report. In this case, instead of not participating in the first stage, LeBron could submit a ranking that only lists Lakers in $t = 1$. Doing so guarantees that he will remain unmatched in $t = 1$, and be matched with Lakers in $t = 2$. That is, even if the period-1 matching is determined by side-A DA, LeBron is better off by misreporting his preferences.

LeBron's deviation in the previous paragraph is consistent with the recommendation received by German students in the college admissions procedure: they should truncate their preferences in the first round if they want to be considered for the next round (see Westkamp, 2013). Indeed, Braun et al. (2010) report that students with high grades truncate their preferences substantially since, given their grades, they have good chances during the second stage, where there are more options to choose from. As the example illustrates, agents may benefit from truncating their preferences even if a stable and strategy-proof mechanism is used.

Two lessons follow from Example 4. First, existing mechanisms used in sequential assignment problems fail to deliver dynamically stable matchings. Second, whenever this is the case, agents' incentives to either participate or truthfully report their preferences might be hindered.

Theorem 2 shows that agents' forward-looking behavior is enough to overcome the inability of spot mechanisms to produce dynamically stable matchings. Indeed, Theorem 2 shows that only dynamically stable matchings can arise as the outcomes of pure strategy SPNE of the game induced by a sequence of spot mechanisms that implement stable matchings. However, as the proof of Theorem 2 and Example 4 above illustrate, achieving dynamically stable matchings maybe at odds with truthful behavior: agents may truncate their preferences to avoid being matched to a matching partner that is worse than what they would obtain by waiting to be matched.

To state Theorem 2, I formally define a spot stable mechanism and, given a sequence of such mechanisms, the non-cooperative game induced by the sequence.

A spot stable mechanism, denoted in what follows by ν, is a mapping that takes two sets of agents, one on each side, and their reported preferences and outputs a matching that is stable given the reported preferences. Formally, for an agent a a rank-ordered list (henceforth, ROL) is a ranking over $\mathscr{B} \cup \{a\}$. Similarly, for an agent b a ROL is a ranking over $\mathscr{A} \cup \{b\}$. Let $\succ_k$ denote agent k's ROL. Thus, a spot mechanism, ν, takes a tuple $(\hat{A}, \hat{B}, \bar{\succ})$ and outputs $\nu(\hat{A}, \hat{B}, \bar{\succ}) \in \mathcal{S}(\hat{A}, \hat{B}, \bar{\succ})$, where $\hat{A} \subseteq \mathscr{A}$, $\hat{B} \subseteq \mathscr{B}$ and $\bar{\succ} = (\succ_k)_{k \in \hat{A} \cup \hat{B}}$ is a profile of ROLs. Note that, in a slight abuse of notation, I index the set of stable matchings, $\mathcal{S}$, both by the set of agents and their reported preferences.

A sequence of spot mechanisms, $\{\nu_t\}_{t=1}^{T}$, induces the following extensive-form game. In each period t, the remaining unmatched agents and the new arriving agents observe who has matched through period $t-1$. They decide simultaneously whether to participate in the mechanism and, if they do, the ROLs to submit. The decision to participate and the submitted ROLs are not observable. Given the set of participants and their ROLs, the spot mechanism ν_t outputs a matching. Matched agents and their partners exit.[9] The game proceeds to period $t+1$.

Theorem 2 considers two versions of this game. In the first version, denoted by Γ_1, the spot mechanisms coincide with side-A DA. Moreover, the agents on side B are

[9]Thus, the game mimics how the mechanisms in the National Resident Matching Program, college admissions in Germany, school choice in Turkey, and in some localities in Sweden operate: only unmatched agents participate in the upcoming rounds.

non-strategic: they automatically participate and submit their true rankings. Theorem 2 studies the matchings that can result as the outcome of SPNE. They satisfy a property denoted *side-A dynamic stability*: the matching satisfies (D_3) and no single agent on side A finds it optimal to wait to be matched. Formally, side-A dynamic stability is the solution concept defined recursively by conditions (D_1) and (D_3) in Definition 6. In particular, when agents on side A consider waiting to be matched, they anticipate that the continuation matching is side A-dynamically stable.

In the second version, denoted by Γ_2, the spot mechanisms can be any stable matching mechanism and both sides are strategic. Theorem 2 studies the matchings that can result as the outcome of *pairwise SPNE* (see Definition B.1).

We are now ready to state Theorem 2:

Theorem 2. *Only side-A dynamically stable matchings for E^T can be the outcome of pure strategy SPNE in Γ_1. Similarly, only dynamically stable matchings for E^T can be the outcome of pure strategy pairwise SPNE in Γ_2.*

The proof is in Section B.1. I highlight here the main insights that follow from the proof and, in particular, how the different components of Definition 6 arise in the application.

First, building on the results in Roth and Vande Vate (1991), I show that it is without loss of generality to focus on equilibrium strategies in which agents' ROLs are either truncations of the rankings induced by their Bernoulli utility functions or only list themselves, that is, an *empty* ROL. Agents may need to submit empty ROLs if their most preferred matching partner is not available in a given round to avoid being matched to someone that is worse than waiting to be matched.(Alternatively, agents can always list their most preferred matching partner, even if they are not available in a given round.) It follows that when agents are allowed to submit different ROLs in each stage (as in the applications described so far), unmatched agents can always participate in the mechanism.

Second, I show that for all periods t and for all matchings that may have ensued through period t, the outcome of equilibrium play starting from period t is a dynamically stable matching. The properties of deferred acceptance in Γ_1 and the possibility of joint deviations in Γ_2 imply that the matching satisfies condition (D_3). To show that no agent benefits from remaining unmatched in a given period, I show that the matching that would result when an agent deviates and submits a ROL that leaves them unmatched is a valid conjecture. Therefore, if the matching is not dynamically stable, the agent would have a deviation. To see why such a deviation induces a matching that is a valid conjecture, fix a period t and an agent k who can match in period t. Suppose one has already shown that equilibrium play from stage $t+1$ onward leads to a dynamically stable matching. Therefore, conditional on remaining unmatched in period t, k expects that the matching starting from period $t+1$ satisfies condition (ii) in Equation 1. Because agents may not report their true preferences, the period-t matching that results when k deviates is not necessarily stable with respect to the true preferences. However, the period-t matching induced by k's deviation always satisfies Definition 5: among the agents that do match in period t, the presence of a blocking pair contradicts that agents submit truncations of their true preference ranking.

Theorem 2 complements the results in Westkamp (2013) and Dur and Kesten (2019). Westkamp (2013) shows that the mechanism used in German college admissions fails to produce (static) stable matchings and proposes a one-shot mechanism that respects the priority of those students with either high-grades and high-waiting times. In a two-period model, Dur and Kesten (2019) show that sequential assignment may be at odds with (static) stability and/or truthful behavior even when using stable and strategy-proof mechanisms in each round. They also characterize the spot mechanisms for which the outcome of pure strategy Nash equilibria is a stable matching.

Theorem 2 contributes to their analysis by identifying the stability property satisfied by the outcomes of equilibrium play regardless of the sequence of stable mechanisms used in each round. Furthermore, it illustrates the form that manipulations take in dynamic environments. Even if the spot mechanism is strategy-proof for one side, agents should be expected to truncate their preferences to ensure their assignments reflect the payoff that the agents expect to be able to guarantee should they stay for an extra round. These two observations can inform applied researchers that study sequential assignment problems (e.g., Narita, 2018; Neilson et al., 2020). First, Theorem 2 warns against the use of the reported preferences as the true preferences, even if a strategy-proof mechanism is used. This complements the works of Shorrer and Sóvágó (2018) and Hassidim et al. (2020) who document that, even in static settings, agents misreport their preferences in strategy-proof mechanisms. Second, stability notions are oftentimes used for preference identification and Theorem 2 identifies dynamic stability as the solution concept for these applications.

Whereas Theorem 2 identifies dynamic stability as the property satisfied by matchings that arise from equilibrium behavior in sequential assignment problems, it should not be interpreted as an implementation result for dynamically stable matchings.[10]

[10]Instead, the game in Lagunoff (1994) can be used to provide an implementation of dynamically stable matchings.

Indeed, a given sequence of spot mechanisms may not implement all dynamically stable matchings in a given economy. Moreover, Theorem 2 does not assert that a pure strategy SPNE exists. Instead, Theorem 2 should be interpreted as stating that dynamic stability is a necessary condition: whenever the matching that results from sequential assignment is not dynamically stable, either a pair of agents would prefer to match outside the algorithm, or an agent will find it optimal to delay the time at which they are available to match.

6 Further directions

Although stability is a key property in the analysis of static matching markets, the analysis of dynamic matching markets has been confined to either equilibrium models or done through the lens of static notions of stability. This paper fills this gap by formulating a stability notion for dynamic matching markets. As such, this paper opens several avenues for further research. First, as the discussion at the end of Section 3 suggests, one could consider refinements of dynamic stability by strengthening condition (iii) in Equation 1. Indeed, one such refinement is proposed in that section. Second, developing an algorithm that implements dynamically stable matchings is definitely of interest. As discussed in Section 4, whether an algorithm to find dynamically stable matchings exists for $T \geq 3$ is an open question. Finally, the analysis in Section 5.2 identifies dynamic stability as the solution concept in sequential assignment problems. Since many sequential assignment applications involve many-to-one matching markets, extending Definition 6 to many-to-one markets is natural. Altinok (2019) is a step in this direction.

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A PROOF OF THEOREM 1

To prove Theorem 1, I consider the following (equivalent) version of Definition 6:

Definition A.1. Given the correspondences $(\mathcal{D}_t)_{t\leq T-1}$, matching m is *dynamically stable* for E^T if the following hold:

1. For all $a \in A_1$, there exists $\overline{m} \in M_{\mathcal{D}}(a, m^0)$ such that $U_1(a, m) \geq U_1(a, \overline{m})$,
2. For all $b \in B_1$, there exists $\overline{m} \in M_{\mathcal{D}}(b, m^0)$ such that $V_1(a, m) \geq V_1(a, \overline{m})$,
3. There is no pair $(a, b) \in A_1 \times B_1$ such that $u(a, b) > U_1(a, m)$ and $v(a, b) > V_1(b, m)$,
4. $(m_s)_{s=2}^T \in \mathcal{D}_{T-1}(E_2^T(m^1))$.

Let $\mathbf{P}(T)$ denote the following inductive statement:

$\mathbf{P}(T)$**:** For all economies of length T, the correspondence $\mathcal{D}_T$ is non-empty.

As I argued in Section 3, that $\mathbf{P}(1) = 1$ follows from Gale and Shapley (1962). To show the inductive step, assume that $\mathbf{P}(T') = 1$ for all $T' < T$. I show that $\mathbf{P}(T) = 1$.

Fix an economy E^T. For each agent $k \in A_1 \cup B_1$ the set $M_{\mathcal{D}}(k, \varnothing)$ is non-empty. To see this, suppose $k = a$ and consider the following matching, m^a. Let m_1^a coincide with an element of $\mathcal{S}(A_1 \setminus \{a\}, B_1)$ for $k' \neq a$ and set $m_1^a(a) = a$. Furthermore, let $(m_s^a)_{s\geq 2} \in \mathcal{D}_{T-1}(m_1^a)$, which is non-empty by the inductive hypothesis. It is immediate to show that $m^a \in M_{\mathcal{D}}(a, \varnothing)$. Similarly, for all $b \in B_1$, $M_{\mathcal{D}}(b, \varnothing) \neq \varnothing$.

Let $m^\star$ denote the following matching in $\mathcal{M}_T$. The period-1 matching, $m_1^\star$ is a stable matching in the following one-period economy.[11] First, the agents are $A_1 \cup B_1$. Second, for each $k \in A_1 \cup B_1$, let $\hat{m}^k$ denote k's least preferred element in $M_{\mathcal{D}}(k, \varnothing)$

[11] If there are ties, fix a tie-breaking procedure.

and define k's preferences $\succ_k$ as follows. For $a \in A_1$, $b \succeq_a a$ only if $u(a,b) \geq U_1(a, \hat{m}^a)$. Moreover, $b \succeq_a b'$ only if $u(a,b) \geq u(a,b')$. Define $\succeq_b$ similarly for $b \in B_1$. The matching starting from period 2 onward, $(m_s^\star)_{s=2}^T$, is an element of $\mathcal{D}_{T-1}(E_2^T(m_1^\star))$. The latter exists by the inductive hypothesis. By construction, $m^\star \in M_{\mathcal{D}}(k, \varnothing)$ whenever $m_1^\star(k) = k$, since $m_1^\star$ was chosen to be a stable matching for the agents who match in $t = 1$. Moreover, $m^\star$ satisfies condition 4 in Definition A.1.

By construction, $m^\star$ satisfies conditions 1-2 in Definition A.1: for any agent k such that $m_1^\star(k) = k$, $\overline{m}$ can be taken to be $m^\star$. Furthermore, for any agent k such that $m_1^\star(k) \neq k$, $\overline{m}$ can be taken to be $\hat{m}^k$; by definition of $m_1^\star$, k is matched to someone who is preferred to $\hat{m}^k$. Finally, suppose that there exists $(a,b) \in A_1 \times B_1$ such that $u(a,b) > U_1(a, m^\star)$ and $v(a,b) > V_1(b, m^\star)$. It must be that at least one of a or b are unmatched; without loss of generality, assume that $m_1^\star(b) = b$. Furthermore, it must be that b prefers $\overline{m}^b$ to matching with a; otherwise, this would lead to a contradiction to the definition of $m_1^\star$. It then follows that $V_1(b, \hat{m}^b) \geq v(a,b) > V_1(b, m^\star)$. However, $m^\star \in M_{\mathcal{D}}(b, \varnothing)$, so that this contradicts the definition of $\hat{m}^b$. Thus, $m^\star$ is dynamically stable for E^T. This concludes the proof of the inductive step.

A.1 Failure of the Lone Wolf theorem

The result in Proposition 1 is based on the following example:

Example A.1. Let $T = 2$. Arrivals are as follows: $A_1 = \{a_{11}, a_{12}\}$, $A_2 = \{a_{21}, a_{22}\}$, $B_1 = \{b_{11}\}$, $B_2 = \{b_{21}, b_{22}\}$. Preferences are given by

$$\begin{array}{lllll} a_{11}: & (b_{21},1) & (b_{22},0) & (b_{11},0) & (b_{22},1) \\ a_{12}: & (b_{11},0) & & & \\ a_{21}: & (b_{21},0) & & & \\ a_{22}: & (b_{22},0) & (b_{21},0) & & \end{array} \qquad \begin{array}{llll} b_{11}: & (a_{11},0) & (a_{12},0) & \\ b_{21}: & (a_{22},0) & (a_{11},0) & (a_{21},0) \\ b_{22}: & (a_{11},0) & (a_{22},0) & \end{array}$$

The following two matchings are dynamically stable:

$$m^A = \left(\begin{array}{ccc} a_{11} & \_ & b_{11} \\ \hline a_{21} & \_ & b_{21} \\ a_{22} & \_ & b_{22} \\ a_{12} & \_ & \varnothing \end{array} \right) \qquad m^B = \left(\begin{array}{ccc} a_{12} & \_ & b_{11} \\ \hline a_{11} & \_ & b_{21} \\ a_{22} & \_ & b_{22} \\ a_{21} & \_ & \varnothing \end{array} \right),$$

Agent a_{12} is unmatched under m^A, while agent a_{21} is unmatched under m^B, so that both the Lone Wolf Theorem and the lattice property fail in this economy. Note that both matchings satisfy condition (D_3). In particular, b_{11} prefers a_{11} to a_{12} so that a_{12} has no possibility of being matched under m^A, while b_{21} prefers a_{11} to a_{21}, so that a_{21} has no possibility of matching under m^B. Below, $\overline{m}^A$ (resp., $\overline{m}^B$) denotes the conjecture that dissuades a_{11} (resp., b_{11}) from blocking m^A (resp., m^B) in $t = 1$:

$$\overline{m}^A = \left(\begin{array}{ccc} a_{12} & \_ & b_{11} \\ \hline a_{11} & \_ & b_{22} \\ a_{22} & \_ & b_{21} \\ a_{21} & \_ & \varnothing \end{array}\right) \qquad \overline{m}^B = \left(\begin{array}{ccc} & \varnothing & \\ \hline a_{11} & \_ & b_{21} \\ a_{22} & \_ & b_{22} \\ a_{12} & \_ & b_{11} \\ a_{21} & \_ & \varnothing \end{array}\right).$$

It follows that neither a_{11} may block m^A by remaining unmatched, nor can b_{11} block m^B by remaining unmatched.

B Proofs of Section 5

Proof of Proposition 2. Suppose it is a pairwise SPNE for the agents in E^T to follow the matchmaker's recommendation for all $t, \hat{m}^{t-1}$. Since an agent can always guarantee the payoff from being single through period T by always rejecting the matchmaker's recommendation, $m_\mu(\hat{m}^{t-1})$ must be individually rational. Furthermore, $m_\mu(\hat{m}^{t-1})$ satisfies condition (D_3) for $E_t^T(\hat{m}^{t-1})$; otherwise, there is a pair of agents who can match in the same period and can jointly deviate and match together, improving on the matchmaker's recommendation. It follows that for all $\hat{m}^{T-1}$, $\mu_T(\hat{m}^{T-1})$ satisfies the static notion of stability (Definition 4) for $E_T^T(\hat{m}^{T-1})$.

Toward a contradiction assume that there exists t, $\hat{m}^{t-1}$, such that $m_\mu(\hat{m}^{t-1})$ is not dynamically stable. Let $t \leq T-1$ denote the largest $s \leq T-1$ such that there exists $\hat{m}^{s-1}$ such that $m_\mu(\hat{m}^{s-1})$ fails either (D_1) or (D_2) in Definition 6. Thus, there exists $\hat{m}^{t-1}$ and $k \in \mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1})$ such that k prefers all elements of $M_{\mathcal{D}}(k, \hat{m}^{t-1})$ to their outcome under $m_\mu(\hat{m}^{t-1})$. By assumption, all agents in $\mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1}) \setminus \{k\}$ are following the matchmaker's recommendation. Thus, if k were to deviate and choose to remain single when the matchmaker recommends $\mu_t(\hat{m}^{t-1})$, and then follow the equilibrium strategy, everyone remains unmatched this period (denote this matching by $m_t^\varnothing$) and from tomorrow onward the matching $m_\mu(\hat{m}^{t-1}, m_t^\varnothing)$ would ensue. By definition of $t, \hat{m}^{t-1}$, $m_\mu(m^{t-1}, m_t^\varnothing) \in \mathcal{D}_{T-t}(E_{t+1}^T(\hat{m}^{t-1}, m_t^\varnothing))$. Thus, $(\hat{m}^{t-1}, m_t^\varnothing, m_\mu(\hat{m}^{t-1}, m_t^\varnothing)) \in M_{\mathcal{D}}(k, \hat{m}^{t-1})$. Then, k has a one-shot deviation to reject the matchmaker's recommendation, contradicting the definition of SPNE. □

Proof of Proposition 3. Let E^2 and m^{A-DA} be as in the statement of Proposition 3. Let $a \in A_1$ be such that $m_1^{A-DA}(a) \neq a$. Construct a matching $\overline{m}$ as follows. Run deferred acceptance as if the economy was given by$(A_1 \setminus \{a\}, B_1, A_2 \cup \{a\}, B_2)$, that is:

(a) a makes proposals as if a arrived in $t = 2$ That is, a proposes to $b \in \overline{B}_2$ before

$b' \in \overline{B}_2$ if, and only if, $u(a,b) > u(a,,b')$

(b) $b \in \overline{B}_1$ accepts a's offer over $a' \in A_1$ only if $\delta_b v(a,b) > v(a',b)$.

Note that $\overline{m} \in M_{\mathcal{D}}(a, m^{A-DA})$. Toward a contradiction, suppose that $U_1(a,\overline{m}) > U_1(a, m^{A-DA})$. Let A^+ denote the set of agents on side A who prefer $\overline{m}$ to m^{A-DA}. According to a lemma by J.S. Hwang (see Gale and Sotomayor (1985) for a proof), there exists $a' \notin A^+$ and $b \in \overline{m}_2(A^+)$ such that $\delta_{a'}^{\mathbb{1}[b \in B_2, a' \in A_1]} u(a',b) > U_{\cdot}(a',\overline{m})$ and $\delta_b^{\mathbb{1}[b \in B_1, a' \in A_2]} v(a',b) > V_{\cdot}(b,\overline{m})$. This contradicts the definition of $\overline{m}$. Thus, it cannot be that a strictly prefers $\overline{m}$ to m^{A-DA}. □

B.1 Proof of Theorem 2

I introduce notation to define the agents' strategies and the solution concept. A public history at the beginning of period t, h^t, is a matching through period $t-1$, $\hat{m}^{t-1}$.

Let $s \geq 1$ and let $k \in A_s \cup B_s$. For any period $t \geq s$, agent k would have observed a public history $\hat{m}^{t-1}$, together with their decision to participate and their reported preferences. While agent k can condition their period t strategy both on h^t and their past actions, it will be clear from the proof that it is without loss of generality to focus on strategies that condition on the public history alone. Thus, to keep notation simple, I define agent k's behavioral strategy in period t as a mapping $\sigma_{k,t}$, that takes a public history h^t such that k is unmatched in period t and outputs k's decision to participate and a ROL, $\succ_{k,t}$. Let $\sigma_k = (\sigma_{k,t})_{t \geq s}$ denote k's strategy profile.

A pure strategy profile $\sigma = (\sigma_k)_{k \in \overline{A}_T \cup \overline{B}_T}$ determines a terminal history $h_\sigma^{T+1} = (\hat{m}_1^\sigma, \ldots, \hat{m}_T^\sigma)$, where letting $\hat{A}_t^\sigma \cup \hat{B}_t^\sigma$ denote the set of agents who participate in the spot mechanism in period t under σ we have that

$$\hat{m}_t^\sigma|_{\hat{A}_t^\sigma \cup \hat{B}_t^\sigma} = \nu_t(\hat{A}_t^\sigma, \hat{B}_t^\sigma, (\sigma_k(h_\sigma^t))_{k \in \hat{A}_t^\sigma \cup \hat{B}_t^\sigma}).$$

Also, $\hat{m}_t^\sigma(k) = k$ for all $k \in \mathcal{A}(\hat{m}^{\sigma,t-1}) \cup \mathcal{B}(\hat{m}^{\sigma,t-1}) \setminus (\hat{A}_t^\sigma \cup \hat{B}_t^\sigma)$. Finally, $\hat{m}_t^\sigma(k) = \hat{m}_{t-1}^\sigma(k)$ for $k \in (\overline{A}_t \cup \overline{B}_t) \setminus (\mathcal{A}(\hat{m}^{\sigma,t-1}) \cup \mathcal{B}(\hat{m}^{\sigma,t-1}))$. Similarly, a pure strategy profile σ together with a public history $h^t = \hat{m}^{t-1}$ determine a continuation matching $m^\sigma(\hat{m}^{t-1})$. An agent's payoff from strategy profile σ at public history $\hat{m}^{t-1}$ is determined by the payoff from the matching it induces. That is, letting $a \in \mathcal{A}(\hat{m}^{t-1})$

$$U(a,\sigma|\hat{m}^{t-1}) = U_t(a, \hat{m}^{t-1}, \hat{m}^\sigma(\hat{m}^{t-1})),$$

and similarly for $b \in \mathcal{B}(\hat{m}^{t-1})$.

Definition B.1. A pure strategy profile $\sigma = (\sigma_k)_{k \in \overline{A}_T \cup \overline{B}_T}$ is a *pairwise SPNE* of Γ_2 if it is a SPNE of Γ_2 and the following holds: There is no period $t \geq 1$, public history $h^t = \hat{m}^{t-1}$, and pair $(a,b) \in \mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1})$ such that there exists σ'_a, σ'_b such that $U(a,(\sigma_{-(a,b)},\sigma'_a,\sigma'_b)|h^t) > U(a,\sigma|h^t)$ and $V(b,(\sigma_{-(a,b)},\sigma'_a,\sigma'_b)|h^t) > V(b,\sigma|h^t)$.

In what follows, I assume that preferences over matchings are strict.[12]

Lemma B.1. In both games, it is without loss of generality to focus on strategy profiles where the agents participate in the mechanism whenever they are unmatched.

This follows from noting that (i) participating is not observable, and (ii) agents who participate can submit a ROL that only includes themselves, i.e., an empty ROL.

Lemma B.2. In both games, without loss of generality, agents either submit an empty ROL or a truncation of the ranking induced by their Bernoulli utility function.

Proof. Fix a history $h^t = \hat{m}^{t-1}$ and an agent $k \in \mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1})$. There are two cases to consider. First, assume that given the equilibrium strategies at h^t, k is matched at the end of period t. Then the result in Roth and Vande Vate (1991) implies that k can do weakly better by submitting a truncation. Second, suppose instead that given the equilibrium strategy at h^t, k remains unmatched at the end of period t. Then it is a property of stable mechanisms that the set of agents other than k who are unmatched is independent of the ROL submitted by k, as long as k is unmatched. To see this, let $\succ$ denote the submitted ROLs under σ at h^t. Furthermore, let $\succ'$ coincide with $\succ$ for $k' \in \mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1}) \setminus \{k\}$. Let $m_t = \nu_t(\mathcal{A}(\hat{m}^{t-1}), \mathcal{B}(\hat{m}^{t-1}), \succ)$ and $\overline{m}_t = \nu_t(\mathcal{A}(\hat{m}^{t-1}), \mathcal{B}(\hat{m}^{t-1}), \succ')$. Assume that $m_t(k) = \overline{m}_t(k) = k$. Toward a contradiction, suppose there exists $a \in \mathcal{A}(\hat{m}^{t-1})$ such that $m_t(a) \neq a = \overline{m}_t(a)$. Let $A^- = \{a' \in \mathcal{A}(\hat{m}^{t-1}) : m_t(a') \succ'_{a'} \overline{m}_t(a')\}$ and $B^+ = \{b \in \mathcal{B}(\hat{m}^{t-1}) : \overline{m}_t(b) \succ'_b m_t(b)\}$. The Decomposition Lemma (Roth and Sotomayor, 1992) implies that $m_t(A^-) = \overline{m}_t(A^-) = B^+$, a contradiction. Thus, without loss of generality, k can submit an empty list in that case. Since the ROL that k submits at h^t is unobserved, in both cases one can then change k's strategy at h^t without affecting the equilibrium. $\square$

Lemma B.3. In both games, for all t and public history $\hat{m}^{t-1}$, the continuation matching $m^\sigma(\hat{m}^{t-1})$ is individually rational and satisfies condition (D_3) in $E_t^T(\hat{m}^{t-1})$.

Proof. Individual rationality follows from the ability of the agents to remain single through period T by always submitting empty ROLs. In Γ_2, condition (D_3) follows from the possibility of joint deviations: otherwise, there would be a period $s \geq t$ and

[12] This is relevant in the proof of Lemma B.3 for Γ_1.

a pair $(a,b) \in \mathcal{A}(\hat{m}^{t-1}, (m_r^\sigma(\cdot))_{r\in\{t,\dots,s-1\}}) \cup \mathcal{B}(\cdot)$ that can deviate by submitting lists that only list each other. Any stable mechanism matches (a,b) together. In Γ_1, condition (D_3) follows from the properties of deferred acceptance. Suppose there is a period $s \geq t$ and a pair $(a,b) \in \mathcal{A}(\hat{m}^{t-1}, (m_r^\sigma(\cdot))_{r\in\{t,\dots,s-1\}}) \cup \mathcal{B}(\cdot)$ that prefer to match with each other over their outcome under $(\hat{m}^{t-1}, m^\sigma(\hat{m}^{t-1}))$. If $m_s^\sigma(\hat{m}^{t-1})(a) \neq a$, then this contradicts either that a submitted a truncation of their true ranking or the stability of the deferred acceptance algorithm with respect to the reported preferences. Suppose then that $m_s^\sigma(\hat{m}^{t-1})(a) = a$. The stability of the outcome of DA with respect to the reported preferences implies that a did not include b in their ROL. Consider the following strategy for a: a submits $\tilde{\succ}_a$ listing all $b' \in \mathscr{B}$ that are preferred to a's outcome under $m^\sigma(\hat{m}^{t-1})$. Under this ROL, which includes b, it cannot be that a is unmatched in period s. Otherwise, b would still be matched to $m_s^\sigma(\hat{m}^{t-1})(b)$, a contradiction (recall the proof of Lemma B.2). Then, a is matched at the end of period s, so that a has a profitable deviation, contradicting the definition of SPNE. □

A corollary of this is that $m^\sigma(\hat{m}^{T-1})$ satisfies Definition 4 for $\mathcal{A}(\hat{m}^{T-1}) \cup \mathcal{B}(\hat{m}^{T-1})$.

Lemma B.4. In Γ_1, for any period t and public history $\hat{m}^{t-1}$, the matching $m^\sigma(\hat{m}^{t-1})$ cannot be improved upon by agents on side A waiting to be matched. Similarly, in Γ_2, for any period t and public history $\hat{m}^{t-1}$, the matching $m^\sigma(\hat{m}^{t-1}) \in \mathcal{D}_{T-(t-1)}(E_t^T(\hat{m}^{t-1}))$.

Proof. The proof is similar for both games so I focus on Γ_2. Let $t \leq T-1$ denote the largest $s \leq T-1$ such that there exists $\hat{m}^{s-1}$ such that $m^\sigma(\hat{m}^{s-1})$ fails either (D_1) or (D_2) in Definition 6. Let $k \in \mathcal{A}(\hat{m}^{t-1}) \cup \mathcal{B}(\hat{m}^{t-1})$ be such that k prefers all matchings in $M_{\mathcal{D}}(k, \hat{m}^{t-1})$ over $(\hat{m}^{t-1}, m^\sigma(\hat{m}^{t-1}))$. Consider the matching $(\hat{m}^{t-1}, \overline{m}_t, \dots, \overline{m}_T)$ that arises when k deviates and submits an empty ROL and then continues to play the equilibrium strategy. By definition of t, $(\overline{m}_s)_{s\geq t+1} \in \mathcal{D}_{T-t}(E_{t+1}^T(\hat{m}^{t-1}, \overline{m}_t))$. (Note that $m^\sigma(\hat{m}^{t-1}, \overline{m}_t) = (\overline{m}_s)_{s\geq t+1}$.) It remains to show that $\overline{m}_t$ satisfies Definition 5. Towards a contradiction, suppose there exists a pair (a,b) such that $\overline{m}_t(a) \neq a$ and $\overline{m}_t(b) \neq b$, but $u(a,b) > u(a, \overline{m}_t(a))$ and $v(a,b) > v(\overline{m}_t(b), b)$. Then, it must be that the ROLs $(\succ_a, \succ_b)$ satisfy that either $\overline{m}_t(a) \succ_a b$ or $\overline{m}_t(b) \succ_b a$. This contradicts Lemma B.2: under truncations, the agents do not switch the order of agents on the other side relative to their true preferences. It follows that $\overline{m}_t$ satisfies Definition 5 and hence, $(\hat{m}^{t-1}, \overline{m}_t, \dots, \overline{m}_T) \in M_{\mathcal{D}}(k, \hat{m}^{t-1})$. Thus, k has a deviation at public history $\hat{m}^{t-1}$, contradicting the definition of SPNE. □