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# Indifference, indecisiveness, experimentation, and stochastic choice

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Among the reasons behind the choice behavior of an individual taking a stochastic form are her potential indifference or indecisiveness between certain alternatives, and/or her willingness to experiment in the sense of occasionally deviating from choosing a best alternative to give a try to other options. We introduce methods of identifying if and when a stochastic choice model may be thought of as arising due to any one of these three reasons. Each of these methods furnishes a natural way of making deterministic welfare comparisons within any model that is "rationalized" as such. In turn, we apply these methods, and characterize the associated welfare orderings, in the case of several well-known classes of stochastic choice models.

Keywords. Stochastic choice, indifference, incomplete preferences, experimentation, the general Luce model, random utility, additive perturbed utility, individual welfare.

JEL classification. D01, D11, D81, D91.

## 1. Introduction

Individuals often make different choices in apparently identical situations, even when the interval between the successive choices is very short (cf. Tversky (1969), Ballinger and Wilcox (1997), Hey (2001), and Agranov and Ortoleva (2017).) At the outset, one can imagine a few "simple" reasons for this phenomenon. Perhaps the simplest of all is that a person may be indifferent across some alternatives which she deems best in a menu $S$, and each time she needs to make a choice from $S$, she chooses one of these items randomly; or she may be able to readily eliminate some alternatives in $S$ from contention, but may find the remaining options difficult to compare—that is, she may be indecisive among those alternatives—whence, again, she resorts to randomization. Or perhaps,

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she knows what she likes best in $S$, but nevertheless gives a "try" to some options occasionally by way of experimentation. There are, of course, other explanations such as randomly fluctuating tastes, being prone to making mistakes, having random attention spans, or being deliberately stochastic, etc.; we do not study these alternate sources of randomization in this paper.

Apart from their simplicity, what makes the first three "explanations" interesting is that they accord exceptionally well with the traditional way of modeling a "rational decision maker" as one who maximizes her preference relation over feasible alternatives. In the first case, the individual is entirely standard. Her preferences are complete and transitive, and she randomizes only when she is indifferent between the options that maximize her preferences in a given feasible menu. Her random choice behavior is thus only a consequence of her *indifference* across some alternatives. The second case is only slightly less orthodox. In that case too the subject has well-behaved preferences, except these are now allowed to be incomplete. This person identifies all maximal options in a menu relative to her preferences, and randomizes only over those. Thus, her stochastic choice behavior originates exclusively from her *lack of strict preference* across these options. The third case is less standard, but also consistent with the decisions of the agent being guided by a deterministic preference relation. We can think of such a person as making her most frequent choices by maximizing her preferences, and less frequent ones by some form of *experimentation* where she "checks out" suboptimal options in a menu every now and again. To give more content to the surmise that she is ultimately guided by her deterministic preferences, we may also require that the lower ranked options in a menu are "tried out" less frequently. In other words, for such a decision maker, the order of choice probabilities of alternatives in a menu is determined entirely by her preferential ranking of these items.

These three potential motives behind one's random choice behavior are neither exclusive nor exhaustive. One may randomize both due to lack of strict preference and experimentation, and as we have noted above, there are other potential explanations for one's choices to have a stochastic form. However, these three motives can be formally applied to any random choice model, thereby providing economically interesting ways of classifying, whence comparing, the inner structure of such models. In addition, the very structure of these motives necessitates a well-defined, and essentially unique, deterministic preference relation. Consequently, any stochastic choice model that is compatible with any one of these motives is naturally furnished with a deterministic welfare ordering across all alternatives. This is an advantage because it is in general quite difficult to make unambiguous welfare comparisons in the context of stochastic choice theory. It also starkly distinguishes the three potential motives behind making randomized choices from other explanations such as being prone to making mistakes, having randomly changing preferences, or having random attention spans.

In this paper, we introduce a method of identifying if, and when, a stochastic choice model is induced by a deterministic (possibly incomplete) preference relation for each of the above three motivations. We then apply these three methods, and characterize the associated welfare orderings, in the case of several well-known classes of stochastic choice models.

In Section 3, we introduce necessary criteria for random choices to arise due to indifference, or more generally, due to lack of strict preference. This is quite straightforward in the case of indifference, but some care is needed in the second case to distinguish between when an agent is indifferent between two alternatives and when she is indecisive instead. Once this is done (using a method that was introduced by Eliaz and Ok (2006)), we obtain simple axiomatic characterizations of such models, and in each case, deduce a unique preference relation relative to which the model is induced by indifference or by lack of strict preference. This relation is complete in the case of indifference and potentially incomplete in the case of lack of strict preference, and it is explicitly characterized in terms of the given stochastic choice rule. As noted above, it provides one with a natural method of making welfare comparisons (across models that are induced by indifference or lack of strict preference). In fact, this relation is a close reflection of the "unambiguously chosen over" relation of Bernheim and Rangel (2007, 2009). In addition, in quite a few models, it possesses the structure of a threshold preorder that was introduced by Luce (1956) to capture choice behavior that is subject to imperfect discrimination.

In Section 3, we also explore the third motive suggested above, and formally define what it means for one's random choices to be compatible with experimentation. In general, there may be more than one preference relation that "rationalizes" a given stochastic choice function as such. However, under certain conditions that are satisfied by many models, this preference too is unique, and again provides a crisp method of making individual welfare comparisons. In addition, we characterize compatibility with experimentation explicitly, and deduce that this concept is distinct from being induced by lack of strict preference, while, of course, every model that is induced by indifference is induced by lack of strict preference.

The remaining sections of the paper study if, and when, certain well-known models of stochastic choice can be viewed "as if" they are induced by indifference, lack of strict preference, and/or experimentation. Leaving the precise definitions of these models (and the credits/citations concerning them) to the main body of the paper, the following list provides a sample of our findings:

- In Section 4, we study Luce-type models. We find that the Luce model is induced by indifference (trivially) and is compatible with experimentation. More generally, the general Luce model may or may not be induced by indifference and/or lack of strict preference, and it may or may not be compatible with experimentation. In Section 4, we identify exactly when any one of these cases holds. (As a corollary, we find that any 2-stage Luce model is induced by lack of strict preference, but it need not be compatible with experimentation; every threshold Luce model is compatible with experimentation.)

- A random utility model may or may not be induced by lack of strict preference; we characterize exactly when this holds in Section 5. In particular, we find that Tversky's model of elimination-by-aspects, as well as the additive random utility model (ARUM), is induced by lack of strict preference. When the error terms are independent and identically distributed (i.i.d.), ARUM is compatible with experimentation as well.

- A weak additive perturbed utility model may or may not be induced by lack of strict preference as we show in Section 6. By contrast, we find that every weak additive perturbed utility model is compatible with experimentation.
- The positive results cited above are accompanied by the full characterization of the unique preference relation that "rationalizes" the model in the way the result indicates.

Some of these results are fairly straightforward, but a few of them highlight certain inapparent aspects of these models. We show that threshold Luce model, Tversky's model, and additive random utility model are consistent with maximization of incomplete preference relation (i.e., lack of strict preference). Hence, one can view these models as three different ways to resolve incomparability between alternatives. Moreover, the corresponding incomplete relations are Luce threshold preorders in the first two models. Hence, our concept of being induced by lack of strict preference catches a commonality between these seemingly very different models. It is also worth noting that any weak additive perturbed utility model is compatible with experimentation. This sits squarely with the common interpretation of this model, and, in essence, provides a formal justification for that reading.

## 2. Preliminaries

Throughout this paper, $X$ stands for an arbitrary finite set with $|X| \geq 2$, and $\mathfrak{X}$ for the collection of all nonempty subsets of $X$. Any one member of $\mathfrak{X}$ is thought of as a feasible menu of mutually exclusive choice alternatives.

*Binary Relations* By a binary relation on $X$, we mean any subset $\mathbf{R}$ of $X \times X$, but often write $x \mathbf{R} y$ instead of $(x, y) \in \mathbf{R}$. For any $S \subseteq X$, by $x \mathbf{R} S$, we mean $x \mathbf{R} y$ for every $y \in S$; the expression $S \mathbf{R} x$ is similarly understood. When either $x \mathbf{R} y$ or $y \mathbf{R} x$, we say that $x$ and $y$ are $\mathbf{R}$-comparable. If any two $x, y \in X$ are $\mathbf{R}$-comparable, we say that $\mathbf{R}$ is complete. The properties of reflexivity, antisymmetry, and transitivity of $\mathbf{R}$ are defined in the usual way. If $\mathbf{R}$ is reflexive and transitive, we refer to it as a **preorder**, or as a **preference relation**, on $X$, and if, in addition, it is antisymmetric, we call it a partial order on $X$. Moreover, a complete partial order is referred to as a linear order. Finally, for any binary relation $\mathbf{R}$ on $X$, $\mathrm{tran}(\mathbf{R})$ stands for the transitive closure of $\mathbf{R}$.

The asymmetric (or strict) part of a preorder $\succsim$ on $X$ is the binary relation $\succ$ on $X$ defined by $x \succ y$ if, and only if $x \succsim y$ but not $y \succsim x$. In turn, $\succsim \setminus \succ$ is the symmetric part of $\succsim$, and is denoted by $\sim$. For any $S \subseteq X$ and $x \in S$, we say that $x$ is a $\succsim$**-maximum** in $S$ if $x \succsim S$, and that it is $\succsim$**-maximal** in $S$ if there is no $y \in S$ with $y \succ x$. We denote the set of all $\succsim$-maximum and $\succsim$-maximal elements in $S$ by $\max(S, \succsim)$ and $\mathbf{MAX}(S, \succsim)$, respectively. It is plain to see that $\max(S, \succsim) \subseteq \mathbf{MAX}(S, \succsim)$, but this inequality may hold strictly.

*Weak Axiom of Revealed Preference* A **choice correspondence** on $\mathfrak{X}$ is any self-map $C$ on $\mathfrak{X}$ such that $C(S) \subseteq S$ for each $S \in \mathfrak{X}$. For any $x, y \in X$, we say that $x$ is $C$-preferred to $y$ if there is a $T \in \mathfrak{X}$ such that $x \in C(T)$ and $y \in T$. In turn, $C$ is said to satisfy the **weak axiom of revealed preference** (WARP) if for every $S \in \mathfrak{X}$ and $x \in S$, we have $x \in C(S)$

provided that $x$ is $C$-preferred to $y$ for *some* $y \in C(S)$. This is the same thing as saying that $C(T) \cap S = C(S)$ for every $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $C(T) \cap S \neq \emptyset$.

*The Luce Threshold Preorder* Let $U : X \to \mathbb{R}$ be any function, which we interpret as a utility function of an individual, and take any $\delta \in [0, \infty]$. We define the **Luce threshold preorder** $\succsim_{U,\delta}$ on $X$ (induced by $U$ and $\delta$) as $x \succ_{U,\delta} y$ if and only if $U(x) - U(y) > \delta$, and $x \sim_{U,\delta} y$ if and only if

$$\big[\big|U(x) - U(z)\big| \leq \delta \quad \text{if and only if} \quad \big|U(y) - U(z)\big| \leq \delta\big] \quad \text{for every } z \in X.$$

In words, $x \succ_{U,\delta} y$ means that the utility of $x$ is larger than that of $y$ beyond a given magnitude of *noticeable difference* (i.e., $\delta$), and $x \sim_{U,\delta} y$ means that (i) the utility difference between $x$ and $y$ is not noticeable; and (ii) the utilities of $x$ and $y$ are so close that whenever the utility difference between $x$ and any other alternative is not noticeable, neither is the utility difference between $y$ and that alternative.

The Luce threshold preorder will play an important role in this paper. We note that close relatives of this preorder were considered widely in the literature on decision theory to capture the idea of imperfect discrimination; see, for instance, Luce (1956) and Beja and Gilboa (1992). (The asymmetric part of $\succsim_{U,\delta}$, which is a semiorder on $X$, is identical to that of those considered in the literature, but its symmetric part is more demanding.) Obviously, $\succsim_{U,0}$ is the complete preorder on $X$ that is represented by $U$, and $\succsim_{U,\infty} = X \times X$.

## 3. Origins of stochastic choice

### 3.1 *Stochastic choice functions*

By a **stochastic choice function** on $\mathfrak{X}$, we mean a function $\mathbb{P} : X \times \mathfrak{X} \to [0, 1]$ such that

$$\sum_{x \in S} \mathbb{P}(x, S) = 1 \quad \text{and} \quad \mathbb{P}(y, S) = 0$$

for every $S \in \mathfrak{X}$ and $y \in X \backslash S$. The set of all such functions is denoted as $\textbf{scf}(X)$.

From the *individualistic* perspective, there are two ways of interpreting $\mathbb{P}(\cdot, S)$. First, we can imagine that an agent has been observed to make choices from the feasible menu $S$ multiple times, and $\mathbb{P}(x, S)$ is simply the relative frequency of the times that she has chosen $x$ from $S$. This is a rather *empirical* interpretation of $\mathbb{P}$, and is in line with the classical revealed preference theory. Alternatively, one can imagine that the choice behavior of the individual is intrinsically probabilistic (a viewpoint favored by psychologists), or that the choice behavior of the agent is deterministic over $S$, but the modeler has only partial information about it. This viewpoint is pioneered by Manski (1977), and is favored by many economists. As such, $\mathbb{P}(x, S)$ is considered as the actual probability of the agent choosing $x$ in $S$. This is a rather *theoretical* interpretation of $\mathbb{P}$.

### 3.2 *Stochastic choice induced by indifference*

One of the reasons why we may perceive the choice behavior of an individual as random is due to potential indifferences. Indeed, a rational person (who maximizes a complete preference relation) would choose $x$ from $\{x, y\}$ some of the times, and $y$ from $\{x, y\}$ at others, if she is truly indifferent between the options $x$ and $y$. This sits well with the empirical interpretation of a stochastic choice rule as well as Manski's interpretation of it, and, at the abstract, it prompts the following definition.

Definition. A stochastic choice function $\mathbb{P}$ on $\mathfrak{X}$ is said to be **induced by indifference** if there exists a complete preorder $\succsim$ on $X$ such that for every $S \in \mathfrak{X}$,

$$\mathbb{P}(x, S) > 0 \quad \text{if and only if} \quad x \in \max(S, \succsim). \tag{1}$$

Thus, a stochastic choice function $\mathbb{P}$ on $\mathfrak{X}$ that is induced by indifference can be thought of as modeling the behavior of a rational individual. It is "as if" this individual has a standard preference relation on $X$. She may indeed make different choices from the same menu at different times, but this is only because (again in the "as if" sense) she is indifferent between the best options in that menu.

Example 3.1. Let $\succsim$ be a complete preorder on $X$, and define $\mathbb{P} \in \textsf{scf}(X)$ by

$$\mathbb{P}(x, S) := \begin{cases} \dfrac{1}{|\max(S, \succsim)|} & \text{if } x \in \max(S, \succsim) \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbb{P}$ is induced by indifference; it models an individual who always chooses a best alternative in a menu (with respect to $\succsim$), and when there are more than one best alternatives, randomizes among these uniformly. More generally, for any $S \in \mathfrak{X}$, let $\mu_{S,\succsim}$ be a probability distribution on $\max(S, \succsim)$ with full support. Then, setting $\mathbb{P}(x, S) := \mu_{S,\succsim}\{x\}$ for each $x \in \max(S, \succsim)$ and $S \in \mathfrak{X}$ yields a stochastic choice function on $\mathfrak{X}$, that is induced by indifference. ◊

It is worth mentioning that the complete preorder $\succsim$ in the definition of being induced by indifference (if it exists) is unique, and it is readily characterized by looking at the choice behavior of the individual over pairwise choice situations. In particular, the strict part of this relation is found as

$$x \succ y \quad \text{if and only if} \quad \mathbb{P}\big(x, \{x, y\}\big) = 1, \tag{2}$$

and hence its symmetric part is found as $x \sim y$ if and only if $0 < \mathbb{P}(x, \{x, y\}) < 1$. Combining these, we see that

$$x \succsim y \quad \text{if and only if} \quad \mathbb{P}\big(x, \{x, y\}\big) > 0. \tag{3}$$

It makes sense to use this relation to make welfare assessments for the subject individual. After all, if $\mathbb{P}$ is induced by indifference, "$x \succ y$" simply means "whenever $x$ and

$y$ are available, $y$ is never chosen," and "$x \succsim y$" means "whenever $x$ and $y$ are both available, $y$ is not chosen unless $x$ is as well." Thus, $\succ$ and $\succsim$ coincide with the "strictly unambiguously chosen over" and "weakly unambiguously chosen over" relations of Bernheim and Rangel (2007, 2009), respectively. But note that $\succsim$ is a meaningful ordering of welfare only when $\mathbb{P}$ is induced by indifference. In this sense, unlike that of Bernheim and Rangel, our approach to measuring welfare by using $\succsim$ is not model-free; it applies only to a class of models. In this sense, at least conceptually, this outlook is more in line with the model-based approach of Rubinstein and Salant (2012), but it nevertheless allows us to look at large classes of models rather than specific ones.

In passing, we would like to emphasize that the notion of "being induced by indifference" provides an admittedly crude evaluation of a stochastic choice function, for it ignores the actual probabilities of being chosen in a menu, and concentrates only on whether or not those probabilities are positive. As such, it should best be thought of as the first test of checking whether or not a person randomizes "only" because she is indifferent between some alternatives. (So, perhaps, "being *possibly* induced by indifference" would be a more appropriate term to use here.) If one's choice behavior does not pass this test, we can be sure that there are other motives behind her random choice behavior. If her behavior passes this test, however, we cannot be certain that her randomizations are only across alternatives that she is indifferent between. After all, a positive stochastic choice function is, trivially, induced by indifference, but it would obviously be a reach to say that every such function models the same type of behavior. The following examples aims to illustrate this point.

Example 3.2. In the following examples, we take $X := \{x, y, z\}$.

(a) Let $\mathbb{P} \in \textbf{scf}(X)$ satisfy $\mathbb{P}(x, \{x, y\}) = \frac{1}{2} = \mathbb{P}(x, \{x, y, z\}) = \mathbb{P}(y, \{x, y, z\})$ and $\mathbb{P}(x, \{x, z\}) = 1 = \mathbb{P}(y, \{y, z\})$. This stochastic choice function is induced by indifference, and this conclusion seems entirely intuitive. Its description corresponds to a rational person who is indifferent between $x$ and $y$, but surely prefers either of these options to $z$.

(b) Let $\mathbb{P} \in \textbf{scf}(X)$ satisfy $\mathbb{P}(x, \{x, y\}) = 0.99 = \mathbb{P}(x, \{x, y, z\})$, $\mathbb{P}(y, \{x, y, z\}) = 0.01$, and $\mathbb{P}(x, \{x, z\}) = 1 = \mathbb{P}(y, \{y, z\})$. This stochastic choice function is also induced by indifference, but it is difficult in this case to imagine that the agent is indifferent between $x$ and $y$. True, it is possible that $x$ is chosen over $y$ 99 percent of the time because the agent's tie-breaking procedure heavily favors $x$ over $y$. However, intuitively speaking, it seems more plausible to imagine that the agent prefers $x$ over $y$ strictly, and the seldom times that $y$ is chosen over $x$ is due to mistakes or changing moods.

(c) Let $\mathbb{P} \in \textbf{scf}(X)$ satisfy $\mathbb{P}(x, \{x, y\}) = \frac{1}{2} = \mathbb{P}(x, \{x, z\}) = \mathbb{P}(x, \{x, y, z\}) = \mathbb{P}(y, \{x, y, z\})$ and $\mathbb{P}(y, \{y, z\}) = 1$. This stochastic choice function is not induced by indifference. We can thus be sure that $\mathbb{P}$ does not correspond to the choice behavior of a rational person with a complete preference relation who only randomizes between the best alternatives in a menu. Indeed, any such relation must declare $x$ and $z$, as well as $x$ and $y$, indifferent, while ranking $y$ strictly above $z$, violating transitivity. It is much more reasonable in this instance to presume that the person is indecisive about the comparative appeal of $x$ and $z$, as well as $x$ and $y$, but she is certain that $y$ is better than $z$. (This interpretation will be formalized in Section 3.3.) $\Diamond$

We conclude that one should really consider the notion of "being induced by indifference" as a necessary, but not sufficient, condition to qualify one's random choice behavior as arising only due to potential indifferences. A similar caveat applies to the other two motives for randomization that we consider in Sections 3.3 and 3.4.

*Characterization* It is quite easy to provide an axiomatic characterization of stochastic choice functions that are induced by indifference. To this end, we consider the following.

STOCHASTIC CHERNOFF AXIOM. *For every $S, T \in \mathfrak{X}$ with $S \subseteq T$, and $x \in S$,*

$$\mathbb{P}(x, T) > 0 \quad \textit{implies} \quad \mathbb{P}(x, S) > 0.$$

This property is a substantial weakening of the classical *regularity* axiom (which requires that the probability of choosing an alternative in a menu does not decrease as the menu contracts), and is a straightforward adaptation of the classical *Chernoff axiom* (also known as Sen's property $\alpha$). Its empirical interpretation is straightforward. Its justification from the theoretical viewpoint à la Manski is also clear: If the modeler is certain that the agent views $x$ "choosable" in a menu $T$, and she thinks the agent is rational, she must assign positive probability to the event that the agent chooses $x$ in a submenu of $S$.

The next property we consider is also familiar.

STOCHASTIC $\beta$-AXIOM. *For every $S \in \mathfrak{X}$ and $x \in S$, if $\mathbb{P}(x, \{x, y\}) > 0$ for* some *$y \in S$ with $\mathbb{P}(y, S) > 0$, then $\mathbb{P}(x, S) > 0$.*

In words, this property says that if the agent has deemed an alternative $x$ in a menu $S$ choosable against some potential choice $y$ in $S$—a "potential choice in $S$" being any alternative whose choice probability in $S$ is positive, or empirically speaking, has been chosen from $S$ at least once—then she would deem $x$ itself choosable from $S$.

These two properties characterize stochastic choice functions induced by indifference.[1]

THEOREM 3.1. *A stochastic choice function $\mathbb{P}$ on $\mathfrak{X}$ is induced by indifference if, and only if, it satisfies the Stochastic Chernoff Axiom and Stochastic $\beta$-Axiom.*

PROOF. The "only if" assertion is readily verified. Conversely, take any $\mathbb{P} \in \textbf{scf}(X)$, and define $C : \mathfrak{X} \to \mathfrak{X}$ by $C(S) := \{x \in S : \mathbb{P}(x, S) > 0\}$. If $\mathbb{P}$ satisfies the Stochastic Chernoff Axiom and Stochastic $\beta$-Axiom, then $C$ satisfies WARP, so the fundamental theorem of revealed preference entails that $C = \max(\cdot, \succsim)$ for some complete preorder $\succsim$ on $X$. $\square$

[1]While motivated by different objectives, Theorem 1 of Fishburn (1978) provides an alternative characterization of stochastic choice functions that are induced by indifference.

### 3.3 *Stochastic choice induced by indecisiveness*

A decision maker may choose between two alternatives in a random manner also when she finds it difficult to compare these items. This may be because she lacks information about the alternatives or about her own tastes, or because the alternatives have multiple attributes which makes their comparison a difficult problem. In economics alone, numerous authors have argued that there is no reason for a rational person's preferences to be complete, and suggested ways to analyze such preferences from a decision-theoretic perspective. (See, among many others, Aumann (1962), Bewley (1986), Ok (2002), and Eliaz and Ok (2006).) Furthermore, the recent experimental literature documents evidence for incompleteness of preferences (see, for instance, Danan and Ziegelmeyer (2006), Sautua (2017), Costa-Gomes et al. (2020), Cettolin and Riedl (2019)). In fact, in their experimental work, Agranov and Ortoleva (2020) suggest explicitly that randomization between two or more alternatives may be a consequence of the inability of an individual to render a crisp comparison between those alternatives.

This discussion suggests concentrating on stochastic choice functions $\mathbb{P}$ on $\mathfrak{X}$ for which there exists a preorder $\succsim$ on $X$ such that

$$\mathbb{P}(x, S) > 0 \quad \text{if and only if} \quad x \in \mathbf{MAX}(S, \succsim) \tag{4}$$

for every $S \in \mathfrak{X}$. However, unlike (1), there may be multiple preference relations that satisfy (4) for every $S \in \mathfrak{X}$, which, in turn, causes difficulties for making welfare comparisons.

To address this issue, let us first note that the strict part of $\succsim$ is uniquely identified when (4) holds for every $S \in \mathfrak{X}$; it is again characterized by (2). The trouble is that when neither $x \succ y$ nor $y \succ x$ hold, there is a certain arbitrariness in the way we choose to declare $x$ and $y$ indifferent or incomparable. Put differently, (4) alone does not distinguish between the "indifference" and "incomparability" parts of $\succsim$.

Fortunately, the literature on incomplete preferences (more specifically, Eliaz and Ok (2006)) suggests a way of sorting this out. Let us take a (possibly incomplete, but transitive) preference relation $\succsim$ on $X$, and ask when two alternatives $x$ and $y$ are rendered *indifferent* by $\succsim$. (Please note that when $\succsim$ is not complete, the notion of "indifference" has no formal meaning; what we are trying to do here is precisely to search for the proper definition for this concept.) Obviously, if we were to view $x$ and $y$ as indifferent, there should not be any strict preference between them; that is, neither $x \succ y$ nor $y \succ x$ may hold. But more should be true. If a person whose preferences are modeled by $\succsim$ is truly indifferent between $x$ and $y$, then one would expect that she would think of these as equivalent objects insofar as her preferences are concerned. In particular, she would assign the same preferential standing to these alternatives relative to any other alternative. More formally, indifference of $x$ and $y$ with respect to $\succsim$ should entail that $x \succ z$ if and only if $y \succ z$, as well as $z \succ x$ if and only if $z \succ y$, for any $z \in X$. This suggests a way of extracting what may be called the "indifference part" of the preference relation $\succsim$, which we denote as $\text{ind}(\succsim)$. Put precisely, $\text{ind}(\succsim)$ is the binary relation on $X$ defined by $(x, y) \in \text{ind}(\succsim)$ if and only if

$$x \succ z \quad \text{if and only if} \quad y \succ z \quad \text{and} \quad z \succ x \quad \text{if and only if} \quad z \succ y$$

for every $z \in X$. It is immediate from this definition that $\text{ind}(\succsim)$ is an equivalence relation on $X$, and $\sim \subseteq \text{ind}(\succsim)$. If $\succsim$ is complete, then this holds as an equality, but in general, it may well hold properly.[2] Those preorders whose symmetric parts match their indifference parts exactly are thus of immediate interest.

Definition. A preorder $\succsim$ on $X$ is said to be **regular** if $\sim = \text{ind}(\succsim)$.[3]

A regular preorder on $X$ thus distinguishes between the behavioral notions of indifference and indecisiveness. When $x \sim y$, and only then, we understand that the associated individual is indifferent between $x$ and $y$ (in the sense that these items are treated identically when compared to any other alternative). When $x$ and $y$ are not $\succsim$-comparable, we understand that the agent is indecisive about the appeal of $x$ and $y$ (so it is possible that another item (such as $x$ plus $\varepsilon$ dollars) be strictly preferred to $x$ but not to $y$).

This discussion culminates in the following definition.

Definition. A stochastic choice function $\mathbb{P}$ on $\mathfrak{X}$ is said to be **induced by lack of strict preference** if there exists a regular preorder $\succsim$ on $X$ such that for every $S \in \mathfrak{X}$,

$$\mathbb{P}(x, S) > 0 \quad \text{if and only if} \quad x \in \mathbf{MAX}(S, \succsim).$$

This property is a relaxation of what we considered in Section 3.2: If $\mathbb{P}$ is induced by indifference, then it is induced by lack of strict preference. (The regularity requirement is automatically satisfied in the definition of being induced by indifference, because every complete preorder is regular.) Moreover, the regular preorder $\succsim$ in the definition above (if it exists) is unique. This is one of the main advantages of working with regular preorders.

Lemma 3.2. *Let $\succsim_1$ and $\succsim_2$ be two regular preorders on $X$ such that $\boldsymbol{MAX}(S, \succsim_1) = \boldsymbol{MAX}(S, \succsim_2)$ for every doubleton $S \in \mathfrak{X}$. Then $\succsim_1 = \succsim_2$.*

Proof. For any distinct $x$ and $y$ in $X$, we have $x \succ_i y$ if and only if $\{x\} = \mathbf{MAX}(\{x, y\}, \succsim_i)$ for $i = 1, 2$. By hypothesis, therefore, $\succ_1 = \succ_2$. But then, by definition of $\text{ind}(\cdot)$, we have $\text{ind}(\succsim_1) = \text{ind}(\succsim_2)$ as well. Since both $\succsim_1$ and $\succsim_2$ are regular, it follows that $\sim_1 = \sim_2$. $\square$

Corollary 3.3. *For any $\mathbb{P} \in \boldsymbol{scf}(X)$ that is induced by lack of strict preference, there is a unique regular preorder $\succsim$ on $X$ that satisfies (4) for every $S \in \mathfrak{X}$.*

We can again determine the preorder $\succsim$ in this result by looking at choices over doubleton menus. In particular, the asymmetric part of $\succsim$ satisfies $x \succ y$ if and only if $\mathbb{P}(x,$

[2] *An example.* Let $X$ consist of the 2-vectors $x = (0, 5)$, $y = (5, 0)$, and $z = (6, 1)$, and let $\succsim$ be the coordinatewise ordering on $X$. Then $\text{ind}(\succsim)$ contains all elements of $X \times X$ except $(y, z)$ and $(z, y)$, while $\sim$ consists only of $(x, x)$, $(y, y)$, and $(z, z)$.

[3] This definition is equivalent to the one given in Eliaz and Ok (2006). See Ribeiro and Riella (2017) for an excellent analysis of regular preorders and their role in choice theory.

$\{x, y\}) = 1$, while its symmetric part is given as $x \sim y$ if and only if $0 < \mathbb{P}(x, \{x, y\}) < 1$ *and for every* $z \in X \backslash \{x, y\}$,

$$\mathbb{P}\big(x, \{x, z\}\big) = \begin{cases} 1 \\ 0 \end{cases} \quad \text{if and only if} \quad \mathbb{P}\big(y, \{y, z\}\big) = \begin{cases} 1 \\ 0. \end{cases}$$

We will later find that for a variety of stochastic choice models, this preorder is actually a particular type of Luce threshold preorder.

Once again, we may use $\succsim$ to draw intrapersonal welfare comparisons for the subject individual. Indeed, using (4), we see that $\succ$ is none other than the "strictly unambiguously chosen over" relation of Bernheim and Rangel (2007, 2009). In this case, $\succsim$ is a subrelation of the "weakly unambiguously chosen over" ordering, but not conversely.

By way of caution, we emphasize again that the preorder $\succsim$ is meaningful only when $\mathbb{P} \in \textbf{scf}(X)$ is induced by lack of strict preference. In this sense, while it applies to a fairly large class of stochastic choice functions, our approach to welfare measurement is certainly not model-free (unlike that of Bernheim and Rangel).

*Characterization* To characterize stochastic choice functions that are induced by lack of strict preference, it is enough to relax the Stochastic $\beta$-Axiom to the following property which says that if the agent has deemed an alternative $x$ in a menu $S$ choosable against every potential choice $y$ in $S$, then she would deem $x$ itself choosable from $S$.

Stochastic Condorcet Axiom. *For every* $S \in \mathfrak{X}$ *and* $x \in S$, *if* $\mathbb{P}(x, \{x, y\}) > 0$ *for every* $y \in S$ *with* $\mathbb{P}(y, S) > 0$, *then* $\mathbb{P}(x, S) > 0$.

To streamline the proof of the following characterization theorem, we note that a self-map $C$ on $\mathfrak{X}$ with $C(S) \subseteq S$ for every $S \in \mathfrak{X}$ is said to satisfy the **weak axiom of revealed noninferiority** (WARNI) if for every $S \in \mathfrak{X}$ and $x \in S$, we have $x \in C(S)$ provided that $x$ is $C$-preferred to $y$ (i.e., if there is a $T \in \mathfrak{X}$ with $x \in C(T)$ and $y \in T$) for *all* $y \in C(S)$. This property was introduced by Eliaz and Ok (2006) to characterize deterministic choice correspondences that arise from the maximization of a regular preorder.

Theorem 3.4. *A stochastic choice function* $\mathbb{P}$ *on* $\mathfrak{X}$ *is induced by lack of strict preference if, and only if, it satisfies the Stochastic Chernoff Axiom and Stochastic Condorcet Axiom.*

Proof. The proof of the "only if" part is routine. Conversely, take any $\mathbb{P} \in \textbf{scf}(X)$ that satisfies the Stochastic Chernoff Axiom and Stochastic Condorcet Axiom. Define the map $C : \mathfrak{X} \to \mathfrak{X}$ by $C(S) := \{x \in S : \mathbb{P}(x, S) > 0\}$, and note that these two axioms jointly entail that $C$ satisfies the Stochastic Chernoff Axiom and in the following: For every $S \in \mathfrak{X}$ and $x \in S$, if $x \in C\{x, y\}$ for every $y \in C(S)$, then $x \in C(S)$. It is an easy exercise to show that these two properties imply WARNI. By Theorem 1 and Proposition 2 of Eliaz and Ok (2006), then there exists a regular preorder $\succsim$ on $X$ with $C = \textbf{MAX}(\cdot, \succsim)$. $\square$

Theorems 3.1 and 3.4 are straightforward translations of standard results of deterministic choice theory to the stochastic choice framework. As we shall see below, however, when these results are applied to specific stochastic choice models, deriving characterizing conditions in terms of model parameters is not a trivial exercise (with the exception of Proposition 4.1). Besides, applications of these simple results reveal inapparent aspects of well-known stochastic choice models such as the random utility model (see Theorem 5.1 and Propositions 5.3 and 5.5 below).

### 3.4 *Stochastic choice and experimentation*

One other reason for a rational individual to exhibit stochastic choice behavior is *experimentation*. A person who makes her choices by maximizing a complete preference relation $\succsim$ may sometimes "try out" some options in a menu $S$ even though these are not among the best choices in $S$ according to $\succsim$. There is no obvious way to pin down such choice behavior in exact terms, but it seems intuitively clear that the most frequently chosen alternatives by this person in $S$ would correspond to the $\succsim$-maximal members of $S$, while less frequently chosen ones may come about due to experimentation. More generally, the lower the ranking of an item in $S$ with respect to $\succsim$, we would expect the less is the probability that this person would indeed give that item a "try." This prompts the following definition.

Definition. A stochastic choice function $\mathbb{P}$ on $\mathfrak{X}$ is said to be **compatible with experimentation** if there exists a complete preorder $\succsim$ on $X$ such that

- $\arg\max_{z\in S} \mathbb{P}(z, S) \subseteq \max(S, \succsim)$ for every $S \in \mathfrak{X}$
- if $x \succ y$, then $\mathbb{P}(x, S) \geq \mathbb{P}(y, S)$ for every $S \in \mathfrak{X}$ with $x, y \in S$
- if $\mathbb{P}(x, S) > \mathbb{P}(y, S)$ for every $S \in \mathfrak{X}$ with $x, y \in S$ and $\mathbb{P}(y, S) > 0$, then $x \succ y$.

If $\mathbb{P} \in \mathbf{scf}(X)$ is compatible with experimentation, it corresponds to the behavior of a rational individual who gives a try to some of the items in a menu $S$ every now and again. Her most frequently chosen items in $S$ are the best elements in that menu according to her preference relation $\succsim$. Moreover, her preferences guide her in experimentations as well, for her "trial" probabilities of each item in $S$ is increasing in the ranking of those items relative to $\succ$. And if an option $x$ is consistently chosen more frequently than another option $y$ in the sense that $\mathbb{P}(x, S) > \mathbb{P}(y, S)$ for every menu $S$ that contains $x$ and $y$—$x$ is "tried out" more often than $y$ in every possible situation—then $x$ must be ranked higher than $y$ by the agent. (We actually allow $\mathbb{P}(x, S) = \mathbb{P}(y, S)$ in this conclusion as long as these probabilities vanish. This is because it may be that there is an option in $S$ that dominates both $x$ and $y$, so even though $x \succ y$ (whence $\mathbb{P}(x, \{x, y\}) > \mathbb{P}(y, \{x, y\})$), $x$ and $y$ are chosen from $S$ with the same probability, namely, 0.)

The complete preorder $\succsim$ in the definition above is, in general, not unique. Nonetheless, any such preorder relates closely to the behavior of $\mathbb{P}$ on pairwise choice situations. Indeed, if we set $S := \{x, y\}$ in the first condition of the definition, we see that $x \succ y$ implies $\arg\max_{z\in S} \mathbb{P}(z, S) = \{x\}$. Thus, $x \succ y$ implies $\mathbb{P}(x, \{x, y\}) > \frac{1}{2}$. Consequently, if $\mathbb{P}(x, \{x, y\}) = \frac{1}{2}$, neither $x \succ y$ nor $y \succ x$ may hold, which means $x \sim y$ (as $\succsim$ is complete). Thus, any complete preorder $\succsim$ on $X$ that "rationalizes" $\mathbb{P}$ as compatible with experimentation satisfies

$$x \succ y \quad \text{implies} \quad \mathbb{P}\big(x, \{x, y\}\big) > \frac{1}{2} \quad \text{and} \quad \mathbb{P}\big(x, \{x, y\}\big) = \frac{1}{2} \quad \text{implies} \quad x \sim y$$

for every $x, y \in X$. The converses to these implications do not hold. (For example, where $X = \{x, y, z\}$, if $\mathbb{P} \in \mathbf{scf}(X)$ satisfies $\mathbb{P}(x, \{x, y\}) = \frac{2}{3}$, $\mathbb{P}(x, \{x, z\}) = \mathbb{P}(y, \{y, z\}) = \mathbb{P}(x, X) = 0$, and $\mathbb{P}(y, X) = \frac{1}{4}$, then it is compatible with experimentation, but the rationalizing preorder $\succsim$ on $X$ for this conclusion satisfies $z \succ x \sim y$.)

*Uniqueness of the Rationalizing Welfare Ordering* The above mentioned nonuniqueness issue does not often arise in practice. Indeed, every model that we consider in this paper that is compatible with experimentation satisfies the following:

$$\mathbb{P}\big(x, \{x, y\}\big) > \frac{1}{2} \quad \text{and} \quad \mathbb{P}(x, S) \geq \mathbb{P}(y, S) > 0 \quad \text{imply} \quad \mathbb{P}(x, S) > \mathbb{P}(y, S) \tag{5}$$

for all $S \in \mathfrak{X}$ and $x, y \in S$. If $\mathbb{P}$ satisfies this condition, it may or may not be compatible with experimentation, but if it is, then there is a *unique* complete preorder $\succsim$ on $X$ that rationalizes $\mathbb{P}$ as such: $x \succsim y$ if and only if $\mathbb{P}(x, \{x, y\}) \geq \frac{1}{2}$. (Note that, without (5), this preorder need not be transitive. For example, the stochastic choice function $\mathbb{P}$ on $X = \{x, y, z\}$ with $\mathbb{P}(x, \{x, y\}) = \mathbb{P}(y, \{y, z\}) = \frac{1}{2}$ and $\mathbb{P}(x, \{x, z\}) = \mathbb{P}(y, X) = \mathbb{P}(z, X) = \frac{1}{3}$ is compatible with experimentation, but the previous preorder is not transitive.)

*Experimentation versus Lack of Strict Preference* The following examples demonstrate that the notion of being compatible with experimentation is logically distinct from that of being induced by lack of strict preference.

Example 3.3. Let $\succcurlyeq$ be a linear order on $X$. Take any $\frac{1}{2} > p > 0$, and define $\mathbb{P} \in \mathbf{scf}(X)$ as

$$\mathbb{P}(x, S) = \begin{cases} (1-p), & \text{if } \{x\} = \max(S, \succcurlyeq) \\ p, & \text{if } \{x\} = \max\big(S \backslash \max(S), \succcurlyeq\big) \\ 0, & \text{otherwise.} \end{cases}$$

In words, this is the stochastic choice function of an individual who mainly behaves according to the preferences $\succcurlyeq$, but occasionally (with probability $p$), she "tries out" the second best alternative in a menu relative to $\succcurlyeq$. As long as $|X| \geq 3$, this choice function is not induced by lack of strict preference, but it is compatible with experimentation. $\Diamond$

Example 3.4. Let $X := \{x, y, z\}$, and take any positive $\mathbb{P} \in \mathbf{scf}(X)$ with $\mathbb{P}(x, \{x, y\}) = \frac{1}{2} = \mathbb{P}(x, \{x, z\})$, $\mathbb{P}(y, \{y, z\}) > \frac{1}{2}$ and $\mathbb{P}(y, X) > \mathbb{P}(z, X)$. Then $\mathbb{P}$ is induced by indifference (and hence by lack of strict preference) simply because $\mathbb{P}$ is positive. However, if $\mathbb{P}$ were to be compatible with experimentation, the first two equalities would entail $x \sim y$ and $x \sim z$ for the associated complete preorder $\succsim$ on $X$. We would then have $x \sim y$, while the latter two inequalities entail $y \succ z$, contradicting the transitivity of $\succsim$. Thus, $\mathbb{P}$ is not compatible with experimentation. $\Diamond$

*Characterization* For any $\mathbb{P} \in \mathbf{scf}(X)$, let us first define the binary relation $R_{\mathbb{P}}$ on $X$ by $x \, R_{\mathbb{P}} \, y$ if and only if there exists an $S \in \mathfrak{X}$ such that $x, y \in S$ and either $\mathbb{P}(x, S) > \mathbb{P}(y, S)$ or $\mathbb{P}(x, S) = \max_{z \in S} \mathbb{P}(z, S)$. Next, we consider the strict partial order $\rhd_{\mathbb{P}}$ on $X$ where $x \rhd_{\mathbb{P}} y$ if and only if $\mathbb{P}(x, S) > \mathbb{P}(y, S)$ for every $S \in \mathfrak{X}$ with $x, y \in S$ and $\mathbb{P}(y, S) > 0$. The following result utilizes these two relations to characterize all stochastic choice functions that are compatible with experimentation.

Theorem 3.5. *A stochastic choice function $\mathbb{P}$ on $\mathfrak{X}$ is compatible with experimentation if, and only if,*

$$x \text{ tran}(R_{\mathbb{P}})\ y \quad \textit{implies} \quad \textit{not } y \rhd_{\mathbb{P}} x. \tag{6}$$

Proof. Suppose $\mathbb{P}$ is compatible with experimentation. Let $\succsim$ be a complete preorder on $X$ such that the three requirements of compatibility of $\mathbb{P}$ with experimentation are satisfied. Take any $x, y \in X$ with $x$ tran$(R_{\mathbb{P}})$ $y$. Then there exist a positive integer $k$ and $x_0, \dots, x_k \in X$ with $x = x_0\ R_{\mathbb{P}}\ x_1\ R_{\mathbb{P}} \cdots R_{\mathbb{P}}\ x_k = y$. By definition of $R_{\mathbb{P}}$, therefore, for each $i \in \{1, \dots, k\}$ there is an $S_i \in \mathfrak{X}$ such that $x_{i-1}, x_i \in S$ and either $\mathbb{P}(x_{i-1}, S_i) > \mathbb{P}(x_i, S_i)$ (whence $x_{i-1} \succsim x_i$ by the second requirement and completeness of $\succsim$) or $\mathbb{P}(x_{i-1}, S_i) = \max_{z \in S_i} \mathbb{P}(z, S_i)$ (whence $x_{i-1} \succsim x_i$ by the first requirement). It follows that $x_0 \succsim \cdots \succsim x_k$, and thus $x \succsim y$, by transitivity of $\succsim$. Since the third requirement implies that $\rhd_{\mathbb{P}} \subseteq \succ$, we conclude that $y \rhd_{\mathbb{P}} x$ cannot hold. Conversely, assume (6). Put $\succsim := \text{tran}(R_{\mathbb{P}})$ which is a preorder (as $R_{\mathbb{P}}$ is reflexive). It is also complete because $\mathbb{P}(x, \{x, y\}) \geq \frac{1}{2}$ implies $x$ $R_{\mathbb{P}}$ $y$ for any $x, y \in X$. Verifying that the three requirements of compatibility of $\mathbb{P}$ with experimentation hold with respect to $\succsim$ is then a routine exercise. □

*Checking for the Experimentation Motive* While Theorem 3.5 provides a complete characterization, checking the condition (6) is not always easy. Fortunately, we can use this theorem to obtain easy-to-check sufficient conditions for compatibility with experimentation. The following is particularly useful in this regard.

Ordinal Independence of Irrelevant Alternatives. *For every $S, T \in \mathfrak{X}$ with $S \subseteq T$, and every $x, y \in S$ with $\mathbb{P}(x, T) > 0$,*

$$\mathbb{P}(x, T) \geq \mathbb{P}(y, T) \quad \textit{implies} \quad \mathbb{P}(x, S) \geq \mathbb{P}(y, S).$$

This property, which we henceforth refer to as *Ordinal IIA*, is a substantial relaxation of Luce's choice axiom.[4] It merely asks the ordering of the choice probabilities of any two alternatives in a menu $T$ remain unchanged in any submenu of $T$ that contains those two alternatives (provided that at least one of these alternatives is chosen with positive probability in $T$), but it does not put any restriction on how the relative magnitude of these probabilities change across submenus.

We recall that $\mathbb{P} \in \textbf{scf}(X)$ is said to be *weakly stochastically transitive* if $\mathbb{P}(x, \{x, y\}) \geq \frac{1}{2}$ and $\mathbb{P}(y, \{y, z\}) \geq \frac{1}{2}$ imply $\mathbb{P}(x, \{x, z\}) \geq \frac{1}{2}$ for every $x, y, z \in X$. We next show that Ordinal IIA implies compatibility with experimentation for weakly stochastically transitive stochastic choice models.

Corollary 3.6. *Every weakly stochastically transitive $\mathbb{P} \in \textbf{scf}(X)$ that satisfies Ordinal IIA is compatible with experimentation.*

[4] Fudenberg, Iijima, and Strzalecki (2015) use the term "ordinal IIA" for a much more demanding condition that also weakens Luce's choice axiom.

Proof. Take any weakly stochastically transitive $\mathbb{P} \in \textsf{scf}(X)$ that satisfies Ordinal IIA, and define the binary relation $\succsim$ on $X$ by $x \succsim y$ if and only if $\mathbb{P}(x, \{x, y\}) \geq \frac{1}{2}$. If $x \; R_{\mathbb{P}} \; y$, there is an $S \in \mathfrak{X}$ with $x, y \in S$ and $\mathbb{P}(x, S) \geq \mathbb{P}(y, S)$ and $\mathbb{P}(x, S) > 0$, so, by Ordinal IIA, $\mathbb{P}(x, \{x, y\}) \geq \frac{1}{2}$, that is, $x \succsim y$. Thus, $R_{\mathbb{P}} \subseteq \succsim$. As $\succsim$ is transitive (because $\mathbb{P}$ is weakly stochastically transitive), this implies $\text{tran}(R_{\mathbb{P}}) \subseteq \succsim$, and (6) follows from this fact. Apply Theorem 3.5. □

### 3.5 *Discussion*

We reiterate that each of the three motives considered above uses limited parts of the information provided by a stochastic choice function $\mathbb{P}$. As such, none of them can, or is expected to, uncover the behavioral origins of $\mathbb{P}$ entirely. Instead, they are primed to provide partial, but potentially useful, information about the overall $\mathbb{P}$, not unlike how summary statistics (such as the mean, variance, etc.) of a probability distribution provide partial information about that distribution. The notions of being induced by indifference and lack of strict preference concern only the *support* of $\mathbb{P}(\cdot, S)$ for any menu $S$. It would thus be a mistake to consider these notions as one-all even for the particular behaviors that they are after. As we have argued earlier, it is best to consider these properties as necessary, but not sufficient, conditions for the behavioral origins that they intend to capture. The case of compatibility with experimentation is also similar, but note that this notion uses the stochastic choice function more fully than the previous two. In particular, it pays attention to all choice probabilities (as opposed to only positive or maximum probabilities), at least in the ordinal sense. While it is a bit more complicated than the previous two, this gives an advantage to this property.

In the following sections, we apply the randomization motives introduced in this section to three well-known classes of stochastic choice models. We show that threshold Luce model, Tversky's model, and additive random utility model are induced by lack of strict preference. We may thus think of behavior that abides by any of these models "as if" it arises from the maximization of an incomplete preference relation; the models differ from each other in the way they stipulate how the individual resolves the incomparability between alternatives consistently. This points to a subtle connection between incompleteness of preferences and stochastic choice behavior.

We also show below that threshold Luce model, additive random utility model with i.i.d. errors, and weak additive perturbed utility model are compatible with experimentation. One may thus interpret the behavior that corresponds to these models "as if" it arises from the maximization of a complete preference relation along with an experimentation procedure; the models are distinguished from each other in the procedure they stipulate. In all of the above cases, the individual will have a unique preference relation that may serve as an unambiguous welfare criterion. In some cases, the preference relation is Luce's threshold preorder, revealing some commonality between seemingly unrelated models.

## 4. Origins of Luce-type models

Throughout this section, unless stated otherwise, $u : X \to (0, \infty)$ is an arbitrary function and $\Gamma : \mathfrak{X} \to \mathfrak{X}$ is any function with $\Gamma(S) \subseteq S$ for every $S \in \mathfrak{X}$.

### 4.1 *The general Luce model*

The **general Luce model** with $(u, \Gamma)$ is the stochastic choice function $\mathbb{P}$ on $\mathfrak{X}$ such that

$$\mathbb{P}(x, S) := \frac{u(x)}{\sum_{\omega \in \Gamma(S)} u(\omega)} \quad \text{for any } x \in \Gamma(S),$$

and $\mathbb{P}(x, S) := 0$ for any $x \in X \backslash \Gamma(S)$; this model was introduced, independently, by Ahumada and Ulku (2018) and Echenique and Saito (2019). We may think of $\Gamma(S)$ as the set of alternatives that survive a preliminary stage of elimination, or as the set of all alternatives in a feasible menu $S$ to which the agent somehow limits her attention. The first interpretation, favored by Echenique and Saito (2019), is in line with two-stage choice procedures (as in Masatlioglu and Ok (2005) and Manzini and Mariotti (2007)), whereas the second is more in line with the revealed attention model of Masatlioglu et al. (2012). Whichever interpretation one adopts, the choice probability of any alternative outside $\Gamma(S)$ is zero. The choice probability of an alternative $x$ in $\Gamma(S)$ is, alternatively, positive, and found as the ratio of $u(x)$ to the sum of utilities of all elements in $\Gamma(S)$.

Let $\mathbb{P}$ be the general Luce model with $(u, \Gamma)$. Then $\mathbb{P}(x, S) > 0$ if and only if $x \in \Gamma(S)$ for any $S \in \mathfrak{X}$, so $\mathbb{P}$ satisfies the Stochastic Chernoff Axiom and Stochastic $\beta$-Axiom if and only if $\Gamma$ satisfies WARP, and it satisfies the Stochastic Chernoff Axiom and Stochastic Condorcet Axiom if and only if $\Gamma$ satisfies WARNI. By Theorems 3.1 and 3.4, therefore, we get the following result.

Proposition 4.1. *The general Luce model with* $(u, \Gamma)$ *is induced by indifference (resp., lack of strict preference) if, and only if,* $\Gamma$ *satisfies WARP (resp., WARNI).*

Interestingly, we can also think of a person whose choice behavior is represented by a general Luce model with a consideration correspondence that satisfies WARP "as if" her random choice behavior arises from experimentation.

Proposition 4.2. *The general Luce model with* $(u, \Gamma)$ *is compatible with experimentation, provided that* $\Gamma$ *satisfies WARP.*

Proof. Take any $(u, \Gamma)$ such that $\Gamma$ satisfies WARP, and denote by $\mathbb{P}$ the general Luce model with $(u, \Gamma)$. By the fundamental theorem of revealed preference, there exists a complete preorder $\succsim^*$ on $X$ such that $\Gamma = \max(\cdot, \succsim^*)$. As $X$ is finite, there is a map $f : X \to \mathbb{R}$ such that $x \succsim^* y$ if and only if $f(x) \geq f(y)$, so we have $\Gamma(S) = \arg\max_{z \in S} f(z)$ for every $S \in \mathfrak{X}$. We define the binary relation $\succsim$ on $X$ by $x \succsim y$ if and only if either $f(x) > f(y)$, or $f(x) = f(y)$ and $u(x) \geq u(y)$. As it is the lexicographic sum of two complete preorders on $X$, $\succsim$ is a complete preorder on $X$. Moreover, it is plain from this description that $\arg\max_{z \in S} \mathbb{P}(z, S) \subseteq \max(S, \succsim)$ for any $S \in \mathfrak{X}$.

Now take any $x, y \in X$. Suppose first that $x \succ y$, that is, either $f(x) > f(y)$, or $f(x) = f(y)$ and $u(x) > u(y)$. For any $S \in \mathfrak{X}$ with $x, y \in S$, if $f(x) > f(y)$, then $y$ does not belong to $\Gamma(S)$, so we trivially have $\mathbb{P}(x, S) \geq 0 = \mathbb{P}(y, S)$, while if $f(x) = f(y)$ and

$u(x) > u(y)$, then $y$ may or may not belong to $\Gamma(S)$, but if it does, so does $x$ as well (because $f(x) = f(y)$), and hence $\mathbb{P}(x, S) > \mathbb{P}(y, S)$. This verifies the second condition of being compatible with experimentation. To verify the third, suppose $\mathbb{P}(x, S) > \mathbb{P}(y, S)$ for every $S \in \mathfrak{X}$ with $x, y \in S$ and $\mathbb{P}(y, S) > 0$. Choosing $S := \{x, y\}$ here, we find $\mathbb{P}(x, \{x, y\}) > \mathbb{P}(y, \{x, y\})$. This can happen if and only if either $\{x\} = \Gamma\{x, y\}$, or $\{x, y\} = \Gamma\{x, y\}$ and $u(x) > u(y)$, that is, if and only if either $f(x) > f(y)$, or $f(x) = f(y)$ and $u(x) > u(y)$, that is, if and only if $x \succ y$. This concludes the proof. $\square$

*The Luce Model* The two axioms that Luce (1959) imposed on a $\mathbb{P} \in \textsf{scf}(X)$ are *positivity* (i.e., $\operatorname{supp}(\mathbb{P}(\cdot, S)) = S$ for each $S \in \mathfrak{X}$), and what is now known as *Luce's choice axiom* (i.e., $\mathbb{P}(x, T) = \mathbb{P}(x, S)\mathbb{P}(S, T)$ for every $x \in X$ and $S, T \in \mathfrak{X}$ with $x \in S \subseteq T$). These axioms entail that $\mathbb{P}$ is a general Luce model with $(u, \Gamma)$ where $\Gamma(S) = S$ for all $S \in \mathfrak{X}$.

Positivity of the Luce model trivially renders this model as induced by indifference. As positivity is only a simplifying assumption—Horan (2021), for instance, classifies any $\mathbb{P} \in \textsf{scf}(X)$ that satisfies Luce's choice axiom as a *Luce model*—it is of interest to see if Luce's choice axiom alone would ensure the same conclusion. It turns out that this is indeed the case. For example, Cerreia-Vioglio et al. (2021) have recently showed that any $\mathbb{P} \in \textsf{scf}(X)$ that satisfies Luce's choice axiom is a general Luce model with some $(u, \Gamma)$ where $\Gamma$ satisfies WARP. Combining this result with Propositions 4.1 and 4.2, we may thus conclude that, even when nonpositive, any Luce model is induced by indifference, and is compatible with experimentation.

*Rationalization of the General Luce Model* Let $\mathbb{P}$ be a general Luce model with $(u, \Gamma)$. If $\Gamma$ satisfies WARP, then this model is induced by indifference (Proposition 4.1), and the complete preorder $\succsim$ on $X$ that "rationalizes" this conclusion is found simply as

$$x \succsim y \quad \text{if and only if} \quad x \in \Gamma\{x, y\}.$$

In this case, the model is also compatible with experimentation (Proposition 4.2), and the complete preorder $\succsim$ on $X$ that "rationalizes" this conclusion is given as

$$x \succsim y \quad \text{if and only if} \quad \text{either } \Gamma\{x, y\} = \{x\}, \quad \text{or} \quad \Gamma\{x, y\} = \{x, y\} \quad \text{and} \quad u(x) \geq u(y).$$

Finally, if $\Gamma$ satisfies WARNI, then $\mathbb{P}$ is induced by lack of strict preference (Proposition 4.1), and the preorder $\succsim$ on $X$ that "rationalizes" this conclusion is given as $x \succ y$ if and only if $\{x\} = \Gamma\{x, y\}$, and $x \sim y$ if and only if (i) $\{x, y\} = \Gamma\{x, y\}$; and (ii) $\{x, z\} = \Gamma\{x, z\}$ if and only if $\{y, z\} = \Gamma\{y, z\}$, for any $z \in X \backslash \{x, y\}$.

### 4.2 *The 2-stage Luce model*

For any partial order $\trianglerighteq$ on $X$, the general Luce model with $(u, \Gamma)$, where

$$\Gamma(S) = \mathbf{MAX}(S, \trianglerighteq) \quad \text{for every } S \in \mathfrak{X},$$

is said to be a **2-stage Luce model** with $(u, \trianglerighteq)$. This is a particularly interesting general Luce model, which presumes that the agent has a certain type of dominance relation $\trianglerighteq$

in her mind, and she does not pay any attention to those alternatives in a menu that are dominated in terms of this relation. Instead, she first contracts the menu by eliminating the dominated alternatives, and then chooses in accordance with the Luce model from the resulting set. 2-stage Luce models are thus more in line with two-stage rational choice procedures (as opposed to limited attention models).

Given that $X$ is finite, the self-map $S \mapsto$ **MAX**$(S, \trianglerighteq)$ on $\mathfrak{X}$ satisfies WARNI as long as $\trianglerighteq$ is a partial order on $X$. Thus, by Proposition 4.1, we have the following result.

Proposition 4.3. *Every 2-stage Luce model is induced by lack of strict preference.*

The following examples illustrate that a 2-stage Luce model need not be induced by indifference nor be compatible with experimentation.

Example 4.1. Define the map $u$ on $X := \{x, y, z\}$ by $u(x) := 2$, $u(y) := 3$, and $u(z) := 4$. Let $\mathbb{P}$ be the 2-stage Luce model with $(u, \trianglerighteq)$, where $a \trianglerighteq b$ if and only if $a = b$ or $u(a) > \frac{3}{2}u(b)$. This model is induced by lack of strict preference, but not by indifference. ♢

Example 4.2. Take any map $u$ on $X := \{x_1, \dots, x_5\}$ with $u(x_1) > \cdots > u(x_5)$. Let $\trianglerighteq$ be a partial order whose asymmetric part is given as $x_1 \rhd x_3 \rhd x_5$ and $x_2 \rhd x_4$. Let $\mathbb{P}$ stand for the 2-stage Luce model with $(u, \trianglerighteq)$. To derive a contradiction, suppose $\mathbb{P}$ is compatible with experimentation, and take any complete preorder $\succsim$ that rationalizes $\mathbb{P}$ as such. Since

$$\mathbb{P}(x_3, \{x_3, x_4\}) > \mathbb{P}(x_4, \{x_3, x_4\}) \quad \text{and} \quad \mathbb{P}(x_3, \{x_1, x_3, x_4\}) = 0 < \mathbb{P}(x_4, \{x_1, x_3, x_4\}),$$

neither $x_3 \succ x_4$ nor $x_4 \succ x_3$ may hold. Thus, $x_3 \sim x_4$. Alternatively,

$$\mathbb{P}(x_4, \{x_4, x_5\}) > \mathbb{P}(x_5, \{x_4, x_5\}) \quad \text{and} \quad \mathbb{P}(x_4, \{x_2, x_4, x_5\}) = 0 < \mathbb{P}(x_5, \{x_2, x_4, x_5\}),$$

so $x_4 \sim x_5$. But since $x_3 \rhd x_5$, there is no $S \in \mathfrak{X}$ with $x_3, x_5 \in S$ and $\mathbb{P}(x_5, S) > 0$. It thus follows from the second part of the definition of being compatible with experimentation that $x_3 \succ x_5$. This contradicts transitivity of $\succsim$. ♢

*Rationalization of the 2-Stage Luce Model* Let $\mathbb{P}$ be a 2-stage Luce model with $(u, \trianglerighteq)$. Proposition 4.3 says that this model can be thought of as arising from the maximization of a preference relation $\succsim$, with randomization in a menu $S$ arising only when there are multiple $\succsim$-maximal elements in $S$. This relation is given as (i) $\succ = \rhd$, and (ii) $x \sim y$ if and only if the strict upper and lower contour sets of $x$ and $y$ (with respect to $\rhd$) coincide. Moreover, if $\succsim$ is complete, then $\succ = \rhd$ while $x \sim y$ if and only if either $x = y$ or $x$ and $y$ are not $\trianglerighteq$-comparable. It follows that $\mathbb{P}$ cannot be induced by indifference, unless the incomparable part of the dominance relation $\trianglerighteq$ is a linear order, which is an extreme requirement.

### 4.3 *The threshold Luce model*

For any $\alpha > 0$, the general Luce model with $(u, \Gamma)$, where

$$\Gamma(S) := \left\{ x \in S : u(x) \geq \frac{\max u(S)}{1+\alpha} \right\},$$

is said to be a **threshold Luce model** with $(u, \alpha)$. (This model was also characterized by Echenique and Saito (2019).) On one hand, this is a 2-stage Luce model, whence it is induced by lack of strict preference. On the other hand, unlike an arbitrary 2-stage Luce model, and even though $\Gamma$ does not satisfy WARP (so Proposition 4.2 does not apply), any threshold Luce model is compatible with experimentation.

Proposition 4.4. *Every threshold Luce model is compatible with experimentation.*

Proof. It is an easy exercise to verify that every threshold Luce model is weakly stochastically transitive and satisfies Ordinal IIA. The claim thus follows from Corollary 3.6. □

The main results of this section are summarized in Figure 1.

*Rationalization of the Threshold Luce Model* Let $\mathbb{P}$ be a threshold Luce model with $(u, \alpha)$. Being a 2-stage Luce model, $\mathbb{P}$ is induced by lack of strict preference (Proposition 4.3). Only a bit of computation shows that the "rationalizing" preorder $\succsim$ of this representation is quite intuitive: it is the *Luce threshold preorder* $\succsim_{U,\delta}$ on $X$ where $U := \log u$ and $\delta := \log(1+\alpha)$ (Section 2). This welfare ordering seems particularly appealing as it fits to the interpretation of the threshold Luce model perfectly. (A similar connection between the threshold Luce model and the threshold Luce preorder was found also by Horan (2021).)

By Proposition 4.4, $\mathbb{P}$ is also compatible with experimentation, while it is readily checked that $\mathbb{P}$ satisfies (5) for all $S \in \mathfrak{X}$ and $x, y \in S$. Thus, as we have noted in Section 3.4, this model's compatibility by experimentation is "rationalized" by a unique

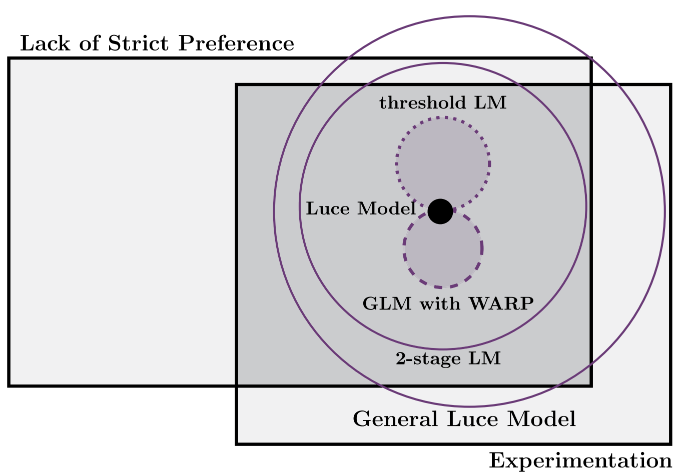


Figure 1. General Luce model, lack of strict preference, and experimentation.

complete preorder $\succsim$ on $X$. This preorder is given simply as $x \succsim y$ if and only if $u(x) \geq u(y)$.

## 5. Origins of random utility models

Throughout this section, $\mathcal{U}$ stands for a nonempty collection of bijections from $X$ onto $\{1, \dots, |X|\}$, and $\mu$ for a probability measure on $2^{\mathcal{U}}$ with $\mu(u) > 0$ for each $u \in \mathcal{U}$.

### 5.1 *The general random utility model*

The **random utility model** (RUM) with $(\mathcal{U}, \mu)$ is the $\mathbb{P} \in \mathbf{scf}(X)$ with

$$\mathbb{P}(x, S) := \sum_{u \in \mathcal{U}} \mu(u) \mathbf{1}_{\arg\max u(S)}(x). \tag{7}$$

Most stochastic choice functions that are used in econometric models of discrete choice are of this form. The class of RUMs, which was characterized first by Falmagne (1978), intersects that of general Luce models, but neither is nested in the other. However, it was shown by Block and Marschak (1960) that the Luce model is, in fact, a RUM.

Our first question is, "Can we think of any RUM as modeling the choice behavior of a rational individual who makes her choices from a menu by maximizing a (possibly incomplete) preference relation, and who only randomizes among the maximal options in that menu?" In other words, can we think of any RUM as induced by lack of strict preference? The short answer is no. The long answer requires determining exactly which sort of RUMs can be thought of as such. This is done in the next result.

Theorem 5.1. *The random utility model $\mathbb{P}$ with $(\mathcal{U}, \mu)$ is induced by lack of strict preference if, and only if,*

$$\min_{u \in \mathcal{U}} \max_{z \in S} \big(u(z) - u(x)\big) > 0 \quad \textit{implies} \quad \max_{z \in S} \min_{u \in \mathcal{U}} \big(u(z) - u(x)\big) > 0 \tag{8}$$

*for any $S \in \mathfrak{X}$ and $x \in S$.*

Proof. Being regular (in the sense of Luce), any RUM satisfies the Stochastic Chernoff Axiom. By Theorem 3.4, therefore, we only need to show that $\mathbb{P}$ satisfies the Stochastic Condorcet Axiom if and only if (8) holds for any $S \in \mathfrak{X}$ and $x \in S$. Assume first that $\mathbb{P}$ satisfies the Stochastic Condorcet Axiom. Take any $S \in \mathfrak{X}$ and $x \in S$ with

$$\min_{u \in \mathcal{U}} \max_{z \in S} \big(u(z) - u(x)\big) > 0, \tag{9}$$

but suppose (8) fails. Then, for every $y \in S \backslash \{x\}$, there is a $u \in \mathcal{U}$ with $u(x) > u(y)$. It follows that $\mathbb{P}(x, \{x, y\}) > 0$ for every $y \in S$, and hence, by the Stochastic Condorcet Axiom, $\mathbb{P}(x, S) > 0$, but this contradicts (9). Conversely, take any $S \in \mathfrak{X}$ and $x \in S$ with $\mathbb{P}(x, \{x, y\}) > 0$ for each $y \in S$. To derive a contradiction, suppose $\mathbb{P}(x, S) = 0$, which means that for every $u \in \mathcal{U}$, there is a $y \in S$ (that depends on $u$) with $u(y) > u(x)$. It follows that (9) holds, and hence, if (8) is valid, there exists a $y^* \in S$ with $u(y^*) > u(x)$ for every $u \in \mathcal{U}$. But this means $\mathbb{P}(x, \{x, y^*\}) = 0$, a contradiction. $\square$

Theorem 5.1 is useful for determining whether or not a particular RUM is induced by lack of strict preference. A major example of this will be encountered in the next section. For the moment, we use it to give concrete examples that show that the notions of being induced by lack of strict indifference and being compatible with experimentation are logically distinct even within the class of RUMs.

Example 5.1. Define the real maps $u$ and $v$ on $X := \{x, y, z\}$ by $(u(x), u(y), u(z)) := (2, 3, 1)$ and $(v(x), v(y), v(z)) := (2, 1, 3)$. Then, for any nondegenerate $\mu \in \Delta(\{u, v\})$, an easy application of Theorem 5.1 shows that the random utility model $\mathbb{P}$ with $(\mathcal{U}, \mu)$ is not induced by lack of strict preference. However, it is readily checked that this model is compatible with experimentation (for any $\mu$). ♢

Example 5.2. Define the real maps $u$, $v$, and $w$ on $X := \{x, y, z\}$ by $(u(x), u(y), u(z)) := (3, 2, 1)$, $(v(x), v(y), v(z)) := (2, 1, 3)$, and $(w(x), w(y), w(z)) := (1, 3, 2)$. Then, for any nondegenerate $\mu \in \Delta(\{u, v, w\})$, we can use Theorem 5.1 to show readily that the random utility model $\mathbb{P}$ with $(\mathcal{U}, \mu)$ is induced by lack of strict preference. However, when $\mu(u) = \frac{4}{9}$ and $\mu(v) = \frac{3}{9}$, it is not compatible with experimentation. ♢

The following result shows that a RUM may be induced by indifference only under special circumstances.

Proposition 5.2. *The random utility model* $\mathbb{P}$ *with* $(\mathcal{U}, \mu)$ *is induced by indifference if, and only if, there is a partition* $\{X_1, \dots, X_n\}$ *of* $X$ *such that* $X_i \subseteq \bigcup_{u \in \mathcal{U}} \arg\max u(X_i)$ *for each* $i$, *and* $\min_{u \in \mathcal{U}}(u(x) - u(y)) > 0$ *for each* $(x, y) \in X_i \times X_j$ *with* $i < j$.

Proof. Suppose there exists a complete preorder $\succsim$ on $X$ such that (1) holds. Then $u(x) = \max u(S)$ for some $u \in \mathcal{U}$ if and only if $x \in \max(S, \succsim)$ for every $S \in \mathfrak{X}$. In particular, $x \succ y$ if and only if $\min_{u \in \mathcal{U}}(u(x) - u(y)) > 0$. Enumerate the quotient set $X/\sim$ as $\{X_1, \dots, X_n\}$ where $x \succ y$ for each $(x, y) \in X_i \times X_j$ with $i < j$, and note that $x_i \in \max(X_i, \succsim)$, whence $x_i \in \bigcup_{u \in \mathcal{U}} \arg\max u(X_i)$ for any $x_i \in X_i$ and $i = 1, \dots, n$. Conversely, suppose there is a partition $\{X_1, \dots, X_n\}$ of $X$ with the said two properties. Consider the binary relation $\succsim := \bigcup(X_i \times X_j)$ where the union is taken over all $(i, j)$ pairs with $i \leq j$. (Note that, by the second property, $x \succ y$ implies $u(x) > u(y)$ for all $u \in \mathcal{U}$.) Now take any $S \in \mathfrak{X}$ and $x \in S$, and suppose $x \in X_i$. If there is a $y \in S$ with $y \succ x$, then $u(y) > u(x)$ for all $u \in \mathcal{U}$, whence $u(x) < \max u(S)$ for all $u \in \mathcal{U}$. Conversely, if $x \succsim S$, then $S \subseteq X_i \cup \dots \cup X_n$. If $y \in S \cap X_j$ with $i < j$, then $x \succ y$, so $u(x) > u(y)$ for all $u \in \mathcal{U}$. If instead $y \in S \cap X_i$, then $u(x) \geq u(y)$ for some $u \in \mathcal{U}$ (because $x \in X_i \subseteq \bigcup_{u \in \mathcal{U}} \arg\max u(X_i)$). Thus, $x \in \max(S, \succsim)$ if and only if $u(x) = \max u(S)$ for some $u \in \mathcal{U}$ (i.e., $\mathbb{P}(x, S) > 0$). □

A RUM may or may not be compatible with experimentation (Example 5.2), and we are not aware of a general result about when it is. As we shall see below, this problem may be better dealt with in the context of the particular RUMs that one is interested in.

*Rationalization of a RUM* Let $\mathbb{P}$ be a RUM with $(\mathcal{U}, \mu)$. Suppose (8) holds for any $S \in \mathfrak{X}$ and $x \in S$ so that, by Theorem 5.1, $\mathbb{P}$ is induced by lack of strict preference. To identify the "rationalizing" preorder $\succsim$ of this representation clearly, let us introduce the map $\Phi_{\mathcal{U}} : X \times X \to \mathbb{R}$ with $\Phi_{\mathcal{U}}(x, y) := \min_{u\in\mathcal{U}}(u(x) - u(y))$. (Notice that $\Phi_{\mathcal{U}}(x, y) \neq 0$ for any distinct $x$ and $y$.) Then the asymmetric part of $\succsim$ is characterized simply as $x \succ y$ if and only if $\Phi_{\mathcal{U}}(x, y) > 0$. Alternatively, we have $x \sim y$ if and only if (i) $\Phi_{\mathcal{U}}(x, y) < 0$ and $\Phi_{\mathcal{U}}(y, x) < 0$, and (ii) $\Phi_{\mathcal{U}}(x, z)\Phi_{\mathcal{U}}(y, z) > 0$ and $\Phi_{\mathcal{U}}(z, x)\Phi_{\mathcal{U}}(z, y) > 0$ for every $z \in X\backslash\{x, y\}$.

The rationale behind viewing $\succsim$ as a welfare ordering was abstractly discussed in Section 3.3. In the special case of RUMs, we see that $\succsim$ enjoys an additional interpretation. Indeed, here $x \succ y$ means simply that *every* utility in $\mathcal{U}$ ranks $x$ strictly above $y$; it is unexceptionable that we indeed conclude in this case that the individual is better off with $x$ than with $y$. The case $x \sim y$ is a bit more subtle. In this case, at least one utility of this person says $x$ is better than $y$, and at least one other says the opposite. In addition, the standing of $x$ and $y$ relative to any other alternative $z$ is the same in the sense that if $x$ dominates (or is dominated by) $z$, this happens for $y$ as well, and vice versa.

### 5.2 *Tversky's model of elimination by aspects*

An important generalization of the Luce model is the model of *elimination by aspects* of Tversky (1972a, 1972b). This model takes as primitives a nonempty finite set $A$, a map $v : A \to (0, \infty)$, and a set-valued function $\mathrm{Att} : X \rightrightarrows A$ that maps each alternative to a nonempty subset of $A$. We interpret $A$ as the set of attributes (aspects) that an alternative in $X$ may or may not possess, $v$ describes the utility scales of the attributes, and $\mathrm{Att}(x)$ stands for the set of all attributes that an alternative $x$ possesses. For any $S \in \mathfrak{X}$ and $a \in A$, we define

$$\mathrm{Att}(S) := \bigcup_{x\in S} \mathrm{Att}(x) \quad \text{and} \quad S_a := \{x \in S : a \in \mathrm{Att}(x)\}.$$

That is, $\mathrm{Att}(S)$ is the set of all attributes that are possessed by at least one member of $S$, and $S_a$ is the set of all alternatives in $S$, if any, that possess the attribute $a$.

The model is described as follows. For any feasible menu $S$, we select the attribute $a \in \mathrm{Att}(S)$ with probability $v(a)/\sum_{b\in\mathrm{Att}(S)} v(b)$, and eliminate all alternatives in $S$ that do not possess $a$. The choice problem then becomes $S_a$. If $\mathrm{Att}(S_a) = \mathrm{Att}(x)$ for every $x \in S_a$, that is, if every $x$ in $S_a$ possesses the same attributes, then the process stops, and all alternatives in $S_a$ are chosen with probability $1/|S_a|$. Otherwise, with probability $v(b)/\sum_{c\in\mathrm{Att}(S_a)\backslash\{a\}} v(c)$, we select the attribute $b \in \mathrm{Att}(S_a)\backslash\{a\}$, and eliminate all alternatives in $S_a$ that do not possess $b$. This process is then continued inductively until it stops. Put formally, we recursively define $\mathbb{P} : X \times \mathfrak{X} \to [0, 1]$ as

$$\mathbb{P}(x, S) := \sum_{a\in\mathrm{Att}(S)} \left( \frac{v(a)}{\sum\limits_{b\in\mathrm{Att}(S)} v(b)} \right) \mathbb{P}(x, S_a) \tag{10}$$

if $x \in S$, and $\mathbb{P}(x, S) := 0$ otherwise. Not only is $\mathbb{P}$ a stochastic choice function on $\mathfrak{X}$, but it is a RUM. In what follows, we refer to it as the **Tversky model** with $(A, v, \mathrm{Att})$.

The Tversky model has a procedural flavor, and as such, one may not expect it to be tied to an individual who ultimately maximizes a deterministic (albeit, incomplete) preference relation. The following result may thus come as a bit of a surprise.

Proposition 5.3. *Every Tversky model is induced by lack of strict preference.*

Proof. Let $\mathbb{P}$ be the Tversky model with $(A, v, \mathrm{Att})$. Then, being a RUM, $\mathbb{P}$ satisfies the Stochastic Chernoff Axiom. By Theorem 3.4, therefore, we only need to show that $\mathbb{P}$ satisfies the Stochastic Condorcet Axiom. We first prove the following two claims.

Claim 1. *For any $S \in \mathfrak{X}$ and $x \in S$, $\mathbb{P}(x, S) = 0$ if and only if $\mathrm{Att}(x) \subset \mathrm{Att}(y)$ for some $y \in S \backslash \{x\}$.*

Proof. Suppose there is a $y \in S\backslash\{x\}$ with $\mathrm{Att}(x) \subset \mathrm{Att}(y)$. Then, obviously, $\mathbb{P}(x, \{x, y\}) = 0$. It is also plain that, according to any Tversky model, the probability of choosing an alternative in a menu cannot decrease as the menu shrinks. (That is, any such model satisfies Luce's regularity axiom.) It follows that $\mathbb{P}(x, \{x, y\}) \geq \mathbb{P}(x, S)$, whence $\mathbb{P}(x, S) = 0$. Conversely, suppose $\mathrm{Att}(x) \subset \mathrm{Att}(y)$ is false for every $y \in S$. We proceed by induction on the size of $S$. At the base step, suppose $|S| = 2$, so $S = \{x, y\}$ for some $y \in X\backslash\{x\}$. By hypothesis, there is an attribute that $x$ possesses but $y$ does not, so if elimination starts with this attribute (which happens with positive probability), $y$ is eliminated, but $x$ is not; we thus have $\mathbb{P}(x, S) > 0$ in this case. (If $|X| = 2$, we are done, so we assume henceforth that $|X| \geq 3$.) Now take any integer $k \in \{2, \ldots, |X| - 1\}$, and suppose that our assertion holds whenever $|S| \leq k$. Finally, suppose $|S| = k + 1$ and $\mathrm{Att}(x) \subset \mathrm{Att}(y)$ is false for every $y \in S$. If there is no $a \in A$ with $|S_a| \leq k$, then $S_a = S$ for every $a \in \mathrm{Att}(S)$, which would entail that the process stops, and $\mathbb{P}(x, S) = 1/|S| > 0$. Otherwise, that is, there is an attribute $a \in A$ with $|S_a| \leq k$, we apply our induction hypothesis to get $\mathbb{P}(x, S_a) > 0$. By (10), then $\mathbb{P}(x, S) > 0$, as we sought. □

Claim 2. *For any $S \in \mathfrak{X}$ and $x \in S$, $\mathbb{P}(x, S) = 0$ if and only if $\mathrm{Att}(x) \subset \mathrm{Att}(y)$ for some $y \in S \backslash \{x\}$ with $\mathbb{P}(y, S) > 0$.*

Proof. Suppose $\mathbb{P}(x, S) = 0$. Then, by Claim 1, there is a $y_1 \in S$ with $\mathrm{Att}(x) \subset \mathrm{Att}(y_1)$. If $\mathbb{P}(y_1, S) > 0$, we are done. Otherwise, apply Claim 1 again (this time $y_1$ playing the role of $x$) to find a $z_1 \in S$ with $\mathrm{Att}(y_1) \subset \mathrm{Att}(z_1)$. Thus, $\mathrm{Att}(x) \subset \mathrm{Att}(z_1)$. If $\mathbb{P}(z_1, S) > 0$, we are done; otherwise, continue inductively. Since $S$ is finite, this process yields a $z \in S$ such that $\mathrm{Att}(x) \subset \mathrm{Att}(z)$ and $\mathrm{Att}(z)$ is not a proper subset of $\mathrm{Att}(y)$ for any $y \in S\backslash\{z\}$. So, by Claim 1 (applied to $z$), we have $\mathbb{P}(z, S) > 0$, and we are done. □

Now, suppose $\mathbb{P}(x, \{x, y\}) > 0$ for every $y \in S$ with $\mathbb{P}(y, S) > 0$. Then, applying Claim 1 to the set $S = \{x, y\}$, we see that $\mathrm{Att}(x)$ is not a proper subset of $\mathrm{Att}(y)$ for any $y \in S\backslash\{x\}$ with $\mathbb{P}(y, S) > 0$. It then follows from Claim 2 that $\mathbb{P}(x, S) > 0$. This proves that $\mathbb{P}$ obeys the Stochastic Condorcet Axiom. □

The Tversky model is built on procedural elements, so it is surprising that it can, in fact, be written as a random utility model (Tversky (1972a)). Proposition 5.3 in a sense further enforces the idea that there is quite a bit of "rationality" underlying this model. Apparently, we can think of an agent whose choice behavior is modeled by a Tversky model as a rational person who maximizes her (possibly incomplete) preferences on every menu, and randomizes in a certain way when there is more than one maximal element in a given menu.

A Tversky model is, in general, not induced by indifference, unless it satisfies a demanding partition requirement.

Proposition 5.4. *The Tversky model* $\mathbb{P}$ *with* $(A, v, \text{Att})$ *is induced by indifference if, and only if, there is a partition* $\{X_1, \dots, X_n\}$ *of* $X$ *such that*

$$\begin{cases} \mathit{Att}(x) \supset \mathit{Att}(y), & \text{if } i < j \\ \mathit{Att}(x) \text{ and } \mathit{Att}(y) \text{ are not properly nested}, & \text{if } i = j \end{cases}$$

*for any distinct* $x, y \in X$ *with* $(x, y) \in X_i \times X_j$, $i, j = 1, \dots, n$.

Proof. Recall that any Tversky model is a RUM; let $\mathbb{P}$ be a RUM with $(\mathcal{U}, u)$. By Proposition 5.2, $\mathbb{P}$ is induced by indifference if and only if there is a there is a partition $\{X_1, \dots, X_n\}$ of $X$ such that (i) $X_i \subseteq \bigcup_{u \in \mathcal{U}} \arg\max u(X_i)$ for each $i$, and (ii) $\min_{u \in \mathcal{U}}(u(x) - u(y)) > 0$ for each $(x, y) \in X_i \times X_j$ with $i < j$. But (i) means for every $i$ and $x \in X_i$, there is a $u \in \mathcal{U}$ with $u(x) = \max u(X_i)$. Thus, (i) holds if and only if $\mathbb{P}(x, X_i) > 0$ for each $x \in X_i$, $i = 1, \dots, n$, while, by Claim 1 above, the latter statement holds if and only if $\text{Att}(x)$ and $\text{Att}(y)$ are not properly nested for any distinct $x, y \in X_i$, $i = 1, \dots, n$. Alternatively, (ii) holds if and only if $\mathbb{P}(x, \{x, y\}) = 1$ for every $(x, y) \in X_i \times X_j$ with $i < j$, and by Claim 2, this is the same thing as saying that $\text{Att}(x) \supset \text{Att}(y)$ for every $(x, y) \in X_i \times X_j$ with $i < j$. □

In contrast to Proposition 5.3, there is no reason for a Tversky model to be compatible with experimentation.

Example 5.3. Let $X := \{x_1, \dots, x_5\}$, $A := \{a, b, c, d, e\}$, and take any map $v$ on $A$ such that $v(a) < v(d) < v(a) + v(b)$. Let $\text{Att}(x_1) := \{a, b, c\}$, $\text{Att}(x_2) := \{e, d\}$, $\text{Att}(x_3) := \{a, b\}$, $\text{Att}(x_4) := \{d\}$, and $\text{Att}(x_5) := \{a\}$. (Note that $\text{Att}(x_1) \supset \text{Att}(x_3) \supset \text{Att}(x_5)$ and $\text{Att}(x_2) \supset \text{Att}(x_4)$.) Then the argument we gave in Example 4.2 applies verbatim to establish that the Tversky model with $(A, v, \text{Att})$ is not compatible with experimentation. ◊

*Rationalization of a Tversky Model* Let $\mathbb{P}$ be a Tversky model with $(A, v, \text{Att})$. By Proposition 5.3, this model is induced by lack of strict preference. Using Claim 1 of the proof of Proposition 5.3, we can compute the preference relation $\succsim$ on $X$ that rationalizes $\mathbb{P}$ as such in the following way: $x \succ y$ if and only if $\text{Att}(y) \subset \text{Att}(x)$, and $x \sim y$ if and only if for every $z \in X$, $\text{Att}(x)$ and $\text{Att}(z)$ are not properly nested whenever $\text{Att}(y)$ and $\text{Att}(z)$ are not properly nested. When $\mathbb{P}$ is induced by indifference, this characterization simplifies. In that case, $x \succ y$ if and only if $\text{Att}(y) \subset \text{Att}(x)$, and $x \sim y$ if and only if $\text{Att}(x)$ and $\text{Att}(y)$ are not properly nested.

### 5.3 *Additive random utility models*

*ARUMs* Let $v$ be an injection from $X$ into $\mathbb{R}$, which is interpreted as the part of the utility function of the individual that is known to the modeler. (As usual, the injectivity of $v$ is imposed to remove potential ties in the ranking of the alternatives.) The actual utility function of the individual, however, may differ from $v$, but the modeler knows this discrepancy only up to a probability. To model this situation, let us take a nonempty finite subset $I$ of real numbers such that $v(x) + s \neq v(y) + t$ for every $s, t \in I$ and every distinct $x, y \in X$. In turn, let $\{\varepsilon_x\}_{x \in X}$ be a collection of $I$-valued random variables (on some fixed probability space) such that $\text{Prob}\{\varepsilon_x = t\} > 0$ for every $t \in I$ and $x \in X$. (As usual, we think of $\varepsilon_x$ as the modeler's "error" of the identification of the agent's utility scale of the alternative $x$.) Our choice of the discrete interval $I$ guarantees that $v + s$ is a real-valued injection on $X$, and thus ranks the alternatives in $X$ strictly, for every map $s : X \to I$. Consequently, for each $s \in I^X$, there is a unique bijection $u_s$ from $X$ onto $\{1, \ldots, |X|\}$ such that the rankings of the alternatives in $X$ by the maps $u_s$ and $v + s$ are the same. Finally, put $\mathcal{U} := \{u_s : s \in I^X\}$, and let $\mu$ be the probability measure on $2^{\mathcal{U}}$ with $\mu(u_s) := \text{Prob}\{\varepsilon_x = s(x) \text{ for each } x \in X\}$. We call the RUM with this $(\mathcal{U}, \mu)$ as the **additive random utility model** (ARUM) with $(v, \{\varepsilon_x\}_{x \in X})$. (The origins of this model go back to the path-breaking work of Thurstone (1945) in psychophysics, and hence, it is often referred to as the **Thurstone model** in the psychology literature.) Where $\mathbb{P}$ stands for this model, and $S$ is any feasible menu, we have

$$\mathbb{P}(x, S) = \text{Prob}\Big\{v(x) + \varepsilon_x = \max_{z \in S}\big(v(z) + \varepsilon_z\big)\Big\}$$

if $x \in S$, and $\mathbb{P}(x, S) := 0$ otherwise. Thus, the likelihood of choosing $x$ in $S$ is the probability that the (random) utility of $x$ is the largest among all achievable (random) utilities in $S$.[5]

ARUMs are the workhorses of discrete choice theory, and are often studied under special assumptions on the distributions of the $\varepsilon_x$s. And since they are built around the random perturbations of a complete preference relation (represented by a utility function), one might expect that they may correspond to the behavior of an agent who never chooses items that are dominated relative to an incomplete preference relation. Indeed, even without invoking any distributional assumption, we can use Theorem 5.1 to show that any ARUM is induced by lack of strict preference.

Proposition 5.5. *Every ARUM is induced by lack of strict preference.*

Proof. Let $v$, $I$, and $\{\varepsilon_x\}_{x \in X}$ be as specified above, and let $\mathbb{P}$ stand for the ARUM with $(v, \{\varepsilon_x\}_{x \in X})$. To show (8) for $\mathcal{U} := \{u_s : s \in I^X\}$, take any $S \in \mathfrak{X}$ and $x \in S$, and suppose that for every $s \in I^X$, there is an alternative $y_s \in S \backslash \{x\}$ such that $v(y_s) + s(y_s) > v(x) +$

[5]We should emphasize that this model departs slightly from the usual formulation of ARUMs in that it assumes that the space of shocks is the same for each alternative and that every combination of shocks has positive probability.

$s(x)$. Now consider the map $t : X \to I$ defined by $t(z) := \max I$ if $z = x$, and $t(z) := \min I$ otherwise. Clearly, for every $s \in I^X$,

$$v(y_t) + s(y_t) \geq v(y_t) + \min I = v(y_t) + t(y_t) > v(x) + t(x) = v(x) + \max I \geq v(x) + s(x).$$

This yields (8) for $\mathcal{U} := \{u_s : s \in I^X\}$, so, by Theorem 5.1, $\mathbb{P}$ is induced by lack of strict preference. $\square$

Example 5.4. An ARUM need not be induced by indifference. To see this, let $X := \{x_1, x_2, x_3\}$, define $v : X \to \mathbb{R}$ by $v(x_i) := i$, and let $\varepsilon_{x_i}$, $i = 1, 2, 3$, be independently and uniformly distributed random variables with values $-0.7$ and $0.7$. Then, according to the associated ARUM $\mathbb{P}$, we have $\mathbb{P}(x_1, \{x_1, x_2\}) > 0$ and $\mathbb{P}(x_2, \{x_2, x_3\}) > 0$, but $\mathbb{P}(x_1, \{x_1, x_3\}) = 0$. It follows that $\mathbb{P}$ fails the Stochastic $\beta$-Axiom, so it cannot be induced by indifference. $\lozenge$

*ARUMs with i.i.d. Errors* Consider the ARUM with $(v, \{\varepsilon_x\}_{x \in X})$, where $v : X \to \mathbb{R}$ is an injection, and for each $x \in X$, $\varepsilon_x$ is an $I$-valued random variable on a probability space with $\text{Prob}\{\varepsilon_x = t\} > 0$ for every $t \in I$ and $x \in X$. Widely used instances of this model presuppose that $\varepsilon_x$s are identically and independently distributed. (In this case, we refer to the model as an **ARUM with i.i.d. errors**.) Proposition 5.5 does not require this assumption. But, interestingly, the i.i.d. hypothesis buys compatibility with experimentation.

Proposition 5.6. *Every ARUM with i.i.d. errors is compatible with experimentation.*

Proof. Let $v$, $I$, and $\{\varepsilon_x\}_{x \in X}$ be as specified above, with $\varepsilon_x$s being i.i.d., and let $\mathbb{P}$ stand for the ARUM with $(v, \{\varepsilon_x\}_{x \in X})$. It is well known that $\mathbb{P}$ is weakly stochastically transitive. Let us show that $\mathbb{P}$ satisfies Ordinal IIA as well. Take any $S \in \mathfrak{X}$ and $x, y \in S$, and assume $v(x) \geq v(y)$. In what follows, we denote the common distribution of $\varepsilon_z$s by $\mathbf{p}$ (which is a probability measure on $I$ with full support). Note that

$$\begin{aligned}
\mathbb{P}(x, S) &= \text{Prob}\big\{v(z) + \varepsilon_z < v(x) + \varepsilon_x \text{ for each } z \in S \backslash \{x\}\big\} \\
&= \sum_{s \in I} \prod_{z \in S \backslash \{x\}} \text{Prob}\big\{\varepsilon_z < v(x) - v(z) + s\big\} \\
&= \sum_{s \in I} \prod_{z \in S \backslash \{x,y\}} \mathbf{p}\big\{t < v(x) - v(z) + s\big\}\, \mathbf{p}\big\{t < v(x) - v(y) + s\big\} \\
&\geq \sum_{s \in I} \prod_{z \in S \backslash \{x,y\}} \mathbf{p}\big\{t < v(y) - v(z) + s\big\}\, \mathbf{p}\big\{t < v(y) - v(x) + s\big\} \\
&= \sum_{s \in I} \prod_{z \in S \backslash \{y\}} \text{Prob}\big\{\varepsilon_z < v(y) - v(z) + s\big\} \\
&= \mathbb{P}(y, S),
\end{aligned}$$

where we use independence to get the second and fourth equalities, and the identical distribution hypothesis, along with $v(x) \geq v(y)$, to get the inequality. Conclusion: $v(x) \geq$

$v(y)$ implies $\mathbb{P}(x, S) \geq \mathbb{P}(y, S)$ for every $S \in \mathfrak{X}$ and $x, y \in S$. Next, notice that the argument above can be modified in the obvious way to show that $v(x) > v(y)$ implies $\mathbb{P}(x, S) > \mathbb{P}(y, S)$ for every $S \in \mathfrak{X}$ and $x, y \in S$, provided that $\mathbb{P}(x, S) > 0$. We have thus proved that $v(x) \geq v(y)$ if and only if $\mathbb{P}(x, S) \geq \mathbb{P}(y, S)$ for every $S \in \mathfrak{X}$ and $x, y \in S$ with $\mathbb{P}(x, S) > 0$. It follows from this observation that $\mathbb{P}$ satisfies Ordinal IIA, and applying Corollary 3.6 completes the proof. $\square$

An ARUM with i.i.d. errors (e.g., the Luce model, independent probit, the general Luce model with $(u, \Gamma)$ where $\Gamma$ satisfies WARP) is thus quite special in that it can be "rationalized" in a variety of ways. It is not only consistent with the choice behavior of a person who maximizes a preference relation (albeit an incomplete one), and randomizes only over the maximal elements in feasible menus, but it is also compatible with experimentation. As such there are multiple welfare orderings that one may utilize in the context of such models.

Example 5.5. Proposition 5.6 is not valid when $\varepsilon_x$s are merely independently distributed. To see this, let $X := \{x_1, x_2, x_3\}$ and define $v : X \to \mathbb{R}$ by $v(x_i) := i$. Consider the following stochastic matrix:

$$[p_{ij}] := \begin{bmatrix} 0.1 & 0.1 & 0.7 & 0.1 \\ 0.1 & 0.5 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.1 & 0.6 \end{bmatrix}.$$

Now put $I := \{t_1, \ldots, t_4\}$, where $t_1 = 10$, $t_2 = 5$, $t_3 = 0$, and $t_4 = -5$, and let $\varepsilon_{x_i}$, $i = 1, 2, 3$, be independent $I$-valued random variables such that $\mathrm{Prob}\{\varepsilon_{x_i} = t_j\} = p_{ij}$ for each $i$ and $j$. Then, where $\mathbb{P}$ is the ARUM with $(v, \{\varepsilon_{x_i}\}_{i=1,2,3})$, one calculates that $\mathbb{P}(x_1, \{x_1, x_3\}) > \frac{1}{2}$, $\mathbb{P}(x_3, \{x_2, x_3\}) > \frac{1}{2}$, while $\mathbb{P}(x_2, X) > \mathbb{P}(x_3, X) > \mathbb{P}(x_1, X) > 0$. To derive a contradiction, suppose there is a complete preorder $\succsim$ on $X$ relative to which $\mathbb{P}$ is compatible with experimentation. Then $x_1 \sim x_3 \sim x_2$ by the second condition of compatibility by experimentation. And yet $\mathbb{P}(x_1, \{x_1, x_2\}) = 0.34 < \frac{1}{2}$, so $x_1 \sim x_2$ cannot hold by the third condition of compatibility by experimentation. This contradicts transitivity of $\succsim$. $\Diamond$

The main results of this section are summarized in Figure 2.

*Rationalization of ARUMs* Let $\mathbb{P}$ be the ARUM with $(v, \{\varepsilon_x\}_{x \in X})$. Proposition 5.5 says that $\mathbb{P}$ is induced by lack of strict preference. The unique preference relation $\succsim$ on $X$ with respect to which this happens is precisely the Luce threshold preorder $\succsim_{v,\delta}$ on $X$ where $\delta := \max I - \min I$. So, just as in the case of any threshold Luce model, in any ARUM as well, we arrive at a method of making deterministic (and unambiguous) welfare judgements in the form of a Luce threshold preorder.

Proposition 5.6 shows that if $\{\varepsilon_x\}_{x \in X}$ is i.i.d., then $\mathbb{P}$ is compatible with experimentation. As this model satisfies (5) for all $S \in \mathfrak{X}$ and $x, y \in S$, there is a unique complete preference relation $\succsim$ on $X$ that "rationalizes" this conclusion; this relation is given simply as $x \succsim y$ if and only if $v(x) \geq v(y)$. We note that this relation is distinct from the Luce threshold preorder $\succsim_{v,\delta}$. We have $\succ_{v,\delta} \subseteq \succ$ and $\sim \subseteq \sim_{v,\delta}$, but these containments may well hold strictly.

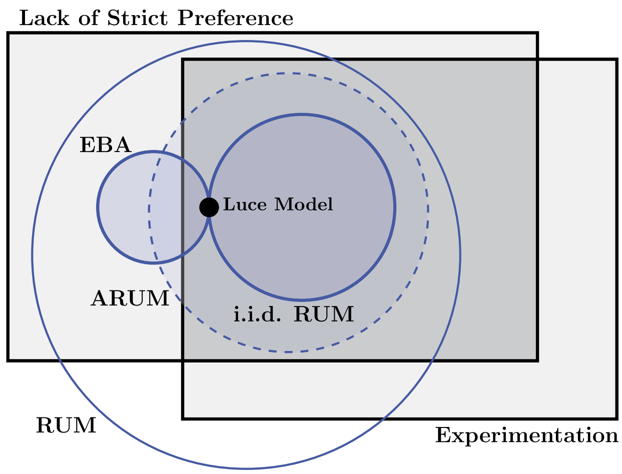


FIGURE 2. RUM, lack of strict preference, and experimentation.

## 6. ORIGINS OF ADDITIVE PERTURBED UTILITY MODELS

In this section, for any $S \in \mathfrak{X}$, we denote the set of all probability measures on $X$ whose supports are contained in $S$ by $\Delta_S(X)$, but write $\Delta(X)$ for $\Delta_X(X)$.

*The Additive Perturbed Utility Model* Following Fudenberg, Iijima, and Strzalecki (2015), we say that a function $c : [0, 1] \to (-\infty, \infty]$ is a **weak cost function** if it is strictly convex and $c|_{(0,1)}$ is a continuously differentiable real function. If, in addition, $c'_+(0) = -\infty$, we say that $c$ is a **cost function**.

In the remaining part of this section, $u$ stands for an arbitrarily given real map on $X$, and $c$ a weak cost function. For any $p \in \Delta(X)$ and $S \in \mathfrak{X}$, and any $u : X \to \mathbb{R}$, we write $\mathbb{E}(u, p)$ for $\sum_{z\in S} p(z)u(z)$ and $\mathbb{C}_S(p)$ for $\sum_{z\in S} c(p(z))$. (For $p \in \Delta_S(X)$, we interpret $\mathbb{C}_S(p)$ as the total cost of randomizing according to $p$ in $S$.) A stochastic choice function $\mathbb{P}$ on $X$ is called a **(weak) additive perturbed utility** (APU) model with $(u, c)$ where $u$ is a real map on $X$, $c$ is a (weak) cost function, and

$$\{\mathbb{P}(\cdot, S)\} = \underset{p\in\Delta_S(X)}{\arg\max} \mathbb{E}(u, p) - \mathbb{C}_S(p) \quad \text{for every } S \in \mathfrak{X}.$$

This model was introduced to the literature by Fudenberg, Iijima, and Strzalecki (2015). It belongs to the family of *deliberately stochastic choice* models which was studied by Machina (1985) and Cerreia-Vioglio et al. (2019), among others. We note that $c'_+(0) = -\infty$ implies that any APU model is positive, and hence, it is, trivially, induced by indifference. As the compatibility with experimentation properties of APU and weak APU models coincide, we will thus work only with weak APU models below. The following observation describes the binary choice probabilities in this context.

LEMMA 6.1. *Let $\mathbb{P}$ be the weak APU model with $(u, c)$. Then, for any $x, y \in X$,*

$$\mathbb{P}(x, \{x, y\}) = 1 \quad \textit{if and only if} \quad u(x) - u(y) \geq c'_-(1) - c'_+(0), \tag{11}$$

*while $\mathbb{P}(x, \{x, y\}) \geq \frac{1}{2}$ if and only if $u(x) \geq u(y)$.*

Let $\mathbb{P}$ be a weak APU model with $(u, c)$, and assume that it is induced by lack of strict preference. Then Lemma 6.1 tells us that the (unique) preorder $\succsim$ that rationalizes $\mathbb{P}$ as such is simply the Luce threshold preorder $\succsim_{u,\delta}$ where $\delta := c'_-(1) - c'_+(0)$. (If either $c'_-(1) = \infty$ or $c'_+(0) = -\infty$, this preorder equals $X \times X$ (declaring all options in $X$ indifferent), which is only natural as $\mathbb{P}$ is then positive.)

Taking stock, we find that the preference relation that "rationalizes" a stochastic choice function $\mathbb{P}$ on $X$ as induced by lack of strict preference is invariably a Luce threshold preorder as long as $\mathbb{P}$ is a threshold Luce model, an ARUM, or a weak APU model (that happens to be induced as such). This points to a somewhat unexpected commonality between these models which appear quite distinct from each other.

*Lack of Strict Preference and the APU Model* The applicability of $\succsim_{u,\delta}$ as an individual welfare ordering depends on whether or not the weak APU model at hand is induced by lack of strict preference. The following example illustrates the fact that a weak APU may exhibit all possible cases in terms of lack of strict preference.

Example 6.1. For any positive real number $\eta$, define the real map $c_\eta$ on $[0, 1]$ by $c_\eta(p) := \eta p^2$, and let $\mathbb{P}_\eta$ stand for the weak APU model with $(u, c_\eta)$. (This specification was adopted by Rosenthal (1989).) Then $\mathbb{P}_\eta$ is induced by lack of strict preference if and only if

$$\sum_{\substack{y \in X \\ 0 < u(y) - u(x) < 2\eta}} \big(u(y) - u(x)\big) < 2\eta \quad \text{for every } x \in X.$$

For instance, let $X := \{x, y, z\}$ and $(u(x), u(y), u(z)) := (3, 2, 1)$. Then the above characterization shows that $\mathbb{P}_\eta$ is induced by lack of strict preference when $\frac{1}{2} < \eta \leq 1$ and not induced as such when $1 < \eta \leq \frac{3}{2}$. One can also check that $\mathbb{P}_\eta$ is induced by indifference for all $\eta$ outside the interval $(\frac{1}{2}, \frac{3}{2}]$. ◊

In the Appendix, Proposition A.2 provides a characterization of weak APU models that are induced by lack of strict preference. The claims in Example 6.1 directly follow from that result.

*Experimentation and the APU Model* Compatibility of weak APU models with experimentation is striking.

Proposition 6.2. *Every weak APU model is compatible with experimentation.*

Not only is any weak APU model $\mathbb{P}$ compatible with experimentation, but it also satisfies (5) for every $S \in \mathfrak{X}$ and every $x, y \in S$. Thus, the complete preorder $\succsim$ on $X$ that rationalizes $\mathbb{P}$ as such is uniquely defined as $x \succsim y$ if and only if $\mathbb{P}(x, \{x, y\}) \geq \frac{1}{2}$ (Section 3.4). In view of Lemma 6.1, this preorder is precisely the one that is represented by $u$. Thus, an individual whose choice behavior is modeled by means of a weak APU model $\mathbb{P}$ can be thought of as one who maximizes $u$ in the sense that the most frequently chosen alternatives by this person in any menu $S$ are among the "best" elements in $S$

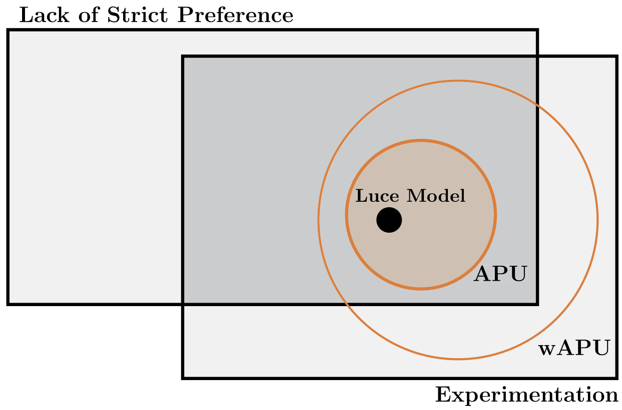


Figure 3. Weak APU, lack of strict preference, and experimentation.

with respect to $\succsim$. In fact, in this instance, we have $\arg\max_{z\in S} \mathbb{P}(z, S) = \max(S, \succsim)$, that is,

$$\arg\max_{z\in S} \mathbb{P}(z, S) = \arg\max_{z\in S} u(z).$$

In other words, the most frequently chosen alternatives in any menu correspond precisely to the deterministic choices of a utility-maximizing decision maker. Alternatively, this person also gives a chance to some alternatives that are not best in $S$ relative to the utility function $u$. That is, she "experiments" now and then, but the lower the utility ranking of an item in a menu $S$, the less is this chance.

The main results of this section are summarized in Figure 3.

## Appendix

This Appendix contains the proofs of the results that were omitted in Section 6 and a characterization of weak APU models that are induced by lack of strict preference.

Lemma A.1. *Let $\mathbb{P}$ be the weak APU model with $(u, c)$. For any $S \in \mathfrak{X}$ and $x, y \in S$,*

$$u(x) \geq u(y) \quad \textit{implies} \quad \mathbb{P}(x, S) \geq \mathbb{P}(y, S),$$

*and if $\mathbb{P}(x, S) > 0$,*

$$u(x) > u(y) \quad \textit{implies} \quad \mathbb{P}(x, S) > \mathbb{P}(y, S).$$

Proof. Take any $S \in \mathfrak{X}$ and $x, y \in S$ with $u(x) \geq u(y)$. Let $p := \mathbb{P}(\cdot, S)$ and note that, by definition,

$$\mathbb{E}(u, p) - \mathbb{C}_S(p) \geq \mathbb{E}(u, q) - \mathbb{C}_S(q) \quad \text{for every } q \in \Delta_S(X). \tag{12}$$

Assume first that $p(x) = 0$. We wish to show that $p(y) = 0$. To derive a contradiction, suppose $p(y) > 0$. Put $t := p(y)/2$, and define $q \in \Delta_S(X)$ by $q(z) := t$ if $z \in \{x, y\}$, and

$q(z) := p(z)$ otherwise. Then (12) entails

$$p(y)u(y) - c(0) - c\big(p(y)\big) \geq tu(x) + tu(y) - 2c(t).$$

Since $c(t)$ is a real number, this inequality cannot hold when $c(0) = \infty$. If $c(0) < \infty$, then

$$0 \geq u(y) - u(x) \geq \frac{c(2t) - c(t)}{t} - \frac{c(t) - c(0)}{t},$$

but this contradicts the strict convexity of $c$.

Let us now consider the case where $p(x) > 0$. To avoid the trivial cases, we assume $1 > p(x)$ and $p(y) > 0$. Take any $\varepsilon$ in the interval $(0, \min\{p(x), 1 - p(y)\})$, and consider the lottery $q \in \Delta_S(X)$ with $q(x) := p(x) - \varepsilon$, $q(y) := p(y) + \varepsilon$, and $q(z) := p(z)$ for every $\in X \backslash \{x, y\}$. By (12),

$$\begin{aligned}&p(x)u(x) + p(y)u(y) - c\big(p(x)\big) - c\big(p(y)\big)\\&\quad\geq \big(p(x) + \varepsilon\big)u(x) + \big(p(y) - \varepsilon\big)u(y) - c\big(p(x) + \varepsilon\big) - c\big(p(y) - \varepsilon\big),\end{aligned}$$

and hence,

$$u(y) - u(x) \geq \frac{1}{\varepsilon}\Big(c\big(p(y)\big) - c\big(p(y) - \varepsilon\big)\Big) - \frac{1}{\varepsilon}\Big(c\big(p(x) + \varepsilon\big) - c\big(p(x)\big)\Big).$$

Since $c$ is differentiable on $(0, 1)$, letting $\varepsilon \downarrow 0$ then yields $u(y) - u(x) \geq c'(p(y)) - c'(p(x))$. Since $c$ is strictly convex, $c'$ is strictly increasing, and hence, $u(x) \geq u(y)$ implies $p(x) \geq p(y)$ and $u(x) > u(y)$ implies $p(x) > p(y)$. □

Proof of Lemma 6.1. By convexity of $c$, we have

$$c'_{-}(1) \geq \frac{c(1) - c(\lambda)}{1 - \lambda} \quad \text{and} \quad c'_{+}(0) \leq \frac{c(0) - c(1 - \lambda)}{1 - \lambda}$$

for every $\lambda \in [0, 1)$. Consequently, for any $x, y \in X$, the right-hand side of (11) implies that

$$u(x) - u(y) \geq \frac{c(1) - c(\lambda) - c(0) + c(1 - \lambda)}{1 - \lambda}, \tag{13}$$

which means

$$u(x) - c(1) - c(0) \geq \lambda u(x) + (1 - \lambda)u(y) - c(\lambda) - c(1 - \lambda)$$

for every $\lambda \in [0, 1)$, that is, $\delta_x$ is a maximizer of the map $p \mapsto \mathbb{E}(u, p) - \mathbb{C}_{\{x,y\}}(p)$ on $p \in \Delta_{\{x,y\}}(X)$. Since there is a unique such maximizer, we conclude that $\mathbb{P}(x, \{x, y\}) = 1$. Conversely, $\mathbb{P}(x, \{x, y\}) = 1$ implies (13), and letting $\lambda \uparrow 1$ yields the right-hand side of (11). The second assertion of Lemma 6.1 follows from applying Lemma A.1 to doubleton sets.

Our next result provides a characterization of weak APU models that are induced by lack of strict preference.

Proposition A.2. *Let $\mathbb{P}$ be the weak APU model with $(u, c)$. Then $\mathbb{P}$ is induced by lack of strict preference if, and only if, for every $x \in X$,*

$$\sum_{y \in \mathcal{A}(x)} (c')^{-1}\big(u(y) - u(x) + c'_+(0)\big) < 1,$$

*where $\mathcal{A}(x) := \{y \in X : 0 < u(y) - u(x) < c'_-(1) - c'_+(0)\}$.*[6]

Proof of Proposition A.2. It is readily checked that every weak APU model satisfies the Stochastic Chernoff Axiom. Thus, by Theorem 3.4, $\mathbb{P}$ is induced by lack of strict preference if and only if it satisfies the Stochastic Condorcet Axiom. Consequently, it is enough to prove that $\mathbb{P}$ satisfies the Stochastic Condorcet Axiom if and only if

$$\sum_{y \in \mathcal{A}(x)} (c')^{-1}\big(u(y) - u(x) + c'_+(0)\big) < 1 \tag{14}$$

for every $x \in X$.

Before we begin the demonstration of this, let us note that (14) is a well-defined statement. Indeed, $c'$ is a strictly increasing and continuous function on $[0, 1]$ by hypothesis, and hence, it is invertible on the set $c'([0, 1])$. Consequently, $(c')^{-1}$ is a strictly increasing function on the interval $[c'_+(0), c'_-(1)]$ with $(c')^{-1}(c'_+(0)) = 0$. But for any $x \in X$ and $y \in \mathcal{A}(x)$, we have $c'_-(1) > u(y) - u(x) + c'_+(0) > c'_+(0)$, so $u(y) - u(x) + c'_+(0)$ belongs to the domain of $(c')^{-1}$. Moreover, since $(c')^{-1}$ is strictly increasing, we have

$$(c')^{-1}\big(u(y) - u(x) + c'_+(0)\big) > 0 \tag{15}$$

for any $x \in X$ and $y \in \mathcal{A}(x)$.

Now, assume first that $\mathbb{P}$ satisfies the Stochastic Condorcet Axiom, and take any $x \in X$. By Lemma 6.1, $\mathbb{P}(x, \{x, y\}) > 0$ for every $y \in \mathcal{A}(x)$, so, by the Stochastic Condorcet Axiom, $\mathbb{P}(x, \mathcal{A}(x) \cup x) > 0$. Enumerate $\mathcal{A}(x)$ as $\{y_1, \dots, y_m\}$, and put $p_i^* := \mathbb{P}(y_i, \mathcal{A}(x) \cup x)$ for each $i = 1, \dots, m$, and let $p_{m+1}^* := \mathbb{P}(x, \mathcal{A}(x) \cup x)$. Since all these numbers are positive, we see that $(p_1^*, \dots, p_{m+1}^*)$ is an interior solution to the smooth optimization problem

$$\text{Maximize } \sum_{i=1}^{m} p_i u(y_i) + p_{m+1} u(x) - \sum_{i=1}^{m} c(p_i) - c(p_{m+1})$$

such that $(p_1, \dots, p_{m+1})$ is a probability distribution. It follows that $u(y_i) - c'(p_i^*) = 0$, $i = 1, \dots, m$, and $u(x) - c'(p_{m+1}^*) = 0$, whence

$$u(y_i) - u(x) - c'\big(p_i^*\big) + c'\big(p_{m+1}^*\big) = 0 \quad \text{for each } i = 1, \dots, m.$$

Since $c'$ is strictly increasing, therefore, $c'(p_i^*) = u(y_i) - u(x) + c'(p_{m+1}^*) > u(y_i) - u(x) + c'_+(0)$, whence

$$p_i^* > (c')^{-1}\big(u(y_i) - u(x) + c'_+(0)\big) \quad \text{for each } i = 1, \dots, m.$$

[6]By convention, summation over the empty set is understood as 0.

Summing these inequalities verifies (14):

$$1 > 1 - p^*_{m+1} = \sum_{i=1}^{m} p^*_i > \sum_{i=1}^{m} (c')^{-1}\big(u(y_i) - u(x) + c'_+(0)\big).$$

Conversely, assume now that (14) holds for every $x \in X$. Take any $S \in \mathfrak{X}$ and $x \in S$ such that $\mathbb{P}(x, \{x, y\}) > 0$ for every $y \in S\backslash\{x\}$ with $\mathbb{P}(y, S) > 0$. (We wish to show that $\mathbb{P}(x, S) > 0$.)

Claim 1. *For every* $y \in S\backslash\{x\}$, $\mathbb{P}(x, \{x, y\}) > 0$.

Proof. Take any $z \in X$ with $\mathbb{P}(z, S) = 0$, and to derive a contradiction, suppose $\mathbb{P}(z, \{x, z\}) = 1$. Then, by Lemma 6.1, $u(z) - u(x) \geq c'_-(1) - c'_+(0)$. Pick any $y \in S$ with $\mathbb{P}(y, S) > 0$. Then, by the choice of $x$, $\mathbb{P}(x, \{x, y\}) > 0$, so, by Lemma 6.1, $u(y) - u(x) < c'_-(1) - c'_+(0)$. Moreover, by Lemma A.1, we have $u(y) > u(z)$. Thus, $c'_-(1) - c'_+(0) > u(y) - u(x) > u(z) - u(x) \geq c'_-(1) - c'_+(0)$, a contradiction. □

By Claim 1 and Lemma 6.1, we have $u(y) - u(x) < c'_-(1) - c'_+(0)$ for every $y \in S\backslash\{x\}$. So, by (15),

$$\begin{aligned} 1 &> \sum_{y \in \mathcal{A}(x)} (c')^{-1}\big(u(y) - u(x) + c'_+(0)\big) \\ &\geq \sum_{y \in \mathcal{A}(x) \cap S} (c')^{-1}\big(u(y) - u(x) + c'_+(0)\big) = \sum_{\substack{y \in S \\ u(y) > u(x)}} (c')^{-1}\big(u(y) - u(x) + c'_+(0)\big). \end{aligned}$$

To derive a contradiction, we suppose $\mathbb{P}(x, S) = 0$. Let us enumerate $S$ as $\{y_1, \dots, y_k, x, z_1, \dots, z_l\}$ such that $u(y_1) \geq \cdots \geq u(y_k) \geq u(x) \geq u(z_1) \geq \cdots \geq u(z_l)$. Note that, by Lemma A.1, we have $\mathbb{P}(z_i, S) = 0$ for each $i = 1, \dots, l$. Now put $p^*_i := \mathbb{P}(y_i, S)$, $i = 1, \dots, k$. Clearly, $(p^*_1, \dots, p^*_k)$ is a solution to the smooth optimization problem

$$\text{Maximize } \sum_{i=1}^{k} p_i u(y_i) + \left(1 - \sum_{i=1}^{k} p_i\right) u(x) - \sum_{i=1}^{k} c(p_i) - c\left(1 - \sum_{i=1}^{k} p_i\right)$$

such that $p_1, \dots, p_k \geq 0$ and $p_1 + \cdots + p_k \leq 1$. So, by the Karush–Kuhn–Tucker theorem, there exist a nonnegative $(k+1)$-vector $(\mu_1, \dots, \mu_k, \lambda)$ such that

$$u(y_i) - u(x) - c'\big(p^*_i\big) + c'\left(1 - \sum_{i=1}^{k} p^*_i\right) + \mu_i - \lambda = 0 \quad \text{and} \quad \mu_i p^*_i = 0$$

for each $i = 1, \dots, k$, and $\lambda(1 - p^*_1 - \cdots - p^*_k) = 0$. Moreover, by Lemma A.1, we have $p^*_1 \geq \cdots \geq p^*_k$. Here not all $p^*_i$s may vanish, for otherwise $\mathbb{P}(x, S) = 1$, a contradiction. Define $k^* := \max\{i \in \{1, \dots, k\} : p^*_i > 0\}$, and note that $1 < k^* \leq k$. We also know that $p^*_1 + \cdots + p^*_k = 1$ (because $\mathbb{P}(x, S) = 0$), so the equations above yield $c'(p^*_i) = u(y_i) -$

$u(x) + c'_+(0) - \lambda = 0$ for each $i = 1, \ldots, k^*$. But then, since $(c')^{-1}$ is a strictly increasing function,

$$1 = \sum_{i=1}^{k} p_i^* = \sum_{i=1}^{k^*} p_i^* = \sum_{i=1}^{k^*} (c')^{-1}\big(u(y_i) - u(x) + c'_+(0) - \lambda\big) \le \sum_{i=1}^{k^*} (c')^{-1}\big(u(y_i) - u(x) + c'_+(0)\big),$$

that is, $1 \ge \sum_{y \in \mathcal{A}(x)} (c')^{-1}(u(y) - u(x) + c'_+(0))$, contradicting (14). Thus, $\mathbb{P}(x, S) > 0$, as we sought. Given the arbitrary choice of $x$ and $S$, we conclude that $\mathbb{P}$ satisfies the Stochastic Condorcet Axiom. $\square$

Proof of Proposition 6.2. Let $\mathbb{P}$ be the weak APU model induced by $(u, c)$. It is well known that $\mathbb{P}$ is weakly stochastically transitive. We now show that it also satisfies Ordinal IIA. To this end, take any $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $x, y \in S$, and suppose $\mathbb{P}(x, T) \ge \mathbb{P}(y, T)$ and $\mathbb{P}(x, T) > 0$. If $\mathbb{P}(y, T) = 0$, we then have $\mathbb{P}(x, T) > \mathbb{P}(y, T)$, so Lemma A.1 entails that $u(x) > u(y)$. If $\mathbb{P}(y, T) > 0$, then $u(x) \ge u(y)$ must hold, because, otherwise, Lemma A.1 would entail $\mathbb{P}(y, T) > \mathbb{P}(x, T)$. Thus, in any contingency, $u(x) \ge u(y)$, and applying Lemma A.1 one more time yields $\mathbb{P}(x, S) \ge \mathbb{P}(y, S)$. Conclusion: $\mathbb{P}$ satisfies Ordinal IIA. We may thus apply Corollary 3.6 to complete the proof. $\square$

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